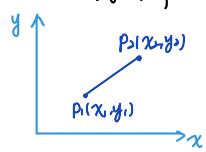
§6. 微分中值定理、泰勒公式

一、微分中值定理

(1) 一元回顾

(Lagrange) $\exists \xi \in (x_1, x_2)$, $f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2)$

(2) 二元分析



考虑 $\varphi(t) := f(x_2 + t(x_1 - x_2),\ y_2 + t(y_1 - y_2))$

$\varphi(1) = f(x_1, y_1)$ $\quad \varphi(0) = f(x_2, y_2)$ $\Rightarrow f(x_1, y_1) - f(x_2, y_2) = \varphi(1) - \varphi(0)$

$\varphi'(\theta) = f_x[x_2 + \theta(x_1 - x_2), y_2 + \theta(y_1 - y_2)](x_1 - x_2) + f_y[x_2 + \theta(x_1 - x_2), y_2 + \theta(y_1 - y_2)](y_1 - y_2)$

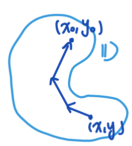
$\Rightarrow f(x_1, y_1) - f(x_2, y_2) = \varphi'(\theta) = \cdots$

(3) $\boxed{\text{定理}}$ $z = f(x,y)$ 定义于区域 $D \subset \mathbb{R}^2$ 上，$P_1 = (x_1, y_1)$. $P_2(x_2, y_2)$. 线段 $\overline{P_1P_2} \subset D$，$f(x,y) \in C^1(D)$

则 $\exists \theta \in (0,1)$, s.t. $f(x_1, y_1) - f(x_2, y_2) = f_x(x_2 + \theta(x_1 - x_2), y_2 + \theta(y_1 - y_2))(x_1 - x_2) + f_y(x_2 + \theta(x_1 - x_2), y_2 + \theta(y_1 - y_2))(y_1 - y_2)$

$\boxed{\text{推论1}}$ $\forall (x,y) \in D$, $f_x(x,y) = f_y(x,y) \equiv 0 \Rightarrow f(x,y) \equiv C$

证：取 $P_0(x_0, y_0) \in D$. $\forall P = (x,y) \in D$. $(x,y) \neq (x_0, y_0)$. $\exists P_j = (x_j, y_j) \in D$, $1 \le j \le n$, s.t. $\overline{P_0P_1}$, $\overline{P_1P_2}$ $\cdots$, $\overline{P_nP} \subset D$



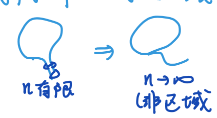
利用Lagrange中值定理

$\Rightarrow f(x,y) = f(x_n, y_n) = \cdots = f(x_1, y_1)$

$= f(x_0, y_0) \equiv C$

需证有限步可以实现

区域除非一条曲线才需无限步



$\boxed{\text{推论2}}$ $f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = df(x_0 + \theta\Delta x, y_0 + \theta\Delta y)$. $\theta \in (0,1)$

二、Taylor公式

(1) $\boxed{\text{定理}}$ (Lagrange余项) 区域 $D \subset \mathbb{R}^2$. $f(x,y) \in C^{n+1}(D)$, $P_0 = (x_0, y_0)$, $P = (x,y)$. $\overline{P_0P} \subset D$

则 $f(x,y) = \sum_{k=0}^{n} \frac{1}{k!} d^k f(x_0, y_0) + \frac{1}{(n+1)!} d^{n+1} f(x_0 + \theta(x - x_0), y_0 + \theta(y - y_0))$ $(*)$

证：构造 $\varphi(t) = f(x_0 + t(x - x_0), y_0 + t(y - y_0))$. $t \in [0,1]$. $\varphi(1) = f(x_0, y_0)$ 且 $\varphi(t) \in C^{n+1}[0,1]$

得 $\varphi(1) = \sum_{k=0}^{n} \frac{1}{k!} \varphi^{(k)}(0) + \frac{1}{(n+1)!} \varphi^{(n+1)}(\theta)$. 此即带Lagrange余项Taylor公式

(2) $d^k f(x_0, y_0) = \left(\frac{\partial}{\partial x}\Delta x + \frac{\partial}{\partial y}\Delta y\right)^k f(x_0, y_0)$. (其中 $\Delta x = x - x_0$, $\Delta y = y - y_0$)

$= \sum_{i+j=k} \binom{k}{i\ j} \frac{\partial^k f(x_0, y_0)}{\partial x^i \partial x^j} \Delta x^i \Delta x^j$

