Junjie Cheng
1002770539
UTORid: chenjunj

Output and answers for Q1, Q2c, Q4:

```
>> Question1a
gamma           error(1)         error(2)
1.000000e-02    8.881784e-16     0.000000e+00
1.000000e-04    -1.101341e-13    0.000000e+00
1.000000e-06    2.875566e-11     0.000000e+00
1.000000e-08    5.024759e-09     0.000000e+00
1.000000e-10    8.274037e-08     0.000000e+00
1.000000e-12    -2.212172e-05    0.000000e+00
1.000000e-14    -7.992778e-04    0.000000e+00
1.000000e-16    1.102230e-01     0.000000e+00
1.000000e-18    -1.000000e+00    0.000000e+00
1.000000e-20    -1.000000e+00    0.000000e+00
```

The error of the first component is increasing as gamma approaches zero, until the error gets to -1.
The error of the second component is always 0, regardless of the value of gamma.

The computed solution becomes more inaccurate as the absolute value of gamma decreases, until the error of the first component gets to -1. At this time, the computed solution is very inaccurate, as the absolute value of the relative error is 1 if we choose infinity-norm. After this stage, the accuracy will not drop further when gamma decreases.

```
>> Question1b
gamma           error(1)         error(2)
1.000000e-02    0.000000e+00     0.000000e+00
1.000000e-04    0.000000e+00     0.000000e+00
1.000000e-06    0.000000e+00     0.000000e+00
1.000000e-08    0.000000e+00     0.000000e+00
1.000000e-10    0.000000e+00     0.000000e+00
1.000000e-12    0.000000e+00     0.000000e+00
1.000000e-14    0.000000e+00     0.000000e+00
1.000000e-16    0.000000e+00     0.000000e+00
1.000000e-18    0.000000e+00     0.000000e+00
1.000000e-20    0.000000e+00     0.000000e+00
```

The computed solution is very accurate. The relative error of x is always 0 for gamma as small as 10^(-20). The accuracy is not changed when the absolute value of gamma decreases.

Pivoting avoids the rounding errors caused by extremely small leading digonal entry, making the computed solution much more accurate.

```
>> Question1c
gamma           error(1)         error(2)
1.000000e-02    0.000000e+00     0.000000e+00
1.000000e-04    0.000000e+00     0.000000e+00
1.000000e-06    0.000000e+00     0.000000e+00
1.000000e-08    0.000000e+00     0.000000e+00
1.000000e-10    0.000000e+00     0.000000e+00
```

```
1.000000e-12      0.000000e+00      0.000000e+00
1.000000e-14      0.000000e+00      0.000000e+00
1.000000e-16      0.000000e+00      0.000000e+00
1.000000e-18      0.000000e+00      0.000000e+00
1.000000e-20      0.000000e+00      0.000000e+00
```

The "better" approximate solution is also accurate. The relative error of x is always 0 for gamma as small as 10^(-20). The accuracy is not changed when the absolute value of gamma decreases.

The iterative refinement is very effective in this context. It can recover the full accuracy for badly scaled systems by repeatedly reducing the residue.

```
>> Question2c
The element with largest absolute value in U2 is 2.
norm(x-x1, inf) = 1
norm(x-x2, inf) = 0
```

Using infinity-norm, the relative error of the poor approximation x1 is 1, which means some entry of x1 is 100% deviated from the exact solution.

The relative error of the good approximation x2 is 0, which means all entry are exact (i.e. same as the exace solution).
x2 corrects all inaccurate entries in x1.

```
>> Question4

y1 =

     3
     5
     9
     4
    10
     8
     7
     1
     2
     6

q =

     3     5     9     4    10     8     7     1     2     6

y2 =

     3
     5
     9
     4
    10
     8
     7
     1
     2
     6
```

2.a) There is no need to interchange the first row.

