2.a) There is no need to interchange the first row.

$P_1 = I, \quad P_1 A = A.$

$$M_1 = I + \begin{pmatrix} 0\\1\\1\\1\\1 \end{pmatrix} \cdot e_1^T = \begin{pmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 1&0&1&0&0\\ 1&0&0&1&0\\ 1&0&0&0&1 \end{pmatrix}$$

$$M_1 P_1 A = \begin{pmatrix} 1&0&0&0&1\\ 0&1&0&0&2\\ 0&-1&1&0&2\\ 0&-1&-1&1&2\\ 0&-1&-1&-1&2 \end{pmatrix}$$

There is no need to interchange the second row

$P_2 = I,$

$$M_2 = I + \begin{pmatrix} 0\\0\\1\\1\\1 \end{pmatrix} e_2^T = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&1&1&0&0\\ 0&1&0&1&0\\ 0&1&0&0&1 \end{pmatrix}.$$

$$M_2 P_2 M_1 P_1 A = \begin{pmatrix} 1&0&0&0&1\\ 0&1&0&0&2\\ 0&0&1&0&4\\ 0&0&-1&1&4\\ 0&0&-1&-1&4 \end{pmatrix}.$$

There is no need to interchange the third row

$P_3 = I$

$$M_3 = I + \begin{pmatrix} 0\\0\\0\\1\\1 \end{pmatrix} e_3^T = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&1&1&0\\ 0&0&1&0&1 \end{pmatrix}.$$

$$M_3 P_3 M_2 P_2 M_1 P_1 A = \begin{pmatrix} 1&0&0&0&1\\ 0&1&0&0&2\\ 0&0&1&0&4\\ 0&0&0&1&8\\ 0&0&0&-1&8 \end{pmatrix}$$

There is no need to interchange the forth row.

$P_4 = I.$

$$M_4 = I + e_5 e_3^T = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&1&1 \end{pmatrix}$$

$$M_4 P_4 M_3 P_3 M_2 P_2 M_1 P_1 A = \begin{pmatrix} 1&0&0&0&1\\ 0&1&0&0&2\\ 0&0&1&0&4\\ 0&0&0&1&8\\ 0&0&0&0&16 \end{pmatrix} = U.$$

Then we have.

$$P_4 P_3 P_2 P_1 A = \hat{M}_1^{-1} \hat{M}_2^{-1} \hat{M}_3^{-1} M_4^{-1} U.$$

where $\hat{M}_i = P_{i+1} M_i P_{i+1}^T$ for $i = 1, 2, 3$

Note that $P_4 = P_3 = P_2 = P_1 = I$,

$\hat{M}_i = M_i$

So, we have $P_4 P_3 P_2 P_1 A = M_1^{-1} M_2^{-1} M_3^{-1} M_4^{-1} U.$

$$P = P_4 P_3 P_2 P_1 = I \cdot I \cdot I \cdot I = I.$$

$$L = M_1^{-1} M_2^{-1} M_3^{-1} M_4^{-1}$$

$$= \left(I - \begin{pmatrix} 0\\1\\1\\1\\1 \end{pmatrix} e_1^T\right) \left(I - \begin{pmatrix} 0\\0\\1\\1\\1 \end{pmatrix} e_2^T\right) \left(I - \begin{pmatrix} 0\\0\\0\\1\\1 \end{pmatrix} e_3^T\right) \left(I - \begin{pmatrix} 0\\0\\0\\0\\1 \end{pmatrix} e_4^T\right)$$

$$= I - \begin{pmatrix} 0\\1\\1\\1\\1 \end{pmatrix} e_1^T - \begin{pmatrix} 0\\0\\1\\1\\1 \end{pmatrix} e_2^T - \begin{pmatrix} 0\\0\\0\\1\\1 \end{pmatrix} e_3^T - \begin{pmatrix} 0\\0\\0\\0\\1 \end{pmatrix} e_4^T.$$

$$= \begin{pmatrix} 1&0&0&0&0\\ -1&1&0&0&0\\ -1&-1&1&0&0\\ -1&-1&-1&1&0\\ -1&-1&-1&-1&1 \end{pmatrix}$$