$\boxed{\text{推论}}$ $f(x,y) = \underbrace{\sum_{k=0}^{n} \frac{1}{k!} \sum_{i+j=k} \binom{k}{i\ j} \frac{\partial^k f(x_0, y_0)}{\partial x^i \partial x^j} \Delta x^i \Delta x^j}_{\text{Taylor多项式}} + \underbrace{\frac{1}{(n+1)!} \sum_{i+j=n+1} \binom{n+1}{i\ j} \frac{\partial^{n+1} f(x_0 + \theta\Delta x, y_0 + \theta\Delta y)}{\partial x^i\ \partial x^j} \Delta x^i \Delta x^j}_{\text{Lagrange余项}}$

(3) 推广至多元

$\boxed{\text{推论}}$ $f(x_1, x_2, \cdots, x_m) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{i_1 + i_2 + \cdots + i_m = k} \binom{k}{i_1\ i_2 \cdots i_m} \frac{\partial^k f(x_1^0, x_2^0, \cdots, x_m^0)}{\partial x_1^{i_1}\ \partial x_2^{i_2} \cdots \partial x_m^{i_m}} \Delta x_1^{i_1} \Delta x_2^{i_2} \cdots \Delta x_m^{i_m}$

$+ \frac{1}{(n+1)!} \sum_{i_1 + \cdots + i_m = n+1} \binom{n+1}{i_1\ i_2 \cdots i_m} \frac{\partial^{n+1} f(x_1^0 + \theta\Delta x_1, \cdots, x_m^0 + \theta\Delta x_m)}{\partial x_1^{i_1}\ \partial x_2^{i_2} \cdots \partial x_m^{i_m}} \Delta x_1^{i_1} \cdots \Delta x_m^{i_m}$

(3) $\boxed{\text{定理}}$ (Pieno余项) 区域 $D \subset \mathbb{R}^2$. $f(x,y) \in C^{n+1}(D)$, $P_0 = (x_0, y_0)$, $P = (x,y)$. $\overline{P_0P} \subset D$

则 $f(x,y) = \sum_{k=0}^{n} \frac{1}{k!} d^k f(x_0, y_0) + o(\rho^n)$. $\rho = \sqrt{\Delta x^2 + \Delta y^2}$

证：

$$\lim_{(\Delta x, \Delta y) \to 0} \frac{\frac{1}{(n+1)!} \sum_{i+j=n+1} \binom{n+1}{i\ j} \frac{\partial^{n+1} f(x_0 + \theta\Delta x, y_0 + \theta\Delta y)}{\partial x^i\ \partial x^j} \Delta x^i \Delta x^j}{\rho^n} = \lim_{(\Delta x, \Delta y) \to 0} A \frac{\Delta x^i \Delta y^j}{\rho^{i+j-1}} = 0$$

例题

例1. 求 $f(x,y)=\sin(\frac{\pi}{2}x^2y)$ 在 $(1,1)$ 二阶带 Pieno 余项型 Taylor 公式

解: $f(x,y)=\sum_{k=0}^{2}\frac{1}{k!}\sum_{i+j=k}\binom{k}{i}\frac{\partial^k f(1,1)}{\partial x^i\partial y^j}\Delta x^i\Delta y^j+o(\Delta x^2+\Delta y^2)$

$=f(1,1)+(\frac{\partial f(1,1)}{\partial x}\Delta x+\frac{\partial f(1,1)}{\partial y}\Delta y)+\frac{1}{2!}(\frac{\partial^2 f(1,1)}{\partial x^2}\Delta x^2+2\frac{\partial^2 f(1,1)}{\partial x\partial y}\Delta x\Delta y+\frac{\partial^2 f(1,1)}{\partial y^2}\Delta y^2)+o(\Delta x^2+\Delta y^2)$

$f(1,1)=1$ $\frac{\partial f(1,1)}{\partial x}=\pi xy\cos(\frac{\pi}{2}x^2y)\big|_{(x,y)=(1,1)}=0$. $\frac{\partial f(1,1)}{\partial y}=\frac{\pi}{2}x^2\cos(\frac{\pi}{2}x^2y)$

$\frac{\partial^2 f(1,1)}{\partial x^2}=\pi y(\cos(\frac{\pi}{2}x^2y)-\pi x^2y\sin(\frac{\pi}{2}x^2y))\big|_{(x,y)=(1,1)}=-\pi^2$

$\frac{\partial^2 f(1,1)}{\partial x\partial y}=\pi x(\cos(\frac{\pi}{2}x^2y)-\frac{\pi}{2}x^2y\sin(\frac{\pi}{2}x^2y))\big|_{(x,y)=(1,1)}=-\frac{\pi^2}{2}$

$\frac{\partial^2 f(1,1)}{\partial y^2}=-\frac{\pi^2}{4}x^4\sin(\frac{\pi}{2}x^2y)=-\frac{\pi^2}{4}$

$\therefore f(x,y)=1-\frac{\pi^2}{8}(4\Delta x^2+4\Delta x\Delta y+\Delta y^2)+o(\Delta x^2+\Delta y^2)$ $(\Delta x=x-1,\ \Delta y=y-1)$

