Appendix

In this appendix, we give the detailed proofs of the main theorems and lemmas.

A.1 Proof of Lemma 1

Proof. We just analyze S_i for a fixed $i \in [K]$. Let the times of removing operation be J. Denote by $B = \alpha\mathcal{R}$, $\mathcal{J} = \{t_r, r \in [J]\}$, $T_r = \{t_{r-1}+1, \ldots, t_r\}$ and $t_0 = 0$. For any $t \in T_r$, if $\nabla_{t,i} \neq 0$, $\neg\mathrm{con}(a(i))$ and $b_{t,i} = 1$, then $(\boldsymbol{x}_t, y_t)$ will be added into S_i. For simplicity, we define a new notation $\nu_{t,i}$ as follows,

$$\nu_{t,i} = \mathbb{I}_{y_t f_{t,i}(\boldsymbol{x}_t)<1} \cdot \mathbb{I}_{\neg\mathrm{con}(a(i))} \cdot b_{t,i}.$$

At the end of the t_r-th round, the following equation can be derived,

$$|S_i| = |S_i(t_{r-1}+1)| + \sum_{t=t_{r-1}+1}^{t_r} \nu_{t,i} = \frac{B}{K},$$

where $|S_i(t_{r-1}+1)|$ is defined the initial size of S_i.

Let $s_r = t_{r-1}+1$. Assuming that there is no budget. We will present an expected bound on $\sum_{t=s_r}^{\bar{t}} \nu_{t,i}$ for any $\bar{t} > s_r$. In the first epoch, $s_1 = 1$ and $|S_i(s_1)| = 0$. Taking expectation w.r.t. $b_{t,i}$ gives

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=s_1}^{\bar{t}} \nu_{t,i}\right] =& \sum_{t=s_1}^{\bar{t}} \frac{\|\nabla_{t,i} - \hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i} \cdot \mathbb{I}_{\nabla_{t,i}\neq 0}}{\|\nabla_{t,i} - \hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i} + \|\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}} \\
\leq& \frac{2}{k_1} \underbrace{\left(1 + \sum_{t=2}^{\bar{t}} \left\| y_t\kappa_i(\boldsymbol{x}_t,\cdot) - \frac{\sum_{(\boldsymbol{x},y)\in V_t} y\kappa_i(\boldsymbol{x},\cdot)}{|V_t|}\right\|^2_{\mathcal{H}_i}\right)}_{\tilde{\mathcal{A}}_{[s_1,\bar{t}],\kappa_i}} \\
=& \frac{2}{k_1}\tilde{\mathcal{A}}_{[s_1,\bar{t}],\kappa_i},
\end{aligned}$$

where we use the fact $\kappa_i(\boldsymbol{x}_t, \boldsymbol{x}_t) \geq k_1$. Let t_1 be the minimal $\bar{t}$ such that

$$\frac{2}{k_1}\tilde{\mathcal{A}}_{[s_1,t_1],\kappa_i} \geq \frac{B}{K}. \tag{A1}$$

The first epoch will end at t_1 in expectation. We define $\tilde{\mathcal{A}}_{T_1,\kappa_i} := \tilde{\mathcal{A}}_{[s_1,t_1],\kappa_i}$.

Next we consider $r \geq 2$. It must be $|S_i(s_r)| = \frac{B}{2K}$. Similar to $r = 1$, we can obtain

$$\mathbb{E}\left[\sum_{t=s_r}^{\bar{t}} \nu_{t,i}\right] \leq \frac{2}{k_1} \underbrace{\sum_{t=s_r}^{\bar{t}} \left\| y_t\kappa_i(\boldsymbol{x}_t,\cdot) - \frac{\sum_{(\boldsymbol{x},y)\in V_t} y\kappa_i(\boldsymbol{x},\cdot)}{|V_t|}\right\|^2_{\mathcal{H}_i}}_{\tilde{\mathcal{A}}_{[s_r,\bar{t}],\kappa_i}} = \frac{2}{k_1}\tilde{\mathcal{A}}_{[s_r,\bar{t}],\kappa_i}.$$

Let t_r be the minimal $\bar{t}$ such that

$$\frac{2}{k_1}\tilde{\mathcal{A}}_{[s_r,\bar{t}],\kappa_i} \geq \frac{B}{2K}, \tag{A2}$$

Let $\tilde{\mathcal{A}}_{T_r,\kappa_i} = \tilde{\mathcal{A}}_{[s_r,\bar{t}],\kappa_i}$. Combining (A1) and (A2), and summing over $r = 1, \ldots, J$ yields

$$\begin{aligned}
\frac{B}{K} + \frac{B(J-1)}{2K} \leq& \frac{2}{k_1}\tilde{\mathcal{A}}_{T_1,\kappa_i} + \sum_{r=2}^{J} \frac{2}{k_1}\tilde{\mathcal{A}}_{T_r,\kappa_i} \\
\leq& \frac{2}{k_1} \underbrace{\sum_{t=s_1}^{T} \left\| y_t\kappa_i(\boldsymbol{x}_t,\cdot) - \frac{\sum_{(\boldsymbol{x},y)\in V_t} y\kappa_i(\boldsymbol{x},\cdot)}{|V_t|}\right\|^2_{\mathcal{H}_i}}_{\tilde{\mathcal{A}}_{T,\kappa_i}} \\
\leq& \frac{2}{k_1}\tilde{\mathcal{A}}_{T,\kappa_i}.
\end{aligned}$$

Arranging terms gives

$$J \leq \frac{4K\tilde{\mathcal{A}}_{T,\kappa_i}}{Bk_1} - 1 \leq \frac{4K\tilde{\mathcal{A}}_{T,\kappa_i}}{Bk_1}. \tag{A3}$$

Taking expectation w.r.t. the randomness of reservoir sampling gives

$$\mathbb{E}[J] \leq \frac{4K}{Bk_1} \cdot \mathbb{E}[\tilde{\mathcal{A}}_{T,\kappa_i}] \leq \frac{12K}{Bk_1}\mathcal{A}_{T,\kappa_i} \cdot \left(1 + \frac{\ln T}{M}\right) + \frac{32K}{Bk_1},$$

where the last inequality comes from Lemma A.1.1. Omitting the last constant term concludes the proof. □

Lemma A.1.1. *The reservoir sampling guarantees*

$$\forall i \in [K], \quad \mathbb{E}\left[\tilde{\mathcal{A}}_{T,\kappa_i}\right] \leq 3\mathcal{A}_{T,\kappa_i} + 8 + \frac{3\mathcal{A}_{T,\kappa_i}\ln T}{M}.$$

Proof. Let $\mu_{t,i} = -\frac{1}{t}\sum_{\tau=1}^{t} y_\tau \kappa_i(\boldsymbol{x}_\tau, \cdot)$ and $\tau_0 = M$. For $t \leq \tau_0$, it can be verified that

$$\begin{aligned}
\tilde{\mathcal{A}}_{\tau_0,\kappa_i} =& 1 + \sum_{t=2}^{\tau_0} \|-y_t\kappa_i(\boldsymbol{x}_t, \cdot) - \mu_{t-1,i}\|_{\mathcal{H}_i}^2 \\
=& 1 + \sum_{t=2}^{\tau_0} \|-y_t\kappa_i(\boldsymbol{x}_t, \cdot) - \mu_{t,i} + \mu_{t,i} - \mu_{t-1,i}\|_{\mathcal{H}_i}^2 \\
\leq& 1 + 2\mathcal{A}_{[2:\tau_0],\kappa_i} + 2\sum_{t=2}^{\tau_0} \|\mu_{t,i} - \mu_{t-1,i}\|_{\mathcal{H}_i}^2,
\end{aligned}$$

where $\mu_{0,i} = 0$. Let V_t be the reservoir at the beginning of round t. Next we consider the case $t > \tau_0$.

$$\begin{aligned}
\tilde{\mathcal{A}}_{[\tau_0:T],\kappa_i} =& \sum_{t=\tau_0+1}^{T} \|-y_t\kappa_i(\boldsymbol{x}_t, \cdot) - \tilde{\mu}_{t-1,i}\|_{\mathcal{H}_i}^2 \\
\leq& \sum_{t=\tau_0+1}^{T} 3\left[\|y_t\kappa_i(\boldsymbol{x}_t, \cdot) + \mu_{t,i}\|_{\mathcal{H}_i}^2 + \|\mu_{t,i} - \mu_{t-1,i}\|_{\mathcal{H}_i}^2 + \|\mu_{t-1,i} - \tilde{\mu}_{t-1,i}\|_{\mathcal{H}_i}^2\right] \\
=& 3\mathcal{A}_{[\tau_0:T],\kappa_i} + 3\sum_{t=\tau_0+1}^{T} \|\mu_{t,i} - \mu_{t-1,i}\|_{\mathcal{H}_i}^2 + 3\sum_{t=\tau_0+1}^{T} \left\|\mu_{t-1,i} + \frac{1}{|V_t|}\sum_{(\boldsymbol{x},y)\in V_t} y\kappa_i(\boldsymbol{x}, \cdot)\right\|_{\mathcal{H}_i}^2.
\end{aligned}$$

