Preprocessing

I. Gridding 网格

一. exact interpolation 精确插值

$y(t_i) = a_1 q_1(t_i) + a_2 q_2(t_i) + \cdots + a_n q_n(t_i)$ $n = m$ = numbers of unknowns

$$\underset{n\times 1}{\begin{bmatrix} y_{01} \\ y_{02} \\ \vdots \\ y_{0n} \end{bmatrix}} = \underset{n\times n}{\begin{bmatrix} q_1(t_1) & q_2(t_1) & \cdots & q_n(t_1) \\ q_1(t_2) & q_2(t_2) & & q_n(t_2) \\ q_1(t_3) & \vdots & \ddots & \vdots \\ \vdots & & & \\ q_1(t_n) & q_2(t_n) & \cdots & q_n(t_n) \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}$$

$$\underline{y} \qquad \underline{Q} \qquad \underline{a}$$

$$\hat{y}(t) = y_{01} q_1(t) + y_{02} q_2(t) + \cdots + y_{0n} q_n(t)$$

1. Langrange polynomial interpolation

$$y(t) = a_1 + a_2 t + a_3 t^2 + \cdots + a_n t^{n-1}$$

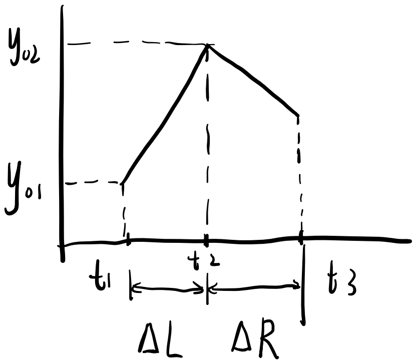
$$\begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix}$$

$$q_i(t) = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(t - t_j)}{(t_i - t_j)} = \frac{(t-t_1)(t-t_2)\cdots(t-t_{i-1})(t-t_{i+1})\cdots(t-t_n)}{(t_i-t_1)(t_i-t_2)\cdots(t_i-t_{i-1})(t_i-t_{i+1})\cdots(t_i-t_n)}$$

$$\hat{y}(t) = y_{01} q_1(t) + y_{02} q_2(t) + \cdots + y_{0n} q_n(t)$$

2D Langrange

2. Linear spline (1D)

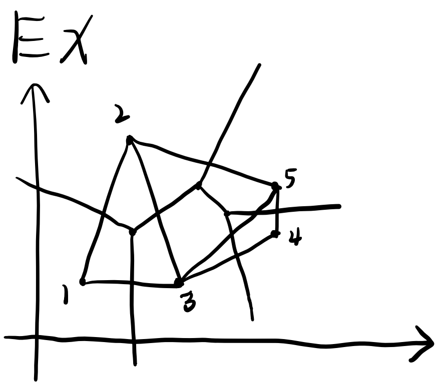


$$S(t, \Delta L, \Delta R) = \begin{cases} 1 + \frac{t}{\Delta L} & -\Delta L \le t \le 0 \\ 1 - \frac{t}{\Delta R} & 0 \le t \le \Delta R \\ 0 & \end{cases}$$

$$\hat{y}(t) = \sum_i y_{0i} \, S(t - t_i, \Delta L_i, \Delta R_i)$$

3. TIN: Triangular Irregular Network

Ex



	x	y	z
P_1	1	1	2.1
P_2	2	4	3.2
P_3	3	1	2.6
P_4	5	2	3.2
P_5	5	3	3.6

$P(3,3) = ?$ by TIN

is included in triangle 2, 3, 5

$$z = ax + by + c$$

$$\begin{cases} 3.2 = a \cdot 2 + b \cdot 4 + c & ① \\ 2.6 = a \cdot 3 + b \cdot 1 + c & ② \\ 3.6 = a \cdot 5 + b \cdot 3 + c & ③ \end{cases}$$

① − ② $\quad 0.6 = -a + 3b \quad \times 2$

③ − ② $\quad 1.0 = 2a + 2b$

$+ \rightarrow 3.2 = 8b \longrightarrow \hat{b} = 0.4$

$\hat{a} = 0.6 \quad \hat{c} = 0.3$

$$z(3,3) = 0.6 \cdot 3 + 0.4 \cdot 3 + 0.4 = 3.4$$

covariance propagation

$$\underset{Z_0}{\begin{bmatrix} 3.2 \\ 2.6 \\ 3.6 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 2 & 4 & 1 \\ 3 & 1 & 1 \\ 5 & 3 & 1 \end{bmatrix}} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

all observations have the same accuracy $\sigma v = 0.1$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^{-1} Z_0$$

