## §3 定积分的分部积分法则与换元法则

### 一、换元法

(1) $\boxed{定义}$ $C^k[a,b]=\{f(x): f、\cdots f^{(k)}在[a,b]连续\}$

$\boxed{定理}$ $f\in C[a,b]$，$\varphi\in C'[\alpha,\beta]$，$\varphi(\alpha)=a$，$\varphi(\beta)=b$，$a\le\varphi(t)\le b$

则 $\int_a^b f(x)dx=\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$

证：设 $F(x)$ 是 $f(x)$ 一个原函数，

则 $[F(\varphi(t))]'=f(\varphi(t))\varphi'(t)$

故 $\int_a^b f(x)dx=F(x)\Big|_a^b=F(\varphi(t))\Big|_\alpha^\beta=\int_\alpha^\beta f(\varphi(t))dt$

(2) 例1. $I=\int_0^1 x^2\sqrt{1-x^2}\,dx$

解：令 $x=\sin t$，$t\in[0,\frac{\pi}{2}]$，$dx=\cos t\,dt$。令 $x=0\Rightarrow t_0=0$，令 $x=1\Rightarrow t_1=\frac{\pi}{2}$

$\therefore I=\int_0^{\frac{\pi}{2}}\sin^2t\cos^2t\,dt=\frac{1}{4}\int_0^{\frac{\pi}{2}}\sin^2 2t\,dt=\frac{1}{8}\int_0^{\frac{\pi}{2}}dt-\frac{1}{8}\int_0^{\frac{\pi}{2}}\cos4t\,dt=\frac{\pi}{16}-\frac{1}{32}\sin4t\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{16}$

小结：1. $\int_a^b f(x)dx$ 中 $dx$ 是微分：$x=\varphi(t)\Rightarrow dx=\varphi'(t)dt$

2. 不定积分换元法要求 $\varphi(t)$ 严格单调有反函数且要用之代换

定积分则只要 $\varphi(t)$ 连续，即 $\varphi\in C'[a,b]$，且不必代回。

例2. $\boxed{推论}$ 1. $f\in C[-a,a]$ 为偶函数，$g\in C[-a,a]$ 为奇函数

则 $\int_{-a}^a f(x)dx=\int_{-a}^0 f(t)dt+\int_0^a f(x)dx\overset{t=-x}{=\!=}-\int_a^0 f(x)dx+\int_0^a f(x)dx=2\int_0^a f(x)dx$

$\int_{-a}^a g(x)dx=\int_{-a}^0 g(t)dt+\int_0^a g(x)dx\overset{t=-x}{=\!=}\int_a^0 g(x)dx+\int_0^a g(x)dx=0$

2. $f\in C(-\infty,+\infty)$，$f(x+T)=f(x)$，则 $\int_a^{a+T}f(x)dx=\int_0^T f(x)dx$

证：$\int_a^{a+T}f(x)dx=\int_a^T f(x)dx+\int_T^{a+T}f(t)dt\overset{t=x+T}{=\!=}\int_a^T f(x)dx+\int_0^a f(x)dx=\int_0^T f(x)dx$

### 二、分部积分法

(1) $\boxed{定理}$ 设 $u,v\in C'[a,b]$，则 $\int_a^b u(x)v'(x)dx=u(x)v(x)\Big|_a^b-\int_a^b v(x)u'(x)dx$

即 $\int_a^b u\,dv=uv\Big|_a^b-\int_a^b v\,du$

证：$(uv)'=u'v+uv'\Rightarrow uv\Big|_a^b=\int_a^b uv'dx+\int_a^b vu'dx=\int_a^b u\,dv+\int_a^b v\,du$

(2) 例1. $I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\int_0^{\frac{\pi}{2}}\cos^n x\,dx\longrightarrow$ 几何理解

解：$I_n=-\int_0^{\frac{\pi}{2}}\sin^{n-1}x\,d\cos x=-\sin^{n-1}x\cos x\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos^2x(n-1)\sin^{n-2}x\,dx$

