§2. Taylor公式

一、Taylor公式

(1) 比较 $\begin{cases} \text{微分：} f(x_0+h) = f(x_0) + f'(x_0)h + o(h) & \text{有误差，以直代曲} \\ \text{Lagrange：} f(x_0+h) = f(x_0) + f'(x_0+\theta h)h \quad (0<\theta<1) & \text{无误差，}\theta\text{未定} \longrightarrow \text{Lagrange余项} \end{cases}$

(2) 推广至以多项式逼近：$f(x_0+h) = \underbrace{f(x_0) + f'(x_0)h + \_\_\_ h^2 + \cdots + \_\_\_ h^n}_{n\text{阶多项式}} + \underbrace{o(h^n)}_{\text{Pieno余项}}$

(3) $\boxed{\text{定理}}$ 1.(唯一性) $f(x) = a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)^n + o((x-x_0)^n)$

$= b_0 + b_1(x-x_0) + \cdots + b_n(x-x_0)^n + o((x-x_0)^n)$

则 $a_i = b_i \ (i=0,1,2,\cdots,n)$

证：公式两边 $\to x_0 \Rightarrow a_0 = b_0$

消去 $a_0=b_0$，约去 $x-x_0$，两边 $\to x_0 \Rightarrow a_1 = b_1$

...

$\Rightarrow a_i = b_i \ (i=0,1,2,\cdots,n)$

2.(Pieno余项Taylor公式)

$f(x)$ 在 x_0 处 n 阶可导，则 $f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + o((x-x_0)^n)$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + o((x-x_0)^n)$$

证：$$\lim_{x\to x_0} \frac{f(x) - \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}{(x-x_0)^n} \xlongequal[\text{L'Hospital法则}]{\text{运用}n-1\text{次}} \lim_{x\to x_0}\left[\frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{n!(x-x_0)} - f^{(n)}(x_0)\right] = 0$$

$$\therefore f(x) - \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = o((x-x_0)^n)$$

$\boxed{\text{定义}}$ $\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ 为 $f(x)$ 在 x_0 处 Taylor多项式

$o((x-x_0)^n)$ 为 Pieno余项式

特别地，$x_0=0$，$\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k$ 称为 $f(x)$ 的 Maclaurin多项式

3.(Lagrange余项Taylor公式)

$f\in C^n[a,b]$，在 (a,b) 内有 $n+1$ 阶导数

则 $\forall x, x_0 \in [a,b]$，$f(x) = \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$，$\xi$ 介于 x_0, x 之间

证：令 $F(t) = f(x) - \sum_{k=0}^{n}\frac{f^{(k)}(t)}{k!}(x-t)^k$，$G(t) = (x-t)^{n+1}$.

不妨 $x > x_0$，F, G 在 $[x_0,x]$ 连续，在 (x_0,x) 可导

由Cauchy中值定理 $\frac{F(x_0)}{G(x_0)} = \frac{F(x_0)-F(x)}{G(x_0)-G(x)} \xlongequal{\exists\xi\in(x_0,x)} \frac{F'(\xi)}{G'(\xi)} = \frac{f^{(n+1)}(\xi)}{(n+1)!}$

$$\therefore f(x_0) = \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

$\boxed{\text{定义}}$ $\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$ 为 Lagrange余项

误差估计：$|R_n(x)| = \frac{|f^{(n+1)}(\xi)|}{(n+1)!}|x-x_0|^{n+1} \leqslant \frac{M|x-x_0|^{n+1}}{(n+1)!}$，其中 M 为 $|f^{(n+1)}(x)|$ 上界

二、常见函数 Taylor(Maclaurin)公式

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{e^{\theta x}\cdot x^{n+1}}{(n+1)!} \quad (0<\theta<1)$$

$$\begin{cases} \operatorname{sh}x = \dfrac{e^x - e^{-x}}{2} = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots + \dfrac{x^{2n-1}}{(2n-1)!} + \dfrac{\operatorname{ch}\theta x \cdot x^{2n+1}}{(2n+1)!} \\ \operatorname{ch}x = \dfrac{e^x + e^{-x}}{2} = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots + \dfrac{x^{2n}}{(2n)!} + \dfrac{\operatorname{sh}\theta x \cdot x^{2n+2}}{(2n+2)!} \end{cases}$$

$$\begin{cases} \sin x = \dfrac{e^{ix} - e^{-ix}}{2i} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots + (-1)^{n-1}\dfrac{x^{2n-1}}{(2n-1)!} + (-1)^n \dfrac{\cos\theta x \cdot x^{2n+1}}{(2n+1)!} \\ \cos x = \dfrac{e^{ix} + e^{-ix}}{2} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots + (-1)^n \dfrac{x^{2n}}{(2n)!} + (-1)^{n+1}\dfrac{\cos\theta x \cdot x^{2n+2}}{(2n+2)!} \end{cases}$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + (-1)^n \frac{x^{n+1}}{(n+1)(1+\theta x)^{n+1}} \quad (x>-1)$$

$$(x+1)^\alpha = 1 + \alpha x + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}(1+\theta x)^{\alpha-n-1}x^{n+1} \quad (x>-1)$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + o(x^{2n+1})$$

