Assignment 7
MATH 515

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Assigned: November 13, 2023
Due: December 4, 2022

1 Warm-up Problems

Problem 1: Warm-up Problem 1
Axler 8C.6

Problem 2: Warm-up Problem 2
Axler 8C.11

Problem 3: Warm-up Problem 3
Axler 8D.3

Problem 4: Warm-up Problem 4
Axler 8D.7

2 Training Problems

Problem 5: Training Problem 1 10 points
What is a linear maps? How are linear maps related to matrices? Suppose that an operator T has zero as an eigenvalue. Why can't T be inverted?

Problem 6: Training Problem 2 10 points
What does it mean for a operator to be *diagonalizable*? Note, I am not asking when an operator is *diagonal*. What are some types of matrices that are diagonalizable?

Problem 7: Training Problem 3 10 points
Let T be a symmetric operator, and v_1 and v_2 two eigenvector with different eigenvalues. Prove that v_1 and v_2 are orthogonal.

Problem 8: Training Problem 4 10 points
What is the difference between a singular value, and an eigenvalue of a matrix? What kinds of matrices have eigenvalues, which singular values, which both, and which neither? Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. How do you solve

$$\operatorname*{argmin}_{x\in\mathbb{R}^n} \|Ax - b\|_2? \tag{1}$$

Problem 9: Training Problem 5 10 points
What does it mean for a matrix $A \in \mathbb{R}^{m\times n}$ to be sparse? How does your answer to the

previous question change if A is sparse?

Problem 10: Training Problem 6 10 points
Let $C([0,1]^2)$ denote the space of continuous functions $[0,1]^2 \to \mathbb{R}$. For each $x \in [0,1]^2$, let $p(x)$ parameterize a path in $[0,1]^2$. Define the operator $L\colon C([0,1]^2) \to C([0,1]^2)$ as

$$(Lf)(x) \coloneqq \int_{p(x)} f(s)ds$$

where the integral in the above is a path integral along the path $p(x)$.

Suppose that you ran across the following problem in your research. You have access to a discretization of Lf and wish to recover f. How would you try and solve it?

3 Programming Problems

In this assignment, you'll be continuing your investigation of the inverse radon transform on a quest to provide the best possible inverses. In order to get the best-quality results possible, you'll be using some code and algorithms that go way beyond the scope of this class. For this assignment, don't worry about understand every last detail of the code and algorithms. Going through every last detail could easily occupy one (or two!) more graduate classes!

The code for this assignment is several tiers more advanced than any matlab code that you've used for past assignments. In particular, you will have to use some external code[1], something called 'mex,' and some code written by me. Mex files are a specific kind of file that allows Matlab code to call c code directly. If you are using SDSU's version of Matlab, the 'mex-ing' process should go swimmingly, but if you are using Octave, or an older version of Matlab, you may have some minor problems. Don't be shy about reaching out to me if you're having trouble!

Problem 11: Programming Problem 7 10 points

1. Download and unzip the code L-BFGS-B-C.zip from D2L. Unzip the file contents.
2. Run the code called

   ```
   compile_mex.m
   ```

   in the folder L-BFGS-B-C-master/Matlab. Running this function *should* compile the c code in L-BFGS-B-C-master/src. *You do not have to understand what the c code is doing for this assignment! You just have to make sure it compiles correctly!* Running the code should generate the files

   ```
   lbfgsb_wrapper.mexa64
   lbfgsb_wrapper.mexmaxi64
   lbfgsb_wrapper.mexw64
   ```

   If your computer generates files with different extensions, do not worry. The file extension depends on your computers internal parameters.

   If you get an error that no suitable compiler is installed then install the MinGW compiler. MinGW is open source c compiler. The installation is easy, you don't even need to leave matlab. Go to Home, Add-Ons, Get Add-Ons, and searching for the MinGW compiler support. See Figure 1.

[1] Written by Stephen Becker of University of Colorado, https://amath.colorado.edu/faculty/becker/

