Homework (150)

Due date: the 2nd/3rd of Dec.

Problem 1. (70pt)

The solutions of the following differential equation are called the Bessel functions of order ν.

$$x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0\,, \qquad ' \equiv \frac{d}{dx}\,.$$

(a) Show that the function $J_n(x)$ in the following expansion satisfies the above differential equation when the index ν is given by the integer n (5pt)

$$e^{\frac{z}{2}(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(z)\,,$$

where $J_n(z)$ is called the Bessel function (of the first kind) of the integer order. Obtain the series expansion form of $J_n(z)$ by expanding two exponentials $e^{zt/2} \cdot e^{-z/(2t)}$ and by collecting the coefficients of t^n. (5pt)

(b) (10pt) Verify that the following integral representation of the Bessel function satisfies the above Bessel equation

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i} \int_C \frac{1}{t^{\nu+1}} e^{t - z^2/4t}\, dt\,, \tag{1}$$

where the path C encircles the branch cut of $1/t^{\nu+1}$ anti-clockwise and $arg\ (t) \to \pm\pi$ as the path goes to infinity as is given in the following Fig. 1:

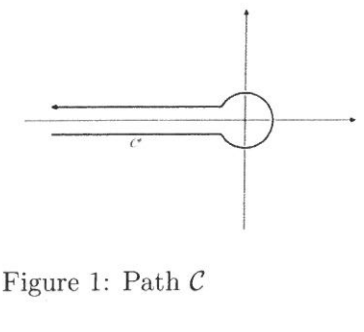

Figure 1: Path $\mathcal{C}$

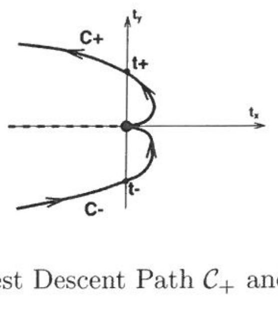

Figure 2: Steepest Descent Path $\mathcal{C}_+$ and $\mathcal{C}_-$

(c) (20pt) Derive the following asymptotic expression of the Bessel function $J_\nu(x)$ by using the steepest descent method

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).$$

[**Hint**: Use the steepest descent path $\mathcal{C}_+ + \mathcal{C}_-$ shown in Fig. 2 and note that $t_\pm = \pm ix/2$.]

(d) (10pt) Prove the following expansion formula

$$e^{i\vec{k}\cdot\vec{x}} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta)\,,$$

where the spherical coordinates are taken as $\vec{k}\cdot\vec{x} = kr\cos\theta$. Here, $j_\ell(x)$ denotes the spherical Bessel function and $P_\ell(x)$ does the Legendre polynomial. Note that the spherical Bessel function is defined by

$$j_\ell(x) \equiv \sqrt{\frac{\pi}{2x}} J_{\ell+\frac{1}{2}}(x)\,,$$

where $J_\nu(x)$ is the Bessel function. **Note** that the spherical Bessel functions can be written in terms of elementary functions; Ex. $j_0(x) = \sin x/x$.

(e) (10pt) Derive the following asymptotic behavior of the spherical Bessel function by using the result in part (b) and (c)

$$j_\ell(x) \sim \frac{\sin(x-\ell\pi/2)}{x}\,.$$

(f) By using the change of variable $t \to zt/2$ in Eq. (1), one can see that the integral representation of the Bessel function can be written as

$$J_\nu(z) = \frac{1}{2\pi i}\int_C \frac{1}{t^{\nu+1}}\, e^{\frac{z}{2}(t-\frac{1}{t})} dt\,.$$

For $\nu = n$, an integer, explain that the contour C can be deformed to a contour C' encircling the origin $t=0$. (5pt) By using this deformed contour C', derive the following representation (5pt)

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\; e^{-in\theta} e^{iz\sin\theta}\,.$$

Problem **2**. (20pt) Neumann function(=Bessel function of the second kind) is defined by

$$N_\nu(x) \equiv \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}\,,$$

which has no regular series expansion about $x=0$. Hankel functions are defined by

$$H_\nu^{(1)}(z) \equiv J_\nu(z) + iN_\nu(z)\,, \qquad H_\nu^{(2)}(z) \equiv J_\nu(z) - iN_\nu(z)\,,$$

and spherical Hankel functions are defined by

$$h_n^{(1)}(z) \equiv \sqrt{\frac{\pi}{2z}} H_{n+\frac{1}{2}}^{(1)}(z)\,, \qquad h_n^{(2)}(z) \equiv \sqrt{\frac{\pi}{2z}} H_{n+\frac{1}{2}}^{(2)}(z)\,,$$

