Fused Optimal Transport Plan

Junhyoung Chung[*]
Department of Statistics
Seoul National University
Seoul 08826, Republic of Korea
junhyoung0534@gmail.com

Abstract

The abstract paragraph should be indented ½ inch (3 picas) on both the left- and right-hand margins. Use 10 point type, with a vertical spacing (leading) of 11 points. The word **Abstract** must be centered, bold, and in point size 12. Two line spaces precede the abstract. The abstract must be limited to one paragraph.

1 Introduction

2 Fused Optimal Transport Plan

Notations. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (S, d_S) a compact metric space. A measurable map $X : \Omega \to S$ is called a random element with distribution $\mathbb{P}_X \coloneqq \mathbb{P} \circ X^{-1}$. We also introduce a feature space $M \subset \mathbb{R}^d$ which is compact, and call any one-to-one and continuous $f : S \to M$ a feature function. Throughout this study, we assume that $\mathrm{diam}(S) = \mathrm{diam}(M) = 1$, where $\mathrm{diam}(A) \coloneqq \sup_{x,x' \in A} d_A(x, x')$. For two probability measures μ, ν on S, denote by

$$\Pi(\mu, \nu) \ = \ \{\pi \text{ on } S \times S : \text{ the marginals are } \mu \text{ and } \nu\}$$

the set of all couplings between μ and ν. Lastly, we say a measurable map $T : S \to S$ pushes forward μ to ν if $\mu(T^{-1}(A)) = \nu(A)$ for all $A \in \mathcal{B}(S)$, where $\mathcal{B}(S)$ is the Borel σ-algebra of S. We denote $T_{\#}\mu = \nu$ if T pushes forward μ to ν.

Fused optimal transport plan. For $0 \leq \alpha \leq 1$ and a feature function f, we consider

$$\begin{aligned} \inf_{\pi \in \Pi(\mathbb{P}_X, \mathbb{P}_Y)} & (1-\alpha)\mathbb{E}_{(X,Y)\sim\pi}\left[\|f(X) - f(Y)\|_2^2\right] \\ & + \alpha \mathbb{E}_{\substack{(X,Y)\sim\pi \\ (X',Y')\sim\pi}} \left[K_h(X, X') \big| d_S(X, X') - d_S(Y, Y')\big|^2\right], \end{aligned} \tag{1}$$

where $K_h : S \times S \to [0, \infty)$ is a bounded, continuous, and symmetric kernel with a bandwidth $h > 0$. We write $\mathcal{S}_{\alpha,h}$ for the set of minimizers of (1).

Proposition 1 (Existence of a minimizer)**.** *For each $0 \leq \alpha \leq 1$ and $h > 0$, the problem* (1) *admits at least one minimizer; that is, $\mathcal{S}_{\alpha,h} \neq \emptyset$.*

3 Local Geometric Preserving Optimal Transport

This section discusses another method to learn an optimal transport while preserving local geometry, which actually coincides with (1) as $h \to 0$. We will investigate this later. We first state some technical assumptions.

[*]https://junhyoung-chung.github.io/

39th Conference on Neural Information Processing Systems (NeurIPS 2025).

Assumption 1. *We assume the followings:*

(A1) S *is an* m*-dimensional compact Riemannian manifold.*

(A2) The density of X*, denoted by* p_X*, is twice-differentiable* ($p_X \in \mathcal{C}^2$) *and* $c \le p_X \le C$ *for some constants* $c, C > 0$*.*

(A3) S *and* M *are* $\mathcal{C}^2$*-diffeomorphic, and* f *is a* $\mathcal{C}^2$*-diffeomorphism.*

(A4) $K_h(x, x') = h^{-(m+2)}\kappa(d_S(x, x')/h)$ *for some* $\kappa : [0, \infty) \to [0, \infty)$*, where* κ *is (i) continuous, (ii) bounded, (iii) non-increasing, (iv) compactly supported, and (v)* $\int_{\mathbb{R}^m} \kappa(\|u\|)du = 1$*.*

Now, consider the following problem:

$$\min_{T \in \mathcal{C}^1 : T_\# \mathbb{P}_X = \mathbb{P}_Y} (1-\alpha)\mathbb{E}_{X \sim \mathbb{P}_X}\left[\|f(X) - f(T(X))\|_2^2\right] + \alpha \int_S \|(\mathcal{J}T)^\top(\mathcal{J}T) - I\|_F^2. \tag{2}$$

Theorem 1. *For* $h > 0$*, let* $R^\pi(h)$ *be defined as*

$$R^\pi(h) \coloneqq \mathbb{E}_{\substack{(X,Y)\sim\pi \\ (X',Y')\sim\pi}}\Big[K_h(X, X')\big|d_S(X, X') - d_S(Y, Y')\big|^2\Big].$$

Under Assumption 1, $R^\pi(h)$ *converges to a limit* $R^\pi(0)$ *as* $h \to 0$*. In particular,*

$$R^\pi(0) = ??$$

References

A Appendix