§6. Maxwell电磁理论

§6.1. Maxwell方程组

一、位移电流

(1) 回顾已有规律

Coulomb $\Rightarrow \begin{cases} \oiint_{\partial\Omega} \vec{D}(\vec{r})\cdot d^2\vec{r} = \iiint_\Omega \rho(\vec{r})\, d^3r \\ \oint_{\partial S^+} \vec{E}(\vec{r})\cdot d\vec{r} = 0 \end{cases}$

Biot-Savart $\Rightarrow \begin{cases} \oiint_{\partial\Omega} \vec{B}(\vec{r})\cdot d^2\vec{r} = 0 \\ \oint_{\partial S^+} \vec{H}(\vec{r})\cdot d\vec{r} = \iint_{S^+} \vec{j}(\vec{r})\cdot d^2\vec{r} \end{cases}$

Faraday: $\mathcal{E} = -\iint_{S^+} \frac{\partial \vec{B}}{\partial t}\cdot d^2\vec{r} + \oint_{\partial S^+} (\vec{v}\times\vec{B})\cdot d\vec{r}$

(2) 感生电场对电场环路定理修正: $\oint_{\partial S^+} \vec{E}\cdot d\vec{r} = -\iint_{S^+} \frac{\partial \vec{B}}{\partial t}\cdot d^2\vec{r}$

(3) 非恒电流界面不连续

① 修正: $\oiint_{\partial\Omega} \vec{j}\cdot d^2\vec{r} = -\frac{dq}{dt} = -\frac{d}{dt}\oiint_{\partial\Omega} \vec{D}\cdot d^2\vec{r} \Rightarrow \oiint_{\partial\Omega} (\vec{j}+\frac{\partial \vec{D}}{\partial t})\cdot d^2\vec{r} = 0$

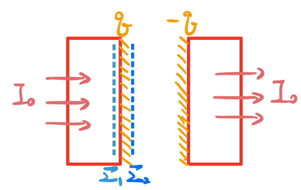
② 引入位移电流 $\frac{d\Phi_D}{dt} = \iint_{S^+} \frac{\partial \vec{D}}{\partial t}\cdot d^2\vec{r}$, 其与传导电流 $\iint_{S^+} \vec{j}\cdot d^2\vec{r}$ 之和全电流连续.

e.g. 充电平行板电容器全电流分析

Sol: 通过Σ_1的全电流近似为I_0.

通过Σ_2的全电流近似为 $\iint_{\Sigma_2} \frac{\partial \vec{D}}{\partial t}\cdot d^2\vec{r} = \frac{dq}{dt} = I_0$

∴全电流连续



③ 电介质中位移电流 $\frac{d\Phi_D}{dt} = \iint_{S^+} \frac{\partial \vec{D}}{\partial t}\cdot d^2\vec{r} = \iint_{S^+} (\underbrace{\varepsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\text{电场变化等效电流}} + \underbrace{\frac{\partial \vec{P}}{\partial t}}_{\text{极化电流密度}\ \vec{j}_p})\cdot d^2\vec{r}$

二、Maxwell方程组

(1) 真空电磁场

Jhm-Maxwell方程组(真空):

① 积分形式 $\begin{cases} \oiint_{\partial\Omega} \vec{D}\cdot d^2\vec{r} = \iiint_\Omega \rho_{e0}\, d^3r \\ \oint_{\partial S^+} \vec{E}\cdot d\vec{r} = -\iint_{S^+} \frac{\partial \vec{B}}{\partial t}\cdot d^2\vec{r} \\ \oiint_{\partial\Omega} \vec{B}\cdot d^2\vec{r} = 0 \\ \oint_{\partial S^+} \vec{H}\cdot d\vec{r} = \iint_{S^+} (\vec{j}_0 + \frac{\partial \vec{D}}{\partial t})\cdot d^2\vec{r} \end{cases}$ $\Longleftrightarrow$ ② 积分形式 $\begin{cases} \vec{\nabla}\cdot\vec{D} = \rho_{e0} \\ \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{H} = \vec{j}_0 + \frac{\partial \vec{D}}{\partial t} \end{cases}$

$\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho_{e0}$

$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{B} = \mu_0\vec{j}_0 + \varepsilon_0\mu_0\frac{\partial \vec{E}}{\partial t}$