例2. 在 $(0,0)$ 邻域内求 $f(x,y)=e^x\cos y$ 的二阶带 Pieno 余项型 Taylor 公式

解: $f(0,0)=1$. $\frac{\partial f(0,0)}{\partial x}=e^x\cos y\big|_{(x,y)=(0,0)}=1$, $\frac{\partial f(0,0)}{\partial y}=-e^x\sin y\big|_{(x,y)=(0,0)}=0$

$\frac{\partial^2 f(0,0)}{\partial x^2}=1$. $\frac{\partial^2 f(0,0)}{\partial y^2}=-e^x\cos y\big|_{(x,y)=(0,0)}=-1$. $\frac{\partial^2 f(0,0)}{\partial x\partial y}=-e^x\sin y\big|_{(x,y)=(0,0)}=0$

$\therefore f(x,y)=1+(\Delta x)+\frac{1}{2!}(\Delta x^2-\Delta y^2)+o(\Delta x^2+\Delta y^2)$

$=1+x+\frac{x^2-y^2}{2}+o(x^2+y^2)$

习题 6.7

T1. 求 $f(x,y)=xy-y$ 在 $(1,1)$ 的二阶 Taylor 多项式

解: $f(1,1)=0$. $\frac{\partial f(1,1)}{\partial x}=1$. $\frac{\partial f(1,1)}{\partial y}=0$ $\frac{\partial^2 f(1,1)}{\partial x^2}=\frac{\partial^2 f(1,1)}{\partial y^2}=0$. $\frac{\partial^2 f(1,1)}{\partial x\partial y}=1$

$\therefore f(x,y)=\Delta x+\frac{1}{2}\Delta x\Delta y=(x-1)+(x-1)(y-1)$

T2. 在 $(0,0)$ 的邻域内将下列函数展开为二阶带 Pieno 余项的 Taylor 公式

(1) $f(x,y)=\frac{\cos x}{\cos y}$

解: $f(0,0)=1$

$\frac{\partial f(0,0)}{\partial x}=\frac{-\sin x}{\cos y}\big|_{(0,0)}=0$. $\frac{\partial f(0,0)}{\partial y}=\frac{\cos x\sin y}{\cos^2 y}\big|_{(0,0)}=0$

$\frac{\partial^2 f(0,0)}{\partial x^2}=\frac{-\cos x}{\cos y}\big|_{(0,0)}=-1$. $\frac{\partial^2 f(0,0)}{\partial y^2}=\cos x\frac{2\sin^2 y+\cos^2 y}{\cos^3 y}\big|_{(0,0)}=1$. $\frac{\partial^2 f(0,0)}{\partial x\partial y}=\frac{-\sin y\cos x}{\cos^2 y}\big|_{(0,0)}=0$

$\therefore f(x,y)=1+\frac{1}{2}(y^2-x^2)+o(x^2+y^2)$

(2) $f(x,y)=\ln(1+x+y)$

解: $f(0,0)=0$.

$\frac{\partial f(0,0)}{\partial x}=\frac{1}{1+x+y}\big|_{(0,0)}=1$, $\frac{\partial f(0,0)}{\partial y}=\frac{1}{1+x+y}\big|_{(0,0)}=1$

$\frac{\partial^2 f(0,0)}{\partial x^2}=-\frac{1}{(1+x+y)^2}\big|_{(0,0)}=-1$, $\frac{\partial^2 f(0,0)}{\partial y^2}=-\frac{1}{(1+x+y)^2}\big|_{(0,0)}=-1$. $\frac{\partial^2 f(0,0)}{\partial x\partial y}=-\frac{1}{(1+x+y)^2}\big|_{(0,0)}=-1$

$\therefore f(x,y)=(x+y)-\frac{1}{2}(x^2+2xy+y^2)+o(x^2+y^2)$

(3) $f(x,y)=\sqrt{1-x^2-y^2}$

解: $f(0,0)=1$

$\frac{\partial f(0,0)}{\partial x}=\frac{-x}{\sqrt{1-x^2-y^2}}\big|_{(0,0)}=0$, $\frac{\partial f(0,0)}{\partial y}=\frac{-y}{\sqrt{1-x^2-y^2}}\big|_{(0,0)}=0$

$\frac{\partial^2 f(0,0)}{\partial x^2}=-\frac{\sqrt{1-x^2-y^2}-x\frac{-x}{\sqrt{1-x^2-y^2}}}{1-x^2-y^2}\big|_{(0,0)}=-1$. $\frac{\partial^2 f(0,0)}{\partial y^2}=-\frac{\sqrt{1-x^2-y^2}-y\frac{-y}{\sqrt{1-x^2-y^2}}}{1-x^2-y^2}\big|_{(0,0)}=-1$ $\frac{\partial^2 f(0,0)}{\partial x\partial y}=\frac{x}{2}\frac{-2y}{(1-x^2-y^2)^{3/2}}\big|_{(0,0)}=0$

