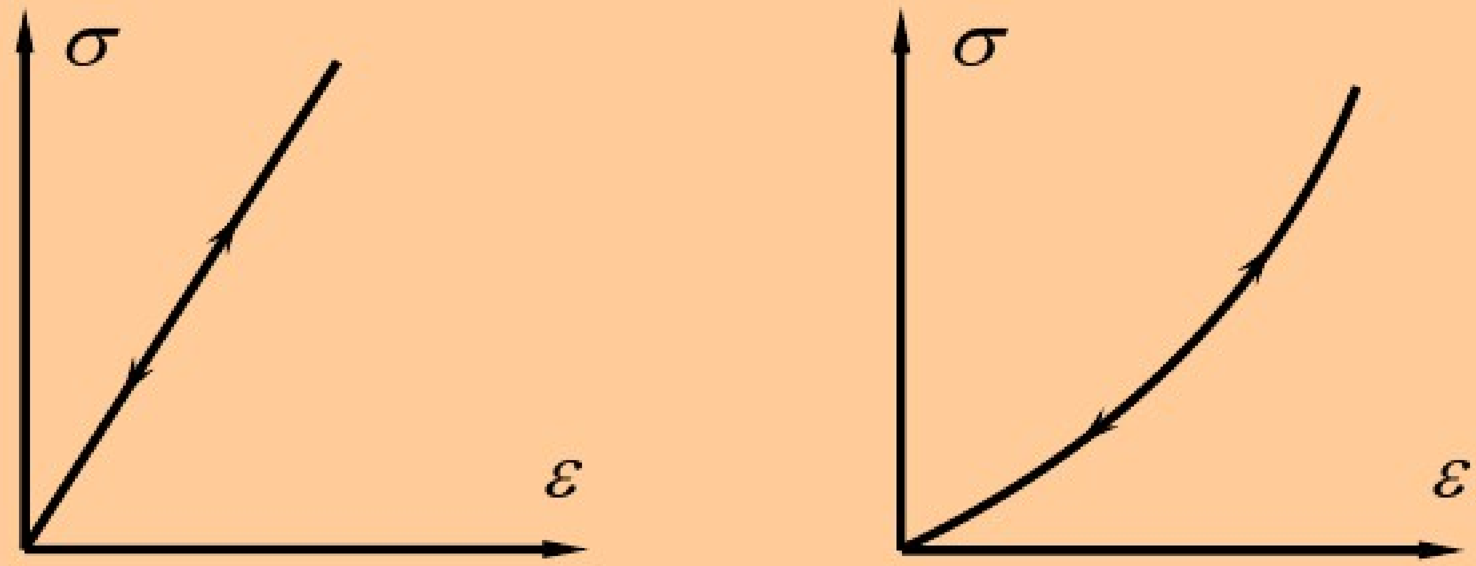
6 弹性固体



(a) 线性弹性　　(b) 非线性弹性

加载曲线和卸载曲线是重合的

6.1 线弹性体的本构关系

等温条件下各向同性弹性体的广义虎克（Hooke）定律

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{31} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{1-\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{1-\nu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{1-\nu} \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_{11} \\ \boldsymbol{E}_{22} \\ \boldsymbol{E}_{33} \\ \boldsymbol{E}_{12} \\ \boldsymbol{E}_{23} \\ \boldsymbol{E}_{31} \end{bmatrix}$$

写成工程应变的形式

$$
\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{31} \end{bmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{bmatrix}
$$

$$
\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}
$$

令: $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$ $\mu = \dfrac{E}{2(1+\nu)}$

$$
\mathbf{D}=\begin{bmatrix}
\lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\
\lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\
\lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\
0 & 0 & 0 & \mu & 0 & 0\\
0 & 0 & 0 & 0 & \mu & 0\\
0 & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}
$$

Lamé常数： $\lambda \quad \mu$

弹性矩阵：D

各向同性线弹性体： $\boldsymbol{T} = \lambda \boldsymbol{I}\ \mathrm{tr}\boldsymbol{E} + 2\mu\boldsymbol{E}$

两端取迹： $\frac{1}{3}\mathrm{tr}\boldsymbol{T} = \frac{3\lambda + 2\mu}{3}\mathrm{tr}\boldsymbol{E}$ $\mathrm{tr}\boldsymbol{E} = \frac{1-2\nu}{E}\mathrm{tr}\boldsymbol{T}$

$\mathrm{tr}\boldsymbol{T}/3$：平均正应力

$\mathrm{tr}\boldsymbol{E}$ ：微元体积的相对变化量

体积弹性模量： $\boxed{K = \frac{3\lambda + 2\mu}{3}} = \frac{\boldsymbol{E}}{3(1-2\nu)}$