Note $U = \begin{pmatrix} 1&0&0&0&1\\ 0&1&0&0&2\\ 0&0&1&0&4\\ 0&0&0&1&8\\ 0&0&0&0&16 \end{pmatrix}$, $LU = \begin{pmatrix} 1&0&0&0&0\\ -1&1&0&0&0\\ -1&-1&1&0&0\\ -1&-1&-1&1&0\\ -1&-1&-1&-1&1 \end{pmatrix} \begin{pmatrix} 1&0&0&0&1\\ 0&1&0&0&2\\ 0&0&1&0&4\\ 0&0&0&1&8\\ 0&0&0&0&16 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&1\\ -1&1&0&0&1\\ -1&-1&1&0&1\\ -1&-1&-1&1&1\\ -1&-1&-1&-1&1 \end{pmatrix} = A$

1 2 3 4 5
5 2 3 4 1
5 1 3 4 2
5 1 2 4 3
5 1 2 3 4

b). ◦ There is no need to interchange row 1. $P_1 = I$.

Interchange col 1 and col 5. $Q_1 = \begin{pmatrix} 0&0&0&0&1\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0 \end{pmatrix}$

$P_1AQ_1 = \begin{pmatrix} 1&0&0&0&1\\ 1&1&0&0&-1\\ 1&-1&1&0&-1\\ 1&-1&-1&1&-1\\ 1&-1&-1&-1&-1 \end{pmatrix}$. $M_1 = I - \begin{pmatrix}0\\1\\1\\1\\1\end{pmatrix}e_1^T = \begin{pmatrix} 1&0&0&0&0\\ -1&1&0&0&0\\ -1&0&1&0&0\\ -1&0&0&1&0\\ -1&0&0&0&1 \end{pmatrix}$ $M_1P_1AQ_1 = \begin{pmatrix} 1&0&0&0&1\\ 0&1&0&0&-2\\ 0&-1&1&0&-2\\ 0&-1&-1&1&-2\\ 0&-1&-1&-1&-2 \end{pmatrix}$

◦ There is no need to interchange row 2. $P_2 = I$,

Interchange col 2 and col 5 $Q_2 = \begin{pmatrix} 1&0&0&0&0\\ 0&0&0&0&1\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0 \end{pmatrix}$

$P_2M_1P_1AQQ_2 = \begin{pmatrix} 1&1&0&0&0\\ 0&-2&0&0&1\\ 0&-2&1&0&-1\\ 0&-2&-1&1&-1\\ 0&-2&-1&-1&-1 \end{pmatrix}$ $M_2 = I - \begin{pmatrix}0\\0\\1\\1\\1\end{pmatrix}e_2^T = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&-1&1&0&0\\ 0&-1&0&1&0\\ 0&-1&0&0&1 \end{pmatrix}$ $M_2P_2M_1P_1AQ_1Q_2 = \begin{pmatrix} 1&1&0&0&0\\ 0&-2&0&0&1\\ 0&0&1&0&-2\\ 0&0&-1&1&-2\\ 0&0&-1&-1&-2 \end{pmatrix}$

◦ There is no need to interchange row 3, $P_3 = I$.