Taking expectation w.r.t. the reservoir sampling yields

$$\begin{aligned}
&\mathbb{E}[\tilde{\mathcal{A}}_{T,\kappa_i}] \\
=& \tilde{\mathcal{A}}_{\tau_0,\kappa_i} + \mathbb{E}[\tilde{\mathcal{A}}_{[\tau_0:T],\kappa_i}] \\
\leq& 1 + 3\mathcal{A}_{T,\kappa_i} + 3\sum_{t=2}^{T} \|\mu_{t,i} - \mu_{t-1,i}\|_{\mathcal{H}_i}^2 + 3\sum_{t=\tau_0+1}^{T} \mathbb{E}\left[\left\|\mu_{t-1,i} + \frac{1}{|V_t|}\sum_{(\boldsymbol{x},y)\in V_t} y\kappa_i(\boldsymbol{x}, \cdot)\right\|_{\mathcal{H}_i}^2\right] \qquad \text{by Lemma } A.8.1 \\
\leq& 1 + 3\mathcal{A}_{T,\kappa_i} + 3\sum_{t=2}^{T} \|\mu_{t,i} - \mu_{t-1,i}\|_{\mathcal{H}_i}^2 + \sum_{t=\tau_0+1}^{T} \frac{3\mathcal{A}_{t-1,\kappa_i}}{(t-1)|V_t|} \\
\leq& 1 + 3\mathcal{A}_{T,\kappa_i} + \sum_{t=2}^{T} \frac{12}{t^2} + \frac{3\mathcal{A}_{T,\kappa_i}\ln T}{M} \\
\leq& 1 + 3\mathcal{A}_{T,\kappa_i} + 7 + \frac{3\mathcal{A}_{T,\kappa_i}\ln T}{M},
\end{aligned}$$

where $|V_t| = M$ for all $t \geq \tau_0$. □

A.2 Proof of Theorem 1

Proof. By the convexity of the Hinge loss function, we decompose the regret as follows

$$
\begin{aligned}
\mathrm{Reg}(f) =& \sum_{t=1}^{T} \ell\left(\sum_{j=1}^{K} p_{t,j} f_{t,j}(\boldsymbol{x}_t), y_t\right) - \sum_{t=1}^{T} \ell\left(f(\boldsymbol{x}_t), y_t\right) \\
\leq& \sum_{t=1}^{T}\sum_{j=1}^{K} p_{t,j}\ell\left(f_{t,j}(\boldsymbol{x}_t), y_t\right) - \sum_{t=1}^{T} \ell\left(f(\boldsymbol{x}_t), y_t\right) \\
\leq& \underbrace{\sum_{t=1}^{T}\left[\sum_{j=1}^{K} p_{t,j}\ell(f_{t,j}(\boldsymbol{x}_t), y_t) - \ell(f_{t,i}(\boldsymbol{x}_t), y_t)\right]}_{\mathcal{T}_1} + \underbrace{\sum_{t\in E_{T,i}} [\ell(f_{t,i}(\boldsymbol{x}_t), y_t) - \ell(f(\boldsymbol{x}_t), y_t)]}_{\mathcal{T}_2},
\end{aligned}
$$

where $E_{T,i} = \{t \in [T], \nabla_{t,i} \neq 0\}$.

A.2.1 Analyzing $\mathcal{T}_1$

The following analysis is same with the proof of Theorem 3.1 in [A1]. Let $c_{t,i} := \ell(f_{t,i}(\boldsymbol{x}_t), y_t)$. The updating of probability is as follows,

$$
p_{t+1,i} = \frac{w_{t+1,i}}{\sum_{j=1}^{K} w_{t+1,j}}, \quad w_{t+1,i} = \exp\left(-\eta_{t+1}\sum_{\tau=1}^{t} c_{\tau,i}\right).
$$

Similar to the analysis of Exp3 [A1], we define a potential function $\Gamma_t(\eta_t)$ as follows,

$$
\Gamma_t(\eta_t) := \frac{1}{\eta_t}\ln\sum_{i=1}^{K} p_{t,i}\exp(-\eta_t c_{t,i}) \leq -\sum_{i=1}^{K} p_{t,i}c_{t,i} + \frac{1}{2}\eta_t\sum_{i=1}^{K} p_{t,i}c_{t,i}^2,
$$

where we use the following two inequalities

$$
\ln x \leq x - 1, \forall x > 0, \quad \exp(-x) \leq 1 - x + \frac{x^2}{2}, \forall x \geq 0.
$$

Summing over $t \in [T]$ yields

$$
\sum_{t=1}^{T}\Gamma_t(\eta_t) \leq -\sum_{t=1}^{T}\langle \boldsymbol{p}_t, \boldsymbol{c}_t\rangle + \sum_{t=1}^{T}\sum_{i=1}^{K}\frac{\eta_t}{2}p_{t,i}c_{t,i}^2. \tag{A4}
$$

On the other hand, by the definition of $p_{t,i}$, we have

$$
\begin{aligned}
\Gamma_t(\eta_t) =& \frac{1}{\eta_t}\ln\frac{\sum_{i=1}^{K}\exp\left(-\eta_t\sum_{\tau=1}^{t-1}c_{\tau,i}\right)\exp(-\eta_t c_{t,i})}{\sum_{j=1}^{K}\exp\left(-\eta_t\sum_{\tau=1}^{t-1}c_{\tau,j}\right)} \\
=& \frac{1}{\eta_t}\ln\frac{\frac{1}{K}\sum_{i=1}^{K}\exp\left(-\eta_t\sum_{\tau=1}^{t}c_{\tau,i}\right)}{\frac{1}{K}\sum_{j=1}^{K}\exp\left(-\eta_t\sum_{\tau=1}^{t-1}c_{\tau,j}\right)} \\
=& \bar{\Gamma}_t(\eta_t) - \bar{\Gamma}_{t-1}(\eta_t),
\end{aligned}
$$

where $\bar{\Gamma}_t(\eta) = \frac{1}{\eta}\ln\frac{1}{K}\sum_{j=1}^{K}\exp\left(-\eta\sum_{\tau=1}^{t}c_{\tau,j}\right)$.
Without loss of generality, let $\bar{\Gamma}_0(\eta) = 0$. Summing over $t = 1, \ldots, T$ yields

$$
\sum_{t=1}^{T}\Gamma_t(\eta_t) = \bar{\Gamma}_T(\eta_T) - \bar{\Gamma}_0(\eta_1) + \sum_{t=1}^{T-1}\left[\bar{\Gamma}_t(\eta_t) - \bar{\Gamma}_t(\eta_{t+1})\right],
$$

where $\bar{\Gamma}_T(\eta_T) \geq \frac{1}{\eta_T} \ln \frac{1}{K} - \sum_{\tau=1}^{T} c_{\tau,i}$. Combining with the upper bound (A4), we obtain

$$\sum_{t=1}^{T} \langle \boldsymbol{p}_t, \boldsymbol{c}_t \rangle - \sum_{\tau=1}^{T} c_{\tau,i} \leq \frac{1}{\eta_T} \ln K + \sum_{t=1}^{T-1} \left[\bar{\Gamma}_t(\eta_{t+1}) - \bar{\Gamma}_t(\eta_t) \right] + \sum_{t=1}^{T} \sum_{i=1}^{K} \frac{\eta_t}{2} p_{t,i} c_{t,i}^2.$$

For simplicity, let $\bar{C}_{t,j} := \sum_{\tau=1}^{t} c_{\tau,j}$. The first derivative of $\bar{\Gamma}_t(\eta)$ w.r.t. η is as follows

$$\begin{aligned}
\frac{\mathrm{d}\, \bar{\Gamma}_t(\eta)}{\mathrm{d}\, \eta} =& \frac{-\ln \sum_{j=1}^{K} \frac{\exp\left(-\eta \bar{C}_{t,j}\right)}{K}}{\eta^2} - \frac{\frac{1}{K} \sum_{j=1}^{K} \bar{C}_{t,j} \exp\left(-\eta \bar{C}_{t,j}\right)}{\frac{\eta}{K} \sum_{j=1}^{K} \exp\left(-\eta \bar{C}_{t,j}\right)} \\
=& \frac{1}{\eta^2} \mathrm{KL}(\tilde{p}_t, \frac{1}{K}) \\
\geq& 0
\end{aligned}$$

where $\tilde{p}_{t,j} = \frac{\exp\left(-\eta \bar{C}_{t,j}\right)}{\sum_{i=1}^{K} \exp\left(-\eta \bar{C}_{t,i}\right)}$. Since $\eta_{t+1} \leq \eta_t$, we have $\bar{\Gamma}_t(\eta_{t+1}) \leq \bar{\Gamma}_t(\eta_t)$. Combining all results, we have

$$\begin{aligned}
& \sum_{t=1}^{T} \langle \boldsymbol{p}_t, \boldsymbol{c}_t \rangle - \sum_{\tau=1}^{T} c_{\tau,i} \\
\leq & \frac{\ln K}{\eta_T} - \frac{\ln K}{\eta_1} + \sum_{t=1}^{T} \sum_{i=1}^{K} \frac{\eta_t}{2} p_{t,i} c_{t,i}^2 \\
\leq & \frac{\sqrt{\ln K}}{\sqrt{2}} \cdot \sqrt{1 + \sum_{\tau=1}^{T-1} \langle \boldsymbol{p}_\tau, \boldsymbol{c}_\tau^2 \rangle} - \frac{\sqrt{\ln K}}{\sqrt{2}} + \sqrt{\ln K} \left(\sqrt{2 \sum_{\tau=1}^{T} \langle \boldsymbol{p}_\tau, \boldsymbol{c}_\tau^2 \rangle} + \frac{\max_{t,j} c_{t,j}}{\sqrt{2}} \right) \qquad \text{by Lemma } A.8.2 \\
\lesssim & \frac{3}{\sqrt{2}} \sqrt{\max_{t,j} c_{t,j} \cdot \sum_{\tau=1}^{T} \langle \boldsymbol{p}_\tau, \boldsymbol{c}_\tau \rangle \ln K}. \qquad \text{(A5)}
\end{aligned}$$

