Appendix

In this appendix, we give the detailed proofs of the main theorems and lemmas.

A.1 Proof of Lemma 1

Proof. We just analyze S_i for a fixed $i \in [K]$. Let the times of removing operation be J. Denote by $B = \alpha\mathcal{R}$, $\mathcal{J} = \{t_r, r \in [J]\}$, $T_r = \{t_{r-1}+1, \ldots, t_r\}$ and $t_0 = 0$. For any $t \in T_r$, if $\nabla_{t,i} \neq 0$, $\neg\mathrm{con}(a(i))$ and $b_{t,i} = 1$, then $(\boldsymbol{x}_t, y_t)$ will be added into S_i. For simplicity, we define a new notation $\nu_{t,i}$ as follows,

$$\nu_{t,i} = \mathbb{I}_{y_t f_{t,i}(\boldsymbol{x}_t)<1} \cdot \mathbb{I}_{\neg\mathrm{con}(a(i))} \cdot b_{t,i}.$$

At the end of the t_r-th round, the following equation can be derived,

$$|S_i| = |S_i(t_{r-1}+1)| + \sum_{t=t_{r-1}+1}^{t_r} \nu_{t,i} = \frac{B}{K},$$

where $|S_i(t_{r-1}+1)|$ is defined the initial size of S_i.

Let $s_r = t_{r-1}+1$. Assuming that there is no budget. We will present an expected bound on $\sum_{t=s_r}^{\bar{t}} \nu_{t,i}$ for any $\bar{t} > s_r$. In the first epoch, $s_1 = 1$ and $|S_i(s_1)| = 0$. Taking expectation w.r.t. $b_{t,i}$ gives

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=s_1}^{\bar{t}} \nu_{t,i}\right] =& \sum_{t=s_1}^{\bar{t}} \frac{\|\nabla_{t,i} - \hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i} \cdot \mathbb{I}_{\nabla_{t,i}\neq 0}}{\|\nabla_{t,i} - \hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i} + \|\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}} \\
\leq& \frac{2}{k_1} \underbrace{\left(1 + \sum_{t=2}^{\bar{t}} \left\| y_t\kappa_i(\boldsymbol{x}_t, \cdot) - \frac{\sum_{(\boldsymbol{x},y)\in V_t} y\kappa_i(\boldsymbol{x}, \cdot)}{|V_t|} \right\|^2_{\mathcal{H}_i}\right)}_{\tilde{\mathcal{A}}_{[s_1,\bar{t}],\kappa_i}} \\
=& \frac{2}{k_1} \tilde{\mathcal{A}}_{[s_1,\bar{t}],\kappa_i},
\end{aligned}$$

where we use the fact $\kappa_i(\boldsymbol{x}_t, \boldsymbol{x}_t) \geq k_1$. Let t_1 be the minimal $\bar{t}$ such that

$$\frac{2}{k_1} \tilde{\mathcal{A}}_{[s_1,t_1],\kappa_i} \geq \frac{B}{K}. \tag{A1}$$

The first epoch will end at t_1 in expectation. We define $\tilde{\mathcal{A}}_{T_1,\kappa_i} := \tilde{\mathcal{A}}_{[s_1,t_1],\kappa_i}$.

Next we consider $r \geq 2$. It must be $|S_i(s_r)| = \frac{B}{2K}$. Similar to $r = 1$, we can obtain

$$\mathbb{E}\left[\sum_{t=s_r}^{\bar{t}} \nu_{t,i}\right] \leq \frac{2}{k_1} \underbrace{\sum_{t=s_r}^{\bar{t}} \left\| y_t\kappa_i(\boldsymbol{x}_t, \cdot) - \frac{\sum_{(\boldsymbol{x},y)\in V_t} y\kappa_i(\boldsymbol{x}, \cdot)}{|V_t|} \right\|^2_{\mathcal{H}_i}}_{\tilde{\mathcal{A}}_{[s_r,\bar{t}],\kappa_i}} = \frac{2}{k_1} \tilde{\mathcal{A}}_{[s_r,\bar{t}],\kappa_i}.$$

Let t_r be the minimal $\bar{t}$ such that

$$\frac{2}{k_1} \tilde{\mathcal{A}}_{[s_r,\bar{t}],\kappa_i} \geq \frac{B}{2K}, \tag{A2}$$

Let $\tilde{\mathcal{A}}_{T_r,\kappa_i} = \tilde{\mathcal{A}}_{[s_r,\bar{t}],\kappa_i}$. Combining (A1) and (A2), and summing over $r = 1, \ldots, J$ yields

$$\begin{aligned}
\frac{B}{K} + \frac{B(J-1)}{2K} \leq& \frac{2}{k_1} \tilde{\mathcal{A}}_{T_1,\kappa_i} + \sum_{r=2}^{J} \frac{2}{k_1} \tilde{\mathcal{A}}_{T_r,\kappa_i} \\
\leq& \frac{2}{k_1} \underbrace{\sum_{t=s_1}^{T} \left\| y_t\kappa_i(\boldsymbol{x}_t, \cdot) - \frac{\sum_{(\boldsymbol{x},y)\in V_t} y\kappa_i(\boldsymbol{x}, \cdot)}{|V_t|} \right\|^2_{\mathcal{H}_i}}_{\tilde{\mathcal{A}}_{T,\kappa_i}} \\
\leq& \frac{2}{k_1} \tilde{\mathcal{A}}_{T,\kappa_i}.
\end{aligned}$$

