Precoding for Multiple Antenna Gaussian Broadcast Channels With Successive Zero-Forcing

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***Abstract*—In this paper, we consider the multiuser Gaussian broadcast channel with multiple transmit antennas at the base station and multiple receive antennas at each user. Assuming full knowledge of the channel state information at the transmitter and the different receivers, a new transmission scheme that employs partial interference cancellation at the transmitter with dirty-paper encoding and decoding is proposed. The maximal achievable throughput of this system is characterized, and it is shown that given any ordered set of users the proposed scheme is asymptotically optimal in the high signal-to-noise ratio (SNR) regime. In addition, with optimal user ordering, the proposed scheme is shown to be optimal in the low-SNR regime. We also consider a linear transmission scheme which employs only partial interuser interference cancellation at the base station without dirty-paper coding. Given a transmit power constraint at the base station, the sum-rate capacity of this scheme is characterized and a suboptimal precoding algorithm is proposed. In several cases, it is shown that, for all values of the SNR, the achievable throughput of this scheme is strictly larger than a system which employs full interference cancellation at the base station [21]. In addition, it is shown that, in some cases, the linear transmission scheme can support simultaneously an increased number of users while achieving a larger system throughput.**

***Index Terms*—Controlled interference, dirty-paper coding, multiple-input multiple-output (MIMO) broadcast channels (BCs), precoding.**

I. Introduction

THE downlink of multiuser multiple-input multiple-output (MIMO) broadcast channels has been an extensive area of research in the last few years. With an increased interest in data throughput and quality-of-service (QoS), MIMO wireless communication systems have played a fundamental role in providing such benefits. In particular, the capacity limits of the Gaussian multiuser broadcast channel with multiple transmit antennas at the base station and multiple receive antennas at each user have captured a large amount of research in recent years [2], [27], [28], [30]. In general, most of the previous research work has focused on the achievable system throughput using Costa's dirty-paper coding (DPC) technique [4]. In [2], it was first shown that the DPC technique achieves the sum-rate capacity of the Gaussian broadcast channel (BC) for a system employing multiple transmit antennas at the base station and users with single receive antenna. A generalization of this result for the multiple receive antennas case was given in [27] and [30], and, finally, in [28], it was shown that the dirty-paper achievable region is actually the capacity region of the Gaussian multiuser MIMO BC.

The DPC technique is based on noncausal knowledge of each user's interfering signal at the base station. With such knowledge, the base station can apply dirty-paper encoding, and quite remarkably without users knowing the interfering signals, the capacity of the system is as if there were no interference at all. The main drawback, however, of the DPC technique is that it is a highly nonlinear technique which makes its implementation a very challenging problem. Construction of practical codes that achieve close to capacity performance of the dirty-paper technique is the subject of current research. A well-known general approach for achieving the capacity of DPC is based on multidimensional lattice quantization with minimum mean-square error (MMSE) scaling [6]. Different practical realizations of the multidimensional lattice-based dirty-paper schemes were proposed in [7], [8], and [18]. Another practical implementation of a DPC-based scheme can be found in [26].

Because of its complexity, several authors have considered a reduced complexity suboptimal DPC scheme that is known as the zero-forcing dirty-paper coding (ZFDPC) scheme and was first proposed in [2]. In [25], a greedy user-selection scheme named greedy zero-forcing dirty-paper coding was suggested. In [5], analysis of the performance of the greedy zero-forcing dirty-paper coding scheme was provided. In [13], a MIMO BC with multiple receive antennas at each receiver was considered, and a comparison of a generalized zero-forcing dirty-paper coding scheme with V-BLAST [9] was given. In [19], the zero-forcing and matched zero-forcing (MMSE precoding) transmission schemes employed in a MIMO BC with single receive antenna users were considered, and a comparison with the ZFDPC technique was made.

Based on the DPC technique, in this paper, we propose a new transmission scheme that assumes full channel knowledge at the transmitter and at each receiver. In this scheme, a set of precoding matrices at the base station are designed such that only partial interuser interference cancellation is performed. Similar to the asymptotic performance (when each user has a single receive antenna) of the ZFDPC scheme in the high and low signal-to-noise ratio (SNR) regimes, it is shown that with

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any ordered set of users the sum-rate capacity of the proposed scheme in the high-SNR regime is asymptotically equal to the sum-rate capacity of the optimal DPC scheme. In the low-SNR regime, it is also shown that with optimal user ordering, the sum-rate capacity of the proposed scheme is asymptotically equal to the sum-rate capacity of the DPC scheme. This scheme can be thought of as a generalization for the ZFDPC scheme for the multiple receive antenna case where we perform successive block-diagonalization of the channel.

In general, the channel inversion technique can not be directly generalized for the case when the users have multiple receive antennas without sacrificing much of the multiplexing and diversity benefits of the single user MIMO channel. In [21], a block-diagonalization (BD) transmission scheme for the multiuser downlink channel was proposed. This scheme can be viewed as a generalization of the channel inversion technique where users work under interference free conditions, however, the inherent multiplexing and diversity gains of each user's MIMO channel can be still utilized up to a certain limit. The main idea of the proposed technique is based on the design of a set of precoding matrices corresponding to each user's transmitted signal such that the interuser interference is canceled. A main limitation of this scheme is that the number of transmit antennas at the base station has to be larger than the total number of receive antennas of all users supported simultaneously in the system. The case when the constraint on the number of transmit antennas is not satisfied was considered in [29], and different receive antenna selection methods were proposed. Another transmission approach that was suggested in [21] is based on the successive optimization of the set of precoding matrices such that the transmission power at the base station is minimized given the set of transmission rates of all the users. However, in [21], it was also shown that compared to the BD scheme the proposed successive optimization scheme does not yield any throughput improvement except in the low-SNR regime. In [10], another linear precoding scheme for the downlink with multiple transmit and receive antennas at each receiver was suggested. Assuming MMSE receivers, it was shown that performing a controlled interuser interference cancellation by maximizing the ratio between the transmitted signal power of each user and the total interference caused by the same user leads to the maximization of a lower bound on the sum-rate capacity.

In this paper, we also consider a linear transmission technique for the multiuser Gaussian MIMO broadcast channel with controlled interuser interference cancellation similar to [10] and [21]. In this scheme, we only perform partial interference cancellation at the base station; however, we do not perform DPC. We characterize the sum-rate capacity of this transmission approach and propose a suboptimal precoding algorithm that determines a set of precoding matrices for the sum-rate capacity maximization problem under a total power constraint at the base station. In this case, it turns out that user ordering can potentially improve the system throughput, and in several cases, it is shown that, for all values of the SNR, the achievable system throughput of this scheme is strictly larger compared to other previously reported transmission techniques with linear complexity. Furthermore, it is shown that in some cases the linear transmission scheme can support an increased number of users while simultaneously having additional gain in the system throughput.

This paper is organized as follows. In Section II, we present the system model under consideration and give a short overview of the main precoding techniques proposed in previous literature. In Section III, the dirty-paper-based transmission scheme is proposed and the sum-rate capacity is determined. In Section IV, the linear precoding transmission scheme is considered and analysis of the sum-rate capacity with a precoding algorithm is provided. In Section V, a discussion on the optimal solution for the linear transmission scheme is given. In Section VI, simulation results are presented, and we conclude with future points of emphasis in Section VII.

II. System Model and Overview

A. System Model

In this paper,[1] we consider the flat fading multiuser MIMO BC with K users having $N_1, N_2, \ldots, N_K$ receive antennas and a base station with M transmit antennas. According to this model, the received signal at user k is given by

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x} + \mathbf{z}_k$$

where $\mathbf{y}_k$ is the $N_k \times 1$ received vector, $\mathbf{H}_k$ is the $N_k \times M$ channel matrix, and $\mathbf{z}_k \sim \mathcal{CN}(0, \mathbf{I})$ is the additive white complex Gaussian noise vector at each receiver. In general, it is assumed that for each user k the channel matrix $\mathbf{H}_k$ is fixed during multiple transmission epochs and is fully known both to the base station and to the kth user. Given an ordered set of users $\mathcal{S}$ with an order π, we can write

$$\begin{bmatrix} \mathbf{y}_{\pi(1)} \\ \mathbf{y}_{\pi(2)} \\ \vdots \\ \mathbf{y}_{\pi(|\mathcal{S}|)} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{\pi(1)} \\ \mathbf{H}_{\pi(2)} \\ \vdots \\ \mathbf{H}_{\pi(|\mathcal{S}|)} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{z}_{\pi(1)} \\ \mathbf{z}_{\pi(2)} \\ \vdots \\ \mathbf{z}_{\pi(|\mathcal{S}|)} \end{bmatrix} \tag{1}$$

and we let $\mathbf{H}_{\mathcal{S}}$ given by

$$\mathbf{H}_{\mathcal{S}} = \left[\mathbf{H}^T_{\pi(1)}, \mathbf{H}^T_{\pi(2)}, \ldots, \mathbf{H}^T_{\pi(|\mathcal{S}|)}\right]^T \tag{2}$$

denote the channel matrix. The transmitted signal vector at the base station is assumed to have the following general structure

$$\mathbf{x} = \sum_{k \in \mathcal{S}} \mathbf{B}_k \mathbf{u}_k \tag{3}$$

where $\mathbf{B}_k$ is the precoding matrix corresponding to the kth user vector $\mathbf{u}_k$. In the following, we present short reviews of previously known transmission schemes proposed for the multiuser MIMO BCs.