$P_1 = I, \quad P_1 A = A.$

$$M_1 = I + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot e_1^T = \begin{pmatrix} 1&0&0&0&0 \\ 1&1&0&0&0 \\ 1&0&1&0&0 \\ 1&0&0&1&0 \\ 1&0&0&0&1 \end{pmatrix}$$

$$M_1 P_1 A = \begin{pmatrix} 1&0&0&0&1 \\ 0&1&0&0&2 \\ 0&-1&1&0&2 \\ 0&-1&-1&1&2 \\ 0&-1&-1&-1&2 \end{pmatrix}$$

There is no need to interchange the second row

$P_2 = I,$

$$M_2 = I + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} e_2^T = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&1&1&0&0 \\ 0&1&0&1&0 \\ 0&1&0&0&1 \end{pmatrix}.$$

$$M_2 P_2 M_1 P_1 A = \begin{pmatrix} 1&0&0&0&1 \\ 0&1&0&0&2 \\ 0&0&1&0&4 \\ 0&0&-1&1&4 \\ 0&0&-1&-1&4 \end{pmatrix}.$$

There is no need to interchange the third row

$P_3 = I$

$$M_3 = I + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} e_3^T = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&1&1&0 \\ 0&0&1&0&1 \end{pmatrix}.$$

$$M_3 P_3 M_2 P_2 M_1 P_1 A = \begin{pmatrix} 1&0&0&0&1 \\ 0&1&0&0&2 \\ 0&0&1&0&4 \\ 0&0&0&1&8 \\ 0&0&0&-1&8 \end{pmatrix}$$

There is no need to interchange the forth row

$P_4 = I.$

$$M_4 = I + e_5 e_3^T = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&1&1 \end{pmatrix}$$

$$M_4 P_4 M_3 P_3 M_2 P_2 M_1 P_1 A = \begin{pmatrix} 1&0&0&0&1 \\ 0&1&0&0&2 \\ 0&0&1&0&4 \\ 0&0&0&1&8 \\ 0&0&0&0&16 \end{pmatrix} = U.$$

Then we have

$$P_4 P_3 P_2 P_1 A = \hat{M}_1^{-1} \hat{M}_2^{-1} \hat{M}_3^{-1} M_4^{-1} U.$$

where $\hat{M}_i = P_{i+1} M_i P_{i+1}^T$ for $i = 1, 2, 3$

Note that $P_4 = P_3 = P_2 = P_1 = I$,

$\hat{M}_i = M_i$

So, we have $P_4 P_3 P_2 P_1 A = M_1^{-1} M_2^{-1} M_3^{-1} M_4^{-1} U.$

$$P = P_4 P_3 P_2 P_1 = I \cdot I \cdot I \cdot I = I.$$

$$L = M_1^{-1} M_2^{-1} M_3^{-1} M_4^{-1}$$

$$= \left(I - \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} e_1^T\right)\left(I - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} e_2^T\right)\left(I - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} e_3^T\right)\left(I - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e_4^T\right)$$

$$= I - \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} e_1^T - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} e_2^T - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} e_3^T - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e_4^T.$$

$$= \begin{pmatrix} 1&0&0&0&0 \\ -1&1&0&0&0 \\ -1&-1&1&0&0 \\ -1&-1&-1&1&0 \\ -1&-1&-1&-1&1 \end{pmatrix}.$$

Note $U = \begin{pmatrix} 1&0&0&0&1 \\ 0&1&0&0&2 \\ 0&0&1&0&4 \\ 0&0&0&1&8 \\ 0&0&0&0&16 \end{pmatrix}$, $LU = \begin{pmatrix} 1&0&0&0&0 \\ -1&1&0&0&0 \\ -1&-1&1&0&0 \\ -1&-1&-1&1&0 \\ -1&-1&-1&-1&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&1 \\ 0&1&0&0&2 \\ 0&0&1&0&4 \\ 0&0&0&1&8 \\ 0&0&0&0&16 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&1 \\ -1&1&0&0&1 \\ -1&-1&1&0&1 \\ -1&-1&-1&1&1 \\ -1&-1&-1&-1&1 \end{pmatrix} = A$

b). ∘ There is no need to interchange row 1. $P_1 = I$

Interchange col 1 and col 5. $Q_1 = \begin{pmatrix} 0&0&0&0&1\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0 \end{pmatrix}$

$P_1AQ_1 = \begin{pmatrix} 1&0&0&0&1\\ 1&1&0&0&-1\\ 1&-1&1&0&-1\\ 1&-1&-1&1&-1\\ 1&-1&-1&-1&-1 \end{pmatrix}$. $\quad M_1 = I - \begin{pmatrix}0\\1\\1\\1\\1\end{pmatrix} e_1^T = \begin{pmatrix} 1&0&0&0&0\\ -1&1&0&0&0\\ -1&0&1&0&0\\ -1&0&0&1&0\\ -1&0&0&0&1 \end{pmatrix} \quad M_1P_1AQ_1 = \begin{pmatrix} 1&0&0&0&1\\ 0&1&0&0&-2\\ 0&-1&1&0&-2\\ 0&-1&-1&1&-2\\ 0&-1&-1&-1&-2 \end{pmatrix}$

