

# Randomized algorithms for linear algebraic computations

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**Slides:** [http://users.oden.utexas.edu/~pgm/main\\_talks.html](http://users.oden.utexas.edu/~pgm/main_talks.html)

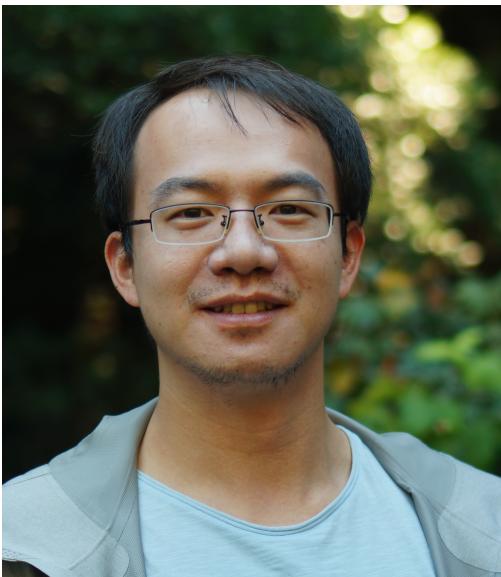
*Research support by:*



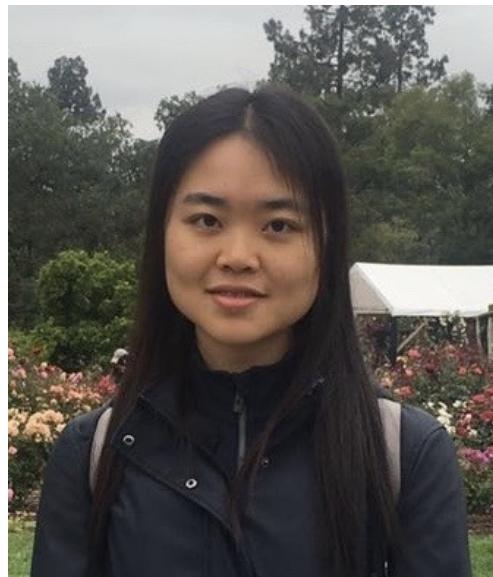
## All recent work is joint with my current research group:



*Yunhui Cai, PhD '24*



*Chao Chen, PDR*  
→ *TT at NCSU*



*Yijung Dong, PhD '23*  
→ *Courant instructor*



*Kate Pearce, PDR*



*Heather Wilber, PDR*



*Anna Yesypenko, PhD '23*

→ *TT at U. Wash.*

Recent mentees: Abi Gopal, James Levitt, Ke Chen, Bowei Wu.

## **Outline of talk**

### **(1) Randomized low rank approximation**

“Randomized singular value decomposition” or “RSVD”.

Techniques based on randomized embeddings.

Relatively well established material within numerical linear algebra.

### **(2) Samples of current research directions**

Finding spanning row and columns.

Matrix approximation via sampling.

Randomized compression of rank structured matrices.

## Low rank approximation — problem formulation:

Let  $\mathbf{A}$  be a given  $m \times n$  matrix, and let  $k$  be an integer such that  $1 \leq k \ll n \leq m$ .

We seek to compute approximate factors  $\mathbf{E}$  and  $\mathbf{F}$  such that

$$\begin{array}{ccc} \mathbf{A} & \approx & \mathbf{E} \quad \mathbf{F}^* \\ m \times n & & m \times k \quad k \times n \end{array}$$

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## Why?

- Fitting a hyperplane to a given set of points. Or fitting a multivariate normal distribution to measurements (“principal component analysis”).
- Model reduction in scientific computing.
- Spectral algorithms in data analysis.
- “Fast” algorithms of various types: Fast Multipole Methods, generalizations of the Fast Fourier Transform, fast direct solvers, etc.
- Many, many, many more.

Observe that from  $\mathbf{E}$  and  $\mathbf{F}$  you can compute approximate singular vectors, find dominant eigenvectors (when  $\mathbf{A}$  is normal), find spanning rows/columns, etc.

We seek only to control the residual error  $\|\mathbf{A} - \mathbf{EF}^*\|$ .

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$$\mathbf{A} \approx \mathbf{E} \mathbf{F}^*. \\ m \times n \quad m \times k \quad k \times n$$

**Existing methods for this task are well established. Textbook methods include:**

1. Compute the full singular value decomposition of  $\mathbf{A}$ , and then truncate:
  - Resulting approximation is in many regards "optimal" — best possible fit.
  - Expensive! Cost is  $O(mn^2)$ . Good for small  $n$ , or "expensive" data.
2. Krylov methods:
  - Standard technique for large sparse matrices.
  - Interacts with  $\mathbf{A}$  only through its action on vectors.
  - Theoretically optimal in important regards.
3. Execute Gram-Schmidt on the columns of  $\mathbf{A}$  ("column pivoted QR"):
  - Simple and practical for medium size dense matrices.
  - Not entirely optimal, but often good enough.
  - Cost is  $O(mnk)$  since you can stop after  $k$  steps.

These methods work great! But room for improvement in important environments.

## Randomized SVD:

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**Objective:** Given an  $m \times n$  matrix  $\mathbf{A}$ , find an approximate rank- $k$  partial SVD:

$$\begin{array}{cccccc} \mathbf{A} & \approx & \mathbf{U} & \mathbf{D} & \mathbf{V}^* \\ m \times n & & m \times k & k \times k & k \times n \end{array}$$

where  $\mathbf{U}$  and  $\mathbf{V}$  are orthonormal, and  $\mathbf{D}$  is diagonal. (We assume  $k \ll \min(m, n)$ .)

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### (A) *Randomized sketching:*

A.1 Draw an  $n \times k$  Gaussian random matrix  $\Omega$ .

`Omega = randn(n, k)`

A.2 Form the  $m \times k$  sample matrix  $\mathbf{Y} = \mathbf{A}\Omega$ .

`Y = A * Omega`

A.3 Form an  $m \times k$  orthonormal matrix  $\mathbf{Q}$  such that  $\text{ran}(\mathbf{Q}) = \text{ran}(\mathbf{Y})$ .

`[Q, ~] = qr(Y)`

### (B) *Deterministic post-processing:*

B.1 Form the  $k \times n$  matrix  $\mathbf{B} = \mathbf{Q}^*\mathbf{A}$ .

`B = Q' * A`

B.2 Form the full SVD of the small matrix  $\mathbf{B}$ :  $\mathbf{B} = \hat{\mathbf{U}}\mathbf{D}\mathbf{V}^*$ .

`[Uhat, Sigma, V] = svd(B, 'econ')`

B.3 Form the matrix  $\mathbf{U} = \mathbf{Q}\hat{\mathbf{U}}$ .

`U = Q * Uhat`

The objective of Stage A is to compute an ON-basis that approximately spans the column space of  $\mathbf{A}$ . The matrix  $\mathbf{Q}$  holds these basis vectors and  $\mathbf{A} \approx \mathbf{Q}\mathbf{Q}^*\mathbf{A}$ .

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Stage B is exact:  $\|\mathbf{A} - \mathbf{Q}\underbrace{\mathbf{Q}^*\mathbf{A}}_{=\mathbf{B}}\| = \|\mathbf{A} - \mathbf{Q}\underbrace{\mathbf{B}}_{=\hat{\mathbf{U}}\mathbf{D}\mathbf{V}^*}\| = \|\mathbf{A} - \underbrace{\mathbf{Q}\hat{\mathbf{U}}}_{=\mathbf{U}}\mathbf{D}\mathbf{V}^*\| = \|\mathbf{A} - \mathbf{UDV}^*\|$ .

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We claim that the columns of  $\mathbf{Y}$  form a good approximate basis for  $\text{ran}(\mathbf{A})$ .

Observe that  $\text{ran}(\mathbf{Y}) \subseteq \text{ran}(\mathbf{A})$  automatically.

Loss of accuracy can happen if  $\text{ran}(\mathbf{Y})$  does not capture important directions.

To avoid this, we draw  $p$  extra samples, for, say,  $p = 5$  or  $p = 10$ .

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**Important:** You only need to ensure that you do not undersample.

Over-sampling is unproblematic, since excess data gets “filtered out” in Stage B.

## Randomized SVD:

*Input:* An  $m \times n$  matrix  $\mathbf{A}$ , a target rank  $k$ , and an over-sampling parameter  $p$  (say  $p = 5$ ).

*Output:* Rank- $(k + p)$  factors  $\mathbf{U}$ ,  $\mathbf{D}$ , and  $\mathbf{V}$  in an approximate SVD  $\mathbf{A} \approx \mathbf{UDV}^*$ .

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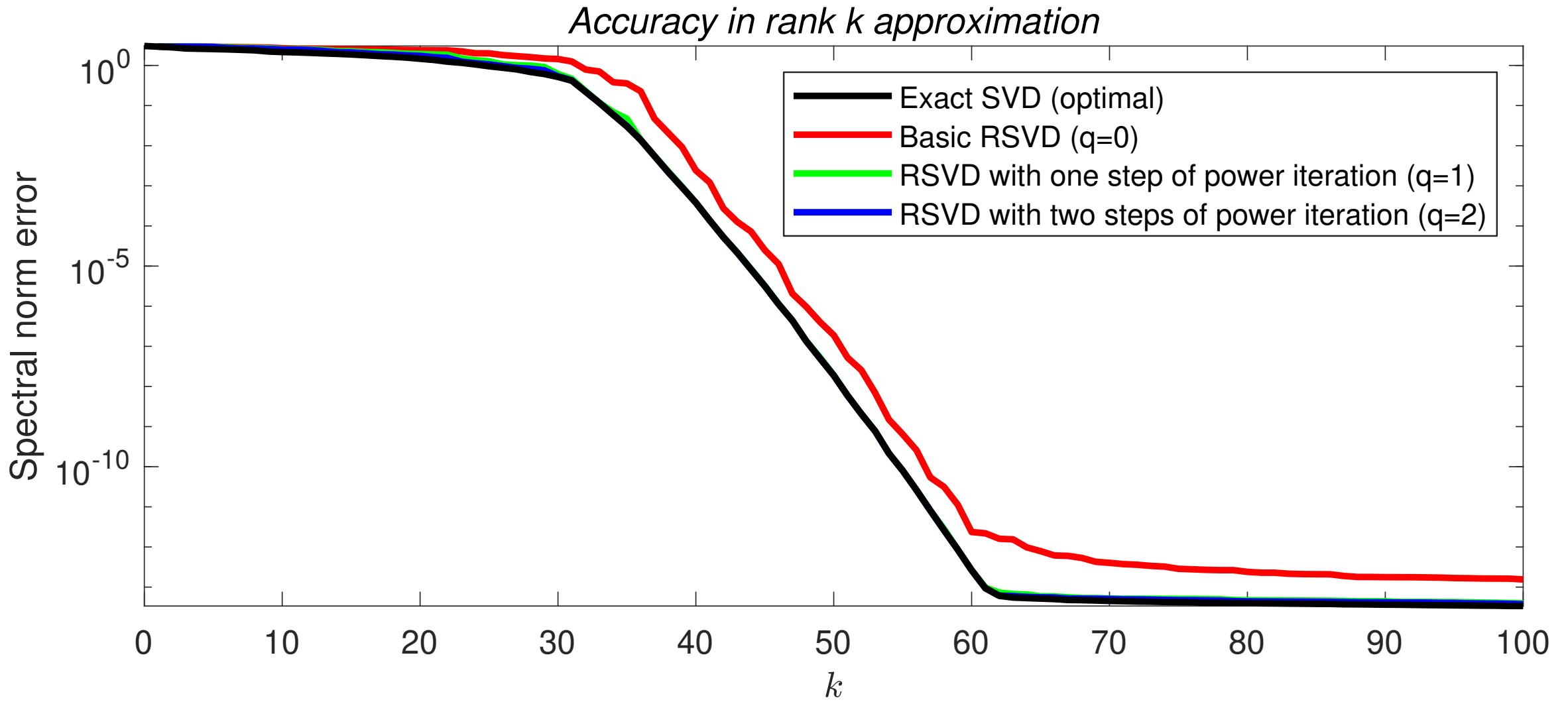
Let us next investigate the accuracy of the method.

To illustrate the errors, we set  $p = 0$  (no over-sampling), and then define

$$e_k = \|\mathbf{A} - \mathbf{UDV}^*\| = \|\mathbf{A} - \mathbf{QQ}^* \mathbf{A}\|.$$

**Eckart-Young theorem:**  $e_k \geq \sigma_{k+1}$ , where  $\sigma_j$  is the  $j$ 'th singular value of  $\mathbf{A}$ .

## Randomized SVD:



The plot shows the errors from the randomized SVD. To be precise, we plot

$$e_k = \|\mathbf{A} - \mathbf{P}_k \mathbf{A}\|,$$

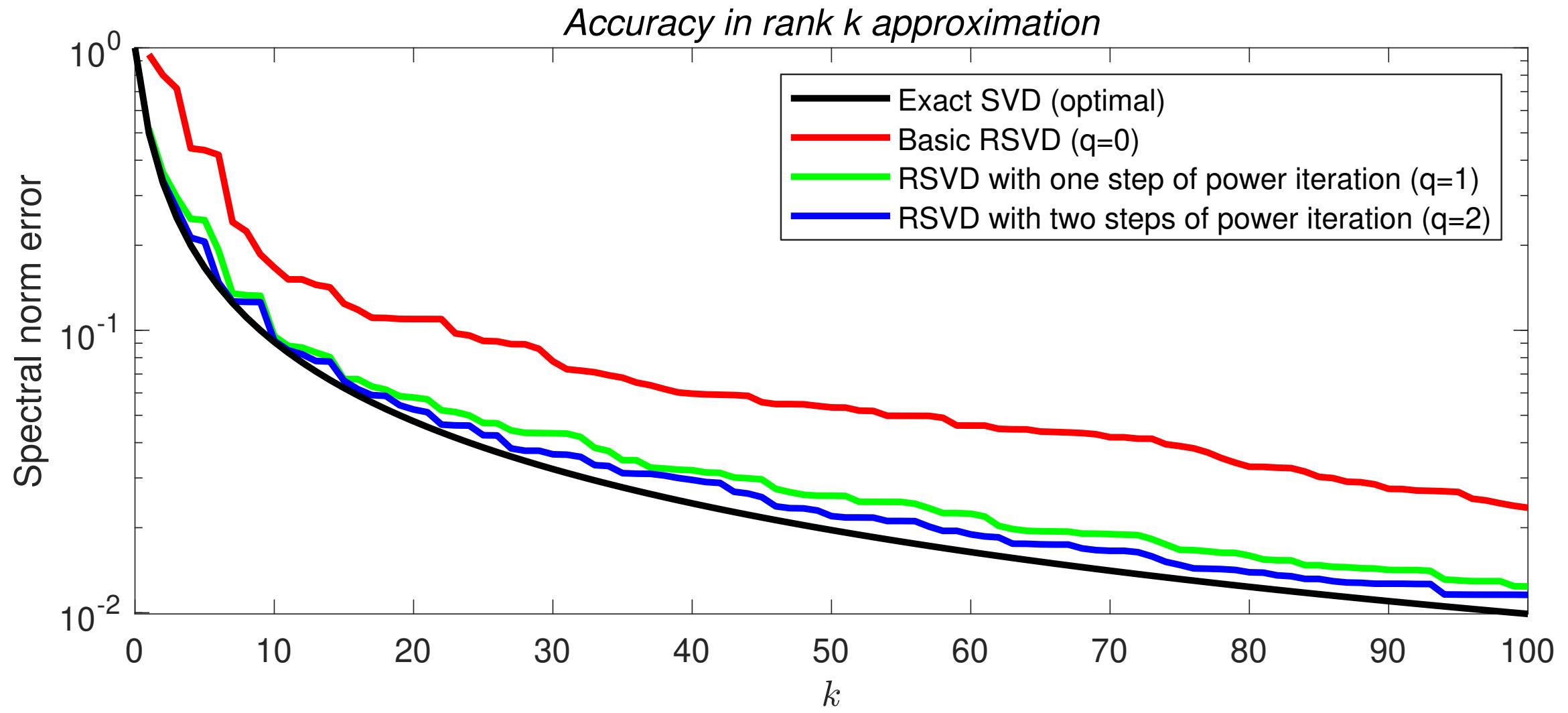
where  $\mathbf{P}_k$  is the orthogonal projection onto the first  $k$  columns of

$$\mathbf{Y} = (\mathbf{A}\mathbf{A}^*)^q \mathbf{A}\Omega,$$

and where  $\Omega$  is a Gaussian random matrix. (For clarity, no oversampling is done.)