B. Zero-Forcing Dirty-Paper Coding

The zero-forcing dirty-paper coding technique [2] is based on the QR decomposition of the channel where it is assumed that

[1]We use boldface to denote matrices and vectors. For any set $\mathcal{S}$, $|\mathcal{S}|$ denotes the cardinality of the set. $\mathbf{I}$ denotes the identity matrix. For any matrix $\mathbf{A}$, $(\mathbf{A})^*$ denotes the conjugate transpose, $(\mathbf{A})^T$ denotes the matrix transpose, $\|\mathbf{A}\|_2$ denotes the maximal singular value of $\mathbf{A}$, $\mathrm{vec}(\mathbf{A})$ denotes the stacking of the columns of $\mathbf{A}$ into a vector and $\mathrm{unvec}(\mathbf{A})$ denotes the reverse operation. $[\mathbf{A}]_{i,j}$ denotes the (i,j)th entry of $\mathbf{A}$. For a square matrix $\mathbf{A}$, $\det(\mathbf{A})$ denotes the matrix determinant, $\mathbf{A}^{-1}$ denotes the matrix inverse, $\mathbf{A}^{1/2}$ denotes the matrix square root, and $\mathbf{A} \geq 0$ denotes that $\mathbf{A}$ is positive semidefinite. For any real number x, $[x]_+ \equiv \max(x, 0)$.

users have a single receive antenna. If we let $m = \mathrm{rank}(\mathbf{H}_\mathcal{S})$, then the channel matrix $\mathbf{H}_\mathcal{S} = \mathbf{GQ}$, where $\mathbf{G} \in \mathbb{C}^{|\mathcal{S}| \times m}$ is a lower triangular matrix having zeros above its main diagonal and $\mathbf{Q} \in \mathbb{C}^{m \times M}$ has orthonormal rows. For simplicity, we assume that after performing the QR decomposition we have the set of users $\{1, 2, \ldots, m\}$. For each user, $\mathbf{u}_k$ is a scalar and the precoding matrix $\mathbf{B}_k$ is given by the corresponding column vector in $\mathbf{Q}^*$. The received signal at the kth receiver is given by

$$y_k = g_{k,k} u_k + \sum_{j<k} g_{k,j} u_j + z_k, \quad k = 1, \ldots, m$$

where $g_{k,j}$ denotes the (k,j)th entry in $\mathbf{G}$. In this case, the interfering signal of the kth user caused by the $(m-k)$ users' symbols $\{u_j, k < j \le m\}$ is canceled by the decomposition structure, while the interfering signal component $\sum_{j<k} g_{k,j} u_j$ is assumed noncausally known at the transmitter. By generating the components of $\{u_k, 1 \le k \le m\}$ according to successive dirty-paper encoding, the performance of user k is the same as if the interfering signal from users $1, 2, \ldots, k-1$ did not exist.

The problem with ZFDPC is that it is not easy to generalize when the number of receive antennas of each user is greater than one. In [13], a generalization of the ZFDPC technique is made; however, in this case, different antennas at each receiver are treated as if they belong to independent users. The main problem with this kind of generalization is related to the fact that in any point-to-point MIMO communication system if we do not consider the asymptotic case of the high-SNR regime, then attempting to diagonalize the channel via inversion like techniques at the transmitter, is strictly suboptimal since it does not exploit the multiplexing and diversity gains existing in any MIMO communication system.

C. Channel Inversion

Assuming full channel knowledge at the transmitter, different types of precoding methods can be applied. A well-known method is the channel inversion or what is known by zero-forcing beamforming [2], [22]. In the channel inversion method, the channel matrix is inverted at the transmitter in order to create orthogonal channels between the transmitter and receivers. In this case, as before, it is assumed that the users have single receive antenna and the input vector $\mathbf{u} = [u_{\pi(1)}, \ldots, u_{\pi(|\mathcal{S}|)}]^T$ is multiplied by the Moore–Penrose pseudoinverse [12] of the matrix $\mathbf{H}_\mathcal{S}$. Assuming that $M \ge |\mathcal{S}|$ and $\mathbf{H}_\mathcal{S}$ is of full rank, the transmitted signal is given by

$$\mathbf{x} = \mathbf{H}_\mathcal{S}^* (\mathbf{H}_\mathcal{S} \mathbf{H}_\mathcal{S}^*)^{-1} \mathbf{u} \tag{4}$$

and the received signal of each user is given by $y_k = u_k + z_k$ for any $k \in \mathcal{S}$. Now given the set of users $\mathcal{S}$ and total power constraint P on the transmitted signal, the throughput of the channel inversion method is found by the waterfilling solution and is given by

$$C_\mathcal{S}^{CI} = \sum_{i=1}^{|\mathcal{S}|} \left[\log \frac{\xi}{\left[(\mathbf{H}_\mathcal{S} \mathbf{H}_\mathcal{S}^*)^{-1} \right]_{i,i}} \right]_+$$

where ξ solves

$$\sum_{i=1}^{|\mathcal{S}|} \left[\xi - \frac{1}{\left[(\mathbf{H}_\mathcal{S} \mathbf{H}_\mathcal{S}^*)^{-1} \right]_{i,i}} \right]_+ = P.$$

The price that is paid here is in the amount of power needed to do channel inversion which causes a significant noise enhancement, especially when the channel is highly correlated or close to singular. However, as in the ZFDPC method, the channel inversion scheme can not be easily generalized to the case of users with multiple receive antennas. Again, if we directly apply the channel inversion method to the MIMO multiuser channel, this will be equivalent to treating each receiver antenna as an independent user without being able to exploit the main benefits of the MIMO point-to-point channel. We note here that, in [17], a precoding technique based on regularizing the inverse in the channel inversion formula was suggested. As before, the proposed scheme can not be generalized to the case of users with multiple receive antennas, and by treating the different receive antennas as independent users the regularized channel inversion technique yields (assuming $\sum_{j \in |\mathcal{S}|} N_j \gg 1$)

$$\mathbf{x} = \mathbf{H}_\mathcal{S}^* \left(\mathbf{H}_\mathcal{S} \mathbf{H}_\mathcal{S}^* + \frac{\sum_{j \in |\mathcal{S}|} N_j}{\rho} \mathbf{I} \right)^{-1} \mathbf{u}$$

where ρ is the SNR value. This precoding technique is also known as MMSE precoding [15], and as can be easily seen in the high-SNR regime it has similar performance as the channel inversion technique.

D. Block-Diagonalization

For the MIMO multiuser downlink channel where users can have more than one receive antenna, a generalization of the channel inversion method was proposed in [21]. According to this method, each user's transmitted signal is precoded such that when multiplied by other users' channels it does not generate any interference. In order to cancel the interuser interference, each precoding matrix $\mathbf{B}_{\pi(j)}$ is constrained to lie in the null space of

$$\mathbf{H}_\mathcal{S}^j = \left[\mathbf{H}_{\pi(1)}^T \cdots \mathbf{H}_{\pi(j-1)}^T \mathbf{H}_{\pi(j+1)}^T \cdots \mathbf{H}_{\pi(|\mathcal{S}|)}^T \right]^T.$$

Therefore, the effective channel has a block diagonal structure, and in order to satisfy the above constraint the null space dimension of $\mathbf{H}_\mathcal{S}^j$ should be greater than zero. This implies that the rank of these matrices should be less than M, and, in general, in a rich scattering environment, we get that the number of transmit antennas should be greater than or equal to the sum of all users' receive antennas (more precisely, $M > \sum_{i=1, i \ne j}^{|\mathcal{S}|} N_{\pi(i)}$, for all j) [21]. In order to maximize the throughput of the system under the BD method, the set of precoding matrices should satisfy the total power constraint on the transmitted signal in addition to the constraint on the null space mentioned above. If we let $\mathbf{V}_\mathcal{S}^{(j,0)}$ denote the matrix of $(M - \mathrm{rank}(\mathbf{H}_\mathcal{S}^j))$ vectors that form a basis for the null space of $\mathbf{H}_{\mathcal{S}}^{j}$, the throughput of the BD method is given by

$$C_{\mathcal{S}}^{\mathrm{BD}} = \max_{\{\mathbf{Q}_k : \mathbf{Q}_k \geq 0, \sum_{k\in\mathcal{S}} \mathrm{Tr}(\mathbf{Q}_k) \leq P\}} \sum_{j=1}^{|\mathcal{S}|} \log\det\left(\mathbf{I} + \mathbf{H}_{\pi(j)}\mathbf{V}_{\mathcal{S}}^{(j,0)}\mathbf{Q}_{\pi(j)}\left(\mathbf{H}_{\pi(j)}\mathbf{V}_{\mathcal{S}}^{(j,0)}\right)^*\right) \quad (5)$$

and the optimal set of covariance matrices $\{\mathbf{Q}_k\}_{k\in\mathcal{S}}$ is obtained by waterfilling over the effective block diagonal channel matrix

$$\mathbf{H}_{\mathcal{S}}^{\mathrm{eff}} = \begin{pmatrix} \mathbf{H}_{\pi(1)}\mathbf{V}_{\mathcal{S}}^{(1,0)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{H}_{\pi(|\mathcal{S}|)}\mathbf{V}_{\mathcal{S}}^{(|\mathcal{S}|,0)} \end{pmatrix} \quad (6)$$

with total power constraint P. Note that the matrix $\mathbf{V}_{\mathcal{S}}^{(j,0)}$ can be obtained using the singular value decomposition (SVD); however, in [3], a more efficient and stable method for computing the above matrix based on the QR decomposition was proposed.

We should note here that multiuser diversity can be achieved by maximizing over the different possible sets of users with different cardinalities. In this case, in order to maximize the total throughput in the system, only the users in the set S but not the ordering of users matters. In [20], a suboptimal algorithm that reduces the search complexity over the different sets of users with relatively small loss in performance can be found.

III. Successive Zero-Forcing Dirty-Paper Coding

In general, interuser interference cancellation at the base station puts several restrictions on the transmitted signal in addition to the constraints on the number of transmit and receive antennas in the system. These constraints generally lead to significant degradation in the system throughput and restrict the capability of supporting multiple users simultaneously. In the channel inversion method with a single receive antenna at each user, orthogonal channels are created for interference free communication. The main disadvantages of this technique are the enhancement of the noise power and that it can not be easily generalized to the case of users with multiple receive antennas. In the BD technique proposed in [21], complete zero forcing of the interuser interference is performed; however, compared to a system that employs DPC, the achievable throughput is reduced significantly.

In the following, a transmission scheme that employs DPC with only partial interference cancellation is proposed. According to [30], we have the following lemma.

Lemma 1: (Yu, Cioffi): Consider a channel with $\mathbf{y}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{s}_k + \mathbf{z}_k$, where $\mathbf{y}_k$ is the received vector, $\mathbf{x}_k$ is the transmitted vector, $\mathbf{s}_k$ is the Gaussian interference vector, and $\mathbf{z}_k$ is the white Gaussian vector noise. If $\mathbf{s}_k$ and $\mathbf{z}_k$ are independent and noncausal knowledge of $\mathbf{s}_k$ is available at the transmitter but not at the receiver, then the capacity of the channel is the same as if $\mathbf{s}_k$ was not present.