例：展开 $\ln(1+\sin^2 x)$ 到 x^4 项的带Pieno余项 Taylor 公式

解：$f(x) = \sin^2 x - \frac{\sin^4 x}{2} + o(x^4)$

$\sin x = x - \frac{x^3}{6} + o(x^3)$

$\therefore f(x) = \left(x - \frac{x^3}{6} + o(x^3)\right)^2 - \frac{(x+o(x))^4}{2}$

$= x^2 - \frac{5}{6}x^4 + o(x^4)$

例题

一、Pieno余项型 Taylor公式

例1. $y = e^x$ 在 $x=0$ 处 Taylor 多项式：$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^n)$

例2. $y = \sin x$ 在 $x=0$ 处 Taylor多项式：$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + o(x^{2k+1})$

例3. $y = \cos x$ 在 $x=0$ 处 Taylor多项式：$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + o(x^{2k})$

例4. $y = (x+1)^\alpha$ 在 $x=0$ 处 Taylor多项式：$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + o(x^n)$

例5. $y = \ln(x+1)$ 在 $x=0$ 处 Taylor多项式：$\ln x = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n+1}x^n}{n} + o(x^n)$

例6. $y = e^{-x^2}$ 在 $x=0$ 处 Taylor多项式：$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + o(x^{2n})$

例7. 求 $\lim\limits_{x\to 0} \frac{e^x - 1 - x - \frac{x}{2}\sin x}{\sin x - x\cos x}$

解：$\lim\limits_{x\to 0} \frac{e^x - 1 - x - \frac{x}{2}\sin x}{\sin x - x\cos x} = \lim\limits_{x\to 0} \frac{\frac{x^2}{2} + \frac{1}{6}x^3 + o(x^3) - \frac{x}{2}\left(x - \frac{1}{6}x^3 + o(x^3)\right)}{x - \frac{1}{6}x^3 + o(x^3) - x\left(1 - \frac{1}{2}x^2 + o(x^2)\right)} = \lim\limits_{x\to 0} \frac{\frac{1}{12}x^4 + \frac{1}{6}x^3 + o(x^4)}{\frac{1}{3}x^3 + o(x^3)} = \frac{1}{2}$

例8 设 $m>1$，求 $\lim\limits_{x\to\infty}\left[(x^m + x^{m-1})^{1/m} - (x^m - x^{m-1})^{1/m}\right]$

解：$\lim\limits_{x\to\infty}\left[(x^m + x^{m-1})^{1/m} - (x^m - x^{m-1})^{1/m}\right] = \lim\limits_{x\to\infty} x\left((1+\frac{1}{x})^{\frac{1}{m}} - (1-\frac{1}{x})^{\frac{1}{m}}\right) = \lim\limits_{x\to\infty} x\left(1 + \frac{1}{mx} + o(\frac{1}{x}) - (1 - \frac{1}{mx} + o(\frac{1}{x}))\right) = \frac{2}{m}$

二、Lagrange余项

例1. $-\frac{\pi}{2} < x < \frac{\pi}{2}$，用带Lagrange余项Taylor公式计算 $\sin x$ 时，为使误差小于 5×10^{-7}，应取多少项？

解：$|R_n(x)| = \frac{|\cos\theta x|\cdot x^{2n+1}}{(2n+1)!} \leq \frac{1}{(2n+1)!}$

令 $\frac{1}{(2n+1)!} < 5\times10^{-7} \Rightarrow n \geq 5$ 即可

习题4.3

T1. 求下列函数在 $x=0$ 处局部Taylor公式

(1) $\operatorname{sh}x = \frac{e^x-e^{-x}}{2} = x + \frac{x^3}{3!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$

(2) $\frac{1}{2}\ln\frac{1+x}{1-x} = \frac{1}{2}(\ln(1+x)-\ln(1-x)) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2k+1}}{2k+1} + o(x^{2k+1})$

(3) $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2}\left(1-\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\cdots+\frac{(-1)^n(2x)^{2n}}{(2n)!}\right)\right) = \frac{(2x)^2}{2\cdot 2} - \frac{(2x)^4}{2\cdot 4!} + \cdots + \frac{(-1)^{n+1}(2x)^{2n}}{2\cdot(2n)!} + o(x^{2n})$

(4) $\frac{x^2+2x-1}{x-1}$

$f(x) = x+3+\frac{2}{x-1} \Rightarrow f(0)=1$

$f'(x) = 1-\frac{2}{(x-1)^2} \Rightarrow f'(0)=-1$

$f''(x) = 2\cdot\frac{2}{(x-1)^3} \Rightarrow f''(0)=-4$

$f^{(n)}(x) = 2\cdot\frac{(-1)^n n!}{(x-1)^{n+1}} \Rightarrow f^{(n)}(0) = -2n! \ (n\geq 2)$

$\therefore f(x) = 1-x-2(x^2+x^3+\cdots+x^n)+o(x^n)$

(5) $\cos x^3 = 1-\frac{(x^3)^2}{2}+\frac{(x^3)^4}{4!}-\cdots+\frac{(-1)^n x^{6n}}{(2n)!}+o(x^{6n})$