表示增加体积的单位变化量所需要的平均正应力

	λ, μ	E, ν	μ, ν	E, μ	K, ν
λ	λ	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{2\mu\nu}{1-2\nu}$	$\frac{\mu(E-2\mu)}{3\mu-E}$	$\frac{3K\nu}{1+\nu}$
μ	μ	$\frac{E}{2(1+\nu)}$	μ	μ	$\frac{3K(1-2\nu)}{2(1+\nu)}$
K	$\frac{1}{3}(3\lambda+2\mu)$	$\frac{E}{3(1-2\nu)}$	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$	$\frac{\mu E}{3(3\mu-E)}$	K
E	$\frac{3\lambda+2\mu}{2(\lambda+\mu)}$	E	$2\mu(1+\nu)$	E	$3K(1-2\nu)$
ν	$\frac{\lambda}{2(\lambda+\mu)}$	ν	ν	$\frac{E}{2\mu}-1$	ν

6.2 线弹性静力学问题

(1) 连续性方程 $\frac{\mathrm{D}\rho}{\mathrm{D}t}+\rho(\nabla\cdot\boldsymbol{v})=0$

$\rho_0=J\rho\approx(1+\mathrm{tr}\boldsymbol{E})\rho$

(2) 运 动 方 程 $\boldsymbol{T}\cdot\nabla+\rho\boldsymbol{b}=\rho\dot{\boldsymbol{v}}$

(3) 动量矩平衡定律 $\boldsymbol{T}=\boldsymbol{T}^{T}$

(4) 能量方程 $\rho\dot{\varepsilon}=\boldsymbol{T}:\boldsymbol{D}-\nabla\cdot\boldsymbol{q}+\rho\varsigma$

等温，自由能＝应变能

(5) 熵定理 $\rho\tau\chi=\rho\tau\dot{\eta}+\boldsymbol{T}:\boldsymbol{D}-\rho\dot{\varepsilon}-\frac{1}{\tau}\boldsymbol{q}\cdot\nabla\tau\geq 0$

可逆

① 平衡方程

$$\boldsymbol{T}\cdot\nabla+\rho_0\boldsymbol{b}=\boldsymbol{0}$$

② 本构方程

$$\boldsymbol{T}=\lambda\boldsymbol{I}\,\mathrm{tr}\boldsymbol{E}+2\mu\boldsymbol{E}$$

③ 几何方程

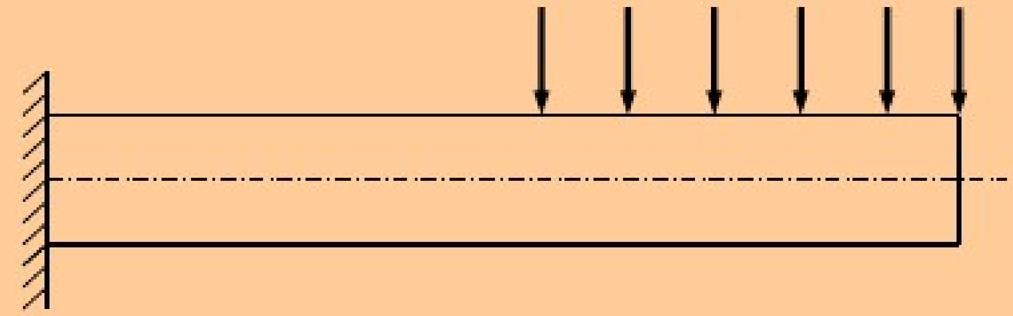
$$\boldsymbol{E}=\frac{1}{2}(\boldsymbol{u}\nabla+\nabla\boldsymbol{u})$$

① 几何边界条件

$$\boldsymbol{u}\big|_{S_1}=\overline{\boldsymbol{u}}$$

② 力学边界条件

$$\boldsymbol{T}\cdot\boldsymbol{n}\big|_{S_2}=\bar{\boldsymbol{t}}$$



$$\boldsymbol{S}_1\cup\boldsymbol{S}_2=\boldsymbol{S}$$

$$\boldsymbol{S}_1\ \mathrm{I}\ \boldsymbol{S}_2=\mathrm{O}$$

③ 几何方程 ⟶ ② 本构方程 ⟶ ① 平衡方程

$$(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho_0 b_x=0$$

$$(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho_0 b_y=0$$

$$(\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho_0 b_z=0$$

$$\mathbf{T}=\begin{bmatrix} T_{11}(x_1,x_2) & T_{12}(x_1,x_2) & 0 \\ T_{21}(x_1,x_2) & T_{22}(x_1,x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{E}=\begin{bmatrix} E_{11}(x_1,x_2) & E_{12}(x_1,x_2) & 0 \\ E_{21}(x_1,x_2) & E_{22}(x_1,x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

