We sometimes should be able to express a transcendental function in a form of a product of infinitely many linear polynomials which become 0 at each zero. For examples, if we want to derive a set of eigenstates of 1-dim quantum harmonic oscillator using Feynmann path integral (Das. Lectures on quantum mechanics. p.515), or want to deduce a partition function of a system consists of N ideal spin-0 bosons through Euclidean path integral (Brown. Quantum field theory. p.96), expressing some trigonometric functions in forms of products of infinite linear polynomials.

0. Motivation

From the "Fundamental Theorem of Algebra", we know the fact that

"Any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

of degree n where each a_n is nonzere has at least one zero."
(Can be deriven from Liouville's theorem)

Using the mathematical induction, we may generalize our previous statement to

"Any polynomial $P(z)$ of degree n can be expressed as a product of n linear factors :

$$P(z) = c(z - z_1)(z - z_2)\cdots(z - z_n)$$

where c and z_k are complex constants."
(Actually, we can directly derive this statement from Rouche's theorem.)

Then we may want to ask a question :
"Can any entire function, say $f(z)$, be factorized as

$$f(z) = c\prod_{i=0}^{\infty}(z - z_i)$$

where c is a complex constant and $\{z_i\}$ is a set of all zeros of $f(z)$?"

In order to answer on this question, we need to discuss the convergence of the above infinite product.

This notion is the first step toward the Weierstrass Factorization Theorem !

1. Proof

Definition. Weierstrass elementary factor
Let n be a positive integer. An elementary factor E_n (which is a function) is defined as equations below.

$$E_0(z) = 1 - z$$
$$E_n(z) = (1 - z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^n}{n}\right) \qquad , n \geq 1$$

Observe that the function $E_n(z/a)$ has an unique zero of order 1 in which $z = a$.

Theorem 1
Let $|z| \leq 1$ and $p \geq 0$. Then $|E_n(z) - 1| \leq |z|^{n+1}$.

proof. It is trivial for $n = 0$ since $|E_n(z) - 1| = |(1 - z) - 1| = |z|$. Hence, let us focus our attention on positive n. Consider a Laurent series expansion of $E_n(z)$ about $z = 0$. Since $E_n(z)$ is non-singular at $z = 0$ (bcs it is entire),

$$E_n(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$$

By differentiating $E_n(z)$ with respect to z we obtain

$$E_n'(z) = \sum_{k=1}^{\infty} k\, a_k\, z^{k-1} = -z^n \exp\left(z + \cdots + \frac{z^n}{n}\right)$$

The second equality comes from the definition of $E_n(z)$. By comparing the two expressions of $E_n'(z)$ we obtain $a_1 = a_2 = \cdots = a_n = 0$ since the lowest order of $E_n'(z)$ is 1. Furthermore, since the coefficients of the expantions of $\exp(z + \cdots + z^p/p)$, $a_k \leq 0$ for $k \geq n+1$.
Observe that

$$E_n(1) = 0 = 1 + \sum_{k=n+1}^{\infty} a_k = -\sum_{k=n+1}^{\infty}|a_k| \qquad \longrightarrow \qquad \sum_{k=n+1}^{\infty}|a_k| = 1$$

Hence, for the region in which $|z| \leq 1$,

$$|E_n(z) - 1| = \left|\sum_{k=n+1}^{\infty} a_k z^k\right| = |z|^{n+1}\left|\sum_{k=n+1}^{\infty} a_k z^{n-p-1}\right| \leq |z|^{n+1}\sum_{k=n+1}^{\infty}|a_n| = |z|^{n+1}$$

and this is what we wanted. ■

Theorem 2
Let $\{a_n\}$ be a complex number sequence such that $\lim_{n\to\infty}|a_n| = \infty$ and anyone is nonzero.
Let $\{\xi_n\}$ be any sequence of non-negative integers for which

$$\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{\xi_n+1} \qquad (2)$$

converges for all $r > 0$.

Then

$$f(z) = \prod_{n=1}^{\infty} E_{\xi_n}\left(\frac{z}{a_n}\right)$$

converges uniformly absolutely and entire with zeros only at points a_n.

proof. Suppose that there are integers ξ_n such that eqn(1) is satisfied. Then, according to the previous theorem,

$$\left|1 - E_{\xi_n}\left(\frac{z}{a_n}\right)\right| \leq \left|\frac{z}{a_n}\right|^{\xi_n+1} \leq \left(\frac{r}{|a_n|}\right)^{\xi_n+1} \qquad (3)$$

whenever $|z| \leq r$ and $r/|a_n| \leq 1$.
Consider an infinite sum which is defined by

$$\sum_{n=1}^{\infty}\left[1 - E_{\xi_n}\left(\frac{z}{a_n}\right)\right].$$

From the just previous inequality(2) and our hypothesis(3), the infinite sum above is convergent uniformly according to the Weierstrass M-test.

WILL BE FILLED SOON ■

Before proceeding the present proof, however, we need a slight red herring into another topic.

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Weierstrass Factorization Theorem
Let f be an entire function.
Let $\{a_n\}$ be the set of all non-zero zeros of f repeated according to multiplicity.
Let f has a zero at $z = 0$ of order $m \geq 0$.
Then, there is an entire function f and a sequance of integers $\{\xi_n\}$ such that

$$f(z) = z^m e^{g(z)}\prod_{n=1}^{\infty} E_{\xi_n}\left(\frac{z}{a_n}\right).$$

proof. ALSO WILL BE FILLED SOON...... ■

2. Applications

Trigonometric Functions

blahblah

Gamma Function

blahblah

References

- John B. Conway, Functions of One Complex Variable II, Springer