例：行列式应用

例1. Fibonacci数列：$F_n=F_{n-1}+F_{n-2}\ (n\geq 3)$，$F_1=1$，$F_2=2$

(1) 证明：
$$F_n=\begin{vmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{vmatrix}$$

(2) 求 F_n.

证：(1) $F_n=F_{n-1}-1\cdot(-1)\cdot F_{n-2}=F_{n-1}+F_{n-2}$ $\longrightarrow \lambda^2-\lambda-1=0,\ \lambda_{1,2}=\frac{1\pm\sqrt5}{2}\triangleq\alpha,\beta$

$F_n=A(\frac{1-\sqrt5}{2})^n+B(\frac{1+\sqrt5}{2})^n$

$$\begin{cases} F_1=\frac{1-\sqrt5}{2}A+\frac{1+\sqrt5}{2}B=1 \\ F_2=\frac{3-\sqrt5}{2}A+\frac{3+\sqrt5}{2}B=2 \end{cases} \Rightarrow \begin{cases} A=\frac{1+\sqrt5}{2\sqrt5} \\ B=-\frac{1-\sqrt5}{2\sqrt5} \end{cases}$$

解：(2) (i) $n=1$. $F_n=1$　　$n=2$. $F_n=2$

$n=3$. $F_3=1-(-1-1)=3$

(ii) 设 $\forall n\leq k-1$，$F_n=\frac{1}{\sqrt5}(\alpha^{n+1}-\beta^{n+1})$

则 $F_k=F_{k-1}+F_{k-2}=\frac{1}{\sqrt5}(\alpha^k-\beta^k+\alpha^{k-1}-\beta^{k-1})$

$=\frac{1}{\sqrt5}(\frac{1+\alpha}{\alpha^2}\alpha^{k+1}-\frac{1+\beta}{\beta^2}\beta^{k+1})$

$\frac{1+\alpha}{\alpha^2}=\frac{1+\frac{1+\sqrt5}{2}}{\frac{3+\sqrt5}{2}}=1$，$\frac{1+\beta}{\beta^2}=\frac{1+\frac{1-\sqrt5}{2}}{\frac{3-\sqrt5}{2}}=1$

$\therefore F_k=\frac{1}{\sqrt5}(\alpha^{k+1}-\beta^{k+1})$ 成立.

综上. $F_n=\frac{1}{\sqrt5}((\frac{1+\sqrt5}{2})^{n+1}-(\frac{1-\sqrt5}{2})^{n+1})$.

例2. 设 $f_{ij}(t)$ 可微，$1\leq i,j\leq n$，令 $F(t)=\det(f_{ij}(t))$

求证：
$$\frac{d}{dt}F(t)=\sum_{j=1}^{n}\begin{vmatrix} f_{11}(t) & f_{12}(t) & \cdots & f_{1,j-1}(t) & \frac{d}{dt}f_{1j}(t) & f_{1,j+1}(t) & \cdots & f_{1n}(t) \\ f_{21}(t) & f_{22}(t) & \cdots & f_{2,j-1}(t) & \frac{d}{dt}f_{2j}(t) & f_{2,j+1}(t) & \cdots & f_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n1}(t) & f_{n2}(t) & \cdots & f_{n,j-1}(t) & \frac{d}{dt}f_{nj}(t) & f_{n,j+1}(t) & \cdots & f_{nn}(t) \end{vmatrix}$$

