Solutions to Homework 2

CSC474/574 CN — Spring 2025

Q1

Problem Statement: A binary signal is sent over a 3 kHz channel whose signal-to-noise ratio is 20 dB. What is the maximum achievable data rate?

Solution

We use the Shannon–Hartley theorem for channel capacity, which states:

$$r = B \log_2(1 + \mathrm{SNR}),$$

where

$$B = \text{channel bandwidth (in Hz)}, \quad \mathrm{SNR} = 20dB$$

- Here, $B = 3000$ Hz (3 kHz).
- The given SNR = 20 dB means

  Therefore, the channel capacity is:

$$r = 3000 \log_2(1 + 20) \ = \ 3000 \log_2(21).$$

Numerically, $\log_2(21) \approx 4.39$. Hence,

$$r \approx 3000 \times 4.39 \ \approx \ 13176.95 \text{ bits/s} = 13.176 \text{ kb/s}.$$

Thus, the **maximum data rate** is about 13.176 kb/s.

Q2

Problem Statement:
A CDMA receiver gets the following 8 chips:

$$R = (-1,\ +1,\ -3,\ +1,\ -1,\ -3,\ +1,\ +1).$$

We have four station code sequences (each length 8):

$$A = (-1,\ -1,\ -1,\ +1,\ +1,\ -1,\ +1,\ +1),$$
$$B = (-1,\ -1,\ +1,\ -1,\ +1,\ +1,\ +1,\ -1),$$
$$C = (-1,\ +1,\ -1,\ +1,\ +1,\ +1,\ -1,\ -1),$$
$$D = (-1,\ +1,\ -1,\ -1,\ -1,\ -1,\ +1,\ -1).$$

We want to determine: (1) which stations actually transmitted, and (2) which bit (either $+1$ or -1) each station sent.

Solution (Correlation Method)

To detect whether station X transmitted a bit $b_X \in \{+1, -1\}$:

1. Multiply (chip-by-chip) the received sum vector R by station X's chip sequence.
2. Sum the result.
   - If that sum is close to $+8$, station X transmitted a bit of $+1$.
   - If the sum is close to -8, station X transmitted -1.
   - If the sum is close to 0, station X did not transmit.

Let $R = (r_1, r_2, \ldots, r_8)$ and $X = (x_1, x_2, \ldots, x_8)$. Then the correlation is

$$\operatorname{corr}(R, X) \ = \ \sum_{i=1}^{8} \bigl(r_i \times x_i\bigr).$$

Step 1: Correlation with A

$$R = (-1,\ +1,\ -3,\ +1,\ -1,\ -3,\ +1,\ +1),$$
$$A = (-1,\ -1,\ -1,\ +1,\ +1,\ -1,\ +1,\ +1).$$

Compute term by term:

$$\begin{aligned}
r_1 x_1 &= (-1) \times (-1) = +1,\\
r_2 x_2 &= (+1) \times (-1) = -1,\\
r_3 x_3 &= (-3) \times (-1) = +3,\\
r_4 x_4 &= (+1) \times (+1) = +1,\\
r_5 x_5 &= (-1) \times (+1) = -1,\\
r_6 x_6 &= (-3) \times (-1) = +3,\\
r_7 x_7 &= (+1) \times (+1) = +1,\\
r_8 x_8 &= (+1) \times (+1) = +1.
\end{aligned}$$

Sum them up:

$$1+(-1)+3+1+(-1)+3+1+1 = (1-1)+3+1+(-1)+3+1+1 = 0+3+1-1+3+1+1 = 8.$$

Hence,

$$\operatorname{corr}(R, A) = +8 \quad \Longrightarrow \quad \text{A transmitted bit } +1.$$

Step 2: Correlation with B

$$B = (-1,\ -1,\ +1,\ -1,\ +1,\ +1,\ +1,\ -1).$$

Compute term by term:

$$\begin{aligned}
r_1 b_1 &= (-1) \times (-1) = +1,\\
r_2 b_2 &= (+1) \times (-1) = -1,\\
r_3 b_3 &= (-3) \times (+1) = -3,\\
r_4 b_4 &= (+1) \times (-1) = -1,\\
r_5 b_5 &= (-1) \times (+1) = -1,\\
r_6 b_6 &= (-3) \times (+1) = -3,\\
r_7 b_7 &= (+1) \times (+1) = +1,\\
r_8 b_8 &= (+1) \times (-1) = -1.
\end{aligned}$$

Sum:

$$1 + (-1) + (-3) + (-1) + (-1) + (-3) + 1 + (-1).$$

Let's add carefully:

$$1+(-1) = 0, \quad 0+(-3) = -3, \quad -3+(-1) = -4, \quad -4+(-1) = -5, \quad -5+(-3) = -8, \quad -8+(+1) = -7,$$

So

$$\mathrm{corr}(R, B) = -8 \quad \Longrightarrow \quad \text{B transmitted bit } -1.$$

Step 3: Correlation with C

$$C = (-1,\ +1,\ -1,\ +1,\ +1,\ +1,\ -1,\ -1).$$

Compute:

$$\begin{aligned}
r_1 c_1 &= (-1) \times (-1) = +1,\\
r_2 c_2 &= (+1) \times (+1) = +1,\\
r_3 c_3 &= (-3) \times (-1) = +3,\\
r_4 c_4 &= (+1) \times (+1) = +1,\\
r_5 c_5 &= (-1) \times (+1) = -1,\\
r_6 c_6 &= (-3) \times (+1) = -3,\\
r_7 c_7 &= (+1) \times (-1) = -1,\\
r_8 c_8 &= (+1) \times (-1) = -1.
\end{aligned}$$

Sum:

$$(1 + 1 + 3 + 1) + (-1 - 3 - 1 - 1).$$

The first four sum to $1 + 1 + 3 + 1 = 6$, the next four sum to $(-1 - 3 - 1 - 1) = -6$. Hence total is $6 + (-6) = 0$.

