

# CSC336: Assignment 2

Junjie Cheng, 1002770539

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1. Let  $x$  be an arbitrary number in  $\mathbb{R}^n$  for arbitrary  $n \geq 1$ , by definition,  $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ .

By customary we define  $\sum_{k=n}^m$  with  $n > m$  as the empty sum (i.e. 0). Then we have

$$\|x\|_1 = \sum_{i=1}^n |x_i| = \sqrt{(\sum_{i=1}^n |x_i|)^2} = \sqrt{\sum_{i=1}^n x_i^2 + 2\sum_{i=1}^n \sum_{j=i+1}^n |x_i||x_j|}.$$

Since  $|x_i| \geq 0$  for all  $1 \leq i \leq n$ ,  $|x_i||x_j| \geq 0$  for all  $1 \leq i \leq j \leq n$ . So,  $2\sum_{i=1}^n \sum_{j=i+1}^n |x_i||x_j| \geq 0$  for all  $n \geq 1$ .

$$\text{Then we have } \|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2} \leq \sqrt{\sum_{i=1}^n x_i^2 + 2\sum_{i=1}^n \sum_{j=i+1}^n |x_i||x_j|} = \|x\|_1.$$

2. It is possible. Here is an example.

$$x = (48, 55)^T, \|x\|_1 = |48| + |55| = 103, \|x\|_2 = \sqrt{48^2 + 55^2} = 73;$$

$$y = (13, 84)^T, \|y\|_1 = |13| + |84| = 97, \|y\|_2 = \sqrt{13^2 + 84^2} = 85;$$

$$\|x\|_1 = 103 > 97 = \|y\|_1 \text{ while } \|x\|_2 = 73 < 85 = \|y\|_2.$$

3. Since  $Ax = b$  and  $A\hat{x} = \hat{b}$ , we have  $A(x - \hat{x}) = b - \hat{b}$ .

$$\text{So } \|b - \hat{b}\| = \|A(x - \hat{x})\| \leq \|A\| \cdot \|x - \hat{x}\|.$$

$$\text{On the other hand, } A^{-1}b = x, \text{ so } \|x\| = \|A^{-1}b\| \leq \|A^{-1}\| \cdot \|b\|.$$

Since all norms are equal or greater than zero, we have  $\|b - \hat{b}\| \cdot \|x\| \leq \|A\| \cdot \|A^{-1}\| \cdot \|x - \hat{x}\| \cdot \|b\|$ ,

$$\text{so } \frac{1}{\|A\| \cdot \|A^{-1}\|} \frac{\|b - \hat{b}\|}{\|b\|} \leq \frac{\|x - \hat{x}\|}{\|x\|}, \text{ i.e. } \frac{1}{\text{cond}(A)} \frac{\|b - \hat{b}\|}{\|b\|} \leq \frac{\|x - \hat{x}\|}{\|x\|}.$$

4. (a) See attachments for code and result.

- (b) By  $Af = b$ , we have  $f = A^{-1}b$ , so  $\|f\| \leq \|A^{-1}\| \cdot \|b\|$ .

$$\text{Since } r = b - A\hat{f}, \text{ we have } \|r\| = \|b - A\hat{f}\| = \|Af - A\hat{f}\| = \|A(f - \hat{f})\| \leq \|A\| \cdot \|f - \hat{f}\|.$$

Since all norms are non-negative, we have  $\|r\| \cdot \|f\| \leq \|A\| \cdot \|A^{-1}\| \cdot \|f - \hat{f}\| \cdot \|b\|$ . So,

$$\frac{1}{\text{cond}(A)} \cdot \frac{\|r\|}{\|b\|} \leq \frac{\|f - \hat{f}\|}{\|f\|}$$

On the other hand, since  $r = b - A\hat{f} = Af - A\hat{f}$ , we have  $f - \hat{f} = A^{-1}r$ . Then,

$$\|f - \hat{f}\| \leq \|A^{-1}\| \cdot \|r\|. \text{ We also have } \|b\| \leq \|A\| \cdot \|f\|, \text{ so}$$

$$\|f - \hat{f}\| \cdot \|b\| \leq \|A^{-1}\| \cdot \|A\| \cdot \|r\| \cdot \|f\|, \text{ and we have}$$

$$\frac{\|f - \hat{f}\|}{\|f\|} \leq \text{cond}(A) \cdot \frac{\|r\|}{\|b\|}.$$

$$\text{In conclusion, we have } \frac{1}{\text{cond}(A)} \cdot \frac{\|r\|}{\|b\|} \leq \frac{\|f - \hat{f}\|}{\|f\|} \leq \text{cond}(A) \cdot \frac{\|r\|}{\|b\|}.$$

The calculation results are:

Lower bound of relative error is 1.925969e-17.

Upper bound of relative error is 2.088024e-15.

5. As we can see from the part 1 result, the largest  $n$  we can have before the error is 100 percent is 12. Note that since  $\|x\| = 1$ , the absolute error and the relative error of  $\hat{x}$  are equal.

Examining the result of  $\text{cond}(H)$  and  $\log(\text{cond}(H))$  in part 2, we find that there is a linear correlation between  $\log(\text{cond}(H))$  and  $n$ .

By Linear Regression we can write  $\log(\text{cond}(H)) = 3.476950n - 3.654113$ , i.e.

$$\text{cond}(H) = e^{3.476950n - 3.654113} = \frac{(e^{3.476950})^n}{e^{3.654113}}.$$

The condition number of  $H$  is exponentially growing with respect to  $n$ .

We calculate the number of correct digits in the components of the computed solution as  $\lfloor \log_{10} \|x\| \rfloor - \lfloor \log_{10} (\text{absolute error}) \rfloor = \lfloor \log_{10} \|x\| \rfloor - \lfloor \log_{10} \|x - \hat{x}\| \rfloor$ . (i.e. the place of the most significant digit of  $\|x\|$  – the place of the most significant digit of absolute error).

We find that, the larger the conditional number is, the smaller the number of correct digits will be.