The matrix  $\mathbf{A}$  is an approximation to a scattering operator for a Helmholtz problem.

## Randomized SVD:



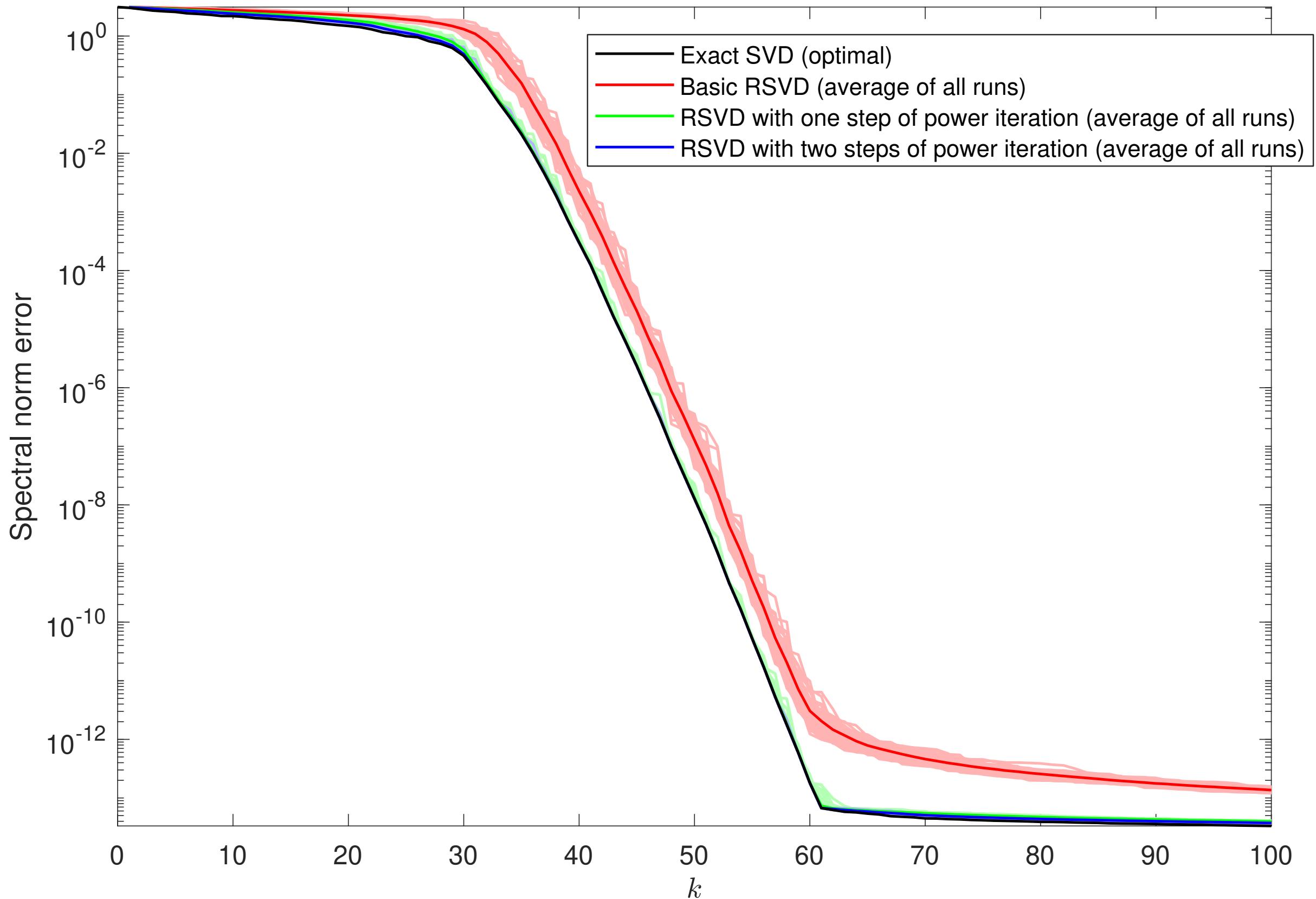
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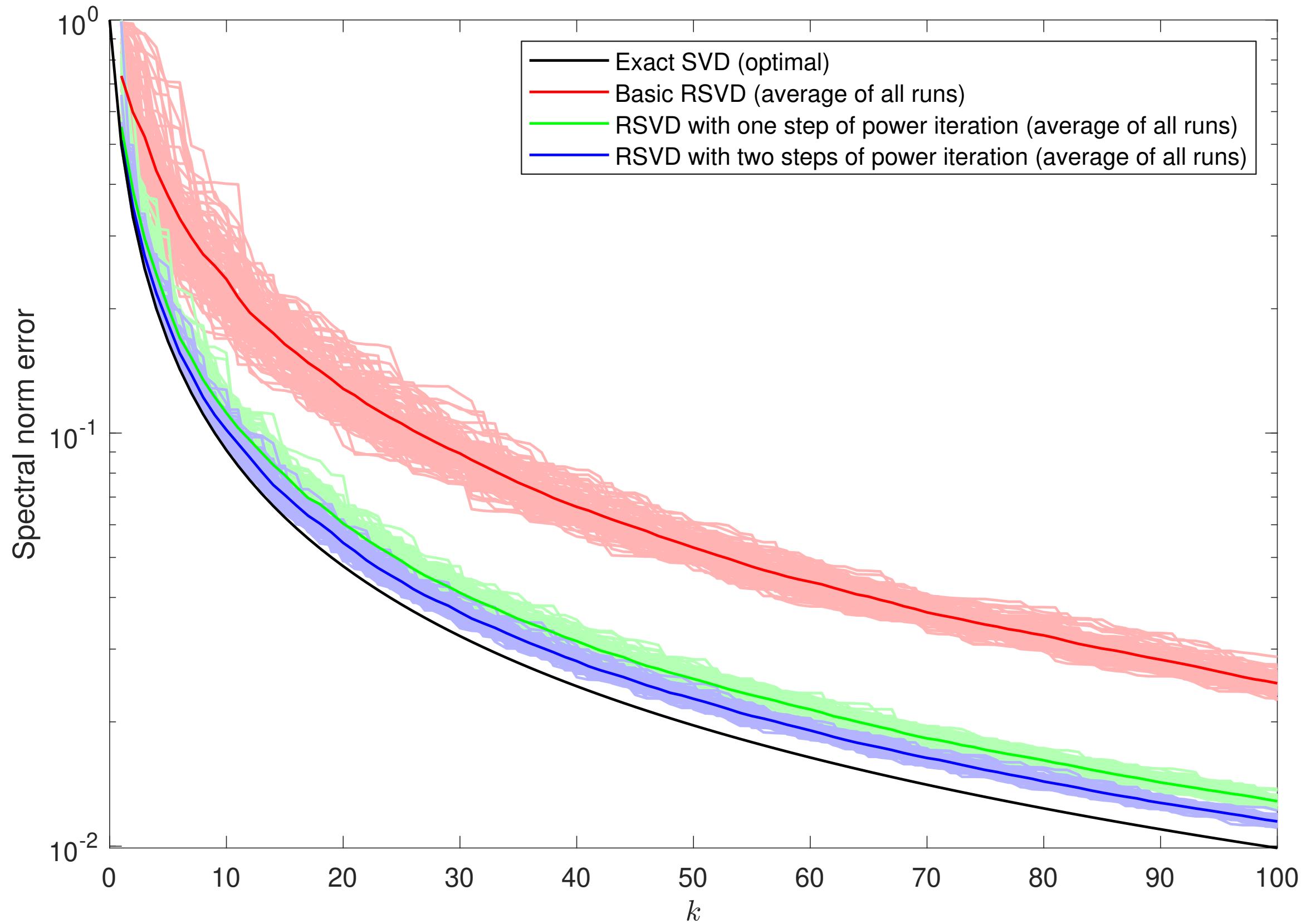
The matrix  $\mathbf{A}$  now has singular values that decay slowly.

**Randomized SVD:** The same plot as before, but now showing 100 instantiations.



*The darker lines show the mean errors across the 100 experiments.*

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Since the error in the RSVD is a random variable (it depends on the draw of  $\Omega$ ), any theoretical analysis needs to describe the *probability distribution* of the error.

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For instance, we can bound the expectation of the error:

**Theorem:** Let  $\mathbf{A}$  be an  $m \times n$  matrix with singular values  $\{\sigma_j\}_{j=1}^{\min(m,n)}$ . Let  $k$  be a target rank, and let  $p$  be an over-sampling parameter such that  $p \geq 2$  and  $k + p \leq \min(m, n)$ .

Let  $\Omega$  be a Gaussian random matrix of size  $n \times (k + p)$  and set  $\mathbf{Q} = \text{orth}(\mathbf{A}\Omega)$ . Then the average error satisfies

$$\mathbb{E}[\|\mathbf{A} - \mathbf{QQ}^*\mathbf{A}\|_{\text{Fro}}] \leq \left(1 + \frac{k}{p-1}\right)^{1/2} \left(\sum_{j=k+1}^{\min(m,n)} \sigma_j^2\right)^{1/2},$$

$$\mathbb{E}[\|\mathbf{A} - \mathbf{QQ}^*\mathbf{A}\|] \leq \left(1 + \sqrt{\frac{k}{p-1}}\right) \sigma_{k+1} + \frac{e\sqrt{k+p}}{p} \left(\sum_{j=k+1}^{\min(m,n)} \sigma_j^2\right)^{1/2}.$$

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Since the error in the RSVD is a random variable (it depends on the draw of  $\Omega$ ), any theoretical analysis needs to describe the *probability distribution* of the error.

There are also bounds on the likelihood of a large deviation from the expectation.  
(It turns out to decay super-exponentially fast as  $p$  increases!)

References (*very incomplete!!*):

- Martinsson, Rokhlin, Tygert, *Yale-CS-1361*, 2006.
- Halko, Martinsson, Tropp, *SIREV*, 2011. Survey, focus on RSVD.
- Witten, Candès, *Algorithmica*, 2015.
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**Current work:** A posteriori error analysis – use only information known to the user.

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## Key points:

- High practical speed — interacts with  $\mathbf{A}$  only through matrix-matrix multiplication.

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- Order of magnitude acceleration for data stored *out-of-core*.
- Highly efficient for GPU computing, or mobile computing (phones, etc).

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- High practical speed — interacts with  $\mathbf{A}$  only through matrix-matrix multiplication.
- Order of magnitude acceleration for data stored *out-of-core*.
- Highly efficient for GPU computing, or mobile computing (phones, etc).
- Consider the problem of computing the dominant  $k$  eigenvectors/eigenvalues of a dense matrix of size  $m \times n$ . Reduction in complexity from  $O(mnk)$  to  $O(mn \log k)$ .

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| (3) Compute an <b>ON matrix</b> $\mathbf{Q}$ s.t. $\mathbf{Y} = \mathbf{QR}$ .         | (6) Form $\mathbf{U} = \mathbf{Q}\hat{\mathbf{U}}$ .                                |

## Key points:

- High practical speed — interacts with  $\mathbf{A}$  only through matrix-matrix multiplication.
- Order of magnitude acceleration for data stored *out-of-core*.
- Highly efficient for GPU computing, or mobile computing (phones, etc).
- Consider the problem of computing the dominant  $k$  eigenvectors/eigenvalues of a dense matrix of size  $m \times n$ . Reduction in complexity from  $O(mnk)$  to  $O(mn \log k)$ .

The key is to use a *Fast Johnson-Lindenstrauss transform*.

- Randomized trigonometric transforms (FFT, Hadamard, etc). Cost is  $O(mn \log(k))$ .
- Chains of Given's rotations (“Kac’s random walk”). Cost is  $O(mn \log(k))$ .
- “Sparse sign matrix”. Place  $r$  random entries in each row of  $\Omega$ . (Say  $r = 2$  or  $r = 4$ .) Cost is now  $O(mn)$ !

Practical acceleration is achieved at ordinary matrix sizes.

## Randomize

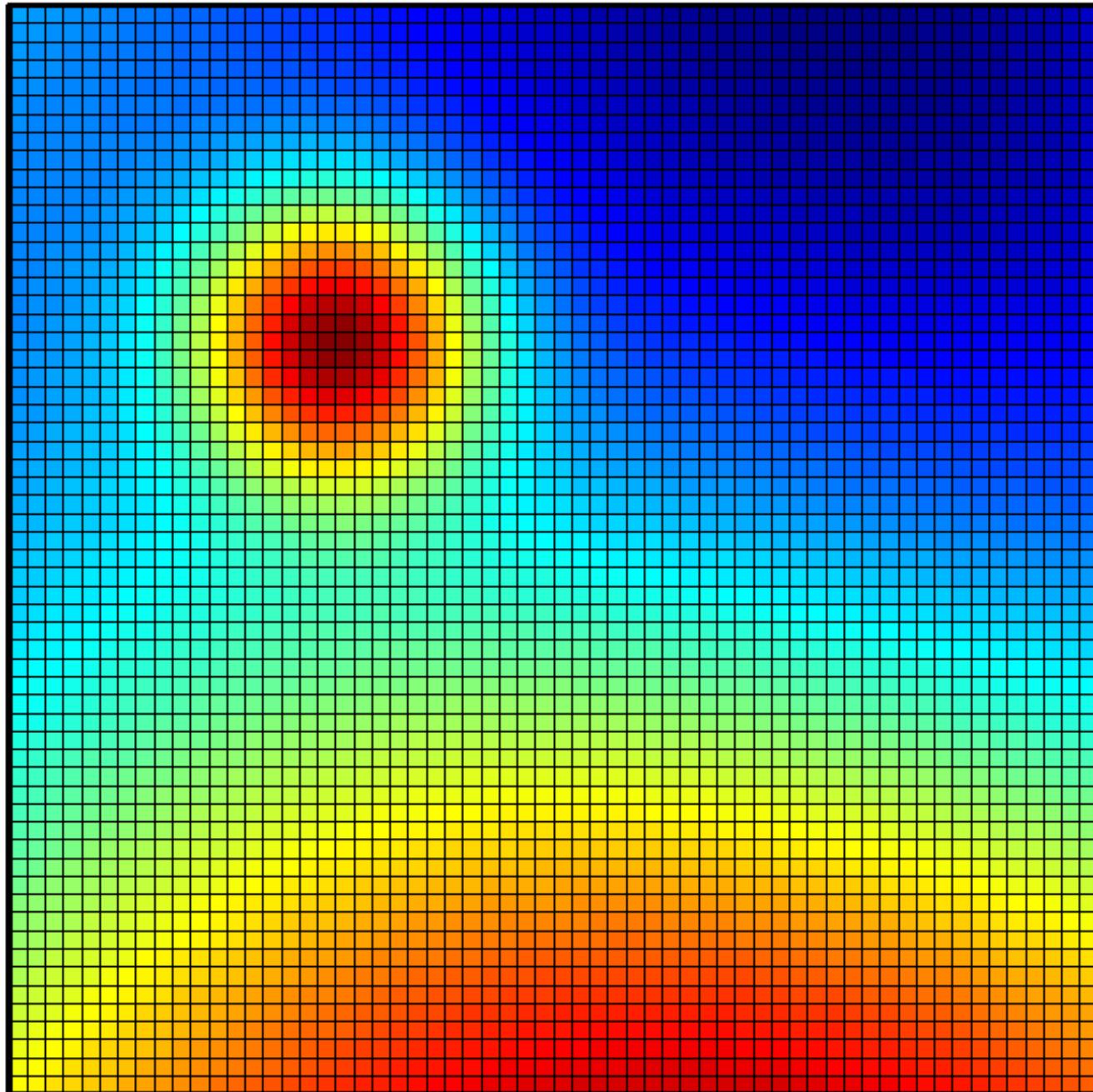
*Input:* An  $n \times n$  matrix  $\mathbf{A}$ .

*Output:* Randomized matrix  $\mathbf{A}\Omega$ .

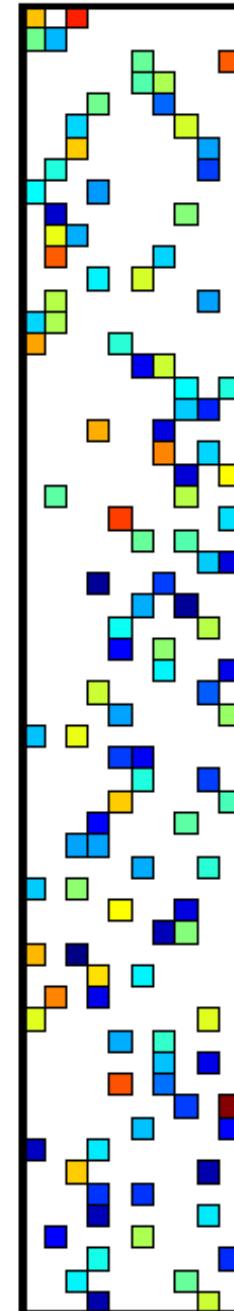
- (1) Draw a sparse random matrix  $\Omega$ .
- (2) Form the product  $\mathbf{A}\Omega$ .
- (3) Compute the QR factorization  $\mathbf{A}\Omega = \mathbf{Q}^* \mathbf{A}\hat{\mathbf{U}} \mathbf{V}^*$ .