$Z_0 = Z + V$

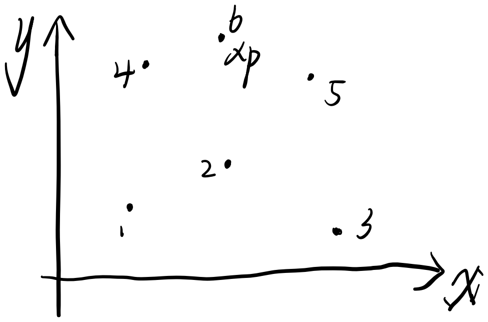
$C_{pp} = (A^{-1}) C_{vv} (A^{-1})^T =$

$C_{zz} = C_{vv} = (0.1)^2 I$

二. other

IDW: Inverse Distance Weighting

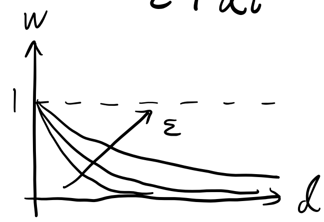
distance↑ weight↓



$\mu = \frac{1}{N}\sum_i z_i$ $\underset{(Avg)}{\mu_w} = \frac{\sum_1^b w_i z_i}{\sum_i w_i} = \hat{z}_p$ $w_i = \frac{1}{d(p_i, p)}$

d: distance

$w_i = \frac{1}{\varepsilon + d_i} \longrightarrow \frac{\varepsilon}{\varepsilon + d_i} \longrightarrow \frac{\varepsilon^\alpha}{\varepsilon^\alpha + d_i^\alpha}$



Ex:

$$w(d) = \begin{cases} \frac{0.1}{0.1 + d^2} & d \le 1 \\ 0 & d > 1 \end{cases}$$

P_i	x	y	z_{obs}
P_1	-1	1	1
P_2	-0.5	0.5	3
P_3	1	1	2
P_4	0.2	0.3	1

Q: $\hat{z}(0,0) = ?$

$\gamma(\sigma_v^2) = 0.01$ (white noise)

$\sigma^2_{\hat{z}}(0,0) = ?$ by L1O

A: $P_0 = (0,0)$

P_i	$d^2(P_0, P_i)$	w_i	$w_i z_{0i}$
P_1	2	0	0
P_2	0.5	$\frac{1}{6}$	$\frac{1}{2}$
P_3	2	0	0
P_4	0.13	$\frac{10}{23}$	$\frac{10}{23}$

$$\hat{z}(P_0) = \frac{\sum_i w_i z_i + \sum_i w_i v_i}{\sum_i w_i}$$

$$= \frac{\frac{1}{2} + \frac{10}{23}}{\ldots}$$

→ L1O

P_i	$d^2(P_1, P_i)$
P_2	0.5
P_3	> 1
P_4	> 1

$\hat{z}(P_1) = 3$

⋮

$\hat{z}(P_2) = 1$ $\hat{z}(P_3) = 0$ $\hat{z}(P_4) = 3$

$\hat{v}_1 = 1 - 3 = -2$

$$= \frac{\quad}{\frac{1}{6}+\frac{10}{23}}$$

$$b^2\hat{\Sigma}(0,0) = \frac{\sum_i W_i^2 (bV_i)^2}{(\sum_i W_i)^2}$$

$$= \frac{(\frac{1}{6})^2 0.01 + (\frac{10}{23})^2 0.01}{(\frac{1}{6}+\frac{10}{23})^2}$$

$$b\hat{\Sigma}(0,0) = \sqrt{\quad}$$

$\hat{V}_2 =$ $\hat{V}_3 =$ $\hat{V}_4 =$

$$rms = \sqrt{\frac{\sum_i V_i^2}{n}} =$$

三. LS interpolation

II. Processing

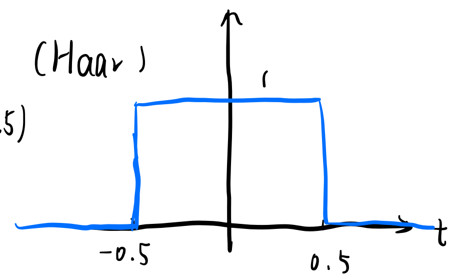
一. deterministic model 确定性模型

1. spline (1D) piecewise polynomials with bounded support

Parent splines

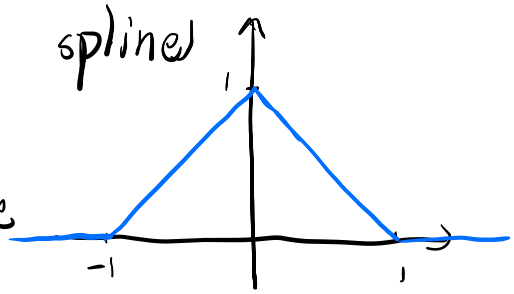
A: spline of order 0 (Haar)

$$\varphi_0(t) = \begin{cases} 1 & t \in (-0.5, 0.5) \\ 0 & \text{otherwise} \end{cases}$$



