Transmit Beamforming for Multiuser Downlink with Per-Antenna Power Constraints

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***Abstract*—We consider the transmit beamforming design for a multi-user downlink with multiple transmit antennas at the base station. Different from the conventional sum-power constraint across the transmit antennas, we assume individual power constraints per antenna. Assuming that perfect channel state information (CSI) is available at the base station, we develop an efficient algorithm to find the optimal beamforming scheme for the classic max-min signal-to-interference-plus-noise ratio (SINR) problem based on solving a sequence of "dual" per-antenna power balancing problems as second-order cone programs. It is proven that the proposed algorithm can find the max-min SINR beamforming solution with guaranteed global optimality and fast convergence speed. Relying on robust optimization techniques, the approach is also generalized to obtain the robust beamforming design that maximizes the worst-case user SINR when the channel uncertainty is bounded by a spherical region. Numerical results are provided to demonstrate the merits of the proposed transmit beamforming schemes.**

Keywords: Multiuser downlink, per-antenna power constraints, transmit beamforming, second-order cone program, semidefinite program.

I. Introduction

For a multiuser wireless downlink with multiple transmit antennas at the base station, transmit beamforming is a low-complexity solution to provide spatial diversity and reduce multipath fading effects. The attention of classic beamforming problems was placed on achieving certain signal-to-interference-plus-noise ratio (SINR) targets with minimum total transmit-power, since an SINR higher than a prescribed value is not helpful in user-perceived quality-of-service for traditional voice services. To this end, optimal beamforming approaches were developed in [1]–[4] to maximize the minimum user SINR under a sum-power constraint across all transmit antennas.

In physical implementation of a multi-antenna base station, however, each antenna usually has its own power amplifier in analog front-end, and it is individually limited by the linearity of this amplifier. Hence, instead of the sum-power constraint, this leads to power constraints imposed on a per-antenna basis. Under such a more realistic power constraint, the transmitter optimization was addressed in [5], where the elegant uplink-downlink duality under a sum-power constraint was extended to downlink problems with per-antenna power constraints via a Lagrangian duality approach. Downlink beamforming designs under per-antenna power constraints were also addressed by another few works. Zero-forcing based beamforming scheme was derived to maximize the minimum user rate in [6]. Based on block coordinate ascend and signomial programming methods, beamforming designs were put forth to maximize the weighted sum rate for a multiuser downlink with power constraints per antenna groups in [7]. For a large antenna array at the base station, an achievable rate for single-user multiple-input-single-output (MISO) beamforming under per-antenna constant-envelop constraints was derived in [8]. With per-antenna and per-antenna array power constraints, the surrogate duality of the max-min beamforming was investigated in [9].

In this paper, we develop a novel approach to max-min SINR beamforming designs for a multi-user downlink with per-antenna power constraints. Assuming that perfect channel state information (CSI) is available at the base station, we show that the optimal max-min SINR beamforming scheme can be found by a one-dimension bisectional search building on the solutions to a sequence of min-max per-antenna power problems. Since such per-antenna power balancing problems can be efficiently solved by a second-order cone programming (SOCP) approach [5], the max-min SINR solution can be then obtained with guaranteed global optimality and fast convergence speed. Relying on the robust optimization techniques [10]–[14], our approach is then generalized to the imperfect CSI case where the channel uncertainty is bounded by a spherical region. It is shown that the robust beamforming design maximizing the worst-case user SINR can be also obtained by solving a sequence of per-antenna power balancing problems via semidefinite programming (SDP). Numerical results are presented to demonstrate the merits of the proposed beamforming solutions.

The rest of this paper is organized as follows. Section II presents the system model under consideration. Section III develops the proposed approach to optimal max-min SINR beamforming designs under per-antenna power constraints when perfect CSI is available. The approach is generalized to the imperfect CSI case in Section IV. The proposed schemes are tested and compared with existing alternatives in Section V, followed by the conclusion in Section VI.

II. Modeling Preliminaries

Consider a downlink scenario where a base station equipped with M antennas transmits independent signal s_k to single-

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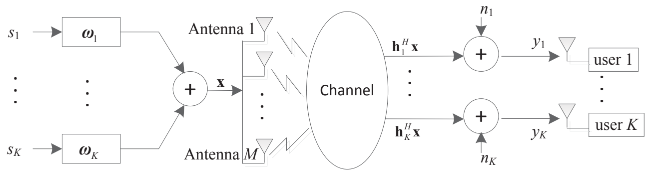

Fig. 1. The system model.