This lemma is actually a generalization of the DPC result [2] for the vector case. Based on this result, we propose the following transmission scheme. The transmitted signal $\mathbf{x}$ at the base station will have the same general structure as in (3), but for each $j \in \{2, 3, \ldots, |\mathcal{S}|\}$, the precoding matrix $\mathbf{B}_{\pi(j)}$ is now constrained to lie only in the null space of

$$\overline{\mathbf{H}}_{\mathcal{S}}^{j-1} = \left[\mathbf{H}_{\pi(1)}^T\mathbf{H}_{\pi(2)}^T \cdots \mathbf{H}_{\pi(j-1)}^T\right]^T \quad (7)$$

and for $j = 1$, we assume $\overline{\mathbf{H}}_{\mathcal{S}}^{0}$ to be a zero matrix (i.e., $\mathbf{B}_{\pi(1)}$ has no such subspace constraint). Now, the received signal of each user is given by

$$\mathbf{y}_{\pi(j)} = \mathbf{H}_{\pi(j)}\left(\mathbf{B}_{\pi(j)}\mathbf{u}_{\pi(j)} + \sum_{i<j}\mathbf{B}_{\pi(i)}\mathbf{u}_{\pi(i)} + \sum_{i>j}\mathbf{B}_{\pi(i)}\mathbf{u}_{\pi(i)}\right) + \mathbf{z}_{\pi(j)}. \quad (8)$$

According to the proposed method, for each j, the term $(\mathbf{H}_{\pi(j)}\sum_{i>j}\mathbf{B}_{\pi(i)}\mathbf{u}_{\pi(i)})$ is canceled by the above constraint on the precoding matrices. By applying successive dirty-paper coding with noncausal knowledge of the interfering signals according to Lemma 1, dirty-paper coding allows the users to operate at a rate as if the interference term $(\mathbf{H}_{\pi(j)}\sum_{i<j}\mathbf{B}_{\pi(i)}\mathbf{u}_{\pi(i)})$ did not exist. We will refer to this scheme as successive zero-forcing dirty-paper coding (SZFDPC). We note here that, in [21], given the transmission rates of all the users, successive optimization of the different users' precoding matrices was performed under a similar subspace constraint.

The SZFDPC transmission scheme can be viewed as an extension of ZFDPC for users with multiple receive antennas. In general, interference cancellation at the base station simplifies the decoding process at each of the receivers. In the SZFDPC scheme, the interference term $(\mathbf{H}_{\pi(j)}\sum_{i>j}\mathbf{B}_{\pi(i)}\mathbf{u}_{\pi(i)})$ is canceled at the base station and knowledge of this interference power is not required at the different receivers. Furthermore, similar to ZFDPC, the optimal precoding matrices at the base station can be easily computed (as it will be later shown) in this case without the need of more computationally complex optimization algorithms, such as the iterative waterfilling algorithm [14].

In the following, we are interested in characterizing the achievable sum-rate capacity of the SZFDPC transmission scheme and of comparing it with other transmission schemes found in the literature. Moreover, similar to [2], the performance of the proposed scheme is of special interest in the two cases of high and low-SNR regimes, where it was shown that in the high-SNR regime, the ZFDPC scheme is optimal in the sense of achieving the cooperative channel capacity (where all users are allowed to cooperate and joint decoding of all users' received signals is possible), whereas in the low-SNR regime the ZFDPC scheme has a similar performance as the channel inversion scheme.

For a given set of ordered users, let

$$\overline{\mathbf{H}}_{\mathcal{S}}^{j-1} = \overline{\mathbf{U}}_{\mathcal{S}}^{j-1}\overline{\mathbf{\Sigma}}_{\mathcal{S}}^{j-1}\left[\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,1)}\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}\right]^* \quad (9)$$

be the singular value decomposition of the matrix $\overline{\mathbf{H}}_{\mathcal{S}}^{j-1}$ for $j \in \{1, 2, \ldots, |\mathcal{S}|\}$, where $\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,1)}$ denotes the matrix of rank($\overline{\mathbf{H}}_{\mathcal{S}}^{j-1}$) right singular vectors of $\overline{\mathbf{H}}_{\mathcal{S}}^{j-1}$ and $\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}$ is the set of vectors consisting a basis for the null space of $\overline{\mathbf{H}}_{\mathcal{S}}^{j-1}$. Note that, for $j = 1$, we let $\overline{\mathbf{V}}_{\mathcal{S}}^{(0,0)}$ be the $M \times M$ identity matrix. Since, for each j the precoding matrix, $\mathbf{B}_{\pi(j)}$ is constrained to lie in the subspace spanned by $\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}$, the achievable throughput of the system is given by

$$C_{\mathcal{S}}^{\mathrm{SZFDPC}} = \max_{\{\mathbf{Q}_k:\mathbf{Q}_k\geq 0,\sum_{k\in\mathcal{S}}\mathrm{Tr}(\mathbf{Q}_k)\leq P\}} \sum_{j=1}^{|\mathcal{S}|} \log\det\left(\mathbf{I} + \left(\mathbf{H}_{\pi(j)}\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}\right)\mathbf{Q}_{\pi(j)}\left(\mathbf{H}_{\pi(j)}\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}\right)^*\right). \quad (10)$$

In order to determine the optimal set of precoding matrices given the set of users $\mathcal{S}$, a similar algorithm as in [21] can be followed; however, the basis for the null space of each user is now given by $\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}$. Given the ordered set of users $\mathcal{S}$, let

$$\mathbf{B}_{\mathcal{S}} \equiv \left[\mathbf{B}_{\pi(1)}\mathbf{B}_{\pi(2)}\cdots\mathbf{B}_{\pi(|\mathcal{S}|)}\right] \quad (11)$$

denote the optimal set of precoding matrices. Then

$$\mathbf{B}_{\mathcal{S}} = \left[\overline{\mathbf{V}}_{\mathcal{S}}^{(0,0)}\widetilde{\mathbf{V}}_{\mathcal{S}}^{(1,1)}\cdots\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)}\widetilde{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|,1)}\right]\hat{\mathbf{\Gamma}}_{\mathcal{S}}^{1/2} \quad (12)$$

where for each $j \in \{1, 2, \ldots, |\mathcal{S}|\}$, $\widetilde{\mathbf{V}}_{\mathcal{S}}^{(j,1)}$ is obtained from the singular value decomposition

$$\mathbf{H}_{\pi(j)}\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)} = \widetilde{\mathbf{U}}_{\mathcal{S}}^{j}\widetilde{\mathbf{\Sigma}}_{\mathcal{S}}^{j}\left[\widetilde{\mathbf{V}}_{\mathcal{S}}^{(j,1)}\widetilde{\mathbf{V}}_{\mathcal{S}}^{(j,0)}\right]$$

and $\hat{\mathbf{\Gamma}}_{\mathcal{S}}$ is the matrix obtained by waterfilling over the singular values of the matrix

$$\widehat{\mathbf{H}}_{\mathcal{S}} = \begin{pmatrix} \mathbf{H}_{\pi(1)}\overline{\mathbf{V}}_{\mathcal{S}}^{(0,0)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{H}_{\pi(|\mathcal{S}|)}\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)} \end{pmatrix} \quad (13)$$

with power constraint P.

In general, multiuser diversity can be achieved if the base station maximizes the sum-rate capacity over different ordered sets of users. In this case, we need to consider different users' orderings which can affect the achievable throughput because of the null space constraints; however, we can only consider sets of users with cardinality equal to K due to the waterfilling algorithm which determines the optimal number of users that should be supported in any ordered set of users $\mathcal{S}$. Therefore, the sum-rate capacity of the SZFDPC transmission scheme with multiuser diversity is given by

$$C^{\mathrm{SZFDPC}} = \max_{\{\mathcal{S}:|\mathcal{S}|=K\}} C_{\mathcal{S}}^{\mathrm{SZFDPC}}. \quad (14)$$

In order to achieve multiuser diversity, we need to consider the different ordered sets as in (14). The problem with the maximization in (14) is that the number of different sets scales exponentially with the number of users in the system. Therefore, an efficient user selection technique is generally required especially when the number of users is large.

Given the set of ordered users $\mathcal{S}$, let $C_{\mathcal{S}}^{\mathrm{COOP}}$ denote the sum-rate capacity of the cooperative transmission scheme. In the following, the optimality of the SZFDPC scheme in the high-SNR regime is established.

Theorem 1: In the high-SNR regime, given an arbitrary set of ordered users $\mathcal{S}$ and assuming that $\mathbf{H}_{\mathcal{S}}$ and $\widehat{\mathbf{H}}_{\mathcal{S}}$ are of full row rank, the sum-rate capacity of the successive zero-forcing dirty-paper coding transmission scheme is asymptotically equal to the sum-rate capacity of the cooperative transmission scheme, i.e.,

$$\lim_{P\to\infty}(C_{\mathcal{S}}^{\mathrm{SZFDPC}} - C_{\mathcal{S}}^{\mathrm{COOP}}) = 0. \quad (15)$$

Proof: First note that when the users are allowed to cooperate, the effective multiuser channel is actually the single user point-to-point MIMO channel. If we let r denote the rank of $\mathbf{H}_{\mathcal{S}}$, the capacity of this channel is given by [23]

$$C_{\mathcal{S}}^{\mathrm{COOP}} = \sum_{i=1}^{r}\left[\log\left(\xi^c\lambda_i^c\right)\right]_+ \quad (16)$$

where $\{\lambda_i^c\}_{i=1}^r$ is the set of eigenvalues of the matrix $\mathbf{H}_{\mathcal{S}}\mathbf{H}_{\mathcal{S}}^*$ and ξ^c is the solution of

$$\sum_{i=1}^{r}\left[\xi - 1/\lambda_i^c\right]_+ = P.$$

Similarly, the sum-rate capacity of the SZFDPC scheme is given by

$$C_{\mathcal{S}}^{\mathrm{SZFDPC}} = \sum_{i=1}^{r}\left[\log\left(\xi^b\lambda_i^b\right)\right]_+ \quad (17)$$

where $\{\lambda_i^b\}_{i=1}^r$ is the set of eigenvalues of the matrix $\widehat{\mathbf{H}}_{\mathcal{S}}\widehat{\mathbf{H}}_{\mathcal{S}}^*$ and ξ^b is the solution of

$$\sum_{i=1}^{r}\left[\xi - 1/\lambda_i^b\right]_+ = P.$$

In order to show the asymptotic optimality of the SZFDPC transmission scheme, we need to show that the difference between the capacity expressions given in (16) and (17) goes to zero as $P \to \infty$. In order to do this, we will first show that

$$\det(\mathbf{H}_{\mathcal{S}}\mathbf{H}_{\mathcal{S}}^*) = \det\left(\widehat{\mathbf{H}}_{\mathcal{S}}\widehat{\mathbf{H}}_{\mathcal{S}}^*\right). \quad (18)$$

This will imply that the geometric means given by

$$M_g^c \equiv \left(\prod_{i=1}^{r}\frac{1}{\lambda_i^c}\right)^{1/r}, \quad M_g^b \equiv \left(\prod_{i=1}^{r}\frac{1}{\lambda_i^b}\right)^{1/r}$$

are equal and similar argument as in [2] can be used to prove (15).