Solving for $\sum_{t=1}^{T} \langle \boldsymbol{p}_t, \boldsymbol{c}_t \rangle$ gives

$$\mathcal{T}_1 = \sum_{t=1}^{T} [\langle \boldsymbol{p}_t, \boldsymbol{c}_t \rangle - c_{t,i}] \leq \frac{3}{\sqrt{2}} \sqrt{\max_{t,j} c_{t,j} \cdot \sum_{\tau=1}^{T} c_{\tau,i} \ln K} + \frac{9}{2} \max_{t,j} c_{t,j} \cdot \ln K. \qquad \text{(A6)}$$

A.2.2 Analyzing $\mathcal{T}_2$

We decompose $E_{T,i}$ as follows.

$$\begin{aligned}
E_i =& \{t \in E_{T,i} : \mathrm{con}(a(i))\}, \\
\mathcal{J}_i =& \{t \in E_{T,i} : |S_i| = \alpha \mathcal{R}_i, b_{t,i} = 1\}, \\
\bar{E}_i =& E_{T,i} \setminus (E_i \cup \mathcal{J}_i).
\end{aligned}$$

We separately analyze the regret in E_i, $\mathcal{J}_i$ and $\bar{E}_i$.
Case 1: regret in E_i
For any $f \in \mathbb{H}_i$, the convexity of loss function gives

$$\begin{aligned}
& \ell(f_{t,i}(\boldsymbol{x}_t), y_t) - \ell(f(\boldsymbol{x}_t), y_t) \\
\leq & \langle f_{t,i} - f, \nabla_{t,i} \rangle \\
= & \underbrace{\langle f_{t,i} - f'_{t,i}, \hat{\nabla}_{t,i} \rangle}_{\Xi_1} + \underbrace{\langle f'_{t,i} - f, \nabla_{i(s_t),i} \rangle}_{\Xi_2} + \langle f'_{t,i} - f, \nabla_{t,i} - \nabla_{i(s_t),i} \rangle + \langle f_{t,i} - f'_{t,i}, \nabla_{t,i} - \hat{\nabla}_{t,i} \rangle \\
= & \Xi_1 + \Xi_2 + \langle f_{t,i} - f'_{t,i}, \nabla_{i(s_t),i} - \hat{\nabla}_{t,i} \rangle + \langle f_{t,i} - f, \nabla_{t,i} - \nabla_{i(s_t),i} \rangle \\
\leq & \left[\mathcal{B}_{\psi_i}(f, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f, f'_{t,i}) \right] + \underbrace{\|f_{t,i} - f\| \cdot \gamma_{t,i}}_{\Xi_3} + \underbrace{\left\langle f_{t,i} - f'_{t,i}, \nabla_{i(s_t),i} - \hat{\nabla}_{t,i} \right\rangle - \mathcal{B}_{\psi_i}(f'_{t,i}, f_{t,i})}_{\Xi_4},
\end{aligned}$$

where the standard analysis of OMD [A2] gives

$$\begin{aligned}\Xi_1 \leq & \mathcal{B}_{\psi_i}(f'_{t,i}, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f'_{t,i}, f_{t,i}) - \mathcal{B}_{\psi_i}(f_{t,i}, f'_{t-1,i}),\\ \Xi_2 \leq & \mathcal{B}_{\psi_i}(f, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f, f'_{t,i}) - \mathcal{B}_{\psi_i}(f'_{t,i}, f'_{t-1,i}).\end{aligned}$$

Substituting into $\gamma_{t,i}$ and summing over $t \in E_i$ gives

$$\begin{aligned}\sum_{t\in E_i} \Xi_3 \leq & \sum_{t\in E_i} \frac{\max_t \|f_{t,i} - f\|_{\mathcal{H}_i} \cdot \left\|\nabla_{t,i} - \hat{\nabla}_{t,i}\right\|^2_{\mathcal{H}_i}}{\sqrt{1+\sum_{\tau\leq t}\left\|\nabla_{\tau,i} - \hat{\nabla}_{\tau,i}\right\|^2_{\mathcal{H}_i} \cdot \mathbb{I}_{\nabla_{\tau,i}\neq 0}}}\\ \leq & 2(U+\lambda_i)\cdot \sum_{t\in E_i} \frac{\left\|\nabla_{t,i} - \hat{\nabla}_{t,i}\right\|^2_{\mathcal{H}_i}}{\sqrt{1+\sum_{\tau\leq t}\left\|\nabla_{\tau,i} - \hat{\nabla}_{\tau,i}\right\|^2_{\mathcal{H}_i} \cdot \mathbb{I}_{\nabla_{\tau,i}\neq 0}}}\\ \leq & 4(U+\lambda_i)\sqrt{\tilde{\mathcal{A}}_{T,\kappa_i}},\end{aligned}$$

where $\|f_{t,i}\|_{\mathcal{H}_i} \leq U + \lambda_i$.
According to Lemma A.8.6, we can obtain

$$\sum_{t\in E_i} \Xi_4 \leq \frac{\lambda_i}{2}\sum_{t\in E_i}\left\|\nabla_{i(s_t),i} - \hat{\nabla}_{t,i}\right\|^2_{\mathcal{H}_i} \leq 2\lambda_i \tilde{\mathcal{A}}_{T,\kappa_i}.$$

Case 2: regret in $\bar{E}_i$
We decompose the instantaneous regret as follows,

$$\begin{aligned}& \langle f_{t,i} - f, \nabla_{t,i}\rangle\\ =& \underbrace{\langle f_{t,i} - f'_{t,i}, \hat{\nabla}_{t,i}\rangle}_{\Xi_1} + \underbrace{\langle f'_{t,i} - f, \tilde{\nabla}_{t,i}\rangle}_{\Xi_2} + \underbrace{\langle f_{t,i} - f'_{t,i}, \tilde{\nabla}_{t,i} - \hat{\nabla}_{t,i}\rangle}_{\Xi_3} + \left\langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\right\rangle\\ \leq & \mathcal{B}_{\psi_i}(f, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f, f'_{t,i}) + \langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle + \Xi_3 - \left[\mathcal{B}_{\psi_i}(f'_{t,i}, f_{t,i}) + \mathcal{B}_{\psi_i}(f_{t,i}, f'_{t-1,i})\right]\\ =& \mathcal{B}_{\psi_i}(f, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f, f'_{t,i}) + \langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle + \Xi_3 - \mathcal{B}_{\psi_i}(f'_{t,i}, f_{t,i}) - \frac{\lambda_i}{2}\|\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}\\ \leq & \mathcal{B}_{\psi_i}(f, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f, f'_{t,i}) + \langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle - \frac{\lambda_i}{2}\|\tilde{\nabla}_{t,i} - \hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i} - \frac{\lambda_i}{2}\|\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i} \quad \text{by Lemma } A.8.6\\ =& \mathcal{B}_{\psi_i}(f, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f, f'_{t,i}) + \langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle + \frac{\lambda_i}{2}\left(\frac{\|\nabla_{t,i} - \hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}}{(\mathbb{P}[b_{t,i}=1])^2}\mathbb{I}_{b_{t,i}=1} - \|\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}\right),\end{aligned}$$

where $\Xi_1 + \Xi_2$ follows the analysis in **Case 1**.
Case 3: regret in $\mathcal{J}_i$
Recalling that the second mirror updating is

$$f'_{t,i} = \mathop{\arg\min}_{f\in\mathbb{H}_i}\left\{\langle f, \tilde{\nabla}_{t,i}\rangle + \mathcal{B}_{\psi_i}(f, \bar{f}'_{t-1,i}(1))\right\}.$$

We still decompose the instantaneous regret as follows

$$\langle f_{t,i} - f, \nabla_{t,i}\rangle = \underbrace{\langle f_{t,i} - f'_{t,i}, \hat{\nabla}_{t,i}\rangle}_{\Xi_1} + \underbrace{\langle f'_{t,i} - f, \tilde{\nabla}_{t,i}\rangle}_{\Xi_2} + \underbrace{\langle f_{t,i} - f'_{t,i}, \tilde{\nabla}_{t,i} - \hat{\nabla}_{t,i}\rangle}_{\Xi_3} + \left\langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\right\rangle.$$

We reanalyze Ξ_1 and Ξ_2 as follows

$$\begin{aligned}\Xi_1 \leq & \mathcal{B}_{\psi_i}(f'_{t,i}, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f'_{t,i}, f_{t,i}) - \mathcal{B}_{\psi_i}(f_{t,i}, f'_{t-1,i}),\\ \Xi_2 \leq & \mathcal{B}_{\psi_i}(f, \bar{f}'_{t-1,i}(1)) - \mathcal{B}_{\psi_i}(f, f'_{t,i}) - \mathcal{B}_{\psi_i}(f'_{t,i}, \bar{f}'_{t-1,i}(1)).\end{aligned}$$