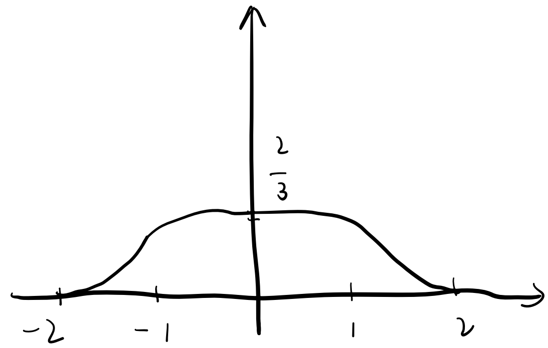
B. spline of order 1 (linear spline)

$$\varphi_1(t) = \begin{cases} 1-|t| & |t|<1 \\ 0 & \text{otherwise} \end{cases}$$



C. spline of order 3 (cubic spline)

$$\varphi_3(t) = \begin{cases} \frac{1}{6}\left[(2-|t|)^3 - 4(1-|t|)^3\right] & |t| \le 1 \\ \frac{1}{6}(2-|t|)^3 & 1<|t|<2 \\ 0 & \text{otherwise} \end{cases}$$



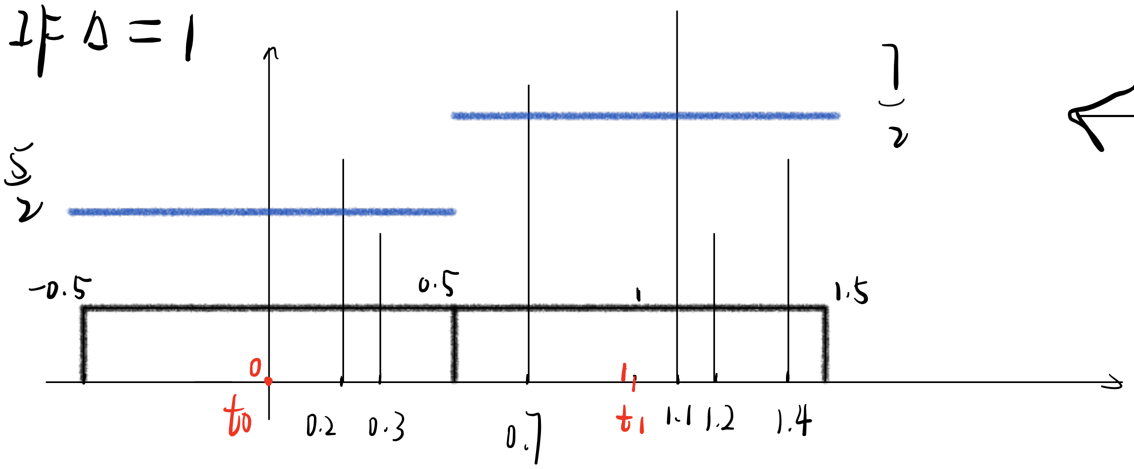
Grid Lag size $T = \frac{t-t_0}{\Delta}$

Example

Order 0 $\quad z(t) = \sum_i a_i S_0\left(\frac{t-t_i}{\Delta}\right)$ $\quad \Delta$: step of the grid

If $\Delta = 1$

$t=0.2$ $z_1=3$
$t=0.3$ $z_2=2$
$t=0.7$ $z_3=4$
$t=1.1$ $z_4=5$
$t=1.2$ $z_5=2$
$t=1.4$ $z_6=3$



$$z(t) = \sum_i a_0 s(t-t_0) + a_1 s(t-t_1)$$

$$\qquad\qquad 0.2-0 \qquad\qquad 0.2-1$$

$$x = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \quad A = \begin{bmatrix} 1 \leftarrow s(0.2-0) & 0 \leftarrow s(0.2-1) \\ 1 \leftarrow s(0.3-0) & 0 \leftarrow s(0.3-1) \\ 0 \quad s(0.7-0) & 1 \quad s(0.7-1) \\ 0 \quad s(1.1-0) & 1 \quad s(1.1-1) \\ 0 \quad s(1.2-0) & 1 \quad s(1.2-1) \\ 0 \quad s(1.4-0) & 1 \quad s(1.4-1) \end{bmatrix} \quad y_0 = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 5 \\ 2 \\ 3 \end{bmatrix}$$

$$\hat{x} = (A^TA)^{-1}A^Ty_0 \qquad N = A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad N^{-1} = \frac{1}{8}\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$A^T y_0 = \begin{bmatrix} \\ 14 \end{bmatrix}$$

$$\hookrightarrow \hat{x} = \begin{bmatrix} \frac{5}{2} \\ \frac{14}{4} \end{bmatrix}$$

