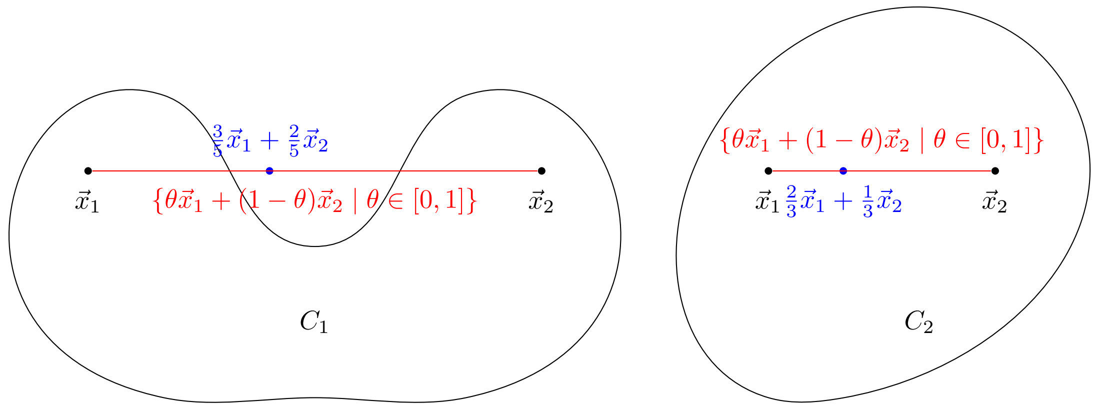


Figure 5.1: Two sets $C_1, C_2 \subseteq \mathbb{R}^2$. C_1 is not convex, but C_2 is. To visualize the behavior of the line segments, we also plot a point on each line segment along with its associated θ.

Algebraically, a set C is convex if for any $\vec{x}_1, \ldots, \vec{x}_k \in C$, any convex combination of $\vec{x}_1, \ldots, \vec{x}_k$ is contained in C.

One way to generate a convex set from any (possibly non-convex) set, including finite and infinite sets, is to take its convex hull.

Definition 92 (Convex Hull)

Let $S \subseteq \mathbb{R}^n$ be a set. The *convex hull* of S, denoted $\operatorname{conv}(S)$, is the set of all convex combinations of points in S, i.e.,

$$\operatorname{conv}(S) = \left\{ \sum_{i=1}^{k} \theta_i \vec{x}_i \;\middle|\; k \in \mathbb{N}, \theta_1, \ldots, \theta_k \geq 0, \sum_{i=1}^{k} \theta_i = 1, \vec{x}_1, \ldots, \vec{x}_k \in S \right\}. \tag{5.2}$$

Here are some properties of the convex hull; the proof is left as an exercise.

Proposition 93

Let $S \subseteq \mathbb{R}^n$ be a set.

(a) $\operatorname{conv}(S)$ is a convex set.

(b) $\operatorname{conv}(S)$ is the minimal convex set which contains S, i.e.,

$$\operatorname{conv}(S) = \bigcap_{\substack{C \supseteq S \\ C \text{ is a convex set}}} C. \tag{5.3}$$

Thus if S is convex then $\operatorname{conv}(S) = S$.

(c) $\operatorname{conv}(S)$ is the union of convex hulls of all finite subsets of S, i.e.,

$$\operatorname{conv}(S) = \bigcup_{\substack{A \subseteq S \\ A \text{ is a finite set}}} \operatorname{conv}(A). \tag{5.4}$$

Actually, the last statement can be strengthened to a separate, more quantitative result, which gives a fundamental characterization of convex sets.

Theorem 94 (Carathéodory's Theorem)
Let $S \subseteq \mathbb{R}^n$ be a set. Then $\operatorname{conv}(S)$ is the union of convex hulls of all finite subsets of S *of size at most* $n+1$, i.e.,

$$\operatorname{conv}(S) = \bigcup_{\substack{A \subseteq S \\ |A| \leq n+1}} \operatorname{conv}(A). \tag{5.5}$$

The proof of this theorem is left as an exercise; interested students can reference the proof in Bertsekas [5, Proposition B.6], for example.

Below, we visualize the convex hull of a *finite set* S. By the above proposition, the convex hull of an infinite set S' is the union of convex hulls of all finite subsets of S'.

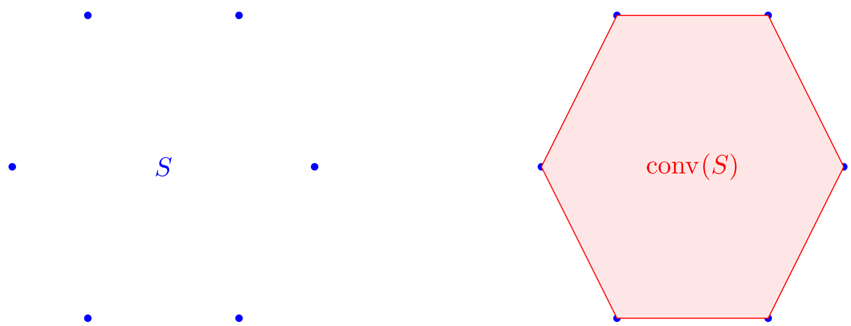


Figure 5.2: A finite set S and its convex hull.

The following content is optional/out of scope for this semester. Regardless, it may be helpful to read it to gain context, or get a deeper understanding of various results.

5.1.2 (OPTIONAL) Conic Hull, Affine Hull, and Relative Interior

As stated above, given a set S that is not necessarily convex, taking arbitrary convex combinations of vectors in S generates the convex hull $\operatorname{conv}(S)$ of S, which is the smallest convex set containing S. Below, we present two other methods of generating convex sets containing S that look similar to the definition of the convex hull. Each method has its own geometric interpretation.

Definition 95 (Conic Hull)
Let $S \subseteq \mathbb{R}^n$ be a set. The *conic hull* of S, denoted $\operatorname{conic}(S)$, is defined as the set of *conic combinations* of vectors in S, i.e., linear combinations of vectors in S with non-negative coefficients:

$$\operatorname{conic}(S) = \left\{ \sum_{i=1}^{k} \theta_i \vec{x}_i \;\middle|\; k \in \mathbb{N}, \theta_1, \ldots, \theta_k \geq 0, \vec{x}_1, \ldots, \vec{x}_k \in S \right\}. \tag{5.6}$$

Geometrically, the conic hull of a set S is the set of all rays from the origin that pass through $\operatorname{conv}(S)$.

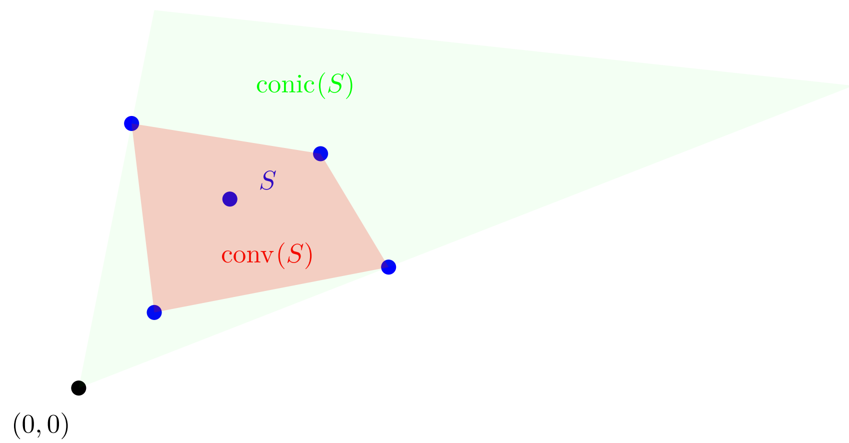


Figure 5.3: A finite set S and its convex and conic hulls.