$=(n-1)I_{n-2}-(n-1)I_n$

$\Rightarrow I_n=\frac{n-1}{n}I_{n-2}$，$I_0=\frac{\pi}{2}$，$I_1=1$

(i) $I_{2k}=\frac{(2k-1)!!}{(2k)!!}I_0=\frac{\pi}{2}\cdot\frac{(2k-1)!!}{(2k)!!}$ $\longrightarrow$ $\begin{cases}(2k+1)!!=(2k+1)(2k-1)(2k-3)\cdots3\cdot1\\(2k)!!=2k(2k-2)(2k-4)\cdots4\cdot2\end{cases}$

(ii) $I_{2k+1}=\frac{(2k)!!}{(2k+1)!!}I_1=\frac{(2k)!!}{(2k+1)!!}$

$\Rightarrow \boxed{I_n=\frac{(n-1)!!}{n!!}\begin{cases}\frac{\pi}{2} & n偶\\1 & n奇\end{cases}}$

$\boxed{推广}$ (J. Wallis公式) $\lim\limits_{n\to\infty}\left[\frac{(2n)!!}{(2n-1)!!}\right]^2\frac{1}{2n+1}=\frac{\pi}{2}$

证：$I_{2n+1}<I_{2n}<I_{2n-1}$ $\longrightarrow$ $\sin x\in(0,1)$，$\sin^n x$ 随 $n\uparrow$ 而 $\downarrow$ $\Rightarrow$ $I_n$ 随 $n\uparrow$ 而 $\downarrow$

$\Leftrightarrow\frac{(2n)!!}{(2n+1)!!}<\frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}<\frac{(2n-2)!!}{(2n-1)!!}$

$\Leftrightarrow\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n+1}<\frac{\pi}{2}<\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n}$

$\lim\limits_{n\to\infty}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\left(\frac{1}{2n}-\frac{1}{2n+1}\right)=\lim\limits_{n\to\infty}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n(2n+1)}\le\lim\limits_{n\to\infty}\frac{1}{2n}\frac{\pi}{2}=0$

$\therefore\lim\limits_{n\to\infty}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n+1}=\frac{\pi}{2}$

## 例题

### 一、分部积分

例1. $\int_1^2 x\ln x\,dx = \frac{1}{2}\left(x^2\ln x\Big|_1^2 - \int_1^2 x\,dx\right) = \frac{1}{2}x^2\ln x\Big|_1^2 - \frac{1}{4}x^2\Big|_1^2 = 2\ln 2 - \frac{3}{4}$

例2. $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \begin{cases} \frac{(2k-1)!!}{(2k)!!}\cdot\frac{\pi}{2} & n=2k \\ \frac{(2k)!!}{(2k+1)!!} & n=2k+1 \end{cases}$

### 二、换元法

例3. $I = \int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$

解：令 $t=\sqrt{x}$，$dx = 2t\,dt$

则 $I = \int_0^1 \frac{2t^2dt}{1+t} = 2\left(\int_0^1 (t-1)dt + \int_0^1 \frac{dt}{1+t}\right) = (t-1)^2\Big|_0^1 + 2\ln|t+1|\Big|_0^1 = 2\ln 2 - 1$

例4. 求椭圆面积

解：$S = 2\frac{b}{a}\int_{-a}^{a}\sqrt{a^2-x^2}\,dx$

令 $x = a\sin t$，$-\frac{\pi}{2}\le t\le\frac{\pi}{2}$，$dx = a\cos t\,dt$

则 $S = 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt = ab\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos 2t\,dt\right) = ab\left(\pi + \frac{1}{2}\sin 2t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) = \pi ab$

例5. 证明：$\int_0^{\frac{\pi}{2}}\cos^n x\,dx = \int_0^{\frac{\pi}{2}}\sin^n x\,dx$

证：由对称性，令 $x = \frac{\pi}{2} - t$

哑变量替换

则 $\int_0^{\frac{\pi}{2}}\cos^n x\,dx = \int_{\frac{\pi}{2}}^{0}\cos^n\left(\frac{\pi}{2}-t\right)(-dt) = \int_0^{\frac{\pi}{2}}\sin^n t\,dt = \int_0^{\frac{\pi}{2}}\sin^n x\,dx$

例6. $I = \int_0^a x^4\sqrt{a^2-x^2}\,dx$

解：令 $x = a\sin t$，$0\le t\le\frac{\pi}{2}$，$dx = a\cos t\,dt$

则 $I = \int_0^{\frac{\pi}{2}} a^4\sin^4 t\,(a\cos t)(a\cos t\,dt)$

$= a^6\int_0^{\frac{\pi}{2}}\sin^4 t\cos^2 t\,dt$

$= a^6\int_0^{\frac{\pi}{2}}\sin^4 t\,dt - a^6\int_0^{\frac{\pi}{2}}\sin^6 t\,dt$

