Problem Set 3

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Exercise 1

The condition that characterizes the optimal amount of cake to eat in Period 1 is:

$$W_2 \in [0, W_1] \quad c_1 \in [0, W_1]$$

The optimization problem can be described as:

$$\max_{(c_1)} = U(c_1) = U(W_1 - W_2) \quad s.t. c_1 \in [0, W_1]$$

The solution to the period 1 problem is $c_1 = W_1$ and svae nothing for the next period

Exercise 2

The condition that characterizes the optimal amount of cake to eat in Period 2 is:

$$W_3 \in [0, W_2] \quad c_2 \in [0, W_2]$$

With the same logic, the consumer will left nothing for the period 3, thus $c_2 = W_2$ and $W_3 = 0$ The optimization problem can be described as:

$$\max_{W_2} = U(c_1) + \beta U(c_2) = U(W_1 - W_2) + \beta U(W_2)$$

The solution to the period 1 problem is $c_2 = W_2$ and save nothing for the next period

Exercise 3

The condition that characterizes the optimal amount of cake to eat in Period 3 is:

$$W_4 \in [0, W_3] \quad c_3 \in [0, W_3]$$

With the same logic, the consumer will left nothing for the period 3, thus $c_3 = W_3$ and $W_4 = 0$ The optimization problem can be described as:

$$\max_{(W_2, W_3)} = U(W_1 - W_2) + \beta U(W_2 - W_3) + \beta^2 U((W_3)$$

Sine we have $W_1 = 1$ and $\beta = 0.9$, taking the first order condition and we get:

The solution to the period 1 problem is $c_2 = W_2$ and save nothing for the next period

Solving the problem, we have $W_1 = 1, W_2 = 0.66, W_3 = 0.31, W_4 = 0$. The whole cake left is decreasing, the consumption first increases then decreases as time goes by.

Exercise 4

As the analysis above, we have $W_{T+1} = \Phi_T(W_T) = min\{W_T\} = 0$ where T is the last period and this W_T can optimize the utility at period T. So the individual entering the last period with utility given by the size of cake W_T is equal to the utility given by eating the whole left cake W_T.

$$V_T(W_T) = u(W_T)$$

Back to the period $T-1$, we have the optimizing problem, which is:

$$V_{T-1}(W_{T-1}) = \max_{W_T, W_{T-1}} U(W_{T-1} - W_T) + \beta U(W_T - W_{T-1})$$

According to the envelope theorem, we can simplify the equation into :

$$V_{T-1}(W_{T-1}) = \max_{W_T, W_{T-1}} U(W_{T-1} - W_T) + \beta V_T(W_T)$$

Since we also have $W_{T+1} = \Phi_T(W_T)$, the above equation can be written as:

$$V_{T-1}(W_{T-1}) = \max_{W_T, W_{T-1}} U(W_{T-1} - \Phi_{T-1}(W_{T-1})) + \beta V_T(\Phi_{T-1}(W_{T-1}))$$

Exercise 5

According to the equation above, we have :

$$\begin{aligned} V_T(W_T) &= u(W_T) = ln(W_T) \\ V_{T-1}(W_{T-1}) &= \max_{W_T, W_{T-1}} U(W_{T-1} - W_T) + \beta V_T(W_T) \\ &= \max_{W_T, W_{T-1}} ln(W_{T-1} - W_T) + \beta V_T(W_T) \end{aligned}$$

Since we have $W_{T+1} = \Phi_T(W_T) = min\{W_T\} = 0$, we know that in the last period of his life T, he would leave nothing to consume, which means:

$$\phi_T(\bar{W}) = W_{T+1} = 0$$

In the period $T-1$, $\phi_{T-1}(\bar{W}) = W_T$ is the amount he wants to consume in the last period, in the optimizing problem, we have:

$$V_{T-1}(\bar{W}) = \max_{W_T} ln(\bar{W} - W_T) + \beta ln(W_T)$$

Taking the first order condition, we have $\beta \bar{W} = (\beta + 1) W_T$, which gives us $\phi_{T-1}(\bar{W}) = \frac{\beta}{\beta+1}\bar{W}$. Hence, $\phi_{T-1}(\bar{W})$ is not the same as $\phi_T(\bar{W})$

Exercise 6

The Bellman equation is:

$$V_{T-2}(W_{T-2}) = \max_{W_T, W_{T-1}} ln(W_{T-2} - W_{T-1}) + \beta ln(W_{T-1} - W_T) + \beta^2 ln(W_T)$$
$$s.t. W_{T-1} \in [0, W_{T-2}], W_T \in [0, W_{T-1}]$$

Taking the first order condition with the respect to W_{T-1}, we are able to get:

$$\frac{\beta}{W_{T-1}} + \frac{\beta^2}{W_{T-1}} = \frac{1}{W_{T-2} - W_{T-1}}$$
$$W_{T-1} = \phi_{T-2}(W_{T-2}) = \frac{\beta + \beta^2}{1 + \beta + \beta^2} W_{T-2}$$

We may plug the equation above into the value function:

$$V_{T-2}(W_{T-2}) = \max_{W_T, W_{T-1}} ln(\frac{W_{T-2}}{1 + \beta + \beta^2}) + \beta ln(\frac{\beta W_{T-2}}{1 + \beta + \beta^2}) + \beta^2 ln(\frac{\beta^2 W_{T-2}}{1 + \beta + \beta^2})$$

Exercise 7

As the results from the previous exercise, we may induce the general form of policy function and value function,

$$\phi_{T-s}(W_{T-s}) = \frac{\beta + \beta^2 \cdots + \beta^s}{1 + \beta + \beta^2 \cdots + \beta^s} W_{T-s}$$
$$V_{T-s}(W_{T-s}) = \max_{W_T, W_{T-1}} ln(\frac{W_{T-s}}{1 + \beta + \beta^2 \cdots + \beta^s}) + \beta ln(\frac{\beta W_{T-2}}{1 + \beta + \beta^2 \cdots + \beta^s})$$
$$+ \beta^2 ln(\frac{\beta^2 W_{T-2}}{1 + \beta + \beta^2 \cdots + \beta^s}) + \cdots + \beta^s ln(\frac{\beta^s W_{T-2}}{1 + \beta + \beta^2 \cdots + \beta^s})$$

Exercise 8

In the infinite horizon, we have Bellman equation as follows:

$$V(W) = \max_{W' \in [0, W]} u(W - W') + \beta V(W')$$