In the following, we prove (18) by induction. Let us consider the matrix $\mathbf{H}_{\mathcal{S}}$ as given in (2), and let us write

$$\mathbf{H}_{\mathcal{S}}^T = \left[\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\right)^T \ \mathbf{H}_{\pi(|\mathcal{S}|)}^T\right]. \quad (19)$$

From (19), it follows that

$$\mathbf{H}_{\mathcal{S}}\mathbf{H}_{\mathcal{S}}^{*} = \begin{pmatrix} \overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\right)^{*} & \overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\mathbf{H}_{\pi(|\mathcal{S}|)}^{*} \\ \mathbf{H}_{\pi(|\mathcal{S}|)}\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\right)^{*} & \mathbf{H}_{\pi(|\mathcal{S}|)}\mathbf{H}_{\pi(|\mathcal{S}|)}^{*} \end{pmatrix}.$$

According to [12, p. 22] and assuming that the matrix $\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}$ is of full row rank

$$\begin{aligned} \det(\mathbf{H}_{\mathcal{S}}\mathbf{H}_{\mathcal{S}}^{*}) &= \det\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\right)^{*}\right) \\ &\quad\times\det\left(\mathbf{H}_{\pi(|\mathcal{S}|)}\mathbf{H}_{\pi(|\mathcal{S}|)}^{*}-\mathbf{H}_{\pi(|\mathcal{S}|)}\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\right)^{*}\right. \\ &\quad\times\left.\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\right)^{*}\right)^{-1}\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\mathbf{H}_{\pi(|\mathcal{S}|)}^{*}\right). \end{aligned} \tag{20}$$

Now let

$$\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)} = \overline{\mathbf{U}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\overline{\mathbf{\Sigma}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\left(\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\right)^{*} \tag{21}$$

be the singular value decomposition of the matrix $\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}$. Then, from (20) and (21)

$$\begin{aligned} \det(\mathbf{H}_{\mathcal{S}}\mathbf{H}_{\mathcal{S}}^{*}) &= \det\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\right)^{*}\right) \\ &\quad\times\det\left(\mathbf{H}_{\pi(|\mathcal{S}|)}\mathbf{H}_{\pi(|\mathcal{S}|)}^{*}-\mathbf{H}_{\pi(|\mathcal{S}|)}\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\left(\overline{\mathbf{\Sigma}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\right)^{*}\right. \\ &\quad\times\left(\overline{\mathbf{\Sigma}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\left(\overline{\mathbf{\Sigma}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\right)^{*}\right)^{-1} \\ &\quad\times\left.\overline{\mathbf{\Sigma}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\left(\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\right)^{*}\mathbf{H}_{\pi(|\mathcal{S}|)}^{*}\right). \end{aligned} \tag{22}$$

Note that, if we let $\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,1)}$ denote the set of right singular vectors corresponding to the first nonzero singular values of $\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}$, i.e.,

$$\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)} = \overline{\mathbf{U}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\overline{\mathbf{\Sigma}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\left[\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,1)}\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)}\right]^{*}$$

then from (22), we get

$$\begin{aligned} \det(\mathbf{H}_{\mathcal{S}}\mathbf{H}_{\mathcal{S}}^{*}) &= \det\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\right)^{*}\right) \\ &\quad\times\det\left(\mathbf{H}_{\pi(|\mathcal{S}|)}\mathbf{H}_{\pi(|\mathcal{S}|)}^{*}-\mathbf{H}_{\pi(|\mathcal{S}|)}\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,1)}\right. \\ &\quad\times\left.\left(\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,1)}\right)^{*}\mathbf{H}_{\pi(|\mathcal{S}|)}^{*}\right). \end{aligned}$$

Finally, since $\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,1)}(\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,1)})^{*} + \overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)}(\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)})^{*} = \mathbf{I}$

$$\begin{aligned} \det(\mathbf{H}_{\mathcal{S}}\mathbf{H}_{\mathcal{S}}^{*}) &= \det\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\right)^{*}\right) \\ &\quad\times\det\left(\mathbf{H}_{\pi(|\mathcal{S}|)}\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)}\left(\mathbf{H}_{\pi(|\mathcal{S}|)}\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)}\right)^{*}\right). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\det\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-1)}\right)^{*}\right) \\ &\quad= \det\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-2)}\left(\overline{\mathbf{H}}_{\mathcal{S}}^{(|\mathcal{S}|-2)}\right)^{*}\right) \\ &\qquad\times\det\left(\mathbf{H}_{\pi(|\mathcal{S}|-1)}\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-2,0)}\left(\mathbf{H}_{\pi(|\mathcal{S}|-1)}\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-2,0)}\right)^{*}\right) \end{aligned}$$

and if we continue in the same way for each j, we get

$$\det(\mathbf{H}_{\mathcal{S}}\mathbf{H}_{\mathcal{S}}^{*}) = \prod_{j=1}^{|\mathcal{S}|}\det\left(\mathbf{H}_{\pi(j)}\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}\left(\mathbf{H}_{\pi(j)}\overline{\mathbf{V}}_{j}^{(j-1,0)}\right)^{*}\right).$$

Therefore, (18) follows immediately from the definition of $\widehat{\mathbf{H}}_{\mathcal{S}}$.

Now, similar to [2], let

$$M_a^c \equiv \frac{1}{r}\sum_{i=1}^{r}\frac{1}{\lambda_i^c}, \quad M_a^b \equiv \frac{1}{r}\sum_{i=1}^{r}\frac{1}{\lambda_i^b}$$

then, for some $P_0 < \infty$, the waterfilling equation

$$\sum_{i=1}^{r}[\xi^c - 1/\lambda_i^c]_+ = P_0$$

has solution $\xi^c = P_0/r + M_a^c$, and

$$\sum_{i=1}^{r}\left[\xi^b - 1/\lambda_i^b\right]_+ = P_0$$

has solution $\xi^b = P_0/r + M_a^b$. Therefore, for $P \geq P_0$, the sum-rate capacities of the cooperative and SZFDPC schemes are given by

$$C_{\mathcal{S}}^{\mathrm{COOP}} = r\log\left(\frac{P/r + M_a^c}{M_g^c}\right)$$

and

$$C_{\mathcal{S}}^{\mathrm{SZFDPC}} = r\log\left(\frac{P/r + M_a^b}{M_g^b}\right)$$

and in the limit, we get

$$\lim_{P\to\infty}\left(C_{\mathcal{S}}^{\mathrm{COOP}} - C_{\mathcal{S}}^{\mathrm{SZFDPC}}\right) = \lim_{P\to\infty} r\log\frac{1 + rM_a^c/P}{1 + rM_a^b/P} = 0.$$

■

Let the sum-rate capacity of the optimal DPC transmission scheme be denoted by $C_{\mathcal{S}}^{\mathrm{DPC}}$. The following corollary follows directly from Theorem 1 and from the fact that the sum-rate capacity of the optimal DPC scheme is both lower bounded and upper bounded by the sum-rate capacities of the SZFDPC and cooperative schemes, respectively.

Corollary 1: In the high-SNR regime

$$\lim_{P\to\infty}\left(C_{\mathcal{S}}^{\mathrm{SZFDPC}} - C_{\mathcal{S}}^{\mathrm{DPC}}\right) = 0.$$

From the above, we note that in the high-SNR regime, the DPC, as well as the SZFDPC schemes, are optimal in the sense of achieving the cooperative channel sum-rate capacity. Also, note that the above results are true with any user ordering. However, user ordering is important in the low-SNR regime and as it will be later shown, in the low-SNR regime, optimal user ordering is required for the SZFDPC scheme to achieve the sum-rate capacity of the optimal DPC scheme.

IV. Precoding With Successive Zero-Forcing

The DPC technique generally requires nonlinear operations and is currently considered very complex to implement in a practical communication system. In this section, we consider a linear transmission scheme that is based on partial interuser interference cancellation. The motivation for considering this kind of transmission scheme is two-fold. First, compared to other linear transmission schemes, our goal is to improve the performance of the system by increasing the achievable system throughput given a transmit power constraint. Second, given that zero-forcing of the interuser interference is required, we are interested in relaxing the constraint on the number of antennas at the base station versus the number of users' receive antennas. This can increase the number of users that can be supported simultaneously and in return increase the achievable throughput. For example, let us consider the following case where the base station is equipped with four transmit antennas and there are three users in the system with two receive antennas each. Furthermore, it is assumed that the first and second users' channels are highly correlated; however, all other pairs of users' channels are independent. An example for such a channel can be if two of the three users are located close to the base station and the third user is far away. Now, if we consider the nulling constraint given in Section III, there exists a certain ordering of users where the base station can support the three users simultaneously; however, it is not possible in this case to support the three users in an interference free environment. Therefore, this is a simple case where optimization of the sum-rate capacity can be performed over the total number of users in the system. This kind of optimization can not be performed if complete interference cancellation is required.

We will consider the following transmission scheme. Given an ordered set of users $\mathcal{S}$, the base station designs the precoding matrix of each user according to the null space constraint as in Section III, i.e., the precoding matrix $\mathbf{B}_{\pi(j)}$, which corresponds to user $\pi(j)$ in $\mathcal{S}$, is constrained to lie in the null space of $\overline{\mathbf{H}}_{\mathcal{S}}^{j-1}$. However, in contrast to the SZFDPC transmission scheme, the base station does not employ a DPC scheme and users design their receivers such that the remaining interference from other users' transmitted signals is treated as a Gaussian noise that is independent of the desired signal. Because users' precoding matrices are designed successively based on the null space of $\overline{\mathbf{H}}_{\mathcal{S}}^{j-1}$ where complete interference cancellation is not required, and we do not employ any DPC technique for combating the remaining interference from other users, we refer to this scheme as successive zero-forcing (SZF).