证：
$$\frac{d}{dt}F(t)=\frac{d}{dt}\sum_{i_1\cdots i_n}(-1)^{\tau(i_1\cdots i_n)}f_{i_1 1}(t)f_{i_2 2}(t)\cdots f_{i_n n}(t)$$
$$=\sum_{i_1\cdots i_n}(-1)^{\tau(i_1\cdots i_n)}\sum_{j=1}^{n}f_{i_1 1}(t)\cdots f_{i_{j-1},j-1}(t)\left(\frac{d}{dt}f_{i_j j}(t)\right)f_{i_{j+1},j+1}(t)\cdots f_{i_n n}(t)$$
$$=\sum_{j=1}^{n}\sum_{i_1\cdots i_n}(-1)^{\tau(i_1\cdots i_n)}f_{i_1 1}(t)\cdots f_{i_{j-1},j-1}(t)\left(\frac{d}{dt}f_{i_j j}(t)\right)f_{i_{j+1},j+1}(t)\cdots f_{i_n n}(t)$$
$$=\sum_{j=1}^{n}\begin{vmatrix} f_{11}(t) & f_{12}(t) & \cdots & f_{1,j-1}(t) & \frac{d}{dt}f_{1j}(t) & f_{1,j+1}(t) & \cdots & f_{1n}(t) \\ f_{21}(t) & f_{22}(t) & \cdots & f_{2,j-1}(t) & \frac{d}{dt}f_{2j}(t) & f_{2,j+1}(t) & \cdots & f_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n1}(t) & f_{n2}(t) & \cdots & f_{n,j-1}(t) & \frac{d}{dt}f_{nj}(t) & f_{n,j+1}(t) & \cdots & f_{nn}(t) \end{vmatrix}$$

例3. 实系数三元多项式 $f(x,y,z)=x^3+y^3+z^3-3xyz$ 有没有一次因式？

解：$f(x,y,z)=\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}=(x+y+z)\begin{vmatrix} 1 & y & z \\ 1 & x & y \\ 1 & z & x \end{vmatrix}$

$\therefore f(x,y,z)$ 有一次因式 $(x+y+z)$

引理：$\begin{vmatrix} 1 & y & z \\ 1 & x & y \\ 1 & z & x \end{vmatrix}=x^2+y^2+z^2-xy-xz-yz$ 不能表成一次因式乘积

证：令 $g(x,y,z)=x^2+y^2+z^2-xy-xz-yz$

若 $g(x,y,z)=P_1(x,y,z)Q_1(x,y,z)$，则曲面 $g(x,y,z)=0$ 至少有一个平面解

又 $g(x,y,z)=\frac12((x-y)^2+(y-z)^2+(z-x)^2)$ 有且只有一直线解 $x=y=z$

故 $g(x,y,z)$ 不可表成两一次因式之积

$\therefore f(x,y,z)$ 有且只有一个一次因式

例4. 因式分解 $\begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}$

解: $f(x,y,z)=(x+y+z)\begin{vmatrix} 1 & x & y & z \\ 1 & 0 & z & y \\ 1 & z & 0 & x \\ 1 & y & x & 0 \end{vmatrix}=(x+y+z)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & -x & z-y & y-z \\ 1 & z-x & -y & x-z \\ 1 & y-x & x-y & -z \end{vmatrix}=(x+y+z)\begin{vmatrix} -x & z-y & y-z \\ z-x & -y & x-z \\ y-x & x-y & -z \end{vmatrix}$

$=-(x+y+z)\begin{vmatrix} x & z-y & y-z \\ y & -y & x-z \\ z & x-y & -z \end{vmatrix}=-(x+y+z)\begin{vmatrix} x & x-y+z & x+y-z \\ y & 0 & x+y-z \\ z & x-y+z & 0 \end{vmatrix}$

$=-(x+y+z)(x-y+z)(x+y-z)\begin{vmatrix} x & 1 & 1 \\ y & 0 & 1 \\ z & 1 & 0 \end{vmatrix}$

$=(x+y+z)(x-y+z)(x+y-z)(x-y-z)$

例5. 计算实数域上n阶三对角线行列式 $\det D_n=\begin{vmatrix} a & b & 0 & 0 & \cdots & 0 & 0 \\ c & a & b & 0 & \cdots & 0 & 0 \\ 0 & c & a & b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c & a \end{vmatrix}$

解: $\det D_n=a\det D_{n-1}-bc\det D_{n-2}$

(i) $bc=0$: $\det D_n=a^n$

(ii) $bc\neq 0$:

特征方程 $\lambda^2-a\lambda+bc=0$.

