Expected Error of Rounded HRF Estimation

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Case I

Assume 1) there is a single stimuli, and 2) two basis functions are orthogonal

$$
\begin{gathered}
1)\ n(t) = I(t_0) \\
2)\ h(t|\beta) = \phi_1(t)\beta_1 + \phi_2(t)\beta_2 \\
\text{where } \int \phi_1'(t)\phi_2(t)dt = 0
\end{gathered}
$$

Let us define $f(t|\beta)$ to be the convolution between the hemodynamic response, denoted by $h(t|\beta)$, and a known stimulus function, $n(t)$.

$$
\begin{aligned}
f(t|\beta) &= (n \cdot h)(t) \\
&= \int n(u)h(t-u)du \\
&= h(t-t_0) \\
&= \phi_1(t-t_0)\beta_1 + \phi_2(t-t_0)\beta_2
\end{aligned}
$$

Our linear regression model for the fMRI reponse at time t_i can be written as:

$$y_i = f(t_i|\beta) + \epsilon_i$$

where $\epsilon_i \sim N(0, V\sigma^2)$. In matrix format,

$$Y = F(X|\beta) + E$$

where $Y = (y_1, y_2, \cdots, y_N)'$, $E = (\epsilon_1, \epsilon_2, \cdots, \epsilon_N)'$, and

$$
\begin{aligned}
F(X|\beta) &= (f(t_1|\beta), \cdots f(t_N|\beta))' \\
&= \begin{pmatrix} \phi_1(t_1-t_0) & \phi_2(t_1-t_0) \\ \vdots & \vdots \\ \phi_1(t_N-t_0) & \phi_2(t_N-t_0) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \\
&:= X\beta
\end{aligned}
$$

So here, $Y = X\beta + E$ and $\hat{\beta} = (X'X)^{-1}X'Y$ where $\hat{\beta}$ is fitted by linear regression model.

Let $R(t_0)$ is rounding integer of t_0. Thus, for given t_0, $R(t_0)$ is also determined (deterministic by t_0).

It's well known that $\hat{\beta}$ is unbiased estimator of β.

$$\mathbb{E}(\hat{\beta}|t_0, R(t_0)) = \mathbb{E}(\hat{\beta}|t_0) = \mathbb{E}((X'X)^{-1}X'Y|t_0) = (X'X)^{-1}X'X\beta = \beta$$

Now, assume that we only have incomplete information: we only know $R(t_0)$ and don't know exact stimuli time point t_0.

Then, we use a different design matrix convoluted with rounded version of stimuli function $n(t) = I(R(t_0))$.

$$\begin{aligned} X^* &= \begin{pmatrix} \phi_1(t_1 - R(t_0)) & \phi_2(t_1 - R(t_0)) \\ \vdots & \vdots \\ \phi_1(t_N - R(t_0)) & \phi_2(t_N - R(t_0)) \end{pmatrix} \\ \hat{\beta^*} &= (X^{*'}X^*)^{-1}X^*Y \end{aligned}$$

Next, the expectation of $\hat{\beta^*}$ is derived conditional on $R(t_0)$ and t_0 as shown below.

Here, let's define $T := X^{*'}(X - X^*)$.

$$\begin{aligned} \mathbb{E}(\hat{\beta^*}|t_0, R(t_0)) &= \mathbb{E}((X^{*'}X^*)^{-1}X^*Y|t_0, R(t_0)) \\ &= (X^{*'}X^*)^{-1}X^*\mathbb{E}(Y|t_0, R(t_0)) \\ &= (X^{*'}X^*)^{-1}X^{*'}X\beta \\ &=: (X^{*'}X^*)^{-1}(X^{*'}X^* + T)\beta \\ &= \beta + (X^{*'}X^*)^{-1}T\beta \end{aligned}$$