Arranging terms gives

$$J \leq \frac{4K\tilde{\mathcal{A}}_{T,\kappa_i}}{Bk_1} - 1 \leq \frac{4K\tilde{\mathcal{A}}_{T,\kappa_i}}{Bk_1}. \tag{A3}$$

Taking expectation w.r.t. the randomness of reservoir sampling gives

$$\mathbb{E}[J] \leq \frac{4K}{Bk_1}\cdot\mathbb{E}[\tilde{\mathcal{A}}_{T,\kappa_i}] \leq \frac{12K}{Bk_1}\mathcal{A}_{T,\kappa_i}\cdot\left(1+\frac{\ln T}{M}\right)+\frac{32K}{Bk_1},$$

where the last inequality comes from Lemma A.1.1. Omitting the last constant term concludes the proof. □

Lemma A.1.1. *The reservoir sampling guarantees*

$$\forall i\in[K],\quad \mathbb{E}\left[\tilde{\mathcal{A}}_{T,\kappa_i}\right] \leq 3\mathcal{A}_{T,\kappa_i} + 8 + \frac{3\mathcal{A}_{T,\kappa_i}\ln T}{M}.$$

Proof. Let $\mu_{t,i} = -\frac{1}{t}\sum_{\tau=1}^{t} y_\tau\kappa_i(\boldsymbol{x}_\tau,\cdot)$ and $\tau_0 = M$. For $t\leq\tau_0$, it can be verified that

$$\begin{aligned}
\tilde{\mathcal{A}}_{\tau_0,\kappa_i} =& 1+\sum_{t=2}^{\tau_0}\|-y_t\kappa_i(\boldsymbol{x}_t,\cdot)-\mu_{t-1,i}\|^2_{\mathcal{H}_i}\\
=&1+\sum_{t=2}^{\tau_0}\|-y_t\kappa_i(\boldsymbol{x}_t,\cdot)-\mu_{t,i}+\mu_{t,i}-\mu_{t-1,i}\|^2_{\mathcal{H}_i}\\
\leq&1+2\mathcal{A}_{[2:\tau_0],\kappa_i}+2\sum_{t=2}^{\tau_0}\|\mu_{t,i}-\mu_{t-1,i}\|^2_{\mathcal{H}_i},
\end{aligned}$$

where $\mu_{0,i}=0$. Let V_t be the reservoir at the beginning of round t. Next we consider the case $t>\tau_0$.

$$\begin{aligned}
\tilde{\mathcal{A}}_{[\tau_0:T],\kappa_i} =& \sum_{t=\tau_0+1}^{T}\|-y_t\kappa_i(\boldsymbol{x}_t,\cdot)-\tilde{\mu}_{t-1,i}\|^2_{\mathcal{H}_i}\\
\leq&\sum_{t=\tau_0+1}^{T}3\left[\|y_t\kappa_i(\boldsymbol{x}_t,\cdot)+\mu_{t,i}\|^2_{\mathcal{H}_i}+\|\mu_{t,i}-\mu_{t-1,i}\|^2_{\mathcal{H}_i}+\|\mu_{t-1,i}-\tilde{\mu}_{t-1,i}\|^2_{\mathcal{H}_i}\right]\\
=&3\mathcal{A}_{[\tau_0:T],\kappa_i}+3\sum_{t=\tau_0+1}^{T}\|\mu_{t,i}-\mu_{t-1,i}\|^2_{\mathcal{H}_i}+3\sum_{t=\tau_0+1}^{T}\left\|\mu_{t-1,i}+\frac{1}{|V_t|}\sum_{(\boldsymbol{x},y)\in V_t}y\kappa_i(\boldsymbol{x},\cdot)\right\|^2_{\mathcal{H}_i}.
\end{aligned}$$

Taking expectation w.r.t. the reservoir sampling yields

$$\begin{aligned}
&\mathbb{E}[\tilde{\mathcal{A}}_{T,\kappa_i}]\\
=&\tilde{\mathcal{A}}_{\tau_0,\kappa_i}+\mathbb{E}[\tilde{\mathcal{A}}_{[\tau_0:T],\kappa_i}]\\
\leq&1+3\mathcal{A}_{T,\kappa_i}+3\sum_{t=2}^{T}\|\mu_{t,i}-\mu_{t-1,i}\|^2_{\mathcal{H}_i}+3\sum_{t=\tau_0+1}^{T}\mathbb{E}\left[\left\|\mu_{t-1,i}+\frac{1}{|V_t|}\sum_{(\boldsymbol{x},y)\in V_t}y\kappa_i(\boldsymbol{x},\cdot)\right\|^2_{\mathcal{H}_i}\right] \qquad \text{by Lemma } A.8.1\\
\leq&1+3\mathcal{A}_{T,\kappa_i}+3\sum_{t=2}^{T}\|\mu_{t,i}-\mu_{t-1,i}\|^2_{\mathcal{H}_i}+\sum_{t=\tau_0+1}^{T}\frac{3\mathcal{A}_{t-1,\kappa_i}}{(t-1)|V_t|}\\
\leq&1+3\mathcal{A}_{T,\kappa_i}+\sum_{t=2}^{T}\frac{12}{t^2}+\frac{3\mathcal{A}_{T,\kappa_i}\ln T}{M}\\
\leq&1+3\mathcal{A}_{T,\kappa_i}+7+\frac{3\mathcal{A}_{T,\kappa_i}\ln T}{M},
\end{aligned}$$

