Probablistic Robotics: Gaussian Filters (Chapter 3) Exercise Solutions

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Table 1: The covariance matrix and mean of the Kalman filter before measurement update.

t	μ	Σ
0	$\begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 10^{-4} & 0 \\ 0 & 10^{-4} & 0 \end{bmatrix}$
1	$\begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 0.2502 & 10^{-4} \\ 10^{-4} & 1.0001 \end{bmatrix}$
2	$\begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 1.5005 & 1.0002 \\ 1.0002 & 2.0001 \end{bmatrix}$
3	$\begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 5.751 & 3.0003 \\ 3.0003 & 3.0001 \end{bmatrix}$
4	$\begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 15.0017 & 6.0004 \\ 6.0004 & 4.0001 \end{bmatrix}$
5	$\begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}$	$\begin{bmatrix} 31.2526 & 10.0005 \\ 10.0005 & 5.0001 \end{bmatrix}$

Problem 1.a

The state vector of the Kalman filter should be $\mathbf{x}_t = [x_t, \dot{x}_t]$. The control should be a scalar $u = \ddot{x}_t$, which is the acceleration of the robot.

Problem 1.b

We can write the linear dynamical system of the robot as follows:

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}(u_t + \epsilon), \tag{1}$$

where $\mathbf{A} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix}$, $\Delta t = 1s$, $u_t = 0$, and $\epsilon \sim \mathcal{N}(0, 1)$. Since the system is a linear transformation of the Gaussian variable x_t, the next state distribution $p(x_{t+1}|x_t, u_t)$ is also Gaussian with mean $\mathrm{E}[\mathbf{x}_{t+1}] = \mathbf{A}\mathbf{x}_t$ and the covariance matrix for the motion is $\mathbf{R} = \mathbf{B}\mathbf{B}^T = \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 1 \end{bmatrix}$. [1]. $\mathbf{R}$ not a valid covariance matrix because it is not positive-definite, The reason for this is because that the position can be inferred from the velocity $x = \dot{x}t$. To overcome this issue, I made the covariance between the position and the velocity to be zero. Thus, $\mathbf{R} = \begin{bmatrix} 0.25 & 0.0 \\ 0.0 & 1 \end{bmatrix}$

[1] According to the linear transformation of Gaussian random variable

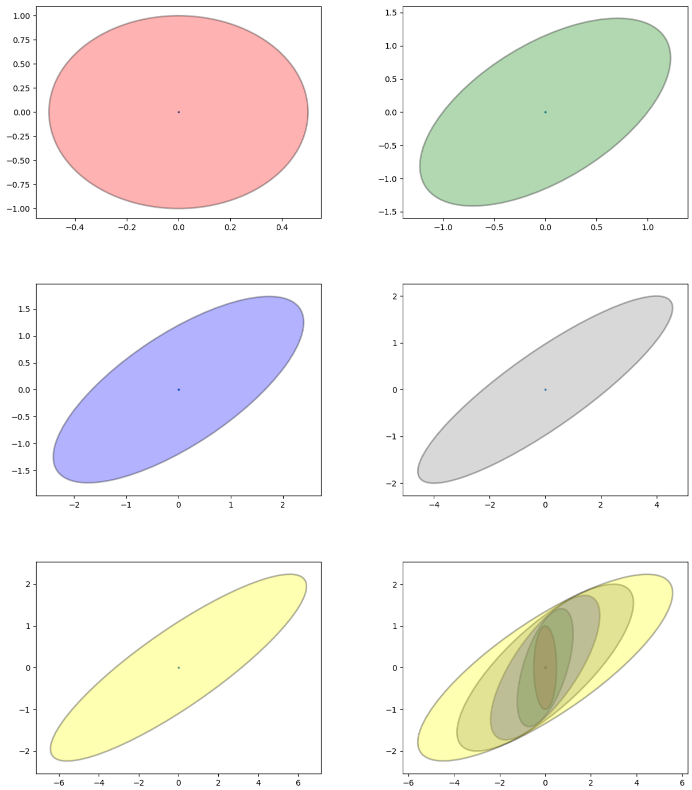


Figure 1: The uncertainty ellipses of the Kalman filter covariance matrices from $t = 1$ to $t = 5$. The horizontal and vertical axes represent the position and velocity, respectively. The last figure shows the covariance matrices at the five time points overlaid on each other.

Problem 1.c

The prediction step of the Kalman filter is given as follows,

$$\begin{aligned}\bar{\mu}_{t+1} &= \mathbf{A}\mu_t \\ \bar{\Sigma}_{t+1} &= \mathbf{A}\Sigma_t\mathbf{A}^T + \mathbf{R}.\end{aligned} \tag{2}$$

The reason that we have zero control input is because that the control of the robot is $0m/s^2$. However, note that the actual acceleration is corrupted by the noise. [2]

Sine the initial state is known, I initialize the covariance matrix with a values $1e^{-4}$ on the diagonal at $t = 0$. We can use this to compute the rest of

[2] I might be wrong about this. The textbook says that the acceleration is randomly set to a Gaussian noise. I'm not sure whether it means that the observed control is always 0, but has some noisy affect or we can observe the control. I'm using the first interpretation.