antenna user k, $k = 1, \ldots, K$; see Fig. 1. The transmitted signal at the base station is given by

$$\boldsymbol{x} = \sum_{k=1}^{K} \boldsymbol{w}_k s_k,$$

where $\boldsymbol{w}_k$ is the linear beamforming (or precoding) vector for user k, $k = 1, ..., K$. Without loss of generality (w.l.o.g.), assume that the information signal s_k has a unit power, i.e., $\mathbb{E}[|s_k|^2] = 1$, where $\mathbb{E}[\cdot]$ denotes the ensemble average. Suppose that each antenna has its own power constraint P_m, $m = 1, \ldots, M$. Given the beamforming vectors $\boldsymbol{w}_k$, $k = 1, \ldots, K$, such per-antenna power constraints correspond to:

$$\left[\sum_{k=1}^{K} \boldsymbol{w}_k \boldsymbol{w}_k^H\right]_{m,m} \le P_m, \quad m = 1, \ldots, M,$$

where $(\cdot)^H$ denotes the conjugate transpose of a vector or a matrix, and $[\cdot]_{m,m}$ denotes the (m,m)-entry of a matrix.

Let $\boldsymbol{h}_k$ denote the downlink channel vector of user k, and n_k denote the additive white Gaussian noise (AWGN) at user k with variance σ_k^2. The signal received at user k is:

$$y_k = \boldsymbol{h}_k^H \boldsymbol{x} + n_k, \quad k = 1, \ldots, K.$$

For notational convenience, let $\boldsymbol{W} := [\boldsymbol{w}_1, \ldots, \boldsymbol{w}_K]$. Then the received SINR at user k is given by

$$\mathrm{SINR}_k(\boldsymbol{W}) = \frac{|\boldsymbol{h}_k^H \boldsymbol{w}_k|^2}{\sum_{l \neq k, l=1}^{K} (|\boldsymbol{h}_k^H \boldsymbol{w}_l|^2) + \sigma_k^2}, \quad k = 1, \ldots, K. \tag{1}$$

III. Max-Min SINR Solution with Perfect CSI

Based on the SINRs in (1), consider the following classic SINR balancing problem [3]:

$$\begin{aligned} \lambda^* = & \max_{\boldsymbol{W}} \min_{1 \le k \le K} \frac{\mathrm{SINR}_k(\boldsymbol{W})}{\gamma_k} \\ \text{s. t.} \quad & \left[\sum_{k=1}^{K} \boldsymbol{w}_k \boldsymbol{w}_k^H\right]_{m,m} \le P_m, \quad m = 1, \ldots, M, \end{aligned} \tag{2}$$

where γ_k denotes the SINR target for user k, $k = 1, ..., K$.

The problem (2) is a non-convex program. Yet, we next show that it can be solved by alternatively solving a sequence of more tractable per-antenna power balancing problems. To this end, for a given $\lambda > 0$, consider the min-max per-antenna power problem:

$$\begin{aligned} \alpha^*(\lambda) = & \min_{\boldsymbol{W}} \max_{1 \le m \le M} \frac{\left[\sum_{k=1}^{K} \boldsymbol{w}_k \boldsymbol{w}_k^H\right]_{m,m}}{P_m} \\ \text{s. t.} \quad & \frac{\mathrm{SINR}_k(\boldsymbol{W})}{\gamma_k} \ge \lambda, \quad k = 1, \ldots, K, \end{aligned} \tag{3}$$

where the power budget P_m serves as a "power target" per antenna. Note that the optimal value $\alpha^*(\lambda)$ is in fact a function of the given $\lambda > 0$. For this function, we can establish the following property:

Lemma 1: $\alpha^*(\lambda)$ is a strictly increasing function for $\lambda > 0$.

Proof: Let $\check{\boldsymbol{W}} = [\check{\boldsymbol{w}}_1, \ldots, \check{\boldsymbol{w}}_K]$ denote the optimal solution for (3) with a given $\lambda > 0$, such that

$$\alpha^*(\lambda) = \max_{1 \le m \le M} \frac{\left[\sum_{k=1}^{K} \check{\boldsymbol{w}}_k \check{\boldsymbol{w}}_k^H\right]_{m,m}}{P_m}.$$