$= \frac{3\times 1}{4\times 2}\times\frac{\pi a^6}{2} - \frac{5\times 3\times 1}{6\times 4\times 2}\times\frac{\pi a^6}{2} = \frac{\pi a^6}{32}$

例7. $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2 x}{1+e^x}\,dx$

解：$\int_{-\frac{\pi}{2}}^{0}\frac{\sin^2 x}{1+e^x}\,dx \overset{t=-x}{=} \int_{\frac{\pi}{2}}^{0}\frac{\sin^2 t}{1+e^{-t}}(-dt) = \int_0^{\frac{\pi}{2}}\frac{e^x}{1+e^x}\sin^2 x\,dx$

$\therefore I = \int_0^{\frac{\pi}{2}}\sin^2 x\,dx = \frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{4}$

### 三、奇偶性、周期性化简

例8. $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^5 x\,dx = 2\int_0^{\frac{\pi}{2}}\cos^5 x\,dx = 2\times\frac{4\times 2}{5\times 3\times 1} = \frac{16}{15}$

例9. $I = \int_0^n (x-[x])\,dx = n\int_0^1 (x-[x])\,dx = n\int_0^1 x\,dx = \frac{n}{2}$

## 习题3.4

T1. 求定积分

(1) $I=\int_{-1}^{1}\frac{x\,dx}{\sqrt{5-4x}}$

解: $I=-\frac{1}{2}\int_{-1}^{1}x\,d\sqrt{5-4x}=-\frac{1}{2}\left(x\sqrt{5-4x}\Big|_{-1}^{1}-\int_{-1}^{1}\sqrt{5-4x}\,dx\right)\xlongequal{t=5-4x}-\frac{1}{2}x\sqrt{5-4x}\Big|_{-1}^{1}-\frac{1}{8}\int_{9}^{1}\sqrt{t}\,dt=-2-\frac{1}{12}t^{\frac{3}{2}}\Big|_{9}^{1}=\frac{1}{6}$

(2) $I=\int_{0}^{\ln 2}xe^{-x}\,dx$

解: $I=-xe^{-x}\Big|_{0}^{\ln 2}+\int_{0}^{\ln 2}e^{-x}\,dx=-(x+1)e^{-x}\Big|_{0}^{\ln 2}=\frac{1-\ln 2}{2}$

(3) $I=\int_{0}^{1}x^2\sqrt{1-x^2}\,dx$

解: 令 $x=\sin t$, $0\le t\le\frac{\pi}{2}$, $dx=\cos t\,dt$

则 $I=\int_{0}^{\frac{\pi}{2}}\sin^2 t\cos^2 t\,dt=\frac{1}{4}\int_{0}^{\frac{\pi}{2}}\sin^2 2t\,dt=\frac{1}{8}\int_{0}^{\frac{\pi}{2}}dt-\frac{1}{8}\int_{0}^{\frac{\pi}{2}}\cos 4t\,dt=\frac{\pi}{16}-\frac{1}{32}\sin 4t\Big|_{0}^{\frac{\pi}{2}}=\frac{\pi}{16}$

(4) $I=\int_{0}^{\pi}x\sin x\,dx$

解: $I=-x\cos x\Big|_{0}^{\pi}+\int_{0}^{\pi}\cos x\,dx=\pi+\sin x\Big|_{0}^{\pi}=\pi$

(5) $I=\int_{0}^{4}\sqrt{x^2+9}\,dx$

解: $I=x\sqrt{x^2+9}\Big|_{0}^{4}-\int_{0}^{4}\frac{x^2\,dx}{\sqrt{x^2+9}}=20-I+9\int_{0}^{4}\frac{dx}{\sqrt{x^2+9}}=20-I+9\ln(x+\sqrt{x^2+9})\Big|_{0}^{4}$