$\therefore f(x,y)=1-\frac{1}{2}(x^2+y^2)+o(x^2+y^2)$

(4) $f(x,y)=\sin(x^2+y^2)$

解: $f(0,0)=0$

$$\frac{\partial f(0,0)}{\partial x}=2x\cos(x^2+y^2)\Big|_{(0,0)}=0 \qquad \frac{\partial f(0,0)}{\partial y}=2y\cos(x^2+y^2)\Big|_{(0,0)}=0$$

$$\frac{\partial^2 f(0,0)}{\partial x^2}=2\left(\cos(x^2+y^2)-2x^2\sin(x^2+y^2)\right)\Big|_{(0,0)}=2,\quad \frac{\partial^2 f(0,0)}{\partial y^2}=2\left(\cos(x^2+y^2)-2y^2\sin(x^2+y^2)\right)\Big|_{(0,0)}=2,$$

$$\frac{\partial^2 f(0,0)}{\partial x\partial y}=-4xy\sin(x^2+y^2)\Big|_{(0,0)}=0$$

$$\therefore\ f(x,y)=(x^2+y^2)+o(x^2+y^2)$$

T3. 在(0,0)邻域内将$f(x,y)=\ln(1+x+y)$展开为一阶带Lagrange余项的Taylor公式

解: 余项为 $\frac{1}{2!}\left(\frac{\partial^2 f(\theta x,\theta y)}{\partial x^2}x^2+\frac{\partial^2 f(\theta x,\theta y)}{\partial x\partial y}2xy+\frac{\partial^2 f(\theta x,\theta y)}{\partial y^2}y^2\right)=\frac{-2}{(1+\theta x+\theta y)^2}$

$$f(x,y)=(x+y)-\frac{(x+y)^2}{2(1+\theta x+\theta y)^2}$$

T4. 用Taylor公式证明当$|x|,|y|,|z|$充分小时，有近似公式

$$\cos(x+y+z)-\cos x\cos y\cos z\approx-(xy+yz+xz)$$

证: 在(0,0,0)邻域内展开 $u(x,y,z)=\cos(x+y+z)-\cos x\cos y\cos z$

$u(0,0,0)=0$

$$\frac{\partial u}{\partial x}(0,0,0)=\left(-\sin(x+y+z)+\sin x\cos y\cos z\right)\Big|_{(0,0,0)}=0.$$

$$\frac{\partial u}{\partial y}(0,0,0)=\left(-\sin(x+y+z)+\cos x\sin y\cos z\right)\Big|_{(0,0,0)}=0.$$

$$\frac{\partial u}{\partial z}(0,0,0)=\left(-\sin(x+y+z)+\cos x\cos y\sin z\right)\Big|_{(0,0,0)}=0.$$

$$\frac{\partial^2 u}{\partial x^2}(0,0,0)=\left(-\cos(x+y+z)+\cos x\cos y\cos z\right)\Big|_{(0,0,0)}=\frac{\partial^2 u}{\partial y^2}(0,0,0)=\frac{\partial^2 u}{\partial z^2}(0,0,0)=0$$

$$\frac{\partial^2 u}{\partial x\partial y}(0,0,0)=\left(-\cos(x+y+z)-\sin x\sin y\cos z\right)\Big|_{(0,0,0)}=\frac{\partial^2 u}{\partial y\partial z}(0,0,0)=\frac{\partial^2 u}{\partial z\partial x}(0,0,0)=-1$$

$$\therefore\ u(x,y,z)=\frac{1}{2!}(-2xy-2xz-2yz)+o(x^2+y^2+z^2)=-(xy+yz+xz)+o(x^2+y^2+z^2)$$

$$\Rightarrow \cos(x+y+z)-\cos x\cos y\cos z\approx-(xy+yz+xz)$$

T5. $\mathbb{D}=\{(x,y):x^2+y^2<1\}$. $f(x,y)\in C^1(\mathbb{D})$ 且 $xf_x(x,y)+yf_y(x,y)\equiv0$. $\forall(x,y)\in\mathbb{D}$. 证明: $f(x,y)\equiv C$

证: $P_0=(0,0)$, $P=(x,y)\in\mathbb{D}\Rightarrow\overline{P_0P}\subset\mathbb{D}$. $\forall P\in\mathbb{D}$

$$f(x,y)-f(0,0)=xf_x(\theta x,\theta y)+yf_y(\theta x,\theta y)\equiv0\Rightarrow f(x,y)=f(x_0,y_0)\equiv C,\ \forall(x,y)\in\mathbb{D}$$