∘ There is no need to interchange row 2. $P_2 = I$,

Interchange col 2 and col 5 $Q_2 = \begin{pmatrix} 1&0&0&0&0\\ 0&0&0&0&1\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0 \end{pmatrix}$

$P_2M_1P_1AQQ_2 = \begin{pmatrix} 1&1&0&0&0\\ 0&-2&0&0&1\\ 0&-2&1&0&-1\\ 0&-2&-1&1&-1\\ 0&-2&-1&-1&-1 \end{pmatrix} \quad M_2 = I - \begin{pmatrix}0\\0\\1\\1\\1\end{pmatrix} e_2^T = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&-1&1&0&0\\ 0&-1&0&1&0\\ 0&-1&0&0&1 \end{pmatrix} \quad M_2P_2M_1P_1AQ_1Q_2 = \begin{pmatrix} 1&1&0&0&0\\ 0&-2&0&0&1\\ 0&0&1&0&-2\\ 0&0&-1&1&-2\\ 0&0&-1&-1&-2 \end{pmatrix}$

∘ There is no need to interchange row 3, $P_3 = I$.

Interchange col 3 and col 5 $\quad Q_3 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0\\ 0&0&1&0&0 \end{pmatrix}$

$P_3M_2P_2M_1P_1AQ_1Q_2Q_3 = \begin{pmatrix} 1&1&0&0&0\\ 0&-2&1&0&0\\ 0&0&-2&0&1\\ 0&0&-2&1&-1\\ 0&0&-2&-1&-1 \end{pmatrix} \quad M_3 = I - \begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix} e_3^T = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&-1&1&0\\ 0&0&-1&0&1 \end{pmatrix} M_3P_3M_2P_2M_1P_1AQ_1Q_2Q_3 = \begin{pmatrix} 1&1&0&0&0\\ 0&-2&1&0&0\\ 0&0&-2&0&1\\ 0&0&0&1&-2\\ 0&0&0&-1&-2 \end{pmatrix}$

∘ There is no need to interchange row 4. $P_4 = I$

Interchange col 4 and col 5, $Q_4 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{pmatrix}$

$P_4M_3P_3M_2P_2M_1P_1AQ_1Q_2Q_3Q_4 = \begin{pmatrix} 1&1&0&0&0\\ 0&-2&1&0&0\\ 0&0&-2&1&0\\ 0&0&0&-2&1\\ 0&0&0&-2&-1 \end{pmatrix} \quad M_4 = I - \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} e_4^T = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&-1&1 \end{pmatrix}$

$M_4P_4M_3P_3M_2P_2M_1P_1AQ_1Q_2Q_3Q_4 = \begin{pmatrix} 1&1&0&0&0\\ 0&-2&1&0&0\\ 0&0&-2&1&0\\ 0&0&0&-2&1\\ 0&0&0&0&-2 \end{pmatrix} = U.$

Then we have

$P_4P_3P_2P_1AQ_1Q_2Q_3Q_4 = \hat{M}_1^{-1}\hat{M}_2^{-1}\hat{M}_3^{-1}M_4^{-1}U$, where $\hat{M}_i = P_{i+1}M_iP_{i+1}^T$ for $i = 1,2,3$

Note $P_1 = P_2 = P_3 = P_4 = I$, $\hat{M}_i = M_i$.

So. $P_4P_3P_2P_1AQ_1Q_2Q_3Q_4 = M_1^{-1}M_2^{-1}M_3^{-1}M_4^{-1}U$.

$P = P_4P_3P_2P_1 = I \cdot I \cdot I \cdot I = I$.

$Q = Q_1Q_2Q_3Q_4 = \begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 1&0&0&0&0 \end{pmatrix}$.

$L = M_1^{-1}M_2^{-1}M_3^{-1}M_4^{-1} = \left(I + \begin{pmatrix}0\\1\\1\\1\\1\end{pmatrix}e_1^T\right)\left(I + \begin{pmatrix}0\\0\\1\\1\\1\end{pmatrix}e_2^T\right)\left(I + \begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}e_3^T\right)\left(I + \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}e_4^T\right)$

$= I + \begin{pmatrix}0\\1\\1\\1\\1\end{pmatrix}e_1^T + \begin{pmatrix}0\\0\\1\\1\\1\end{pmatrix}e_2^T + \begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}e_3^T + \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}e_4^T.$