Then $\Xi_1+\Xi_2+\Xi_3$ can be further bounded as follows,

$$\begin{aligned}\Xi_1+\Xi_2+\Xi_3 \leq & \mathcal{B}_{\psi_i}(f,f'_{t-1,i})-\mathcal{B}_{\psi_i}(f,f'_{t,i})+\left[\mathcal{B}_{\psi_i}(f,\bar{f}'_{t-1,i}(1))-\mathcal{B}_{\psi_i}(f,f'_{t-1,i})\right]+\\ & \left[\mathcal{B}_{\psi_i}(f'_{t,i},f'_{t-1,i})-\mathcal{B}_{\psi_i}(f'_{t,i},\bar{f}'_{t-1,i}(1))\right]-\left[\mathcal{B}_{\psi_i}(f'_{t,i},f_{t,i})+\mathcal{B}_{\psi_i}(f_{t,i},f'_{t-1,i})\right]+\Xi_3.\end{aligned}$$

By Lemma A.8.6, we analyze the following term

$$\begin{aligned}&\Xi_3-\left[\mathcal{B}_{\psi_i}(f'_{t,i},f_{t,i})+\mathcal{B}_{\psi_i}(f_{t,i},f'_{t-1,i})\right]\\ \leq&\frac{\lambda_i}{2}\left[\frac{\|\nabla_{t,i}-\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}}{(\mathbb{P}[b_{t,i}=1])^2}\mathbb{I}_{b_{t,i}=1}-\|\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}\right]-\frac{1}{2\lambda_i}\|f'_{t-1,i}-f'_{t,i}\|^2_{\mathcal{H}_i}+\langle f'_{t-1,i}-f'_{t,i},\tilde{\nabla}_{t,i}\rangle.\end{aligned}$$

Substituting into the instantaneous regret gives

$$\begin{aligned}\langle f_{t,i}-f,\nabla_{t,i}\rangle \leq & \mathcal{B}_{\psi_i}(f,f'_{t-1,i})-\mathcal{B}_{\psi_i}(f,f'_{t,i})+\left\langle f_{t,i}-f,\nabla_{t,i}-\tilde{\nabla}_{t,i}\right\rangle+\langle\tilde{\nabla}_{t,i},f'_{t-1,i}-f'_{t,i}\rangle+\\ & \frac{\|\bar{f}'_{t-1,i}(1)-f\|^2_{\mathcal{H}_i}-\|f'_{t-1,i}-f\|^2_{\mathcal{H}_i}}{2\lambda_i}+\frac{\lambda_i}{2}\frac{\|\nabla_{t,i}-\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}}{(\mathbb{P}[b_{t,i}=1])^2}\mathbb{I}_{b_{t,i}=1}-\frac{\lambda_i}{2}\|\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}.\end{aligned}$$

Combining all

Combining the above three cases, we obtain

$$\begin{aligned}\mathcal{T}_2\leq&\sum_{t\in E_{T,i}}\left[\mathcal{B}_{\psi_i}(f,f'_{t-1,i})-\mathcal{B}_{\psi_i}(f,f'_{t,i})\right]+4(U+\lambda_i)\tilde{\mathcal{A}}^{\frac{1}{2}}_{T,\kappa_i}+\sum_{t\in\mathcal{J}_i}\left[\langle\tilde{\nabla}_{t,i},f'_{t-1,i}-f'_{t,i}\rangle+\frac{2U^2}{\lambda_i}\right]+\\ &\frac{\lambda_i}{2}\sum_{t\in\bar{E}_i\cup\mathcal{J}_i}\left[\frac{\|\nabla_{t,i}-\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}}{(\mathbb{P}[b_{t,i}=1])^2}\mathbb{I}_{b_{t,i}=1}-\|\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}\right]+\sum_{t\in\bar{E}_i\cup\mathcal{J}_i}\left\langle f_{t,i}-f,\nabla_{t,i}-\tilde{\nabla}_{t,i}\right\rangle+2\lambda_i\tilde{\mathcal{A}}_{T,\kappa_i}.\end{aligned}$$

Recalling that $\|f'_{t,i}\|_{\mathcal{H}_i}\leq U$ and $f\leq U$. Conditioned on $b_{s_r,i},\ldots,b_{t-1,i}$, taking expectation w.r.t. $b_{t,i}$ gives

$$\mathbb{E}\left[\mathcal{T}_2\right]\leq\frac{U^2}{2\lambda_i}+\left(2U+\frac{2U^2}{\lambda_i}\right)\cdot J+\frac{5\lambda_i}{2}\tilde{\mathcal{A}}_{T,\kappa_i}+4(U+\lambda_i)\sqrt{\tilde{\mathcal{A}}_{T,\kappa_i}}. \tag{A7}$$

Let $\lambda_i=\frac{\sqrt{K}U}{2\sqrt{B}}$. Assuming that $B\geq K$, we have $\lambda_i\leq\frac{U}{2}$. Then

$$\begin{aligned}\mathbb{E}\left[\mathcal{T}_2\right]=&O\left(\frac{U\sqrt{B}}{\sqrt{K}}+\frac{\sqrt{K}U}{\sqrt{B}k_1}\tilde{\mathcal{A}}_{T,\kappa_i}+U\sqrt{\tilde{\mathcal{A}}_{T,\kappa_i}}\right) && \text{by (A3)}\\ =&O\left(\frac{U\sqrt{B}}{\sqrt{K}}+\frac{\sqrt{K}U\mathcal{A}_{T,\kappa_i}\ln T}{\sqrt{B}k_1}\right), && \text{by Lemma } A.1.1\end{aligned}$$

where we omit the lower order term.

A.2.3 Combining $\mathcal{T}_1$ and $\mathcal{T}_2$

Combining $\mathcal{T}_1$ and $\mathcal{T}_2$, and taking expectation w.r.t. the randomness of reservoir sampling gives

$$\begin{aligned}&\mathbb{E}\left[\mathrm{Reg}(f)\right]\\ =&\mathbb{E}\left[\sum^T_{t=1}\ell(f_t(\boldsymbol{x}_t),y_t)-\sum^T_{t=1}\ell(f_{t,i}(\boldsymbol{x}_t),y_t)\right]+\mathbb{E}\left[\mathcal{T}_2\right]\\ \leq&\frac{3}{\sqrt{2}}\mathbb{E}\left[\sqrt{\max_{t,j}c_{t,j}\cdot\sum^T_{t=1}\ell(f_{t,i}(\boldsymbol{x}_t),y_t)\ln K}\right]+\frac{9}{2}\max_{t,j}c_{t,j}\cdot\ln K+\mathbb{E}\left[\mathcal{T}_2\right] \qquad \text{by (A6)}\\ =&\frac{3}{\sqrt{2}}\mathbb{E}\left[\sqrt{\max_{t,j}c_{t,j}\cdot\left(\sum^T_{t=1}\ell(f(\boldsymbol{x}_t),y_t)+\mathbb{E}\left[\mathcal{T}_2\right]\right)\ln K}\right]+\frac{9}{2}\max_{t,j}c_{t,j}\cdot\ln K+\mathbb{E}\left[\mathcal{T}_2\right]\\ =&O\left(\sqrt{\max_{t,j}c_{t,j}\cdot L_T(f)\ln K}+\frac{U\sqrt{B}}{\sqrt{K}}+\frac{\sqrt{K}U\mathcal{A}_{T,\kappa_i}\ln T}{\sqrt{B}k_1}+\max_{t,j}c_{t,j}\cdot\ln K\right).\end{aligned}$$

For the Hinge loss function, we have $\max_{t,j}c_{t,j}=1+U$. □

A.3 Proof of Theorem 2

Proof. For simplicity, denote by

$$\Lambda_i = \sum_{t\in\mathcal{J}_i}\left[\left\|\bar{f}'_{t-1,i}(1) - f\right\|^2_{\mathcal{H}_i} - \left\|f'_{t-1,i} - f\right\|^2_{\mathcal{H}_i}\right].$$

There must be a constant $\xi_i \in (0,4]$ such that $\Lambda_i \leq \xi_i U^2 J$. We will prove a better regret bound if ξ_i is small enough. Recalling that (A3) gives an upper bound on J. If $\xi_i \leq \frac{1}{J}$, then we rewrite (A7) by

$$\mathcal{T}_2 \leq \frac{U^2}{2\lambda_i} + 2UJ + \frac{U^2}{2\lambda_i} + \frac{5\lambda_i}{2}\tilde{\mathcal{A}}_{T,\kappa_i} + 4(U+\lambda_i)\sqrt{\tilde{\mathcal{A}}_{T,\kappa_i}}.$$

Let $\lambda_i = \frac{\sqrt{2}U}{\sqrt{5\tilde{\mathcal{A}}_{T,\kappa_i}}}$. Taking expectation w.r.t. the reservoir sampling and using Lemma A.1.1 gives

$$\mathbb{E}\left[\mathcal{T}_2\right] = O\left(\frac{UK}{Bk_1}\mathcal{A}_{T,\kappa_i}\ln T + U\sqrt{\mathcal{A}_{T,\kappa_i}\ln T}\right),$$

where we omit the lower order terms. Combining $\mathcal{T}_1$ and $\mathcal{T}_2$ gives