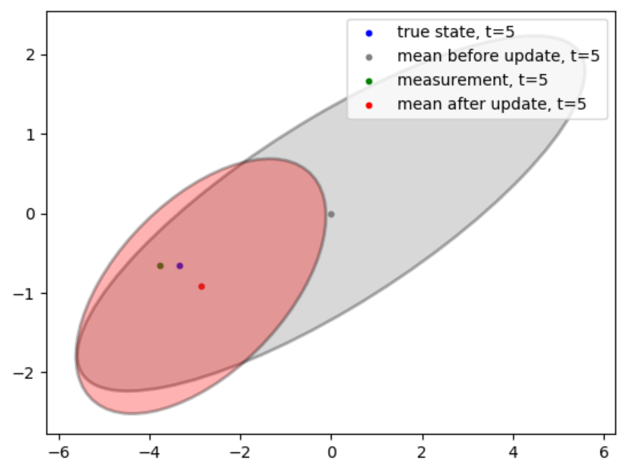


Figure 2: The uncertainty ellipses before (gray) and after (red) update. The blue and green points are the true states and the measurement, respectively. THe gray and red points are the mean before and after seeing the measurements.

the prediction step. Table 1 shows the mean and the covariance matrix of the kalman filter between $t \in [0, 1, 2, 3, 4, 5]$.

Problem 1.d

Figure 1 shows the uncertainty ellipses (2-standard deviation) from $t = 1$ to $t = 5$. The directions and the lengths of the uncertainty ellipses are determined by the eigenvectors and the eigenvalues, respectively.

The uncertainty ellipses increase in both dimensions meaning that as the time passes, the robot is more uncertain about its position and velocity. This can also be reflected by Eq. 2. The next step's covariance matrix is added by the motion's covariance.

Problem 2.a

The linear measurement model satisfies the following equation

$$z_t = \mathbf{C}\mathbf{x}_t + \delta, \tag{3}$$

where $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ is a row vector and $\delta \sim \mathcal{N}(\delta; 0, 10)$. The variance of the measurement model is $Q = 10$.

Problem 2.b

The procedure for this problem is as follows:

1. Initialize the Kalman filter with $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{Q}, \mathbf{R}$ and the initial mean and covariance matrix.

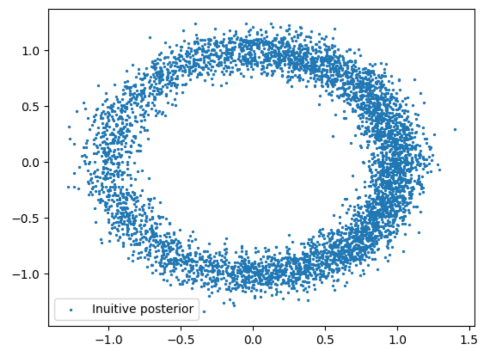


Figure 3: The "intuitive" posterior before incorporating the measurement.

2. Run Kalman filter prediction step based on the control $u = 0.0$.
3. Sample a noisy control $\bar{\mu}_t \sim \mathcal{N}(\bar{\mu}_t; 0, 1)$. ,
4. Evolve the robot's state based on $\bar{\mu}_t$ to get the true state $\bar{\mathbf{x}}_{t+1} = \mathbf{A}\bar{\mathbf{x}}_t + \mathbf{B}\bar{u}_{t+1}$
5. Repeat 2 - 4 for 5 steps.
6. Sample a noisy measurement around the true state $z_t \sim \mathcal{N}(z_t; \bar{\mathbf{x}}_t, 10)$.
7. Run Kalman filter measurement update step based on the measurement z_t.

Fig. 2 shows two uncertainty ellipses before and after seeing the measurement. As we can see that the uncertainty decreases after observing the measurement.

Problem 3 TODO

Problem 4.a

The nonlinear motion model of the robot is given by

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix} = \mathbf{g}(\mathbf{x}_t, d_{t+1}) = \begin{bmatrix} x_t + d_{t+1}\cos\theta_t \\ y_t + d_{t+1}\sin\theta_t \\ \theta_t \end{bmatrix}, \tag{4}$$

where x, y, θ represent the robot pose and d is the forwarding movement. In this question $d = 1$.

The "intuitive" posterior should be a circle with a thick edge as shown in Fig. 4. The thickness of the edge is determined by the variances of x and y.

The procedure for producing the "intuitive" posterior is as follows,

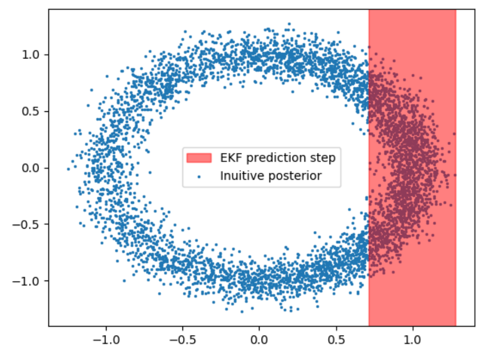


Figure 4: The "intuitive" posterior and the EKF posterior before incorporating the measurement. The scatter plot shows the sampled "intuitive" posterior and the red "bar" shows the EKF belief posterior. The "bar" is actually a long ellipse.