## Key points

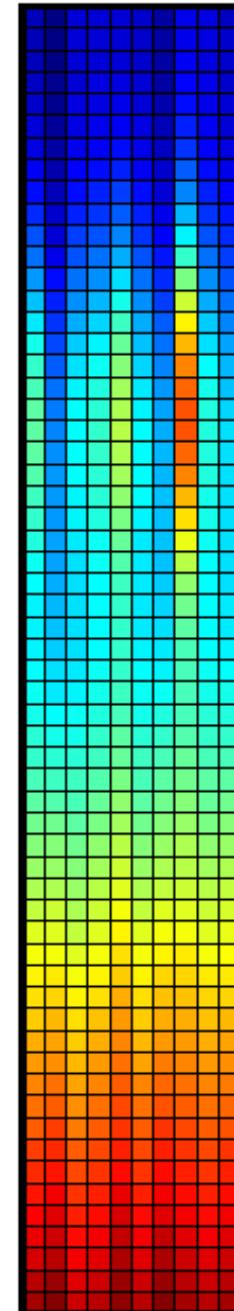
- High probability of success.
- Order of  $\mathcal{O}(mn)$  complexity.
- Highly efficient.
- Considerably faster than dense randomization.



$\mathbf{A}$



$\Omega$



$\mathbf{A}\Omega$

'  $p = 5$ ).

$\mathbf{Q}^* \mathbf{A}$ .

$\hat{\mathbf{U}} \mathbf{D} \mathbf{V}^*$ .

on.

of a  
 $k$ ).

The matrix  $\Omega$  is a sparse random matrix. Two nonzero entries are placed randomly in each row. In consequence, each column of  $\mathbf{A}$  contributes to precisely two columns of the sample matrix  $\mathbf{Y} = \mathbf{A}\Omega$ . This structured random map has  $\mathcal{O}(mn)$  complexity, is easy to work with practically, and often provides good accuracy.

## Randomized SVD:

*Input:* An  $m \times n$  matrix  $\mathbf{A}$ , a target rank  $k$ , and an over-sampling parameter  $p$  (say  $p = 5$ ).

*Output:* Rank- $(k + p)$  factors  $\mathbf{U}$ ,  $\mathbf{D}$ , and  $\mathbf{V}$  in an approximate SVD  $\mathbf{A} \approx \mathbf{UDV}^*$ .

- |  |   |
|--|---|
| (1) Draw an $n \times (k + p)$ <b>random matrix</b> $\Omega$ .                         | (4) Form the small matrix $\mathbf{B} = \mathbf{Q}^* \mathbf{A}$ .                  |
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- Single pass algorithms have been developed for *streaming environments*.  
The idea is that you are allowed to observe each matrix element only once.  
You cannot store the matrix. *Not possible with deterministic methods!*

## Outline of talk

### (1) Randomized low rank approximation

“Randomized singular value decomposition” or “RSVD”.

Techniques based on randomized embeddings.

Relatively well established material within numerical linear algebra.

### (2) Samples of current research directions

Finding spanning row and columns.

Matrix approximation via sampling.

Randomized compression of rank structured matrices.

## Finding spanning rows and columns — CUR/interpolatory/CPQR decompositions

A common task in linear algebra is to find sets of columns/rows that span the column/row space of a matrix. To illustrate, suppose that we are given an  $m \times n$  matrix  $\mathbf{A}$ , a rank  $k < \min(m, n)$ , and seek to compute a factorization

$$\begin{array}{ccc} \mathbf{A} & \approx & \mathbf{C} \quad \mathbf{Z}, \\ m \times n & & m \times k \quad k \times n \end{array}$$

where  $\mathbf{C} = \mathbf{A}(J, :)$  holds a subset of  $k$  columns of  $\mathbf{A}$ , as before.

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### Applications:

- *Data interpretation*: The columns you pick sometimes correspond to specific variables that explain some data set — specific genes, specific stocks, etc.
- *Preserving structure*: If  $\mathbf{A}$  is sparse/non-negative, then so is  $\mathbf{C}$ . Particularly powerful in a “CUR” decomposition  $\mathbf{A} \approx \mathbf{C}\mathbf{U}\mathbf{R}$  where  $\mathbf{R}$  holds a subset of rows.
- *Storage efficiency*: In many environments, there is no need to explicitly store the factors  $\mathbf{R}$  or  $\mathbf{C}$  – just store the index vector and extract / build  $\mathbf{R}$  or  $\mathbf{C}$  when needed.

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### Standard techniques:

- Golub-Businger pivoting strategy: The standard tool. Simple, attractive flop count, generally works well, but can be substantially suboptimal. Challenging to implement efficiently on modern hardware.
- Specialized pivoting strategies such as Gu-Eisenstat: Guaranteed to select columns that are close to optimal. Rarely implemented (too complicated).
- Randomized sampling strategies: Very popular subject in theoretical CS. Powerful asymptotic theory. In practice, competitive only for huge matrices, or matrices where entry evaluation is expensive.

## Finding spanning rows and columns — CUR/interpolatory/CPQR decompositions

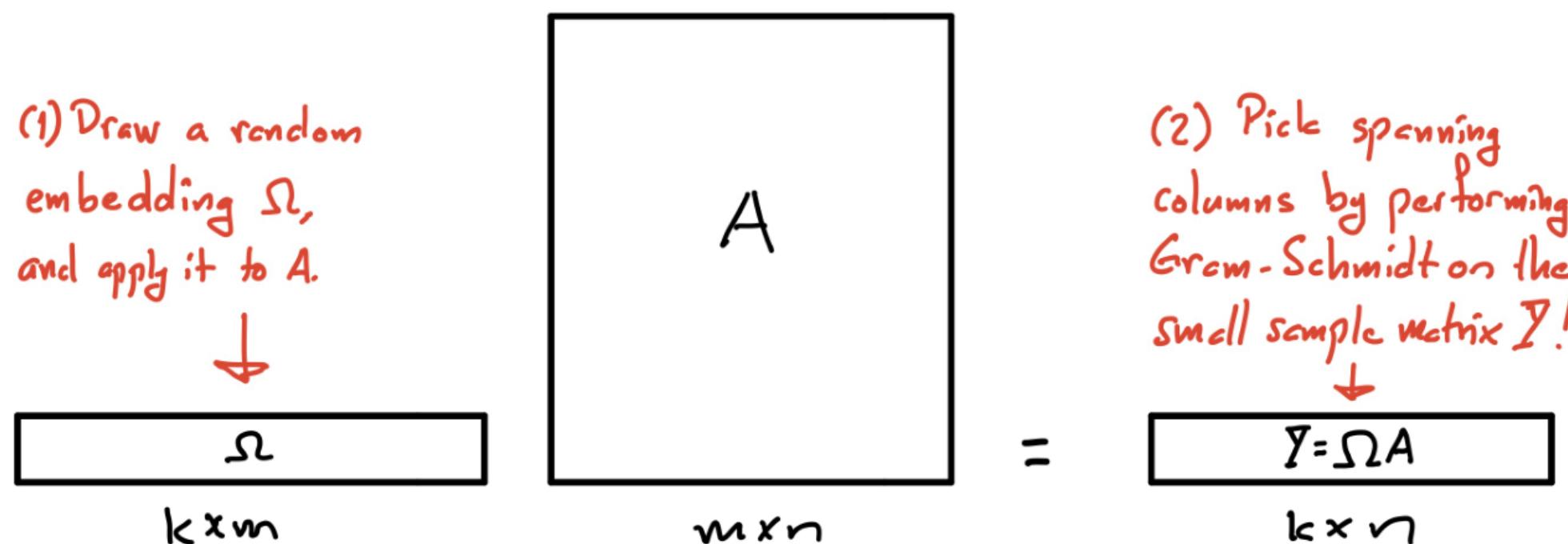
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- (1) Apply a random embedding  $\Omega$  to the columns of  $\mathbf{A}$
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- (1) Apply a random embedding  $\Omega$  to the columns of  $\mathbf{A}$
  - (2) Execute a classical pivoting method on  $\Omega\mathbf{A}$
- Sparse random embeddings work well for step (1).
  - *Partially* pivoted LU can be used for step (2). Fast!! (Very surprising to me!)
  - Simple to implement.
  - Overall complexity as low as  $O(mn + k^2n)$ . (Versus the classical  $O(mnk)$ .)

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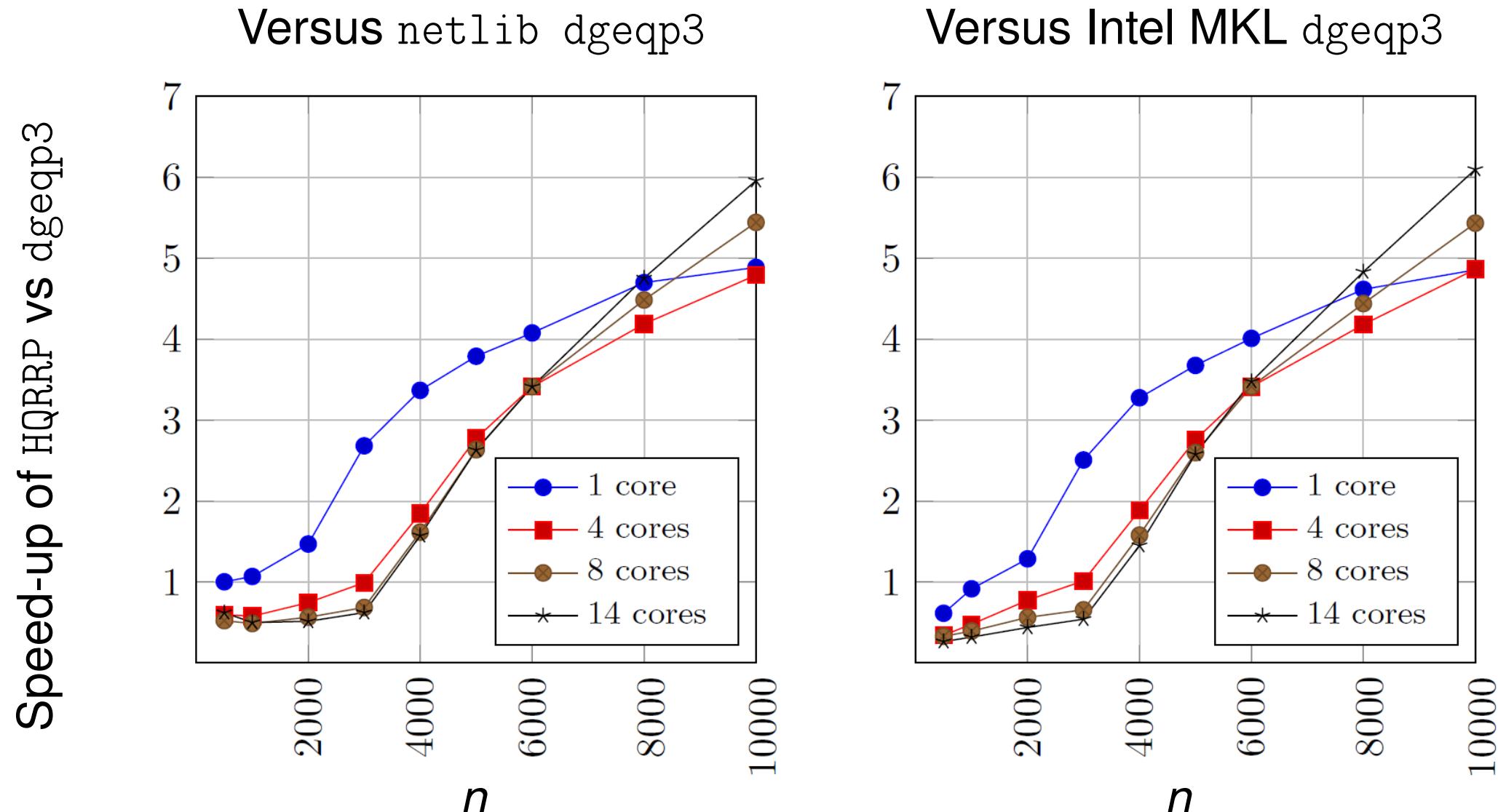
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References: Liberty, Woolfe, Martinsson, Rokhlin and Tygert (2007); Sorensen & Embree (2006); Halko, Martinsson, Tropp (2011); Martinsson, Tropp (2020); Dong & Martinsson, arXiv:2104.05877, 2021; ...

**Computing full factorizations — accelerated column pivoted QR:** The column selection strategy on the previous slide has been applied to resolve a classical problem in numerical linear algebra: How do you pick *groups* of pivots when executing column pivoted QR? The purpose is to move flops from BLAS2 to BLAS3 operations.



*Speedup attained by a randomized algorithm for computing a full column pivoted QR factorization of an  $n \times n$  matrix. The speed-up is measured versus LAPACK's faster routine `dgeqp3` as implemented in Netlib (left) and Intel's MKL (right). Our implementation was done in C, and was executed on an Intel Xeon E5-2695. Joint work with G. Quintana-Ortí, N. Heavner, and R. van de Geijn (SISC 2017). Closely related work by Duersch and Gu, SISC 2017 / SIREV 2020.*

## Randomization as a means to avoid edge cases

## Pivoting revisited

Consider the problem of solving a linear system

$$\sum_{j=1}^N a_{ij}x_j = b_i, \quad i \in \{1, 2, \dots, N\},$$

or

$$\mathbf{Ax} = \mathbf{b},$$

and our old favourite solution algorithm: Gaussian elimination (GE).

Without pivoting, GE can fail. Either outright. Or, more subtly, “numerically”.

If  $\mathbf{A}$  is a *Gaussian random matrix* GE without pivoting works very well, even for large  $N$ .

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If  $\mathbf{A}$  is a *Gaussian random matrix* GE without pivoting works very well, even for large  $N$ .

Randomisation can be used to make the failure modes of GE vanishingly unlikely.

For instance, you may randomly rotate the coefficient matrix to obtain

$$(\mathbf{U}^* \mathbf{A} \mathbf{V}) (\mathbf{V}^* \mathbf{x}) = (\mathbf{U}^* \mathbf{b}),$$

where  $\mathbf{U}$  and  $\mathbf{V}$  are drawn from a Haar distribution (uniform on unitary matrices). This new system can safely be solved without pivoting. Dramatically reduces *communication*.

To make things shine, use *structured* random matrices.

**Question:** How much randomness is required?

## Outline of talk

### (1) Randomized low rank approximation

“Randomized singular value decomposition” or “RSVD”.

Techniques based on randomized embeddings.

Relatively well established material within numerical linear algebra.

### (2) Samples of current research directions (time permitting)

Finding spanning row and columns.

Matrix approximation via sampling.

Randomized compression of rank structured matrices.

# **Matrix approximation by sampling**

To simplify slightly, there are two paradigms for how to use randomization to approximate matrices:

## **Randomized embeddings**

(What we have discussed so far.)

## **Randomized sampling**

(What we will discuss next.)

# Matrix approximation by sampling

To simplify slightly, there are two paradigms for how to use randomization to approximate matrices:

## Randomized embeddings

(What we have discussed so far.)

Often faster than classical deterministic methods.

Highly reliable and robust.

High accuracy is attainable.

Best for scientific computing.

## Randomized sampling

(What we will discuss next.)

Sometimes *far* faster than classical deterministic methods. Faster than matrix-vector multiplication, even.

Can fail in the “general” case.

Typically low accuracy.

Enables solution of large scale problems in “big data” where no other methods work.

## Matrix approximation by sampling

Suppose that  $\mathbf{A} = \sum_{t=1}^T \mathbf{A}_t$  where each  $\mathbf{A}_t$  is “simple” in some sense.

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Suppose that  $\mathbf{A} = \sum_{t=1}^T \mathbf{A}_t$  where each  $\mathbf{A}_t$  is “simple” in some sense.