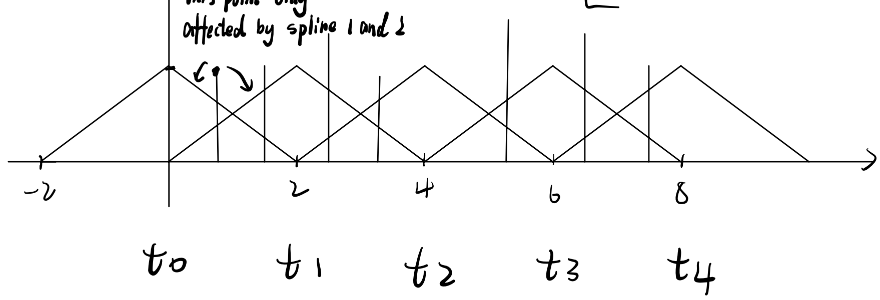
$$\varphi_0\left(\frac{t}{\Delta}\right) = \begin{cases} 1 & t \in \left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

Order 1 $\quad z(t) = \sum_i a_i S_0\left(\frac{t - t_i}{\Delta}\right)$

$\Delta = 2$

5 splines

$$A = \begin{bmatrix} S_1\left(\frac{t_1 - 0}{2}\right) & S_2\left(\frac{t_2 - 2}{2}\right) & 0 & 0 & 0 \\ & & & & \\ & & & & \end{bmatrix}$$



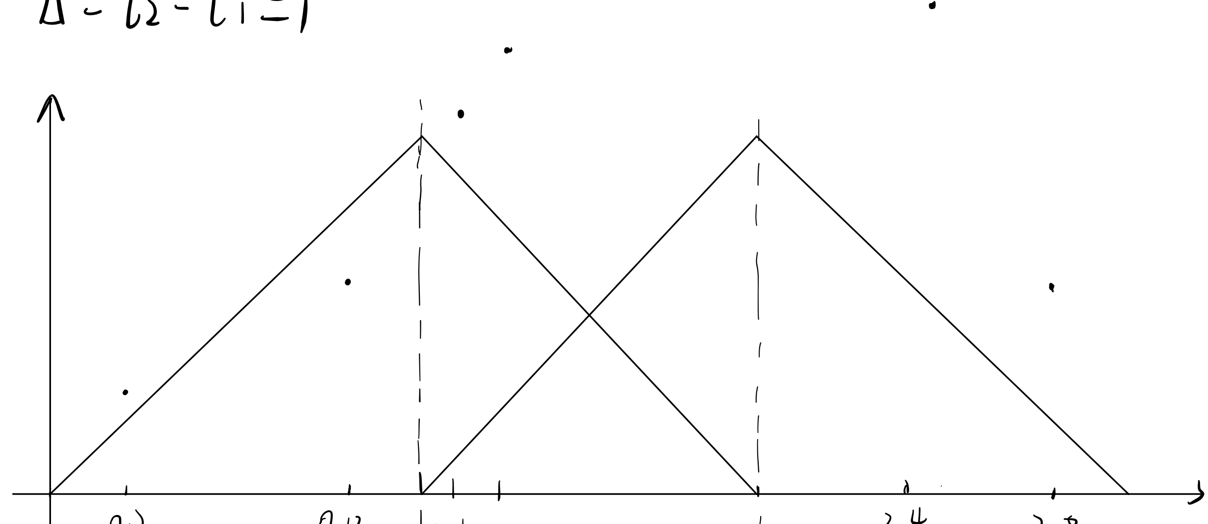
Exercise

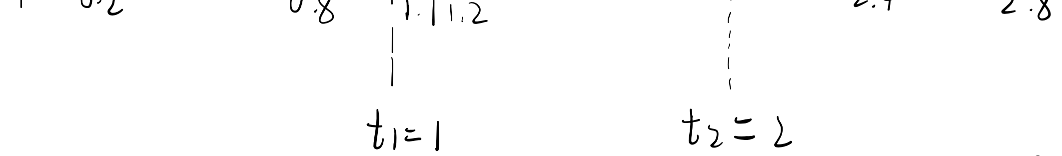
2 linear spline centered in $t_1 = 1$ $t_2 = 2$