where $|V_t|=M$ for all $t\geq\tau_0$. □

A.2 Proof of Theorem 1

Proof. By the convexity of the Hinge loss function, we decompose the regret as follows

$$
\begin{aligned}
\mathrm{Reg}(f) =& \sum_{t=1}^{T} \ell\left(\sum_{j=1}^{K} p_{t,j} f_{t,j}(\boldsymbol{x}_t), y_t\right) - \sum_{t=1}^{T} \ell\left(f(\boldsymbol{x}_t), y_t\right) \\
\leq& \sum_{t=1}^{T}\sum_{j=1}^{K} p_{t,j} \ell\left(f_{t,j}(\boldsymbol{x}_t), y_t\right) - \sum_{t=1}^{T} \ell\left(f(\boldsymbol{x}_t), y_t\right) \\
\leq& \underbrace{\sum_{t=1}^{T}\left[\sum_{j=1}^{K} p_{t,j} \ell(f_{t,j}(\boldsymbol{x}_t), y_t) - \ell(f_{t,i}(\boldsymbol{x}_t), y_t)\right]}_{\mathcal{T}_1} + \underbrace{\sum_{t\in E_{T,i}} [\ell(f_{t,i}(\boldsymbol{x}_t), y_t) - \ell(f(\boldsymbol{x}_t), y_t)]}_{\mathcal{T}_2},
\end{aligned}
$$

where $E_{T,i} = \{t \in [T], \nabla_{t,i} \neq 0\}$.

A.2.1 Analyzing $\mathcal{T}_1$

The following analysis is same with the proof of Theorem 3.1 in [A1]. Let $c_{t,i} := \ell(f_{t,i}(\boldsymbol{x}_t), y_t)$. The updating of probability is as follows,

$$
p_{t+1,i} = \frac{w_{t+1,i}}{\sum_{j=1}^{K} w_{t+1,j}}, \quad w_{t+1,i} = \exp\left(-\eta_{t+1}\sum_{\tau=1}^{t} c_{\tau,i}\right).
$$

Similar to the analysis of Exp3 [A1], we define a potential function $\Gamma_t(\eta_t)$ as follows,

$$
\Gamma_t(\eta_t) := \frac{1}{\eta_t}\ln\sum_{i=1}^{K} p_{t,i}\exp(-\eta_t c_{t,i}) \leq -\sum_{i=1}^{K} p_{t,i}c_{t,i} + \frac{1}{2}\eta_t\sum_{i=1}^{K} p_{t,i}c_{t,i}^2,
$$

where we use the following two inequalities

$$
\ln x \leq x - 1, \forall x > 0, \quad \exp(-x) \leq 1 - x + \frac{x^2}{2}, \forall x \geq 0.
$$

Summing over $t \in [T]$ yields

$$
\sum_{t=1}^{T}\Gamma_t(\eta_t) \leq -\sum_{t=1}^{T}\langle \boldsymbol{p}_t, \boldsymbol{c}_t\rangle + \sum_{t=1}^{T}\sum_{i=1}^{K}\frac{\eta_t}{2}p_{t,i}c_{t,i}^2. \tag{A4}
$$

On the other hand, by the definition of $p_{t,i}$, we have

$$
\begin{aligned}
\Gamma_t(\eta_t) =& \frac{1}{\eta_t}\ln\frac{\sum_{i=1}^{K}\exp\left(-\eta_t\sum_{\tau=1}^{t-1}c_{\tau,i}\right)\exp(-\eta_t c_{t,i})}{\sum_{j=1}^{K}\exp\left(-\eta_t\sum_{\tau=1}^{t-1}c_{\tau,j}\right)} \\
=& \frac{1}{\eta_t}\ln\frac{\frac{1}{K}\sum_{i=1}^{K}\exp\left(-\eta_t\sum_{\tau=1}^{t}c_{\tau,i}\right)}{\frac{1}{K}\sum_{j=1}^{K}\exp\left(-\eta_t\sum_{\tau=1}^{t-1}c_{\tau,j}\right)} \\
=& \bar{\Gamma}_t(\eta_t) - \bar{\Gamma}_{t-1}(\eta_t),
\end{aligned}
$$

where $\bar{\Gamma}_t(\eta) = \frac{1}{\eta}\ln\frac{1}{K}\sum_{j=1}^{K}\exp\left(-\eta\sum_{\tau=1}^{t}c_{\tau,j}\right)$.
Without loss of generality, let $\bar{\Gamma}_0(\eta) = 0$. Summing over $t = 1, \ldots, T$ yields