* 角标 ρ_{e0}, $\vec{j}_0$ 表示自由电荷/电流密度

(2) 电磁介质中完备方程组

Thm—Maxwell方程组(电磁介质中):

基本规律
$$\begin{cases}\nabla\cdot\vec{D}=\rho_{e0}\\ \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\\ \nabla\cdot\vec{B}=0\\ \nabla\times\vec{H}=\vec{j}_0+\frac{\partial\vec{D}}{\partial t}\end{cases}$$

联立介质性质
$$\begin{cases}\vec{D}=\varepsilon_r\varepsilon_0\vec{E}\\ \vec{B}=\mu_r\mu_0\vec{H}\\ \vec{j}_0=\sigma\vec{E}\end{cases}$$

三、边界条件

(1) 磁介质界面
$$\begin{cases}\hat{n}\cdot\vec{B}_1=\hat{n}\cdot\vec{B}_2\\ \hat{n}\times\vec{H}_1=\hat{n}\times\vec{H}_2\end{cases}$$

高频趋肤 ⇓

→ 有面电流，$\hat{n}\times\vec{H}_{out}=\vec{i}_0$

(2) 电介质界面
$$\begin{cases}\hat{n}\cdot\vec{D}_1=\hat{n}\cdot\vec{D}_2\\ \hat{n}\times\vec{E}_1=\hat{n}\times\vec{E}_2\end{cases}$$

(3) 导体界面
$$\begin{cases}\hat{n}\cdot(\vec{D}_2-\vec{D}_1)=\sigma_{e0}\\ \hat{n}\cdot\vec{j}_{01}=\hat{n}\cdot\vec{j}_{02}\end{cases}$$

§6.2.电磁波

一、电磁波解

(1) 自由的空间Maxwell方程组
$$\begin{cases}\vec\nabla\cdot\vec D=0\\ \vec\nabla\times\vec E=-\mu_r\mu_0\frac{\partial\vec H}{\partial t}\\ \vec\nabla\cdot\vec B=0\\ \vec\nabla\times\vec H=\varepsilon_r\varepsilon_0\frac{\partial\vec E}{\partial t}\end{cases}$$

(2) 规定 $\vec E\parallel\hat i$，$\vec H\parallel\hat j$，$\hat k=\hat i\times\hat j$

得
$$\begin{cases}\frac{\partial E}{\partial z}\hat j=-\mu_r\mu_0\frac{\partial H}{\partial t}\hat j\\ -\frac{\partial H}{\partial z}\hat i=\varepsilon_r\varepsilon_0\frac{\partial E}{\partial t}\hat i\end{cases}\Rightarrow\begin{cases}\frac{\partial^2E}{\partial z^2}=-\mu_r\mu_0(-\varepsilon_r\varepsilon_0)\frac{\partial^2E}{\partial t^2}\text{，即}\\ \frac{\partial^2H}{\partial z^2}=-\varepsilon_r\varepsilon_0(-\mu_r\mu_0)\frac{\partial^2H}{\partial t^2}\end{cases}$$

Thm—平面电磁波波动方程：
$$\begin{cases}\frac{\partial^2E}{\partial z^2}=\varepsilon_r\varepsilon_0\mu_r\mu_0\frac{\partial^2E}{\partial t^2}\\ \frac{\partial^2H}{\partial z^2}=\varepsilon_r\varepsilon_0\mu_r\mu_0\frac{\partial^2H}{\partial t^2}\end{cases}$$

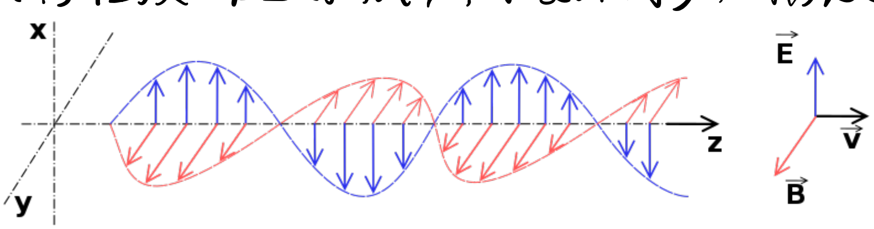
(3) 解
$$\begin{cases}\vec E=\vec E_0e^{i(\omega t-kz)}=E_0e^{i(\omega t-kz+\varphi_E)}\\ \vec H=\vec H_0e^{i(\omega t-kz)}=H_0e^{i(\omega t-kz+\varphi_H)}\end{cases}$$
→ 振幅 $\sqrt{\varepsilon_r\varepsilon_0}E_0=\sqrt{\mu_r\mu_0}H_0$