By using the following integral representation of the Hankel function $H^{(1)}(z)$

$$H_\nu^{(1)}(z) = \frac{1}{\pi i}\int_0^{\infty e^{i\pi}} \frac{1}{t^{\nu+1}} e^{\frac{z}{2}(t-\frac{1}{t})} dt\,,$$

obtain its leading order asymptotic behavior. [**Hint**: We can deform the above contour into the steepest one given by the contour $\mathcal{C}_+$ in Prob. 1 (c) (See Fig. 2)]

Problem **3**. (30pt) The Gamma-function is defined by

$$\Gamma(z) \equiv \int_0^\infty dt\ t^{z-1} e^{-t} ,$$

(a) Derive that the following identity (**reflection formula**) (15pt)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} .$$

Hint: By using the change of variables $(x, y) \to (u = x/y, y)$ for

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty \int_0^\infty dxdy\ e^{-x} x^{z-1} e^{-y} y^{-z} .$$

show that

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty du \frac{u^{z-1}}{u+1} ,$$

and, then, use an appropriate contour integration on the complex u-plane to show that

$$\int_0^\infty du \frac{u^{z-1}}{u+1} = \frac{1}{z} - \sum_{k=1}^{\infty} (-1)^k \frac{2z}{k^2 - z^2} = \frac{\pi}{\sin(\pi z)} .$$

(b) Show that the integral representation of the reciprocal of the Gamma function is given by

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\mathcal{C}} dt\ t^{-z} e^t ,$$

where $\mathcal{C}$ denotes the same contour given in Fig.1 in Prob. 1. (15pt)
[**Hint**: Note that the contour $\mathcal{C}$ in Fig. 1 can be decomposed into $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$, where $\mathcal{C}_1$ runs from $-\infty, \arg(t) = -\pi$, $\mathcal{C}_2$ encircles the origin in the counter clockwise, and $\mathcal{C}_3$ terminates at $-\infty, \arg(t) = \pi$. On $\mathcal{C}_1$, set $t = xe^{-i\pi}$, on $\mathcal{C}_2$, set $t = xe^{i\theta}$ and on $\mathcal{C}_3$, set $t = xe^{i\pi}$. Evaluate the integral by this decomposition and then use the identity given in part (a).]

Problem **4**. (30pt) There are various ways to obtain Green's function for the Helmholtz equation, which is defined by

$$(\nabla^2 + k^2) G(\mathbf{x}, \mathbf{x}') = \delta^3(\mathbf{x} - \mathbf{x}') .$$

In the class, we have obtained this Green's function in the closed form by using the following representation

$$G(\mathbf{x}, \mathbf{x}') \equiv \frac{\hbar^2}{2m} \left\langle \mathbf{x} \left| \frac{1}{E - \hat{H}_0 + i\epsilon} \right| \mathbf{x}' \right\rangle , \qquad E \equiv \frac{\hbar^2 k^2}{2m} , \qquad \hat{H}_0 = \frac{\hat{\mathbf{p}}^2}{2m} ,$$

and an appropriate contour integral. (See also Sakurai's book Chap. 6.2 or Arfken's book Chap. 10) We would like to rederive the same result by using the free particle propagator $K(\mathbf{x},t \mid \mathbf{x}',0) \equiv \langle \mathbf{x}| e^{-\frac{i}{\hbar}t\hat{H}_0}|\mathbf{x}'\rangle$, which was obtained in the class of path integrals.

(a) From the one-dimensional result for the propagator, which was obtained in the class (see Sakurai's book (6.16))

$$K(x,t \mid x',0) = \sqrt{\frac{m}{2\pi i\hbar t}}e^{i\frac{m(x-x')^2}{2\hbar t}},$$

derive the three-dimensional result for the propagator $K(\mathbf{x},t \mid \mathbf{x}',0)$. (10pt)
[**Hint**: Using the definition of the propagator, it would be straightforward to obtain the result.]