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{Reg}(f)\right]\\
=&\frac{3}{\sqrt{2}}\mathbb{E}\left[\sqrt{\max_{t,j} c_{t,j}\cdot\left(\sum_{t=1}^{T}\ell(f(\boldsymbol{x}_t),y_t) + \mathbb{E}\left[\mathcal{T}_2\right]\right)\ln K}\right] + \frac{9}{2}\max_{t,j} c_{t,j}\cdot\ln K + \mathbb{E}\left[\mathcal{T}_2\right]\\
=&O\left(\sqrt{\max_{t,j} c_{t,j}\cdot L_T(f)\ln K} + \frac{UK}{Bk_1}\mathcal{A}_{T,\kappa_i}\ln T + U\sqrt{\mathcal{A}_{T,\kappa_i}\ln T} + \max_{t,j} c_{t,j}\cdot\ln K\right),
\end{aligned}$$

which concludes the proof. □

A.4 Proof of Lemma 2

Proof. Recalling the definition of $\mathcal{J}$ and T_r in Section A.1. For any $t \in T_r$, $(\boldsymbol{x}_t, y_t)$ will be added into S only if $b_t = 1$. At the end of the t_r-th round, we have

$$|S| = \frac{B}{2}\mathbb{I}_{r\neq 1} + \sum_{t=t_{r-1}+1}^{t_r} b_t = B.$$

We remove $\frac{B}{2}$ examples from S at the end of the t_r-th round.

Assuming that there is no budget. For any $t_0 > t_{r-1}+1$, we will prove an upper bound on $\sum_{t=t_{r-1}+1}^{t_0} b_t$. Define a random variable X_t as follows,

$$X_t = b_t - \mathbb{P}[b_t = 1], \quad |X_t| \leq 1.$$

Under the condition of $b_{t_{r-1}+1},\ldots,b_{t-1}$, we have $\mathbb{E}_{b_t}[X_t] = 0$. Thus $X_{t_{r-1}+1},\ldots,X_{t_0}$ form bounded martingale difference. Let $\hat{L}_{a:b} := \sum_{t=a}^{b}\ell(f_t(\boldsymbol{x}_t),y_t)$ and $\hat{L}_{1:T} \leq N$. The sum of conditional variances satisfies

$$\Sigma^2 \leq \sum_{t=t_{r-1}+1}^{t_0}\frac{|\ell'(f_t(\boldsymbol{x}_t),y_t)|}{|\ell'(f_t(\boldsymbol{x}_t),y_t)| + G_1} \leq \frac{G_2}{G_1}\hat{L}_{t_{r-1}+1:t_0},$$

where the last inequality comes from Assumption 3. Since $\hat{L}_{t_{r-1}+1:t_0}$ is a random variable, Lemma A.8.3 can give an upper bound on $\sum_{t=t_{r-1}+1}^{t_0} b_t$ with probability at least $1 - 2\lceil\log N\rceil\delta$. Let t_r be the minimal t_0 such that

$$\frac{G_2}{G_1}\hat{L}_{t_{r-1}+1:t_r} + \frac{2}{3}\ln\frac{1}{\delta} + 2\sqrt{\frac{G_2}{G_1}\hat{L}_{t_{r-1}+1:t_r}\ln\frac{1}{\delta}} \geq \frac{B}{2}\cdot\mathbb{I}_{r\geq 2} + B\cdot\mathbb{I}_{r=1}.$$

The r-th epoch will end at t_r. Summing over $r \in \{1, \dots, J\}$, with probability at least $1 - 2J\lceil \log N \rceil \delta$,

$$\sum_{r=1}^{J} \sum_{t=t_{r-1}+1}^{t_r} b_t \le \sum_{r=1}^{J} \left(\frac{G_2}{G_1} \hat{L}_{t_{r-1}+1:t_r} + \frac{2}{3} \ln \frac{1}{\delta} + 2\sqrt{\frac{G_2}{G_1} \hat{L}_{t_{r-1}+1:t_r} \ln \frac{1}{\delta}} \right),$$

which is equivalent to

$$\frac{B}{2} + \frac{JB}{2} \le \frac{G_2}{G_1} \hat{L}_{1:T} + \frac{2}{3} J \ln \frac{1}{\delta} + 2\sqrt{J \frac{G_2}{G_1} \hat{L}_{1:T} \ln \frac{1}{\delta}}.$$

Solving the above inequality yields,

$$J \le \frac{2G_2}{G_1} \frac{\hat{L}_{1:T}}{B - \frac{4}{3} \ln \frac{1}{\delta}} + \frac{16G_2}{G_1} \frac{\hat{L}_{1:T}}{(B - \frac{4}{3} \ln \frac{1}{\delta})^2} \ln \frac{1}{\delta} + \frac{4\sqrt{2}}{(B - \frac{4}{3} \ln \frac{1}{\delta})^{\frac{3}{2}}} \frac{G_2}{G_1} \hat{L}_{1:T} \sqrt{\ln \frac{1}{\delta}}.$$

Let $B \ge 21 \ln \frac{1}{\delta}$. Simplifying the above result concludes the proof. □

A.5 Proof of Theorem 3

Proof. Let $\boldsymbol{p} \in \Delta_{K-1}$ satisfy $p_i = 1$. By the convexity of loss function, we have

$$\begin{aligned}
\mathrm{Reg}(f) \le & \sum_{t=1}^{T} \langle \ell'(f_t(\boldsymbol{x}_t), y_t), f_t(\boldsymbol{x}_t) - f(\boldsymbol{x}_t) \rangle \\
= & \sum_{t=1}^{T} \left\langle \ell'(f_t(\boldsymbol{x}_t), y_t), \sum_{i=1}^{K} p_{t,i} f_{t,i}(\boldsymbol{x}_t) - f(\boldsymbol{x}_t) \right\rangle \\
= & \sum_{t=1}^{T} \left\langle \ell'(f_t(\boldsymbol{x}_t), y_t), \sum_{i=1}^{K} p_{t,i} f_{t,i}(\boldsymbol{x}_t) - \sum_{i=1}^{K} p_i f_{t,i}(\boldsymbol{x}_t) + \sum_{i=1}^{K} p_i f_{t,i}(\boldsymbol{x}_t) - f(\boldsymbol{x}_t) \right\rangle \\
= & \underbrace{\sum_{t=1}^{T} \ell'(f_t(\boldsymbol{x}_t), y_t) \sum_{i=1}^{K} (p_{t,i} - p_i) f_{t,i}(\boldsymbol{x}_t)}_{\mathcal{T}_1} + \underbrace{\sum_{t=1}^{T} \langle \nabla_{t,i}, f_{t,i} - f \rangle}_{\mathcal{T}_2}.
\end{aligned}$$

We first analyze $\mathcal{T}_1$. We have

$$\begin{aligned}
\mathcal{T}_1 = & \sum_{t \in T^1} \sum_{i=1}^{K} (p_{t,i} - p_i) \cdot \ell'(f_t(\boldsymbol{x}_t), y_t) \cdot \left(f_{t,i}(\boldsymbol{x}_t) - \min_{j \in [K]} f_{t,j}(\boldsymbol{x}_t) \right) + \\
& \sum_{t \in T^2} \sum_{i=1}^{K} (p_{t,i} - p_i) \cdot \ell'(f_t(\boldsymbol{x}_t), y_t) \cdot \left(f_{t,i}(\boldsymbol{x}_t) - \max_{j \in [K]} f_{t,j}(\boldsymbol{x}_t) \right) \\
= & \sum_{t=1}^{T} \langle \boldsymbol{p}_t - \boldsymbol{p}, \boldsymbol{c}_t \rangle \qquad \text{by (16)} \\
\le & \frac{3}{\sqrt{2}} \sqrt{\max_{t,j} c_{t,j} \sum_{\tau=1}^{T} \langle \boldsymbol{p}_\tau, \boldsymbol{c}_\tau \rangle \ln K} \qquad \text{by (A5)} \\
\le & \frac{3}{\sqrt{2}} \sqrt{\max_{t,j} c_{t,j} \sum_{\tau=1}^{T} |\ell'(f_t(\boldsymbol{x}_t), y_t)| \cdot 2U \ln K} \\
\le & \frac{6}{\sqrt{2}} U \sqrt{G_2 G_1 \hat{L}_{1:T} \ln K}. \qquad \text{by Assumption (3)}
\end{aligned}$$

Next we analyze $\mathcal{T}_2$. We decompose $[T]$ as follows,

$$\begin{aligned}
T_1 = & \{t \in [T] : \mathrm{con}(a)\}, \\
\mathcal{J} = & \{t \in [T] : |S| = \alpha \mathcal{R}, b_t = 1\}, \\
\bar{T}_1 = & [T] \setminus (T_1 \cup \mathcal{J}).
\end{aligned}$$