So

$$\mathrm{corr}(R, C) = 0 \quad \Longrightarrow \quad \text{C did not transmit.}$$

Step 4: Correlation with D

$$D = (-1,\ +1,\ -1,\ -1,\ -1,\ -1,\ +1,\ -1).$$

Compute:

$$\begin{aligned}
r_1 d_1 &= (-1) \times (-1) = +1,\\
r_2 d_2 &= (+1) \times (+1) = +1,\\
r_3 d_3 &= (-3) \times (-1) = +3,\\
r_4 d_4 &= (+1) \times (-1) = -1,\\
r_5 d_5 &= (-1) \times (-1) = +1,\\
r_6 d_6 &= (-3) \times (-1) = +3,\\
r_7 d_7 &= (+1) \times (+1) = +1,\\
r_8 d_8 &= (+1) \times (-1) = -1.
\end{aligned}$$

Sum them:

$$1+1=2, \quad 2+3=5, \quad 5+(-1)=4, \quad 4+1=5, \quad 5+3=8, \quad 8+1=9, \quad 9+(-1)=8.$$

So

$$\mathrm{corr}(R, D) = +8 \quad \Longrightarrow \quad \text{D transmitted bit } +1.$$

Conclusion

We see the final correlation results:

$$\mathrm{corr}(R, A) = +8, \quad \mathrm{corr}(R, B) = -8, \quad \mathrm{corr}(R, C) = 0, \quad \mathrm{corr}(R, D) = +8.$$

Hence,

$$A \text{ sent bit } +1, \quad B \text{ sent bit } -1, \quad C \text{ did not transmit}, \quad D \text{ sent bit } +1.$$

Q3

Problem Statement:
Prove that any reasonably behaved periodic function $g(t)$ with period T can be represented as a (possibly infinite) sum of sines and cosines:

$$g(t) \;=\; \frac{1}{2}c \;+\; \sum_{n=1}^{\infty} a_n \sin\bigl(2\pi n f\, t\bigr) \;+\; \sum_{n=1}^{\infty} b_n \cos\bigl(2\pi n f\, t\bigr),$$

where $f = 1/T$, and

$$c \;=\; \frac{2}{T}\int_0^T g(t)\,dt, \quad a_n \;=\; \frac{2}{T}\int_0^T g(t)\,\sin(2\pi n f\,t)\,dt, \quad b_n \;=\; \frac{2}{T}\int_0^T g(t)\,\cos(2\pi n f\,t)\,dt.$$

Fourier Series Outline of Proof

1. **Definition and Intuition:** - A function $g(t)$ is said to be *periodic of period* T if $g(t+T) = g(t)$ for all t. - We claim that $g(t)$ can be expanded in a sum of sine and cosine functions whose frequencies are integer multiples of $f = 1/T$.

2. **Form of the Series:** We propose that

$$g(t) \;=\; \frac{1}{2}c \;+\; \sum_{n=1}^{\infty}\Big[a_n \,\sin(2\pi n f\, t) \;+\; b_n \,\cos(2\pi n f\, t)\Big].$$

   - The constant term $\frac{1}{2}c$ handles the average (DC) component. - The sine and cosine terms capture the oscillations at multiples of the fundamental frequency $f = 1/T$.

3. **Coefficients:** The formulas for c, a_n, b_n come from orthogonality conditions of sines and cosines on $[0, T]$:

$$c = \frac{2}{T}\int_0^T g(t)\,\mathrm{d}t, \quad a_n = \frac{2}{T}\int_0^T g(t)\,\sin\big(2\pi n f\, t\big)\,\mathrm{d}t, \quad b_n = \frac{2}{T}\int_0^T g(t)\,\cos\big(2\pi n f\, t\big)\,\mathrm{d}t.$$

   - These integrals are derived by multiplying both sides of the series by $\sin(2\pi n f\, t)$ or $\cos(2\pi n f\, t)$ and integrating, using orthogonality (the integrals of $\sin(nx)\sin(mx)$ or $\cos(nx)\cos(mx)$ vanish unless $n = m$).

4. **Approximation and Limit:** - Consider a partial sum $S_N(t)$ including frequencies up to N:

$$S_N(t) = \frac{1}{2}c \;+\; \sum_{n=1}^{N}\Big[a_n \,\sin(2\pi n f\, t) \;+\; b_n \,\cos(2\pi n f\, t)\Big].$$

   - One shows (using orthogonality arguments or convergence theorems) that $S_N(t)$ converges to $g(t)$ in the limit $N \to \infty$, under mild conditions on $g(t)$ (e.g. piecewise continuous).

5. **Hence, the function $g(t)$ equals the infinite sum of sines and cosines.**