Suppose we only know $R(t_0)$, and we don't know the exact value of t_0. Additionally, we assume:

$$t_0|R(t_0) \sim U(R(t_0) - 1/2, R(t_0) + 1/2)$$

$$\begin{aligned} \mathbb{E}(\hat{\beta^*}|R(t_0)) &= \mathbb{E}_{t_0|R(t_0)}(\hat{\beta^*}|t_0, R(t_0)) \\ &= \mathbb{E}_{t_0|R(t_0)}[\beta + (X^{*'}X^*)^{-1}T\beta] \\ &= \beta + (X^{*'}X^*)^{-1}\mathbb{E}_{t_0|R(t_0)}[T]\beta \end{aligned}$$

Here,

We can apply Taylor expansion to express $\phi_k(t - t_0)$ as an infinite sum of $\phi_k^{(n)}(t - R(t_0))$ for $k = 1, 2$, allowing us to calculate the expectation with respect to t_0 conditional on $R(t_0)$.

$$\begin{aligned} \phi_k(t - t_0) &= \phi_k(t - R(t_0)) + (R(t_0) - t_0)\phi_k^{'}(t - R(t_0)) + \frac{1}{2}(R(t_0) - t_0)^2\phi_k^{''}(t - R(t_0)) + \cdots \\ &= \phi_k(t - R(t_0)) + \sum_{n=1}^{\infty}\frac{1}{n!}(R(t_0) - t_0)^n\phi_k^{(n)}(t - R(t_0)) \end{aligned}$$

$$
\begin{aligned}
\mathbb{E}_{t_0|R(t_0)}[\phi_k(t-t_0)] &= \phi_k(t-R(t_0)) + \sum_{n=1}^{\infty}\frac{1}{n!}\mathbb{E}_{t_0|R(t_0)}[(R(t_0)-t_0)^n]\phi_k^{(n)}(t-R(t_0)) \\
&= \phi_k(t-R(t_0)) + \sum_{n=0}^{\infty}\frac{1}{2^{2n+1}(2n+2)(2n+1)!}\phi_k^{(2n+1)}(t-R(t_0)) \\
&= \phi_k(t-R(t_0)) + \sum_{n=0}^{\infty}\frac{1}{2^{2n+1}(2n+2)!}\phi_k^{(2n+1)}(t-R(t_0))
\end{aligned}
$$

Note:

$$
\begin{aligned}
\mathbb{E}_{t_0|R(t_0)}[(R(t_0)-t_0)^n] &= \int_{R_0-1/2}^{R_0+1/2}(R(t_0)-x)^n dx \\
&= \int_{-1/2}^{1/2} s^n ds = \frac{(1/2)^{n+1}-(-1/2)^{n+1}}{n+1}
\end{aligned}
$$

Therefore, expectation of T can be expressed as below:

$$
\begin{aligned}
\mathbb{E}_{t_0|R(t_0)}[T] &= \mathbb{E}_{t_0|R(t_0)}[X^{*'}(X-X^*)] \\
&= X^{*'}\ \mathbb{E}_{t_0|R(t_0)}[(X-X^*)] \\
&= X^{*'}\ \mathbb{E}_{t_0|R(t_0)}\begin{pmatrix} \phi_1(t_1-t_0)-\phi_1(t_1-R(t_0)) & \phi_2(t_1-t_0)-\phi_2(t_1-R(t_0)) \\ \vdots & \vdots \\ \phi_1(t_N-t_0)-\phi_1(t_N-R(t_0)) & \phi_2(t_N-t_0)-\phi_2(t_N-R(t_0)) \end{pmatrix} \\
&= X^{*'}\mathbb{E}_{t_0|R(t_0)}\begin{pmatrix} \sum_{n=1}^{\infty}\frac{1}{n!}(R(t_0)-t_0)^n\phi_1^{(n)}(t_1-R(t_0)) & \sum_{n=1}^{\infty}\frac{1}{n!}(R(t_0)-t_0)^n\phi_2^{(n)}(t_1-R(t_0)) \\ \vdots & \vdots \\ \sum_{n=1}^{\infty}\frac{1}{n!}(R(t_0)-t_0)^n\phi_1^{(n)}(t_N-R(t_0)) & \sum_{n=1}^{\infty}\frac{1}{n!}(R(t_0)-t_0)^n\phi_2^{(n)}(t_N-R(t_0)) \end{pmatrix} \\
&= \begin{pmatrix} \phi_1(t_1-R(t_0)) & \cdots & \phi_1(t_N-R(t_0)) \\ \phi_2(t_1-R(t_0)) & \cdots & \phi_2(t_N-R(t_0)) \end{pmatrix} \\
&\times \begin{pmatrix} \sum_{n=0}^{\infty}\frac{1}{2^{2n+1}(2n+2)!}\phi_1^{2n+1}(t_1-R(t_0)) & \sum_{n=0}^{\infty}\frac{1}{2^{2n+1}(2n+2)!}\phi_2^{2n+1}(t_1-R(t_0)) \\ \vdots & \vdots \\ \sum_{n=0}^{\infty}\frac{1}{2^{2n+1}(2n+2)!}\phi_1^{2n+1}(t_N-R(t_0)) & \sum_{n=0}^{\infty}\frac{1}{2^{2n+1}(2n+2)!}\phi_2^{2n+1}(t_N-R(t_0)) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{i=1}^{N}(\phi_1(t_i-R(t_0))\sum_{n=0}^{\infty}\frac{\phi_1^{(2n+1)}(t_i-R(t_0))}{2^{(2n+1)}(2n+2)!}) & \sum_{i=1}^{N}(\phi_1(t_i-R(t_0))\sum_{n=0}^{\infty}\frac{\phi_2^{(2n+1)}(t_i-R(t_0))}{2^{(2n+1)}(2n+2)!}) \\ \sum_{i=1}^{N}(\phi_2(t_i-R(t_0))\sum_{n=0}^{\infty}\frac{\phi_1^{(2n+1)}(t_i-R(t_0))}{2^{(2n+1)}(2n+2)!}) & \sum_{i=1}^{N}(\phi_2(t_i-R(t_0))\sum_{n=0}^{\infty}\frac{\phi_2^{(2n+1)}(t_i-R(t_0))}{2^{(2n+1)}(2n+2)!}) \end{pmatrix} \\
&=: \begin{pmatrix} R_{\phi_1\phi_1}(\tilde{t},R(t_0)) & R_{\phi_1\phi_2}(\tilde{t},R(t_0)) \\ R_{\phi_2\phi_1}(\tilde{t},R(t_0)) & R_{\phi_2\phi_2}(\tilde{t},R(t_0)) \end{pmatrix}
\end{aligned}
$$

Therefore, the expectation of the fitted $\hat{\beta}^*$ conditional on $R(t_0)$ is expressed as follows:

$$
\begin{aligned}
\mathbb{E}(\hat{\beta}^*|R(t_0)) &= \beta + (X^{*'}X^*)^{-1}\mathbb{E}_{t_0|R(t_0)}(T)\beta \\
&= \beta + (X^{*'}X^*)^{-1}\begin{pmatrix} R_{\phi_1\phi_1}(\tilde{t},R(t_0)) & R_{\phi_1\phi_2}(\tilde{t},R(t_0)) \\ R_{\phi_2\phi_1}(\tilde{t},R(t_0)) & R_{\phi_2\phi_2}(\tilde{t},R(t_0)) \end{pmatrix}\beta
\end{aligned}
$$