A. Sum Rate Capacity of SZF

Given the above transmission scheme, we need to characterize the sum-rate capacity of the SZF scheme and to find a way of designing the optimal set of precoding matrices. Note that from the above constraint on the set of precoding matrices $\{\mathbf{B}_{\pi(j)}\}_{j=1}^{|\mathcal{S}|}$, the received signal for each $j \in \{1, 2, \ldots, |\mathcal{S}|\}$ is given by

$$\mathbf{y}_{\pi(j)} = \mathbf{H}_{\pi(j)} \left(\mathbf{B}_{\pi(j)}\mathbf{u}_{\pi(j)} + \sum_{i<j} \mathbf{B}_{\pi(i)}\mathbf{u}_{\pi(i)} \right) + \mathbf{z}_{\pi(j)}. \tag{23}$$

Now, assuming that the set of vectors $\{\mathbf{u}_{\pi(j)}\}_{j=1}^{|\mathcal{S}|}$ have Gaussian distribution and are independent, it follows that the interference term $\sum_{i<j} \mathbf{B}_{\pi(i)}\mathbf{u}_{\pi(i)}$ is also Gaussian. By treating this spatially correlated interference signal as noise, and by applying a whitening filter on the received signal, a minimum Euclidean distance decoding strategy is optimal [16], and the maximal achievable rate of each user is given by (24), shown at the bottom of the page, for a given ordered set of users $\mathcal{S}$ and set of covariance matrices such that for each j

$$\mathbf{B}_{\pi(j)}\mathbf{B}_{\pi(j)}^* = \overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)} \mathbf{Q}_{\pi(j)} \left(\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)} \right)^*.$$

Hence, given the above rates, for each $\mathcal{S}$ the achievable system throughput of the SZF scheme is given by

$$C_{\mathcal{S}}^{\mathrm{SZF}} = \max_{\{\mathbf{Q}_k\}_{k\in\mathcal{S}}:\mathbf{Q}_k\geq 0, \sum_{k\in\mathcal{S}} \mathrm{Tr}(\mathbf{Q}_k)\leq P} \sum_{j=1}^{|\mathcal{S}|} R_{\pi(j)} \tag{25}$$

and when there are K users in the system, the maximal achievable throughput with multiuser diversity is

$$C^{\mathrm{SZF}} = \max_{\{\mathcal{S}:|\mathcal{S}|\leq K\}} C_{\mathcal{S}}^{\mathrm{SZF}}. \tag{26}$$

In order to find the maximal achievable throughput and the optimal set of precoding matrices, we need to solve the optimization problem given in (26). Therefore, we need first to be able to solve the optimization problem in (25). In general, because of the nonconvexity of the rate equations in (24) in the set of covariance matrices, solving (25) optimally is a difficult problem. We note that from the rate equations in (24), the sum-rate capacity optimization problem of the SZF scheme is similar to the sum-rate capacity optimization problem of the BC with added subspace constraints on the set of covariance matrices. In what follows, a suboptimal optimization algorithm that is based on the existing duality [27] between the BC and the MAC is proposed, and as it turns out, *for several antenna system configurations, the achievable system throughput of the SZF scheme is strictly larger (for all SNR values) than the sum-rate capacity of the block-diagonalization scheme [21], where complete interuser interference cancellation is performed at the base station.*

$$R_{\pi(j)} = \log \frac{\det\left(\mathbf{I} + \mathbf{H}_{\pi(j)} \left(\sum_{i=1}^{j} \overline{\mathbf{V}}_{\mathcal{S}}^{(i-1,0)} \mathbf{Q}_{\pi(i)} \left(\overline{\mathbf{V}}_{\mathcal{S}}^{(i-1,0)}\right)^*\right) \mathbf{H}_{\pi(j)}^*\right)}{\det\left(\mathbf{I} + \mathbf{H}_{\pi(j)} \left(\sum_{i=1}^{j-1} \overline{\mathbf{V}}_{\mathcal{S}}^{(i-1,0)} \mathbf{Q}_{\pi(i)} \left(\overline{\mathbf{V}}_{\mathcal{S}}^{(i-1,0)}\right)^*\right) \mathbf{H}_{\pi(j)}^*\right)} \tag{24}$$

In [27], it was shown that any rate in the capacity region of the BC (achieved with DPC) is in the capacity region of the dual MAC (achieved with successive decoding) under a sum power constraint. Therefore, the sum-rate capacities of the BC and the dual MAC are also equal with

$$C_{\mathrm{BC}}(\mathbf{H}_{\pi(1)}, \ldots, \mathbf{H}_{\pi(|\mathcal{S}|)}, P) = C_{\mathrm{MAC}}(\mathbf{H}^*_{\pi(1)}, \ldots, \mathbf{H}^*_{\pi(|\mathcal{S}|)}, P). \quad (27)$$

The sum-rate capacity of the dual MAC under a sum power constraint is given by (28), shown at the bottom of the page [27], [30], where $\{\mathbf{P}_k\}_{k\in\mathcal{S}}$ is the set of covariance matrices for the MAC. Before we give the optimization algorithm for the SZF transmission scheme, in the following we give a short overview of the iterative waterfilling algorithm [14] that optimally solves the sum-rate capacity of the BC based on the above duality between the BC and the MAC.

BC Iterative Waterfilling Algorithm: In order to solve the optimization problem in (28), an iterative waterfilling algorithm was proposed in [14]. An important feature of the algorithm is the linear complexity with the number of users supported simultaneously in the system. In fact, three algorithms were suggested with different linear complexities and rates of convergence. For the two-user case, the least complex algorithm converges for all channel realizations and with the fastest rate. For the more general case, a hybrid algorithm was suggested which trades off between rate of convergence and optimal convergence point. In this case, a good starting point for rapid convergence is found using the two-user algorithm, and then the iterative algorithm with the least complexity (for $|\mathcal{S}| > 2$) is used to find the optimal sum-rate capacity.

The nth iteration of the iterative waterfilling algorithm [14].

1) Generate effective channels

$$\mathbf{G}_j^{(n)} = \mathbf{H}_{\pi(j)} \left(\mathbf{I} + \sum_{i\neq j} \left(\mathbf{H}_{\pi(i)}\right)^* \mathbf{P}_{\pi(i)}^{(n-1)} \mathbf{H}_{\pi(i)} \right)^{-1/2} \quad j \in \{1, 2, \ldots, |\mathcal{S}|\}.$$

2) Perform waterfilling over an effective block diagonal channel matrix with power constraint P and obtain

$$\left\{\mathbf{S}_j^{(n)}\right\}_{j=1}^{|\mathcal{S}|} = \arg \max_{\{\mathbf{S}_j\}_{j=1}^{|\mathcal{S}|}:\mathbf{S}_j\geq 0, \sum_{j=1}^{|\mathcal{S}|} \mathrm{Tr}(\mathbf{S}_j)\leq P} \sum_{j=1}^{|\mathcal{S}|} \log\det\left(\mathbf{I} + \left(\mathbf{G}_j^{(n)}\right)^* \mathbf{S}_j \mathbf{G}_j^{(n)}\right).$$

3) Update the covariance matrices as

$$\mathbf{P}_{\pi(j)}^{(n)} = \frac{1}{|\mathcal{S}|}\mathbf{S}_j^{(n)} + \frac{|\mathcal{S}|-1}{|\mathcal{S}|}\mathbf{P}_{\pi(j)}^{(n-1)}, \quad j \in \{1, 2, \ldots, |\mathcal{S}|\}.$$

We note here that, initially, the covariance matrices are all initialized with the identity matrix $(P/(\sum_{j\in|\mathcal{S}|} N_j))\mathbf{I}$, and for the first few iterations, the covariance matrices in step 3 are updated by $\mathbf{S}_j^{(n)}$ without using the exponential filter.

After solving (28) using the above optimization algorithm, we obtain the sum-rate capacity with the optimal set of covariance matrices $\{\mathbf{P}_k\}_{k\in\mathcal{S}}$ for the MAC. In order to find the corresponding set of covariance matrices for the BC that is denoted by $\{\overline{\mathbf{Q}}_k\}_{k\in\mathcal{S}}$, we use the transformations provided in [27]. If we consider the simple case when $\mathcal{S} = \{1, 2, \ldots, K\}$, and let

$$\mathbf{D}_j \equiv \left(\mathbf{I} + \mathbf{H}_j \left(\sum_{l=1}^{j-1} \overline{\mathbf{Q}}_l\right) \mathbf{H}_j^*\right)$$

$$\mathbf{E}_j \equiv \left(\mathbf{I} + \sum_{l=j+1}^{K} \mathbf{H}_l^* \mathbf{P}_l \mathbf{H}_l\right)$$

the jth user BC covariance matrix

$$\overline{\mathbf{Q}}_j = \mathbf{E}_j^{-1/2}\mathbf{F}_j\mathbf{G}_j^*\mathbf{D}_j^{1/2}\mathbf{P}_j\mathbf{D}_j^{1/2}\mathbf{G}_j\mathbf{F}_j^*\mathbf{E}_j^{-1/2}.$$

The matrices $\mathbf{F}_j$ and $\mathbf{G}_j$ are obtained from the SVD of

$$\mathbf{E}_j^{-1/2}\mathbf{H}_j^*\mathbf{D}_j^{-1/2} = \mathbf{F}_j\mathbf{\Gamma}_j\mathbf{G}_j^*$$

where $\mathbf{\Gamma}_j$ is the square diagonal singular value matrix. In the following, the sum-rate capacity optimization algorithm for the SZF scheme is given.

SZF Scheme Optimization Algorithm: Given the set of ordered users $\mathcal{S}$, the set of channel realizations $\{\mathbf{H}_k\}_{k\in\mathcal{S}}$, and the power constraint P, do the following.