$$
\sum_{t=1}^{T}\Gamma_t(\eta_t) = \bar{\Gamma}_T(\eta_T) - \bar{\Gamma}_0(\eta_1) + \sum_{t=1}^{T-1}\left[\bar{\Gamma}_t(\eta_t) - \bar{\Gamma}_t(\eta_{t+1})\right],
$$

where $\bar{\Gamma}_T(\eta_T) \geq \frac{1}{\eta_T}\ln\frac{1}{K} - \sum_{\tau=1}^{T} c_{\tau,i}$. Combining with the upper bound (A4), we obtain

$$\sum_{t=1}^{T}\langle \boldsymbol{p}_t, \boldsymbol{c}_t\rangle - \sum_{\tau=1}^{T} c_{\tau,i} \leq \frac{1}{\eta_T}\ln K + \sum_{t=1}^{T-1}\left[\bar{\Gamma}_t(\eta_{t+1}) - \bar{\Gamma}_t(\eta_t)\right] + \sum_{t=1}^{T}\sum_{i=1}^{K}\frac{\eta_t}{2}p_{t,i}c_{t,i}^2.$$

For simplicity, let $\bar{C}_{t,j} := \sum_{\tau=1}^{t} c_{\tau,j}$. The first derivative of $\bar{\Gamma}_t(\eta)$ w.r.t. η is as follows

$$\begin{aligned}\frac{\mathrm{d}\,\bar{\Gamma}_t(\eta)}{\mathrm{d}\,\eta} =& \frac{-\ln\sum_{j=1}^{K}\frac{\exp\left(-\eta\bar{C}_{t,j}\right)}{K}}{\eta^2} - \frac{\frac{1}{K}\sum_{j=1}^{K}\bar{C}_{t,j}\exp\left(-\eta\bar{C}_{t,j}\right)}{\frac{\eta}{K}\sum_{j=1}^{K}\exp\left(-\eta\bar{C}_{t,j}\right)}\\ =&\frac{1}{\eta^2}\mathrm{KL}(\tilde{p}_t, \frac{1}{K})\\ \geq& 0\end{aligned}$$

where $\tilde{p}_{t,j} = \frac{\exp\left(-\eta\bar{C}_{t,j}\right)}{\sum_{i=1}^{K}\exp\left(-\eta\bar{C}_{t,i}\right)}$. Since $\eta_{t+1} \leq \eta_t$, we have $\bar{\Gamma}_t(\eta_{t+1}) \leq \bar{\Gamma}_t(\eta_t)$. Combining all results, we have

$$\begin{aligned}&\sum_{t=1}^{T}\langle \boldsymbol{p}_t, \boldsymbol{c}_t\rangle - \sum_{\tau=1}^{T} c_{\tau,i}\\ \leq&\frac{\ln K}{\eta_T} - \frac{\ln K}{\eta_1} + \sum_{t=1}^{T}\sum_{i=1}^{K}\frac{\eta_t}{2}p_{t,i}c_{t,i}^2\\ \leq&\frac{\sqrt{\ln K}}{\sqrt{2}}\cdot\sqrt{1+\sum_{\tau=1}^{T-1}\langle \boldsymbol{p}_\tau, \boldsymbol{c}_\tau^2\rangle} - \frac{\sqrt{\ln K}}{\sqrt{2}} + \sqrt{\ln K}\left(\sqrt{2\sum_{\tau=1}^{T}\langle \boldsymbol{p}_\tau, \boldsymbol{c}_\tau^2\rangle} + \frac{\max_{t,j} c_{t,j}}{\sqrt{2}}\right) \qquad \text{by Lemma } A.8.2\\ \lesssim&\frac{3}{\sqrt{2}}\sqrt{\max_{t,j} c_{t,j}\cdot\sum_{\tau=1}^{T}\langle \boldsymbol{p}_\tau, \boldsymbol{c}_\tau\rangle\ln K}. \end{aligned} \tag{A5}$$

Solving for $\sum_{t=1}^{T}\langle \boldsymbol{p}_t, \boldsymbol{c}_t\rangle$ gives

$$\mathcal{T}_1 = \sum_{t=1}^{T}[\langle \boldsymbol{p}_t, \boldsymbol{c}_t\rangle - c_{t,i}] \leq \frac{3}{\sqrt{2}}\sqrt{\max_{t,j} c_{t,j}\cdot\sum_{\tau=1}^{T} c_{\tau,i}\ln K} + \frac{9}{2}\max_{t,j} c_{t,j}\cdot\ln K. \tag{A6}$$

A.2.2 Analyzing $\mathcal{T}_2$

We decompose $E_{T,i}$ as follows.

$$\begin{aligned}E_i =& \{t \in E_{T,i} : \mathrm{con}(a(i))\},\\ \mathcal{J}_i =& \{t \in E_{T,i} : |S_i| = \alpha\mathcal{R}_i, b_{t,i} = 1\},\\ \bar{E}_i =& E_{T,i} \setminus (E_i \cup \mathcal{J}_i).\end{aligned}$$