Then, the bias from the true β is:

$$
\begin{aligned}
&\text{Bias}(\hat{\beta}^*|R(t_0)) \\
&\quad = (X^{*'}X^*)^{-1} \begin{pmatrix} R_{\phi_1\phi_1}(\tilde{t}, R(t_0)) & R_{\phi_1\phi_2}(\tilde{t}, R(t_0)) \\ R_{\phi_2\phi_1}(\tilde{t}, R(t_0)) & R_{\phi_2\phi_2}(\tilde{t}, R(t_0)) \end{pmatrix} \beta \\
&\quad = \begin{pmatrix} \sum_{i=1}^N \phi_1^2(t_i - R(t_0)) & \sum_{i=1}^N \phi_1(t_i - R(t_0))\phi_2(t_i - R(t_0)) \\ \sum_{i=1}^N \phi_2(t_i - R(t_0))\phi_1(t_i - R(t_0)) & \sum_{i=1}^N \phi_2^2(t_i - R(t_0)) \end{pmatrix}^{-1} \\
&\quad \begin{pmatrix} R_{\phi_1\phi_1}(\tilde{t}, R(t_0)) & R_{\phi_1\phi_2}(\tilde{t}, R(t_0)) \\ R_{\phi_2\phi_1}(\tilde{t}, R(t_0)) & R_{\phi_2\phi_2}(\tilde{t}, R(t_0)) \end{pmatrix} \beta
\end{aligned}
$$

This seems reasonable because the bias is zero when $R(t_0) = t_0$.

Case II

Assume 1) there is a single stimuli, and 2) thre are k basis functions

$$
\begin{aligned}
&1)\ n(t) = I(t_0) \\
&2)\ h(t|\beta) = \phi_1(t)\beta_1 + \cdots + \phi_k(t)\beta_k
\end{aligned}
$$

Let us define $f(t|\beta)$ to be the convolution between the hemodynamic response, denoted by $h(t|\beta)$, and a known stimulus function, $n(t)$.

$$
\begin{aligned}
f(t|\beta) &= (n \cdot h)(t) \\
&= \int n(u)h(t-u)du \\
&= h(t-t_0) \\
&= \phi_1(t-t_0)\beta_1 + \cdots \phi_k(t-t_0)\beta_k
\end{aligned}
$$

Our linear regression model for the fMRI reponse at time t_i can be written as:

$$
y_i = f(t_i|\beta) + \epsilon_i
$$

where $\epsilon_i \sim N(0, V\sigma^2)$. In matrix format,

$$
Y = F(X|\beta) + E
$$

where $Y = (y_1, y_2, \cdots, y_N)'$, $E = (\epsilon_1, \epsilon_2, \cdots, \epsilon_N)'$, and

$$
\begin{aligned}
F(X|\beta) &= (f(t_1|\beta), \cdots f(t_N|\beta))' \\
&= \begin{pmatrix} \phi_1(t_1-t_0) & \cdots & \phi_k(t_1-t_0) \\ \vdots & \cdots & \vdots \\ \phi_1(t_N-t_0) & \cdots & \phi_k(t_N-t_0) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} \\
&:= X\beta
\end{aligned}
$$

So here, $Y = X\beta + E$ and $\hat{\beta} = (X'X)^{-1}X'Y$ where $\hat{\beta}$ is fitted by linear regression model.

Let $R(t_0)$ is rounding integer of t_0. Thus, for given t_0, $R(t_0)$ is also determined (deterministic by t_0).

It's well known that $\hat{\beta}$ is unbiased estimator of β.

$$
\mathbb{E}(\hat{\beta}|t_0, R(t_0)) = \mathbb{E}(\hat{\beta}|t_0) = \mathbb{E}((X'X)^{-1}X'Y|t_0) = (X'X)^{-1}X'X\beta = \beta
$$

Now, assume that we only have incomplete information: we only know $R(t_0)$ and don't know exact stimuli time point t_0.