t_0	y_0
0.2	0.3191
0.7	0.5798
1.1	1.1980
1.2	1.3843
2.4	1.6376
2.8	0.6257

$a: \hat{a}_i = ?$ $\hat{\sigma}^2_{\hat{a}_i} = ?$

$\Delta = t_2 - t_1 = 1$



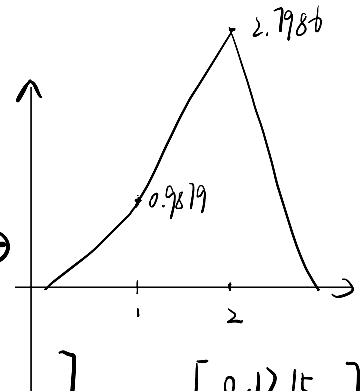


$$A=\begin{bmatrix} s(t_1-1) & s(t_2-1) \\ s(t_2-1) & s(t_2-1) \\ \vdots & \vdots \\ s(t_6-1) & s(t_6-1) \end{bmatrix} = \begin{bmatrix} s(0.2-1) & s(0.2-2) \\ s(0.7-1) & s(0.7-2) \\ \vdots & \vdots \\ s(2.8-1) & s(2.8-2) \end{bmatrix} = \begin{bmatrix} 0.2 & 0 \\ 0.7 & 0 \\ 0.9 & 0.1 \\ 0.8 & 0.2 \\ 0 & 0.6 \\ 0 & 0.2 \end{bmatrix}$$

$$N=A^TA=\begin{bmatrix} 1.98 & 0.25 \\ 0.25 & 0.45 \end{bmatrix} \qquad A^Ty_0=\begin{bmatrix} 2.6533 \\ 1.5063 \end{bmatrix}$$

$$N^{-1}=\frac{1}{1.98\cdot 0.45-0.25^2}\begin{bmatrix} 0.45 & -0.25 \\ -0.25 & 1.98 \end{bmatrix}$$

$$\hat{x}=N^{-1}(A^Ty_0)=\begin{bmatrix} 0.9879 \\ 0.2768 \end{bmatrix}$$



$$\hat{v}=y_0-\hat{y}=\begin{bmatrix} y_{01}-0.2\cdot 0.9879 \\ y_{02}-0.7\cdot 0.9879 \\ y_{03}-(0.9\cdot 0.9879+0.1\cdot 2.7968) \\ \vdots \end{bmatrix} = \begin{bmatrix} 0.1215 \\ -0.1118 \\ 0.0292 \\ 0.0396 \\ -0.0385 \\ 0.0663 \end{bmatrix}$$

$$\hat{\sigma}_0^2=\frac{\hat{v}^T\cdot\hat{v}}{6-2}=0.0088$$

$$C_{xx}=\hat{\sigma}_0^2\cdot N^{-1}=\begin{bmatrix} 0.0048 & -0.0027 \\ 0 & 0.0210 \end{bmatrix}$$

$$\hat{\sigma}_{\hat{a}_1}^2=0.0048 \qquad \hat{\sigma}_{\hat{a}_2}^2=0.0210$$

* Regularized Least-square adjustment

Hybrid Norm

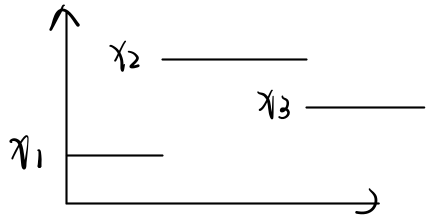
$$\phi = (\underline{y_0} - A\underline{x} - \underline{b})^T Q^{-1} (\underline{y_0} - Ax - \underline{b}) + \lambda (x^T x) \qquad \left(\lambda = \frac{\sigma^2 v}{\sigma^2 y} \right)$$

Regularized LS

$$\hat{x} = (A^T Q^{-1} A + \lambda I)^{-1} A^T Q^{-1} (y_0 - b)$$

① order 0

$$\begin{cases} y_{0i} = \underbrace{a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + b_i}_{\hat{y}} + v_i \\ 0 = x_i + y_i \end{cases}$$



② order 1

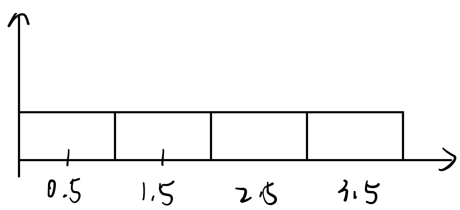
$$\begin{cases} y_{0i} = \cdots \\ 0 = x_{i+1} - x_i + y_i \end{cases}$$