解得 $I=10+\frac{9}{2}\ln 3$

(6) $I=\int_{0}^{\frac{1}{2}}\frac{x^2\,dx}{\sqrt{1-x^2}}$

解: $I=-x\sqrt{1-x^2}\Big|_{0}^{\frac{1}{2}}+\int_{0}^{\frac{1}{2}}\sqrt{1-x^2}\,dx\xlongequal{x=\sin t}-\frac{\sqrt{3}}{4}+\frac{1}{2}\int_{0}^{\frac{\pi}{6}}dt+\frac{1}{2}\int_{0}^{\frac{\pi}{6}}\cos 2t\,dt=-\frac{\sqrt{3}}{4}+\frac{\pi}{12}+\frac{1}{4}\sin 2t\Big|_{0}^{\frac{\pi}{6}}=\frac{\pi}{12}-\frac{\sqrt{3}}{8}$

(7) $I=\int_{0}^{1}\sqrt{4-x^2}\,dx$

解: 令 $x=2\sin t$, $0\le t\le\frac{\pi}{6}$, $dx=2\cos t\,dt$

则 $I=4\int_{0}^{\frac{\pi}{6}}\cos^2 t\,dt=2\int_{0}^{\frac{\pi}{6}}dt+2\int_{0}^{\frac{\pi}{6}}\cos 2t\,dt=\frac{\pi}{3}+\sin 2t\Big|_{0}^{\frac{\pi}{6}}=\frac{\pi}{3}+\frac{\sqrt{3}}{2}$

(8) $I=\int_{0}^{3}x\sqrt[3]{1-x^2}\,dx$

解: $I\xlongequal{u=1-x^2}-\frac{1}{2}\int_{1}^{-8}u^{\frac{1}{3}}\,du=-\frac{3}{8}u^{\frac{4}{3}}\Big|_{1}^{-8}=-\frac{45}{8}$

(9) $I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\cos x-\cos^3 x}\,dx$

解: $I=2\int_{0}^{\frac{\pi}{2}}\sqrt{\cos x}\sin x\,dx\xlongequal{u=\cos x}-2\int_{1}^{0}\sqrt{u}\,du=\frac{4}{3}u^{\frac{3}{2}}\Big|_{0}^{1}=\frac{4}{3}$

(10) $I=\int_{0}^{\frac{\pi}{2}}\cos^n 2x\,dx$

解: $I\xlongequal{t=2x}\frac{1}{2}\int_{0}^{\pi}\cos^n t\,dt\xlongequal{u=t-\frac{\pi}{2}}\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^n u\,du$

$n=2k$, $I=\int_{0}^{\frac{\pi}{2}}\sin^{2k}u\,du=\frac{(2k-1)!!}{(2k)!!}\frac{\pi}{2}$ ; $n=2k+1$, $I=0$

$$\therefore I=\begin{cases}\frac{(n-1)!!}{n!!}\frac{\pi}{2}, & n=2k\\ 0, & n=2k+1\end{cases}$$

(11) $I=\int_{0}^{a}(a^2-x^2)^{\frac{n}{2}}\,dx$

解: 令 $x=a\sin t$, $dx=a\cos t\,dt$

$$\text{则 } I=\int_{0}^{\frac{\pi}{2}}a^{n+1}\cos^{n+1}x\,dx=\begin{cases}\frac{n!!}{(n+1)!!}\frac{\pi a^{n+1}}{2}, & n=2k+1\\ \frac{n!!}{(n+1)!!}a^{n+1}, & n=2k\end{cases}$$

(12) $I_{11}=\int_{0}^{\frac{\pi}{2}}\sin^{11}x\,dx=\frac{10\times8\times6\times4\times2}{11\times9\times7\times5\times3\times1}=\frac{256}{693}$

(13) $I=\int_{0}^{\pi}\sin^6\frac{x}{2}\,dx\xlongequal{t=\frac{x}{2}}2\int_{0}^{\frac{\pi}{2}}\sin^6 t\,dt=\frac{5\times3\times1}{6\times4\times2}\pi=\frac{5\pi}{16}$

(14) $I=\int_0^{\pi}(x\sin x)^2dx$

解: $I=\frac{1}{2}\int_0^{\pi}x^2(1-\cos 2x)dx=\frac{1}{2}\int_0^{\pi}x^2dx-\frac{1}{2}\int_0^{\pi}x^2\cos 2x\,dx$