Interchange col 3 and col 5 $Q_3 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0\\ 0&0&1&0&0 \end{pmatrix}$

$P_3M_2P_2M_1P_1AQ_1Q_2Q_3 = \begin{pmatrix} 1&1&0&0&0\\ 0&-2&1&0&0\\ 0&0&-2&0&1\\ 0&0&-2&1&-1\\ 0&0&-2&-1&-1 \end{pmatrix}$ $M_3 = I - \begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}e_3^T = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&-1&1&0\\ 0&0&-1&0&1 \end{pmatrix}$ $M_3P_3M_2P_2M_1P_1AQ_1Q_2Q_3 = \begin{pmatrix} 1&1&0&0&0\\ 0&-2&1&0&0\\ 0&0&-2&0&1\\ 0&0&0&1&-2\\ 0&0&0&-1&-2 \end{pmatrix}$

◦ There is no need to interchange row 4. $P_4 = I$

Interchange col 4 and col 5, $Q_4 = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&1&0 \end{pmatrix}$

$P_4M_3P_3M_2P_2M_1P_1AQ_1Q_2Q_3Q_4 = \begin{pmatrix} 1&1&0&0&0\\ 0&-2&1&0&0\\ 0&0&-2&1&0\\ 0&0&0&-2&1\\ 0&0&0&-2&-1 \end{pmatrix}$ $M_4 = I - \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}e_4^T = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&-1&1 \end{pmatrix}$

$$M_4P_4M_3P_3M_2P_2M_1P_1AQ_1Q_2Q_3Q_4 = \begin{pmatrix} 1&1&0&0&0\\ 0&-2&1&0&0\\ 0&0&-2&1&0\\ 0&0&0&-2&1\\ 0&0&0&0&-2 \end{pmatrix} = U.$$

Then we have

$P_4P_3P_2P_1AQ_1Q_2Q_3Q_4 = \hat{M}_1^{-1}\hat{M}_2^{-1}\hat{M}_3^{-1}M_4^{-1}U$, where $\hat{M}_i = P_{i+1}M_iP_{i+1}^T$ for $i = 1,2,3$

Note $P_1 = P_2 = P_3 = P_4 = I$, $\hat{M}_i = M_i$.

So. $P_4P_3P_2P_1AQ_1Q_2Q_3Q_4 = M_1^{-1}M_2^{-1}M_3^{-1}M_4^{-1}U$.

$P = P_4P_3P_2P_1 = I \cdot I \cdot I \cdot I = I$.

$Q = Q_1Q_2Q_3Q_4 = \begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 1&0&0&0&0 \end{pmatrix}$.

$$L = M_1^{-1}M_2^{-1}M_3^{-1}M_4^{-1} = \left(I + \begin{pmatrix}0\\1\\1\\1\\1\end{pmatrix}e_1^T\right)\left(I + \begin{pmatrix}0\\0\\1\\1\\1\end{pmatrix}e_2^T\right)\left(I + \begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}e_3^T\right)\left(I + \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}e_4^T\right)$$

$$= I + \begin{pmatrix}0\\1\\1\\1\\1\end{pmatrix}e_1^T + \begin{pmatrix}0\\0\\1\\1\\1\end{pmatrix}e_2^T + \begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}e_3^T + \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}e_4^T.$$

$$= \begin{pmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 1&1&1&0&0\\ 1&1&1&1&0\\ 1&1&1&1&1 \end{pmatrix},$$

$$PAQ = AQ = \begin{pmatrix} 1&0&0&0&1\\ -1&1&0&0&1\\ -1&-1&1&0&1\\ -1&-1&-1&1&1\\ -1&-1&-1&-1&1 \end{pmatrix}\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 1&0&0&0&0 \end{pmatrix} = \begin{pmatrix} 1&1&0&0&0\\ 1&-1&1&0&0\\ 1&-1&-1&1&0\\ 1&-1&-1&-1&1\\ 1&-1&-1&-1&-1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 1&1&1&0&0\\ 1&1&1&1&0\\ 1&1&1&1&1 \end{pmatrix}\begin{pmatrix} 1&1&0&0&0\\ 0&-2&1&0&0\\ 0&0&-2&1&0\\ 0&0&0&-2&1\\ 0&0&0&0&-2 \end{pmatrix} = LU.$$