$= \begin{pmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 1&1&1&0&0\\ 1&1&1&1&0\\ 1&1&1&1&1 \end{pmatrix}$

$PAQ = AQ = \begin{pmatrix} 1&0&0&0&1\\ -1&1&0&0&1\\ -1&-1&1&0&1\\ -1&-1&-1&1&1\\ -1&-1&-1&-1&1 \end{pmatrix}\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 1&0&0&0&0 \end{pmatrix} = \begin{pmatrix} 1&1&0&0&0\\ 1&-1&1&0&0\\ 1&-1&-1&1&0\\ 1&-1&-1&-1&1\\ 1&-1&-1&-1&-1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 1&1&1&0&0\\ 1&1&1&1&0\\ 1&1&1&1&1 \end{pmatrix}\begin{pmatrix} 1&1&0&0&0\\ 0&-2&1&0&0\\ 0&0&-2&1&0\\ 0&0&0&-2&1\\ 0&0&0&0&-2 \end{pmatrix} = LU.$

3.a) First swap row 1 and 2 $P_1 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&1 \end{pmatrix}$ $P_1A = \begin{pmatrix} 4&8&4\\1&4&4\\2&10&5 \end{pmatrix}$

$M_1 = I - \begin{pmatrix} 0\\1/4\\1/2 \end{pmatrix} \cdot e_1^T = \begin{pmatrix} 1&0&0\\-1/4&1&0\\-1/2&0&1 \end{pmatrix}$ $M_1P_1A = \begin{pmatrix} 4&8&4\\0&2&3\\0&6&3 \end{pmatrix}$

Then swap row 2 and 3. $P_2 = \begin{pmatrix} 1&0&0\\0&0&1\\0&1&0 \end{pmatrix}$ $P_2M_1P_1A = \begin{pmatrix} 4&8&4\\0&6&3\\0&2&3 \end{pmatrix}$

$M_2 = I - \begin{pmatrix} 0\\0\\1/3 \end{pmatrix} e_2^T = \begin{pmatrix} 1&0&0\\0&1&0\\0&-1/3&0 \end{pmatrix}$ $M_2P_2M_1P_1A = \begin{pmatrix} 4&8&4\\0&6&3\\0&0&2 \end{pmatrix} = U.$

Then we have

$P_2P_1A = \hat{M}_1^{-1}M_2^{-1}U$. where $\hat{M}_1 = P_2M_1P_2^T = \begin{pmatrix} 1&0&0\\-1/2&1&0\\-1/4&0&1 \end{pmatrix}$

So $P = P_2 \cdot P_1 = \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix}$

$L = \hat{M}^{-1} M_2^{-1}$

$= \begin{pmatrix} 1&0&0\\-1/2&1&0\\-1/4&0&1 \end{pmatrix}^{-1} \begin{pmatrix} 1&0&0\\0&1&0\\0&-1/3&0 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 1&0&0\\1/2&1&0\\1/4&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0\\0&1&0\\0&1/3&0 \end{pmatrix}$

$= \begin{pmatrix} 1&0&0\\1/2&1&0\\1/4&1/3&1 \end{pmatrix}$

Note

$PA = \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix} \begin{pmatrix} 1&4&4\\4&8&4\\2&10&5 \end{pmatrix} = \begin{pmatrix} 4&8&4\\2&10&5\\1&4&4 \end{pmatrix}$

$= \begin{pmatrix} 1&0&0\\1/2&1&0\\1/4&1/3&1 \end{pmatrix} \begin{pmatrix} 4&8&4\\0&6&3\\0&0&2 \end{pmatrix} = LU.$

3. b) $Ax = b$, so $PAx = Pb$

Let $\hat{b} = Pb = \begin{pmatrix} 0&1&0\\0&0&1\\1&0&0 \end{pmatrix} \begin{pmatrix} -1\\0\\3 \end{pmatrix} = \begin{pmatrix} 0\\3\\-1 \end{pmatrix}$

Now we have $LUx = PAx = Pb = \hat{b}$

Let $y = Ux$, solve $Ly = \hat{b}$.

$\begin{pmatrix} 1&0&0\\1/2&1&0\\1/4&1/3&1 \end{pmatrix} \begin{pmatrix} y_1\\y_2\\y_3 \end{pmatrix} = \begin{pmatrix} 0\\3\\-1 \end{pmatrix}$

we have. $y_1 = 0$
$\frac{1}{2}y_1 + y_2 = 3.$
$\frac{1}{4}y_1 + \frac{1}{3}y_2 + y_3 = -1.$

so $y_1 = 0$
$y_2 = 3 - \frac{1}{2} \times 0 = 3$
$y_3 = -1 - \frac{1}{4} \times 0 - \frac{1}{3} \times 3 = -2.$

So $y = \begin{pmatrix} 0\\3\\-2 \end{pmatrix}$

Now solve $Ux = y$.