For another $0 < \lambda' < \lambda$, let $\beta = \lambda'/\lambda$. It is clear: $0 < \beta < 1$. We can show that $\sqrt{\beta}\check{\boldsymbol{W}}$ is in the feasible set of (3) with λ' as the minimum $\frac{\mathrm{SINR}_k(\boldsymbol{W})}{\gamma_k}$ requirement. This is because: $\forall k$,

$$\begin{aligned} \frac{\mathrm{SINR}_k(\sqrt{\beta}\check{\boldsymbol{W}})}{\gamma_k} &= \frac{\beta |\boldsymbol{h}_k^H \check{\boldsymbol{w}}_k|^2}{\sum_{l \neq k, l=1}^{K} (\beta |\boldsymbol{h}_k^H \check{\boldsymbol{w}}_l|^2) + \sigma_k^2} \\ &\ge \frac{\beta |\boldsymbol{h}_k^H \check{\boldsymbol{w}}_k|^2}{\sum_{l \neq k, l=1}^{K} (|\boldsymbol{h}_k^H \check{\boldsymbol{w}}_l|^2) + \sigma_k^2} \ge \beta\lambda = \lambda'. \end{aligned}$$

On the other hand, we have:

$$\begin{aligned} & \max_{1 \le m \le M} \frac{\left[\sum_{k=1}^{K} (\sqrt{\beta}\check{\boldsymbol{w}}_k \sqrt{\beta}\check{\boldsymbol{w}}_k^H)\right]_{m,m}}{P_m} \\ & = \beta \max_{1 \le m \le M} \frac{\left[\sum_{k=1}^{K} (\check{\boldsymbol{w}}_k \check{\boldsymbol{w}}_k^H)\right]_{m,m}}{P_m} = \beta\alpha^*(\lambda). \end{aligned}$$

Therefore, we must have: $\alpha^*(\lambda') \le \beta\alpha^*(\lambda)$. It is easy to see that $\alpha^*(\lambda) > 0$ for $\lambda > 0$. It in turn implies that $\alpha^*(\lambda') \le \beta\alpha^*(\lambda) < \alpha^*(\lambda)$ for any $0 < \lambda' < \lambda$. ■

Let $\boldsymbol{W}^*(\lambda)$ denote the optimal beamforming matrix for (3) with a given λ. Relying on the monotonicity of $\alpha^*(\lambda)$, we can further show the following close relationship between problems (2) and (3):

Lemma 2: If it holds $\alpha^*(\check{\lambda}) = 1$, then $\check{\lambda}$ and the corresponding $\boldsymbol{W}^*(\check{\lambda})$ are the optimal value and optimal solution for (2), respectively.

Proof: Since $\frac{\left[\sum_{k=1}^{K} \check{\boldsymbol{w}}_k^*(\check{\lambda})[\check{\boldsymbol{w}}_k^*(\check{\lambda})]^H\right]_{m,m}}{P_m} \le \alpha^*(\check{\lambda}) = 1, \forall m$, the beamforming matrix $\boldsymbol{W}^*(\check{\lambda})$ is in the feasible set of (2). This implies:

$$\lambda^* \ge \min_{1 \le k \le K} \frac{\mathrm{SINR}_k(\boldsymbol{W}^*(\check{\lambda}))}{\gamma_k} \ge \check{\lambda}. \tag{4}$$

Let $\boldsymbol{W}^*$ denote the optimal solution of (2). We can show that $\boldsymbol{W}^*$ is in the feasible set of (3) with λ^* as the minimum $\frac{\mathrm{SINR}_k(\boldsymbol{W})}{\gamma_k}$ requirement. This is because: $\lambda^* =$

$\min_{1\leq k\leq K} \frac{\text{SINR}_k(\boldsymbol{W}^*)}{\gamma_k}$, thus $\frac{\text{SINR}_k(\boldsymbol{W}^*)}{\gamma_k} \geq \lambda^*, \forall k$. On the other hand, we have: $\left[\sum_{k=1}^K \boldsymbol{w}_k^*[\boldsymbol{w}_k^*]^H\right]_{m,m} \leq P_m, \forall m$. Therefore,

$$\alpha^*(\lambda^*) \leq \max_{1\leq m\leq M} \frac{\left[\sum_{k=1}^K \boldsymbol{w}_k^*[\boldsymbol{w}_k^*]^H\right]_{m,m}}{P_m} \leq 1. \quad (5)$$

By Lemma 1, $\alpha^*(\lambda)$ is a strictly increasing function of λ. The inequality $\lambda^* \geq \check{\lambda}$ in (4) implies:

$$\alpha^*(\lambda^*) \geq \alpha^*(\check{\lambda}) = 1. \quad (6)$$

We have both (5) and (6) only when all the inequalities are satisfied with equality; i.e., $\lambda^* = \check{\lambda}$ and $\boldsymbol{W}^* = \boldsymbol{W}^*(\check{\lambda})$. ■