**Example:** Sparse matrix written as a sum over its nonzero entries

$$\underbrace{\begin{bmatrix} 5 & -2 & 0 \\ 0 & 0 & -3 \\ 1 & 0 & 0 \end{bmatrix}}_{=\mathbf{A}} = \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=\mathbf{A}_1} + \underbrace{\begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{=\mathbf{A}_2} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}}_{=\mathbf{A}_3} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{=\mathbf{A}_4}$$

**Example:** Each  $\mathbf{A}_i$  could be a column of the matrix

$$\underbrace{\begin{bmatrix} 5 & -2 & 7 \\ 1 & 3 & -3 \\ 1 & -1 & 1 \end{bmatrix}}_{=\mathbf{A}} = \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{=\mathbf{A}_1} + \underbrace{\begin{bmatrix} 0 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{=\mathbf{A}_2} + \underbrace{\begin{bmatrix} 0 & 0 & 7 \\ 0 & 0 & -3 \\ 0 & 0 & 1 \end{bmatrix}}_{=\mathbf{A}_3}.$$

**Example:** Matrix-matrix multiplication broken up as a sum of rank-1 matrices:

$$\mathbf{A} = \mathbf{BC} = \sum_t \mathbf{B}(:, t) \mathbf{C}(t, :).$$

## Matrix approximation by sampling

Suppose that  $\mathbf{A} = \sum_{t=1}^T \mathbf{A}_t$  where each  $\mathbf{A}_t$  is “simple” in some sense.

Let  $\{p_t\}_{t=1}^T$  be a probability distribution on the index vector  $\{1, 2, \dots, T\}$ .

Draw an index  $t \in \{1, 2, \dots, T\}$  according to the probability distribution given, and set

$$\mathbf{X} = \frac{1}{p_t} \mathbf{A}_t.$$

Then from the definition of the expectation, we have

$$\mathbb{E}[\mathbf{X}] = \sum_{t=1}^T p_t \times \frac{1}{p_t} \mathbf{A}_t = \sum_{t=1}^T \mathbf{A}_t = \mathbf{A},$$

so  $\mathbf{X}$  is an unbiased estimate of  $\mathbf{A}$ .

Clearly, a single draw is not a good approximation — unrepresentative, *large variance*.

Instead, draw several samples and average:

$$\bar{\mathbf{X}} = \frac{1}{k} \sum_{t=1}^k \mathbf{X}_t,$$

where  $\mathbf{X}_t$  are independent samples from the same distribution.

As  $k$  grows, the variance will decrease, as usual. Various Bernstein inequalities apply.

## Matrix approximation by sampling

As an illustration of the theory, we cite a matrix-Bernstein result from J. Tropp (2015):

**Theorem:** Let  $\mathbf{A} \in \mathbb{R}^{m \times n}$ . Construct a probability distribution for  $\mathbf{X} \in \mathbb{R}^{m \times n}$  that satisfies

$$\mathbb{E}[\mathbf{X}] = \mathbf{A} \quad \text{and} \quad \|\mathbf{X}\| \leq R.$$

Define the per-sample second-moment:  $v(\mathbf{X}) := \max\{\|\mathbb{E}[\mathbf{XX}^*]\|, \|\mathbb{E}[\mathbf{X}^*\mathbf{X}]\|\}$ .

Form the matrix sampling estimator:  $\bar{\mathbf{X}}_k = \frac{1}{k} \sum_{t=1}^k \mathbf{X}_t$  where  $\mathbf{X}_t \sim \mathbf{X}$  are iid.

$$\text{Then } \mathbb{E}\|\bar{\mathbf{X}}_k - \mathbf{A}\| \leq \sqrt{\frac{2v(\mathbf{X}) \log(m+n)}{k}} + \frac{2R \log(m+n)}{3k}.$$

$$\text{Furthermore, for all } s \geq 0: \mathbb{P}[\|\bar{\mathbf{X}}_k - \mathbf{A}\| \geq s] \leq (m+n) \exp\left(\frac{-ks^2/2}{v(\mathbf{X}) + 2Rs/3}\right).$$

---

Suppose that we want  $\mathbb{E}\|\mathbf{A} - \bar{\mathbf{X}}\| \leq 2\epsilon$ . The theorem says to pick

$$k \geq \max\left\{\frac{2v(\mathbf{X}) \log(m+n)}{\epsilon^2}, \frac{2R \log(m+n)}{3\epsilon}\right\}$$

In other words, the number  $k$  of samples should be proportional to both  $v(\mathbf{X})$  and to the upper bound  $R$ .

The scaling  $\textcolor{red}{k} \sim \frac{1}{\epsilon^2}$  is discouraging, and unavoidable (since error  $\epsilon \sim 1/\sqrt{k}$ ).

## Matrix approximation by sampling: Matrix matrix multiplication

Given two matrices  $\mathbf{B}$  and  $\mathbf{C}$ , consider the task of evaluating

$$\begin{array}{ccc} \mathbf{A} & = & \mathbf{B} \quad \mathbf{C} = \sum_{t=1}^T \mathbf{B}(:, t) \mathbf{C}(t, :). \\ m \times n & m \times T & T \times n \end{array}$$

Sampling approach:

1. Fix a probability distribution  $\{p_t\}_{t=1}^T$  on the index vector  $\{1, 2, \dots, T\}$ .
2. Draw a subset of  $k$  indices  $J = \{t_1, t_2, \dots, t_k\} \subseteq \{1, 2, \dots, T\}$ .
3. Use  $\bar{\mathbf{A}} = \frac{1}{k} \sum_{i=1}^k \frac{1}{p_{t_i}} \mathbf{B}(:, t_i) \mathbf{C}(t_i, :)$  to approximate  $\mathbf{A}$ .

You get an unbiased estimator regardless of the probability distribution. But the computational profile depends critically how which one you choose. Common choices:

*Uniform distribution:* Very fast. Not very reliable or accurate.

*Sample according to column/row norms:* Cost is  $O(mnk)$ , which is much better than  $O(mnT)$  when  $k \ll T$ . Better outcomes than uniform, but not great in general case.

In either case, you need  $k \sim \frac{1}{\epsilon^2}$  to attain precision  $\epsilon$ .

## Matrix approximation by sampling:

Low rank approximation.

Given an  $m \times n$  matrix  $\mathbf{A}$ , we seek a rank- $k$  matrix  $\bar{\mathbf{A}}$  such that  $\|\mathbf{A} - \bar{\mathbf{A}}\|$  is small.

Sampling approach:

1. Draw vectors  $J$  and  $I$  holding  $k$  samples from the column and row indices, resp.
2. Form matrices  $\mathbf{C}$  and  $\mathbf{R}$  consisting of the corresponding columns and rows

$$\mathbf{C} = \mathbf{A}(:, J), \quad \text{and} \quad \mathbf{R} = \mathbf{A}(I, : ).$$

3. Use as your approximation

$$\begin{array}{ccccccccc}\bar{\mathbf{A}} & = & \mathbf{C} & \mathbf{U} & \mathbf{R}, \\ m \times n & & m \times k & k \times k & k \times n\end{array}$$

where  $\mathbf{U}$  is computed from information in  $\mathbf{A}(I, J)$ . (It should be an approximation to the optimal choice  $\mathbf{U} = \mathbf{C}^\dagger \mathbf{A} \mathbf{R}^\dagger$ . For instance,  $\mathbf{U} = \mathbf{A}(I, J)^{-1}$ .)

The computational profile depends crucially on the probability distribution that is used.

*Uniform probabilities:* Can be very cheap. But in general not reliable.

*Probabilities from “leverage scores”:* Optimal distributions can be computed using the information in the top left and right singular vectors of  $\mathbf{A}$ . Then quite strong theorems can be proven on the quality of the approximation. Problem: Computing the probability distribution is as expensive as computing a partial SVD.

**Matrix approximation by sampling:** Connections to randomized embedding.

**Task:** Find a rank  $k$  approximation to a given  $m \times n$  matrix  $\mathbf{A}$ .

*Sampling approach:* Draw a subset of  $k$  columns  $\mathbf{Y} = \mathbf{A}(:, J)$  where  $J$  is drawn at random. Let our approximation to the matrix be

$$\mathbf{A}_k = \mathbf{Y}\mathbf{Y}^\dagger\mathbf{A}.$$

As we have seen, this in general does not work very well. But it does work well for the class of matrices for which uniform sampling is optimal.

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As we have seen, this in general does not work very well. But it does work well for the class of matrices for which uniform sampling is optimal. *We can turn  $\mathbf{A}$  into such a matrix!* Let  $\Omega$  be a matrix drawn from a uniform distribution on the set of  $n \times n$  unitary matrices (the “Haar distribution”). Then form

$$\tilde{\mathbf{A}} = \mathbf{A}\Omega.$$

Now each column of  $\tilde{\mathbf{A}}$  has exactly the same distribution! We may as well pick  $J = 1 : k$ , and can then pick a great sample through

$$\mathbf{Y} = \tilde{\mathbf{A}}(:, J) = \mathbf{A}\Omega(:, J).$$

The  $n \times k$  “slice”  $\Omega(:, J)$  is in a sense an optimal random embedding.

**Fact:** Using a Gaussian matrix is mathematically equivalent to using  $\Omega(:, J)$ .

**Question:** What other choices of random projection might mimic the action of  $\Omega(:, J)$ ?

## Matrix approximation by sampling: Structured random embeddings

**Task:** Find a rank  $k$  approximation to a given  $m \times n$  matrix  $\mathbf{A}$ .

**Approach:** Draw an  $n \times k$  random embedding  $\Omega$ , set  $\mathbf{Y} = \mathbf{A}\Omega$ , and then form  $\mathbf{A}_k = \mathbf{Y}\mathbf{Y}^\dagger\mathbf{A}$ .

### Choices of random embeddings:

- *Gaussian (or slice of Haar matrix):* Optimal. Leads to  $O(mnk)$  overall cost.
- *Subsampled randomized Fourier transform (SRFT):* Indistinguishable from Gaussian in practice. Leads to  $O(mn\log(k))$  overall cost. Adversarial counter examples can be built, so supporting theory is weak.
- *Chains of Givens rotations:* Similar profile to an SRFT.
- *Sparse random projections:* Need at least two nonzero entries per row. Works surprisingly well.
- *Additive random projections:* You can use a map with only  $\pm 1$  entries.

## Matrix approximation by sampling: Key points

- These techniques provide a path forwards for problems where traditional techniques are simply unaffordable.

Kernel matrices in data analysis form a prime target. These are dense matrices, and you just cannot form the entire matrix.

- Popular topic for theory papers.
- When techniques based on randomized embeddings that systematically mix all coordinates *are* affordable, they perform far better. Higher accuracy, and less variability in the outcome.

## Outline of talk

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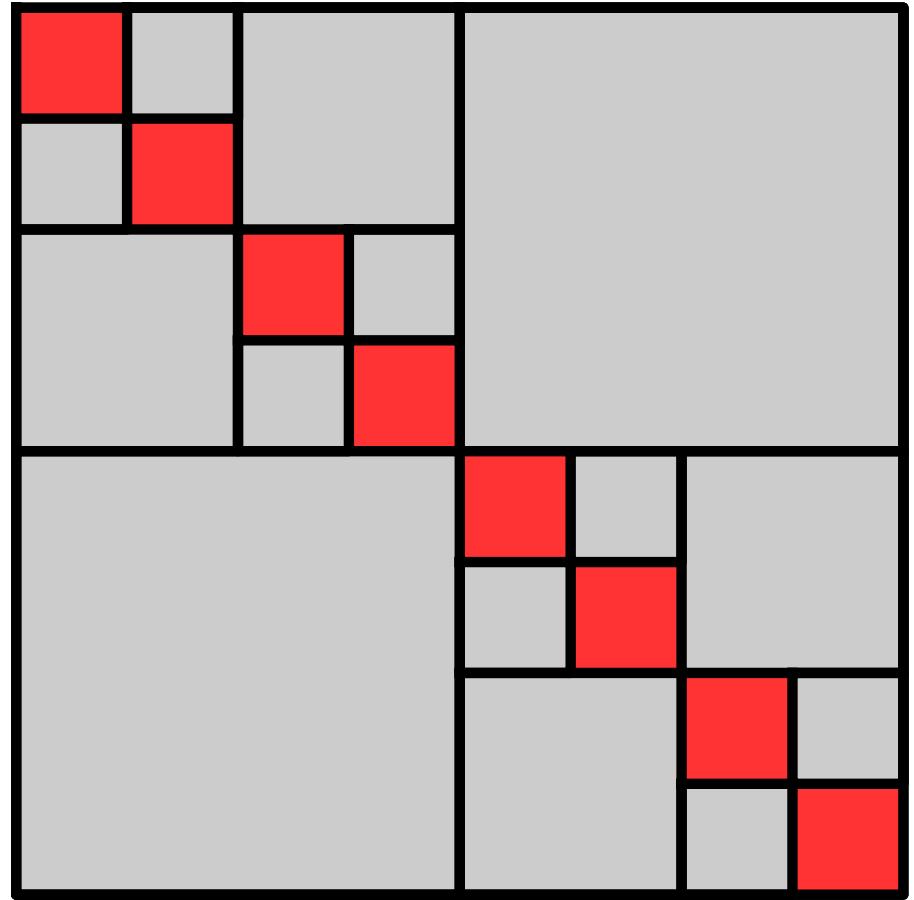
Matrix approximation via sampling.

Randomized compression of rank structured matrices.

## Rank structured matrices

We use the term *rank structured* to refer to matrices that are not themselves of globally low rank, but can be tessellated into sub-blocks in such a way that each block is either small or of low numerical rank.

We focus on “hierarchical” tessellations (as the one shown on the right). Some techniques apply to “flat” formats as well.



*All gray blocks have low rank.*

Hierarchically rank structured matrices often admit linear or close to linear complexity algorithms for the matrix-vector multiply, matrix-matrix multiply, LU factorization, etc.

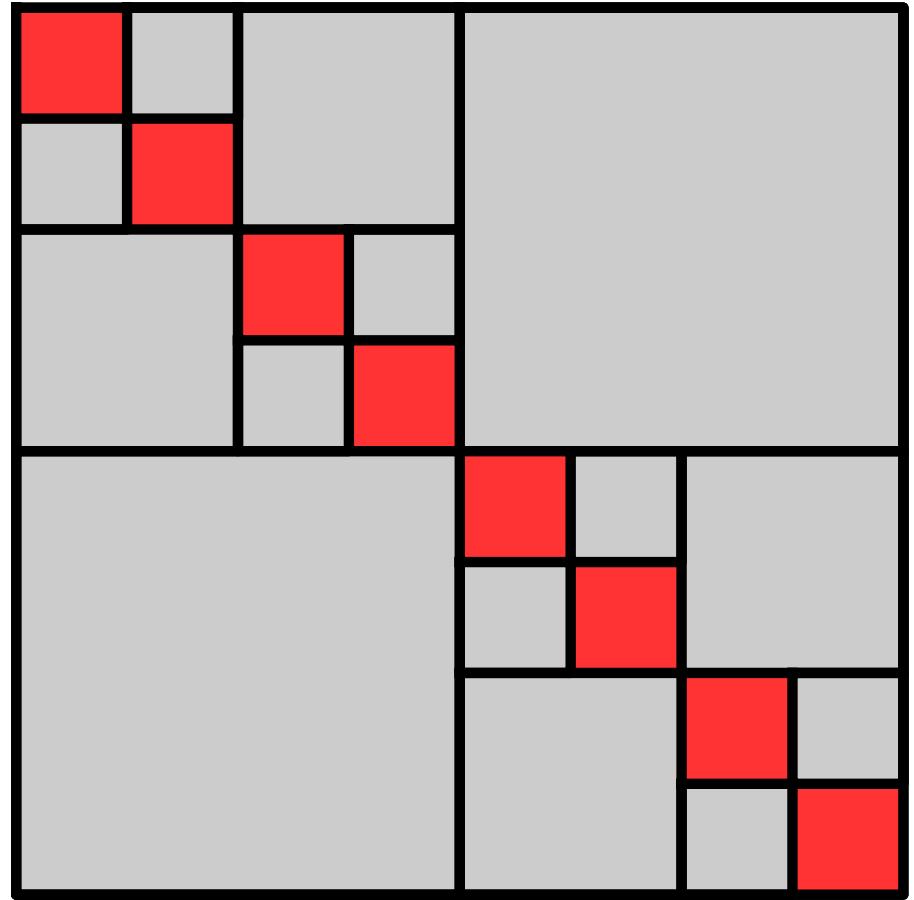
Ubiquitous applications in scientific computing: *Solution operators for elliptic PDEs, DtN operators, scattering matrices, Schur complements in sparse direct solvers, etc.*

More recently, have been shown to arise in data science as well — kernel matrices, covariance matrices, Hessians, etc.