(b) Derive the following result for $Im[A] > 0$ and $Im[B] > 0$ (10pt)

$$\int_0^\infty ds \frac{1}{(is)^{3/2}}e^{i(As+\frac{B}{s})} = -ie^{2i\sqrt{AB}}\sqrt{\frac{\pi}{B}},$$

[**Hint**: Recall the integral representation of Bessel functions in Prob. **1**]

(c) Notice that the following integral expression [Here, the condition $\epsilon > 0$ is also important]

$$\frac{\hbar^2}{2m}\left\langle \mathbf{x}\left|\frac{1}{E-\hat{H}_0+i\epsilon}\right|\mathbf{x}'\right\rangle = \frac{-i\hbar}{2m}\int_0^\infty dt\, \langle \mathbf{x}|e^{\frac{i}{\hbar}t(E-\hat{H}_0+i\epsilon)}|\mathbf{x}'\rangle = \frac{-i\hbar}{2m}\int_0^\infty dt\, e^{\frac{i}{\hbar}t(E+i\epsilon)}\langle \mathbf{x}|e^{-\frac{i}{\hbar}tH_0}|\mathbf{x}'\rangle\,.$$

Using this observation and the results in part (a) and (b), rederive the closed expression of the Green's function $G(\mathbf{x},\mathbf{x}')$ (10pt).

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1. (a)

$$e^{\frac{z}{2}(t-1/t)} = e^{\frac{zt}{2}} \left(= \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n t^n \frac{1}{n!} \right) \otimes e^{-\frac{z}{2t}} \left(= \sum_{m=0}^{\infty} \left(\frac{z}{2}\right)^m (-1)^m t^{-m} \frac{1}{m!} \right)$$

① $n - m = \nu \geq 0$

$n = \nu + m$.

$$\otimes : \sum_{m=0}^{\infty} \left(\frac{z}{2}\right)^{2m+\nu} (-1)^m t^{\nu} \frac{1}{m!} \frac{1}{(m+\nu)!}$$

② $n - m = \nu < 0$.

$$\otimes : \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^{2n-\nu} (-1)^{n-\nu} \frac{1}{m!} \frac{1}{(n-\nu)!} t^{\nu}$$

Set $\alpha = |\nu|$,

$$: \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^{2n+\alpha} t^{\alpha} (-1)^{n+\alpha} \frac{1}{n!} \frac{1}{(n+\alpha)!} = (-1)^{\alpha} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^{2n+\alpha} (-1)^n \frac{1}{n!} \frac{1}{(n+\alpha)!} t^{\alpha}.$$

for integer. $\alpha = |n-m|$, $\nu = n-m$

$$e^{\frac{z}{2}(t-1/t)} = \sum_{\nu=-\infty}^{\infty} \underbrace{(-1)^{\frac{\alpha+\nu}{2}} \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+\alpha)!} \left(\frac{z}{2}\right)^{2m+\alpha} \right)}_{J_\nu(z)} t^{\nu}$$

for $J_\nu(z)$

$$z^2 f''(z) + z f'(z) + (z^2 - \nu^2) \underbrace{f(z)}_{J_\nu(z)} = 0.$$

$$\simeq ((2m+\alpha)(2m+\alpha-1) + (2m+\alpha) - \alpha^2 + z^2) f(z)$$

$$= (4m^2 + 4m\alpha + z^2) f(z)$$

m-order

$: O(z^{\alpha}) \rightarrow \ : \ 0.$

from z^2 term.

$m: k(z^{2k+\alpha}) : \ 4k^2 + 4k\alpha + (-k(k+\alpha) \cdot 4) = 0.$

$\downarrow$

$\therefore J_\nu(z)$ is solution of given eqn.

정정훈

(b) $\rightarrow + \circlearrowleft + \leftarrow \ : C$

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i}\int_C \frac{1}{t^{\nu+1}} e^{t - z^2/4t}\,dt$$

$$\left(\oint_c z^k\,dz = \int_0^{2\pi} e^{i(k+1)\theta}\,d\theta = \begin{cases}2\pi i & \text{if } k=-1\\ 0 & \text{o.w.}\end{cases}\right)$$

음의 실수축을 따라하는 적분은 중간에 특이점이 없고 방향이 반대이므로 상쇄된다.

$$\int_C \frac{1}{t^{\nu+1}} \underline{e^{t - z^2/4t}}\,dt$$

위식에 따라 t^ν 항을 제외한 나머지 항들의 컨투어적분은 0이다.

$$e^{t-z^2/4t} = \left(\sum_n^\infty \frac{1}{n!}t^n\right)\left(\sum_n^\infty \left(\frac{z}{2}\right)^{2n}\frac{(-1)^n}{n!}t^{-n}\right)$$

(a)와 같이 t 항의 차수가 ν인 항은

$$\sum_{n=0}^\infty \left(\frac{z}{2}\right)^{2n} \frac{(-1)^n}{n!\,(n+\nu)!}\,t^\nu$$ 다음과 같으므로.