Lemma 2 clearly indicates that the optimal solution for (2) can be obtained by solving the equation $\alpha^*(\check{\lambda}) = 1$. To solve this equation, we need to find $\alpha^*(\check{\lambda})$ for (3) with any given λ. Introducing an auxiliary variable α, we can rewrite (3) as:

$$\begin{aligned} \alpha^*(\lambda) = &\min_{\boldsymbol{W},\, \alpha}\ \alpha \\ \text{s. t.}\ & \frac{\left[\sum_{k=1}^K \boldsymbol{w}_k\boldsymbol{w}_k^H\right]_{m,m}}{P_m} \leq \alpha, \quad m = 1,\ldots,M, \\ & \frac{\text{SINR}_k(\boldsymbol{W})}{\gamma_k} \geq \lambda, \quad k = 1,\ldots,K. \end{aligned} \quad (7)$$

This problem can be reformulated as a convex second-order cone program (SOCP) [5]. Observe that an arbitrary phase rotation can be added to the beamforming vectors without affecting the SINRs. Hence, we can choose a phase such that $\boldsymbol{h}_k^H\boldsymbol{w}_k$ is real and nonnegative. Furthermore, by proper rearrangement, the SINR constraints in (7) can be rewritten as:

$$\left(1 + \frac{1}{\lambda\gamma_k}\right)|\boldsymbol{w}_k^H\boldsymbol{h}_k|^2 \geq \left\| \begin{matrix} \boldsymbol{h}_k^H\boldsymbol{W} \\ \sigma_k \end{matrix} \right\|^2, \quad k = 1,\ldots,K.$$

Since $\boldsymbol{h}_k^H\boldsymbol{w}_k$ can be assumed to be real and nonnegative, we can take square root for both sides of this inequality. The constraint then becomes a SOCP constraint, which is convex. Now the problem (7) can be readily reformulated as a convex program:

$$\begin{aligned} \alpha^*(\lambda) = &\min_{\boldsymbol{W},\, \alpha}\ \alpha \\ \text{s. t.}\ & \left[\sum_{k=1}^K \boldsymbol{w}_k\boldsymbol{w}_k^H\right]_{m,m} \leq \alpha P_m, \quad m = 1,\ldots,M, \\ & \sqrt{1 + \frac{1}{\lambda\gamma_k}}\boldsymbol{w}_k^H\boldsymbol{h}_k \geq \left\| \begin{matrix} \boldsymbol{h}_k^H\boldsymbol{W} \\ \sigma_k \end{matrix} \right\|, \quad k = 1,\ldots,K. \end{aligned} \quad (8)$$

General convex programming (e.g., interior-point) solvers can be employed to find $\alpha^*(\lambda)$ and the optimal beamforming matrix $\boldsymbol{W}^*(\check{\lambda})$ for (8), and, equivalently, (3).

Building on the solution for (8), we can solve the max-min SINR problem (2) via the following bisectional search:

Algorithm 1: *for max-min SINR problem*
Initialize: select a tolerance level $\delta > 0$,
set $\lambda_{\min} = 0$, $\lambda_{\max} = \max_k\{\frac{\|\boldsymbol{h}_k\|^2}{\gamma_k\sigma_k^2}\}(\sum_{m=1}^M P_m)$.
Repeat: let $\lambda = \frac{\lambda_{\max}+\lambda_{\min}}{2}$; solve (8) to obtain $\alpha^*(\lambda)$ and $\boldsymbol{W}^*(\lambda)$;
if $\alpha^*(\lambda_{\max}) > 1$, $\lambda_{\max} = \lambda$;
elseif $\alpha^*(\lambda_{\max}) < 1$, $\lambda_{\min} = \lambda$.
until $(\lambda_{\max} - \lambda_{\min})/\lambda_{\max} < \delta$.
Output: $\boldsymbol{W}^*(\lambda)$.

Here we set $\lambda_{\min} = 0$, which is an obvious lower bound for our bisectional search. The upper bound $\lambda_{\max}$ is determined as follows. It is clear that

$$\begin{aligned} \frac{\text{SINR}_k(\boldsymbol{W})}{\gamma_k} &= \frac{|\boldsymbol{h}_k^H\boldsymbol{w}_k|^2}{\gamma_k[\sum_{l\neq k,l=1}^K(|\boldsymbol{h}_k^H\boldsymbol{w}_l|^2) + \sigma_k^2]} \\ &\leq \frac{\|\boldsymbol{h}_k\|^2\|\boldsymbol{w}_k\|^2}{\gamma_k\sigma_k^2} \leq \frac{\|\boldsymbol{h}_k\|^2}{\gamma_k\sigma_k^2}(\sum_{m=1}^M P_m), \quad \forall\boldsymbol{W},\ \forall k. \end{aligned}$$

