7.1 Newton流体的本构关系

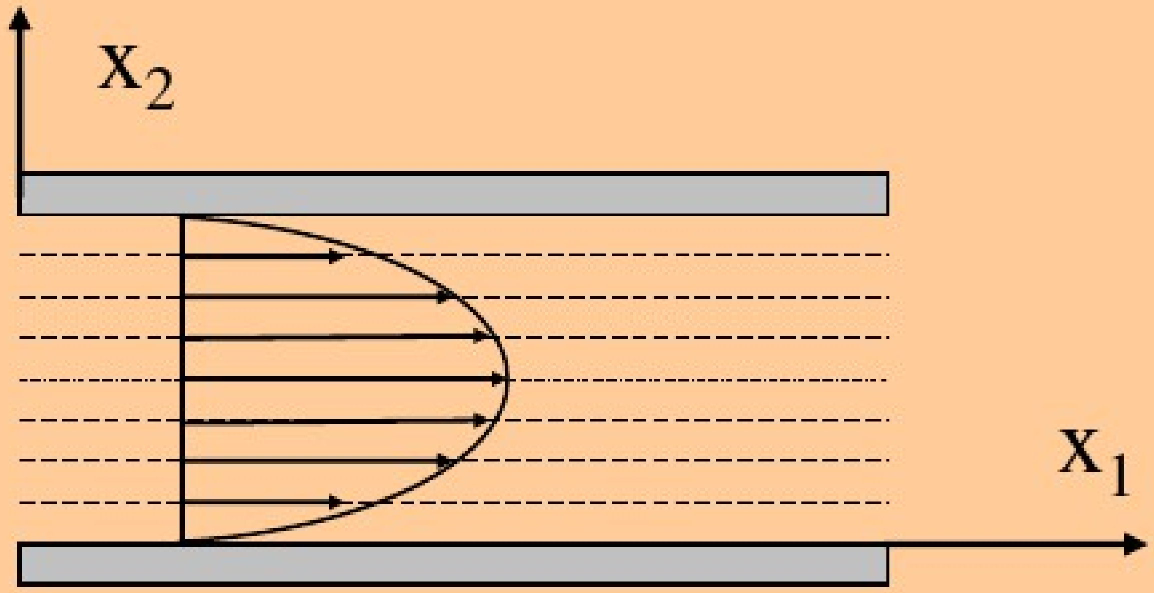
$$\boldsymbol{T} = -\pi\boldsymbol{I} + \lambda(\rho,\theta)\boldsymbol{I}\ \mathrm{tr}\boldsymbol{D} + 2\mu(\rho,\theta)\boldsymbol{D}$$

$$\mathbf{D} = \frac{1}{3}\mathbf{I}\ \mathrm{tr}\mathbf{D} + \mathbf{D}'$$

$$\boldsymbol{T} = -\pi\boldsymbol{I} + 2\mu\left(\boldsymbol{D} - \frac{1}{3}\boldsymbol{I}\ \mathrm{tr}\boldsymbol{D}\right) + \left(\lambda + \frac{2}{3}\mu\right)\boldsymbol{I}\ \mathrm{tr}\boldsymbol{D}$$

$$= \underbrace{-\pi\mathbf{I}}_{\text{可逆应力}\mathbf{T}^{E}} + \underbrace{2\mu\left(\mathbf{D} - \frac{1}{3}\mathbf{I}\ \mathrm{tr}\mathbf{D}\right) + \mu'\mathbf{I}\ \mathrm{tr}\mathbf{D}}_{\text{不可逆应力}\mathbf{T}^{D}}$$

$$\boldsymbol{T} = -\pi\boldsymbol{I} + 2\mu\left(\boldsymbol{D} - \frac{1}{3}\boldsymbol{I}\ \mathrm{tr}\boldsymbol{D}\right) + \mu'\boldsymbol{I}\ \mathrm{tr}\boldsymbol{D}$$



$$\boldsymbol{v} = v_1(x_2)\boldsymbol{e}_1$$

$$\mathbf{D} = \begin{bmatrix} 0 & \mathrm{d}v_1/(2\mathrm{d}x_2) & 0 \\ \mathrm{d}v_1/(2\mathrm{d}x_2) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{tr}\boldsymbol{D} = 0$$

$$\mathbf{T} = \begin{bmatrix} 0 & \mu\mathrm{d}v_1/\mathrm{d}x_2 & 0 \\ \mu\mathrm{d}v_1/\mathrm{d}x_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\tau_{12} = \mu\mathrm{d}\boldsymbol{v}_1/\mathrm{d}\boldsymbol{x}_2$$

μ ：动力学粘性系数

流动克服粘性而引起的能量耗散：

$$\mathbf{T}^{D}=2\mu\left(\mathbf{D}-\frac{1}{3}\mathbf{I}\ \mathrm{tr}\mathbf{D}\right)+\mu'\mathbf{I}\ \mathrm{tr}\mathbf{D}$$

两端取迹: $$\mu'=\left(\frac{1}{3}\mathrm{tr}\boldsymbol{T}^{D}\right)\Big/\mathrm{tr}\boldsymbol{D}$$