Exercise

t	0.2	0.5	0.6	1.4	1.8	3.3	3.5
y_0	0.8126	1.0478	1.0896	3.0731	3.0578	4.0040	4.0677

order 0

$$N = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad A^T y_0 = \begin{bmatrix} y_{01} + y_{02} + y_{03} \\ y_{04} + y_{05} \\ 0 \\ y_{06} + y_{07} \end{bmatrix} \quad \hat{x} = \begin{bmatrix} 1.0465 \\ 2.9935 \\ 3.5020 \\ 4.0104 \end{bmatrix}$$



$\Delta = 1$

此时 t_3 为0，因此引入正则化

$\lambda = 0.1$

$$N' = N + 0.1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 3.1 & & & \\ & 2.1 & & \\ & & 0.1 & \\ & & & 2.1 \end{bmatrix}$$

$$\begin{cases} y_{01} = x_1 \cdot 1 + x_2 \cdot 0 + x_3 \cdot 0 + x_4 \cdot 0 + v_1 \\ y_{02} = x_1 \cdot 1 + x_2 \cdot 0 + x_3 \cdot 0 + x_4 \cdot 0 + v_2 \\ \vdots \\ y_{0n} = \\ 0 = x_1 + y_1 \\ 0 = x_2 + y_2 \\ 0 = x_3 + y_3 \\ 0 = x_4 + y_4 \end{cases}$$

order 1

$$\begin{cases} 0 = x_2 - x_1 + y_{12} \\ 0 = x_3 - x_2 + y_{23} \\ 0 = x_4 - x_3 + y_{34} \end{cases}$$

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad A^T A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$N' = \begin{bmatrix} 3.1 & -0.1 & & 0 \\ -0.1 & 2.2 & -0.1 & \\ & 0 & 0.2 & -0.1 \\ & & -0.1 & 2.1 \end{bmatrix}$$

$\hat{x}_1 = 1.0465$

$\hat{x}_2 = 2.9935$

$\hat{x}_3 = 3.5020$

$\hat{x}_4 = 4.0104$

2. Discrete Fourier Transform (DFT)

$$y(t_k) = a_0 + 2 \sum_{1}^{L}{}_{l} \left(a_l \cos\left(2\pi \frac{k}{n} l\right)\right) + b_l \sin\left(2\pi \frac{k}{n} l\right))$$

$K = 0, 1, \ldots N-1$

Ever N, $L = \frac{N}{2}$

Odd N $L = \frac{N-1}{2}$

$$\underline{V} = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix} \qquad C_{vv} = \sigma_0^2 Q = \sigma_0^2 I$$

$$A = \begin{bmatrix} 1 & \cos(2\pi \frac{0}{N} 1) & \cos(2\pi \frac{0}{N} \cdot 2) & \cdots & \sin(2\pi \frac{0}{N} \cdot 1) \\ 1 & \cos(2\pi \frac{1}{N} \cdot 1) & \cos(2\pi \frac{1}{N} \cdot 2) & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & \cos(2\pi \frac{N-1}{N} \cdot 1) & \cos(2\pi \frac{N-1}{N} \cdot 2) & \cdots & \end{bmatrix}$$

$$N = \begin{bmatrix} N & & & 0 \\ & 2N & & \\ & & 2N & \\ 0 & & & \ddots \end{bmatrix} \qquad \hat{x} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ b_1 \\ b_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{1}{N} \Sigma_k y_{0k} \\ \hat{a}_l = \frac{1}{N} \Sigma_k y_{0k} \cos(2\pi \frac{k}{N} l) \\ \hat{b}_l = \frac{1}{N} \Sigma_k y_{0k} \cos(2\pi \frac{k}{N} l) \end{bmatrix}$$

$$\sigma_0^2 = \frac{\hat{v}^T Q^{-1} \hat{v}}{N-m} = \frac{|y_0|^2 - N a_0^2 - 2N \Sigma_l a_l^2 - 2N \Sigma_l b_l^2}{N-m}$$

Example

$N = 7$

Q: estimate the model

K	K/n	y_{0k}
0	0	5.94
1	1/7	3.18
2	2/7	-6.70
3	3/7	0.08
4	4/7	8.06
5	5/7	-3.55
6	6/7	-6.35