$\begin{pmatrix} 4&8&4\\0&6&3\\0&0&2 \end{pmatrix} \begin{pmatrix} x_1\\x_2\\x_3 \end{pmatrix} = \begin{pmatrix} 0\\3\\-2 \end{pmatrix}.$

$4x_1 + 8x_2 + 4x_3 = 0$
$6x_2 + 3x_3 = 3$
$2x_3 = -2$

so $x_3 = (-2) \times \frac{1}{2} = -1$
$x_2 = [3 - 3 \times (-1)] \times \frac{1}{6} = 1$
$x_1 = (0 - 4 \times (-1) - 8 \times 1) \times \frac{1}{4} = -1$

So $x = \begin{pmatrix} -1\\1\\-1 \end{pmatrix}$

3c). $uv^T = A - \hat{A} = \begin{pmatrix} 0&0&0\\0&0&0\\0&5&0 \end{pmatrix} = \begin{pmatrix} 0\\0\\5 \end{pmatrix} \begin{pmatrix} 0\\1\\0 \end{pmatrix}^T$. $u = \begin{pmatrix} 0\\0\\5 \end{pmatrix}$ $v = \begin{pmatrix} 0\\1\\0 \end{pmatrix}$ satisfies

3d) We have $u=\begin{pmatrix}0\\0\\5\end{pmatrix}$ $v=\begin{pmatrix}0\\1\\0\end{pmatrix}$

According to Algorithm 2.6, we can solve $\hat{A}\hat{x}=b$ in this way

① Solve $Az=u$, so $z=A^{-1}u$.

let $\hat{u}=Pu=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0\\0\\5\end{pmatrix}=\begin{pmatrix}0\\5\\0\end{pmatrix}$

Now we have $LUz=PAz=Pu=\hat{u}=\begin{pmatrix}0\\5\\0\end{pmatrix}$

Let $a=Uz$, Solve $La=\hat{u}$ for a.

$La=\hat{u}$, $\begin{pmatrix}1&0&0\\1/2&1&0\\1/4&1/3&1\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}0\\5\\0\end{pmatrix}$

$a_1=0$ $\qquad a_1=0$

$\frac{1}{2}a_1+a_2=5 \Rightarrow a_2=5-\frac{1}{2}\times0=5$ $\qquad a=\begin{pmatrix}0\\5\\-5/3\end{pmatrix}$

$\frac{1}{4}a_1+\frac{1}{3}a_2+a_3=0 \qquad a_3=0-\frac{1}{4}\times0-\frac{1}{3}\times5=-\frac{5}{3}$

Solve $Uz=a$ for z.

$Uz=a$. $\begin{pmatrix}4&8&4\\0&6&3\\0&0&2\end{pmatrix}\cdot\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}0\\5\\-5/3\end{pmatrix}$

$4z_1+8z_2+4z_3=0 \qquad z_3=\frac{1}{2}\cdot(-\frac{5}{3})=-\frac{5}{6}$

$6z_2+3z_3=5 \Rightarrow z_2=\frac{1}{6}(5-3\times(-\frac{5}{6}))=\frac{5}{4}$

$2z_3=-\frac{5}{3} \qquad z_1=\frac{1}{4}(0-4\times(-\frac{5}{6})-8\times\frac{5}{4})=-\frac{5}{3}$

So, $z=\begin{pmatrix}-5/3\\5/4\\-5/6\end{pmatrix}$

②. from 3b), we have solved $Ax=b$ for x.

