CSC336: Assignment 2

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1. Let x be an arbitrary number in $\mathbb{R}^n$ for arbitrary $n \geq 1$, by definition, $||x||_2 = \sqrt{\Sigma_{i=1}^n x_i^2}$.

   By customary we define $\Sigma_{k=n}^m$ with $n > m$ as the empty sum (i.e. 0). Then we have

   $||x||_1 = \Sigma_{i=1}^n |x_i| = \sqrt{(\Sigma_{i=1}^n |x_i|)^2} = \sqrt{\Sigma_{i=1}^n x_i^2 + 2\Sigma_{i=1}^n \Sigma_{j=i+1}^n |x_i||x_j|}$.

   Since $|x_i| \geq 0$ for all $1 \leq i \leq n$, $|x_i||x_j| \geq 0$ for all $1 \leq i \leq j \leq n$. So, $2\Sigma_{i=1}^n \Sigma_{j=i+1}^n |x_i||x_j| \geq 0$ for all $n \geq 1$.

   Then we have $||x||_2 = \sqrt{\Sigma_{i=1}^n x_i^2} \leq \sqrt{\Sigma_{i=1}^n x_i^2 + 2\Sigma_{i=1}^n \Sigma_{j=i+1}^n |x_i||x_j|} = ||x||_1$.

2. It is possible. Here is an example.

   $x = (48, 55)^T, ||x||_1 = |48| + |55| = 103, ||x||_2 = \sqrt{48^2 + 55^2} = 73;$

   $y = (13, 84)^T, ||y||_1 = |13| + |84| = 97, ||y||_2 = \sqrt{13^2 + 84^2} = 85;$

   $||x||_1 = 103 > 97 = ||y||_1$ while $||x||_2 = 73 < 85 = ||y||_2$.

3. Since $Ax = b$ and $A\hat{x} = \hat{b}$, we have $A(x - \hat{x}) = b - \hat{b}$.

   So $||b - \hat{b}|| = ||A(x - \hat{x})|| \leq ||A|| \cdot ||x - \hat{x}||$.

   On the other hand, $A^{-1}b = x$, so $||x|| = ||A^{-1}b|| \leq ||A^{-1}|| \cdot ||b||$.

   Since all norms are equal or greater than zero, we have $||b-\hat{b}|| \cdot ||x|| \leq ||A|| \cdot ||A^{-1}|| \cdot ||x-\hat{x}|| \cdot ||b||$, so $\frac{1}{||A|| \cdot ||A^{-1}||} \frac{||b-\hat{b}||}{||b||} \leq \frac{||x-\hat{x}||}{||x||}$, i.e. $\frac{1}{cond(A)} \frac{||b-\hat{b}||}{||b||} \leq \frac{||x-\hat{x}||}{||x||}$.

4. (a) See attachments for code and result.

   (b) By $Af = b$, we have $f = A^{-1}b$, so $||f|| \leq ||A^{-1}|| \cdot ||b||$.

   Since $r = b - A\hat{f}$, we have $||r|| = ||b - A\hat{f}|| = ||Af - A\hat{f}|| = ||A(f - \hat{f})|| \leq ||A|| \cdot ||f - \hat{f}||$.

   Since all norms are non-negative, we have $||r|| \cdot ||f|| \leq ||A|| \cdot ||A^{-1}|| \cdot ||f - \hat{f}|| \cdot ||b||$. So, $\frac{1}{cond(A)} \cdot \frac{||r||}{||b||} \leq \frac{||\hat{f}-f||}{||f||}$

   On the other hand, since $r = b - A\hat{f} = Af - A\hat{f}$, we have $f - \hat{f} = A^{-1}r$. Then, $||f - \hat{f}|| \leq ||A^{-1}|| \cdot ||r||$. We also have $||b|| \leq ||A|| \cdot ||f||$, so $||f - \hat{f}|| \cdot ||b|| \leq ||A^{-1}|| \cdot ||A|| \cdot ||r|| \cdot ||f||$, and we have $\frac{||\hat{f}-f||}{||f||} \leq cond(A) \cdot \frac{||r||}{||b||}$.

   In conclusion, we have $\frac{1}{cond(A)} \cdot \frac{||r||}{||b||} \leq \frac{||\hat{f}-f||}{||f||} \leq cond(A) \cdot \frac{||r||}{||b||}$.

   The calculation results are:

   Lower bound of relative error is 1.925969e-17.

   Upper bound of relative error is 2.088024e-15.

5. As we can see from the part 1 result, the largest n we can have before the error is 100 percent is 12. Note that since $||x|| = 1$, the absolute error and the relative error of $\hat{x}$ are equal.

   Examining the result of $cond(H)$ and $log(cond(H))$ in part 2, we find that there is a linear correlation between $log(cond(H))$ and n.