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{\cancel{2\pi i}}{\cancel{2\pi i}} \sum_{n=0}^\infty \left(\frac{z}{2}\right)^{2n}\frac{(-1)^n}{n!\,(n+\nu)!} = \sum_{n=0}^\infty \left(\frac{z}{2}\right)^{2n+\nu}\frac{(-1)^n}{n!\,(n+\nu)!}$$

(c)

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i}\int_C \frac{1}{t^{\nu+1}} e^{t - z^2/4t}\,dt.$$

Steepest Descent Method

$\underline{\int e^{u+iv}}$ 의 saddle point 근처로 근사적 path of Steepest Descent를 따라서 적분.

$t - z^2/4t$, let $z = |z|e^{i\theta}$

$' : 1 + z^2/4t^2 \Rightarrow t_\pm = \pm\frac{iz}{2} = \frac{|z|}{2}e^{i(\theta \pm \frac{\pi}{2})}$

$'' : -\frac{z^2}{2t^3}$

exponential 위의 함수 $f(t)$의 ~~테일러전개~~ 근사. (at saddle point)

$$\therefore \; f(t_\pm + \delta t) \simeq \pm iz + \frac{2}{|z|} e^{-i(\theta \pm \frac{\pi}{2})} \delta t^2 + \cdots O(\delta t^3)$$

Steepest Descent path.

z를 편의상 양의 실수로 두자. 이를 x로 쓰면.

$$f(t_\pm + \delta t) \simeq \pm ix + \frac{2}{|x|} e^{-i(\theta \pm \frac{\pi}{2})} \delta t^2 + O(\delta t^3)$$

f는 지수위의 함수이므로 Steepest Descent 의 방향은 δt^2 텀이 음의 실수 값을 갖는 방향이다.

$$\mathrm{Im}\left[e^{-i(\theta \pm \frac{\pi}{2})} \delta t^2\right] = 0, \quad \frac{2}{x} e^{-i(\theta \pm \frac{\pi}{2})} \delta t^2 < 0.$$

$\delta t = |\delta t| e^{i\phi}$ 라하면,

$$e^{-i\underbrace{(\theta \pm \frac{\pi}{2} - 2\phi)}_{=\pi}}$$

$\delta t =$

$$2\phi = \theta \pm \frac{\pi}{2} + \pi$$

$\underline{\phi_\pm = \frac{\theta}{2} + \frac{\pi}{2} \pm \frac{\pi}{4}}$ 를 만족하면 실수 δt 에대해

$$f(t_\pm + \delta t e^{i\phi_\pm}) \simeq \pm ix - \frac{2}{|x|} \delta t^2 + O(\delta t^3)$$

Steepest Descent Method는 Imaginary 값을 바꾸지 않는 경로이므로 적분상의 한점 t에 대하여

$$\mathrm{Im}[f(t)] = x = \mathrm{Im}[t] - \mathrm{Im}\left[\frac{x^2}{4t}\right]$$ 을 만족하고

$t_\pm = \pm \frac{ix}{2}$ 에서, $t_+ = \frac{ix}{2}$, $t_- = -\frac{ix}{2}$ 인 패스임은 알수있다.

c) Saddle point 근처에서 움직이는 경로가 $|z|$에만 의존하므로.

$$\delta t = \frac{|z|}{2} r e^{i\phi_+}$$

$$dt = \frac{|z|}{2} e^{i\phi_+} dr$$ 로쓰고

함수 $f(t)$가 지수위에 있으므로 $\frac{1}{t^{\nu+1}}$ 은 saddle point 에서의 값을 사용하면.