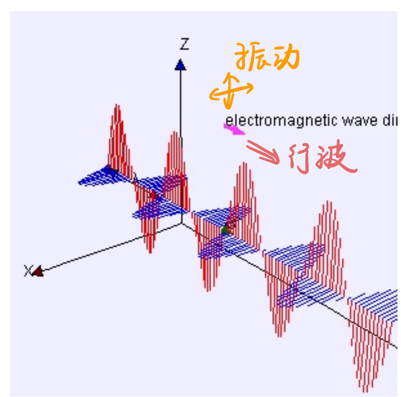
时域振动　空间传播　初相位 $\xrightarrow{\text{同步}}$ $\varphi_E=\varphi_H$

波速 $v=\frac{\omega}{k}=\frac{1}{\sqrt{\varepsilon_r\mu_r\varepsilon_0\mu_0}}$

⟶ 真空波速 $c=\frac{1}{\sqrt{\varepsilon_0\mu_0}}$．折射率 $n=\frac{c}{v}=\sqrt{\varepsilon_r\mu_r}\sim\sqrt{\varepsilon_r}$

(4) 性质：$\vec k\perp\langle\vec E,\vec H\rangle$，$\vec E$&$\vec H$同步，横波．





二、电磁波能动量

(1) 空间电磁场能量密度 $w=\frac12\vec D\cdot\vec E+\frac12\vec B\cdot\vec H$

(2) 推导：
$$\frac{\partial}{\partial t}\iiint_\Omega w\,d^3r=\frac12\iiint_\Omega\frac{\partial}{\partial t}(\vec D\cdot\vec E+\vec B\cdot\vec H)d^3r$$
$$=\frac12\iiint_\Omega\frac{\partial}{\partial t}(\varepsilon_r\varepsilon_0\vec E^2+\mu_r\mu_0\vec H^2)d^3r$$
$$=\iiint_\Omega\left(\varepsilon_r\varepsilon_0\vec E\cdot\frac{\partial\vec E}{\partial t}+\mu_r\mu_0\vec H\cdot\frac{\partial\vec H}{\partial t}\right)d^3r$$
$$=\iiint_\Omega\left[\vec E\cdot(\vec\nabla\times\vec H-\vec j_0)+\vec H\cdot(\vec\nabla\times\vec E)\right]d^3r$$
$$=\iiint_\Omega\left(-\vec\nabla\cdot(\vec E\times\vec H)-\vec j_0\cdot\vec E\right)d^3r$$

化简：$\frac{d}{dt}W(\Omega) = -\oint_{\partial\Omega}(\vec E\times\vec H)\cdot d^2\vec r - \iiint_\Omega \vec j_0\cdot\vec E\, d^3r$

穿越$\partial\Omega$电磁波输运能量

$-\iiint_\Omega(\rho \vec j_0^2+\vec j_0\cdot\vec K)d^3r = -I_0^2R + I_0\Delta\mathcal{E}$

$= -Q + P$

焦耳热损耗　电动势做功

(3) 定义：Poynting矢量 $\vec S \triangleq \vec E\times\vec H$

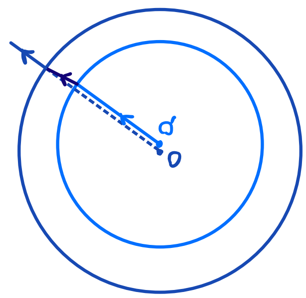
意义：电磁场能流密度矢量

平均：$\langle S\rangle = \frac{1}{2}E_0H_0 = \frac{1}{2}\sqrt{\frac{\varepsilon_r\varepsilon_0}{\mu_r\mu_0}}E_0^2 \propto E_0^2$

(4) 电磁辐射

① 带电粒子获瞬时加速度a，获得速度$u=a\Delta t\ll c$，又经间隔τ.