We separately analyze the regret in E_i, $\mathcal{J}_i$ and $\bar{E}_i$.
Case 1: regret in E_i
For any $f \in \mathbb{H}_i$, the convexity of loss function gives

$$\begin{aligned}&\ell(f_{t,i}(\boldsymbol{x}_t), y_t) - \ell(f(\boldsymbol{x}_t), y_t)\\ \leq&\langle f_{t,i} - f, \nabla_{t,i}\rangle\\ =&\underbrace{\langle f_{t,i} - f'_{t,i}, \hat{\nabla}_{t,i}\rangle}_{\Xi_1} + \underbrace{\langle f'_{t,i} - f, \nabla_{i(s_t),i}\rangle}_{\Xi_2} + \langle f'_{t,i} - f, \nabla_{t,i} - \nabla_{i(s_t),i}\rangle + \langle f_{t,i} - f'_{t,i}, \nabla_{t,i} - \hat{\nabla}_{t,i}\rangle\\ =&\Xi_1 + \Xi_2 + \langle f_{t,i} - f'_{t,i}, \nabla_{i(s_t),i} - \hat{\nabla}_{t,i}\rangle + \langle f_{t,i} - f, \nabla_{t,i} - \nabla_{i(s_t),i}\rangle\\ \leq&\left[\mathcal{B}_{\psi_i}(f, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f, f'_{t,i})\right] + \underbrace{\|f_{t,i} - f\|\cdot\gamma_{t,i}}_{\Xi_3} + \underbrace{\left\langle f_{t,i} - f'_{t,i}, \nabla_{i(s_t),i} - \hat{\nabla}_{t,i}\right\rangle - \mathcal{B}_{\psi_i}(f'_{t,i}, f_{t,i})}_{\Xi_4},\end{aligned}$$

where the standard analysis of OMD [A2] gives

$$\begin{aligned}
\Xi_1 \leq& \mathcal{B}_{\psi_i}(f'_{t,i}, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f'_{t,i}, f_{t,i}) - \mathcal{B}_{\psi_i}(f_{t,i}, f'_{t-1,i}),\\
\Xi_2 \leq& \mathcal{B}_{\psi_i}(f, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f, f'_{t,i}) - \mathcal{B}_{\psi_i}(f'_{t,i}, f'_{t-1,i}).
\end{aligned}$$

Substituting into $\gamma_{t,i}$ and summing over $t \in E_i$ gives

$$\begin{aligned}
\sum_{t\in E_i} \Xi_3 \leq& \sum_{t\in E_i} \frac{\max_t \|f_{t,i} - f\|_{\mathcal{H}_i} \cdot \left\|\nabla_{t,i} - \hat{\nabla}_{t,i}\right\|^2_{\mathcal{H}_i}}{\sqrt{1 + \sum_{\tau\leq t} \left\|\nabla_{\tau,i} - \hat{\nabla}_{\tau,i}\right\|^2_{\mathcal{H}_i} \cdot \mathbb{I}_{\nabla_{\tau,i}\neq 0}}}\\
\leq& 2(U+\lambda_i) \cdot \sum_{t\in E_i} \frac{\left\|\nabla_{t,i} - \hat{\nabla}_{t,i}\right\|^2_{\mathcal{H}_i}}{\sqrt{1 + \sum_{\tau\leq t} \left\|\nabla_{\tau,i} - \hat{\nabla}_{\tau,i}\right\|^2_{\mathcal{H}_i} \cdot \mathbb{I}_{\nabla_{\tau,i}\neq 0}}}\\
\leq& 4(U+\lambda_i)\sqrt{\tilde{\mathcal{A}}_{T,\kappa_i}},
\end{aligned}$$

where $\|f_{t,i}\|_{\mathcal{H}_i} \leq U + \lambda_i$.
According to Lemma A.8.6, we can obtain

$$\sum_{t\in E_i} \Xi_4 \leq \frac{\lambda_i}{2} \sum_{t\in E_i} \left\|\nabla_{i(s_t),i} - \hat{\nabla}_{t,i}\right\|^2_{\mathcal{H}_i} \leq 2\lambda_i \tilde{\mathcal{A}}_{T,\kappa_i}.$$

Case 2: regret in $\bar{E}_i$
We decompose the instantaneous regret as follows,

$$\begin{aligned}
&\langle f_{t,i} - f, \nabla_{t,i}\rangle\\
=& \underbrace{\langle f_{t,i} - f'_{t,i}, \hat{\nabla}_{t,i}\rangle}_{\Xi_1} + \underbrace{\langle f'_{t,i} - f, \tilde{\nabla}_{t,i}\rangle}_{\Xi_2} + \underbrace{\langle f_{t,i} - f'_{t,i}, \tilde{\nabla}_{t,i} - \hat{\nabla}_{t,i}\rangle}_{\Xi_3} + \left\langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\right\rangle\\
\leq& \mathcal{B}_{\psi_i}(f, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f, f'_{t,i}) + \langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle + \Xi_3 - \left[\mathcal{B}_{\psi_i}(f'_{t,i}, f_{t,i}) + \mathcal{B}_{\psi_i}(f_{t,i}, f'_{t-1,i})\right]\\
=& \mathcal{B}_{\psi_i}(f, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f, f'_{t,i}) + \langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle + \Xi_3 - \mathcal{B}_{\psi_i}(f'_{t,i}, f_{t,i}) - \frac{\lambda_i}{2}\|\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}\\
\leq& \mathcal{B}_{\psi_i}(f, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f, f'_{t,i}) + \langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle - \frac{\lambda_i}{2}\|\tilde{\nabla}_{t,i} - \hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i} - \frac{\lambda_i}{2}\|\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i} \quad \text{by Lemma } A.8.6\\
=& \mathcal{B}_{\psi_i}(f, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f, f'_{t,i}) + \langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\rangle + \frac{\lambda_i}{2}\left(\frac{\|\nabla_{t,i} - \hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}}{(\mathbb{P}[b_{t,i}=1])^2}\mathbb{I}_{b_{t,i}=1} - \|\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}\right),
\end{aligned}$$

