CSC336: Assignment 2

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1. Let x be an arbitrary number in $\mathbb{R}^n$ for arbitrary $n \geq 1$, by definition, $||x||_2 = \sqrt{\Sigma_{i=1}^n x_i^2}$.

   By customary we define $\Sigma_{k=n}^m$ with $n > m$ as the empty sum (i.e. 0). Then we have

   $||x||_1 = \Sigma_{i=1}^n |x_i| = \sqrt{(\Sigma_{i=1}^n |x_i|)^2} = \sqrt{\Sigma_{i=1}^n x_i^2 + 2\Sigma_{i=1}^n \Sigma_{j=i+1}^n |x_i||x_j|}$.

   Since $|x_i| \geq 0$ for all $1 \leq i \leq n$, $|x_i||x_j| \geq 0$ for all $1 \leq i \leq j \leq n$. So, $2\Sigma_{i=1}^n \Sigma_{j=i+1}^n |x_i||x_j| \geq 0$ for all $n \geq 1$.

   Then we have $||x||_2 = \sqrt{\Sigma_{i=1}^n x_i^2} \leq \sqrt{\Sigma_{i=1}^n x_i^2 + 2\Sigma_{i=1}^n \Sigma_{j=i+1}^n |x_i||x_j|} = ||x||_1$.

2. It is possible. Here is an example.

   $x = (48, 55)^T, ||x||_1 = |48| + |55| = 103, ||x||_2 = \sqrt{48^2 + 55^2} = 73;$

   $y = (13, 84)^T, ||y||_1 = |13| + |84| = 97, ||y||_2 = \sqrt{13^2 + 84^2} = 85;$

   $||x||_1 = 103 > 97 = ||y||_1$ while $||x||_2 = 73 < 85 = ||y||_2$.

3. Since $Ax = b$ and $A\hat{x} = \hat{b}$, we have $A(x - \hat{x}) = b - \hat{b}$.

   So $||b - \hat{b}|| = ||A(x - \hat{x})|| \leq ||A|| \cdot ||x - \hat{x}||$.

   On the other hand, $A^{-1}b = x$, so $||x|| = ||A^{-1}b|| \leq ||A^{-1}|| \cdot ||b||$.

   Since all norms are equal or greater than zero, we have $||b - \hat{b}|| \cdot ||x|| \leq ||A|| \cdot ||A^{-1}|| \cdot ||x - \hat{x}|| \cdot ||b||$, so $\frac{1}{||A|| \cdot ||A^{-1}||} \frac{||b-\hat{b}||}{||b||} \leq \frac{||x-\hat{x}||}{||x||}$, i.e. $\frac{1}{cond(A)} \frac{||b-\hat{b}||}{||b||} \leq \frac{||x-\hat{x}||}{||x||}$.

4. (a) See attachments for code and result.

   (b) By $Af = b$, we have $f = A^{-1}b$, so $||f|| \leq ||A^{-1}|| \cdot ||b||$.

   Since $r = b - A\hat{f}$, we have $||r|| = ||b - A\hat{f}|| = ||Af - A\hat{f}|| = ||A(f - \hat{f})|| \leq ||A|| \cdot ||f - \hat{f}||$.

   Since all norms are non-negative, we have $||r|| \cdot ||f|| \leq ||A|| \cdot ||A^{-1}|| \cdot ||f - \hat{f}|| \cdot ||b||$. So, $\frac{1}{cond(A)} \cdot \frac{||r||}{||b||} \leq \frac{||\hat{f}-f||}{||f||}$

   On the other hand, since $r = b - A\hat{f} = Af - A\hat{f}$, we have $f - \hat{f} = A^{-1}r$. Then, $||f - \hat{f}|| \leq ||A^{-1}|| \cdot ||r||$. We also have $||b|| \leq ||A|| \cdot ||f||$, so $||f - \hat{f}|| \cdot ||b|| \leq ||A^{-1}|| \cdot ||A|| \cdot ||r|| \cdot ||f||$, and we have $\frac{||\hat{f}-f||}{||f||} \leq cond(A) \cdot \frac{||r||}{||b||}$.

   In conclusion, we have $\frac{1}{cond(A)} \cdot \frac{||r||}{||b||} \leq \frac{||\hat{f}-f||}{||f||} \leq cond(A) \cdot \frac{||r||}{||b||}$.

   The calculation results are:

   Lower bound of relative error is 1.925969e-17.

   Upper bound of relative error is 2.088024e-15.

5. As we can see from the part 1 result, the largest n we can have before the error is 100 percent is 12. Note that since $||x|| = 1$, the absolute error and the relative error of $\hat{x}$ are equal.

   Examining the result of $cond(H)$ and $log(cond(H))$ in part 2, we find that $log(cond(H))$ and n are collinear.

   By Linear Regression we can write $log(cond(H)) = 3.476950n - 3.654113$, i.e.

   $cond(H) = e^{3.476950n-3.654113} = \frac{(e^{3.476950})^n}{e^{3.654113}}$. The condition number of H is exponentially growing with respect to n.

   We calculate the number of correct digits in the components of the computed solution as $\lfloor \log ||x|| \rfloor - \lfloor \log (\textbf{absolute error}) \rfloor = \lfloor \log ||x|| \rfloor - \lfloor \log ||x - \hat{x}|| \rfloor$. (i.e. the place of the most significant digit of $||x||$ $-$ the place of the most significant digit of absolute error).

   We find that, the larger the conditional number is, the smaller the number of correct digits will be.