1) Find the optimal set of covariance matrices $\{\mathbf{P}_k\}_{k\in\mathcal{S}}$ that solves (28) using the iterative waterfilling algorithm.
2) Use the MAC to BC covariance transformations (28) on the set $\{\mathbf{P}_k\}_{k\in\mathcal{S}}$, and let $\{\overline{\mathbf{Q}}_k\}_{k\in\mathcal{S}}$ denote the transformed set of BC covariance matrices.
3) For $j = 1, 2, \ldots, |\mathcal{S}| - 1$, set

$$\mathbf{Q}_{\pi(j)} = \overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)} \left(\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}\right)^* \overline{\mathbf{Q}}_{\pi(j)} \overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)} \left(\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}\right)^*.$$

4) For $j = |\mathcal{S}|$:
   - find $\overline{\mathbf{Q}}_{\pi(|\mathcal{S})|}$ by waterfilling over the effective channel matrix

$$\mathbf{H}_{\text{eff}} = \left(\mathbf{I} + \mathbf{H}_{\pi(|\mathcal{S}|)} \left(\sum_{i=1}^{|\mathcal{S}|-1} \mathbf{Q}_{\pi(i)}\right) \mathbf{H}^*_{\pi(|\mathcal{S}|)}\right)^{-1/2} \times \mathbf{H}_{\pi(|\mathcal{S}|)} \overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)} \left(\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)}\right)^*$$

$$C_{\mathrm{MAC}}(\mathbf{H}^*_{\pi(1)}, \ldots, \mathbf{H}^*_{\pi(|\mathcal{S}|)}, P) = \max_{\{\mathbf{P}_k\}_{k\in\mathcal{S}}:\mathbf{P}_k\geq 0 \sum_{k\in\mathcal{S}} \mathrm{Tr}(\mathbf{P}_k)\leq P} \log\det\left(\mathbf{I} + \sum_{j=1}^{|\mathcal{S}|} \mathbf{H}^*_{\pi(j)}\mathbf{P}_{\pi(j)}\mathbf{H}_{\pi(j)}\right) \quad (28)$$

with power constraint $P - \sum_{i=1}^{|\mathcal{S}|-1} \mathrm{Tr}(\mathbf{Q}_{\pi(i)})$;
- set

$$\mathbf{Q}_{\pi(|\mathcal{S}|)} = \overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)} \left(\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)}\right)^* \overline{\mathbf{Q}}_{\pi(|\mathcal{S}|)} \overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)} \left(\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)}\right)^*.$$

In the third step of the SZF optimization algorithm, we perform a projection of the optimal solution that is obtained by the iterative waterfilling algorithm, and then we project back to the original subspace. In this case, if the projection was unitary we would get the optimal waterfilling solution. Note that, in the above algorithm, $\{\mathbf{Q}_{\pi(j)}\}_{j=1}^{|\mathcal{S}|}$ denote the covariance matrices which include the subspace projections. This type of optimization is similar to solving a relaxed optimization problem, where the optimal solution can be obtained by solving the BC sum-rate capacity optimization problem, and then in order to satisfy the subspace constraint we perform the projection given in the third step.

In the fourth step of the above optimization algorithm, we can design the covariance matrix $\overline{\mathbf{Q}}_{\pi(|\mathcal{S}|)}$ optimally since the covariance matrices or equivalently the interference term contributed by users $1, 2, \ldots, |\mathcal{S}| - 1$ is already determined in steps $1-3$, and since the achievable rates of users $1, 2, \ldots, |\mathcal{S}| - 1$ are not affected by modifying $\overline{\mathbf{Q}}_{\pi(|\mathcal{S}|)}$ due to the subspace projection. Given the set of covariance matrices $\{\mathbf{Q}_{\pi(i)}\}_{i=1}^{|\mathcal{S}|-1}$, the rate of the last user in $\mathcal{S}$ is given by

$$R_{\pi(|\mathcal{S}|)} = \log\det\left(\mathbf{I} + \left(\mathbf{I} + \mathbf{H}_{\pi(|\mathcal{S}|)}\left(\sum_{j=1}^{|\mathcal{S}|-1}\mathbf{Q}_{\pi(j)}\right)\mathbf{H}_{\pi(|\mathcal{S}|)}^*\right)^{-1} \times \mathbf{H}_{\pi(|\mathcal{S}|)}\mathbf{Q}_{\pi(|\mathcal{S}|)}\mathbf{H}_{\pi(|\mathcal{S}|)}^*\right).$$

From the fact that $\det(\mathbf{I} + \mathbf{AB}) = \det(\mathbf{I} + \mathbf{BA})$ for any two matrices $\mathbf{A}$ and $\mathbf{B}$ with suitable dimensions, and if we let

$$\mathbf{A} \equiv \left(\mathbf{I} + \mathbf{H}_{\pi(|\mathcal{S}|)}\left(\sum_{j=1}^{|\mathcal{S}|-1}\mathbf{Q}_{\pi(j)}\right)\mathbf{H}_{\pi(|\mathcal{S}|)}^*\right)^{-1}$$

and

$$\mathbf{B} \equiv \mathbf{H}_{\pi(|\mathcal{S}|)}\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)}\left(\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)}\right)^* \times \overline{\mathbf{Q}}_{\pi(|\mathcal{S}|)}\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)}\left(\overline{\mathbf{V}}_{\mathcal{S}}^{(|\mathcal{S}|-1,0)}\right)^*\mathbf{H}_{\pi(|\mathcal{S}|)}^*$$

then the fourth step in the SZF optimization algorithm follows from

$$\det(\mathbf{I}+\mathbf{AB}) = \det(\mathbf{I}+\mathbf{A}^{1/2}\mathbf{A}^{1/2}\mathbf{B}) = \det(\mathbf{I}+\mathbf{A}^{1/2}\mathbf{B}\mathbf{A}^{1/2}).$$

In the maximization over the different ordered sets of users as given in (26), it is required to maximize the achievable throughput over

$$N_s = \sum_{n=1}^{K} n! \binom{K}{n} \tag{29}$$

different sets. Also note that, from (25), it is clear that the SZF scheme always maximizes over different ordered sets of users that include users with full channel degrees of freedom (i.e., without any subspace constraint). In [24], it was shown that the optimal transmission strategy for a base station with a single transmit antenna is always to transmit to the best user, but this was shown not to be the case [2], in general, when the base station has multiple transmit antennas. However, in certain cases, especially when the SNR is very low the optimal transmission strategy will be always to transmit to the best user as it will be later shown, and in this case the SZF scheme with optimal user ordering achieves the same sum-rate capacity of the SZFDPC and DPC schemes.

B. SZF in the Low-SNR Regime

In [2], it was shown that in the low-SNR regime the ZFDPC scheme has a similar performance as the channel inversion scheme. It is, therefore, of interest to see if a similar relation exists between the SZF and the SZFDPC transmission schemes.

Theorem 2: For any ordered set of users $\mathcal{S}$

$$\lim_{P\to 0} \frac{C_{\mathcal{S}}^{\mathrm{SZFDPC}}}{C_{\mathcal{S}}^{\mathrm{SZF}}} = 1.$$

Proof: From (13) and (17), the sum-rate capacity of the SZFDPC scheme is

$$C_{\mathcal{S}}^{\mathrm{SZFDPC}} = \sum_{i=1}^{r}\left[\log\left(\xi^b\lambda_i^b\right)\right]_+$$

where $\{\lambda_i^b\}_{i=1}^r$ are the eigenvalues of $\widehat{\mathbf{H}}_{\mathcal{S}}\widehat{\mathbf{H}}_{\mathcal{S}}^*$ and ξ^b is the solution to

$$\sum_{i=1}^{r}\left[\xi - 1/\lambda_i^b\right]_+ = P. \tag{30}$$

For P sufficiently small, only the dominant eigenvalue in (30) will be activated, and, therefore, the sum-rate capacity of the SZFDPC scheme is given by

$$C_{\mathcal{S}}^{\mathrm{SZFDPC}} = \log\left(1 + \lambda_{\mathcal{S}}^b P\right) \tag{31}$$

where

$$\lambda_{\mathcal{S}}^b \equiv \max_{j=1}^{|\mathcal{S}|}\left\|\mathbf{H}_{\pi(j)}\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}\right\|^2.$$

Now consider the sum-rate capacity of the SZF scheme given in (25). The sum-rate capacity of the SZF scheme is first upper bounded by the sum-rate capacity of the SZFDPC scheme. Furthermore, from the optimization problem as given in (25), we note that the sum-rate capacity of the SZF scheme is also lower bounded by the capacity of the single-user channel with effective channel response $\mathbf{H}_{\pi(j)}\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}$ for each $j = 1, \ldots, |\mathcal{S}|$. This is very easily seen if we let each of the covariance matrices be equal to zero, except for the jth user. Hence

$$\begin{aligned}\log\left(1 + \lambda_{\mathcal{S}}^b P\right) &\le \max_{j=1}^{|\mathcal{S}|}\max_{\mathbf{Q}\ge 0\,\mathrm{Tr}(\mathbf{Q})\le P}\log\det\left(\mathbf{I} + \mathbf{H}_{\pi(j)}\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)} \times \mathbf{Q}\left(\mathbf{H}_{\pi(j)}\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}\right)^*\right)\\ &\le C_{\mathcal{S}}^{\mathrm{SZF}} \le C_{\mathcal{S}}^{\mathrm{SZFDPC}}.\end{aligned}$$

Therefore, we have that in the low-SNR regime the capacity of the SZF scheme is lower bounded and upper bounded by (31).■

Theorem 3: In the low-SNR regime, the sum-rate capacity of the SZF scheme with optimal user ordering is asymptotically equal to the sum-rate capacity of the optimal DPC scheme

$$\lim_{P\to 0}\frac{C^{\mathrm{DPC}}}{C^{\mathrm{SZF}}}=1.$$

Proof: From Theorem 2, since the sum-rate capacities of the SZF and SZFDPC schemes are asymptotically equal for any ordered set $\mathcal{S}$, we can equivalently consider the sum-rate capacity of the SZFDPC scheme.

The sum-rate capacity of the optimal DPC scheme is given by the sum-rate capacity of the dual MAC. In this case, given the ordered set of users $\mathcal{S}$

$$C_{\mathcal{S}}^{\mathrm{DPC}}=\max_{\{\mathbf{P}_k\}_{k\in\mathcal{S}}:\mathbf{P}_k\geq 0\sum_{k\in\mathcal{S}}\mathrm{Tr}(\mathbf{P}_k)\leq P}\log\det\left(\mathbf{I}+\sum_{j=1}^{|\mathcal{S}|}\mathbf{H}_{\pi(j)}^{*}\mathbf{P}_{\pi(j)}\mathbf{H}_{\pi(j)}\right).$$

Now similar to the proof in Theorem 2, in the low-SNR regime, the capacity of the DPC scheme is given by the eigenvalue which corresponds to the channel with the largest singular value. From (31), given the ordered set of users $\mathcal{S}$, since for each j

$$\left\|\mathbf{H}_{\pi(j)}\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}\right\|_2\leq\left\|\mathbf{H}_{\pi(j)}\right\|_2\left\|\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}\right\|_2\leq\left\|\mathbf{H}_{\pi(j)}\right\|_2$$

we have that the capacity of the SZFDPC scheme is upper bounded by the capacity given by the channel with the maximal eigenvalue. Therefore, by performing optimal user ordering, where we do not have any subspace constraint for the first user in $\mathcal{S}$ we get the above result. ■

Therefore, we can conclude that in the low-SNR regime the SZF transmission scheme is optimal when user ordering is performed and is asymptotically equal to the SZFDPC scheme for any set of ordered users. Furthermore, with optimal user ordering the sum-rate capacity of the SZFDPC scheme is asymptotically equal to the sum-rate capacity of the DPC scheme.