① $\Delta=a^2-4bc=0$, $\det D_n=(An+B)\alpha^n$

$$\begin{cases} \det D_1=(A+B)\alpha=a, \\ \det D_2=(2A+B)\alpha^2=a^2-bc=\frac{3}{4}a^2 \\ \det D_3=(3A+B)\alpha^3=a^3-2abc=\frac{1}{2}a^3 \end{cases}$$

设 $\alpha=ka$ $\Rightarrow \frac{1}{2k^3}-\frac{3}{4k^3}=\frac{3}{4k^2}-\frac{1}{k}\Rightarrow k=1$(舍)或 $\frac{1}{2}$

$\Rightarrow A=B=1$

$\therefore \det D_n=(n+1)(\frac{a}{2})^n\quad (a^2=4bc)$

② $a^2\neq 4bc$

两特征根 $\lambda_{1,2}$: $\lambda_1+\lambda_2=a$, $\lambda_1\lambda_2=bc$

设 $\det D_n=A\lambda_1^{n+1}+B\lambda_2^{n+1}$

$$\begin{cases} \det D_1=A\lambda_1^2+B\lambda_2^2=a \\ \det D_2=A\lambda_1^3+B\lambda_2^3=a^2-bc \end{cases}$$

$a\det D_1=A\lambda_1^3+B\lambda_2^3+(A\lambda_1+B\lambda_2)\lambda_1\lambda_2=a^2-bc+bc\Rightarrow \det D_0=A\lambda_1+B\lambda_2=1$

$\Rightarrow B\lambda_2(\lambda_1-\lambda_2)=\lambda_1-a\Rightarrow B=\frac{-\lambda_2}{\lambda_2(\lambda_1-\lambda_2)}=\frac{-1}{\lambda_1-\lambda_2}\Rightarrow A=\frac{1}{\lambda_1-\lambda_2}$

$\therefore \det D_n=\frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2}$

综上: $\det D_n=\begin{cases} a^n & bc=0 \\ (n+1)(\frac{a}{2})^n & bc=\frac{a^2}{4} \\ \frac{\lambda_1^{n+1}-\lambda_2^{n+1}}{\lambda_1-\lambda_2} & bc\neq\frac{a^2}{4} \end{cases}$

例6. 计算 $\det D_n=\begin{vmatrix} 2n & n & 0 & \cdots & 0 & 0 \\ n & 2n & n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2n & n \end{vmatrix}$

解: $\det D_n=(n+1)n^n$

例7. 计算 $\det D_n=\begin{vmatrix} 2\cos\alpha & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2\cos\alpha & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\cos\alpha & 1 \end{vmatrix}$

解: $\Delta=4(\cos^2\alpha-1)$

(i) $\alpha=2k\pi$. $\det D_n=(n+1)\cos^n\alpha=(n+1)$

(ii) $\alpha=(2k+1)\pi$. $\det D_n=(n+1)\cos^n\alpha=(-1)^n(n+1)$

(iii) $\alpha\neq m\pi$, $\lambda_{1,2}=\frac{-2\cos\alpha\pm 2i\sin\alpha}{2}=-\cos\alpha\pm i\sin\alpha=-e^{\mp i\alpha}$

$$\Rightarrow \det D_n=\frac{-e^{-i(n+1)\alpha}+e^{i(n+1)\alpha}}{2i\sin\alpha}=\frac{2i\sin(n+1)\alpha}{2i\sin\alpha}=\frac{\sin(n+1)\alpha}{\sin\alpha}$$

$$\therefore \det D_n=\begin{cases} n+1 & \alpha=2k\pi \\ (-1)^n(n+1) & \alpha=(2k+1)\pi \\ \frac{\sin(n+1)\alpha}{\sin\alpha} & \alpha\neq m\pi \end{cases}\quad (k,m\in\mathbb{Z})$$

例8. 设 $a_1,\cdots,a_n$ 是数域 K 中互不相同的数, $b_1,\cdots,b_n$ 是数域 K 中任意一组给定的数.