This implies that $\lambda^* = \max_{\boldsymbol{W}}\min_{1\leq k\leq K}\frac{\text{SINR}_k(\boldsymbol{W})}{\gamma_k} \leq \max_k\{\frac{\|\boldsymbol{h}_k\|^2}{\gamma_k\sigma_k^2}\}(\sum_{m=1}^M P_m) := \lambda_{\max}$. With such $\lambda_{\min}$ and $\lambda_{\max}$, the bisectional search needs $\mathcal{O}(\log_2(\frac{\lambda_{\max}-\lambda_{\min}}{\delta}))$ iterations to solve $\alpha^*(\lambda) = 1$ up to a desired accuracy level, due to the monotonicity of $\alpha^*(\lambda)$.

Using the available convex programming solvers, the optimal solution for (8) can be found in polynomial time. Building on these solutions, the proposed bisectional search can solve the equation $\alpha^*(\check{\lambda}) = 1$ and find the optimal $\boldsymbol{W}^*$ for (2) geometrically fast. Summarizing, we clearly have:

Proposition 1: Algorithm 1 globally converges to the optimal solution $\boldsymbol{W}^*$ for (2) geometrically fast.

Algorithm 1 is an efficient approach to find the max-min SINR beamforming scheme under per-antenna power constraints, with guaranteed global optimality and fast convergence speed. The derived max-min SINR beamforming solution is important and meaningful in quality-of-service provisioning for the conventional (voice) communications.

IV. Robust Beamforming with Imperfect CSI

Our approach can be generalized to obtain the robust beamforming design that maximizes the worst-case user SINR for practical downlink systems where the base station has only uncertain CSI. The channel uncertainty could originate from a variety of sources such as estimation errors, feedback quantization, hardware deficiencies, and delays in CSI acquisition. Motivated by the channel estimation, it is common to assume an additive error model: $\boldsymbol{h}_k = \tilde{\boldsymbol{h}}_k + \boldsymbol{\delta}_k$, where $\tilde{\boldsymbol{h}}_k$ is the estimated channel known at the base station, and $\boldsymbol{\delta}_k$ denotes the channel uncertainty. Similar to [10]–[14], we further assume that channel uncertainty is bounded by a spherical region:

$$\mathcal{H}_k := \left\{\tilde{\boldsymbol{h}}_k + \boldsymbol{\delta}_k \;|\; \|\boldsymbol{\delta}_k\| \leq \epsilon_k\right\}, \quad \forall k, \quad (9)$$

where $\epsilon_k > 0$ specifies the radius of $\mathcal{H}_k$.

For the channel uncertainty region $\mathcal{H}_k$, define the worst-case SINR for user k as

$$\widetilde{\text{SINR}}_k(\boldsymbol{W}) := \min_{\boldsymbol{h}_k\in\mathcal{H}_k} \frac{|\boldsymbol{h}_k^H\boldsymbol{w}_k|^2}{\sum_{l\neq k,l=1}^K|\boldsymbol{h}_k^H\boldsymbol{w}_l|^2 + \sigma_k^2}, \quad \forall k. \quad (10)$$

The optimal robust beamforming design is then to solve the max-min SINR problem:

$$\begin{aligned}\tilde{\lambda}^* =& \max_{\boldsymbol{W}} \min_{1\le k\le K} \frac{\widetilde{\mathrm{SINR}}_k(\boldsymbol{W})}{\gamma_k}\\ \text{s. t.} \quad & \left[\sum_{k=1}^{K} \boldsymbol{w}_k \boldsymbol{w}_k^H\right]_{m,m} \le P_m, \quad m=1,\ldots,M.\end{aligned} \tag{11}$$

Similarly, (11) can be solved by alternatively solving a sequence of the following per-antenna power balancing problems:

$$\begin{aligned}\tilde{\alpha}^*(\lambda) =& \min_{\boldsymbol{W}} \max_{1\le m\le M} \frac{\left[\sum_{k=1}^{K} \boldsymbol{w}_k \boldsymbol{w}_k^H\right]_{m,m}}{P_m}\\ \text{s. t.} \quad & \frac{\widetilde{\mathrm{SINR}}_k(\boldsymbol{W})}{\gamma_k} \ge \lambda, \quad k=1,\ldots,K.\end{aligned} \tag{12}$$