Definition 96 (Affine Set)
Let $S \subseteq \mathbb{R}^n$ be a set. We say that S is an *affine set* if it is closed under affine combinations: for each $\vec{x}_1, \vec{x}_2 \in S$, and any $\theta \in \mathbb{R}$, we have $\theta\vec{x}_1 + (1-\theta)\vec{x}_2 \in S$.

Note the difference between affine and convex sets. In the latter, θ is restricted to $[0, 1]$. Geometrically this restriction corresponds to the (finite) line segment connecting $\vec{x}_1$ and $\vec{x}_2$ being contained in S. In the former, however, θ can be any real number, corresponding to the whole (infinite) line connecting $\vec{x}_1$ and $\vec{x}_2$ being contained in S.

Note that an affine set is a translation of a subspace. This intuition is one of the most helpful ways to understand affine sets.

Proposition 97
For a set $A \subseteq \mathbb{R}^n$, define the translation $A + \vec{x} \doteq \{\vec{a} + \vec{x} \mid \vec{a} \in A\}$.

(a) Let $S \subseteq \mathbb{R}^n$ be a nonempty affine set. Then there is a subspace $U \subseteq \mathbb{R}^n$ such that, for any $\vec{x} \in S$, we have $S = U + \vec{x}$.

(b) For any subspace $U \subseteq \mathbb{R}^n$ and vector $\vec{x} \in \mathbb{R}^n$, the set $U + \vec{x}$ is an affine set.

Proof.

(a) Let $\vec{x} \in S$ be any vector in S, and define $U := S + (-\vec{x}) = \{\vec{s} - \vec{x} \mid \vec{s} \in S\}$. We claim that U is a subspace. Indeed, since $\vec{x} \in S$, we see that $\vec{0} \in U = S + (-\vec{x})$. We show that U is closed under addition. Let $\vec{u}_1, \vec{u}_2 \in U$. By definition of U, there exist $\vec{s}_1, \vec{s}_2 \in S$ such that $\vec{u}_1 = \vec{s}_1 - \vec{x}$ and $\vec{u}_2 = \vec{s}_2 - \vec{x}$. Then

$$\vec{u}_1 + \vec{u}_2 = \vec{s}_1 - \vec{x} + \vec{s}_2 - \vec{x} \tag{5.7}$$

$$= \left[2\left(\frac{\vec{s}_1 + \vec{s}_2}{2}\right) + (1-2)\vec{x}\right] - \vec{x}. \tag{5.8}$$

Now because S is affine it is convex, so $\frac{\vec{s}_1+\vec{s}_2}{2} \in S$. As an affine combination of elements in S, we have $2\left(\frac{\vec{s}_1+\vec{s}_2}{2}\right) + (1-2)\vec{x} \in S$. Thus we have $\vec{u}_1 + \vec{u}_2 \in S + (-\vec{x}) = U$, so that U is closed under vector addition. To show that U is closed under scalar multiplication, let $\alpha \in \mathbb{R}$ and $\vec{u} \in U$. By definition of U, there exists

$\vec{s} \in S$ such that $\vec{u} = \vec{s} - \vec{x}$. Then

$$\begin{aligned} \alpha\vec{u} &= \alpha(\vec{s} - \vec{x}) && (5.9) \\ &= [\alpha\vec{s} + (1-\alpha)\vec{x}] - \vec{x}. && (5.10) \end{aligned}$$

Since S is affine it is convex, so $\alpha\vec{s} + (1-\alpha)\vec{x} \in S$. Thus $\alpha\vec{u} \in S + (-\vec{x}) = U$, so U is closed under scalar multiplication. We have shown that U is closed under linear combinations and contains $\vec{0}$, so U is a subspace and the claim is proved.

(b) Let $\alpha \in \mathbb{R}$ and let $\vec{s}_1, \vec{s}_2 \in U + \vec{x}$. By definition of U, there exist $\vec{s}_1, \vec{s}_2 \in S$ such that $\vec{s}_1 = \vec{u}_1 + \vec{x}$ and $\vec{s}_2 = \vec{u}_2 + \vec{x}$. Then

$$\begin{aligned} \alpha\vec{s}_1 + (1-\alpha)\vec{s}_2 &= \alpha(\vec{u}_1 + \vec{x}) + (1-\alpha)(\vec{u}_2 + \vec{x}) && (5.11) \\ &= [\alpha\vec{u}_1 + (1-\alpha)\vec{u}_2] + \vec{x}. && (5.12) \end{aligned}$$

Since U is a subspace, $\alpha\vec{u}_1 + (1-\alpha)\vec{u}_2 \in U$. Thus, from above, we have $\alpha\vec{s}_1 + (1-\alpha)\vec{s}_2 = [\alpha\vec{u}_1 + (1-\alpha)\vec{u}_2] + \vec{x} \in U + \vec{x}$. We have shown that $U + \vec{x}$ is closed under affine combinations, so it is an affine set.

□

Definition 98 (Affine Hull)

Let $S \subseteq \mathbb{R}^n$ be a set. The *affine hull* of S, denoted $\operatorname{aff}(S)$, is defined as the set of *affine combinations* of vectors in S, i.e., linear combinations of vectors in S with coefficients which sum to 1:

$$\operatorname{aff}(S) = \left\{ \sum_{i=1}^{k} \theta_i \vec{x}_i \;\middle|\; k \in \mathbb{N}, \theta_1, \dots, \theta_k \in \mathbb{R}, \sum_{i=1}^{k} \theta_i = 1, \vec{x}_1, \dots, \vec{x}_k \in S \right\}. \tag{5.13}$$

Here are some properties of the affine hull; the proof is left as an exercise.

Proposition 99

Let $S \subseteq \mathbb{R}^n$ be a set.

(a) $\operatorname{aff}(S)$ is an affine set.

(b) $\operatorname{aff}(S)$ is the minimal affine set which contains S, i.e.,

$$\operatorname{aff}(S) = \bigcap_{\substack{C \supseteq S \\ C \text{ is an affine set}}} C. \tag{5.14}$$

Thus if S is affine then $\operatorname{aff}(S) = S$.

(c) $\operatorname{aff}(S)$ is the union of affine hulls of all finite subsets of S, i.e.,

$$\operatorname{aff}(S) = \bigcup_{\substack{A \subseteq S \\ A \text{ is a finite set}}} \operatorname{aff}(A). \tag{5.15}$$

We can actually get an elementary refinement of (c) above.

Corollary 100. *Let $S \subseteq \mathbb{R}^n$ be a set, and let $\operatorname{aff}(S)$ be the translation of a linear subspace of dimension $d \leq n$. Then $\operatorname{aff}(S)$ is the union of affine hulls of all finite subsets of S of size at most d, i.e.,*

$$\operatorname{aff}(S) = \bigcup_{\substack{A \subseteq S \\ |A| \leq d}} \operatorname{aff}(A). \tag{5.16}$$

Proof. Suppose that $\operatorname{aff}(S) = U + \vec{x}$ where $U \subseteq \mathbb{R}^n$ is a subspace of dimension d. We prove both subset relations, i.e., $A \subseteq B$ and $B \subseteq A$ implies $A = B$.