电场存在过渡区，$E_\theta = \frac{a\tau\sin\theta}{c}E_r$



$u\ll c \Rightarrow E_r \cong \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\cong\frac{1}{4\pi\varepsilon_0}\frac{q}{c^2\tau^2}$

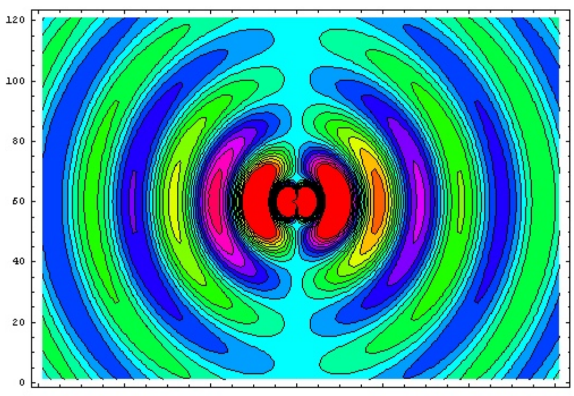
$\Rightarrow E_\theta \cong \frac{1}{4\pi\varepsilon_0}\frac{qa\sin\theta}{c^2 r}$

$\Rightarrow S = \sqrt{\frac{\varepsilon_0}{\mu_0}}E_\theta^2 = \frac{q^2a^2\sin^2\theta}{16\pi^2\varepsilon_0 c^3 r^2}$

总辐射功率 $\frac{dW_\infty}{dt} = \frac{q^2a^2}{6\pi\varepsilon_0c^3}$

② 偶极子 $\vec p = \vec p_0\cos\omega t$ 激发辐射

电磁场线环不断向外扩展，总功率 $\frac{dW_\infty}{dt} = \frac{\omega^4 p_0^2}{12\pi\varepsilon_0c^3}\propto\omega^4$

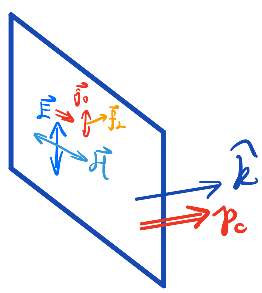


(5) 光压与场动量

① 机制

界面 $\vec E\Rightarrow$ 电子竖直往复运动

同步 $\vec H \Rightarrow$ Lorentz力沿 $\hat k$ 方向 ~ 光压.



② 光压 $\vec P_c = \frac{1}{c}(\vec S_{in}-\vec S_{out})$

③ 电磁波动量密度 $\vec g = \frac{1}{c^2}\vec S = \frac{1}{c^2}\vec E\times\vec H$

取 cdt，$\frac{d\vec G}{dS} = \frac{1}{c^2}(\vec S_{out}-\vec S_{in})\cdot cdt$

Newton III $\Rightarrow -P_c\,dt = \frac{1}{c}(\vec S_{out}-\vec S_{in})dt$

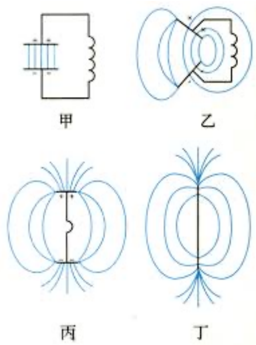
$\Rightarrow P_c = \frac{1}{c}(\vec S_{in}-\vec S_{out})$