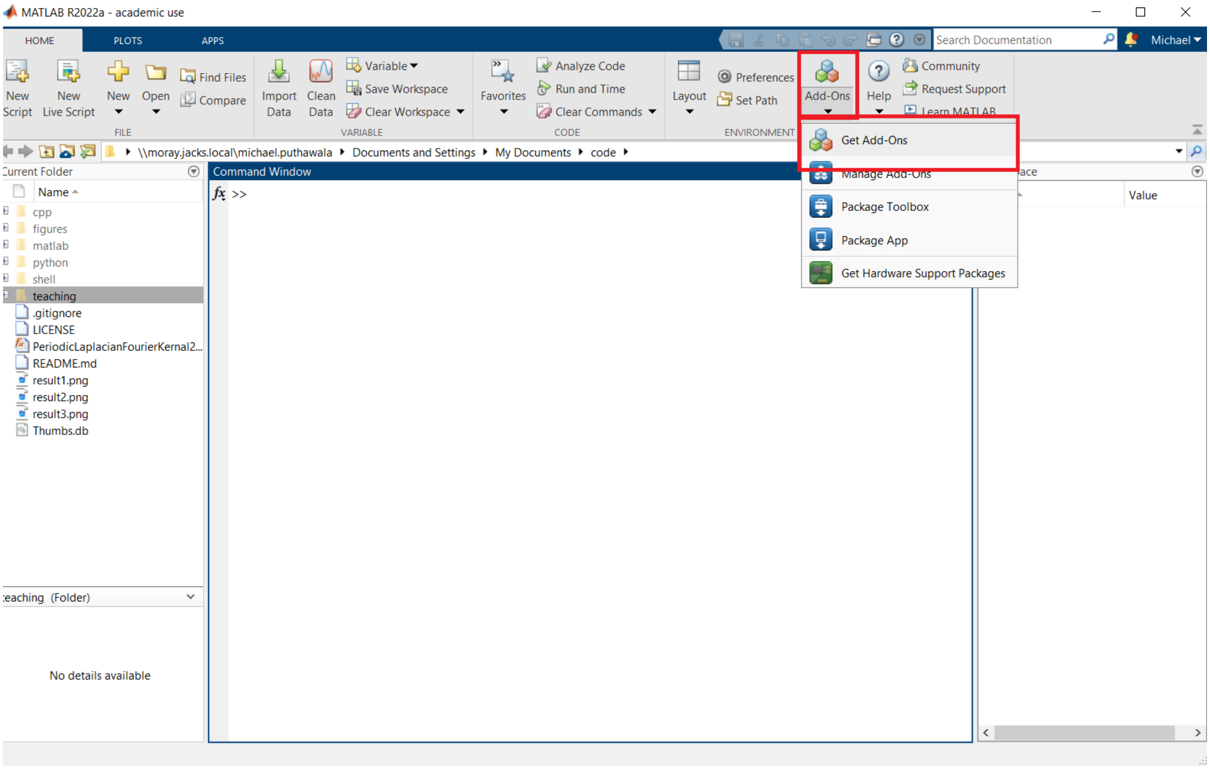

Figure 1: Button to get Add-Ons

After installing a compiler, you may get the following error:

```
Error using mex
Microsoft (R) Manifest Tool
Copyright (c) Microsoft Corporation.
All rights reserved.
mt : general error c101008d: Failed to write the updated
    manifest to the resource of file "lbfgsb_wrapper.mexw64".
     The system cannot open the device or
file specified.
Error in compile_mex (line 40)
     mex('lbfgsb_wrapper.c','-largeArrayDims','-UDEBUG',...
```

Don't worry, this error doesn't matter much. You'll still be able to do the rest of the assignment.

3. Make sure that your mex file was generated correctly by running the file called

```
example_NNLS.m
```

If all went well, you should see the following output:

```
>> example_NNLS
Iter     5, f(x) = 1.006880e+03, ||grad||_infty = 2.12e+02
Iter    10, f(x) = 1.005712e+03, ||grad||_infty = 2.13e+02
Iter    15, f(x) = 1.005707e+03, ||grad||_infty = 2.13e+02
Iter    20, f(x) = 1.005707e+03, ||grad||_infty = 2.13e+02
Iter    25, f(x) = 1.005707e+03, ||grad||_infty = 2.13e+02
t =
      0.3176
```

```
Skipping Newton method because we can't find it, or it is
    too slow
Skipping predCorr method because we can't find it, or it is
    too slow
t =
    3.6946
== Size of problem is 1500 x 1000 ==
     lbfgsb:  obj is 1005.71, min(x) is      0, err is 2.24e
        -06, time is  0.318 s
  lsqnonneg:  obj is 1005.71, min(x) is      0, err is 0.00e
     +00, time is  3.695 s
```

4. Download the code

   ```
   SolveConstrainedTikhProblem.m
   SolveConstrainedL1Problem.m
   regularized_shepp_logan_reconstruction_experiment_driver.m
   ComputeDivMatrix.m
   ```

   and the file

   ```
   shepp_logan_40x40.mat
   ```

   from D2L. Make sure that the variables flag_enable_constraint and flag_do_l1_reconstruc in the script regularized_shepp_logan_reconstruction_experiment_driver.m are **both set to false**.

   Run the script regularized_shepp_logan_reconstruction_experiment_driver.m. That script will generate a figure. Save the figure, and upload it to D2L.

Problem 12: Programming Problem 8 10 points
The emphasis of this assignment is to explore how what you've learned in this class can be used to solve an important real-world problem.