证明: $\exists!$ K 上多项式 $f(x)=c_1+c_2x+\cdots+c_nx^{n-1}$. s.t. $f(a_i)=b_i$, $\forall i=1,2,\cdots,n$

证: 对线性方程组 $\begin{cases} c_1+a_1c_2+\cdots+a_1^{n-1}c_n=b_1 \\ c_1+a_2c_2+\cdots+a_2^{n-1}c_n=b_2 \\ \quad\vdots \\ c_1+a_nc_2+\cdots+a_n^{n-1}c_n=b_n \end{cases}$

$\det A=\det V_n(a_1,\cdots,a_n)=\prod_{1\le i<j\le n}(a_j-a_i)\neq 0$

$\therefore$ 有唯一解 $(c_1,c_2,\cdots,c_n)'$

$\therefore$ $\exists!$ $f(x)=\sum_{i=1}^{n}c_ix^{i-1}$ s.t. $f(a_i)=b_i$, $\forall i=1,2,\cdots,n$

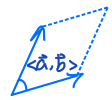
补充题二

T1. 在空间右手直角坐标系 $[O;\vec{e}_1,\vec{e}_2,\vec{e}_3]$ 中，两个非零向量 $\vec{a}=(a_1,a_2,0)'$，$\vec{b}=(b_1,b_2,0)'$

(1) 求以 $\vec{a}$，$\vec{b}$ 为邻边平行四边形面积，用一个行列式表出.

(2) 求以 $\vec{a}$，$\vec{b}$ 为邻边三角形面积，用一个行列式表出.

解：(1)
$$
\begin{aligned}
S_{\square} &= |\vec{a}||\vec{b}|\sqrt{1-\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)^2} \\
&= \sqrt{\vec{a}^2\vec{b}^2-(\vec{a}\cdot\vec{b})^2} \\
&= \sqrt{(a_1^2+a_2^2)(b_1^2+b_2^2)-(a_1b_1+a_2b_2)^2} \\
&= |a_1b_2-a_2b_1| \\
&= \left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|
\end{aligned}
$$

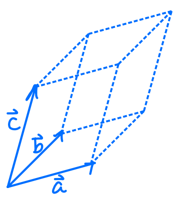


(2) $S_{\triangle}=\frac{1}{2}\left|\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right|$

T2. 在空间右手直角坐标系 $[O;\vec{e}_1,\vec{e}_2,\vec{e}_3]$ 中，三个非零向量 $\vec{a}=(a_1,a_2,a_3)'$，$\vec{b}=(b_1,b_2,b_3)'$，$\vec{c}=(c_1,c_2,c_3)'$

求以 $\vec{a}$，$\vec{b}$，$\vec{c}$ 为棱的平行六面体的体积.

解：



$$
\begin{aligned}
&S_{底} = |\vec{a}\times\vec{b}| \\
&\vec{n}=\vec{a}\times\vec{b} \Rightarrow d=\frac{|\vec{n}\cdot\vec{c}|}{|\vec{n}|} \\
&V = S_{底}\, d = |\vec{n}\cdot\vec{c}| = |(\vec{a}\times\vec{b})\cdot\vec{c}| \\
&\quad = |(a_2b_3-a_3b_2,\ a_3b_1-a_1b_3,\ a_1b_2-a_2b_1)'\cdot(c_1,c_2,c_3)'| \\
&\quad = \left|c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\right| \\
&\quad = \left|\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\right|
\end{aligned}
$$

T3. 求元素为 0 或 1 的三阶行列式最大值.

解：按第一列展开，为最大，$a_{11}=a_{31}=1$，$a_{21}=0$

$$\det A_2 = \begin{vmatrix} \overset{1}{a_{22}} & \overset{1}{a_{23}} \\ \overset{0}{a_{32}} & \overset{1}{a_{33}} \end{vmatrix} + \begin{vmatrix} \overset{1}{a_{12}} & \overset{0}{a_{13}} \\ \overset{1}{a_{22}} & \overset{1}{a_{23}} \end{vmatrix}$$

先令第一项最大 $\Rightarrow a_{22}=a_{33}=1$

再考虑第二项，因已有 $a_{22}=1$，为最大，$a_{13}=0$，$a_{12}=a_{23}=1$

再返回第一项，因已有 $a_{23}=1$，为最大，$a_{32}=0$

$$\therefore \det A_{2max} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2$$

T4. 求元素为 1 或 -1 的三阶行列式最大值.

