**An Integrated Assignment-Routing Network Representation for Solving Multi-Vehicle Routing Problem with Pickup and Delivery with Time Windows**

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**Abstract**

*Keywords:*multi-vehicle routing problem with pickup and delivery with time windows, forward dynamic programming

1. **Introduction**

The vehicle routing problem with pickup and delivery with time windows (VRPPDTW) is a combinatorial optimization problem in which we are looking for an optimal set of routes for a fleet of vehicles to serve a set of transportation requests. Each request is a combination of pickup at the origin and drop-off at the destination within particular time windows. Several algorithms have been suggested for solving the multi-vehicle routing problem with pickup and delivery with time windows (m-VRPPDTW). For instance, Dumas et al. (1991) proposed a column generation algorithm for the m-VRPPDTW to minimize the total travel cost considering tight vehicle capacity constraints, as well as time windows, precedence, and coupling constraints. Ruland (1995) and Ruland and Rodin (1997) proposed a polyhedral approach for the multi-vehicle routing problem with pickup and delivery. Savelsbergh and Sol (1998) developed a branch-and-price algorithm for the m-VRPPDTW to minimize the total number of vehicles needed to serve all transportation requests as the primary objective, and minimize the total distance traveled as the secondary objective. Several branch-and-cut algorithms developed by Lu and Dessouky (2004), Cordeau (2006), and Ropke et al. (2007) for solving the m-VRPPDTW to minimize the total routing cost. Ropke and Cordeau (2009) presented a branch-and-cut-and-price algorithm in which the lower bounds are controlled by a column generation scheme and strengthened by introducing several valid inequalities to the problem. Baldacci et al. (2011) proposed a new exact algorithm for the m-VRPPDTW based on a set-partitioning formulation improved by additional cuts to minimize the total routing cost. Häme and Hakula (2015) suggested a clustering algorithm in which the multi-vehicle solution is based on a recursive single-vehicle algorithm.

A number of studies focused on solving the VRPPDTW by dynamic programming (DP) approach. For instance, Psaraftis (1980) presented an exact backward DP solution algorithm for the single-VRPPDTW to minimize a weighted combination of the total service and waiting time for customers. Psaraftis (1983) further modified the algorithm to a forward DP approach in which the passengers’ service state representation was adapted from the path representation for the Traveling Salesman Problem (TSP) proposed by Bellman (1962) and Held and Karp (1962). Desrosiers et al. (1986) proposed a forward DP algorithm for the single-VRPPDTW to minimize the total distance traveled to serve all customers. Mahmoudi and Zhou (2015) also proposed a forward DP solution algorithm for solving the m-VRPPDTW on a three-dimensional state-space-time network to minimize the total routing cost. Their time-dependent state is jointly defined by the customers’ carrying state, the current node being visited, and the time.

Although several algorithms have been proposed to solve VRPPDTW, this problem even for the single vehicle cases is still classified as one of the toughest problems of combinatorial optimization (Azi et al. 2007; Hernández-Pérez and Salazar-González, 2009; and Häme, 2011). In general, in the column generation (CG), branch-and-cut, or any other algorithm designed for combinatorial optimization problems, generating additional columns or cuts is an exhausting and time-consuming task. In addition, although these algorithms may provide near-optimal solutions, the quality of such solutions are highly dependent on the gap between the upper and lower bounds. In this paper, we intend to find the exact solution for the m-VRPPDTW along a set of predefined paths. Since all effective paths have been automatically generated through constructing the state-space-time network, generating additional paths is not required. In order to define our passengers’ service patterns in the m-VRPPDTW, we utilize the path representation for the Traveling Salesman Problem (TSP) proposed by Bellman (1962) and Held and Karp (1962). We further apply a forward DP solution algorithm to find the exact solution for the m-VRPPDTW in the small-scale transportation networks.

The rest of the paper is organized as follows. Section 2 contains a precise mathematical description of the m-VRPPDTW in the state-space-time networks. In section 3, we present our new integer programming model for the m-VRPPDTW. Section 4 contains our proposed forward DP solution algorithm followed by a comprehensive discussion about the space and time complexity of the algorithm. Section 5 provides computational results of the small-scale transportation networks to demonstrate the solution optimality of our developed algorithm coded by C++. We conclude the paper in section 7 with discussions on possible extensions.

1. **Problem Statement Based on Service State-space-time Network Representation**

Psaraftis (1980) presented an exact backward DP solution algorithm for the single-VRPPDTW to minimize a weighted combination of the total service and waiting time for customers. Psaraftis (1983) further modified the algorithm to a forward DP approach in which the state representation, , consists of , the location currently being visited, and , where is the service status of passenger . In this representation, , , and denote starting depot, passenger ’s origin, and passenger ’s destination, respectively. The status of passenger is chosen from the set , where 3 means passenger is still waiting to be picked up, 2 means passenger has been picked up but the service has not been completed, and 1 means passenger has been successfully delivered. This cumulative passenger service state representation (in terms of ) requires a space complexity of . In fact, Psaraftis (1980) and Psaraftis (1983) adapted the path representation for the Traveling Salesman Problem (TSP) proposed by Bellman (1962) and Held and Karp (1962) to define the passengers’ service patterns.

In this paper, in order to define the passengers’ service patterns in the m-VRPPDTW, we also utilize the path representation for the Traveling Salesman Problem (TSP) proposed by Bellman (1962) and Held and Karp (1962). In the Bellman-Held-Karp algorithm, the path representation consists of two terms: the control/decision term and the cumulative service (only visit) state term . In this research, we extend the control term to and , and the cumulative service (visit) state term to cumulative service (pickup and drop-off) state to handle vehicle capacity constraints, as well as time windows, precedence, and coupling constraints. In section 2.1, we will extensively discuss how to construct a service state-space-time network for the m-VRPPDTW.

* 1. **Description of the m-VRPPDTW in the Service State-space-time Network**

In this paper, we intend to construct a network by which the assignment and routing problem be integrated. First, to distinguish transportation nodes from passengers’ and vehicles’ origin and destination, we add dummy nodes for origin and destination depots and passengers’ pickup and drop-off locations (Mahmoudi and Zhou, 2015). In addition, in order to write the flow balance constraints, a super source and a super sink node have been added to the network.

We also utilize the path representation for the Traveling Salesman Problem (TSP) proposed by Bellman (1962) and Held and Karp (1962) to define the passengers’ service patterns in the m-VRPPDTW. State representation consists of control/decision term , the node currently being visited at time ; and cumulative service state in which the status of passenger , , is chosen from the set , where 0 means passenger is still waiting to be picked up, 1 means passenger has been picked up but the service has not been completed, and 2 means passenger has been successfully delivered.

In addition, for each vehicle, a unique block is generated. Each block consists of two major layers, opening and ending layers, whose task is transmitting the information related to the passengers’ service status from the ending layer of current block to the opening layer of the next block. Moreover, several minor layers of passengers’ service states are built inside each block. Each layer (either minor, or major) contains the whole base network

By building this directed acyclic multi-dimensional network, our proposed model can not only solve the m-VRPPDTW on a transportation network with time-dependent congestion, but also avoid the complex procedure of any sub-tour elimination possibly existing in the optimal solution of many existing formulations.