where $\Xi_1 + \Xi_2$ follows the analysis in **Case 1**.
Case 3: regret in $\mathcal{J}_i$
Recalling that the second mirror updating is

$$f'_{t,i} = \mathop{\arg\min}_{f\in\mathbb{H}_i} \left\{\langle f, \tilde{\nabla}_{t,i}\rangle + \mathcal{B}_{\psi_i}(f, \bar{f}'_{t-1,i}(1))\right\}.$$

We still decompose the instantaneous regret as follows

$$\langle f_{t,i} - f, \nabla_{t,i}\rangle = \underbrace{\langle f_{t,i} - f'_{t,i}, \hat{\nabla}_{t,i}\rangle}_{\Xi_1} + \underbrace{\langle f'_{t,i} - f, \tilde{\nabla}_{t,i}\rangle}_{\Xi_2} + \underbrace{\langle f_{t,i} - f'_{t,i}, \tilde{\nabla}_{t,i} - \hat{\nabla}_{t,i}\rangle}_{\Xi_3} + \left\langle f_{t,i} - f, \nabla_{t,i} - \tilde{\nabla}_{t,i}\right\rangle.$$

We reanalyze Ξ_1 and Ξ_2 as follows

$$\begin{aligned}
\Xi_1 \leq& \mathcal{B}_{\psi_i}(f'_{t,i}, f'_{t-1,i}) - \mathcal{B}_{\psi_i}(f'_{t,i}, f_{t,i}) - \mathcal{B}_{\psi_i}(f_{t,i}, f'_{t-1,i}),\\
\Xi_2 \leq& \mathcal{B}_{\psi_i}(f, \bar{f}'_{t-1,i}(1)) - \mathcal{B}_{\psi_i}(f, f'_{t,i}) - \mathcal{B}_{\psi_i}(f'_{t,i}, \bar{f}'_{t-1,i}(1)).
\end{aligned}$$

Then $\Xi_1+\Xi_2+\Xi_3$ can be further bounded as follows,

$$\begin{aligned}\Xi_1+\Xi_2+\Xi_3 \leq & \mathcal{B}_{\psi_i}(f,f'_{t-1,i})-\mathcal{B}_{\psi_i}(f,f'_{t,i})+\left[\mathcal{B}_{\psi_i}(f,\bar{f}'_{t-1,i}(1))-\mathcal{B}_{\psi_i}(f,f'_{t-1,i})\right]+\\ & \left[\mathcal{B}_{\psi_i}(f'_{t,i},f'_{t-1,i})-\mathcal{B}_{\psi_i}(f'_{t,i},\bar{f}'_{t-1,i}(1))\right]-\left[\mathcal{B}_{\psi_i}(f'_{t,i},f_{t,i})+\mathcal{B}_{\psi_i}(f_{t,i},f'_{t-1,i})\right]+\Xi_3.\end{aligned}$$

By Lemma A.8.6, we analyze the following term

$$\begin{aligned}&\Xi_3-\left[\mathcal{B}_{\psi_i}(f'_{t,i},f_{t,i})+\mathcal{B}_{\psi_i}(f_{t,i},f'_{t-1,i})\right]\\ \leq&\frac{\lambda_i}{2}\left[\frac{\|\nabla_{t,i}-\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}}{(\mathbb{P}[b_{t,i}=1])^2}\mathbb{I}_{b_{t,i}=1}-\|\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}\right]-\frac{1}{2\lambda_i}\|f'_{t-1,i}-f'_{t,i}\|^2_{\mathcal{H}_i}+\langle f'_{t-1,i}-f'_{t,i},\tilde{\nabla}_{t,i}\rangle.\end{aligned}$$

Substituting into the instantaneous regret gives

$$\begin{aligned}\langle f_{t,i}-f,\nabla_{t,i}\rangle \leq & \mathcal{B}_{\psi_i}(f,f'_{t-1,i})-\mathcal{B}_{\psi_i}(f,f'_{t,i})+\left\langle f_{t,i}-f,\nabla_{t,i}-\tilde{\nabla}_{t,i}\right\rangle+\langle\tilde{\nabla}_{t,i},f'_{t-1,i}-f'_{t,i}\rangle+\\ & \frac{\|\bar{f}'_{t-1,i}(1)-f\|^2_{\mathcal{H}_i}-\|f'_{t-1,i}-f\|^2_{\mathcal{H}_i}}{2\lambda_i}+\frac{\lambda_i}{2}\frac{\|\nabla_{t,i}-\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}}{(\mathbb{P}[b_{t,i}=1])^2}\mathbb{I}_{b_{t,i}=1}-\frac{\lambda_i}{2}\|\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}.\end{aligned}$$