We show the quicker subset inequality first. We have from earlier results that

$$\operatorname{aff}(S) = \bigcup_{\substack{A \subseteq S \\ A \text{ is a finite set}}} \operatorname{aff}(A) \supseteq \bigcup_{\substack{A \subseteq S \\ |A| \leq d}} \operatorname{aff}(A). \tag{5.17}$$

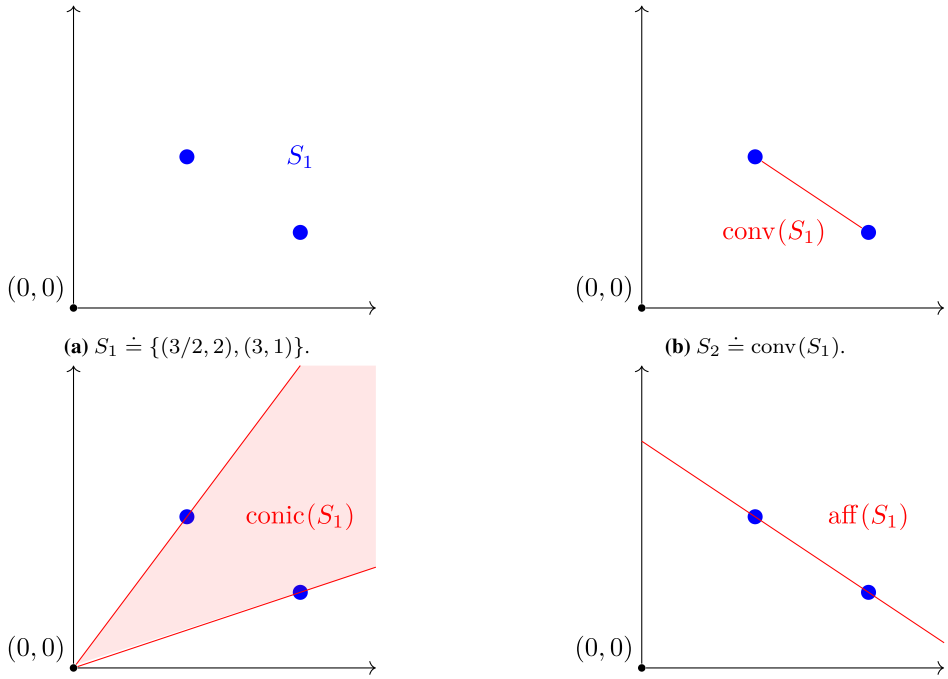
Towards showing the reverse inequality, let $\vec{s}_1, \ldots, \vec{s}_d$ be elements of S such that, if we define $\vec{u}_i \doteq \vec{s}_i - \vec{x}$, then $\vec{u}_1, \ldots, \vec{u}_d$ is a basis for U. (Such $\vec{s}_i$ have to exist; if they don't, then there are no $\vec{s}_1, \ldots, \vec{s}_d$ whose translates by $\vec{x}$ span U, a contradiction with the definition of $\operatorname{aff}(S) = U + \vec{x}$). Now taking $A = \{\vec{s}_1, \ldots, \vec{s}_d\}$, we see that $\operatorname{aff}(A) = \operatorname{aff}(S)$. Thus we have

$$\operatorname{aff}(S) = \operatorname{aff}(\{\vec{s}_1, \ldots, \vec{s}_d\}) \subseteq \bigcup_{\substack{A \subseteq S \\ |A| \leq d}} \operatorname{aff}(A). \tag{5.18}$$

Therefore we have

$$\operatorname{aff}(S) = \bigcup_{\substack{A \subseteq S \\ |A| \leq d}} \operatorname{aff}(A), \tag{5.19}$$

as desired. □



(a) $S_1 \doteq \{(3/2, 2), (3, 1)\}$. **(b)** $S_2 \doteq \mathrm{conv}(S_1)$. **(c)** $S_3 \doteq \mathrm{conic}(S_1)$. **(d)** $S_4 \doteq \mathrm{aff}(S_1)$.

Figure 5.4: (a) The set $S_1 \doteq \{(3/2, 2), (3, 1)\}$ of two points in $\mathbb{R}^2$; (b) The convex hull $S_2 \doteq \mathrm{conv}(S_1)$ of S_1, which is the closed line segment connecting the two points in S_1; (c) The conic hull $S_3 \doteq \mathrm{conic}(S_1)$ of S_1, which is the union of all rays passing through S_1; (d) The affine hull $S_4 \doteq \mathrm{aff}(S_1)$ of S_1, which is the infinite line connecting the two points in S_1. Note that we also have $S_3 = \mathrm{conic}(S_2)$ and $S_4 = \mathrm{aff}(S_2)$; this relationship can be shown to hold in general from definitions.

Next, given a set $S \subseteq \mathbb{R}^n$, we sometimes wish to distinguish points that lie on the "boundary" of S from points that lie in the "interior" of S. For instance, consider the set $S \doteq [0, 1) \subseteq \mathbb{R}$. In this case, 0 and 1 lie on the "boundary" of S, since they are infinitely close to both points inside S and points outside S. On the other hand, $\frac{1}{10}$ lies in the interior of S, since all points within a sufficiently small distance lie in S. Although 0 and 1 can both be geometrically interpreted as points on the boundary of $S = [0, 1)$, note that $0 \in S$ while $1 \notin S$. In general, a set may contain either all, some, or none of the points on its boundary.

Below, we formalize the notion of interior points[1].

Definition 101 (Interior)
Let $S \subseteq \mathbb{R}^n$ and let $\vec{x} \in \mathbb{R}^n$.

(a) (*Open ball.*) Let $r > 0$. We call $N_r(\vec{x}) \doteq \{\vec{y} \in \mathbb{R}^n \mid \|\vec{y} - \vec{x}\|_2 < r\}$ the *open ball* in $\mathbb{R}^n$ of radius r centered at $\vec{x}$.

(b) (*Interior.*) We say that $\vec{x}$ is an *interior point* of S when there exists some $r > 0$ such that $N_r(\vec{x}) \subseteq S$.[a] The set of all interior points of S is called the *interior* of S and denoted $\mathrm{int}(S)$.

[a] In the definition for the interior point, it does not matter whether we use $\subseteq$ or $\subset$. Think about why; this is a good exercise to internalize the definitions.

[1] The definitions provided below can be generalized to spaces more abstract than $\mathbb{R}^n$ or even general finite-dimensional vector spaces, such as metric or topological spaces.

In words, given a set $S \subseteq \mathbb{R}^n$, we say that $\vec{x} \in \mathbb{R}^n$ is an interior point of S if it is contained inside an open ball in $\mathbb{R}^n$ that is in turn entirely contained in S. A mental picture is provided in Figure 5.5.

