# Continuous Fused Optimal Transport

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## Abstract

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## 1 Introduction

## 2 Fused Optimal Transport

**Notations.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(S, d_S)$ be a compact metric space. A measurable map $X : \Omega \to S$ is called a random element, and its distribution is defined as $\mathbb{P}_X := \mathbb{P} \circ X^{-1}$. In addition, we introduce a feature space $M \subset \mathbb{R}^d$, which is also a compact region. Any measurable map $f : S \to M$ is called a feature function. For two random elements $X$ and $Y$, we say a measurable map $T : S \to S$ pushes forward $\mathbb{P}_X$ to $\mathbb{P}_Y$, or simply $X$ to $Y$, if $\mathbb{P}_X(T^{-1}(A)) = \mathbb{P}_Y(A)$ for all $A \in \mathcal{B}(S)$, where $\mathcal{B}(S)$ is the Borel $\sigma$-algebra of $S$. We denote $T_\# \mathbb{P}_X = \mathbb{P}_Y$ if $T$ pushes forward $X$ to $Y$.

**Fused optimal transport.** For some feature function $f$, we consider the following problem: for $0 \leq \alpha \leq 1$,

$$\min_{T : T_\# \mathbb{P}_X = \mathbb{P}_Y} (1-\alpha) \mathbb{E}_{X \sim \mathbb{P}_X} \left[ \| f(X) - f(T(X)) \|_2^2 \right] \\ + \alpha \mathbb{E}_{(X, X') \sim \mathbb{P}_X \otimes \mathbb{P}_X} \left[ K_h(X, X') \left| d_S(X, X') - d_S(T(X), T(X')) \right|^2 \right], \quad (1)$$

where $\| \cdot \|_2$ is a Euclidean norm, and $K_h(\cdot, \cdot) : S \times S \to [0, \infty)$ is a bounded symmetric kernel with a bandwidth $h > 0$. Throughout this study, we assume that there exist solutions for (1).

**Assumption 1.** *For all $0 \leq \alpha \leq 1$, there exists a non-empty solution set $\mathcal{S}_\alpha$ for* (1)*.*

(1) reduces to a conventional OT problem when $\alpha = 0$, which have been widely discussed across several literature. In this case, there exists a $\mathbb{P}_X$-a.e. uniqueness solution, denoted as $T^*$, which is guaranteed by the Brenier theorem under some regular assumptions.

**Assumption 2.** *A feature function $f$ is one-to-one and continuous.*

**Assumption 3.** *The distribution of $f(X)$ is dominated by the Lebesgue measure.*

**Assumption 4.** *$f(X)$ and $f(Y)$ have finite second moments, i.e., $\mathbb{E}[\|f(X)\|_2^2], \mathbb{E}[\|f(Y)\|_2^2] < \infty$.*

**Lemma 1.** *Consider a fused optimal transport problem in* (1) *with $\alpha = 0$. Under Assumptions 2 to 4, there exists a $\mathbb{P}_X$-a.e. unique solution $T^*$.*

*https://junhyoung-chung.github.io/

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Lemma 1 illustrates the $\mathbb{P}_X$-a.e. uniqueness of the solution for the special case of (1) with $\alpha = 0$, which is the simple application of the Brenier theorem.

In contrast, global solutions for (1) with $\alpha > 0$ may not be unique due to the non-convexity of the second term. However, we can still guarantee that there exists a $T^*_\alpha \in \mathcal{S}_\alpha$ near by $T^*$ under some required assumptions. To this end, we first introduce the following alternative problem for (1), which is essentially equivalent: for $Z \coloneqq f(X)$,

$$\min_{U:U_\#\mathbb{P}_Z = f_\#\mathbb{P}_Y} (1-\alpha)\mathbb{E}_{Z\sim\mathbb{P}_Z}\left[\|Z - U(Z)\|_2^2\right] + \alpha R(U), \tag{2}$$

where

$$R(U) \coloneqq \mathbb{E}_{(Z,Z')\sim\mathbb{P}_Z\otimes\mathbb{P}_Z}\left[K_h^f(Z,Z')\left|d_S(g(Z),g(Z')) - d_S(g(U(Z)),g(U(Z')))\right|^2\right]. \tag{3}$$

Here $K_h^f(z,z') \coloneqq K_h(g(z),g(z'))$, and $g \coloneqq f^{-1}$, where $f^{-1}$ is defined on $f(S)$.

# A Appendix

## A.1 Proof for Lemma 1

*Proof.* Since $\alpha = 0$, (1) boils down to the following optimization problem:

$$\min_{T:T_\#\mathbb{P}_X = \mathbb{P}_Y} \mathbb{E}\left[\|f(X) - f(T(X))\|_2^2\right]. \tag{4}$$

Define $Z \coloneqq f(X)$, whose distribution $\mathbb{P}_Z$ is equal to $f_\#\mathbb{P}_X$. In addition, by Assumption 2, we can define $g \coloneqq f^{-1}$, where $f^{-1}$ is defined on $f(S)$. Then, the above problem is equivalent to

$$\min_{U:U_\#\mathbb{P}_Z = f_\#\mathbb{P}_Y} \mathbb{E}\left[\|Z - U(Z)\|_2^2\right]. \tag{5}$$

This is because $(f\circ T\circ g)_\#\mathbb{P}_Z = f_\#\mathbb{P}_Y$ when $T_\#\mathbb{P}_X = \mathbb{P}_Y$. Now, the Brenier theorem guarantees that under Assumptions 3 and 4, there exists a $\mathbb{P}_Z$-a.e. unique solution $U^*$ for (5).

Let $T^* \coloneqq g\circ U^*\circ f$. Observe that $T^*_\#\mathbb{P}_X = g_\#(U^*_\#\mathbb{P}_Z) = \mathbb{P}_Y$ and that $f(T^*(X)) = U^*(Z)$, which implies that $T^*$ is a solution for (4).

What remains is to show that $T^*$ is $\mathbb{P}_X$-a.e. unique. Suppose that there exists another solution $\tilde{T}^*$ such that $\mathbb{P}(T^*(X) \neq \tilde{T}^*(X)) > 0$. Then,

$$\mathbb{P}(T^*(X) \neq \tilde{T}^*(X)) = \mathbb{P}(g(U^*(Z)) \neq (\tilde{T}^*\circ g)(Z)) = \mathbb{P}(U^*(Z) \neq (f\circ\tilde{T}^*\circ g)(Z)) > 0.$$

Considering that $\tilde{U}^* \coloneqq f\circ\tilde{T}^*\circ g$ is also a solution for (5), this contradicts to the fact that $U^*$ is $\mathbb{P}_Z$-a.e. unique. $\square$