Define the $M \times M$ square matrices $\boldsymbol{D}_m$ with $[\boldsymbol{D}_m]_{i,j} = 1$ for $i = j = m$, and $[\boldsymbol{D}_m]_{i,j} = 0$ otherwise. Using $\boldsymbol{D}_m$ and introducing an auxiliary variable α, we can rewrite (12) as:

$$\begin{aligned}\tilde{\alpha}^*(\lambda) =& \min_{\boldsymbol{W},\, \alpha} \ \alpha\\ \text{s. t.} \quad & \sum_{k=1}^{K} \boldsymbol{w}_k^H \boldsymbol{D}_m \boldsymbol{w}_k \le \alpha P_m, \quad m=1,\ldots,M,\\ & \frac{\widetilde{\mathrm{SINR}}_k(\boldsymbol{W})}{\gamma_k} \ge \lambda, \quad k=1,\ldots,K.\end{aligned} \tag{13}$$

By the definitions of $\mathcal{H}_k$ and $\widetilde{\mathrm{SINR}}_k$, the constraint $\frac{\widetilde{\mathrm{SINR}}_k(\boldsymbol{W})}{\gamma_k} \ge \lambda$ can be rewritten as: $\forall \boldsymbol{\delta}_k^H \boldsymbol{\delta}_k \le \epsilon_k^2$,

$$(\tilde{\boldsymbol{h}}_k + \boldsymbol{\delta}_k)^H \Big(\frac{1}{\lambda\gamma_k} \boldsymbol{w}_k \boldsymbol{w}_k^H - \sum_{l\ne k} \boldsymbol{w}_l \boldsymbol{w}_l^H\Big)(\tilde{\boldsymbol{h}}_k + \boldsymbol{\delta}_k) - \sigma_k^2 \ge 0.$$

By the well-known S-procedure in robust optimization [16, Appendix B.2], for Hermitian matrices $\boldsymbol{A}$, $\boldsymbol{B}$, a vector $\boldsymbol{c}$, and a real scalar d, the condition

$$\boldsymbol{x}^H \boldsymbol{A} \boldsymbol{x} + \boldsymbol{c}^H \boldsymbol{x} + \boldsymbol{x}^H \boldsymbol{c} + d \ge 0, \ \forall\ \boldsymbol{x}^H \boldsymbol{B} \boldsymbol{x} \le 1 \tag{14}$$

holds true, if and only if there exists one nonnegative scalar t such that

$$\begin{pmatrix} \boldsymbol{A} + t\boldsymbol{B} & \boldsymbol{c} \\ \boldsymbol{c}^H & d - t \end{pmatrix} \succeq \boldsymbol{0}, \tag{15}$$

i.e., it is positive semi-definite.

Define $\boldsymbol{X}_k := \boldsymbol{w}_k \boldsymbol{w}_k^H$, and $\boldsymbol{X} := [\boldsymbol{X}_1, \ldots, \boldsymbol{X}_K]$. Introducing an auxiliary variable vector $\boldsymbol{t} := [t_1, \ldots, t_K]$ and relying on the S-procedure, we can then reformulate (13) into:

$$\begin{aligned}\tilde{\alpha}^*(\lambda) =& \min_{\boldsymbol{X},\, \boldsymbol{t},\, \alpha} \ \alpha\\ \text{s. t.} \quad & \sum_{k=1}^{K} \mathrm{tr}(\boldsymbol{D}_m \boldsymbol{X}_k) \le \alpha P_m, \quad \forall m,\\ & \begin{pmatrix} \boldsymbol{Y}_k + t_k \boldsymbol{I} & \boldsymbol{Y}_k \tilde{\boldsymbol{h}}_k \\ \tilde{\boldsymbol{h}}_k^H \boldsymbol{Y}_k^H & \tilde{\boldsymbol{h}}_k^H \boldsymbol{Y}_k \tilde{\boldsymbol{h}}_k - \sigma_k^2 - t_k \epsilon_k^2 \end{pmatrix} \succeq \boldsymbol{0}, \quad \forall k,\\ & \boldsymbol{Y}_k := \frac{1}{\lambda\gamma_k} \boldsymbol{X}_k - \sum_{l\ne k} \boldsymbol{X}_l, \quad \forall k,\\ & t_k \ge 0, \ \ \boldsymbol{X}_k \succeq \boldsymbol{0}, \ \ \mathrm{rank}(\boldsymbol{X}_k) = 1, \quad \forall k,\end{aligned} \tag{16}$$

where $\mathrm{tr}(\cdot)$ and $\mathrm{rank}(\cdot)$ denote the trace and rank operators, respectively.