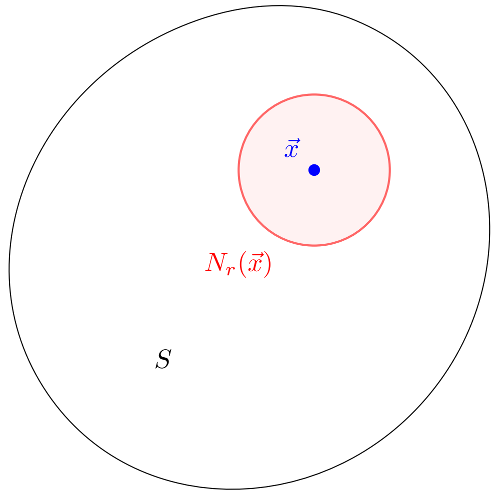


Figure 5.5: The vector $\vec{x}$ is an interior point of the set $S \subseteq \mathbb{R}^2$, since there exists a 2-dimensional ball (red) centered at $\vec{x}$ that is contained in S.

Some sets may represent geometric shapes embedded in Euclidean space of strictly higher dimension, and therefore must have empty interior. As an example, consider the set S_2 defined in Figure 5.4(b), which connects the points $(3/2, 2)$ and $(3, 1)$ in $\mathbb{R}^2$. This is a one-dimensional line segment embedded in an Euclidean space of dimension 2. Indeed, if one claims that a point in S_2, say, the midpoint $(9/4, 3/2)$, is an interior point of S_2, then one would have to show the existence of a two-dimensional open ball centered at $(9/4, 3/2)$ that lies entirely in S_2. This is impossible, since S_1 is a one-dimensional line segment, and so S_2 has empty interior.

However, it may still be geometrically meaningful to classify points in such a set as points on the "edge" of the set, or points "inside" the set. In the context of the line segment S_2, the end points $S_1 = \{(3/2, 2), (3, 1)\}$ appear at the "edge" of S_2, while the remaining points $S_2 \setminus S_1$ are located "inside" S_2. As we explained above, this cannot be captured using the definitions of interior points and the interior presented in Definition 101. Roughly speaking, this is because $S_2 \setminus S_1$ can only be considered points "inside" the line segment S_2 from a one-dimensional perspective, e.g., relative to the line $S_4 = \operatorname{aff}(S_1) = \operatorname{aff}(S_2)$ that contains S_2. This motivates the following definition of *relative interior,* provided below.

Definition 102 (Relative Interior Points, Relative Interior)

Let $S \subseteq \mathbb{R}^n$ be a set, and let $\vec{x} \in S$. We say that $\vec{x}$ is a *relative interior point* of S when there exists some $r > 0$ such that $N_r(\vec{x}) \cap \operatorname{aff}(S) \subseteq S$. The relative interior of S is the set of all relative interior points of S, and is denoted $\operatorname{relint}(S)$.

In words, given a set $S \subseteq \mathbb{R}^n$, we say that $\vec{x} \in \mathbb{R}^n$ is a relative interior point of S if it is contained inside an open ball in $\mathbb{R}^n$ *whose intersection with* $\operatorname{aff}(S)$ is entirely contained in S.

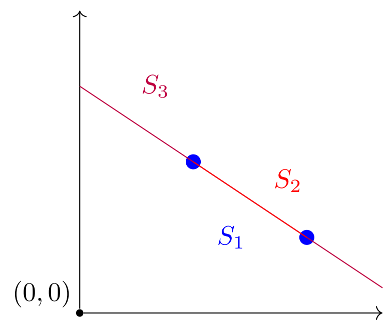


Figure 5.6: A similar setup to Figure 5.4. Here $S_1 \doteq \{(3/2, 2), (3, 1)\}$, $S_2 \doteq \operatorname{conv}(S_1) = \{\theta(3/2, 2)+(1-\theta)(3, 1) \mid \theta \in [0, 1]\}$, and $S_3 \doteq \operatorname{aff}(S_1) = \operatorname{aff}(S_2) = \{\theta(3/2, 2) + (1 - \theta)(3, 1) \mid \theta \in \mathbb{R}\}$. Thus $\operatorname{relint}(S_2) = S_2 \setminus S_1$. In other words, S_2 is the line segment connecting $(3/2, 2)$ and $(3, 1)$, $S_3 = \operatorname{aff}(S_2)$ is the extension of S_2 into a line, and $\operatorname{relint}(S_2)$ is the open (i.e., excluding the endpoints) line segment connecting $(3/2, 2)$ and $(3, 1)$. This illustrates the description of the relative interior of a set as its interior when viewed as a subset of its own affine hull.

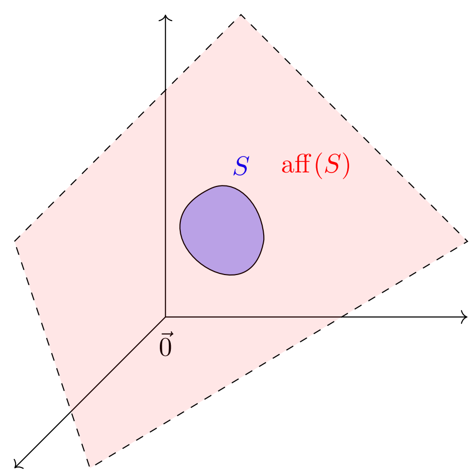


Figure 5.7: A set S and its affine hull. While the interior of S is the empty set, its relative interior is nonempty.

Next, we use the concept of relative interior to characterize *strictly convex sets*.

Definition 103 (Strictly Convex Sets)
Let $C \subseteq \mathbb{R}^n$ be a set. We say that C is a *strictly convex set* if for every $\vec{x}_1, \vec{x}_2 \in C$ and each $\theta \in (0, 1)$, we have $\theta\vec{x}_1 + (1 - \theta)\vec{x}_2 \in \operatorname{relint}(C)$.

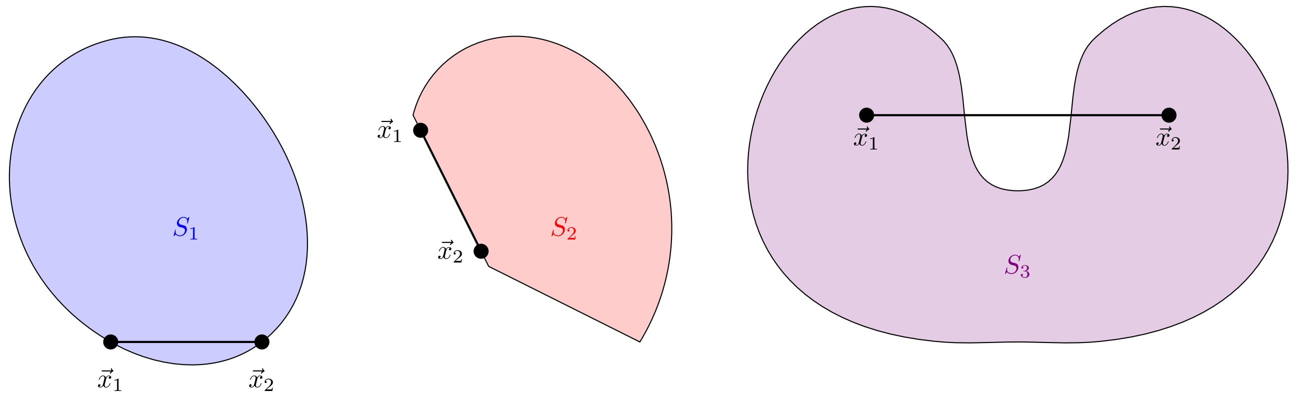


Figure 5.8: *Left:* a strictly convex set S_1. *Middle:* a convex set S_2 which is not strictly convex. *Right:* A non-convex set S_3. All three sets are defined to include their boundaries. In particular, S_2 is not strictly convex because some sections of its boundary consist of line segments. For any two points along the same line segment, each convex combination of these points will lie on the boundary of S_2.