Combining all
Combining the above three cases, we obtain

$$\begin{aligned}\mathcal{T}_2\leq&\sum_{t\in E_{T,i}}\left[\mathcal{B}_{\psi_i}(f,f'_{t-1,i})-\mathcal{B}_{\psi_i}(f,f'_{t,i})\right]+4(U+\lambda_i)\tilde{\mathcal{A}}^{\frac{1}{2}}_{T,\kappa_i}+\sum_{t\in\mathcal{J}_i}\left[\langle\tilde{\nabla}_{t,i},f'_{t-1,i}-f'_{t,i}\rangle+\frac{2U^2}{\lambda_i}\right]+\\ &\frac{\lambda_i}{2}\sum_{t\in\bar{E}_i\cup\mathcal{J}_i}\left[\frac{\|\nabla_{t,i}-\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}}{(\mathbb{P}[b_{t,i}=1])^2}\mathbb{I}_{b_{t,i}=1}-\|\hat{\nabla}_{t,i}\|^2_{\mathcal{H}_i}\right]+\sum_{t\in\bar{E}_i\cup\mathcal{J}_i}\left\langle f_{t,i}-f,\nabla_{t,i}-\tilde{\nabla}_{t,i}\right\rangle+2\lambda_i\tilde{\mathcal{A}}_{T,\kappa_i}.\end{aligned}$$

Recalling that $\|f'_{t,i}\|_{\mathcal{H}_i}\leq U$ and $f\leq U$. Conditioned on $b_{s_r,i},\ldots,b_{t-1,i}$, taking expectation w.r.t. $b_{t,i}$ gives

$$\mathbb{E}\left[\mathcal{T}_2\right]\leq\frac{U^2}{2\lambda_i}+\left(2U+\frac{2U^2}{\lambda_i}\right)\cdot J+\frac{5\lambda_i}{2}\tilde{\mathcal{A}}_{T,\kappa_i}+4(U+\lambda_i)\sqrt{\tilde{\mathcal{A}}_{T,\kappa_i}}. \tag{A7}$$

Let $\lambda_i=\frac{\sqrt{K}U}{2\sqrt{B}}$. Assuming that $B\geq K$, we have $\lambda_i\leq\frac{U}{2}$. Then

$$\begin{aligned}\mathbb{E}\left[\mathcal{T}_2\right]=&O\left(\frac{U\sqrt{B}}{\sqrt{K}}+\frac{\sqrt{K}U}{\sqrt{B}k_1}\tilde{\mathcal{A}}_{T,\kappa_i}+U\sqrt{\tilde{\mathcal{A}}_{T,\kappa_i}}\right) && \text{by (A3)}\\ =&O\left(\frac{U\sqrt{B}}{\sqrt{K}}+\frac{\sqrt{K}U\mathcal{A}_{T,\kappa_i}\ln T}{\sqrt{B}k_1}\right), && \text{by Lemma } A.1.1\end{aligned}$$

where we omit the lower order term.

A.2.3 Combining $\mathcal{T}_1$ and $\mathcal{T}_2$

Combining $\mathcal{T}_1$ and $\mathcal{T}_2$, and taking expectation w.r.t. the randomness of reservoir sampling gives

$$\begin{aligned}&\mathbb{E}\left[\mathrm{Reg}(f)\right]\\ =&\mathbb{E}\left[\sum_{t=1}^T\ell(f_t(\boldsymbol{x}_t),y_t)-\sum_{t=1}^T\ell(f_{t,i}(\boldsymbol{x}_t),y_t)\right]+\mathbb{E}\left[\mathcal{T}_2\right]\\ \leq&\frac{3}{\sqrt{2}}\mathbb{E}\left[\sqrt{\max_{t,j}c_{t,j}\cdot\sum_{t=1}^T\ell(f_{t,i}(\boldsymbol{x}_t),y_t)\ln K}\right]+\frac{9}{2}\max_{t,j}c_{t,j}\cdot\ln K+\mathbb{E}\left[\mathcal{T}_2\right] \qquad \text{by (A6)}\\ =&\frac{3}{\sqrt{2}}\mathbb{E}\left[\sqrt{\max_{t,j}c_{t,j}\cdot\left(\sum_{t=1}^T\ell(f(\boldsymbol{x}_t),y_t)+\mathbb{E}\left[\mathcal{T}_2\right]\right)\ln K}\right]+\frac{9}{2}\max_{t,j}c_{t,j}\cdot\ln K+\mathbb{E}\left[\mathcal{T}_2\right]\\ =&O\left(\sqrt{\max_{t,j}c_{t,j}\cdot L_T(f)\ln K}+\frac{U\sqrt{B}}{\sqrt{K}}+\frac{\sqrt{K}U\mathcal{A}_{T,\kappa_i}\ln T}{\sqrt{B}k_1}+\max_{t,j}c_{t,j}\cdot\ln K\right).\end{aligned}$$