1. Draw the initial pose of the robot $x_0 \sim \mathcal{N}(x_0; 0, 0.01), y \sim \mathcal{N}(y_0; 0, 0.01), \theta \sim \mathcal{N}(\theta_0; 0, 100)$.
2. Evolve the robot pose based on Eq. 4.
3. Record the pose and repeat the steps 1 - 2 until enough samples are collected.

Problem 4.b

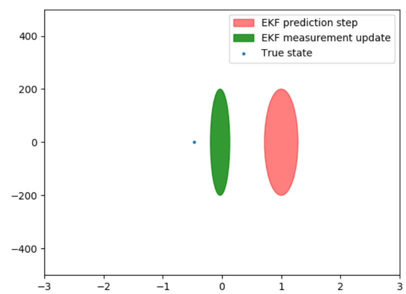
The state transition is the one represented in Eq. 4.

$$\begin{aligned} G_t &= \begin{bmatrix} \partial f_1/\partial x & \partial f_1/y & \partial f_1/\partial\theta \\ \partial f_2/\partial x & \partial f_2/y & \partial f_2/\partial\theta \\ \partial f_3/\partial x & \partial f_3/y & \partial f_3/\partial\theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -\sin\theta_t \\ 0 & 1 & \cos\theta_t \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned} \tag{5}$$

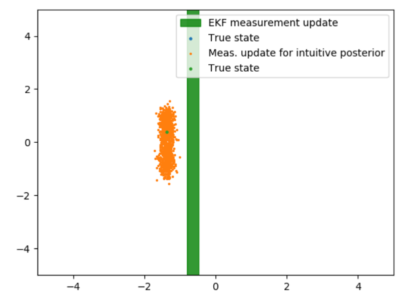
where $f_1(x_t, \theta_t) = x_t + \cos\theta_t, f_2(y_t, \theta_t) = y_t + \sin\theta_t, f_3(\theta_t) = \theta_t$ are the functions specify the robot motions. Since the control is flawless, we assume a covariance matrix with small values on the diagonal $R = \begin{bmatrix} 0.001 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 0.001 \end{bmatrix}$.

Problem 4.c

Fig 4 represents the posterior belief distribution of the position before incorporating the measurement. The red "bar" shows the posterior belief of the EKF prediction step. It is actually an ellipse shown in the zoomed out version in



(a)



(b)

Figure 5: (a): EKF posteriors before (red) and after (green) incorporating the measurement. (b): The "intuitive" posterior (yellow points) and the EKF posterior (green) after incorporating the measurement.

Fig 5. The uncertainty along y-axis is very large. This can also be seen by computing the covariance matrix of the prediction step.

$$\bar{\Sigma} = G_t \Sigma_{t-1} G_t^T + R_t = \begin{bmatrix} 0.02 & 0 & 0 \\ 0.02 & 10000.02 & 0 \\ 0 & 10000 & 10000.0 \end{bmatrix}, \tag{6}$$

where the initial covariance matrix is $\Sigma_0 = \begin{bmatrix} 0.001 & 0 & 0 \\ 0 & 0.001 & 0 \\ 0 & 0 & 10000 \end{bmatrix}$. The resulting covariance has a large variance along the y-axis and on θ. EKF approximates the nonlinear transformation (the motion model) of the Gaussian random variable (the state) by first order Taylor expansion. The difference between the "intuitive" and EKF posteriors is because the approximations of the mean and covariance are less accurate when the uncertainty in the state is higher.

Problem 4.d

In this question, we are asked to incorporate measurement update step and compare (1) the EKF belief before and after update and (2) the "intuitive" belief and EKF belief after update.

The measurement is a linear transformation of the state shown in Eq 7, so there is no need to linearize it.

$$\begin{aligned} z_t &= h(x_t) + \delta_t \\ &= \mathbf{H}_t \mathbf{x}_t + \delta_t, \end{aligned} \tag{7}$$

where $\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $\delta_t \sim \mathcal{N}(\delta_t; 0, 0.01)$.

Fig. 5 shows the result of incorporating the measurement for EKF and the "intuitive" posterior. Note that the difference between two green ellipses in

Fig. 5a and Fig. 5b is because the measurements are random and thus different in the two updates. The mean and covariance of EKF in Fig. 5b are

$$\mu_1 = \begin{bmatrix} -0.629 \\ 0 \\ 0 \end{bmatrix}, \Sigma_1 = \begin{bmatrix} 6.666 \times 10^{-3} & 0 & 0 \\ 0 & 10000.02 & 10000.02 \\ 0 & 10000 & 10000.01 \end{bmatrix}. \tag{8}$$

Since the measurement only senses the x position, only the uncertainty in x-axis goes down.