The above content is optional/out of scope for this semester, but now we resume the required/in scope content.

5.1.3 Hyperplane and Half-Spaces

Definition 104 (Hyperplane)
Let $\vec{a}, \vec{x}_0 \in \mathbb{R}^n$ and $b \in \mathbb{R}$. A *hyperplane* is a set of the form

$$\{\vec{x} \in \mathbb{R}^n \mid \vec{a}^\top \vec{x} = b\} \tag{5.20}$$

or, equivalently, a set of the form

$$\{\vec{x} \in \mathbb{R}^n \mid \vec{a}^\top (\vec{x} - \vec{x}_0) = 0\}. \tag{5.21}$$

The equations $\vec{a}^\top \vec{x} = b$ and $\vec{a}^\top (\vec{x} - \vec{x}_0) = 0$ are connected, because if we define $b = \vec{a}^\top \vec{x}_0$, then the second equation resolves to the first equation; and if we take $\vec{x}_0$ to be any vector such that $\vec{a}^\top \vec{x}_0 = b$, then the first equation resolves to the second equation.

Example 105. Hyperplanes are convex. Consider a hyperplane $H \doteq \{\vec{x} \in \mathbb{R}^n \mid \vec{a}^\top \vec{x} = b\}$. Let $\vec{x}_1, \vec{x}_2 \in H$ and $\theta \in [0, 1]$. Then

$$\begin{aligned} \vec{a}^\top (\theta \vec{x}_1 + (1-\theta)\vec{x}_2) &= \theta \vec{a}^\top \vec{x}_1 + (1-\theta)\vec{a}^\top \vec{x}_2 && (5.22) \\ &= \theta b + (1-\theta) b && (5.23) \\ &= b && (5.24) \end{aligned}$$

so $\theta \vec{x}_1 + (1-\theta)\vec{x}_2 \in H$. Thus H is convex.

To show that a set C is convex, we need to show that for *every* $\vec{x}_1, \vec{x}_2 \in C$ and *every* $\theta \in [0, 1]$, that $\theta \vec{x}_1 + (1-\theta)\vec{x}_2 \in C$.

To show that C is *not* convex, we just need to come up with *one* choice of $\vec{x}_1, \vec{x}_2 \in C$ and *one* $\theta \in [0,1]$ such that $\theta\vec{x}_1 + (1-\theta)\vec{x}_2 \notin C$. Note that even if C is non-convex, there could be *some* choices of $\vec{x}_1, \vec{x}_2 \in C, \theta \in [0,1]$ such that $\theta\vec{x}_1 + (1-\theta)\vec{x}_2 \in C$; but if C is non-convex, there is at least one choice of $\vec{x}_1, \vec{x}_2 \in C, \theta \in [0,1]$ such that $\theta\vec{x}_1 + (1-\theta)\vec{x}_2 \notin C$.

Definition 106 (Half-Space)
Let $\vec{a}, \vec{x}_0 \in \mathbb{R}^n$ and $b \in \mathbb{R}$. A *positive half-space* is a set of the form

$$\{\vec{x} \in \mathbb{R}^n \mid \vec{a}^\top \vec{x} \geq b\} \qquad \text{or} \qquad \{\vec{x} \in \mathbb{R}^n \mid \vec{a}^\top (\vec{x} - \vec{x}_0) \geq 0\}. \tag{5.25}$$

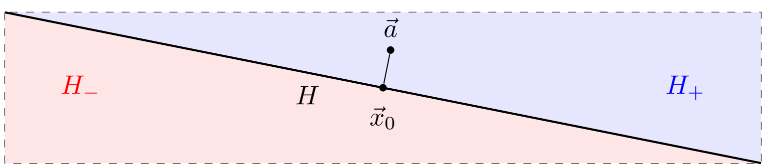
A *negative half-space* is a set of the form

$$\{\vec{x} \in \mathbb{R}^n \mid \vec{a}^\top \vec{x} \leq b\} \qquad \text{or} \qquad \{\vec{x} \in \mathbb{R}^n \mid \vec{a}^\top (\vec{x} - \vec{x}_0) \leq 0\}. \tag{5.26}$$

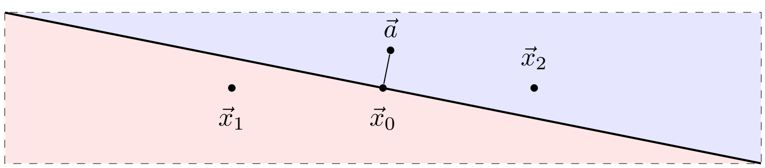
The mental picture we have for these hyperplanes and half-spaces is the following. Let $\vec{x}_0 \in \mathbb{R}^n$ and define

$$H \doteq \{\vec{x} \in \mathbb{R}^n \mid \vec{a}^\top (\vec{x} - \vec{x}_0) = 0\} \tag{5.27}$$
$$H_+ \doteq \{\vec{x} \in \mathbb{R}^n \mid \vec{a}^\top (\vec{x} - \vec{x}_0) \geq 0\} \tag{5.28}$$
$$H_- \doteq \{\vec{x} \in \mathbb{R}^n \mid \vec{a}^\top (\vec{x} - \vec{x}_0) \leq 0\}. \tag{5.29}$$

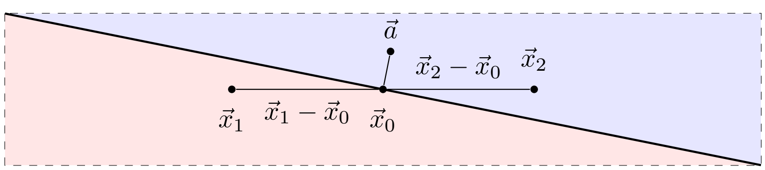
Then the alignment of these objects looks like the following:



In words, the positive and negative half-spaces partition $\mathbb{R}^n$. Looking at some individual vectors, say $\vec{x}_1 \in H_-$ and $\vec{x}_2 \in H_+$, we have the picture



If we draw lines connecting $\vec{x}_0$ with $\vec{x}_1$ and $\vec{x}_2$, they are not themselves representations of $\vec{x}_1$ and $\vec{x}_2$, unless $\vec{x}_0 = \vec{0}$. Instead, they are representations of the *displacements* of $\vec{x}_1$ and $\vec{x}_2$ from $\vec{x}_0$. Thus, we see the following picture:



And this gives us a clearer understanding of what's going on — $\vec{x}_1 - \vec{x}_0$ forms an obtuse angle with $\vec{a}$, indicating a negative dot product, whereas $\vec{x}_2 - \vec{x}_0$ forms an acute angle with $\vec{a}$, indicating a positive dot product. And this is how H_+ and H_- are computed.

This allows us to consider what it means for a hyperplane to separate two sets. It means that for every vector in the first set, the dot product is non-positive, and for every vector in the second set, the dot product is non-negative.

Example 107 (Set of PSD Matrices is Convex)**.** Consider $\mathbb{S}_+^n$, the set of all symmetric positive semidefinite (PSD) matrices. We want to show that $\mathbb{S}_+^n$ is convex. Take $A_1, A_2 \in \mathbb{S}_+^n$ and $\theta \in [0,1]$. We want to show that $\theta A_1 + (1-\theta)A_2 \in \mathbb{S}_+^n$.