Case 1: regret in T_1
We decompose $\langle f_{t,i} - f, \nabla_{t,i}\rangle$ as follows,

$$\begin{aligned}
&\langle f_{t,i} - f, \nabla_{t,i}\rangle \\
=&\langle f_{t+1,i} - f, \nabla_{i(s_t),i}\rangle + \langle f_{t+1,i} - f, \nabla_{t,i} - \nabla_{i(s_t),i}\rangle + \langle f_{t,i} - f_{t+1,i}, \nabla_{i(s_t),i} + \nabla_{t,i} - \nabla_{i(s_t),i}\rangle \\
\leq&\mathcal{B}_{\psi_i}(f, f_{t,i}) - \mathcal{B}_{\psi_i}(f, f_{t+1,i}) - \mathcal{B}_{\psi_i}(f_{t+1,i}, f_{t,i}) + \left\langle f_{t,i} - f_{t+1,i}, \nabla_{i(s_t),i}\right\rangle + \left\langle f_{t,i} - f, \nabla_{t,i} - \nabla_{i(s_t),i}\right\rangle \\
\leq&\mathcal{B}_{\psi_i}(f, f_{t,i}) - \mathcal{B}_{\psi_i}(f, f_{t+1,i}) + \frac{\lambda}{2}\|\nabla_{i(s_t),i}\|^2_{\mathcal{H}_i} + \left\langle f_{t,i} - f, \nabla_{t,i} - \nabla_{i(s_t),i}\right\rangle,
\end{aligned}$$

where the last inequality comes from Lemma A.8.6. Next we analyze the third term.

$$\begin{aligned}
\sum_{t\in T_1}\langle f_{t,i} - f, \nabla_{t,i} - \nabla_{i(s_t),i}\rangle \leq& 2U\cdot\sum_{t\in T_1}\frac{|\ell'(f_t(\boldsymbol{x}_t), y_t)|}{\sqrt{1+\sum_{\tau\in T_1,\tau\leq t}|\ell'(f_\tau(\boldsymbol{x}_\tau), y_\tau)|}} \\
\leq& 4U\sqrt{G_2\hat{L}_{1:T}}.
\end{aligned}$$

Case 2: regret in $\bar{T}_1$
We use a different decomposition as follows

$$\begin{aligned}
&\langle f_{t,i} - f, \nabla_{t,i}\rangle \\
=&\underbrace{\langle f_{t+1,i} - f, \tilde{\nabla}_{t,i}\rangle}_{\Xi_1} + \underbrace{\langle f_{t+1,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle}_{\Xi_2} + \underbrace{\langle f_{t,i} - f_{t+1,i}, \nabla_{t,i}\rangle}_{\Xi_3} \\
\leq&\mathcal{B}_{\psi_i}(f, f_{t,i}) - \mathcal{B}_{\psi_i}(f, f_{t+1,i}) - \mathcal{B}_{\psi_i}(f_{t+1,i}, f_{t,i}) + \langle f_{t+1,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle + \\
&\langle f_{t,i} - f_{t+1,i}, \tilde{\nabla}_{t,i}\rangle + \langle f_{t,i} - f_{t+1,i}, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle \\
=&\mathcal{B}_{\psi_i}(f, f_{t,i}) - \mathcal{B}_{\psi_i}(f, f_{t+1,i}) + \left\langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\right\rangle + \langle f_{t,i} - f_{t+1,i}, \tilde{\nabla}_{t,i}\rangle - \mathcal{B}_{\psi_i}(f_{t+1,i}, f_{t,i}) \\
\leq&\mathcal{B}_{\psi_i}(f, f_{t,i}) - \mathcal{B}_{\psi_i}(f, f_{t+1,i}) + \left\langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\right\rangle + \frac{\lambda_i}{2}\|\tilde{\nabla}_{t,i}\|^2_{\mathcal{H}_i}.
\end{aligned}$$

Case 3: regret in $\mathcal{J}$
We decompose $\langle f_{t,i} - f, \nabla_{t,i}\rangle$ into three terms as in **Case 2**. The second mirror updating is

$$f_{t+1,i} = \mathop{\arg\min}_{f\in\mathbb{H}_i}\left\{\langle f, \tilde{\nabla}_{t,i}\rangle + \mathcal{B}_{\psi_i}(f, \bar{f}_{t,i}(2))\right\}.$$

Similar to the analysis of **Case 2**, we obtain

$$\begin{aligned}
\Xi_1 \leq&\mathcal{B}_{\psi_i}(f, f_{t,i}) - \mathcal{B}_{\psi_i}(f, f_{t+1,i}) - \mathcal{B}_{\psi_i}(f_{t+1,i}, \bar{f}_{t,i}(2)) + [\mathcal{B}_{\psi_i}(f, \bar{f}_{t,i}(2)) - \mathcal{B}_{\psi_i}(f, f_{t,i})], \\
\Xi_3 =&\langle \bar{f}_{t,i}(2) - f_{t+1,i}, \tilde{\nabla}_{t,i}\rangle + \langle f_{t,i} - \bar{f}_{t,i}(2), \tilde{\nabla}_{t,i}\rangle + \langle f_{t,i} - f_{t+1,i}, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle.
\end{aligned}$$

Combining Ξ_1, Ξ_2 and Ξ_3 gives

$$\begin{aligned}
&\langle f_{t,i} - f, \nabla_{t,i}\rangle \\
\leq&\mathcal{B}_{\psi_i}(f, f_{t,i}) - \mathcal{B}_{\psi_i}(f, f_{t+1,i}) + \langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle + \\
&\underbrace{\mathcal{B}_{\psi_i}(f, \bar{f}_{t,i}(2)) - \mathcal{B}_{\psi_i}(f, f_{t,i}) + \langle f_{t,i} - \bar{f}_{t,i}(2), \tilde{\nabla}_{t,i}\rangle}_{\Xi_4} + \langle \bar{f}_{t,i}(2) - f_{t+1,i}, \tilde{\nabla}_{t,i}\rangle - \mathcal{B}_{\psi_i}(f_{t+1,i}, \bar{f}_{t,i}(2)) \qquad \text{(A8)} \\
\leq&\mathcal{B}_{\psi_i}(f, f_{t,i}) - \mathcal{B}_{\psi_i}(f, f_{t+1,i}) + \langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle + \\
&\frac{2U^2}{\lambda_i} + 4UG_1 + \langle \bar{f}_{t,i}(2) - f_{t+1,i}, \tilde{\nabla}_{t,i}\rangle - \mathcal{B}_{\psi_i}(f_{t+1,i}, \bar{f}_{t,i}(2)) \qquad \text{by Lemma } A.8.6 \\
\leq&\mathcal{B}_{\psi_i}(f, f_{t,i}) - \mathcal{B}_{\psi_i}(f, f_{t+1,i}) + \langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle + \frac{2U^2}{\lambda_i} + 4UG_1 + \frac{\lambda_i}{2}\|\tilde{\nabla}_{t,i}\|^2_{\mathcal{H}_i}.
\end{aligned}$$

Combining the regret in T_1, $\mathcal{J}$ and $\bar{T}_1$ gives

$$\begin{aligned}
\mathcal{T}_2 \leq & 4U\sqrt{G_2\hat{L}_{1:T}} + \left(\frac{2U^2}{\lambda_i} + 4UG_1\right)|\mathcal{J}| + \underbrace{\sum_{t\in\bar{T}_1\cup\mathcal{J}} \langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle}_{\Xi_{2,1}} + \\
& \sum_{t=1}^{T} (\mathcal{B}_{\psi_i}(f, f_{t,i}) - \mathcal{B}_{\psi_i}(f, f_{t+1,i})) + \lambda_i \underbrace{\left(\frac{1}{2}\sum_{t\in\bar{T}_1\cup\mathcal{J}} \|\tilde{\nabla}_{t,i}\|^2_{\mathcal{H}_i} + \sum_{t\in T_1}\frac{1}{2}\|\nabla_{i(s_t),i}\|^2_{\mathcal{H}_i}\right)}_{\Xi_{2,2}} \\
\leq & 4U\sqrt{G_2\hat{L}_{1:T}} + \left(\frac{U^2}{2\lambda_i} + 4UG_1\right)|\mathcal{J}| + \Xi_{2,1} + \frac{U^2}{2\lambda_i} + \Xi_{2,2}.
\end{aligned}$$

Lemma A.8.4 gives, with probability at least $1 - \Theta(\lceil \ln T\rceil)\delta$,

$$\begin{aligned}
\Xi_{2,1} \leq & \frac{4}{3}UG_1 \ln\frac{1}{\delta} + 2U\sqrt{2G_2G_1\hat{L}_{1:T}\ln\frac{1}{\delta}}, \\
\Xi_{2,2} \leq & G_1G_2\hat{L}_{1:T} + \frac{2}{3}G_1^2 \ln\frac{1}{\delta} + 2\sqrt{G_1^3G_2\hat{L}_{1:T}\ln\frac{1}{\delta}}.
\end{aligned}$$

Let $\lambda_i = \frac{2U}{\sqrt{B}G_1}$. Using Lemma 2 and combining $\mathcal{T}_1$ and $\mathcal{T}_2$ gives, with probability at least $1 - \Theta(\lceil \ln T\rceil)\delta$,

$$\begin{aligned}
\mathrm{Reg}(f) = & \hat{L}_{1:T} - L_T(f) \\
\leq & \mathcal{T}_1 + \mathcal{T}_2 \\
\leq & 10U\sqrt{G_2G_1\hat{L}_{1:T}\ln\frac{1}{\delta}} + \frac{UG_1\sqrt{B}}{4} + \frac{6UG_2\hat{L}_{1:T}}{\sqrt{B - \frac{4}{3}\ln\frac{1}{\delta}}},
\end{aligned}$$

where we omit the constant terms and the lower order terms. Let $\gamma = \frac{6UG_2}{\sqrt{B-\frac{4}{3}\ln\frac{1}{\delta}}}$ and $U \leq \frac{1}{8G_2}\sqrt{B - \frac{4}{3}\ln\frac{1}{\delta}}$. Then $1 - \gamma \geq \frac{1}{4}$. Solving for $\hat{L}_{1:T}$ concludes the proof.