By dropping the rank constraints $\mathrm{rank}(\boldsymbol{X}_k) = 1$, $\forall k$, (16) becomes a semidefinite program (SDP) which can be efficiently solved by available convex programming solvers in polynomial time. If the rank-one optimal solution $\boldsymbol{X}_k^*(\lambda)$, $\forall k$, is yielded for the relaxed (16), then we can find the optimal beamforming vectors $\boldsymbol{w}_k^*(\lambda)$ as the (scaled) eigenvector with respect to the only positive eigenvalue of $\boldsymbol{X}_k^*(\lambda)$ for the original problem (12). In fact, it was shown in [14, Theorem 1] that the relaxed (16) always has rank-one optimal solution $\boldsymbol{X}_k^*(\lambda)$, $\forall k$, when the uncertainty bounds ϵ_k are sufficiently small. However, for large ϵ_k case, the existence of rank-one optimal solutions for the relaxed (16) cannot be provably guaranteed. Hence, the exact optimal solution for the original problem (12) may not be constructed from $\boldsymbol{X}_k^*(\lambda)$ with possibly a rank greater than one. In this case, randomized rounding is a widely adopted method to obtain a feasible rank-one approximate solution from the SDP relaxation; specifically, a Gaussian randomized rounding strategy [13] can be applied to get a vector $\boldsymbol{w}_k^*(\lambda)$ from $\boldsymbol{X}_k^*(\lambda)$ to nicely approximate the solution of the original problem (12).

Building on the solution for (12), a bisectional search similar to that in Algorithm 1 can be then implemented to find the optimal beamforming design for max-min SINR problem (11) with geometrically fast convergence speed. Such a robust beamforming design can provide the worst-case user SINR guarantees in practical downlink systems where only uncertain CSI is available at the base station.

V. Numerical Results

Consider a downlink scenario where the base station with $M = 4$ transmit antennas serves $K = 4$ users. Each transmit antenna has equal power budget, i.e., $P_m = P/M$, $\forall m$, where P is the total power budget at the base station. Assume that the receiver noise power is 0 dBm. For such a downlink system, we evaluate the proposed beamforming designs and compare their performance with the existing alternatives in terms of worst-case SINR and average bit error rate (BER).

Perfect CSI case: Suppose that a normalized Rayleigh fading channel model is adopted for channel coefficient vectors $\boldsymbol{h}_k$, $\forall k$, i.e., $\boldsymbol{h}_k \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I})$, where $\mathcal{CN}(\bar{\boldsymbol{x}}, \boldsymbol{\Sigma})$ denotes a complex

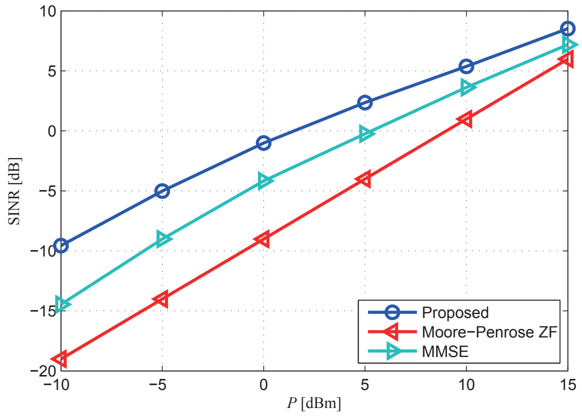

Fig. 2. Average minimal SINR comparison with perfect CSI: one base station with 4 Tx antennas, 4 single-Rx-antenna users.

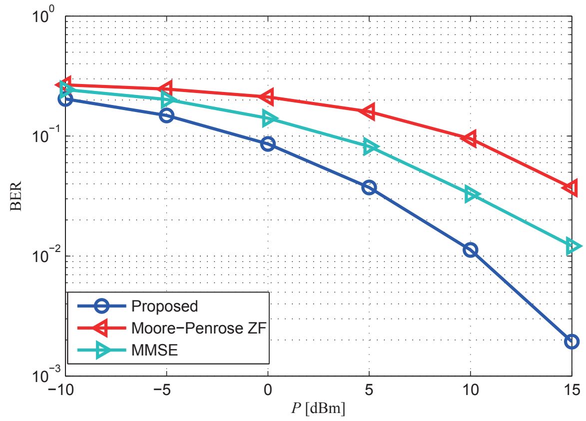

Fig. 3. Average BER comparison with perfect CSI: one base station with 4 Tx antennas, 4 single-Rx-antenna users.