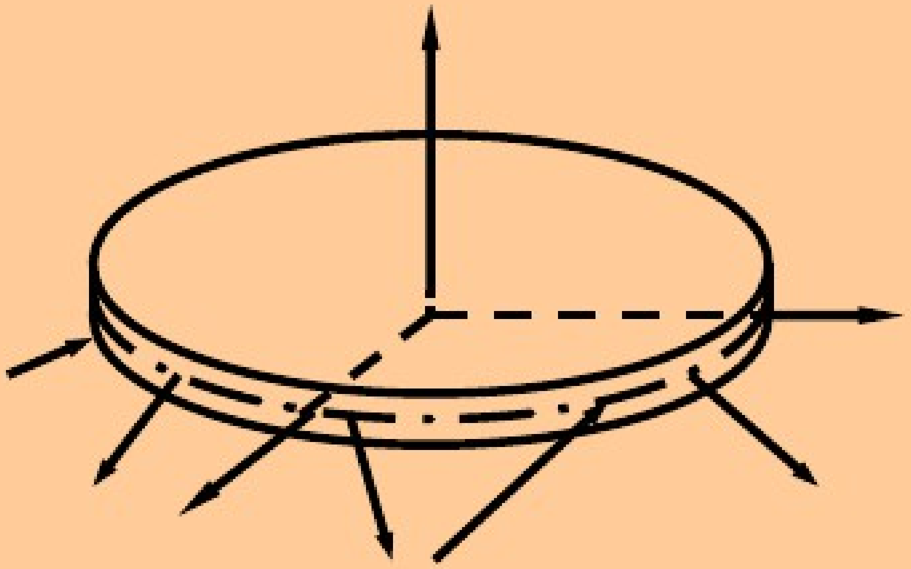


图6.4 平面应力的例子

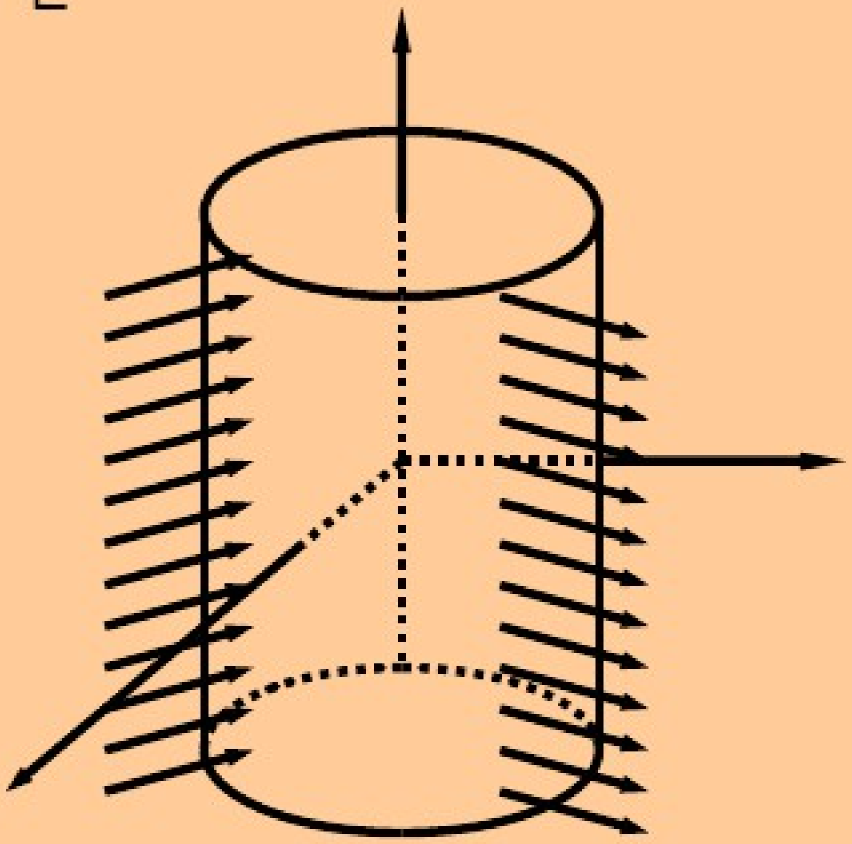


图6.5 平面应变的例子