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i}\int_{C_+} \frac{1}{t^{\nu+1}} \exp\left(t - \frac{z^2}{4t}\right) dt \approx \frac{1}{2\pi} e^{i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})} \int_{-\infty}^{\infty} e^{-|z|r^2/2} dr$$

$$+ \quad '' \quad \int_{C_-} \quad ''$$

$$t_+^{-\nu-1} = \left(\frac{|z|}{2} e^{i\frac{\pi}{2}}\right)^{-\nu-1} = \left(\frac{2}{z}\right)^{\nu+1} e^{-i(\nu+1)\frac{\pi}{2}}$$

$$e^{f(t)} = e^{iz - \frac{|z|}{2} r^2} \qquad dt = \frac{z}{2} e^{i\phi_+} dr.$$

$\left(\frac{z}{2}\right) \to$ cancel.

$$e^{i(z - \frac{\nu\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{4})} = e^{i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})}$$

$$J_\nu(z) = \frac{1}{\sqrt{2\pi z}} e^{i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})} + (\, C_- \,)$$

$$\phi_- = \frac{\pi}{2} - \frac{\pi}{4}, \quad t_- \to \frac{|z|}{2} e^{-i\frac{\pi}{2}} \qquad iz \to -iz.$$

$$C_- = \frac{1}{\sqrt{2\pi z}} e^{-i(z - \frac{\nu\pi}{2} - \frac{\pi}{4})}$$

$$\therefore J_\nu(z) \cong \frac{1}{\sqrt{2\pi z}} \cdot 2\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)(1 + \cdots)$$

$$= \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).$$

(d)

$j_\ell(\rho)$ can be ~~written~~ below

$$j_\ell(\rho) = (-\rho)^\ell \left(\frac{1}{\rho}\frac{d}{d\rho}\right)^\ell \left(\frac{\sin\rho}{\rho}\right).$$

한편, $j_0(\rho)$는 다음과 같이 쓸수있다.

$$j_0(\rho) = \frac{\sin\rho}{\rho} = \frac{1}{2}\int_{-1}^{1} e^{i\rho w} dw.$$

$D \equiv \left(\frac{1}{\rho}\frac{d}{d\rho}\right)$ 에 대해서 적용해보면.

$$D\int_{-1}^{1} e^{i\rho w} dw = \frac{1}{\rho}\frac{d}{d\rho}\int_{-1}^{1} e^{i\rho w} dw = \frac{i}{\rho}\int_{-1}^{1} w e^{i\rho w} dw = \frac{i}{2\rho}\int_{-1}^{1} 2w e^{i\rho w} dw$$

Let C as -1

$$= \frac{i}{2f}\int_{-1}^{1} e^{ifw}\, d(w^2+c) = \frac{i}{2f}\int_{-1}^{1} e^{ifw} d(w^2-1) = \frac{i}{2f}\left(\cancel{(w^2-1)e^{ifw}\Big|_{-1}^{1}}^{\,0} - \int_{-1}^{1} w^2-1\, de^{ifw}\right)$$

$$= -\frac{i}{2f}\int_{-1}^{1} if(w^2-1)e^{ifw}dw = \frac{1}{2}\int_{-1}^{1}(w^2-1)e^{ifw}dw.$$

임의의 차수 l에 대하여, 같은 formalism으로

$$\frac{1}{f}\frac{1}{df}\int_{-1}^{1}(w^2-1)^l e^{ifw}dw = \frac{i}{f}\int_{-1}^{1} w(w^2-1)^l e^{ifw}dw = \frac{i}{2f}\int_{-1}^{1}(w^2-1)^l e^{ifw}d(w^2-1)$$

$$= \frac{-i}{2(l+1)f}\int_{-1}^{1}(w^2-1)^{l+1}d(e^{ifw}) = \frac{1}{2(l+1)}\int_{-1}^{1}(w^2-1)^{l+1}e^{ifw}dw$$ 이므로.

$D\left(=\frac{1}{f}\frac{d}{df}\right)$에 대해 l번 반복하면

$$D^l\int_{-1}^{1} e^{ifw}dw = \frac{1}{2^l l!}\int_{-1}^{1}(w^2-1)^l e^{ifw}dw$$ 임을 알 수 있다.

따라서.

$$j_l(f) = (-f)^l D^l \frac{1}{2}\int_{-1}^{1} e^{ifw}dw = \frac{(-f)^l}{2^{l+1}l!}\int_{-1}^{1}(w^2-1)^l e^{ifw}dw.$$

한편, $-f\int_{-1}^{1} e^{ifw}dw = i\int_{-1}\frac{d}{dw}(e^{ifw})\,dw$ 로 쓸 수 있고,

$$j_l(f) = \frac{i^l}{2^{l+1}l!}\int_{-1}^{1}(w^2-1)^l\frac{d^l}{dw^l}(e^{ifw})dw = \frac{(-i)^l}{2^{l+1}l!}\int_{-1}^{1}\frac{d^l}{dw^l}(w^2-1)e^{ifw}dw$$

$\rightarrow$ integration by part

$$P_l(w) = \frac{1}{2^l l!}\frac{d^l}{dw^l}(w^2-1)^l \quad (\text{Rodrigues formula})$$

Therefore, $j_l(f) = \frac{(-i)^l}{2}\int_{-1}^{1}P_l(w)e^{ifw}dw.$

$$\int_{-1}^{1}P_l(w)e^{ifw}dw = 2i^l j_l(f)$$ 임을 알 수 있다.