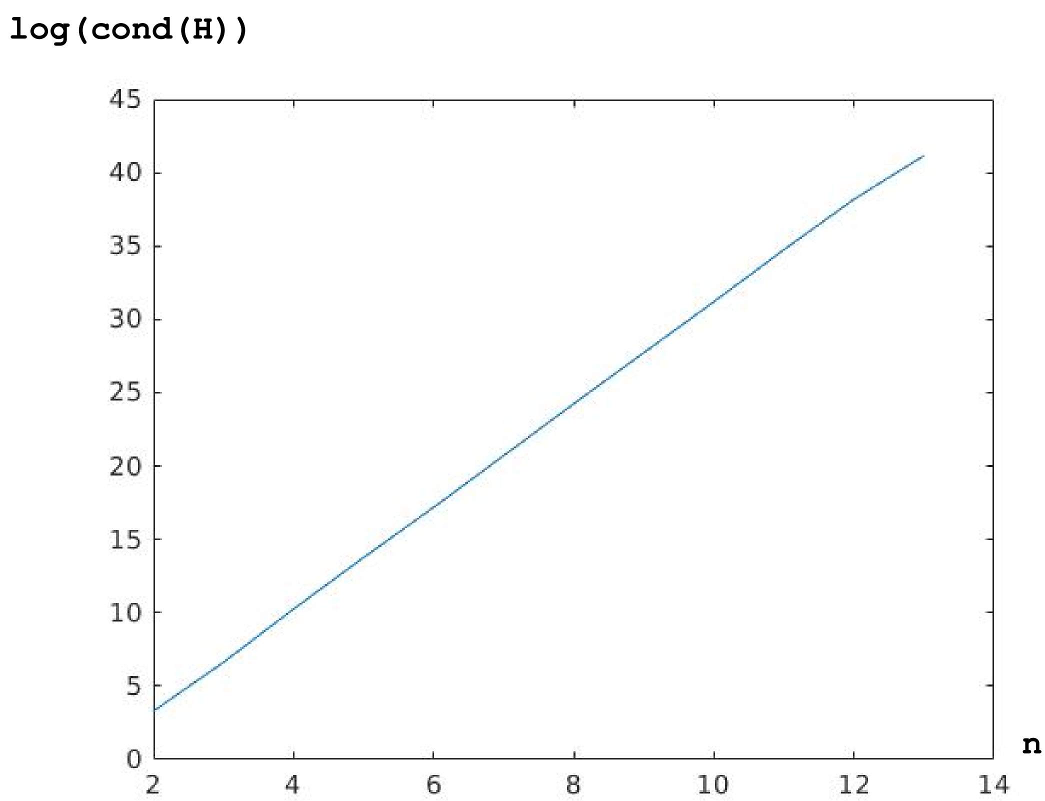
   By Linear Regression we can write $log(cond(H)) = 3.476950n - 3.654113$, i.e.

   $cond(H) = e^{3.476950n-3.654113} = \frac{(e^{3.476950})^n}{e^{3.654113}}$.

   The condition number of H is exponentially growing with respect to n.

   We calculate the number of correct digits in the components of the computed solution as $\lfloor \log_{10} ||x|| \rfloor - \lfloor \log_{10} (\textbf{absolute error}) \rfloor = \lfloor \log_{10} ||x|| \rfloor - \lfloor \log_{10} ||x - \hat{x}|| \rfloor$. (i.e. the place of the most significant digit of $||x||$ − the place of the most significant digit of absolute error).

   We find that, the larger the conditional number is, the smaller the number of correct digits will be.



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```
function Question4(~)

    Alpha = sqrt(2)/2;
    A = zeros(13, 13);

    A(1,2) = 1;
    A(1,6) = -1;
    A(2,3) = 1;
    A(3,1) = Alpha;
    A(3,4) = -1;
    A(3,5) = -Alpha;
    A(4,1) = Alpha;
    A(4,3) = 1;
    A(4,5) = Alpha;
    A(5,4) = 1;
    A(5,8) = -1;
    A(6,7) = 1;
    A(7,5) = Alpha;
    A(7,6) = 1;
    A(7,9) = -Alpha;
    A(7,10) = -1;
    A(8,5) = Alpha;
    A(8,7) = 1;
    A(8,9) = Alpha;
    A(9,10) = 1;
    A(9,13) = -1;
    A(10,11) = 1;
    A(11,8) = 1;
    A(11,9) = Alpha;
    A(11,12) = -Alpha;
    A(12,9) = Alpha;
    A(12,11) = 1;
    A(12,12) = Alpha;
    A(13,13) = 1;
    A(13,12) = Alpha;

    b = zeros(13,1);
    b(2) = 10;
    b(8) = 15;
    b(10) = 20;

    f = A\b %#ok<NOPRT>

    condA = cond(A, 2);

    r = b - A*f;
    r_residue = norm(r,2) / norm(b,2);
    r_error_lower = (1 / condA) *r_residue;
    r_error_upper = condA * r_residue;

    fprintf('Lower bound of relative error is %e.\n', r_error_lower);
    fprintf('Upper bound of relative error is %e.\n', r_error_upper);

end

>> Question4

f =

  -28.2843
   20.0000
   10.0000
  -30.0000
   14.1421
   20.0000
         0
  -30.0000
    7.0711
   25.0000
```

```
   20.0000
  -35.3553
   25.0000

Lower bound of relative error is 1.925969e-17.
Upper bound of relative error is 2.088024e-15.
```

