Fused Optimal Transport Plan

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Abstract

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1 Introduction

2 Fused Optimal Transport Plan

Notations. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (S, d_S) a compact metric space. A measurable map $X : \Omega \to S$ is called a random element with distribution $\mathbb{P}_X \coloneqq \mathbb{P} \circ X^{-1}$. We also introduce a feature space $M \subset \mathbb{R}^d$ which is compact, and call any one-to-one and continuous $f : S \to M$ a feature function. Throughout this study, we assume that $\mathrm{diam}(S) = \mathrm{diam}(M) = 1$, where $\mathrm{diam}(A) \coloneqq \sup_{x,x' \in A} d_A(x, x')$. For two probability measures μ, ν on S, denote by

$$\Pi(\mu, \nu) \;=\; \{\pi \text{ on } S \times S :\; \text{the marginals are } \mu \text{ and } \nu\}$$

the set of all couplings between μ and ν. Lastly, we say a measurable map $T : S \to S$ pushes forward μ to ν if $\mu(T^{-1}(A)) = \nu(A)$ for all $A \in \mathcal{B}(S)$, where $\mathcal{B}(S)$ is the Borel σ-algebra of S. We denote $T_{\#}\mu = \nu$ if T pushes forward μ to ν.

Fused optimal transport plan. For $0 \le \alpha \le 1$ and a feature function f, we consider

$$\begin{aligned}\inf_{\pi \in \Pi(\mathbb{P}_X, \mathbb{P}_Y)} & (1-\alpha)\mathbb{E}_{(X,Y)\sim\pi}\big[\|f(X) - f(Y)\|_2^2\big] \\ & + \alpha \mathbb{E}_{\substack{(X,Y)\sim\pi \\ (X',Y')\sim\pi}}\Big[K_h(X, X')\big|d_S(X, X') - d_S(Y, Y')\big|^2\Big], \end{aligned} \tag{1}$$

where $K_h : S \times S \to [0, \infty)$ is a bounded, continuous, and symmetric kernel with a bandwidth $h > 0$. We write $\mathcal{S}_{\alpha,h}$ for the set of minimizers of (1).

Proposition 1 (Existence of a minimizer)**.** *For each $0 \le \alpha \le 1$ and $h > 0$, the problem* (1) *admits at least one minimizer; that is, $\mathcal{S}_{\alpha,h} \neq \emptyset$.*

An equivalent formulation in feature space. Let $Z \coloneqq f(X)$ and $W \coloneqq f(Y)$, and consider the set of couplings $\Pi(\mathbb{P}_Z, \mathbb{P}_W)$ on $M \times M$. Define $K_h^g(z, z') \coloneqq K_h(g(z), g(z'))$ with $g \coloneqq f^{-1}$ on $f(S)$. Then, (1) is equivalent to

$$\begin{aligned}\inf_{\gamma \in \Pi(\mathbb{P}_Z, \mathbb{P}_W)} & (1-\alpha)\mathbb{E}_{(Z,W)\sim\gamma}\big[\|Z - W\|_2^2\big] \\ & + \alpha \mathbb{E}_{\substack{(Z,W)\sim\gamma \\ (Z',W')\sim\gamma}}\Big[K_h^g(Z, Z')\big|d_S\big(g(Z), g(Z')\big) - d_S\big(g(W), g(W')\big)\big|^2\Big]. \end{aligned} \tag{2}$$

[*]https://junhyoung-chung.github.io/

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Thus, π is feasible for (1) if and only if $\gamma = (f \times f)_{\#}\pi$ is feasible for (2) with the same objective value. Let $\mathcal{S}^f_\alpha$ be the solution set for (2), which is in fact $\mathcal{S}^f_{\alpha,h} = \{(f \times f)_{\#}\pi : \ \pi \in \mathcal{S}_{\alpha,h}\}$. In the sequel, we focus on analyzing (2) for mathematical simplicity.

3 Local Geometric Preserving Optimal Transport

It is important to study the role of K^g_h in (2). To this end, we state some technical assumptions.

Assumption 1. *We assume the followings:*

- *(A1)* *S is an m-dimensional compact Riemannian manifold.*
- *(A2)* *The density of X, denoted by p_X, is twice-differentiable ($p_X \in \mathcal{C}^2$) and $c \leq p_X \leq C$ for some constants $c, C > 0$.*
- *(A3)* *S and M are $\mathcal{C}^2$-diffeomorphic, and f is a $\mathcal{C}^2$-diffeomorphism.*
- *(A4)* *$K_h(x, x') = h^{-(m+2)}\kappa(d_S(x, x')/h)$ for some $\kappa : [0, \infty) \to [0, \infty)$, where κ is (i) continuous, (ii) bounded, (iii) non-increasing, (iv) compactly supported, and (v) $\int_{\mathbb{R}^m} \kappa(\|u\|)du = 1$.*

References

A Appendix