One of the ways to tell if a matrix A is PSD is to check whether $\vec{x}^\top A\vec{x} \geq 0$ for all $\vec{x} \in \mathbb{R}^n$. Checking this for our convex combination, we get

$$\vec{x}^\top(\theta A_1 + (1-\theta)A_2)\vec{x} = \theta \underbrace{\vec{x}^\top A_1\vec{x}}_{\geq 0} + (1-\theta)\underbrace{\vec{x}^\top A_2\vec{x}}_{\geq 0} \tag{5.30}$$

$$\geq 0. \tag{5.31}$$

Note that it is possible to come up with linear combinations of PSD matrices that are not PSD; indeed, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ are PSD, yet their difference $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is not PSD. But all convex combinations of PSD matrices are PSD, as we have confirmed above.

Theorem 108 (Separating Hyperplane Theorem)

Let $C, D \subseteq \mathbb{R}^n$ be two nonempty *disjoint* convex sets, i.e., $C \cap D = \emptyset$. Then there exists a hyperplane that separates C and D, i.e., there exists $\vec{a}, \vec{x}_0 \in \mathbb{R}^n$ such that[a]

$$\vec{a}^\top(\vec{x} - \vec{x}_0) \geq 0, \qquad \forall \vec{x} \in C \tag{5.32}$$

$$\vec{a}^\top(\vec{x} - \vec{x}_0) \leq 0, \qquad \forall \vec{x} \in D. \tag{5.33}$$

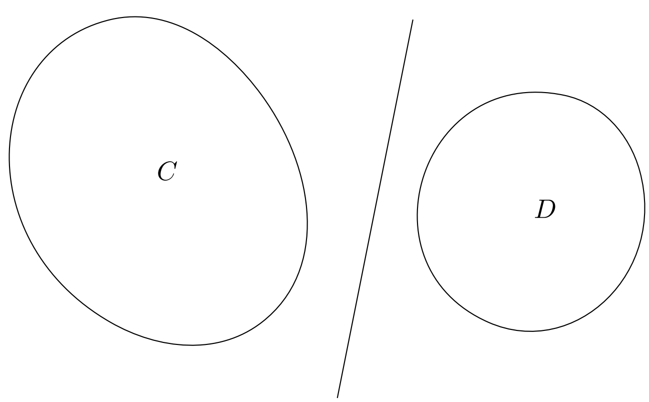
Moreover, if C is closed (containing its boundary points) and D is closed and bounded, then there exists a hyperplane that separates C and D without intersecting either set, i.e., there exists $\vec{a}, \vec{x}_0 \in \mathbb{R}^n$ such that

$$\vec{a}^\top(\vec{x} - \vec{x}_0) > 0, \qquad \forall \vec{x} \in C \tag{5.34}$$

$$\vec{a}^\top(\vec{x} - \vec{x}_0) < 0, \qquad \forall \vec{x} \in D. \tag{5.35}$$

[a] By defining $b \doteq \vec{a}^\top\vec{x}_0$, one can express $\vec{a}^\top(\vec{x} - \vec{x}_0)$ as $\vec{a}^\top\vec{x} - \vec{b}$ if desired.

The mental picture we want to have is the following.



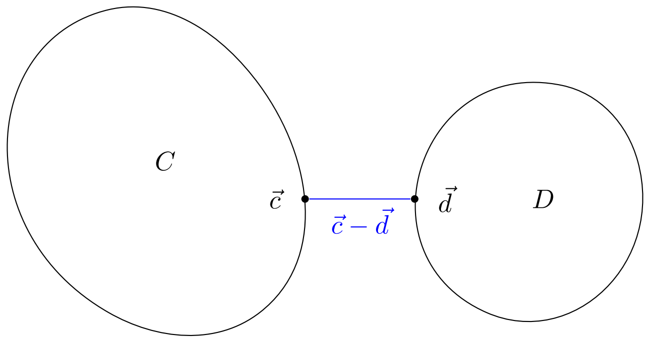
Proof. We prove the part of the theorem statement in the case where C is closed and bounded and D is closed.

Even though our theorem statement concerns existence of such $\vec{a}$ and $\vec{x}_0$, we will prove it by construction, i.e., we will construct a $\vec{a}$ and $\vec{x}_0$ which separate C and D. This proof strategy is very powerful and will show up frequently.

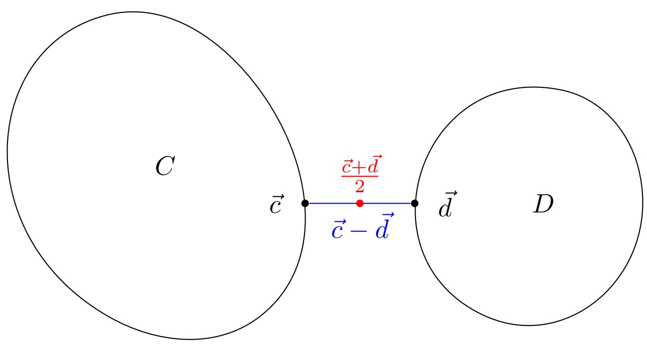
Since C and D are disjoint, any points in C and D are separated by some positive distance; since they are compact, this distance has a finite lower bound.[2] Define

$$\operatorname{dist}(C, D) \doteq \min_{\substack{\vec{c} \in C \\ \vec{d} \in D}} \left\| \vec{c} - \vec{d} \right\|_2 . \tag{5.36}$$

Note that $\operatorname{dist}(C, D) > 0$, and there exists some $c \in C$ and $d \in D$ such that $\left\| \vec{c} - \vec{d} \right\|_2 = \operatorname{dist}(C, D)$.[3]



This signals that we want $\vec{c} - \vec{d}$ to be the *normal* vector of our hyperplane — that is, our $\vec{a}$ vector. To find the other point $\vec{x}_0$ which the hyperplane passes through, we can just have it pass through the midpoint of $\vec{c}$ and $\vec{d}$, i.e., $\frac{\vec{c}+\vec{d}}{2}$. This gives the following diagram.

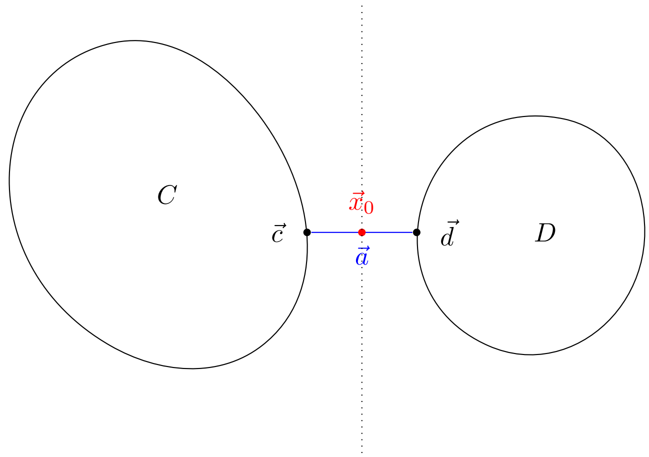


Thus our proposed hyperplane has $\vec{a}$ and $\vec{x}_0$ equal to

$$\vec{a} = \vec{c} - \vec{d}, \qquad \vec{x}_0 = \frac{\vec{c} + \vec{d}}{2}. \tag{5.37}$$

It yields the following picture, where the hyperplane is a dotted line.