For the Hinge loss function, we have $\max_{t,j}c_{t,j}=1+U$. □

A.3 Proof of Theorem 2

Proof. For simplicity, denote by

$$\Lambda_i = \sum_{t \in \mathcal{J}_i} \left[\left\| \bar{f}'_{t-1,i}(1) - f \right\|^2_{\mathcal{H}_i} - \left\| f'_{t-1,i} - f \right\|^2_{\mathcal{H}_i} \right].$$

There must be a constant $\xi_i \in (0, 4]$ such that $\Lambda_i \leq \xi_i U^2 J$. We will prove a better regret bound if ξ_i is small enough. Recalling that (A3) gives an upper bound on J. If $\xi_i \leq \frac{1}{J}$, then we rewrite (A7) by

$$\mathcal{T}_2 \leq \frac{U^2}{2\lambda_i} + 2UJ + \frac{U^2}{2\lambda_i} + \frac{5\lambda_i}{2} \tilde{\mathcal{A}}_{T,\kappa_i} + 4(U + \lambda_i)\sqrt{\tilde{\mathcal{A}}_{T,\kappa_i}}.$$

Let $\lambda_i = \frac{\sqrt{2}U}{\sqrt{5\tilde{\mathcal{A}}_{T,\kappa_i}}}$. Taking expectation w.r.t. the reservoir sampling and using Lemma A.1.1 gives

$$\mathbb{E}\left[\mathcal{T}_2\right] = O\left(\frac{UK}{Bk_1} \mathcal{A}_{T,\kappa_i} \ln T + U\sqrt{\mathcal{A}_{T,\kappa_i} \ln T} \right),$$

where we omit the lower order terms. Combining $\mathcal{T}_1$ and $\mathcal{T}_2$ gives

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{Reg}(f)\right] \\
=&\frac{3}{\sqrt{2}}\mathbb{E}\left[\sqrt{\max_{t,j} c_{t,j} \cdot \left(\sum_{t=1}^{T} \ell(f(\boldsymbol{x}_t), y_t) + \mathbb{E}\left[\mathcal{T}_2\right] \right) \ln K} \right] + \frac{9}{2} \max_{t,j} c_{t,j} \cdot \ln K + \mathbb{E}\left[\mathcal{T}_2\right] \\
=&O\left(\sqrt{\max_{t,j} c_{t,j} \cdot L_T(f) \ln K} + \frac{UK}{Bk_1} \mathcal{A}_{T,\kappa_i} \ln T + U\sqrt{\mathcal{A}_{T,\kappa_i} \ln T} + \max_{t,j} c_{t,j} \cdot \ln K \right),
\end{aligned}$$

which concludes the proof. □

A.4 Proof of Lemma 2

Proof. Recalling the definition of $\mathcal{J}$ and T_r in Section A.1. For any $t \in T_r$, $(\boldsymbol{x}_t, y_t)$ will be added into S only if $b_t = 1$. At the end of the t_r-th round, we have

$$|S| = \frac{B}{2}\mathbb{I}_{r \neq 1} + \sum_{t=t_{r-1}+1}^{t_r} b_t = B.$$

We remove $\frac{B}{2}$ examples from S at the end of the t_r-th round.

Assuming that there is no budget. For any $t_0 > t_{r-1} + 1$, we will prove an upper bound on $\sum_{t=t_{r-1}+1}^{t_0} b_t$. Define a random variable X_t as follows,

$$X_t = b_t - \mathbb{P}[b_t = 1], \quad |X_t| \leq 1.$$

Under the condition of $b_{t_{r-1}+1}, \ldots, b_{t-1}$, we have $\mathbb{E}_{b_t}[X_t] = 0$. Thus $X_{t_{r-1}+1}, \ldots, X_{t_0}$ form bounded martingale difference. Let $\hat{L}_{a:b} := \sum_{t=a}^{b} \ell(f_t(\boldsymbol{x}_t), y_t)$ and $\hat{L}_{1:T} \leq N$. The sum of conditional variances satisfies

$$\Sigma^2 \leq \sum_{t=t_{r-1}+1}^{t_0} \frac{|\ell'(f_t(\boldsymbol{x}_t), y_t)|}{|\ell'(f_t(\boldsymbol{x}_t), y_t)| + G_1} \leq \frac{G_2}{G_1} \hat{L}_{t_{r-1}+1:t_0},$$

where the last inequality comes from Assumption 3. Since $\hat{L}_{t_{r-1}+1:t_0}$ is a random variable, Lemma A.8.3 can give an upper bound on $\sum_{t=t_{r-1}+1}^{t_0} b_t$ with probability at least $1 - 2\lceil \log N \rceil \delta$. Let t_r be the minimal t_0 such that

$$\frac{G_2}{G_1} \hat{L}_{t_{r-1}+1:t_r} + \frac{2}{3} \ln \frac{1}{\delta} + 2\sqrt{\frac{G_2}{G_1} \hat{L}_{t_{r-1}+1:t_r} \ln \frac{1}{\delta}} \geq \frac{B}{2} \cdot \mathbb{I}_{r \geq 2} + B \cdot \mathbb{I}_{r=1}.$$