## Rank structured matrices

We use the term *rank structured* to refer to matrices that are not themselves of globally low rank, but can be tessellated into sub-blocks in such a way that each block is either small or of low numerical rank.

We focus on “hierarchical” tessellations (as the one shown on the right). Some techniques apply to “flat” formats as well.



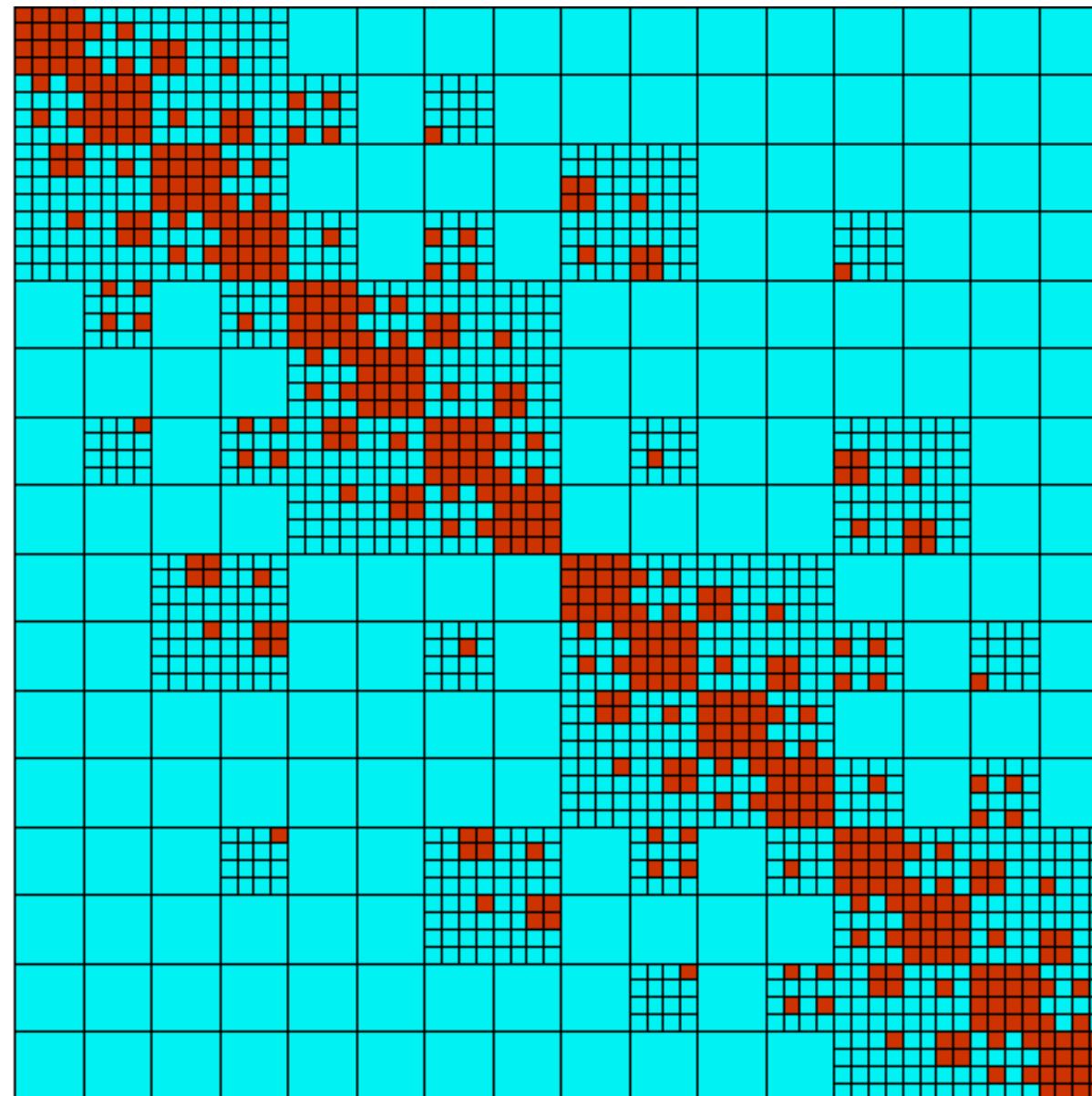
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References: *Fast Multipole Method (Greengard, Rokhlin); Panel Clustering (Hackbusch);  $\mathcal{H}$ - and  $\mathcal{H}^2$ -matrices (Hackbusch et al); Hierarchically Block Separable matrices; Hierarchically Semi Separable matrices (Xia et al); HODLR matrices (Darve et al); BLR matrices (Buttari, Amestoy, Mary, ...); ...*

In real life, tessellation patterns of rank structured matrices tend to be more complex ...



*Image credit: Ambikasaran & Darve, arxiv.org #1407.1572*

## Approximation of rank structured matrices

**Environment:** We are given a rank structured matrix  $\mathbf{A} \in \mathbb{R}^{N \times N}$  (to be precise,  $\mathbf{A}$  is HBS/HSS of rank  $k$ ). We assume that we can evaluate  $\mathbf{x} \mapsto \mathbf{Ax}$  and  $\mathbf{x} \mapsto \mathbf{A}^*\mathbf{x}$  fast.

**Objective:** Construct thin matrices  $\Omega$  and  $\Psi$  such that  $\mathbf{A}$  can be completely reconstructed in  $O(N)$  work from the set  $\{\mathbf{Y}, \Omega, \mathbf{Z}, \Psi\}$  where  $\mathbf{Y} = \mathbf{A}\Omega$  and  $\mathbf{Z} = \mathbf{A}^*\Psi$ ?

*Sample the column space of the matrix:*

$$\begin{array}{ccc} \mathbf{Y} & = & \mathbf{A} \\ \text{You see this} & & ? \\ \hline & & \Omega \\ & & \text{Pick this matrix} \end{array}$$

*If  $\mathbf{A} \neq \mathbf{A}^*$ , then sample the row space too:*

$$\begin{array}{ccc} \mathbf{Z}^* & = & \mathbf{A} \\ \text{YOU SEE THIS} & & ? \\ \hline & & \Psi^* \\ & & \text{PICK THIS MATRIX} \end{array}$$

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**The low rank case:** In the particularly simple case where  $\mathbf{A}$  has *global* rank  $k$ , we revert to the case we considered in the first part of the talk.

In the current framework, the randomized SVD takes the form:

- Set  $s = k$  and draw a “test matrix”  $\Omega \in \mathbb{R}^{N \times s}$  from a Gaussian distribution.
- Form the “sample matrix”  $\mathbf{Y} = \mathbf{A}\Omega$ .
- Build  $\Psi$  to hold an ON basis for  $\text{ran}(\mathbf{Y})$ , e.g.,  $[\Psi, \sim] = \text{qr}(\mathbf{Y}, 0)$ .
- Form  $\mathbf{Z} = \mathbf{A}^*\Psi$ .

Then  $\mathbf{A} = \Psi (\Psi^*\mathbf{A}) = \Psi \mathbf{Z}^*$  with probability 1.

In the more typical case where  $\mathbf{A}$  is only *approximately* of rank  $k$ , some *oversampling* is required to get a reliable scheme. (Say  $s = k + 10$ , or  $s = 2k$ , or some such.)

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**Why generalize from “global low rank” to “rank structured”:**

- Integral operators from classical physics. If you have a legacy method for the matrix-vector multiple (e.g. the Fast Multipole Method), then we could enable a range of operations – LU factorization, matrix inversion, etc.
- Multiplication of operators. Useful for forming Dirichlet-to-Neumann operators, for combining solvers of multi-physics problems, etc.
- Compression of Schur complements that arise in the LU or Cholesky factorization of sparse matrices. This lets us overcome key bottlenecks (e.g. LU factorization of a “finite element” matrix is accelerated from  $O(N^2)$  to close to linear complexity.)

## Approximation of rank structured matrices

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### Available techniques for the rank structured case:

For the most general structured matrix formats (e.g.  $\mathcal{H}$ -matrices), the problem has been solved in principle, and close to linear complexity algorithms exist:

- L. Lin, J. Lu, L. Ying, *Fast construction of hierarchical matrix representation from matrix-vector multiplication*, JCP 2011.
- P.G. Martinsson, SISC, **38**(4), pp. A1959-A1986, 2016.

However, existing methods require  $\sim k \log(N)$  matvecs, and do not have great practical speed. For instance, as dimension  $d$  increases, the bound on flops has an  $8^d$  factor . . .

Recently proposed algorithms have reduced the pre-factors by constructing bespoke random matrices that are designed to be optimal for any given tessellation pattern. The key technical idea is to formulate admissibility criteria that form a graph, and then exploit powerful graph coloring algorithms. This technique also enables compression of kernel matrices that arise in ML. [J. Levitt & P.G. Martinsson, arxiv arXiv: arXiv:2205.03406, 2022.]

## Approximation of rank structured matrices

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### Available techniques for the rank structured case:

The good news is that in the context of *numerical PDEs*, more specialized rank structured formats are often sufficient — hierarchically semi-separable matrices, hierarchically block-separable matrices, “ $\mathcal{H}$ -matrices with weak admissibility”, etc.

For these matrices, algorithms with true linear complexity and high practical speed exist. First generation algorithms were not fully black box, as they required the ability to evaluate a small number of matrix entries explicitly.

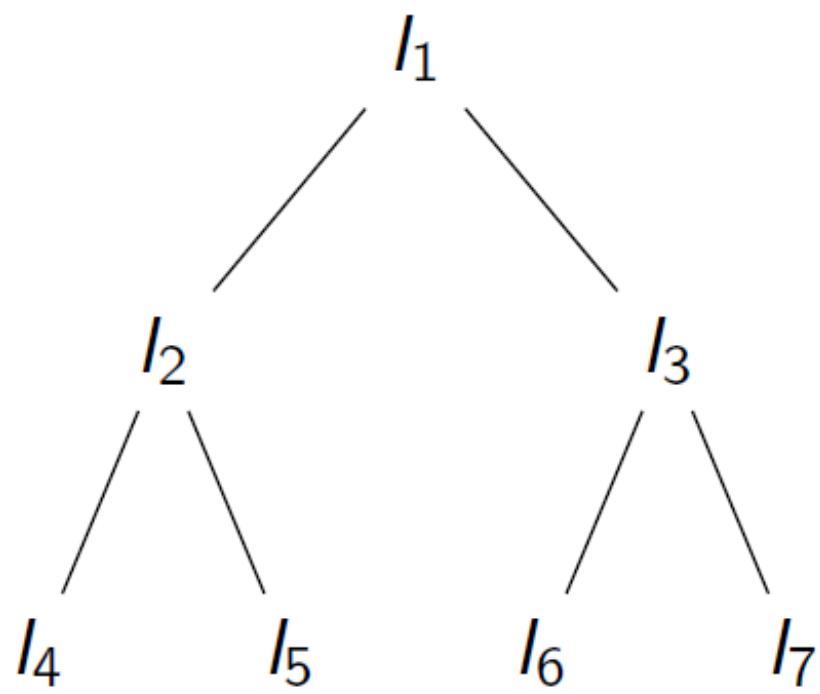
- P.G. Martinsson, *SIAM J. Matrix Anal. Appl.* **32**(4), 2011.
- Later improvements by Jianlin Xia, Sherry Li, and others. Widely used.

However, a fully black box algorithm with true linear complexity and high practical speed is now available:

- J. Levitt & P.G. Martinsson, arxiv arXiv:2205.02990, 2022.

## Approximation of rank-structured matrices – A binary tree structure

An example binary tree structure for a matrix of size  $400 \times 400$ . The levels of the tree represent successively refined partitions of the index vector  $[1, \dots, 400]$ .



Level 0:  $I_1 = [1, 2, \dots, 400]$

Level 1:

$I_2 = [1, \dots, 200], I_3 = [201, \dots, 400]$

Level 2:

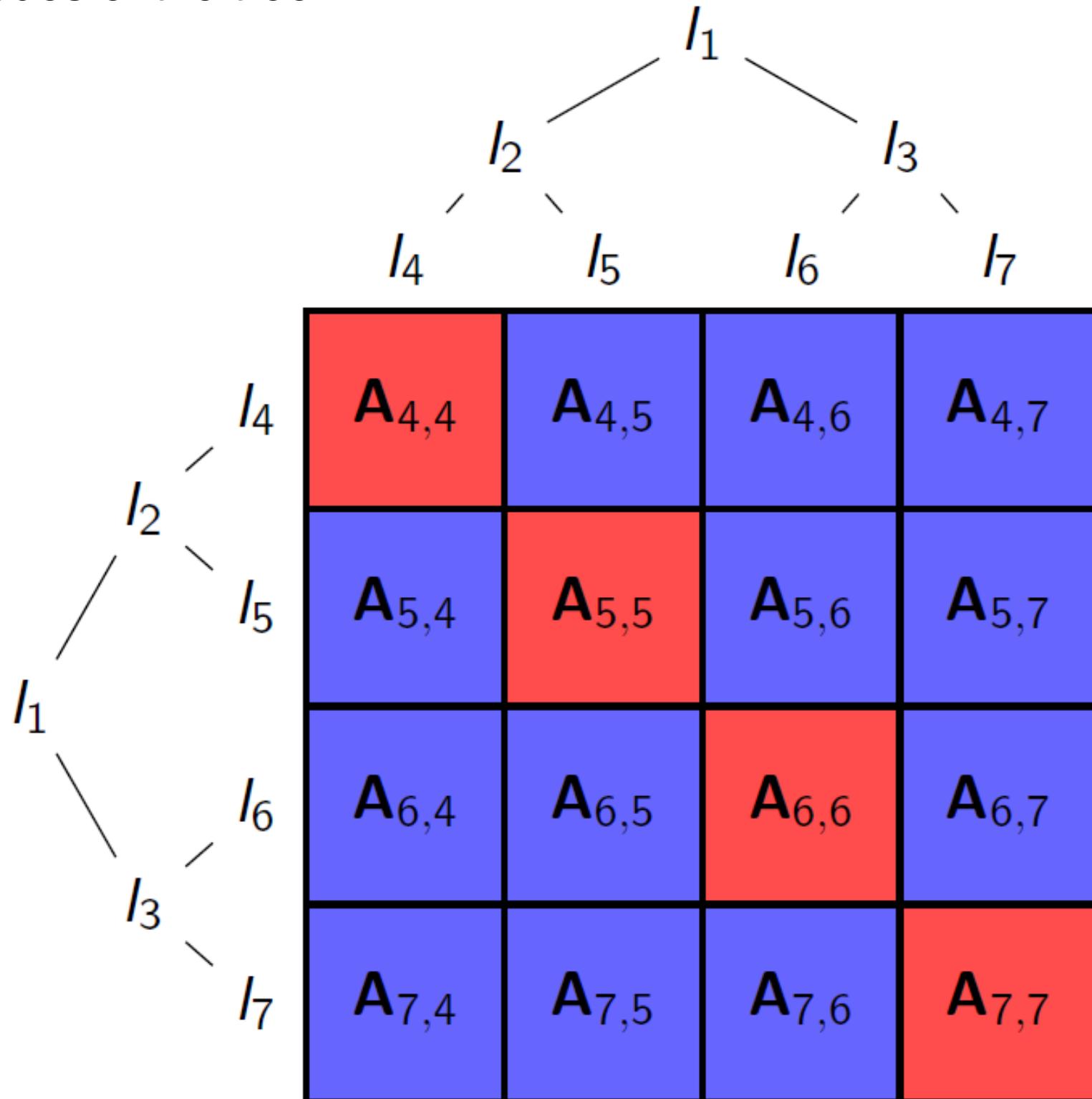
$I_4 = [1, \dots, 100], I_5 = [101, \dots, 200],$   
 $I_6 = [201, \dots, 300], I_7 = [301, \dots, 400]$

Let  $m$  denote the leaf node size.

Let  $L \approx \log(N/m)$  denote the depth of the tree.

## Approximation of rank-structured matrices – HBS (a.k.a. HSS) structure

Consider the following tessellation of a matrix, where each block represents interactions between two leaf nodes of the tree.