So $A^{-1}b=x=\begin{pmatrix}-1\\1\\-1\end{pmatrix}$

③ According to formula (2),

$$\hat{A}^{-1}=(A-uv^T)^{-1}=A^{-1}+\frac{A^{-1}uv^TA^{-1}}{1-v^TA^{-1}u}$$

$$\text{So } \hat{x}=\hat{A}^{-1}b=A^{-1}b+\frac{(A^{-1}u)\cdot v^T\cdot(A^{-1}b)}{1-v^T(A^{-1}u)}$$

$$=\frac{z\cdot v^T\cdot x}{1-v^T\cdot z}+x$$

$$=\frac{\begin{pmatrix}-5/3\\5/4\\-5/6\end{pmatrix}\left[(0\ 1\ 0)\cdot\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right]}{1-(0\ 1\ 0)\begin{pmatrix}-5/3\\5/4\\-5/6\end{pmatrix}}+x=\frac{\begin{pmatrix}-5/3\\5/4\\-5/6\end{pmatrix}}{1-5/4}+x=-4\times\begin{pmatrix}-5/3\\5/4\\-5/6\end{pmatrix}+\begin{pmatrix}-1\\1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}20/3\\-5\\10/3\end{pmatrix}+\begin{pmatrix}-1\\1\\-1\end{pmatrix}=\begin{pmatrix}17/3\\-4\\7/3\end{pmatrix}$$

MatLab code for Q1, Q2c, Q4:

Question1a.m

```
function Question1a(~)
    warning('off');
    b = [1 1]';
    x = [1 -1]';
    fprintf('%s\t\t%s\t%s\n', 'gamma', 'error(1)', 'error(2)');
    for k = 1:10
        gam = 10 ^ (- 2*k);
        %A = [gam -1+gam; 2 1];

        L = [1 0; 2/gam 1];
        U = [gam -1+gam ;0 -1+2/gam];

        y = L\b;
        hat_x= U\y;

        err = hat_x - x;

        fprintf('%e\t%e\t%e\n', gam, err(1), err(2));
    end
end
```

Question1b.m

```
function Question1b(~)
    b = [1 1]';
    x = [1 -1]';
    P = [0 1; 1 0];
    tilde_b = P*b;
    fprintf('%s\t\t%s\t%s\n', 'gamma', 'error(1)', 'error(2)');
    for k = 1:10
        gam = 10 ^ (- 2*k);
        %A = [gam -1+gam; 2 1];
        L = [1 0; gam/2 1];
        U = [2 1; 0 -1+gam/2];

        y = L\tilde_b;
        hat_x= U\y;

        err = hat_x - x;

        fprintf('%e\t%e\t%e\n', gam, err(1), err(2));
    end
end
```

Question1c.m

```
function Question1c(~)
    b = [1 1]';
    x = [1 -1]';
    fprintf('%s\t\t%s\t%s\n', 'gamma', 'error(1)', 'error(2)');
    for k = 1:10
        gam = 10 ^ (- 2*k);
        A = [gam -1+gam; 2 1];
        L = [1 0; 2/gam 1];
        U = [gam -1+gam ;0 -1+2/gam];

        y = L\b;
        hat_x= U\y;
```

```
        r = b - A*hat_x;

        z = L\r;
        e = U\z;
        tilde_x = hat_x + e;

        err = tilde_x - x;

        fprintf('%e\t%e\t%e\n', gam, err(1), err(2));
    end
end
```

Question2c.m

```
function Question2c(~)
    n = 60;
    A = ones(n,n);
    A = A - triu(A);
    A = eye(n) - A;
    A = A + [ones(n-1,1); 0] * [zeros(1,n-1), 1];

    Q = diag(ones(n-1,1),1);
    Q(n,1) = 1;

    [L1, U1, P1] = lu(A);

    [L2, U2] = lu(A*Q);

    fprintf('%s %d.\n', 'The element with largest absolute value in U2 is',
max(max(abs(U2))));

    x = ones(n,1);
    b = A*x;

    y = L1\b;
    x1 = U1\y;

    fprintf('%s %d\n', 'norm(x-x1, inf) =', norm(x-x1, inf));
    y = L2\b;
    z = U2\y;
    x2 = Q * z;
    fprintf('%s %d\n', 'norm(x-x2, inf) =', norm(x-x2, inf));

end
```

Question4.m

```
function Question4(~)
    p = [3 5 9 4 10 8 7 9 10];
    x = [1:10]';

    y1 = perm_a(p,x)
    q = perm_b(p)
    y2 = perm_c(q,x)
end
```

perm_a.m

```
function y = perm_a(p, x)
```

```
    n = length(x);
    y = x;
    for i = 1:n-1
        y([i p(i)]) = y([p(i) i]);
    end
end

perm_b.m
function q = perm_b(p)
    n = length(p) + 1;
    q = 1:n;
    for i = 1:n-1
        q([i p(i)]) = q([p(i) i]);
    end
end

perm_c.m
function y = perm_c(q,x)
    n = length(q);
    y = zeros(n,1);
    for i = 1:n
        y(i) = x(q(i));
    end
end
```