Finally, we explain why it must be satisfied that $K \leq d$. The space complexity of M-OMD-S is $O(KB + dB + K)$. According to Assumption 1, the coefficient α only depends on d. If $K \leq d$, then the space complexity of M-OMD-S is $O(dB)$. In this case, $B = \Theta(\alpha\mathcal{R})$. If $K > d$, then the space complexity is $O(KB)$. M-OMD-S must allocate the memory resource over K hypotheses. For instance, if $K = d^\nu$, $\nu > 1$, then $B = \Theta(K^{\frac{1-v}{v}}\alpha\mathcal{R})$. Thus the regret bound will increase a factor of order $O(K^{\frac{v-1}{2v}})$. □

A.6 Proof of Theorem 4

Proof. Let $\kappa(\boldsymbol{x}, \boldsymbol{v}) = \langle \boldsymbol{x}, \boldsymbol{v}\rangle^p$. The adversary first constructs $\mathcal{I}_T$. For $1 \leq t \leq 3B$, let $\boldsymbol{x}_t = \boldsymbol{e}_t$ where $\boldsymbol{e}_t$ is the standard basis vector in $\mathbb{R}^d$. Let $y_t = 1$ if t is odd. Otherwise, $y_t = -1$. For $3B + 1 \leq t \leq T$, let $(\boldsymbol{x}_t, y_t) \in \{(\boldsymbol{x}_\tau, y_\tau)\}_{\tau=1}^{3B}$ uniformly.

We construct a competitor as follows,

$$\bar{f}_{\mathbb{H}} = \frac{U}{\sqrt{3B}} \cdot \sum_{t=1}^{3B} y_t\kappa(\boldsymbol{x}_t, \cdot).$$

It is easy to prove

$$\begin{aligned}
L_T(\bar{f}_{\mathbb{H}}) = & T \cdot \ln\left(1 + \exp\left(-\frac{U}{\sqrt{3B}}\right)\right), \\
\|\bar{f}_{\mathbb{H}}\|_{\mathcal{H}} = & U.
\end{aligned}$$

Thus $\bar{f}_{\mathbb{H}} \in \mathbb{H}$.
Let $\mathcal{A}$ be an algorithm storing B examples at most. At the beginning of round t, let

$$f_t = \sum_{i\leq B} a_i^{(t)}\kappa(\boldsymbol{x}_i^{(t)}, \cdot)$$

be the hypothesis maintained by $\mathcal{A}$ where $\boldsymbol{x}_i^{(t)} \in \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{t-1}\}$. Besides, it must be satisfied that

$$\|f_t\|_{\mathcal{H}} = \sqrt{\sum_{i \le B} |a_i^{(t)}|^2} \le U. \tag{A9}$$

It is easy to obtain $\sum_{t=1}^{3B} \ell(f_t(\boldsymbol{x}_t), y_t) = 3\ln(2)B$. For any $t \ge 3B+1$, the expected per-round loss is

$$\mathbb{E}\left[\ell(f_t(\boldsymbol{x}_t), y_t)\right] = \frac{2}{3}\ln(2) + \frac{1}{3B}\sum_{i \le B} \ln(1 + \exp(-|a_i^{(t)}|)).$$

Note that $|a_1^{(t)}|, \ldots, |a_B^{(t)}|$ must satisfy (A9). By the Lagrangian multiplier method, the minimum is obtained at $|a_i^{(t)}| = \frac{U}{\sqrt{B}}$ for all i. Then we have

$$\mathbb{E}\left[\ell(f_t(\boldsymbol{x}_t), y_t)\right] \ge \frac{2}{3}\ln(2) + \frac{\ln(1 + \exp(-\frac{U}{\sqrt{B}}))}{3}.$$

It can be verified that

$$\forall 0 < x \le 0.2, \quad \ln(1 + \exp(-x)) \le \ln(2) - 0.45x.$$

Let $B < T$ and $U \le \frac{1}{5}\sqrt{3B}$. The expected regret w.r.t. $\bar{f}_{\mathbb{H}}$ is lower bounded as follows

$$\begin{aligned}
\mathbb{E}\left[\mathrm{Reg}(\bar{f}_{\mathbb{H}})\right] \ge & 3\ln(2)B + (T-3B) \cdot \frac{\ln(1+\exp(-\frac{U}{\sqrt{B}}))}{3} + \frac{2}{3}\ln(2)\cdot(T-3B) - T \cdot \ln\left(1 + \exp\left(-\frac{U}{\sqrt{3B}}\right)\right) \\
= & \left(\frac{2}{3}T + B\right) \cdot \left(\ln(2) - \ln\left(1 + \exp\left(-\frac{U}{\sqrt{3B}}\right)\right)\right) + \frac{1}{3}(T-3B)\ln\frac{1+\exp(-\frac{U}{\sqrt{B}})}{1+\exp(-\frac{U}{\sqrt{3B}})} \\
\ge & \frac{\sqrt{3}}{10} \cdot \frac{UT}{\sqrt{B}} + \frac{1}{3}\left(\frac{\sqrt{3}}{3} - 1\right) \cdot \frac{UT}{\sqrt{B}}.
\end{aligned}$$

It can be verified that $L_T(\bar{f}_{\mathbb{H}}) = \Theta(T)$. Replacing T with $\Theta(L_T(\bar{f}_{\mathbb{H}}))$ concludes the proof. □

A.7 Proof of Theorem 5

Proof. There is a $\xi_i \in (0, 4]$ such that

$$\Lambda_i = \sum_{t \in \mathcal{J}} \left[\left\|\bar{f}_{t,i}(2) - f\right\|^2_{\mathcal{H}_i} - \|f_{t,i} - f\|^2_{\mathcal{H}_i}\right] \le \xi_i U^2 |\mathcal{J}|.$$

We can rewrite Ξ_4 in (A8) as follow,

$$\sum_{t \in \mathcal{J}} \Xi_4 \le \left(\frac{\xi_i U^2}{2\lambda_i} + 4UG_1\right) \cdot |\mathcal{J}|.$$

If $\xi_i \le \frac{1}{|\mathcal{J}|}$, then $\frac{\xi_i U^2}{2\lambda_i} \cdot |\mathcal{J}| \le \frac{U^2}{2\lambda_i}$. Let $\lambda_i = \frac{U}{\sqrt{G_1 G_2 \hat{L}_{1:T}}}$. In this way, we obtain a new upper bound on $\mathcal{T}_2$. Combining $\mathcal{T}_1$ and $\mathcal{T}_2$ gives

$$\hat{L}_{1:T} - L_T(f) \le 12U\sqrt{G_2 G_1 \hat{L}_{1:T} \ln\frac{1}{\delta}} + \frac{16UG_2\hat{L}_{1:T}}{B - \frac{4}{3}\ln\frac{1}{\delta}} + 4UG_1\ln\frac{1}{\delta}.$$

Let $\gamma = 16UG_2(B - \frac{4}{3}\ln\frac{1}{\delta})^{-1}$ and $U < \frac{B - \frac{4}{3}\ln\frac{1}{\delta}}{32G_2}$. We have $\gamma \le \frac{1}{2}$. Solving for $\hat{L}_{1:T}$ concludes the proof. □

A.8 Auxiliary Lemmas

Lemma A.8.1 ([A3]). *$\forall t > M$ and $\forall i \in [K]$, $\mathbb{E}[\|\hat{\nabla}_{t,i} - \mu_{t,i}\|^2_{\mathcal{H}_i}] \le \frac{1}{t|V|}\mathcal{A}_{t,\kappa_i}$.*

Lemma A.8.2. *Let η_t follow* (10) *and $\boldsymbol{p}_1$ be the uniform distribution. Then*

$$\sum_{t=1}^{T}\sum_{i=1}^{K}\frac{\eta_t}{2}p_{t,i}c_{t,i}^2 \le \sqrt{2\ln K}\sqrt{\sum_{\tau=1}^{T}\sum_{i=1}^{K}p_{\tau,i}c_{\tau,i}^2} + \frac{\sqrt{2\ln K}}{2}\max_{t,i} c_{t,i}.$$

Proof. Let $\sigma_\tau = \sum_{i=1}^{K} p_{\tau,i}c_{\tau,i}^2$ and $\sigma_0 = 1$. We decompose the term as follows

$$\begin{aligned}\frac{\sqrt{\ln K}}{\sqrt{2}}\cdot\sum_{t=1}^{T}\frac{\sigma_t}{\sqrt{1+\sum_{\tau=1}^{t-1}\sigma_\tau}} =& \frac{\sqrt{\ln K}}{\sqrt{2}}\cdot\sum_{t=1}^{T}\frac{\sigma_t}{\sqrt{\sum_{\tau=0}^{t-1}\sigma_\tau}}\\ =& \frac{\sqrt{\ln K}}{\sqrt{2}}\cdot\left[\sigma_1 + \sum_{t=2}^{T}\frac{\sigma_{t-1}}{\sqrt{1+\sum_{\tau=1}^{t-1}\sigma_\tau}} + \sum_{t=2}^{T}\frac{\sigma_t - \sigma_{t-1}}{\sqrt{1+\sum_{\tau=1}^{t-1}\sigma_\tau}}\right].\end{aligned}$$

