Quantum field theoretical approaches to two archetypes of second order phase transition : Bose-Einstein condensation and 2-dim Ising model

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Abstract

It is well known that most of second order phase transitions chiefly manifest spontaneous symmetry breakdowns. In this paper, so as to explore these transitions, we cover a basic formalism of non-relativistic quantum field theory and apply it not only to delve into a system of ideal Bose gas as well as the well-known phonon-roton-maxon excitation spectrum of bosonic superfluid ^{4}He, but also to obtain the exact critical exponent of 2-dim Ising model "$\frac{1}{8}$" analytically.

1. Introduction

Albeit second order phase transitions such as the ferromagnetic - paramagnetic phase transition and the superfluid - liquid phase transition have enormously significant rules and implications in physics such as universality and spontaneous symmetry breakdown, it hasn't been taught to undergraduate students with sophisticated explanations due to its difficulty. Hence. the ultimate goal of this paper is to make undergraduate students to be apprehended these phenomena in a language of non-relativistic quantum field theory and let them to apply this theory to their researches.

From section 2 to 3, we cover non-relativistic quantum field theory (in a nutshell level) referring Brown's quantum field theory textbook and basic properties of superfluidity such as two-fluid behaviour and curl-free property. In section C, we delve into gaseous systems of which bosons are ideal, of which bosons exhibit a pair potential, and excitation spectrum of 4He, the well-known phonon-maxon-roton spectrum. In section D, lastly, we cover the way to solve 2-dim Ising model in an analytical manner applying Majorana fermions and Wigner-Jordan transform.

2. Non-relativistic quantum fiend theory

Let $|\Psi\rangle$ denote a system consists of N identical particles. A wave function of this system is written as

$$\Psi(\mathbf{r_1}, \mathbf{r_2}, \cdots, \mathbf{r_N}) = \langle \mathbf{r_1}, \mathbf{r_2}, \cdots, \mathbf{r_N} | \Psi \rangle \ . \tag{2.1}$$

in which each $\mathbf{r_i}$ can denote not only a position of i-th particle but also its spin or other physical variables. For a system consists of bosonic particles, the wave function above should be symmetric under the particle interchange.

$$\Psi(\mathbf{r_1}, \ldots, \mathbf{r_i}, \ldots, \mathbf{r_j}, \ldots, \mathbf{r_N}) = \Psi(\mathbf{r_1}, \ldots, \mathbf{r_j}, \ldots, \mathbf{r_i}, \ldots, \mathbf{r_N}) \tag{2.2}$$

Meanwhile, for a system consists of fermionic particles, the wave function should be antisymmetric under the particle interchange.

$$\Psi(\mathbf{r_1}, \ldots, \mathbf{r_i}, \ldots, \mathbf{r_j}, \ldots, \mathbf{r_N}) = -\Psi(\mathbf{r_1}, \ldots, \mathbf{r_j}, \ldots, \mathbf{r_i}, \ldots, \mathbf{r_N}) \tag{2.3}$$

From now on, upper signs will be used for bosonic systems and lower signs will be used for fermionic systems, respectively. From (2.1), (2.2), and (2.3), we may obtain

$$\langle \mathbf{r_1}, \ldots, \mathbf{r_i}, \ldots, \mathbf{r_j}, \ldots, \mathbf{r_N}| = \pm \langle \mathbf{r_1}, \ldots, \mathbf{r_j}, \ldots, \mathbf{r_i}, \ldots, \mathbf{r_N}|$$

and the normalisation condition is given by

$$\langle \mathbf{r_1}, \ldots, \mathbf{r_N} | \mathbf{r'_1}, \ldots, \mathbf{r'_N} \rangle \quad = \quad \pm \sum_{\text{all permutations}} \left[\prod_{a=1}^{N} \delta(\mathbf{r_a} - \mathbf{r'_{P(a)}}) \right]$$

where each permutation iterated by the sum sends a to P(a). It is, however, not that efficient to utilise this eigenbasis. Instead, consider a collection of states

$$\Big\{ \, |\mathbf{0}\rangle \, , \, |\mathbf{r}\rangle \, , \, |\mathbf{r_1}, \mathbf{r_2}\rangle \, , \, \cdots \, , \, |\mathbf{r'_1}, \ldots, \mathbf{r'_n}\rangle \, , \cdots \Big\}$$

where $|\mathbf{0}\rangle$ denotes a vacuum state in which possesses no particle. Focusing on this eigenbasis (of all the various states with a fixed number of particle) enables us to settle down in grand canonical ensemble, meanwhile pursuing our formalism using the previous basis implies we are in canonical ensemble. In this enlarged system, the inner product is defined by

$$\langle \mathbf{r_1}, \ldots, \mathbf{r_M} | \mathbf{r'_1}, \ldots, \mathbf{r'_N} \rangle = \delta_{\mathrm{MN}} \langle \mathbf{r_1}, \ldots, \mathbf{r_M} | \mathbf{r'_1}, \ldots, \mathbf{r'_M} \rangle \ .$$

For examples,

$$\langle \mathbf{0} | \mathbf{0} \rangle = 1 \, , \ \langle \mathbf{0} | \mathbf{r_1}, \mathbf{r_2} \rangle \ , \ \langle \mathbf{r} | \mathbf{r'} \rangle = \delta(\mathbf{r} - \mathbf{r'}) \, .$$

Consider an operator defined by

Appendix