解：共 6 项：$\overset{A_1}{a_{11}a_{22}a_{33}}$　$\overset{A_2}{a_{21}a_{32}a_{13}}$　$\overset{A_3}{a_{31}a_{12}a_{23}}$　$\overset{B_1}{-a_{31}a_{22}a_{13}}$　$\overset{B_2}{-a_{11}a_{32}a_{23}}$　$\overset{B_3}{-a_{21}a_{12}a_{33}}$

不可能 $A_{1,2,3}=B_{1,2,3}=1$. 否则 $\prod a_{ij}=A_1A_2A_3=1=(-B_1)(-B_2)(-B_3)=-1$ 矛盾！

又 $\det D_3 \equiv |A_1|+|A_2|+|A_3|+|B_1|+|B_2|+|B_3| \pmod 2 \equiv 0 \pmod 2$

则 $\det D_3 \leq 4$. 构造 $\det A = \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 4$

$\therefore \det D_{3max}=4$

T5 设$n\geqslant 3$，证明：元素为1或-1的n阶行列式满足：$|\det D_n|\leqslant (n-1)!(n-1)$

证：(i) $n=3$，由T4，$\det D_3\leqslant 4=(3-1)!\times(3-1)$ 成立。

(ii) 设$|\det D_{k-1}|\leqslant (k-2)!(k-2)$

则$|\det D_k|=\left|\sum_{i=1}^{k}a_{i1}A_{i1}\right|\leqslant\sum_{i=1}^{k}|a_{i1}||A_{i1}|$

每个A_{i1}可看作$\pm\det D_{k-1}$

$\Rightarrow |\det D_k|\leqslant k(k-2)(k-2)!=(k^2-2k)(k-2)!<(k-1)^2(k-2)!=(k-1)!(k-1)$ 成立

综上，$|\det D_n|\leqslant (n-1)!(n-1)$（$\forall n\geqslant 3$，等号当且仅当$n=3$成立）

T6. 求元素为1或-1的四阶行列式最大值.

解：由T5，$\det D_4<3!\times 3=18$

又$2\mid\det D_4$

$\therefore \det D_4\leqslant 16$

又构造 $\begin{vmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{vmatrix}=\begin{vmatrix}1&1&1&1\\0&-2&0&-2\\0&0&-2&-2\\0&-2&-2&0\end{vmatrix}=-8\begin{vmatrix}1&0&1\\0&1&1\\1&1&0\end{vmatrix}=16$

$\therefore \det D_{4\max}=16$.

T7. 设$n\geqslant 2$，证明：元素为1或-1的n阶行列式的值能被2^{n-1}整除.

证：(i) $2\mid\det D_2$ 成立！

(ii) 设$2^{k-2}\mid\det D_{k-1}$ 成立.

考虑$\det D_k$：

若$a_{i1}=a_{11}$，将第1行-1倍加到第i行，否则将第1行加到第i行，$D_k\to\widetilde{D_k}$，$\det D_k=\det\widetilde{D_k}$

此时$\det\widetilde{D_k}=\begin{vmatrix}a_{11}&a_{12}\ \cdots\ a_{1k}\\0&\\\vdots&J_{k-1}\\0&\end{vmatrix}=\pm\det J_{k-1}$，其中$J_{k-1}$的元素$a'_{ij}$ $(1\leqslant i,j\leqslant k-1)$：$a'_{ij}\in\{-2,0,2\}$

$2\mid a'_{ij}\Rightarrow J_{k-1}$每一行可提出公因子2，提出后$a'_{ij}\to\widehat{a}'_{ij}\in\{\pm1,0\}$，$J_{k-1}\to D_{k-1}$

则$\det D_k=\pm\det J_{k-1}=\pm 2^{k-1}\det D_{k-1}$

$\det D_{k-1}\in\mathbb{Z}\Rightarrow 2^{k-1}\mid\det D_k$ 成立！

综上：$2^{n-1}\mid\det D_n$，$\forall n\geqslant 2$.