This problem has you exploring two common methods for inverting the Radon transform. The first method is called Tikhonov regularization, and is discussed further in Qualifying Problem 14. The other method is something called Total Variation (TV) or L1 regularization. Giving a full, rigorous description of TV regularization goes far beyond the scope of this class, but is a common and powerful tool[2]. Briefly, given a $M \in \mathbb{R}^{m\times n}$ and $v \in \mathbb{R}^m$ total variation looks to solve problems of the form

$$\operatorname*{argmin}_{x\in\mathbb{R}^n} \|Mx - v\|_2^2 + \lambda \|Cx\|_1 . \tag{2}$$

where $C\colon \mathbb{R}^{2n\times n}$ is a discrete approximation to the divergence operator form multi-variable calculus, and $\|\cdot\|_1$ denotes the L1 norm.

1. The variables

   ```
   tikh_lambda_0, tikh_lambda_1, tikh_lambda_2, tikh_lambda_3
   ```

   control the strength of the regularization for the solution of the Tikhonov reconstruction problem. λ_0 controls the strength of the regularization when there is no noise in the measurement, and λ_1, λ_2, and λ_3 control the strength of the regularization

[2] TV regularization was key to the reconstruction of the famous first image of a black hole published in 2019. See, e.g., https://www.ipam.ucla.edu/news/rudin-osher-fatemi-model-captures-infinity-and-beyond/

when there is noise in the measurements as specified by snr_level_1, snr_level_1 and snr_level_3 respectively.

Making λ larger will make a reconstruction that is less noisy but more blurry. Making λ smaller will reduce the blur in the reconstruction at the cost of amplifying noise in the measurement.

Running the script regularized_shepp_logan_reconstruction_experiment_driver.m will solve the Tikhonov regularization problem, make plots of the reconstructions that were computed and output the error of the reconstruction.

Experiment with the values of tikh_lambda_0, tikh_lambda_1, tikh_lambda_2, and tikh_lambda_3. What values of these parameters yield the reconstructions that visually look the best to you? Which ones produce the lowest reconstruction error? How the the SNR level change what the 'best' value of the λ s are in each case?

2. Set the value flag_enable_constraint to true. Setting this to true will force each component of the reconstruction to have a value between 0 and 2.

   Rerun the regularized_shepp_logan_reconstruction_experiment_driver.m code. Do the reconstructions look different before or after adding these constraints? Write a few sentences to paragraphs describing your findings.

3. Set the value of flag_do_tv_reconstruc.m to true. This will solve the TV regularized inverse problems (described by Eqn. 2) in addition to the Tikhonov regularization problem. Just as before the values

   ```
   tv_lambda_1 , tv_lambda_2 , tv_lambda_3
   ```

   control the strength of the regularization.

   Rerun the code again[3] compare the quality of the TV regularization and compare it visually to the Tikhoniv reconstruction. Which one looks better to you, visually? Which ones produces a lower reconstruction error?

4. Suppose that you were an applied mathematician working for Siemens, a German company that makes MRIs. The CEO of Siemens want to give doctors the best-quality MRI reconstructions possible, which require inverting the Radon transform. The CEO tasks *you* with writing a report outlining how to design an algorithm to invert the radon transform as reliably, quickly, and with the best quality reconstructions as possible.

   What would you write in your report? Would you do Tikhoniv regularization, TV regularization, or something else entirely? How much regularization would you use? How would you balance the needs to have a reliable algorithm that provides good-quality results with the need for the algorithm to run quickly?

   Write one to three paragraphs detailing your answer to the above questions. Be creative!

4 Qualifying Problems

Problem 13: Qualifying Problem 1 10 points
Let $A \in \mathbb{R}^{n\times n}$ have eigenvalues $\lambda_1, \ldots, \lambda_k$. Prove that

$$\ker(A - \lambda_i I) = \ker((A - \lambda_i I)^2) \text{ for each } i = 1, \ldots, k \tag{3}$$

[3] Note, solving the TV regularized problem will take a lot longer than solving the Tikhonov problem. This is unavoidable, as it's just a more difficult problem to solve mathematically. Don't be surprised if solving the problem one time takes your computer a 1 - 5 minutes.

if and only if A is diagonalizable.

Problem 14: Qualifying Problem 2 10 points
Let $M\colon \mathbb{R}^{m\times n}$ and $v \in \mathbb{R}^m$ be given. Let $C \in \mathbb{R}^{o\times n}$, $u \in \mathbb{R}^o$ and $\lambda \geq 0$ be given. We call the problem of solving

$$\operatorname*{argmin}_{x\in\mathbb{R}^n} \|Mx - v\|_2^2 + \lambda \|Cx - u\|_2^2 \tag{4}$$

the *Tikhonov regularization* of the $Mx = v$ problem with regularizer C, e and parameter λ.

1. Prove that the Tikhonov regularization of the $Mx = v$ problem with regularizer C, e and parameter λ is the variational form of an $Ax = b$ problem, for some matrix A and vector b. What are A and b in terms of M, C, v, u and λ?

2. Prove if $\lambda > 0$, and $\ker(M) \cap \ker(C) = \{\mathbf{0}\}$, then the the Tikhonov regularization problem has a unique solution.