Then, we use a different design matrix convoluted with rounded version of stimuli function $n(t) = I(R(t_0))$.

$$
\begin{aligned}
X^* &= \begin{pmatrix} \phi_1(t_1 - R(t_0)) & \cdots & \phi_k(t_1 - R(t_0)) \\ \vdots & \dots & \vdots \\ \phi_1(t_N - R(t_0)) & \cdots & \phi_k(t_N - R(t_0)) \end{pmatrix} \\
\hat{\beta^*} &= (X^{*'} X^*)^{-1} X^* Y
\end{aligned}
$$

Next, the expectation of $\hat{\beta^*}$ is derived conditional on $R(t_0)$ and t_0 as shown below.

$$
\begin{aligned}
\mathbb{E}(\hat{\beta^*}|t_0, R(t_0)) &= \mathbb{E}((X^{*'} X^*)^{-1} X^* Y | t_0, R(t_0)) \\
&= (X^{*'} X^*)^{-1} X^* \mathbb{E}(Y|t_0, R(t_0)) \\
&= X^{*'} X \beta \\
&= \begin{pmatrix} \phi_1(t_1 - R(t_0)) & \cdots & \phi_1(t_N - R(t_0)) \\ \vdots & \dots & \vdots \\ \phi_k(t_1 - R(t_0)) & \cdots & \phi_k(t_N - R(t_0)) \end{pmatrix} \begin{pmatrix} \phi_1(t_1 - t_0) & \cdots & \phi_k(t_1 - t_0) \\ \vdots & \dots & \vdots \\ \phi_1(t_N - t_0) & \cdots & \phi_k(t_N - t_0) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} \\
&= \begin{pmatrix} \sum_{i=1}^N \phi_1(t_i - R(t_0))\phi_1(t_i - t_0) & \cdots & \sum_{i=1}^N \phi_1(t_i - R(t_0))\phi_k(t_i - t_0) \\ \vdots & \dots & \vdots \\ \sum_{i=1}^N \phi_k(t_i - R(t_0))\phi_1(t_i - t_0) & \cdots & \sum_{i=1}^N \phi_k(t_i - R(t_0))\phi_k(t_i - t_0) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} \\
&= \left(\sum_{i=1}^N \phi_i(t_i - R(t_0))\phi_j(t_i - t_0) \right)_{ij} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}
\end{aligned}
$$

Suppose we only know $R(t_0)$, and we don't know the exact value of t_0. Additionally, we assume:

$$
t_0 | R(t_0) \sim U(R(t_0) - 1/2, R(t_0) + 1/2)
$$

$$
\begin{aligned}
\mathbb{E}(\hat{\beta^*}|R(t_0)) &= \mathbb{E}_{t_0|R(t_0)}(\hat{\beta^*}|t_0, R(t_0)) \\
&= \mathbb{E}_{t_0|R(t_0)}[\left(\sum_{i=1}^N \phi_i(t_i - R(t_0))\phi_j(t_i - t_0) \right)_{ij} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix}]
\end{aligned}
$$

Now, we can streamline the expected bias calculation for $\hat{\beta^*}$ by employing the Taylor expansion of $\phi_j(t - t_0)$.

$$
\begin{aligned}
\mathbb{E}_{t_0|R(t_0)}(\phi_j(t - R(t_0))\phi_j(t - t_0)) &= \phi_j(t - R(t_0))\mathbb{E}_{t_0|R(t_0)}(\phi_j(t - t_0)) \\
&= \phi_j(t - R(t_0))\mathbb{E}_{t_0|R(t_0)}(\phi_j(t - R(t_0)) + \sum_{n=0}^{\infty} \frac{\phi_j^{(2n+1)}(t - R(t_0))}{2^{(2n+1)}(2n+2)!})
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}_{t_0|R(t_0)}(\sum_{i=1}^{N}\phi_j(t_i-R(t_0))\phi_j(t_i-t_0))\\
&\quad=\sum_{i=1}^{N}\phi_j^2(t_i-R(t_0))+\sum_{i=1}^{N}(\phi_j(t_i-R(t_0))\sum_{n=0}^{\infty}\frac{\phi_j^{(2n+1)}(t_i-R(t_0))}{2^{(2n+1)}(2n+2)!})\\
&\quad=1+\sum_{i=1}^{N}(\phi_j(t_i-R(t_0))\sum_{n=0}^{\infty}\frac{\phi_j^{(2n+1)}(t_i-R(t_0))}{2^{(2n+1)}(2n+2)!})\\
&\quad=:1+R_{\phi_j\phi_j}(\tilde{t},t_0)
\end{aligned}
$$