$f = kr$, $w = \cos\theta$ 를 대입해보면.

$$\int_{-1}^{1} P_\ell(\cos\theta)\, e^{ikr\cos\theta}\, d(\cos\theta) = 2 i^\ell j_\ell(kr)$$

한편 $P_\ell(x)$는 다음과 같은 orthogonality condition 을 만족하므로.

$$\int_{-1}^{1} P_{\ell'}(\cos\theta)\, P_\ell(\cos\theta)\, d(\cos\theta) = \frac{2}{2\ell+1}\delta_{\ell,\ell'}$$

$e^{ikr\cos\theta}$ 를 P_ℓ을 기저로 다시 표현하면,

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\, j_\ell(kr)\, P_\ell(\cos\theta)$$ 임을 알 수 있다.

(e)

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}}\, J_{\ell+\frac{1}{2}}(x)$$

$$J_{\ell+\frac{1}{2}}(x) \sim \sqrt{\frac{2}{\pi x}}\, \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$ 이므로

$$j_\ell(x) \sim \frac{1}{x}\cos\left(x - \frac{\ell\pi}{2} - \frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{1}{x}\sin\left(x - \frac{\ell\pi}{2}\right).$$

(f) 함수 $f(z) = u(x,y) + iv(x,y)$ 로 놓아보자. ($z = x+iy$)

$$\frac{df(z)}{dz} = \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$$ 이다. (해석함수 편미분의 정의).

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \Rightarrow \begin{array}{l} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{array}$$ (코시-리만 방정식).

한편 함수가 해석적인 영역에서의 폐곡선 위의 적분을 생각하자.

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+idy) = \oint_C (u\,dx - v\,dy) + i\oint_C (u\,dy + v\,dx)$$

그린정리에 의하여.

$$= \iint_A \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_A \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy = 0.$$ (코시-리만방정식)

∴ 해석함수는 폐곡선 적분이 0 이다.

한편, $J_n(z) = \frac{1}{2\pi i}\int_c \frac{1}{t^{n+1}} e^{\frac{z}{2}(t-\frac{1}{t})} dt$ 는 $t \neq 0$ 에서 해석적이므로.

C를 바꿀수있다. $C \equiv \int_0^{2\pi} e^{i\theta}$ 로 정의하면. $t = e^{i\theta}$ 가 되고, θ를 $0 \to 2\pi$

$i e^{i\theta} d\theta = dt$ 이므로.

$$J_n(z) = \frac{1}{2\pi}\int e^{-i(n+1)\theta} e^{\frac{z}{2}(e^{i\theta} - e^{-i\theta})} e^{i\theta} d\theta.$$

$$= \frac{1}{2\pi}\int e^{-in\theta} e^{iz\sin\theta} d\theta$$ 임을 알수있다.

2.

1. (c) 에서. $J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i}\int_c \frac{1}{t^{\nu+1}} e^{\frac{z}{2}(t-\frac{1}{t})} dt.$

이를 steepest Descent path로 바꾸면

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i}\left(\int_{C_+} \frac{1}{t^{\nu+1}} e^{\frac{z}{2}(t-\frac{1}{t})} + \int_{C_-} \frac{1}{t^{\nu+1}} e^{\frac{z}{2}(t-\frac{1}{t})}\right)$$

로 나눈고, 이길은 saddle point 에 따라 나누어지는 것은 알수있다.