$=\frac{\pi^3}{6}-\frac{1}{4}\left(x^2\sin 2x\Big|_0^{\pi}-2\int_0^{\pi}x\sin 2x\,dx\right)$

$=\frac{\pi^3}{6}-\frac{1}{4}\left(x\cos 2x\Big|_0^{\pi}-\int_0^{\pi}\cos 2x\,dx\right)$

$=\frac{\pi^3}{6}-\frac{\pi}{4}+\frac{1}{8}\sin 2x\Big|_0^{\pi}=\frac{\pi^3}{6}-\frac{\pi}{4}$

(15) $I=\int_0^{\frac{\pi}{4}}tg^4x\,dx$

解: 令 $t=tgx$, $0\le t\le 1$. $dx=\frac{dt}{t^2+1}$

则 $I=\int_0^1\frac{t^4}{1+t^2}dt=\int_0^1(t^2-1)dt+\int_0^1\frac{dt}{t^2+1}=\left(\frac{1}{3}t^3-t+arctg\,t\right)\Big|_0^1=\frac{\pi}{4}-\frac{2}{3}$

(16) $I=\int_0^1\arcsin x\,dx$

解: 令 $x=\sin t$, $0\le t\le\frac{\pi}{2}$, $dx=\cos t\,dt$

则 $I=\int_0^{\frac{\pi}{2}}t\cos t\,dt=t\sin t\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin t\,dt=\frac{\pi}{2}+\cos t\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2}-1$

(17) $I=\int_0^{\pi}\ln(x+\sqrt{x^2+a^2})dx$

解: $I=x\ln(x+\sqrt{x^2+a^2})\Big|_0^{\pi}-\int_0^{\pi}\frac{x\,dx}{\sqrt{x^2+a^2}}$

$\overset{u=x^2+a^2}{=\!=}\pi\ln(\pi+\sqrt{\pi^2+a^2})-\frac{1}{2}\int_{a^2}^{a^2+\pi^2}\frac{du}{\sqrt{u}}$

$=\pi\ln(\pi+\sqrt{\pi^2+a^2})-\sqrt{a^2+\pi^2}+|a|$

T2. $f\in C[a,b]$, 证明: $\int_a^b f(x)dx=(b-a)\int_0^1 f[a+(b-a)x]dx$

证: 令 $x=a+(b-a)t$, $0\le t\le 1$, $dx=(b-a)dt$

则 $\int_a^b f(x)dx=\int_0^1 f[a+(b-a)t](b-a)dt=(b-a)\int_0^1 f[a+(b-a)x]dx$

T3. 证明: $\int_0^a x^3f(x^2)dx=\frac{1}{2}\int_0^{a^2}xf(x)dx$

证: 令 $t=x^2$, $0\le t\le a^2$, $dt=2x\,dx$.

则 $\int_0^a x^3f(x^2)dx=\frac{1}{2}\int_0^{a^2}tf(t)dt=\frac{1}{2}\int_0^{a^2}xf(x)dx$

T4. 证明: $\int_0^1 x^m(1-x)^ndx=\int_0^1 x^n(1-x)^mdx$

证: 令 $t=1-x$, $dx=-dt$, $x=1-t$

则 $\int_0^1 x^m(1-x)^ndx=\int_1^0(1-t)^mt^n(-dt)=\int_0^1 x^n(1-x)^mdx$

T5. 设 $f(x)$ 连续, 证明: $\int_0^x\left[\int_0^t f(x)dx\right]dt=\int_0^x f(t)(x-t)dt$

证: $\int_0^x\left[\int_0^t f(x)dx\right]dt=t\int_0^t f(x)dx\Big|_0^x-\int_0^x tf(t)dt$

$=x\int_0^x f(t)dt-\int_0^x tf(t)dt$

$=\int_0^x(x-t)f(t)dt$

T6. 证明: $\int_0^{\pi}xf(\sin x)dx=\pi\int_0^{\frac{\pi}{2}}f(\sin x)dx$

证: 令 $x=\pi-t$, $dx=-dt$

则 $I=\int_0^{\pi}xf(\sin x)dx=\int_{\pi}^0(\pi-t)f(\sin t)(-dt)=\pi\int_0^{\pi}f(\sin t)dt-\int_0^{\pi}tf(\sin t)dt=\pi\int_0^{\pi}f(\sin t)dt-I$