N=7 is odd $L=\frac{N-1}{2}=3$

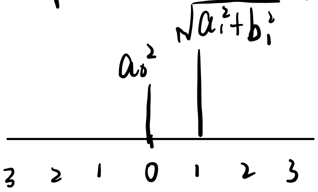
$$\hat{a}_0 = \frac{1}{7}(y_{00}+y_{01}+\cdots+y_{06}) = 0.0943$$

$$\hat{a}_1 = \frac{1}{7}\left(y_{00}\cos(0) + y_{01}\cos(2\pi\cdot\frac{1}{7}\cdot 1) + y_{02}\cos(2\pi\frac{2}{7}\cdot 1) + \cdots + y_{06}\cos(2\pi\frac{6}{7}\cdot 1)\right) = -0.1556$$

$$\hat{a}_2 = \frac{1}{7}\left(y_{00}\cos(0) + y_{01}\cos(2\pi\frac{1}{7}\cdot 2) + y_{02}\cos(2\pi\frac{2}{7}\cdot 2) + \cdots + y_{06}\cos(2\pi\frac{6}{7}\cdot 2)\right) = 2.9936$$

同理 $\hat{a}_3 = 0.0869$ 将cos换为sin得 $\hat{b}_1$ $\hat{b}_2$ $\hat{b}_3$

Q: show the power spectrum



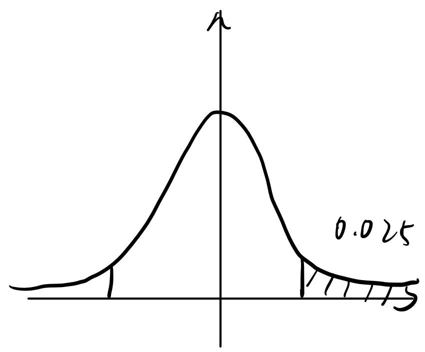
Q: Select only coeff's significantly different from 0 ($\alpha=5$)

H_0 $a_0=0$

$$\hat{\sigma}_0^2 = (|y_0|^2 - 7\hat{a}_0^2 - 14(\hat{a}_1^2+\hat{a}_2^2+\hat{b}_1^2+\hat{b}_2^2))/(7-5) = 0.2501$$

t 卡方检验 $= \frac{\hat{a}_0 - a_0}{\sqrt{\hat{\sigma}_0^2 a_0}} \sim t_{N-m}$

$C_{\hat{x}\hat{x}} = \hat{\sigma}_0 N^{-1}$ $= \frac{0.0943-0}{\sqrt{0.2501/7}} = 0.498 <$



if $\alpha=5 \rightarrow t=4.302$

H_0 is accepted

H_0 $a_1=0$ H_0 is accepted

H_0 $a_2=0$ H_0 is rejected $\Rightarrow a_2 \neq 0$

the residual model is $y(t_k) = a_2\cos(2\pi\frac{k}{7}\cdot 2) + b_2\sin(2\pi\frac{k}{7}\cdot 2)$

Q: evaluate the accuracy

$\hat{\sigma}_0 = (|y_0|^2 - 14(\hat{a}_2^2 + \hat{b}_2)^2) / 7-2 = 0.2289$

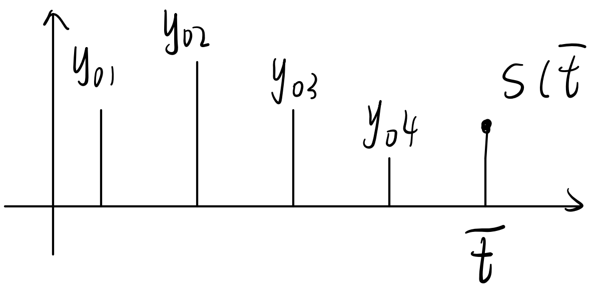
$\hat{\sigma}_{\hat{a}_2} = \sqrt{\frac{\sigma_0^2}{14}}$

二. stochastic modeling 随机性模型

$\hat{\lambda} = C_s(\underline{t}, \bar{t})(C_s(\underline{t}, \underline{t}) + C_v(\underline{t}, \underline{t}))^{-1}$

$S(\bar{t}) = \lambda^T y_0 = C_s(\underline{t}, \bar{t})^T (C_s(\underline{t}, \underline{t}) + C_v(\underline{t}, \underline{t}))^{-1}$

$\sigma_e^2 = \sigma^2_{s(\bar{t})} - C_s(\underline{t}, \bar{t})^T (C_s(\underline{t}, \underline{t}) + C_v(\underline{t}, \underline{t}))^{-1} C_s(\underline{t}, \bar{t})$



$S(\bar{t}) = \lambda_1 y_{01} + \lambda_2 y_{02} + \cdots + \lambda_n y_{0n}$

$C_s(\underline{t}, \bar{t}) = \begin{bmatrix} C_s(t_1, \bar{t}) \\ C_s(t_2, \bar{t}) \\ \vdots \\ C_s(t_n, \bar{t}) \end{bmatrix}$ $\quad C_s(t, t) = \begin{bmatrix} C_s(t_1, t_1) & C_s(t_1, t_2) & \cdots & \cdots \\ C_s(t_2, t_1) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \vdots & \cdots & & C_s(t_n, t_n) \end{bmatrix}$