As mentioned earlier, as a linear transmission scheme, the SZF method improves the achievable system throughput. This will be shown later in the simulation results section where we apply the SZF optimization algorithm. In a system satisfying the transmit-receive antenna constraint, it is also possible to compare the throughput of the SZF scheme with other linear transmission schemes such as the BD scheme proposed in [21] where precoding at the base station is performed for interference cancellation. In this case, we will show that complete cancellation of the interuser interference at the base station is a suboptimal transmission strategy, and by letting the users work under limited interference, an improved system throughput is achievable.

Another important issue is the ability of supporting multiple number of users simultaneously. In general, multiuser precoding-based techniques where interference cancellation is required at the base station are strongly limited by the antenna system configuration, by the rank of each user's channel, and by the correlation between different users' channels. In this case, it will be shown that, in some cases, the SZF scheme can simultaneously support an increased number of users while having a gain in the system throughput.

V. Discussion

In general, because of the suboptimality of the SZF optimization algorithm proposed in the previous section, it might be possible to improve over the performance of this algorithm by considering different constrained optimization techniques where a locally or globally optimum point could be found. However, the sum-rate capacity optimization problem of the SZF scheme can in general be written as

$$\begin{aligned}&\max_{\{\mathbf{Q}_k\}_{k\in\mathcal{S}}}\sum_{j=1}^{|\mathcal{S}|}\log\frac{\det\left(\mathbf{I}+\mathbf{H}_{\pi(j)}\left(\sum_{i=1}^{j}\mathbf{Q}_{\pi(i)}\right)\mathbf{H}_{\pi(j)}^{*}\right)}{\det\left(\mathbf{I}+\mathbf{H}_{\pi(j)}\left(\sum_{i=1}^{j-1}\mathbf{Q}_{\pi(i)}\right)\mathbf{H}_{\pi(j)}^{*}\right)}\\ &\text{subject to }\overline{\mathbf{H}}_{\mathcal{S}}^{j-1}\mathbf{Q}_{\pi(j)}\left(\overline{\mathbf{H}}_{\mathcal{S}}^{j-1}\right)^{*}=0,\quad j=2,3,\ldots,|\mathcal{S}| \qquad (32)\\ &\qquad\sum_{j=1}^{|\mathcal{S}|}\mathrm{Tr}(\mathbf{Q}_{\pi(j)})\leq P\\ &\qquad\mathbf{Q}_{\pi(j)}\geq\mathbf{0},\quad j=1,2,\ldots,|\mathcal{S}|.\end{aligned}$$

This form of the SZF optimization problem is similar to the standard form of the BC sum-rate capacity optimization problem, except that in this case we have in addition a set of subspace constraints given by (32). As mentioned above, by ignoring the set of subspace constraints, this problem can be solved optimally using the duality property between the BC and the MAC. The set of subspace constraints, however, is linear in the optimization matrix variables $\{\mathbf{Q}_k\}_{k\in\mathcal{S}}$, and, therefore, it imposes a new set of linear equalities.

We note that the above form of the SZF optimization problem requires an increased number of optimization variables. Let $t_j\equiv(M-\mathrm{rank}(\overline{\mathbf{H}}_{\mathcal{S}}^{j-1}))$ denote the number of column vectors in $\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}$ as defined in Section III. Since each covariance matrix must be positive semi-definite and Hermitian, each $\mathbf{Q}_{\pi(j)}$ can be generated from a corresponding length $t_j\min(t_j,N_j)$ vector $\mathbf{w}_j$ in the following way.

For each $j=1,2,\ldots,|\mathcal{S}|$,

1) Generate the $t_j\times\min(t_j,N_j)$ matrix $\mathbf{B}_{\pi(j)}$ from $\mathrm{unvec}(\mathbf{w}_j)$.
2) Set $\mathbf{Q}_{\pi(j)}=\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}\mathbf{B}_{\pi(j)}\mathbf{B}_{\pi(j)}^{*}\left(\overline{\mathbf{V}}_{\mathcal{S}}^{(j-1,0)}\right)^{*}$.

Therefore, the actual number of optimization variables should be less than what is given in the above general form, and especially for users ordered last in $\mathcal{S}$, the number of optimization variables significantly decreases with t_j.

VI. Simulation Results

In this section, simulation results for the above proposed transmission schemes are provided.

Dirty-Paper Coding-Based Transmission Schemes: In this part of the simulation results, the SZFDPC transmission scheme proposed in Section III is simulated, where we compare the proposed transmission scheme with the other DPC-based transmission schemes. In particular, we consider the optimal DPC and the QR or ZFDPC transmission schemes.

In Fig. 1, a plot of the average sum-rate capacity versus the SNR is provided for a system with two users equipped with two

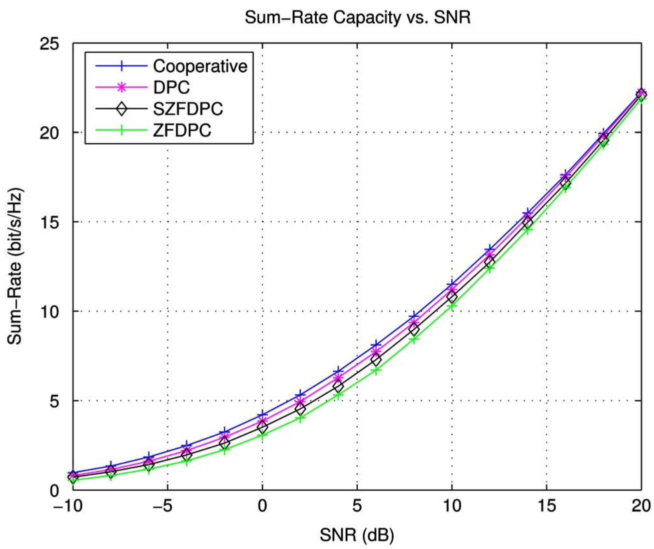

Fig. 1. Dirty-paper coding-based schemes: Sum-rate capacity with arbitrary user ordering, $K = 2, M = 4$, and $N_1 = N_2 = 2$.

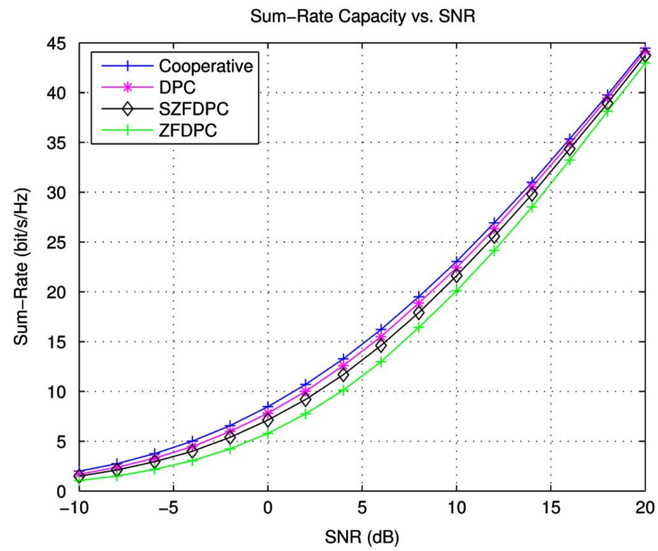

Fig. 2. Dirty-paper coding-based schemes: Sum-rate capacity with arbitrary user ordering, $K = 2, M = 8$, and $N_1 = N_2 = 4$.

receive antennas each and a base station with four transmit antennas. In this case, the channels of all users are generated independently with independent and identically distributed (i.i.d.) complex Gaussian elements $\mathcal{CN}(0, 1)$. First, the performance of the SZFDPC scheme is shown, and it is compared with the ZFDPC scheme and the DPC (optimal noncooperative) transmission scheme. Furthermore, in Fig. 1, we plot the sum-rate capacity of the optimal cooperative transmission scheme. In this case, we do not perform optimal user ordering, and by Monte-Carlo simulation the average sum-rate capacity is computed.

As it can be seen from Fig. 1, and in accordance with Theorem 1, the sum-rate capacity of the SZFDPC scheme is asymptotically optimal in the high-SNR regime, where it approaches the sum-rate capacities of the cooperative and the DPC transmission schemes. In the ZFDPC transmission scheme that is based on the QR decomposition of the channel, each receive antenna is treated as an independent user, and as given in Section II, DPC is performed after the multiplication by a unitary precoding matrix. By comparing the ZFDPC and the SZFDPC transmission schemes, we can see that the achievable throughput is improved for all values of the SNR and the improvement is greater for a medium range of SNR values.

In Fig. 2, a similar simulation as before is performed; however, in this case, we consider a system with two users but with four receive antennas each and a base station equipped with eight transmit antennas. Similar to the previous simulation, we have the same conclusion regarding the asymptotic optimality of the proposed scheme in the high-SNR regime. However, if we compare the performance of the SZFDPC scheme with the ZFDPC scheme, we can see that in this case the improvement in the system throughput is even larger than before. This is an intuitive result because of the fact that in the ZFDPC scheme different users' antennas are treated as independent users. Therefore, changing the antenna configuration in the system such that we increase the degrees of freedom in each user's channel should improve the performance of the SZFDPC scheme relative to the ZFDPC scheme.

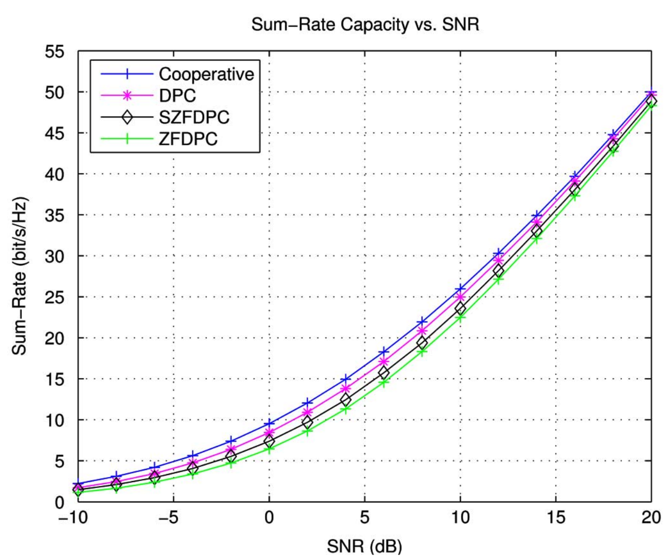

Fig. 3. Dirty-paper coding-based schemes: Sum-rate capacity with arbitrary user ordering, $K = 3, M = 9$, and $N_1 = N_2 = N_3 = 3$.