## Approximation of rank-structured matrices – HBS (a.k.a. HSS) structure

HBS requirements for the finest level: for every leaf node  $\tau$ , there must exist basis matrices  $\mathbf{U}_\tau$  and  $\mathbf{V}_\tau$  such that for every leaf node  $\tau' \neq \tau$ , we have

$$\mathbf{A}_{\tau,\tau'} = \mathbf{U}_\tau \tilde{\mathbf{A}}_{\tau,\tau'} \mathbf{V}_{\tau'}^*.$$

$m \times m \quad m \times k \quad k \times k \quad k \times m$

$\mathbf{A}_{4,4}$	$\mathbf{A}_{4,5}$	$\mathbf{A}_{4,6}$	$\mathbf{A}_{4,7}$
$\mathbf{A}_{5,4}$	$\mathbf{A}_{5,5}$	$\mathbf{A}_{5,6}$	$\mathbf{A}_{5,7}$
$\mathbf{A}_{6,4}$	$\mathbf{A}_{6,5}$	$\mathbf{A}_{6,6}$	$\mathbf{A}_{6,7}$
$\mathbf{A}_{7,4}$	$\mathbf{A}_{7,5}$	$\mathbf{A}_{7,6}$	$\mathbf{A}_{7,7}$

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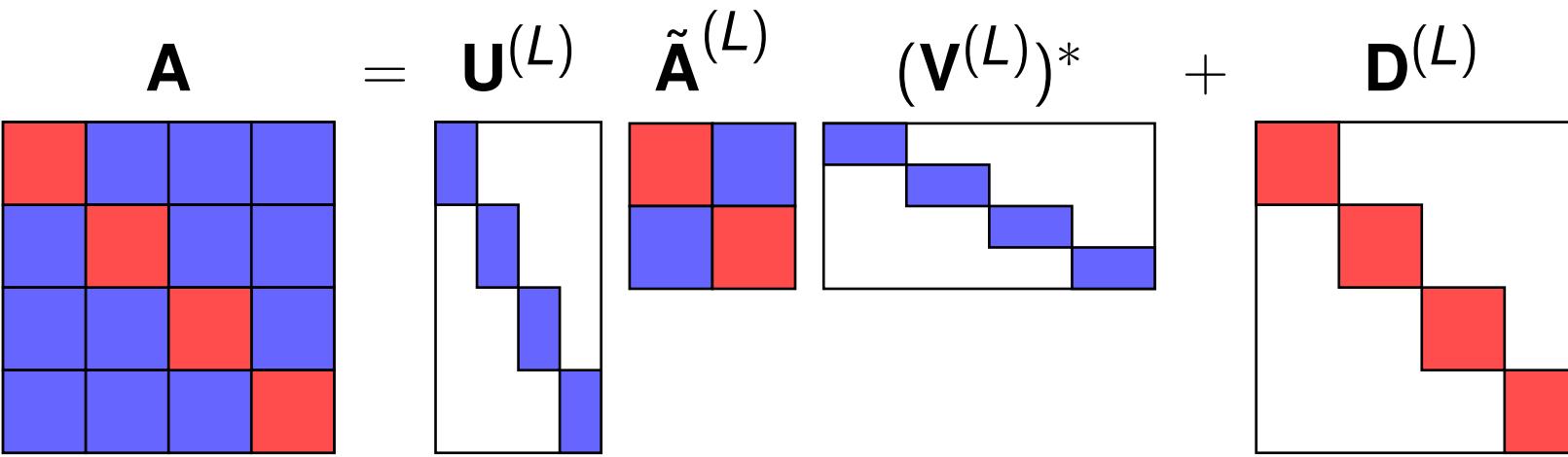
$$\mathbf{A}_{\tau,\tau'} = \mathbf{U}_\tau \tilde{\mathbf{A}}_{\tau,\tau'} \mathbf{V}_{\tau'}^*. \\ m \times m \quad m \times k \quad k \times k \quad k \times m$$

$\mathbf{A}_{4,4}$	$\mathbf{A}_{4,5}$	$\mathbf{A}_{4,6}$	$\mathbf{A}_{4,7}$
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$\mathbf{A}_{7,4}$	$\mathbf{A}_{7,5}$	$\mathbf{A}_{7,6}$	$\mathbf{A}_{7,7}$

The on-diagonal blocks are not assumed to be low-rank, and pose the main challenge for black-box compression.

## Approximation of rank-structured matrices – Telescoping factorization

This leads to a factorization of  $\mathbf{A}$ .

$$\mathbf{A} = \mathbf{U}^{(L)} \tilde{\mathbf{A}}^{(L)} (\mathbf{V}^{(L)})^* + \mathbf{D}^{(L)}$$


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$\tilde{\mathbf{A}}^{(L)}$  is also an HBS matrix, and it can be factorized similarly, leading to a *telescoping factorization*.

For example, a factorization of an HBS matrix with a tree of depth  $L = 3$  takes the form

$$\mathbf{A} = \mathbf{U}^{(3)}(\mathbf{U}^{(2)}(\mathbf{U}^{(1)}\mathbf{D}^{(0)}(\mathbf{V}^{(1)})^* + \mathbf{D}^{(1)})(\mathbf{V}^{(2)})^* + \mathbf{D}^{(2)})(\mathbf{V}^{(3)})^* + \mathbf{D}^{(3)}.$$

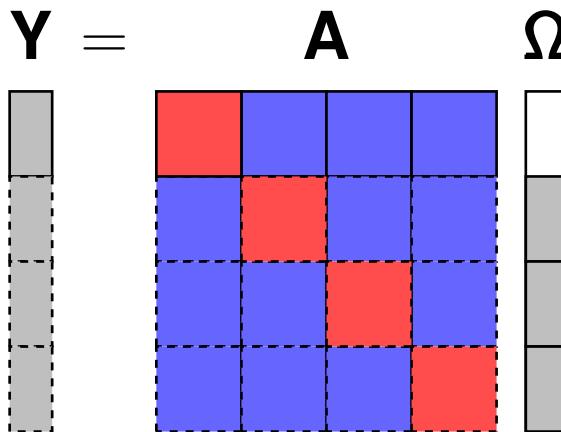
## Approximation of rank-structured matrices – A naive approach

Consider the task of finding basis matrix  $\mathbf{U}_4$  for node 4 using randomized sampling. We seek a sample of  $\mathbf{A}(I_4, I_4^C)$ , the HBS row block of node 4.

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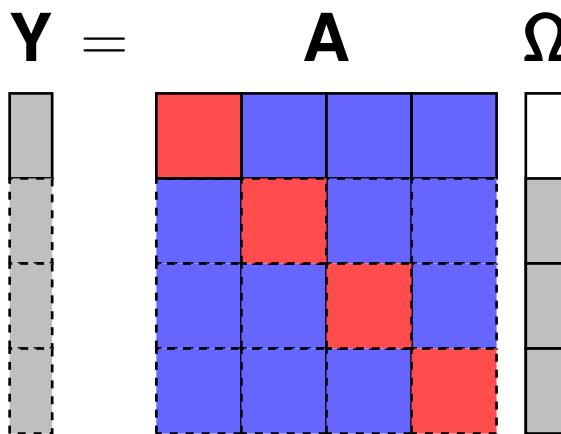
The naive approach is to sample with a random matrix  $\Omega \in \mathbb{R}^{N \times r}$ ,  $r = k + 10$ , that has a block of zeros in rows indexed by  $I_4$ . Then  $\mathbf{Y}(I_4, :)$  will contain a sample of  $\mathbf{A}(I_4, I_4^C)$ .



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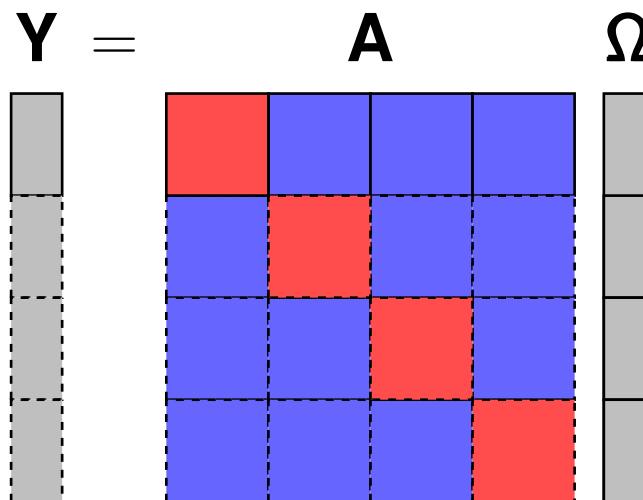
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This scheme requires taking a separate set of  $r$  samples for each leaf node, for a total of  $\sim rN/m$  samples. There is a lot of wasted information in  $\mathbf{Y}$ .

## Approximation of rank-structured matrices – the “not quite black box” approach

Sample  $\mathbf{A}$  with a completely dense random matrix  $\Omega \in \mathbb{R}^{N \times r}$ .



Afterwards, subtract unwanted contributions back out of  $\mathbf{Y}$ .

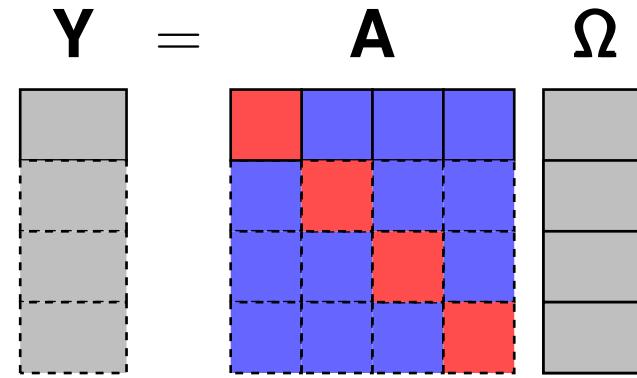
Subtracting the contribution of  $\mathbf{A}_{4,4}$  gives the desired sample of  $\mathbf{A}(I_4, I_4^c)$ ,

$$\mathbf{Y}(I_4, :) - \mathbf{A}_{4,4}\Omega(I_4, : ).$$

This scheme requires only  $r$  samples in total, but it also requires direct access to a small number of entries of  $\mathbf{A}$ .

## Approximation of rank-structured matrices – Finding U

Sample  $\mathbf{A}$  with a completely dense random matrix  $\Omega \in \mathbb{R}^{N \times (r+m)}$ , where  $m$  is the leaf node size.

$$\mathbf{Y} = \mathbf{A} \Omega$$


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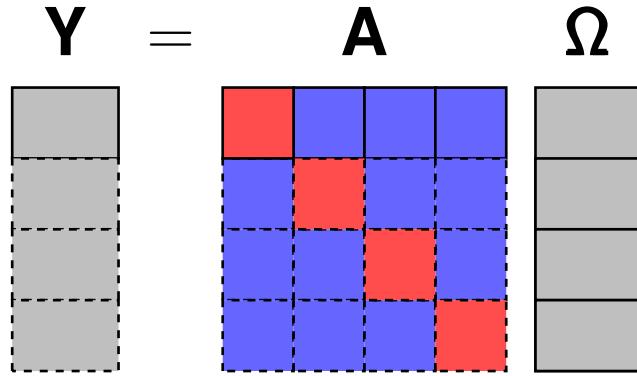
Since  $\Omega(I_4, :)$  is of size  $m \times (r + m)$ , it has a nullspace of dimension at least  $r$ . Let

$$\mathbf{Q}_4 = \text{nullspace}(\Omega(I_4, :), r)$$

be an  $(r + m) \times r$  orthonormal basis of the nullspace of  $\Omega(I_4, :)$ .

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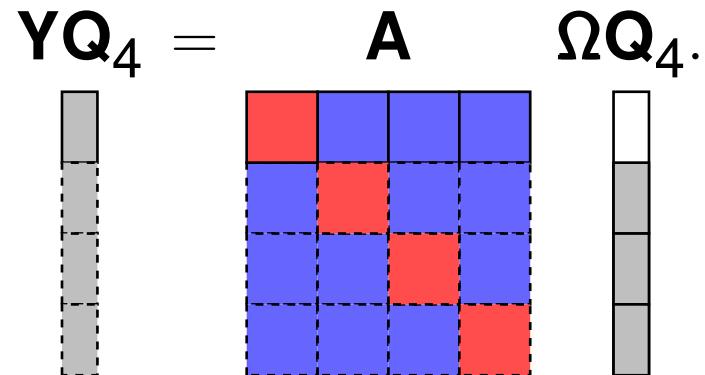
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Then

$$\mathbf{YQ}_4 = \mathbf{A} \quad \Omega\mathbf{Q}_4.$$


## Approximation of rank-structured matrices – Finding U

$$\mathbf{Y} = \mathbf{A} \Omega$$

$$\mathbf{Q}_4 = \text{nullspace}(\Omega(I_4, :), r)$$

$$\mathbf{YQ}_4 = \mathbf{A} \Omega \mathbf{Q}_4$$

We find the sample by multiplying  $\mathbf{Y}(I_4, :) \mathbf{Q}_4$ .

Orthonormalizing the sample gives basis matrix  $\mathbf{U}_4$ ,

$$\mathbf{U}_4 = \text{qr}(\mathbf{Y}(I_4, :) \mathbf{Q}_4).$$

## Approximation of rank-structured matrices – Finding $\mathbf{U}$

- For each leaf node  $\tau$ , we compute

$$\mathbf{Q}_\tau = \text{nullspace}(\Omega(I_\tau, :), r)$$

$$\mathbf{U}_\tau = \text{qr}(\mathbf{Y}(I_\tau, :) \mathbf{Q}_\tau).$$

- $\mathbf{U}_\tau$  only depends on  $\Omega(I_\tau, :)$  and  $\mathbf{Y}(I_\tau, :)$ .
- We only need  $r + m$  samples to find  $\mathbf{U}_\tau$  for every leaf node  $\tau$ .
- $\Omega \mathbf{Q}_\tau$  is a Gaussian random matrix, except for the block intentionally zeroed out.

# Approximation of rank-structured matrices – Compression overview

Recall the telescoping factorization  $\mathbf{A} = \mathbf{U}^{(L)} \tilde{\mathbf{A}}^{(L)} (\mathbf{V}^{(L)})^* + \mathbf{D}^{(L)}$ .

Steps:

1. Find  $\mathbf{U}^{(L)}, \mathbf{V}^{(L)}$ .

2. Find  $\mathbf{D}^{(L)}$ .

3. Compress  $\tilde{\mathbf{A}}^{(L)}$

recursively.

*Compute randomized samples of  $\mathbf{A}$  and  $\mathbf{A}^*$ .*

1: Form Gaussian random random matrices  $\Omega$  and  $\Psi$  of size  $N \times s$ .

2: Multiply  $\mathbf{Y} = \mathbf{A}\Omega$  and  $\mathbf{Z} = \mathbf{A}^*\Psi$ .

*Compress level by level from finest to coarsest.*

3: **for** level  $\ell = L, L-1, \dots, 0$  **do**

4:     **for** node  $\tau$  in level  $\ell$  **do**

5:         **if**  $\tau$  is a leaf node **then**

$$\Omega_\tau = \Omega(I_\tau, :), \quad \Psi_\tau = \Psi(I_\tau, :)$$

6:

$$\mathbf{Y}_\tau = \mathbf{Y}(I_\tau, :), \quad \mathbf{Z}_\tau = \mathbf{Z}(I_\tau, :)$$

7:         **else**

8:             Let  $\alpha$  and  $\beta$  denote the children of  $\tau$ .

$$\Omega_\tau = \begin{bmatrix} \mathbf{V}_\alpha^* \Omega_\alpha \\ \mathbf{V}_\beta^* \Omega_\beta \end{bmatrix}, \quad \Psi_\tau = \begin{bmatrix} \mathbf{U}_\alpha^* \Psi_\alpha \\ \mathbf{U}_\beta^* \Psi_\beta \end{bmatrix}$$

$$9: \quad \mathbf{Y}_\tau = \begin{bmatrix} \mathbf{U}_\alpha^* (\mathbf{Y}_\alpha - \mathbf{D}_\alpha \Omega_\alpha) \\ \mathbf{U}_\beta^* (\mathbf{Y}_\beta - \mathbf{D}_\beta \Omega_\beta) \end{bmatrix}, \quad \mathbf{Z}_\tau = \begin{bmatrix} \mathbf{V}_\alpha^* (\mathbf{Z}_\alpha - \mathbf{D}_\alpha^* \Psi_\alpha) \\ \mathbf{V}_\beta^* (\mathbf{Z}_\beta - \mathbf{D}_\beta^* \Psi_\beta) \end{bmatrix}$$

10:         **if** level  $\ell > 0$  **then**

$$11: \quad \mathbf{Q}_\tau = \text{nullspace}(\Omega_\tau, r), \quad \mathbf{P}_\tau = \text{nullspace}(\Psi_\tau, r)$$

$$11: \quad \mathbf{U}_\tau = \text{qr}(\mathbf{Y}_\tau \mathbf{Q}_\tau, r), \quad \mathbf{V}_\tau = \text{qr}(\mathbf{Z}_\tau \mathbf{P}_\tau, r)$$

$$12: \quad \mathbf{D}_\tau = (\mathbf{I} - \mathbf{U}_\tau \mathbf{U}_\tau^*) \mathbf{Y}_\tau \Omega_\tau^\dagger + \mathbf{U}_\tau \mathbf{U}_\tau^* ((\mathbf{I} - \mathbf{V}_\tau \mathbf{V}_\tau^*) \mathbf{Z}_\tau \Psi_\tau^\dagger)^*$$

13:         **else**

$$14: \quad \mathbf{D}_\tau = \mathbf{Y}_\tau \Omega_\tau^\dagger$$

## Approximation of rank-structured matrices – Finding D

From the telescoping factorization

$$\mathbf{A} = \mathbf{U}^{(L)} \tilde{\mathbf{A}}^{(L)} (\mathbf{V}^{(L)})^* + \mathbf{D}^{(L)}$$

we define  $\tilde{\mathbf{A}}^{(L)}$  and  $\mathbf{D}^{(L)}$  as follows.