[2] Proving this requires some mathematical analysis and is out of scope of the course.
[3] Same as the above footnote. The fact that C is closed and bounded and D is closed will not be used from this point onwards.



Notice that there are many separating hyperplanes, such as the one discussed before the theorem. But we just need to prove that this hyperplane separates C and D.

The equation for this hyperplane is

$$\begin{aligned}\vec{a}^\top(\vec{x}-\vec{x}_0) &= (\vec{c}-\vec{d})^\top\left(\vec{x}-\frac{\vec{c}+\vec{d}}{2}\right) &(5.38)\\ &= (\vec{c}-\vec{d})^\top\vec{x}-\frac{(\vec{c}-\vec{d})^\top(\vec{c}+\vec{d})}{2} &(5.39)\\ &= (\vec{c}-\vec{d})^\top\vec{x}-\frac{\vec{c}^\top\vec{c}-\vec{d}^\top\vec{d}}{2} &(5.40)\\ &= (\vec{c}-\vec{d})^\top\vec{x}-\frac{\|\vec{c}\|_2^2-\left\|\vec{d}\right\|_2^2}{2}. &(5.41)\end{aligned}$$

Thus the given hyperplane is also available in $(\vec{a}, b)$ form as

$$\vec{a}=\vec{c}-\vec{d}, \qquad b=\frac{\|\vec{c}\|_2^2-\left\|\vec{d}\right\|_2^2}{2}. \tag{5.42}$$

Now we prove that it actually separates $\vec{c}$ and $\vec{d}$. Define $f\colon \mathbb{R}^n \to R$ by

$$f(\vec{x})=(\vec{c}-\vec{d})^\top\left(\vec{x}-\frac{\vec{c}+\vec{d}}{2}\right). \tag{5.43}$$

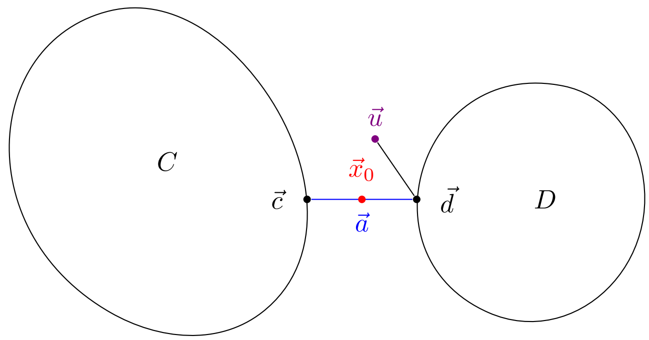
For the sake of contradiction, suppose there exists $\vec{u}\in D$ such that $f(\vec{u})\geq 0$. We can write

$$\begin{aligned}0 &\leq f(\vec{u}) &(5.44)\\ &= (\vec{c}-\vec{d})^\top\left(\vec{u}-\frac{\vec{c}+\vec{d}}{2}\right) &(5.45)\\ &= (\vec{c}-\vec{d})^\top\left(\vec{u}-\vec{d}-\frac{\vec{c}-\vec{d}}{2}\right) &(5.46)\\ &= (\vec{c}-\vec{d})^\top(\vec{u}-\vec{d})-\frac{(\vec{c}-\vec{d})^\top(\vec{c}-\vec{d})}{2} &(5.47)\\ &= (\vec{c}-\vec{d})^\top(\vec{u}-\vec{d})-\frac{1}{2}\left\|\vec{c}-\vec{d}\right\|_2^2. &(5.48)\end{aligned}$$

Thus

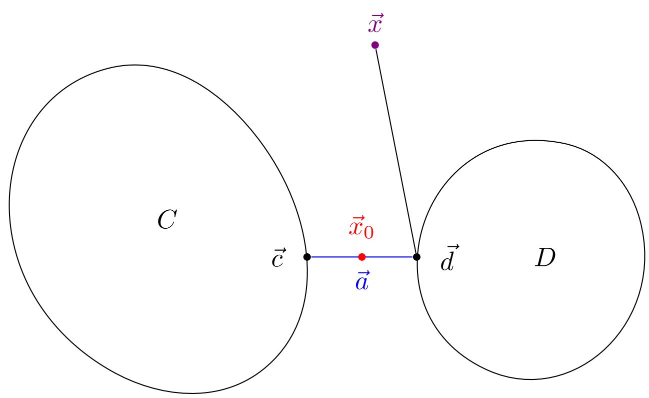
$$0 \leq (\vec{c} - \vec{d})^\top (\vec{u} - \vec{d}) - \frac{1}{2} \left\| \vec{c} - \vec{d} \right\|_2^2 < (\vec{c} - \vec{d})^\top (\vec{u} - \vec{d}). \tag{5.49}$$

This means that $\vec{c} - \vec{d}$ and $\vec{u} - \vec{d}$ form an *acute* angle. It also means that $\vec{u} \neq \vec{d}$, since otherwise the dot product would be 0. Going back to our picture, this means that $\vec{u}$ would have to be positioned similarly to the following:



At least from the diagram, it seems hard to imagine a $\vec{u} \in D$ such that $\vec{u} - \vec{d}$ and $\vec{c} - \vec{d}$ form an acute angle. Namely, any vector $\vec{x} \in \mathbb{R}^n$ (of reasonably small norm, such as the $\vec{u}$ in the figure) such that $\vec{x} - \vec{d}$ and $\vec{c} - \vec{d}$ form an acute angle, seems to be closer to $\vec{c}$ than $\vec{d}$ is to $\vec{c}$.

Why do we need the "reasonably small norm" condition? Consider the following possible $\vec{x}$:



Certainly, this $\vec{x}$ is farther from $\vec{c}$ than $\vec{d}$ is, and so no contradiction would be derived.

If we can prove that our $\vec{u}$, which we assume is in D, is closer to $\vec{c}$ than $\vec{d}$ is, then we can derive a contradiction with the fact that $\vec{d}$ is the closest vector in D to $\vec{c}$. But we can't prove this for our $\vec{u}$ directly, because $\left\| \vec{u} - \vec{d} \right\|_2$ may be large as in the above figure, so instead we take another vector $\vec{x}$ which is close to $\vec{d}$, where the displacement between $\vec{x}$ and $\vec{d}$ points in the direction of $\vec{u}$. We will show that this $\vec{x}$ is in D yet is closer to $\vec{c}$ than $\vec{d}$ is, thus deriving a contradiction.