$\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}f(\sin x)dx$

$\because\ \sin(\frac{\pi}{2}-x)=\sin(\frac{\pi}{2}+x)$

$\therefore\ I=\pi\int_0^{\frac{\pi}{2}}f(\sin x)dx$

T7. $I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$

解: $I=\int_0^{\pi}x\frac{\sin x}{2-\sin^2 x}dx=\pi\int_0^{\frac{\pi}{2}}\frac{\sin x\,dx}{\cos^2 x+1}\xlongequal{u=\cos x}\pi\int_0^1\frac{du}{u^2+1}=\pi\arctan u\Big|_0^1=\frac{\pi^2}{4}$

T8. $f\in C(-\infty,+\infty)$. 周期为 T

证明: (1) $F(x)=\frac{x}{T}\int_0^T f(t)dt-\int_0^x f(t)dt$ 以T为周期

(2) $\lim\limits_{x\to+\infty}\frac{1}{x}\int_0^x f(t)dt=\frac{1}{T}\int_0^T f(t)dt$

证: (1) $F(x+T)=\frac{x}{T}\int_0^T f(t)dt+\int_0^T f(t)dt-\int_0^T f(t)dt-\int_T^{x+T}f(t)dt$

$=\frac{x}{T}\int_0^T f(t)dt-\int_0^x f(t)dt=F(x)$

$\therefore$ F(x) 以 T 为周期

(2) $\lim\limits_{x\to+\infty}\frac{\int_0^x f(t)dt}{x}=\lim\limits_{x\to+\infty}\frac{\frac{x}{T}\int_0^T f(t)dt-F(x)}{x}=\frac{1}{T}\int_0^T f(t)dt-\lim\limits_{x\to+\infty}\frac{F(x)}{x}$

$\because$ F(x) 连续且以T为周期

$\therefore \exists M$, s.t. $|F(x)|\le M$

$\therefore \forall\varepsilon>0$. $\exists A=\frac{M}{\varepsilon}$, s.t. $\forall x>A$, $\left|\frac{F(x)}{x}\right|<\frac{M}{A}<\varepsilon \Rightarrow \lim\limits_{x\to+\infty}\frac{F(x)}{x}=0$

$\therefore \lim\limits_{x\to+\infty}\frac{\int_0^x f(t)dt}{x}=\frac{1}{T}\int_0^T f(t)dt$

T9. f(x) 是以 T 为周期周期函数, $f(x_0)\neq 0$, 且 $\int_0^T f(x)dx=0$

证明: f(x) 在 $(x_0,x_0+T)$ 上至少有两个根

证: 不妨设 $f(x_0)=f(x_0+T)>0$

若 $\forall x\in(x_0,x_0+T)$. $f(x)\geq 0$, 则 $\int_{x_0}^{x_0+T}f(x)dx=\int_0^T f(x)dx>0$. 矛盾!

$\therefore \exists\xi\in(x_0,x_0+T)$, $f(\xi)<0$

又 $f(x_0)f(\xi)=f(x_0+T)f(\xi)<0$ 且 $f\in C[x_0,x_0+T]$

$\therefore$ 至少存在 $x_1\in(x_0,\xi)$, $x_2\in(\xi,x_0+T)$, s.t. $f(x_1)=f(x_2)=0$

T10. $I_m=\int_0^{2m\pi}\frac{dx}{\sin^4 x+\cos^4 x}$

解: $\sin^4(x+\frac{\pi}{2})+\cos^4(x+\frac{\pi}{2})=\cos^4 x+\sin^4 x$

$\therefore I_m=4m\int_0^{\frac{\pi}{2}}\frac{dx}{\sin^4 x+\cos^4 x}$

$=4m\int_0^{\frac{\pi}{2}}\frac{dx}{1-2\sin^2 x\cos^2 x}$

$=8m\int_0^{\frac{\pi}{2}}\frac{dx}{2-\sin^2 2x}$

$=-2\sqrt{2}m\int_0^{\frac{\pi}{2}}\frac{d\sqrt{2}\cot 2x}{2\cot^2 2x+1}$

$\xlongequal{u=\cot 2x}+2\sqrt{2}m\arctan\sqrt{2}u\Big|_{-\infty}^{+\infty}$

$=2\sqrt{2}m\pi$