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$$\mathbf{A} = \mathbf{U}^{(L)} \overbrace{(\mathbf{U}^{(L)})^* \mathbf{A} \mathbf{V}^{(L)}}^{\tilde{\mathbf{A}}^{(L)}} (\mathbf{V}^{(L)})^* + \overbrace{\mathbf{A} - \mathbf{U}^{(L)} (\mathbf{U}^{(L)})^* \mathbf{A} \mathbf{V}^{(L)} (\mathbf{V}^{(L)})^*}^{\mathbf{D}^{(L)}}$$

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Block  $\mathbf{D}_\tau$  of  $\mathbf{D}^{(L)}$  is given by

$$\begin{aligned}\mathbf{D}_\tau &= \mathbf{A}_{\tau,\tau} - \mathbf{U}_\tau \mathbf{U}_\tau^* \mathbf{A}_{\tau,\tau} \mathbf{V}_\tau \mathbf{V}_\tau^* \\ &= \dots \\ &= (\mathbf{I} - \mathbf{U}_\tau \mathbf{U}_\tau^*) \mathbf{Y}_\tau \Omega_\tau^\dagger + \mathbf{U}_\tau \mathbf{U}_\tau^* \left( (\mathbf{I} - \mathbf{V}_\tau \mathbf{V}_\tau^*) \mathbf{Z}_\tau \Psi_\tau^\dagger \right)^*\end{aligned}$$

## Approximation of rank-structured matrices – Compressing $\tilde{\mathbf{A}}^{(L)}$

To compute randomized samples of  $\tilde{\mathbf{A}}^{(L)}$ , we multiply the telescoping factorization with  $\Omega$  to obtain

$$\mathbf{Y} = \mathbf{A}\Omega = (\mathbf{U}^{(L)}\tilde{\mathbf{A}}^{(L)}(\mathbf{V}^{(L)})^* + \mathbf{D}^{(L)})\Omega,$$

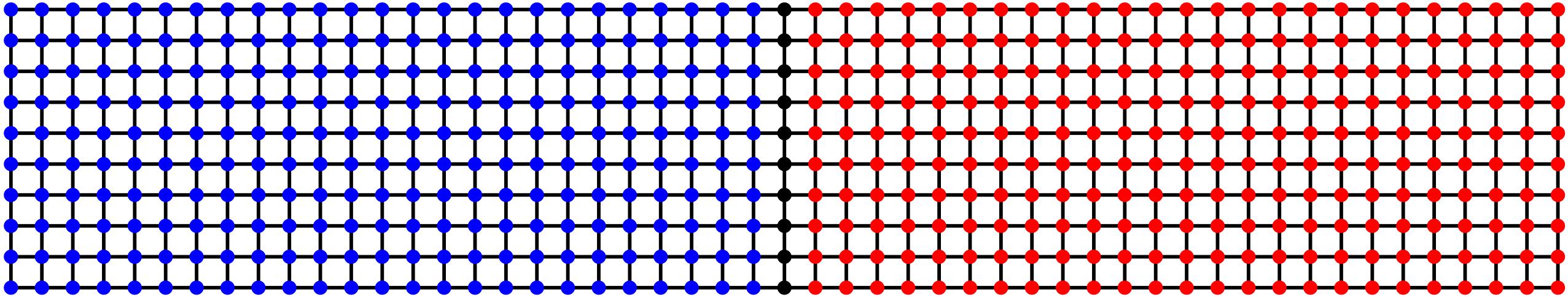
and rearrange to obtain

$$\underbrace{(\mathbf{U}^{(L)})^*(\mathbf{Y} - \mathbf{D}^{(L)}\Omega)}_{\text{sample matrix}} = \tilde{\mathbf{A}}^{(L)} \underbrace{(\mathbf{V}^{(L)})^*\Omega}_{\text{test matrix}}.$$

## Approximation of rank-structured matrices: Sparse LU

Let  $\mathbf{C}$  be the stiffness matrix for the standard five-point stencil finite difference approximation to the Poisson equation on a rectangular grid.

We partition the grid as shown and tessellate  $\mathbf{C}$  accordingly.



$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{0} & \mathbf{C}_{13} \\ \mathbf{0} & \mathbf{C}_{22} & \mathbf{C}_{23} \\ \mathbf{C}_{31} & \mathbf{C}_{32} & \mathbf{C}_{33} \end{bmatrix}$$

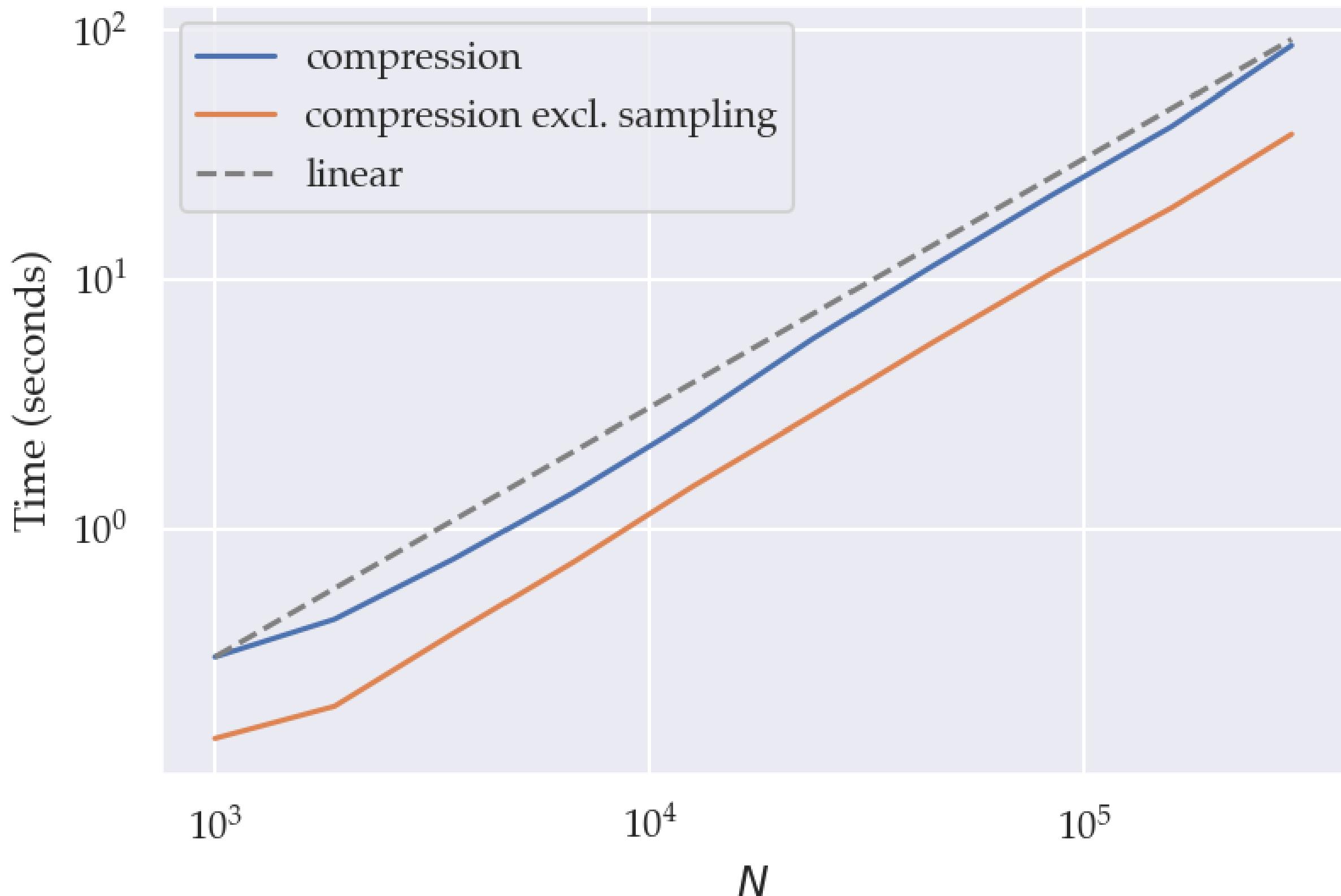
The matrix we seek to compress is the Schur complement

$$\mathbf{A} = \mathbf{C}_{33} - \mathbf{C}_{31}\mathbf{C}_{11}^{-1}\mathbf{C}_{31} - \mathbf{C}_{32}\mathbf{C}_{22}^{-1}\mathbf{C}_{23}.$$

# Approximation of rank-structured matrices: Sparse LU

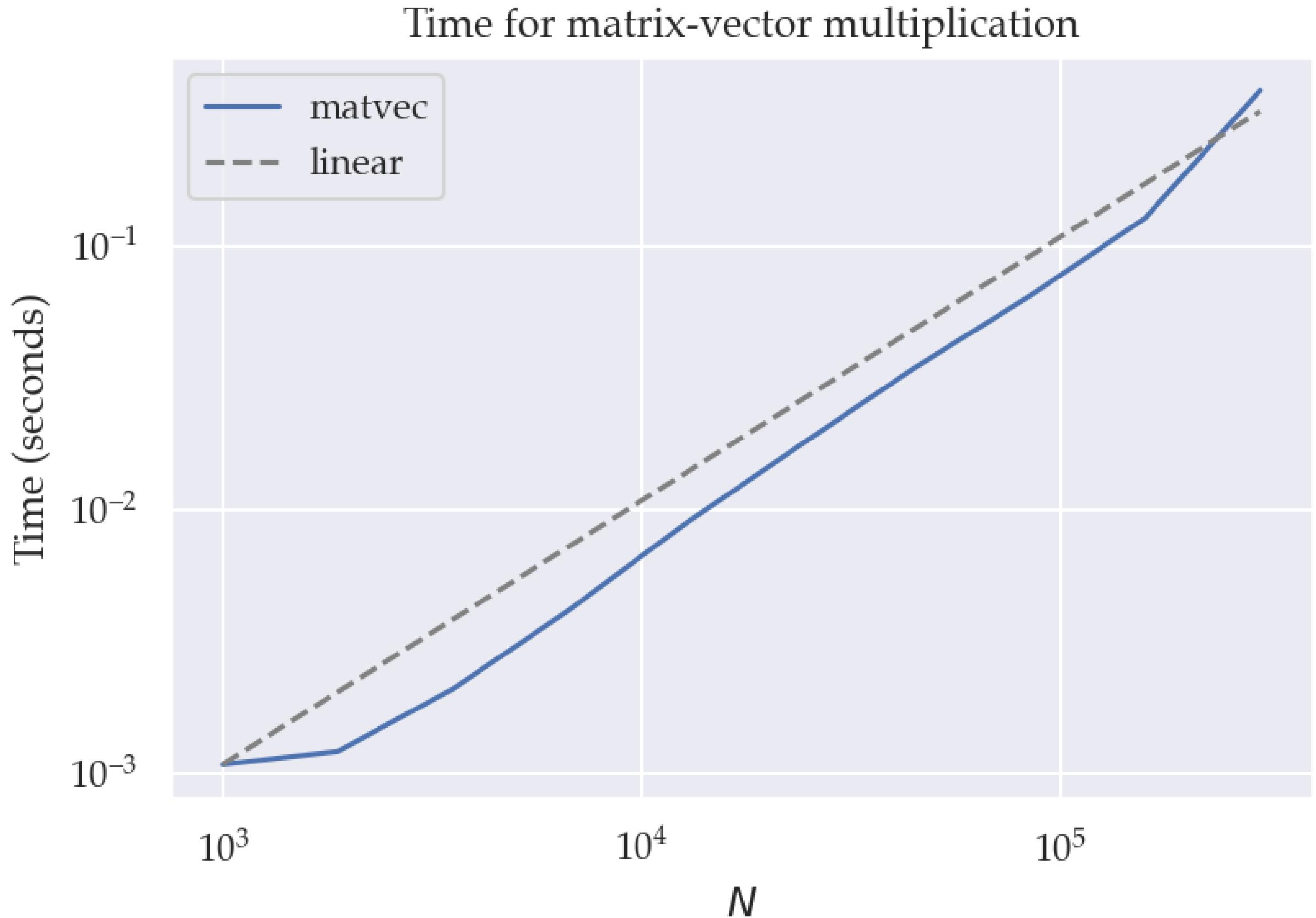
$r = 30, m = 60$

Time for matrix compression



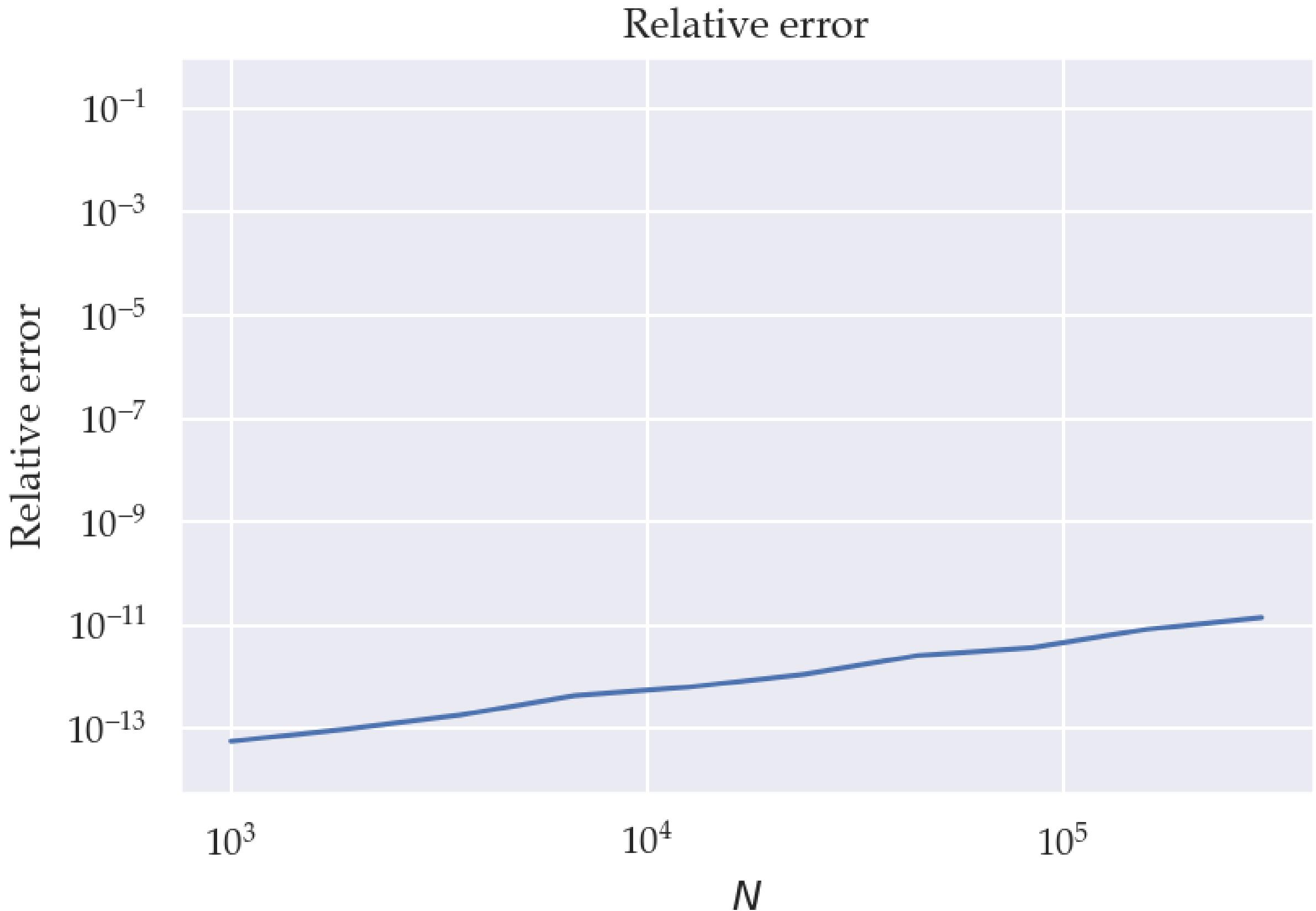
# Approximation of rank-structured matrices: Sparse LU