IF stationary

$\hat{C}(0) = \frac{1}{N} \sum_i y_{0i}^2$ $\quad C(t, \bar{t}) = C(|t - \bar{t}|)$

$\hat{C}(\Delta) = \frac{1}{N_\Delta} \sum_{0 < |t_i - t_j| < \Delta} {}_{ij}\, y_{0i} y_{0j}$ $\quad \Delta$: lag

Exercise

t y $\quad$ assume the process is zero mean

1	5.0752
2	2.7808
3	2.5821
4	-0.7891
5	-1.3083
6	-2.4116
7	-3.8262
8	-2.7029

Q: estimate the cov. at lag 0, 1, 2

$\hat{C}(0) = \frac{1}{8}(5.0752^2 + \cdots) = 9.6243$

$\hat{C}(1) = \frac{1}{7}(5.0752 \cdot 2.7808 + 2.7808 \cdot 2.5821 + \cdots) = 6.7832$

$\hat{C}(2) = \frac{1}{6}(5.0752 \cdot 2.5821 - 2.7808 \cdot 0.7891 + \cdots) = 3.9025$

Q: interpolate with $c(\tau) = Ae^{-a\tau^2}$ using the estimate lags

$$\left\{\begin{array}{l} Ae^{-a \cdot 1} = c(1) \\ Ae^{-a \cdot 2^2} = c(2) \end{array}\right\} \div \frac{e^{-a}}{e^{-4a}} = \frac{c(1)}{c(2)} \Rightarrow a = \frac{\ln(c(1)) - \ln(c(2))}{3}$$

$$\hat{a} = \frac{\ln(\hat{C}(1)) - \ln \hat{C}(2)}{3} = 0.19$$

$$A = \frac{c(1)}{e^{-a}} \Rightarrow \hat{A} = \frac{\hat{C}(1)}{e^{-\hat{a}}} = 8.1606$$

$$\sigma_v^2 = \hat{C}(0) - \hat{A} = A - \hat{A} = 1.4637$$

Exercise

$C_s(\tau) = e^{-|\tau|}$ signal covariance τ = distance

whitenoise $\sigma_v^2 = 0.25$

$\underline{y}_0 = \underline{S} + \underline{v}$

t	y_0
-1	1.2351
1	3.8137

Q: estimate $\hat{s}(0)$ and $\sigma_{\hat{s}(0)}$

$$\hat{s}(0) = C_{s(0,t)}^T (C_s(t,t) + C_v(t,t))^{-1} y_0$$

$$C_{vv} = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}$$

$$C_{ss} = \begin{bmatrix} C_s(-1,-1) & C_s(-1,1) \\ C_s(1,-1) & C_s(1,1) \end{bmatrix} = \begin{bmatrix} C_s(0) & C_s(2) \\ C_s(2) & C_s(0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & e^{-2} \\ e^{-2} & 1 \end{bmatrix}$$

$$\hat{s}(0) = \begin{bmatrix} C_s(0,-1) \\ C_s(0,1) \end{bmatrix}^T \begin{bmatrix} C_{vv} + C_{ss} \end{bmatrix} \begin{bmatrix} 1.2351 \\ 3.813 \end{bmatrix} = 1.3410$$

1×2 $\quad$ 2×2 $\quad$ 2×1

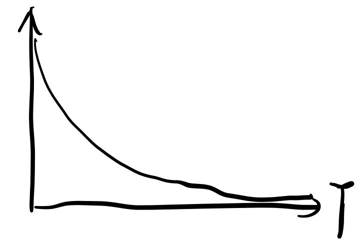
$$\sigma^2_{\hat{s}} = \sigma^2_{s(0)} - C_s^T (C_{ss} + C_{vv})^{-1} C_s$$

$$= 1 - [e^{-1}\ e^{-1}] [\quad] \begin{bmatrix} e^{-1} \\ e^{-1} \end{bmatrix}$$

cov models

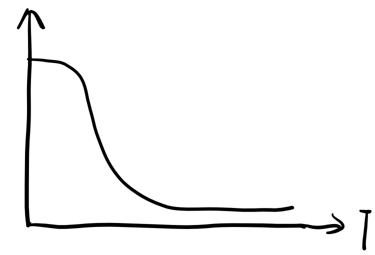
$C(T) = Ae^{-\alpha T}$

expotional



$C(T) = Ae^{-\alpha T^2}$

normal



covalation length T: $C(l) = \frac{A}{2}$