3.a) First swap row 1 and 2 $P_1 = \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}$ $P_1A = \begin{pmatrix} 4&8&4 \\ 1&4&4 \\ 2&10&5 \end{pmatrix}$

$M_1 = I - \begin{pmatrix} 0 \\ 1/4 \\ 1/2 \end{pmatrix} \cdot e_1^T = \begin{pmatrix} 1&0&0 \\ -1/4&1&0 \\ -1/2&0&1 \end{pmatrix}$ $M_1P_1A = \begin{pmatrix} 4&8&4 \\ 0&2&3 \\ 0&6&3 \end{pmatrix}$

Then swap row 2 and 3. $P_2 = \begin{pmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}$ $P_2M_1P_1A = \begin{pmatrix} 4&8&4 \\ 0&6&3 \\ 0&2&3 \end{pmatrix}$

$M_2 = I - \begin{pmatrix} 0 \\ 0 \\ 1/3 \end{pmatrix} e_2^T = \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&-1/3&0 \end{pmatrix}$ $M_2P_2M_1P_1A = \begin{pmatrix} 4&8&4 \\ 0&6&3 \\ 0&0&2 \end{pmatrix} = U.$

Then we have

$$P_2P_1A = \hat{M}_1^{-1}M_2^{-1}U. \quad \text{where } \hat{M}_1 = P_2M_1P_2^T = \begin{pmatrix} 1&0&0 \\ -1/2&1&0 \\ -1/4&0&1 \end{pmatrix}$$

So $P = P_2 \cdot P_1 = \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix}$

Note

$PA = \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix}\begin{pmatrix} 1&4&4 \\ 4&8&4 \\ 2&10&5 \end{pmatrix} = \begin{pmatrix} 4&8&4 \\ 2&10&5 \\ 1&4&4 \end{pmatrix}$

$= \begin{pmatrix} 1&0&0 \\ 1/2&1&0 \\ 1/4&1/3&1 \end{pmatrix}\begin{pmatrix} 4&8&4 \\ 0&6&3 \\ 0&0&2 \end{pmatrix} = LU.$

$L = \hat{M}^{-1} M_2^{-1}$

$= \begin{pmatrix} 1&0&0 \\ -1/2&1&0 \\ -1/4&0&1 \end{pmatrix}^{-1}\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&-1/3&0 \end{pmatrix}^{-1}$

$= \begin{pmatrix} 1&0&0 \\ 1/2&1&0 \\ 1/4&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&1/3&0 \end{pmatrix}$

$= \begin{pmatrix} 1&0&0 \\ 1/2&1&0 \\ 1/4&1/3&1 \end{pmatrix}$

3. b) $Ax = b$, so $PAx = Pb$

Let $\hat{b} = Pb = \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$

Now we have $LUx = PAx = Pb = \hat{b}$

Let $y = Ux$, solve $Ly = \hat{b}$.

$\begin{pmatrix} 1&0&0 \\ 1/2&1&0 \\ 1/4&1/3&1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$

we have. $y_1 = 0$
$\frac{1}{2}y_1 + y_2 = 3.$
$\frac{1}{4}y_1 + \frac{1}{3}y_2 + y_3 = -1.$

so $y_1 = 0$
$y_2 = 3 - \frac{1}{2}\times 0 = 3.$
$y_3 = -1 - \frac{1}{4}\times 0 - \frac{1}{3}\times 3 = -2.$

So $y = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}$

Now solve $Ux = y$.

$\begin{pmatrix} 4&8&4 \\ 0&6&3 \\ 0&0&2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}.$

$4x_1 + 8x_2 + 4x_3 = 0$
$6x_2 + 3x_3 = 3$
$2x_3 = -2$

so $x_3 = (-2) \times \frac{1}{2} = -1$
$x_2 = [3 - 3\times(-1)]\times\frac{1}{6} = 1$
$x_1 = (0 - 4\times(-1) - 8\times 1)\times\frac{1}{4} = -1$

So $x = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

3c). $uv^T = A - \hat{A} = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&5&0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^T$ $u = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$ $v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ satisfies

3d) We have $u=\begin{pmatrix}0\\0\\5\end{pmatrix}$ $v=\begin{pmatrix}0\\1\\0\end{pmatrix}$

According to Algorithm 2.6, we can solve $\hat{A}\hat{x}=b$ in this way

① Solve $Az=u$, so $z=A^{-1}u$.