Gaussian random vector with mean $\bar{\boldsymbol{x}}$ and covariance matrix $\boldsymbol{\Sigma}$. Fig. 2 compares the minimal user SINR for different schemes, where each point is obtained by averaging the results with 1000 randomly generated channel realizations. In addition to the optimal max-min SINR design yielded by Algorithm 1, we include the performance of the Moore-Penrose zero-forcing (ZF) beamforming [6] and the MMSE beamforming [15] schemes. It is clearly shown that the proposed design outperforms the Moore-Penrose ZF and MMSE ones. For example, when the total power budget is 5 dBm, the proposed scheme achieves about 3 dB and 5 dB gain compared to MMSE and Moore-Penrose ZF designs, respectively. Fig. 3 shows the average BER performance comparison, when BPSK is adopted as the modulation scheme. Again, the proposed beamforming design clearly outperforms Moore-Penrose ZF and MMSE ones. For example, when BER $= 10^{-2}$, the proposed design achieves about 5 dB gain compared to MMSE design. Fig. 3 also indicates that the performance gain achieved by the proposed design becomes larger as the total power budget P increases.

Imperfect CSI case: Assume that the channel estimate $\tilde{\boldsymbol{h}}_k \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{I})$, and the channel uncertainty bounds $\epsilon_k = 0.2$ in (9), $\forall k$. Fig. 4 depicts the worst-case SINR among all users for different schemes, where "Robust" denotes the proposed robust beamforming design in Section IV, and "Non-Robust" denotes the proposed max-min SINR beamforming design in Section III based on channel estimate $\tilde{\boldsymbol{h}}_k$ without considering channel uncertainty. For comparison, performance of Moore-Penrose ZF scheme in [6] and MMSE scheme in [15] is also included. Again, each result is obtained as the average of those with 1000 randomly generated channel realizations. As shown in Fig. 4, by taking into account the channel uncertainty, the proposed robust design can significantly outperform the non-robust one in worst-case SINR for low and medium transmit power regimes. Fig. 5 shows the average BER comparison, when BPSK is adopted as the modulation scheme. The proposed robust scheme achieves about 2 dB gain compared to

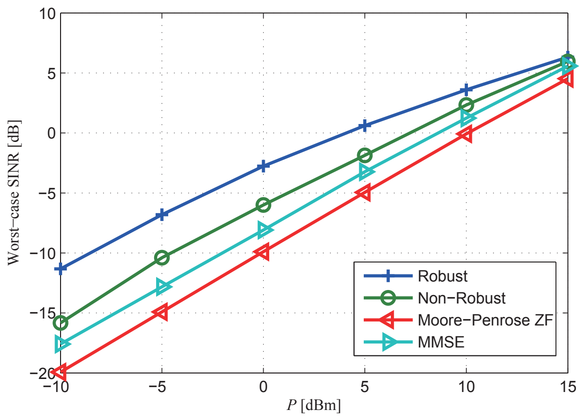

Fig. 4. Average worst-case SINR with imperfect CSI: one base station with 4 Tx antennas, 4 single-Rx-antenna users.

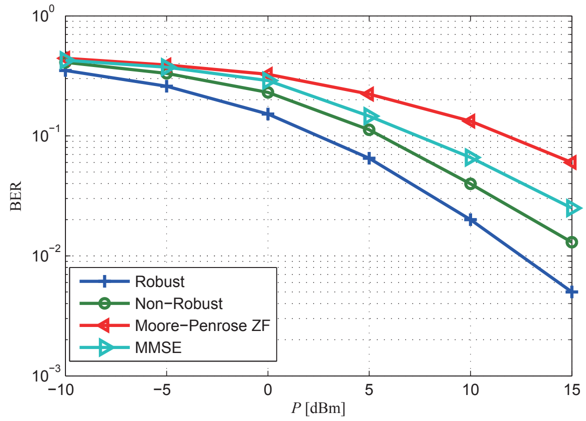

Fig. 5. Average BER comparison with imperfect CSI: one base station with 4 Tx antennas, 4 single-Rx-antenna users.

the non-robust scheme when BER = 10^{-1}. The gain becomes even larger as transmit power P grow large. Compared to Moore-Penrose ZF and MMSE schemes, the proposed robust scheme achieves more than 5 dB gain in BER performance.

VI. Conclusion

An efficient algorithm was proposed to find the optimal max-min SINR beamforming schemes based on solving a sequence of convex per-antenna power balancing problems. The proposed approach applies to both perfect and imperfect CSI cases. Numerical results demonstrated that the proposed transmit beamforming schemes significantly outperform the existing alternatives.

It was shown in our recent work [17] that max-min SINR solution can be used as a corner stone to pursue the optimal beamforming designs based on arbitrary utility functions that are monotonic in the user SINRs in two-way relay systems. Building on the proposed max-min SINR solution, a similar monotonic programming approach could be also developed to obtain per-antenna power constrained downlink beamforming design for other important optimization criteria, such as weighted sum-rate maximization, average symbol-error-rate (SER) or BER minimization, etc. This interesting direction will be investigated in the future work.

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