We analyze the third term.

$$\begin{aligned}\sum_{t=2}^{T}\frac{\sigma_t-\sigma_{t-1}}{\sqrt{1+\sum_{\tau=1}^{t-1}\sigma_\tau}} =& \frac{-\sigma_1}{\sqrt{1+\sigma_1}} + \frac{\sigma_T}{\sqrt{1+\sum_{\tau=1}^{T-1}\sigma_\tau}} + \sum_{t=2}^{T-1}\sigma_t\left[\frac{1}{\sqrt{1+\sum_{\tau=1}^{t-1}\sigma_\tau}} - \frac{1}{\sqrt{1+\sum_{\tau=1}^{t}\sigma_\tau}}\right]\\ \le& -\frac{\sigma_1}{\sqrt{1+\sigma_1}} + \max_{t=1,\ldots,T}\sigma_t\cdot\frac{1}{\sqrt{1+\sigma_1}}.\end{aligned}$$

Now we analyze the second term.

$$\sum_{t=2}^{T}\frac{\sigma_{t-1}}{\sqrt{1+\sum_{\tau=1}^{t-1}\sigma_\tau}} = \frac{\sigma_1}{\sqrt{1+\sigma_1}} + \sum_{t=3}^{T}\frac{\sigma_{t-1}}{\sqrt{1+\sum_{\tau=1}^{t-1}\sigma_\tau}}.$$

For any $a > 0$ and $b > 0$, we have $2\sqrt{a}\sqrt{b} \le a + b$. Let $a = 1 + \sum_{\tau=1}^{t-1}\sigma_\tau$ and $b = 1+\sum_{\tau=1}^{t-2}\sigma_\tau$. Then we have

$$2\sqrt{1+\sum_{\tau=1}^{t-1}\sigma_\tau}\cdot\sqrt{1+\sum_{\tau=1}^{t-2}\sigma_\tau} \le 2\left(1+\sum_{\tau=1}^{t-1}\sigma_\tau\right) - \sigma_{t-1}.$$

Dividing by $\sqrt{a}$ and rearranging terms yields

$$\frac{1}{2}\frac{\sigma_{t-1}}{\sqrt{1+\sum_{\tau=1}^{t-1}\sigma_\tau}} \le \sqrt{1+\sum_{\tau=1}^{t-1}\sigma_\tau} - \sqrt{1+\sum_{\tau=1}^{t-2}\sigma_\tau}.$$

Summing over $t = 3, \ldots, T$, we obtain

$$\sum_{t=3}^{T}\frac{\sigma_{t-1}}{\sqrt{1+\sum_{\tau=1}^{t-1}\sigma_\tau}} \le 2\sqrt{1+\sum_{\tau=1}^{T-1}\sigma_\tau} - 2\sqrt{1+\sigma_1}.$$

Summing over all results, we have

$$\begin{aligned}\sum_{t=1}^{T}\frac{\sigma_t}{\sqrt{1+\sum_{\tau=1}^{t-1}\sigma_\tau}} \le& 2\sqrt{1+\sum_{\tau=1}^{T-1}\sigma_\tau} - 2\sqrt{1+\sigma_1} + \max_t\sigma_t\cdot\frac{1}{\sqrt{1+\sigma_1}} + \sigma_1\\ \le& 2\sqrt{\sum_{\tau=1}^{T}\sigma_\tau} + \max_t\sigma_t,\end{aligned}$$

which concludes the proof. □

Lemma A.8.3 (Improved Bernstein's inequality [A4])**.** *Let $X_1, \ldots, X_n$ be a bounded martingale difference sequence w.r.t. the filtration $\mathcal{F} = (\mathcal{F}_k)_{1\leq k\leq n}$ and with $|X_k| \leq a$. Let $Z_t = \sum_{k=1}^t X_k$ be the associated martingale. Denote the sum of the conditional variances by*

$$\Sigma_n^2 = \sum_{k=1}^n \mathbb{E}\left[X_k^2|\mathcal{F}_{k-1}\right] \leq v,$$

where $v \in [0, V]$ is a random variable and $V \geq 2$ is a constant. Then for all constants $a > 0$, with probability at least $1 - 2\lceil \log V\rceil\delta$,

$$\max_{t=1,\ldots,n} Z_t < \frac{2a}{3}\ln\frac{1}{\delta} + \sqrt{\frac{2}{V}\ln\frac{1}{\delta}} + 2\sqrt{v\ln\frac{1}{\delta}}.$$

Lemma A.8.4. *With probability at least $1 - \Theta(\lceil \ln T\rceil)\delta$,*

$$\sum_{t\in\bar{T}_1\cup\mathcal{J}} \frac{1}{(\mathbb{P}[b_t = 1])^2}\|\nabla_{t,i}\|^2_{\mathcal{H}_i}\cdot\mathbb{I}_{b_t=1} \leq \sum_{t\in\bar{T}_1\cup\mathcal{J}}\frac{\|\nabla_{t,i}\|^2_{\mathcal{H}_i}}{\mathbb{P}[b_t=1]} + \frac{4G_1^2}{3}\ln\frac{1}{\delta} + 4\sqrt{G_1^3G_2\hat{L}_{1:T}\ln\frac{1}{\delta}}.$$

Proof. Define a random variable X_t by

$$X_t = \frac{\|\nabla_{t,i}\|^2_{\mathcal{H}_i}}{(\mathbb{P}[b_t=1])^2}\mathbb{I}_{b_t=1} - \frac{\|\nabla_{t,i}\|^2_{\mathcal{H}_i}}{\mathbb{P}[b_t=1]}, \quad |X_t| \leq 2G_1^2.$$

$\{X_t\}_{t\in\bar{T}_1}$ forms bounded martingale difference w.r.t. $\{b_\tau\}_{\tau=1}^{t-1}$. The sum of conditional variances is

$$\Sigma^2 \leq \sum_{t\in\bar{T}_1\cup\mathcal{J}}\mathbb{E}[X_t^2] \leq 4G_1^3G_2\hat{L}_{1:T}.$$

Using Lemma A.8.3 concludes the proof. □

Lemma A.8.5. *Let $\Delta_t = \langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle$. With probability at least $1 - \Theta(\lceil\ln T\rceil)\delta$,*

$$\sum_{t\in\bar{T}_1}\Delta_t \leq \frac{4UG_1}{3}\ln\frac{1}{\delta} + 2U\sqrt{2G_2G_1\hat{L}_{1:T}\ln\frac{1}{\delta}}.$$

Proof. The proof is similar to that of Lemma A.8.4. □

Lemma A.8.6. *For each $i \in [K]$, let $\psi_i(f) = \frac{1}{2\lambda_i}\|f\|^2_{\mathcal{H}_i}$. Then $\mathcal{B}_{\psi_i}(f,g) = \frac{1}{2\lambda_i}\|f-g\|^2_{\mathcal{H}_i}$ and the solutions of (4) and (5) are as follows*

$$\begin{aligned} f_{t,i} =& f'_{t-1,i} - \lambda_i\hat{\nabla}_{t,i}, \\ f'_{t,i} =& \min\left\{1, \frac{U}{\left\|f'_{t-1,i} - \lambda_i\nabla_{t,i}\right\|_{\mathcal{H}_i}}\right\}\cdot\left(f'_{t-1,i} - \lambda_i\nabla_{t,i}\right). \end{aligned}$$

Similarly, we can obtain the solution of (14)

$$f_{t+1,i} = \min\left\{1, \frac{U}{\left\|f_{t,i} - \lambda_i\tilde{\nabla}_{t,i}\right\|_{\mathcal{H}_i}}\right\}\cdot\left(f_{t,i} - \lambda_i\tilde{\nabla}_{t,i}\right).$$

Besides,

$$\langle f_{t,i} - f_{t+1,i}, \tilde{\nabla}_{t,i}\rangle - \mathcal{B}_{\psi_i}(f_{t+1,i}, f_{t,i}) \leq \frac{\lambda_i}{2}\left\|\tilde{\nabla}_{t,i}\right\|^2_{\mathcal{H}_i}.$$

Proof. We can solve (4) and (5) by Lagrangian multiplier method. Next we prove the last inequality.

$$\begin{aligned} \langle f_{t,i} - f_{t+1,i}, \tilde{\nabla}_{t,i}\rangle - \mathcal{B}_{\psi_i}(f_{t+1,i}, f_{t,i}) =& \left\langle f_{t,i} - f_{t+1,i}, \tilde{\nabla}_{t,i}\right\rangle - \frac{\|f_{t+1,i} - f_{t,i}\|^2_{\mathcal{H}_i}}{2\lambda_i} \\ =& \frac{\lambda_i}{2}\left\|\tilde{\nabla}_{t,i}\right\|^2_{\mathcal{H}_i} - \frac{1}{2\lambda_i}\left\|f_{t+1,i} - f_{t,i} - \lambda_i\hat{\nabla}_i\right\|^2_{\mathcal{H}_i} \\ \leq& \frac{\lambda_i}{2}\left\|\tilde{\nabla}_{t,i}\right\|^2_{\mathcal{H}_i}, \end{aligned}$$

which concludes the proof. □

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