Let $\hat{u}=Pu=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0\\0\\5\end{pmatrix}=\begin{pmatrix}0\\5\\0\end{pmatrix}$

Now we have $LUz=PAz=Pu=\hat{u}=\begin{pmatrix}0\\5\\0\end{pmatrix}$

Let $a=Uz$, Solve $La=\hat{u}$ for a.

$La=\hat{u}$, $\begin{pmatrix}1&0&0\\1/2&1&0\\1/4&1/3&1\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}0\\5\\0\end{pmatrix}$

$a_1=0$ $\qquad$ $u_1=0$

$\frac{1}{2}a_1+a_2=5$ $\quad\Rightarrow\quad$ $a_2=5-\frac{1}{2}\times0=5$ $\qquad$ $a=\begin{pmatrix}0\\5\\-5/3\end{pmatrix}$

$\frac{1}{4}a_1+\frac{1}{3}a_2+a_3=0$ $\qquad$ $a_3=0-\frac{1}{4}\times0-\frac{1}{3}\times5=-\frac{5}{3}$

Solve $Uz=a$ for z.

$Uz=a$. $\begin{pmatrix}4&8&4\\0&6&3\\0&0&2\end{pmatrix}\cdot\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}=\begin{pmatrix}0\\5\\-5/3\end{pmatrix}$

$4z_1+8z_2+4z_3=0$ $\qquad$ $z_3=\frac{1}{2}\cdot(-\frac{5}{3})=-\frac{5}{6}$

$6z_2+3z_3=5$ $\quad\Rightarrow\quad$ $z_2=\frac{1}{6}(5-3\times(-\frac{5}{6}))=\frac{5}{4}$

$2z_3=-\frac{5}{3}$ $\qquad$ $z_1=\frac{1}{4}(0-4\times(-\frac{5}{6})-8\times\frac{5}{4})=-\frac{5}{3}$

So, $z=\begin{pmatrix}-5/3\\5/4\\-5/6\end{pmatrix}$

②. from 3b), we have solved $Ax=b$ for x.

So $A^{-1}b=x=\begin{pmatrix}-1\\1\\-1\end{pmatrix}$

③ According to formula (2),

$$\hat{A}^{-1}=(A-uv^T)^{-1}=A^{-1}+\frac{A^{-1}uv^TA^{-1}}{1-v^TA^{-1}u}$$

So $\hat{x}=\hat{A}^{-1}b=A^{-1}b+\dfrac{(A^{-1}u)\cdot v^T\cdot(A^{-1}b)}{1-v^T(A^{-1}u)}$

$$=\frac{z\cdot v^T\cdot x}{1-v^T\cdot z}+x$$

$$=\frac{\begin{pmatrix}-5/3\\5/4\\-5/6\end{pmatrix}\left[(0\ 1\ 0)\cdot\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right]}{1-(0\ 1\ 0)\begin{pmatrix}-5/3\\5/4\\-5/6\end{pmatrix}}+x=\frac{\begin{pmatrix}-5/3\\5/4\\-5/6\end{pmatrix}}{1-5/4}+x=-4\times\begin{pmatrix}-5/3\\5/4\\-5/6\end{pmatrix}+\begin{pmatrix}-1\\1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}20/3\\-5\\10/3\end{pmatrix}+\begin{pmatrix}-1\\1\\-1\end{pmatrix}=\begin{pmatrix}17/3\\-4\\7/3\end{pmatrix}$$