$r = 30, m = 60$



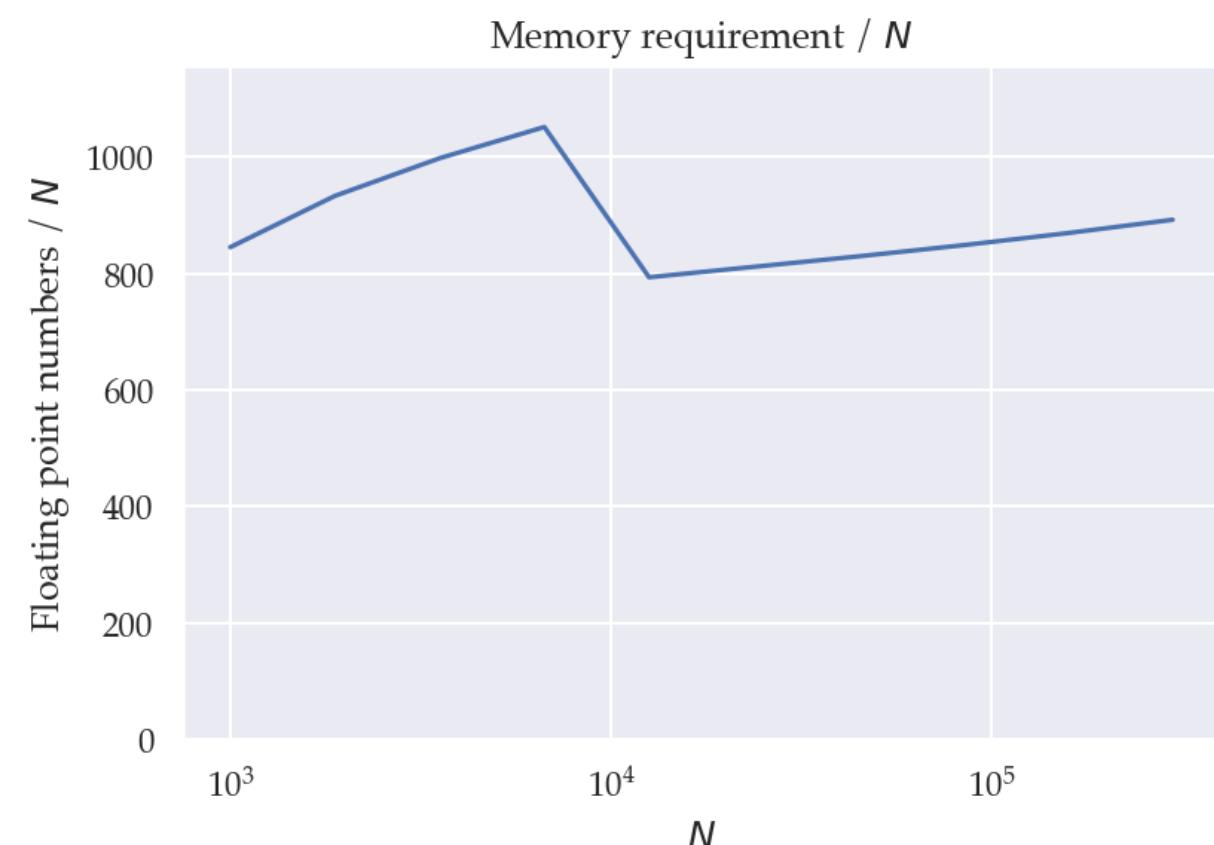
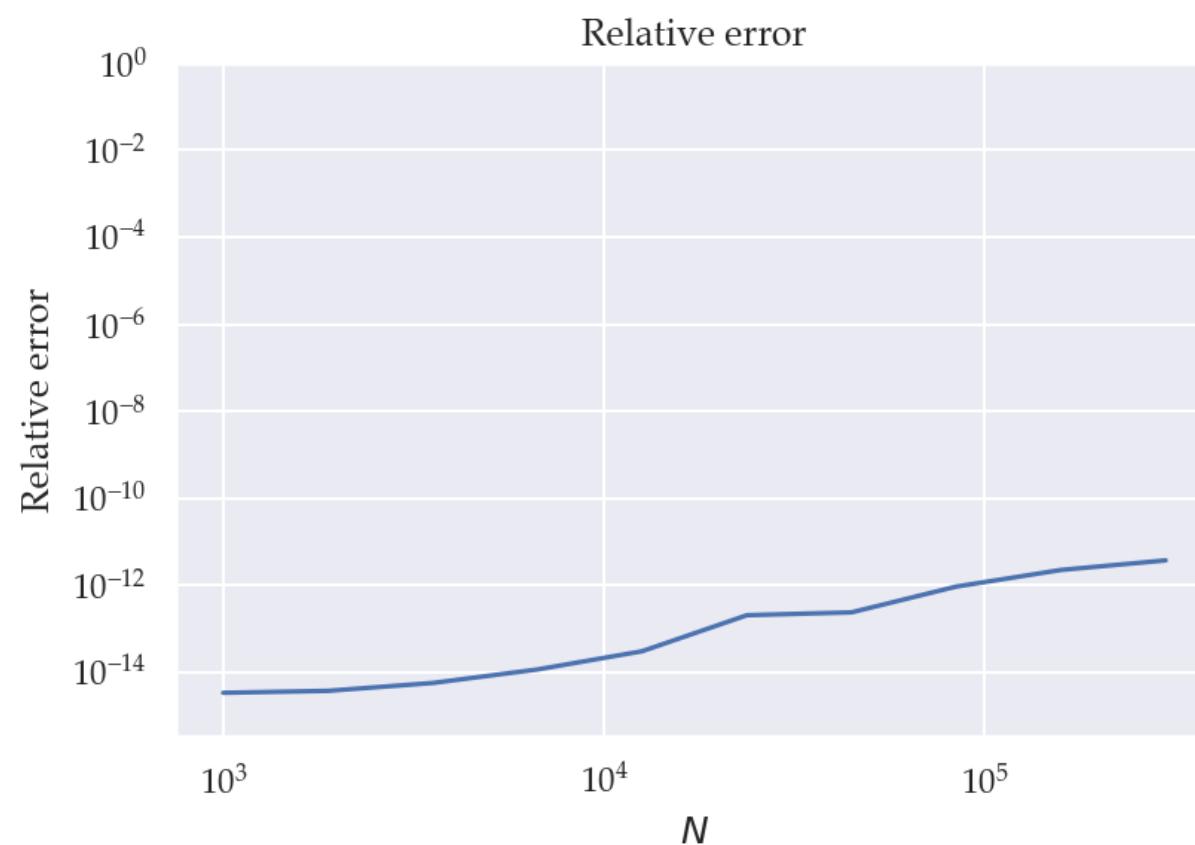
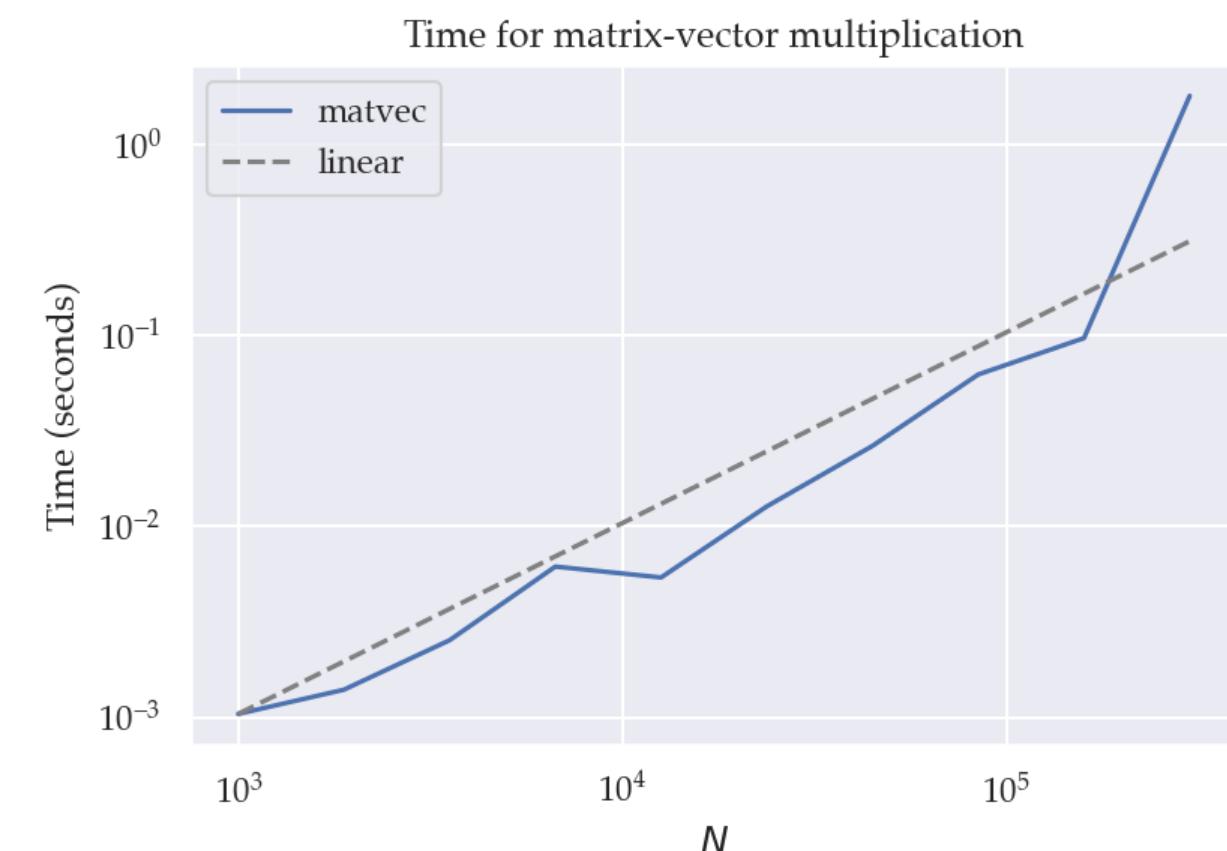
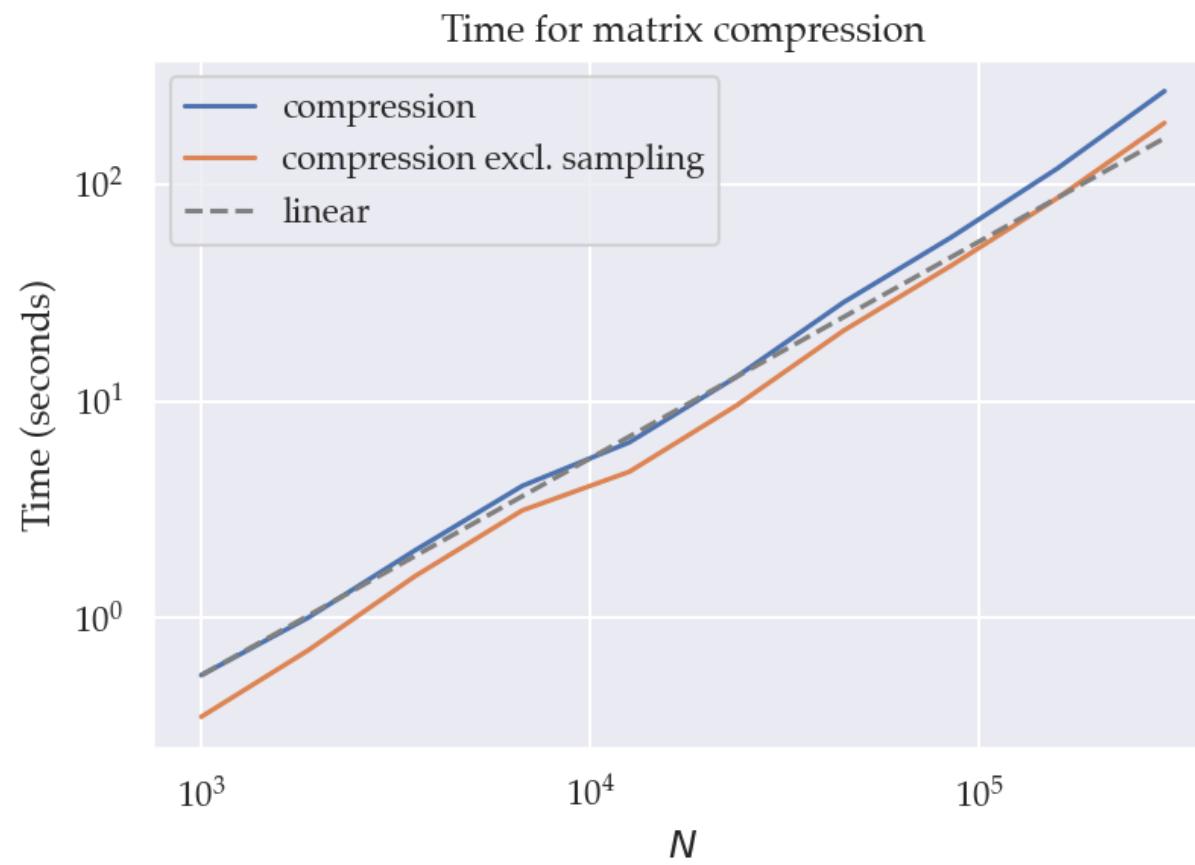
# Approximation of rank-structured matrices: Sparse LU

$r = 30, m = 60$



# Approximation of rank-structured matrices: FMM

$r = 50, m = 100$



## Approximation of rank-structured matrices: Key points

- Fully “black box”. Interacts with  $\mathbf{A}$  only via the matvec.
- True linear complexity. Requires only  $O(k)$  samples from  $\mathbf{A}$  and  $\mathbf{A}^*$ .  
Much faster in practice than existing black box algorithms.  
*(However, prefactor in # samples is slightly suboptimal – unlike Townsend/Halikias.)*
- Ideal tool for acceleration of sparse direct solvers.

## Key points on randomized singular value decomposition (RSVD):

- High practical speed — interacts with  $\mathbf{A}$  only through matrix-matrix multiplication.
- Highly *communication efficient*.  
Acceleration of classical algorithms such as column pivoted QR.  
Particularly efficient for GPUs, out-of-core computing, distributed memory, etc.
- Reduction in complexity from  $O(mnk)$  to  $O(mn\log k)$  or even less via *structured random embeddings*.
- Single pass algorithms have been developed for *streaming environments*.  
*Not possible with deterministic methods!*

## Current research directions:

- High performance implementations.
- Faster algorithms for computing *full* matrix factorizations.
- Solvers for linear systems and for least squares problems.
- Compression of continuum operators and rank structured matrices.
- Applications of randomized embeddings outside of linear algebra: Faster nearest neighbor search, faster clustering algorithms, data compression on the fly, etc.

## Surveys:

- P.G. Martinsson and J. Tropp, “Randomized Numerical Linear Algebra: Foundations & Algorithms”.  
*Acta Numerica*, 2020. (Arxiv report 2002.01387)  
Long survey summarizing major findings in the field in the past decade.
- P.G. Martinsson, “Randomized methods for matrix computations.” *The Mathematics of Data*, IAS/Park City Mathematics Series, 25(4), pp. 187 - 231, 2018.  
Book chapter that is written to be accessible to a broad audience. Focused on practical aspects.
- N. Halko, P.G. Martinsson, J. Tropp, “Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions.” *SIAM Review*, 53(2), 2011, pp. 217-288.  
Survey that describes the randomized SVD and its variations.

## Tutorials, summer schools, etc:

- 2020: 3 lecture mini course on randomized linear algebra, KTH, Stockholm. Videos available.
- 2016: Park City Math Institute (IAS): *The Mathematics of Data*.
- 2014: CBMS summer school at Dartmouth College. 10 lectures on YouTube.
- 2009: NIPS tutorial lecture, Vancouver, 2009. Online video available.

## Software:

- ID: <http://tygert.com/software.html> (ID, SRFT, CPQR, etc)
- RSVDPACK: <https://github.com/sergeyvoronin> (RSVD, randomized ID and CUR)
- HQRRP: <https://github.com/flame/hqrrp/> (LAPACK compatible randomized CPQR)
- Randomized UTV: <https://github.com/flame/randutv>

**DOE report on randomized algorithms:** <https://arxiv.org/abs/2104.11079> (2021)