# GROUP ACTIONS

Exercises

1. Let $\rho : G \mapsto Sym(\Omega)$ be a representation of the group $G$ on the set $\Omega$. Show that this defines an action of $G$ on $\Omega$ by setting $\alpha^x := \alpha^{\rho(x)}$ for all $\alpha \in \Omega$ and $x \in G$, and that $\rho$ is the representation which corresponds to this action.

*Proof.* Since $\rho$ is a representation of $G$ on $\Omega$, $\rho$ is a homomorphism. Let $\alpha^x := \alpha^{\rho(x)}$, then

$$\alpha^1 = \alpha^{\rho(1)} = \alpha;$$
$$(\alpha^x)^y = (\alpha^{\rho(x)})^{\rho(y)} = \alpha^{\rho(xy)} = \alpha^{xy}.$$

Thus, this defines an action of $G$ on $\Omega$.

2. Explain why we do not usually get an action of a group $G$ on itself by defining $a^x := xa$. Show, however, that $a^x := x^{-1}a$ does give an action of $G$ on itself (called the left regular representation of $G$). Similarly, show how to define an action of a group on the set of left cosets $aH(a \in G)$ of a subgroup $H$.

*Proof.* Let $a^x := xa$, then

$$(a^x)^y = (xa)^y = yxa \neq xya = a^{xy}.$$

The equation holds if and only if $G$ is an abelian group.

For all $a \in G$ and $x \in G$, let $a^x := x^{-1}a$, then

$$a^1 = 1^{-1}a = a;$$
$$(a^x)^y = (x^{-1}a)^y = y^{-1}x^{-1}a = (xy)^{-1}a = a^{xy}.$$

Thus, this defines an action of $G$ on itself.

Let $\Omega = \{aH|\ a \in G\}$, defining an action of $G$ on $\Omega$ by setting $(aH)^x := x^{-1}aH$.

3. Show that the kernel of $\rho_H$ in Example 1.3.4 is equal to the largest normal subgroup of $G$ contained in the subgroup $H$.

*Proof.* Let $\Gamma_H := \{Ha|\ a \in G\}$ and define an action if $G$ on $\Gamma_H$ by right multiplication: $(Ha)^x := Hax$. We denote the corresponding representation of $G$ on $\Gamma_H$ by $\rho_H$. We have

$$\ker\ \rho_H = \bigcap_{a \in G} a^{-1}Ha.$$

Assume that $N$ is the subgroup of $H$ and $N \trianglelefteq G$, then we have $N = N^a \leq H^a$ for any $a \in G$. Thus, we obtained that $N \leq \bigcap_{a \in G} H^a = \ker\ \rho_H$.

Hence, by the arbitrariness of $N$, $\ker\ \rho_H$ is equal to the largest normal subgroup of $G$ contained in the subgroup $H$.

4. Use the previous exercise to prove that if $G$ is a group with a subgroup $H$ of finite index $n$, then $G$ has a normal subgroup $K$ contained in $H$ whose index in $G$ is finite and divides $n!$. In particular, if $H$ has index 2 then $H$ is normal in $G$.