T2. 展开下列函数Maclaurin多项式至指定阶数

(1) $e^x\sin x \ (x^4)$

$e^x\sin x = \left(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+o(x^4)\right)\left(x-\frac{x^3}{3!}+o(x^5)\right) = x+x^2+\frac{1}{5}x^3+o(x^4)$

(2) $\sqrt{1+x}\cos x \ (x^4)$

$\sqrt{x+1}\cos x = \left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{125}x^4+o(x^4)\right)\left(1-\frac{x^2}{2}+\frac{x^4}{4!}+o(x^4)\right) = 1+\frac{x}{2}-\frac{5x^2}{8}-\frac{3x^3}{16}+\frac{25x^4}{384}+o(x^4)$

(3) $\sqrt{1-2x+x^3}-\sqrt{1-3x+x^2} \ (x^3)$

$$\sqrt{1+(x^3-2x)}-\sqrt{1+(x^2-3x)} = \left(1+\frac{x^3-2x}{2}-\frac{(x^3-2x)^2}{8}+\frac{(x^3-2x)^3}{16}+o(x^3)\right)-\left(1+\frac{x^2-3x}{2}-\frac{(x^2-3x)^2}{8}+\frac{(x^2-3x)^3}{16}+o(x^3)\right)$$

$$=\left(1+\frac{x^3-2x}{2}-\frac{4x^2}{8}+\frac{-8x^3}{16}+o(x^3)\right)-\left(1+\frac{x^2-3x}{2}-\frac{-6x^3+9x^2}{8}+\frac{-27x^3}{16}+o(x^3)\right)$$

$$=\frac{x}{2}+\frac{x^2}{8}+\frac{15x^3}{16}+o(x^3)$$

变限积分
展开被积项

T3. 求下列函数Maclaurin级数

(1) $\arctan x$

$$\arctan x = \int_0^x \frac{dt}{1+t^2}$$

$$= \int_0^x (1-t^2+t^4-t^6+\cdots+(-1)^k t^{2k}+o(t^{2k}))\,dt$$

$$= x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+\frac{(-1)^k x^{2k+1}}{2k+1}+o(x^{2k+1})$$

(2) $\arcsin x$

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$$

$$= \int_0^x \left(1+\frac{1}{2}t^2+\frac{3}{8}t^4+\frac{5}{16}t^6+\cdots+\frac{(2k-1)!!}{(2k)!!}t^{2k}+o(t^{2k})\right)dt$$

$$= x+\frac{1}{6}x^3+\frac{3}{40}x^5+\cdots+\frac{(2k-1)!!}{(2k+1)(2k)!!}x^{2k+1}+o(x^{2k+1})$$

T4. 用 Taylor 公式求极限

(1) $\lim\limits_{x\to 0}\frac{1-x^2-e^{-x^2}}{x\sin^3 2x} = \lim\limits_{x\to 0}\frac{1-x^2-\left(1-x^2+\frac{x^4}{2}+o(x^4)\right)}{x(2x+o(x))^3} = \lim\limits_{x\to 0}\frac{-\frac{x^4}{2}+o(x^4)}{8x^4+o(x^4)} = -\frac{1}{16}$

(2) $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = \lim\limits_{x\to 0}\frac{e^x-x-1}{x(e^x-1)} = \lim\limits_{x\to 0}\frac{\frac{1}{2}x^2+o(x^2)}{x^2+o(x^2)} = \frac{1}{2}$

(3) $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{\cos x}{\sin x}\right)\frac{1}{\sin x} = \lim\limits_{x\to 0}\frac{\sin x - x\cos x}{x\sin^2 x} = \lim\limits_{x\to 0}\frac{\left(x-\frac{x^3}{6}\right)-x\left(1-\frac{x^2}{2}\right)+o(x^3)}{x^3+o(x^3)} = \lim\limits_{x\to 0}\frac{\frac{1}{3}x^3+o(x^3)}{x^3+o(x^3)} = \frac{1}{3}$

T5. x 较小时，可用 $\sin a + x\cos a$ 近似替代 $\sin(a+x)$. 证明其误差不大于 $\frac{1}{2}|x|^2$

证: $R(x) = \sin(a+x) - (\sin a + x\cos a)$

$R'(0) = 0$

$R'(x) = \cos(a+x) - \cos a \Rightarrow R'(0) = 0$

$R''(x) = -\sin(a+x)$

$\therefore |R(x)| = \left|0+0+\frac{R''(\xi)x^2}{2}\right| = \frac{|\sin(a+\xi)|\cdot x^2}{2} \leqslant \frac{x^2}{2}$

T6. $0\leqslant x\leqslant \frac{1}{3}$，用公式 $1+x+\frac{x^2}{2}+\frac{x^3}{6}$ 计算 e^x，证明其误差不超 8×10^{-4}

证: $R_4(x) = \frac{e^{\theta x}x^4}{24} \leqslant \frac{e^{\frac{1}{3}}}{24\cdot 3^4} \approx 7.18\times 10^{-4} < 8\times 10^{-4}$