④黑体：$p_{c,\text{blackbody}} = \frac{1}{c}\|S_{in}\| = \frac{1}{c}EH$

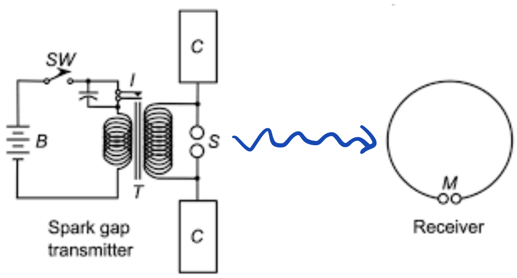
三、电磁波应用

(1) 激发与接收

LC高频回路 → 开放向外传输 → 谐振回路接收

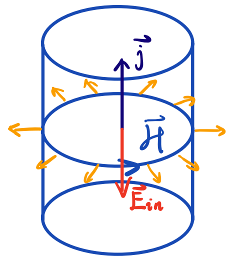


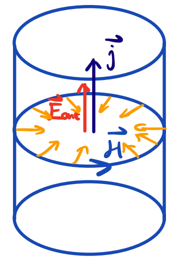
(2) Hertz实验：M电火花 → 电磁波

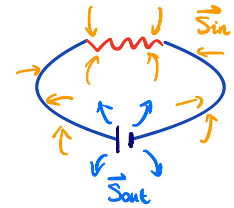


§6.3 电磁场与电路

一、能量在直流电路中传播







(1) 电源内部：$\vec{S}$指向外

(2) 外电路：$\vec{S}$指向内

二、交流趋肤效应

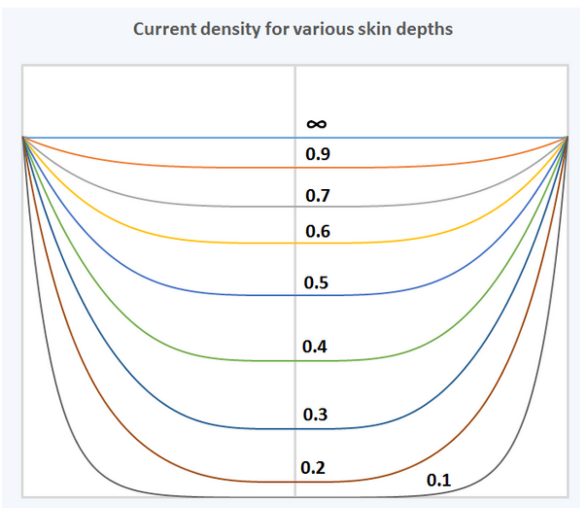
(1) 现象：交流电导线内电流集中分部于表面

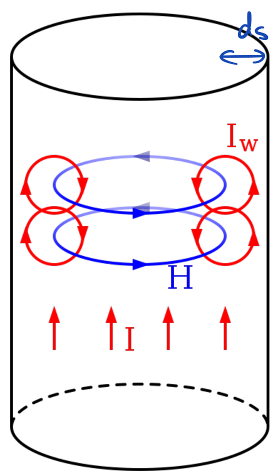
(2) Maxwell方程组
$$\boxed{\begin{cases}\nabla\cdot\vec{E}=0\\ \nabla\times\vec{E}=-\mu_r\mu_0\dfrac{\partial\vec{H}}{\partial t}\\ \nabla\cdot\vec{H}=0\\ \nabla\times\vec{H}=\sigma\vec{E}\end{cases}} \Rightarrow \begin{cases}\dfrac{\partial E_x}{\partial z}=-\mu_r\mu_0\dfrac{\partial H_y}{\partial t}\\ \dfrac{\partial H_y}{\partial z}=\sigma E_x\end{cases}$$

$$\Rightarrow \frac{\partial^2 E_x}{\partial z^2}=-\mu_r\mu_0\sigma\frac{\partial E_x}{\partial t}$$

E衰减至E_0e^{-1}特征深度

$$\Rightarrow \boxed{E_x(z)=E_0e^{-\frac{z}{d_s}}e^{i(\omega t-\frac{z}{d_s})}}\text{，其中 } \boxed{d_s\triangleq\sqrt{\frac{2}{\mu_r\mu_0\sigma\omega}}}\text{ 为趋肤深度}$$





三、交流电的准恒条件

(1) 电磁场波长 $\boxed{\lambda=\frac{c}{\nu}}$ vs. 电路线度 l

① $\lambda\sim l$. 无法定义电压，宏观与微观不再明显区分.

② $l\gg\lambda$ (准恒条件)：适用一般电路规律.

(2) 电报线方程 ($l\gg\lambda$ 传输交流电)

Thm — 电报线方程：

设一对长距传输线单位长度电容C^*、电感L^*.

则传输损耗
$$\boxed{\begin{cases}\dfrac{\partial I}{\partial x}=-C^*\dfrac{\partial U}{\partial t}\\ \dfrac{\partial U}{\partial x}=-L^*\dfrac{\partial I}{\partial t}\end{cases}}$$