In the same way,

$$
\begin{aligned}
\mathbb{E}_{t_0|R(t_0)}(\phi_j(t-R(t_0))\phi_l(t-t_0))&=\phi_j(t-R(t_0))\mathbb{E}_{t_0|R(t_0)}(\phi_l(t-t_0))\\
&=\phi_k(t-R(t_0))\mathbb{E}_{t_0|R(t_0)}(\phi_l(t-R(t_0))+\sum_{n=0}^{\infty}\frac{\phi_j^{(2n+1)}(t-R(t_0))}{2^{(2n+1)}(2n+2)!})
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}_{t_0|R(t_0)}(\sum_{i=1}^{N}\phi_j(t_i-R(t_0))\phi_l(t_i-t_0))\\
&\quad=\sum_{i=1}^{N}\phi_j(t_i-R(t_0))\phi_l(t_i-R(t_0))+\sum_{i=1}^{N}(\phi_j(t_i-R(t_0))\sum_{n=0}^{\infty}\frac{\phi_l^{(2n+1)}(t_i-R(t_0))}{2^{(2n+1)}(2n+2)!})\\
&\quad=\sum_{i=1}^{N}(\phi_j(t_i-R(t_0))\sum_{n=0}^{\infty}\frac{\phi_l^{(2n+1)}(t_i-R(t_0))}{2^{(2n+1)}(2n+2)!})\\
&\quad=:R_{\phi_j\phi_l}(\tilde{t},t_0)
\end{aligned}
$$

Therefore, the expectation of the fitted $\hat{\beta}^*$ conditional on $R(t_0)$ is expressed as follows:

$$
\begin{aligned}
\mathbb{E}(\hat{\beta}^*|R(t_0))&=\mathbb{E}_{t_0|R(t_0)}(\begin{pmatrix}\sum_{i=1}^{N}\ \phi_1(t_i-R(t_0))\phi_1(t_i-t_0) & \cdots & \sum_{i=1}^{N}\ \phi_1(t_i-R(t_0))\phi_k(t_i-t_0)\\ \vdots & \cdots & \vdots\\ \sum_{i=1}^{N}\ \phi_k(t_i-R(t_0))\phi_1(t_i-t_0) & \cdots & \sum_{i=1}^{N}\ \phi_k(t_i-R(t_0))\phi_k(t_i-t_0)\end{pmatrix}\begin{pmatrix}\beta_1\\ \vdots\\ \beta_k\end{pmatrix})\\
&=\begin{pmatrix}\beta_1+R_{\phi_1\phi_1}(\tilde{t},t_0)\beta_1+R_{\phi_1\phi_2}(\tilde{t},t_0)\beta_2+\cdots+R_{\phi_1\phi_k}(\tilde{t},t_0)\beta_k\\ \vdots\\ \beta_k+R_{\phi_2\phi_1}(\tilde{t},t_0)\beta_1+R_{\phi_2\phi_2}(\tilde{t},t_0)\beta_2+\cdots+R_{\phi_2\phi_k}(\tilde{t},t_0)\beta_k\end{pmatrix}
\end{aligned}
$$

Then, the bias from the true β is:

$$
\mathrm{Bias}(\hat{\beta}^*|R(t_0))=\begin{pmatrix}\sum_{j=1}^{k}R_{\phi_1\phi_j}(\tilde{t},t_0)\beta_j\\ \vdots\\ \sum_{j=1}^{k}R_{\phi_k\phi_j}(\tilde{t},t_0)\beta_j\end{pmatrix}
$$

This seems reasonable because the bias is zero when $R(t_0)=t_0$.