增长单位体积率所需要克服的粘性力，称之为**膨胀粘性系数**，也称第二粘性系数

$$\rho\tau\chi = \boldsymbol{T}^{D} : \boldsymbol{D} \geq 0$$

$$\boldsymbol{T}^{D} : \boldsymbol{D} = \lambda(\mathrm{tr}\boldsymbol{D})^{2} + 2\mu\boldsymbol{D} : \boldsymbol{D}$$

$$\mathbf{D}:\mathbf{D} = \left(\frac{1}{3}\mathbf{I}\ \mathrm{tr}\mathbf{D} + \mathbf{D}'\right):\left(\frac{1}{3}\mathbf{I}\ \mathrm{tr}\mathbf{D} + \mathbf{D}'\right)$$

$$= (\frac{1}{3}\mathrm{tr}\mathbf{D})^{2}\mathbf{I}:\mathbf{I} + \frac{2}{3}\mathrm{tr}\mathbf{D}\ \mathbf{I}:\mathbf{D}' + \mathbf{D}':\mathbf{D}'$$

$$= \frac{1}{3}(\mathrm{tr}\mathbf{D})^{2} + \mathbf{D}':\mathbf{D}'$$

$$\boldsymbol{T}^{D} : \boldsymbol{D} = \frac{1}{3}(3\lambda + 2\mu)(\mathrm{tr}\boldsymbol{D})^{2} + 2\mu\boldsymbol{D}' : \boldsymbol{D}' \geq 0$$

$$\mu > 0 \qquad \mu' = \lambda + \frac{2}{3}\mu > 0$$

$$\because \quad \mu' \ll \mu、\ \pi \longrightarrow \mu' = 0$$

假设平均法向应力与形变率无关，满足这一假设的流体称为**Stokes流体**

$$\boldsymbol{T} = -\pi\boldsymbol{I} + 2\mu\left(\boldsymbol{D} - \frac{1}{3}\boldsymbol{I}\ \mathrm{tr}\boldsymbol{D}\right)$$

$\mathrm{tr}\boldsymbol{D} = 0$ **不可压缩流**

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu\boldsymbol{D}$$

$\mu = 0$ **无粘流，理想流体**

$$\boldsymbol{T} = -\pi\boldsymbol{I}$$

7.2 Newton流体的场方程

$$\left[-\pi\boldsymbol{I}+2\mu\left(\boldsymbol{D}-\frac{1}{3}\boldsymbol{I}\ \mathrm{tr}\boldsymbol{D}\right)+\mu'\boldsymbol{I}\ \mathrm{tr}\boldsymbol{D}\right]\cdot\nabla+\rho\boldsymbol{b}=\rho\dot{\boldsymbol{v}}$$

$$\left[-\pi\boldsymbol{I}+\mu(\boldsymbol{v}\nabla+\nabla\boldsymbol{v})-\frac{2}{3}\mu\boldsymbol{I}\ \nabla\cdot\boldsymbol{v}+\mu'\boldsymbol{I}\ \nabla\cdot\boldsymbol{v}\right]\cdot\nabla+\rho\boldsymbol{b}=\rho\dot{\boldsymbol{v}}$$

$$(\boldsymbol{I}\nabla\cdot\boldsymbol{v})\cdot\nabla=\nabla\nabla\cdot\boldsymbol{v} \qquad (-\pi\boldsymbol{I})\cdot\nabla=-\pi\nabla=-\nabla\pi$$

$$(\nabla\boldsymbol{v})\cdot\nabla=\nabla\nabla\cdot\boldsymbol{v} \qquad (\boldsymbol{v}\nabla)\cdot\nabla=\nabla^2\boldsymbol{v}$$

$$-\nabla\pi+\mu\left(\nabla^2\boldsymbol{v}+\frac{1}{3}\nabla\nabla\cdot\boldsymbol{v}\right)+\mu'\nabla\nabla\cdot\boldsymbol{v}+\rho\boldsymbol{b}=\rho\dot{\boldsymbol{v}}$$

$$-\nabla\pi+\mu\left(\nabla^2\boldsymbol{v}+\frac{1}{3}\nabla\nabla\cdot\boldsymbol{v}\right)+\mu'\nabla\nabla\cdot\boldsymbol{v}+\rho\boldsymbol{b}=\rho\left(\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right)$$

① 运动方程

$$-\nabla\pi+\mu\left(\nabla^{2}\boldsymbol{v}+\frac{1}{3}\nabla\nabla\cdot\boldsymbol{v}\right)+\mu'\nabla\nabla\cdot\boldsymbol{v}+\rho\boldsymbol{b}=\rho\dot{\boldsymbol{v}}$$

② 能量方程

$$k\nabla^{2}\theta+\rho\varsigma+\left[2\mu\boldsymbol{D}:\boldsymbol{D}+\left(\mu'-\frac{2}{3}\mu\right)(\mathrm{tr}\boldsymbol{D})^{2}-\tau\frac{\partial\pi}{\partial\theta}\mathrm{tr}\boldsymbol{D}\right]=\rho c_{V}\dot{\theta}$$

③ 连续性方程

$$\dot{\rho}+\rho\nabla\cdot\boldsymbol{v}=0$$

④ 状态方程

$$\pi=\pi(\rho,\theta)$$

Navier-Stokes（NS）方程

$$-\nabla p+\mu\nabla^{2}\boldsymbol{v}+\rho\boldsymbol{b}=\rho\dot{\boldsymbol{v}}$$

$$k\nabla^{2}\theta+\rho\varsigma+2\mu\boldsymbol{D}:\boldsymbol{D}=\rho c_{V}\dot{\theta}$$

$$\nabla\cdot\boldsymbol{v}=0$$

① 无穷远处边界条件

$$\boldsymbol{v}\big|_{\infty}=\boldsymbol{v}_{\infty} \qquad p\big|_{\infty}=p_{\infty} \qquad \theta\big|_{\infty}=\theta_{\infty} \qquad \rho\big|_{\infty}=\rho_{\infty}$$

② 两种介质界面处的边界条件（**无滑移条件** ）

$$v_n^{(1)}=v_n^{(2)} \qquad v_t^{(1)}=v_t^{(2)}$$

固壁边界： $\boldsymbol{v}\big|_{Wall}=0$