In Fig. 3, we perform a simulation with three users equipped with three receive antennas each and a base station equipped with nine transmit antennas. Similar conclusions can be also made here, where we can see the improvement in the system throughput when the SZFDPC is considered, and the asymptotic optimality in the high-SNR regime.

In Fig. 4, we plot the sum-rate capacities of the SZFDPC and the ZFDPC transmission schemes as a function of the number of users for an SNR equal to 5 dB. In this case, we also fix the number of receive antennas of each user to two and we let $M = 2K$. From Fig. 4, we can see the improvement in the achievable throughput for all values of K and notice the linear increase in capacity with the number of users in the system.

SZF Versus Linear-Based Transmission Schemes: In the following, the SZF transmission scheme from Section IV is simulated, and a comparison of this scheme with the BD,

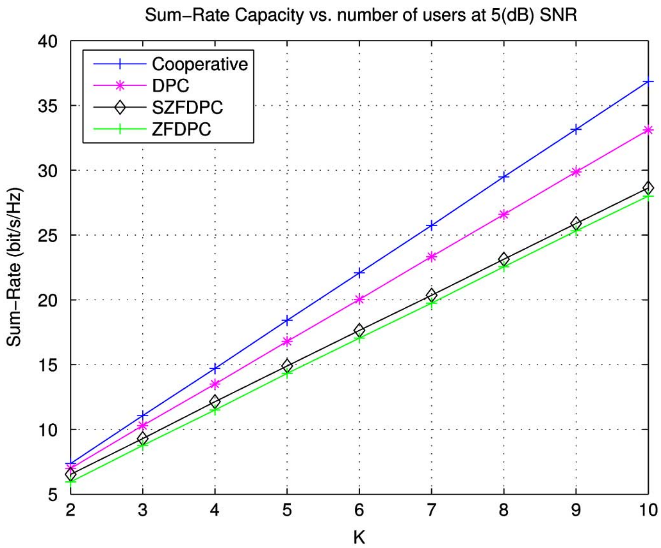

Fig. 4. Dirty-paper coding-based schemes: Sum-rate capacity with arbitrary user ordering, $M = 2K$, and $N_i = 2$ for all i.

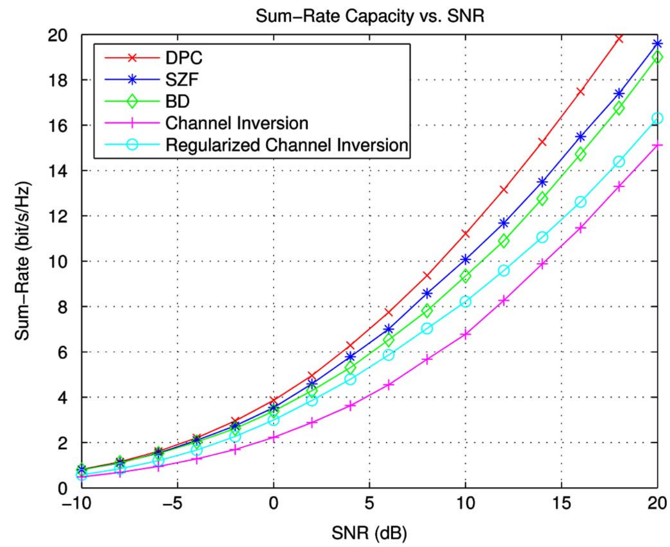

Fig. 5. Achievable throughput of the SZF transmission scheme, $K = 2, M = 4$, and $N_1 = N_2 = 2$.

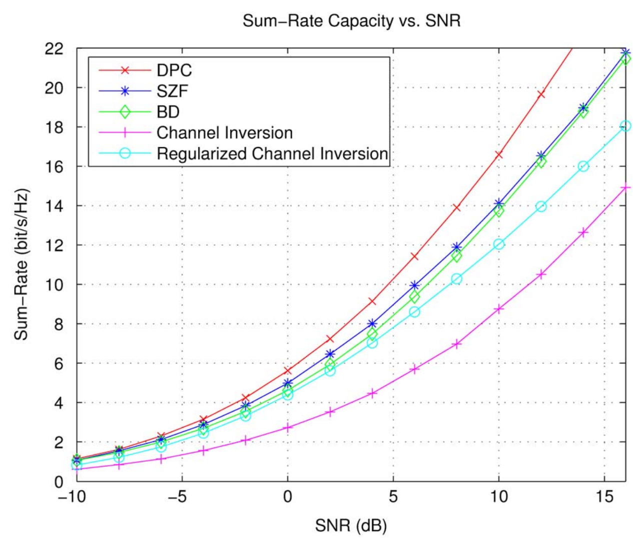

Fig. 6. Achievable throughput of the SZF transmission scheme, $K = 3, M = 6$, and $N_1 = N_2 = N_3 = 2$.

the channel inversion, and the regularized channel inversion schemes is made. We note here that, in both the channel inversion and regularized channel inversion schemes, each user's receive antenna is treated as an independent user and in this case, as given in (4), the total number of receive antennas should be less or equal to the number of transmit antennas. In general, optimal user ordering is performed for each channel realization, and we consider the cases when complete interuser interference cancellation can be performed and when only partial interuser interference cancellation is possible.

In Fig. 5, a simulation of a communication system with two users with two receive antennas each and a base station with four transmit antennas is performed. In this case, it is also assumed that the different users' channels are generated independently with i.i.d. Rayleigh fading elements. For the SZF method, we apply the precoding algorithm given in Section IV where we compute the covariance matrices of the different users and the corresponding achievable rates. The achievable throughput is plotted in Fig. 5. In the same figure, we can also see the sum-rate capacity of the BD transmission scheme and the sum-rate capacity which corresponds to the channel inversion and regularized channel inversion schemes. As mentioned above, in this case for each channel realization we perform optimal user ordering.

As we can see from Fig. 5, with the SZF transmission scheme and for all values of the SNR, the achievable system throughput has improved compared to the BD scheme and the improvement is much more significant compared to the channel inversion scheme. In Fig. 5, we also plot the sum-rate capacity of the DPC scheme. As we can see from the figure, and as given in Theorem 3, the SZF transmission scheme is asymptotically optimal in the low-SNR regime where it approaches the DPC sum-rate capacity.

A similar simulation was performed for a system with three users with two receive antennas each and a base station with six transmit antennas. The results are shown in Fig. 6, and the improvement over the BD and channel inversion schemes is clear. Therefore, we can conclude that *a complete interuser interference cancellation is not necessary and an improved system throughput can be achieved using the SZF method with only partial interference cancellation*.

In Fig. 7, we consider a system with an asymmetric number of users' receive antennas. We assume that there are two users in the system. However, one of the users has two receive antennas and the other has four receive antennas. The base station is equipped with four transmit antennas. In this asymmetric case when the channels are of full rank the BD transmission scheme can not support the two users simultaneously. Using the SZF scheme, the base station can support the two users simultaneously if the user with the two receive antennas is the first user in $\mathcal{S}$. As it can be seen from Fig. 7, the achievable system throughput is larger for all values of the SNR when we compare the SZF scheme with the maximal achievable single user throughput (optimized over the two users). Hence, with the

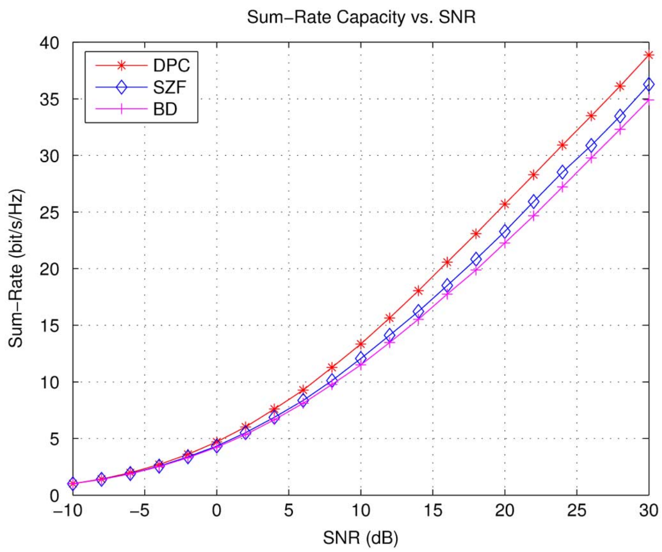

Fig. 7. Achievable throughput of the SZF scheme with asymmetric number of receive antennas, $K = 2, M = 4, N_1 = 2$, and $N_2 = 4$.

suboptimal precoding algorithm proposed in Section IV, it is shown that in spite of the antenna system configuration the two users can be supported simultaneously and an increased system throughput can be achieved.

VII. Conclusion

In this paper, the multiuser broadcast channel with multiple transmit antennas at the base station and multiple receive antennas at each user was considered. Assuming full knowledge of the channel state information at the transmitter and the different receivers, a new transmission scheme based on dirty-paper coding and partial interuser interference cancellation was proposed. In this transmission scheme, it was shown that in the high-SNR regime and for any ordered set of users, the sum-rate capacity is asymptotically equal to the optimal cooperative channel sum-rate capacity. In the low-SNR regime, it was shown that with optimal user ordering the proposed scheme is optimal in the sense of achieving the sum-rate capacity of the dirty-paper coding scheme. In addition, from simulation results, the proposed technique achieves an improved throughput over generalized zero-forcing dirty-paper coding for all values of the SNR.

Because of current practicality issues, a linear transmission scheme was also considered. In this transmission scheme, dirty-paper coding is not implemented and only partial interuser interference cancellation is required. The sum-rate capacity of this approach was characterized, and due to the complexity of the optimal precoding solution a suboptimal optimization algorithm was proposed.

Compared to other linear transmission schemes, such as block-diagonalization and regularized channel inversion, it was shown that the maximum achievable throughput of the proposed transmission scheme is strictly larger and, in some cases, where users' asymmetries or channel correlations are found, this scheme can also support a larger number of users simultaneously in addition to the improvement in the achievable throughput.

Because of the suboptimality of the proposed optimization algorithm, it is of interest to find an improved solutions for the broadcast channel sum-rate capacity optimization problem with arbitrary linear constraints on the set of covariance matrices.

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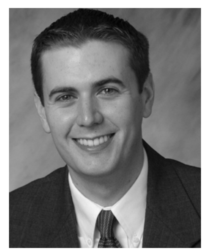

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