```
function Question5(~)

    n = 1;
    relative_error = 0;
    fprintf('Part1: \n');
    fprintf('\tn\trelative_error\n');
    warning('off');

    while (relative_error < 1)
            n = n+1;
        [x_hat, relative_error] = hat_error(n); %relative error and absolute error are the same.
            fprintf('\t%d\t%e\n', n, relative_error);
    end

    fprintf('\nPart2: \n');
    fprintf('\tn\tcond(H)\t\tlog(cond(H))\n');

    n_2 = (2:13)';
    cond_H = zeros(12,1);
    log_cond_H = zeros(12,1);
    correct = zeros(12,1);

    for i = 2:13
        cond_H(i-1) = cond(hilb(i), Inf);
        log_cond_H(i-1) = log(cond_H(i-1));
        fprintf('\t%d\t%e\t%f\n', i, cond_H(i-1), log_cond_H(i-1));
    end

    plot(n_2, log_cond_H);

    slope_intersect = [ones(12,1), n_2] \ log_cond_H;

    fprintf('By Linear Regression we can write log(cond(H)) = %fn%f\n', slope_intersect(2),
slope_intersect(1));

    fprintf('\nPart3: \n');
    fprintf('\tn\tnorm(x_hat)\tcorrect digits\tcond(H)\n');

    for i = 2:13
        [x_hat, relative_error] = hat_error(i);
        x_hat_norm = norm(x_hat, Inf);
        correct(i-1) = 0-floor(log10(relative_error));
        fprintf('\t%d\t%e\t%d\t\t%e\n', i, x_hat_norm, correct(i-1), cond_H(i-1));
    end
end

function [x_hat,err] = hat_error(n)
    H = hilb(n);
    x = ones(n, 1);
    b = H*x;
    x_hat = H\b;
            err = norm(x_hat-x, Inf);
end
```

```
(Below are the results of the script running on CDF)
>> Question5
Part1:
            n           relative_error
            2           7.771561e-16
            3           7.438494e-15
            4           4.130030e-13
            5           1.130429e-12
            6           5.025884e-10
```

7	1.367104e-08
8	6.551379e-07
9	6.465747e-06
10	6.634040e-04
11	7.171209e-03
12	2.687440e-01
13	3.079042e+00

Part2:

n	cond(H)	log(cond(H))
2	2.700000e+01	3.295837
3	7.480000e+02	6.617403
4	2.837500e+04	10.253264
5	9.436560e+05	13.757517
6	2.907028e+07	17.185227
7	9.851949e+08	20.708350
8	3.387279e+10	24.245878
9	1.099649e+12	27.726013
10	3.535233e+13	31.196386
11	1.229477e+15	34.745365
12	3.841961e+16	38.187344
13	7.490658e+17	41.157603

By Linear Regression we can write log(cond(H)) = 3.476950n-3.654113

Part3:

n	norm(x_hat)	correct digits	cond(H)
2	1.000000e+00	16	2.700000e+01
3	1.000000e+00	15	7.480000e+02
4	1.000000e+00	13	2.837500e+04
5	1.000000e+00	12	9.436560e+05
6	1.000000e+00	10	2.907028e+07
7	1.000000e+00	8	9.851949e+08
8	1.000000e+00	7	3.387279e+10
9	1.000006e+00	6	1.099649e+12
10	1.000663e+00	4	3.535233e+13
11	1.007171e+00	3	1.229477e+15
12	1.227167e+00	1	3.841961e+16
13	4.079042e+00	0	7.490658e+17

(Below are the results of the script running on MatLab online, for larger n, the results are quite different!)
>> Question5
Part1:

n	relative_error
2	7.771561e-16
3	7.438494e-15
4	4.678480e-13
5	3.441691e-13
6	3.988309e-10
7	1.539426e-08
8	5.777080e-07
9	2.109568e-05
10	2.773788e-04
11	2.129638e-02
12	1.143963e-01
13	9.452850e+00

Part2:

n	cond(H)	log(cond(H))
2	2.700000e+01	3.295837
3	7.480000e+02	6.617403
4	2.837500e+04	10.253264
5	9.436560e+05	13.757517
6	2.907028e+07	17.185227
7	9.851949e+08	20.708350
8	3.387279e+10	24.245878
9	1.099649e+12	27.726013
10	3.535233e+13	31.196386
11	1.229477e+15	34.745365
12	3.841961e+16	38.187344
13	7.490658e+17	41.157603

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Part3:

n	norm(x_hat)	correct digits	cond(H)
2	1.000000e+00	16	2.700000e+01
3	1.000000e+00	15	7.480000e+02
4	1.000000e+00	13	2.837500e+04
5	1.000000e+00	13	9.436560e+05
6	1.000000e+00	10	2.907028e+07
7	1.000000e+00	8	9.851949e+08
8	1.000001e+00	7	3.387279e+10
9	1.000018e+00	5	1.099649e+12
10	1.000277e+00	4	3.535233e+13
11	1.021296e+00	2	1.229477e+15
12	1.114396e+00	1	3.841961e+16
13	1.045285e+01	0	7.490658e+17