$f(t)$를 다시 써보면,

$$f(t_\pm + \delta t) = \pm iz + \frac{2}{|z|} e^{-i(\theta \pm \frac{z}{2})} \delta t^2 + O(\delta t^3)$$

z를 양의 실수로 ~~정의~~ ~~하고~~ 놓고, 계산하면, (imaginary 가 고정된 채로는 따라)

$t = t_x + i t_y$

$$f(t_+) = ix \quad \mathrm{Im}[f(t)] = t_y + \frac{x^2}{4}\frac{t_y}{t_x^2 + t_y^2} = x.$$

$t \to \infty$ 극한에서

$t_y = x.$

$|t| \to 0$ 인극한에서

$$\cancel{|t|^2 t_y} + \frac{x^2}{4} t_y = x|t|^2 \Rightarrow t_y = \frac{4}{x}(t_x^2 + t_y^2)$$

t_y^3

$t_y - \underbrace{\frac{4}{x} t_y^2}_{\text{기여가 매우 낮다고 하면}} = \frac{4}{x} t_x^2.$ $t_y = \frac{4}{x} t_x^2$ 을 따라 움직이는데.

~~해~~

z가 0부터 ∞ 까지 ~~적~~적분을 코시적분정리를 이용하면 C_+ 패스로 바꿀수 있다.

$J_\nu(z)$와 $H_\nu^{(1)}(z)$ 는 적분구간과 ~~f~~ factor 2 차이만 나므로.

C_+ 패스경로 적분값의 2배가 $H_\nu^{(1)}(z)$ 임을 알수있다.

따라서

$$H_\nu^{(1)}(z) \sim \frac{1}{\pi} e^{i(z-\frac{\nu}{2}\pi-\frac{\pi}{4})} \int_{-\infty}^{\infty} e^{-|z| r^2/2} dr$$

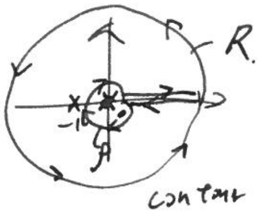
$$= \underline{\sqrt{\frac{2}{\pi z}}\, e^{i(z-\frac{\nu}{2}\pi-\frac{\pi}{4})}}.$$

3. (a) $\Gamma(z)\Gamma(1-z) = \int_0^\infty \int_0^\infty dx\,dy\, e^{-x} x^{z-1} e^{-y} y^{-z}$

change of variable $x \to u = x/y$. $du = \frac{1}{y} dx$. , $x = uy$

$$\Rightarrow \int_0^\infty \int_0^\infty y\,du\,dy\, e^{-uy} (uy)^{z-1} e^{-y} y^{-z} = \int_0^\infty \int_0^\infty dy\,du\, e^{-y(u+1)} u^{z-1}$$

$$= \int_0^\infty du \left(-\frac{u^{z-1}}{u+1} e^{-y(u+1)} \right)\Big|_0^{\infty} = \int_0^\infty du\, \frac{u^{z-1}}{u+1}$$



$$: \int_\rho^R \frac{x^{z-1}}{1+x} dx + \int_R \frac{u^{z-1}}{1+u} du + \int_R^\rho \frac{(xe^{i2\pi})^{z-1}}{1+x} dx + \int_\rho \frac{u^{z-1}}{1+u} du$$

residue theorem

$$\overset{\downarrow}{=} 2\pi i \underbrace{R(-1)}_{\text{residue}} = 2\pi i e^{(z-1)i\pi}.$$

$u' = u-1 \Rightarrow \int \frac{(u'-1)^{z-1}}{u'} du$

$u'^{-1} : (-1)^{z-1} = e^{i\pi(z-1)}$

$\rho \to R$ + $R \to \rho$

$$: \int_\rho^R \frac{x^{z-1}}{1+x} dx - e^{i2\pi(z-1)} \int_\rho^R \frac{x^{z-1}}{1+x} dx$$

$$= \left(1 - e^{2i\pi(z-1)}\right) \int_\rho^R \frac{x^{z-1}}{1+x} dx.$$

$$\int_R \frac{x^{z-1}}{1+x}\,dx \to 0.$$

$R \to \infty$, $\boxed{z<1}$

$$\int_\rho \frac{x^{z-1}}{1+x}\,dx = \frac{x^{z-1}}{1} \to 0.$$

$\rho \to 0$, $\boxed{z>0}$.

$$\therefore\ (1-e^{2i\pi(z-1)})\int_\rho^R \frac{x^{z-1}}{1+x}\,dx = -2ie^{iz\pi}\sin z\pi \int_0^\infty \frac{x^{z-1}}{1+x}\,dx = 2\pi i e^{(z-1)i\pi}$$

$$\int_0^\infty \frac{x^{z-1}}{1+x}\,dx = \frac{\pi}{\sin \pi z}.$$

(b)