Here are the details. Let $\vec{p}\colon [0,1] \to \mathbb{R}^n$ trace out the line from $\vec{d}$ to $\vec{u}$; namely, let $\vec{p}(t) = \vec{d} + t(\vec{u} - \vec{d}) = t\vec{u} + (1-t)\vec{d}$. Since $\vec{u}, \vec{d} \in D$ by assumption, and D is convex, we have that $\vec{p}(t) \in D$ for all $t \in [0,1]$. Now we see that

$$\begin{aligned}
\|\vec{p}(t) - \vec{c}\|_2^2 &= \left\| \vec{d} + t(\vec{u} - \vec{d}) - \vec{c} \right\|_2^2 \\
&= \left\| (\vec{d} - \vec{c}) + t(\vec{u} - \vec{d}) \right\|_2^2 \\
&= \left\| \vec{d} - \vec{c} \right\|_2^2 + 2t(\vec{d} - \vec{c})^\top (\vec{u} - \vec{d}) + t^2 \left\| \vec{u} - \vec{d} \right\|_2^2 \\
&= \left\| \vec{c} - \vec{d} \right\|_2^2 - 2t(\vec{c} - \vec{d})^\top (\vec{u} - \vec{d}) + t^2 \left\| \vec{u} - \vec{d} \right\|_2^2 .
\end{aligned}$$

We want to show that there exists t such that $\|\vec{p}(t) - \vec{c}\|_2^2 < \left\|\vec{c} - \vec{d}\right\|_2^2$, i.e.,

$$-2t(\vec{c} - \vec{d})^\top(\vec{u} - \vec{d}) + t^2 \left\|\vec{u} - \vec{d}\right\|_2^2 < 0, \tag{5.50}$$

i.e.,

$$2\underbrace{(\vec{c} - \vec{d})^\top(\vec{u} - \vec{d})}_{>0} - t\left\|\vec{u} - \vec{d}\right\|_2^2 > 0. \tag{5.51}$$

Now for all $0 < t < \frac{2(\vec{c}-\vec{d})^\top(\vec{u}-\vec{d})}{\left\|\vec{u}-\vec{d}\right\|_2^2}$, i.e., t small enough, the above inequality holds, so we have for this t that

$$\begin{aligned}\|\vec{p}(t) - \vec{c}\|_2^2 &= \left\|\vec{c} - \vec{d}\right\|_2^2 - 2t(\vec{c} - \vec{d})^\top(\vec{u} - \vec{d}) + t^2\left\|\vec{u} - \vec{d}\right\|_2^2 \\ &< \left\|\vec{c} - \vec{d}\right\|_2^2.\end{aligned}$$

However, $\vec{p}(t) \in D$, a contradiction. □

The following content is optional/out of scope for this semester. Regardless, it may be helpful to read it to gain context, or get a deeper understanding of various results.

5.1.4 (OPTIONAL) Cones

Definition 109 (Cones, Proper Cones)
Let $K \subseteq \mathbb{R}^n$.

(a) We call K a *cone* if, for any $\vec{v} \in K$ and $\alpha \in \mathbb{R}_+$[a], we have $\alpha\vec{v} \in K$.

(b) We call K a *convex cone* if it is both a cone and a convex set.

(c) We call K a *pointed cone* if it contains no line through the origin, i.e., if for each nonzero $\vec{v} \in K$, there exists some $\alpha \in \mathbb{R}$ such that $\alpha\vec{v} \notin K$.

(d) We call K a *solid cone* if it has non-empty interior, i.e., if there exists some $\vec{v} \in K$ and some $r > 0$ such that the open ball in $\mathbb{R}^n$ of radius r centered at $\vec{v}$ is contained in K: namely, we have $\{\vec{w} \in \mathbb{R}^n \mid \|\vec{w} - \vec{v}\|_2 < r\} \subseteq K$.

(e) We call K a *closed cone* if it is a closed set, i.e., it contains its boundary points.

(f) We call K a *proper cone* if it is convex, pointed, solid, and closed.

[a] That is, $\alpha \in \mathbb{R}$ and $\alpha \geq 0$

Note that non-empty cones must contain the zero vector, which corresponds to the case of taking $\alpha = 0$ in the definition of a cone.

The definition of proper cones is motivated by their connection to generalized inequalities in convex optimization, which will be discussed later in the course in the context of second-order cone programs (SOCPs) and semidefinite programming (SDPs). For this we require the above definitions to apply to a slightly broader context. We would need to replace $\mathbb{R}^n$ with a generic vector space V and the $\|\cdot\|_2$ norm with any norm $\|\cdot\|_V$ on this vector space. In fact, for the following results to hold we additionally need to have an inner product on this vector space $\langle\cdot, \cdot\rangle_V$ that is compatible with the norm, i.e., $\langle\vec{x}, \vec{x}\rangle_V = \|\vec{x}\|_V^2$; that is, we would need an *inner product space*. One can check (as an exercise) that $\mathbb{R}^n$ is an example of such an inner product space, with the ℓ^2 norm $\|\cdot\|_2$ and usual inner product. Thus, in order to generalize the results introduced in this section, we would replace $\mathbb{R}^n$ with V, replace the norm $\|\vec{x}\|_2$ with $\|\vec{x}\|_V$, and replace the inner product $\vec{x}^\top\vec{y}$ with $\langle\vec{x},\ \vec{y}\rangle_V$.[4]

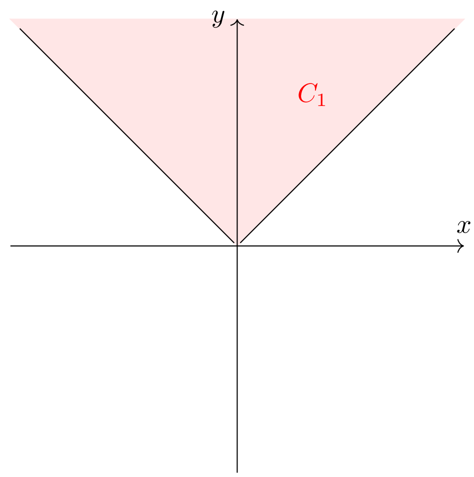
For now, we return to working over $\mathbb{R}^n$ so as to not introduce additional complexity in the definitions.

Example 110. The sets

$$C_1 \doteq \{(\vec{x}, y) \in \mathbb{R}^{n+1} \mid \|\vec{x}\|_2 \leq y\} \tag{5.52}$$

$$C_2 \doteq \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\} \tag{5.53}$$

are cones. If $n = 1$ then C_1 looks like:



Now, we discuss some important classes of cones.

Definition 111

(a) A set of the form

$$C \doteq \{(\vec{x}, t) \in \mathbb{R}^{n+1} \mid A\vec{x} \leq t\vec{y}, t \geq 0\} \tag{5.54}$$

is called a *polyhedral cone*, and in particular corresponds to the polyhedron $\{\vec{x} \in \mathbb{R}^n \mid A\vec{x} \leq \vec{y}\}$.

(b) A set of the form

$$C \doteq \{(\vec{x}, t) \in \mathbb{R}^{n+1} \mid \|A\vec{x} - t\vec{y}\|_2 \leq tz, t \geq 0\} \tag{5.55}$$

is called a *ellipsoidal cone*, and in particular corresponds to the ellipse $\{\vec{x} \in \mathbb{R}^n \mid \|A\vec{x} - \vec{y}\|_2 \leq z\}$.[a]

[a] The ellipsoidal cone corresponding to the unit circle — which is, after all, an ellipse — is the *second order cone*, to be discussed later.

Proposition 112

Polyhedral and ellipsoidal cones are convex cones.

Proof. Left as an exercise. □

[4] In this class, the issue really only comes up when discussing vector spaces where each element is a matrix, where the norm is the Frobenius norm, and the inner product is a corresponding "Frobenius inner product," to be defined later. This is relevant in semidefinite programming, for example.

Proposition 113

Let $K \subseteq \mathbb{R}^n$ be a cone. Define

$$K^\star \doteq \{\vec{y} \in \mathbb{R}^n \mid \vec{y}^\top \vec{x} \geq 0 \text{ for each } \vec{x} \in K\}. \tag{5.56}$$

Then $K^\star$ is a closed convex cone. We call $K^\star$ the *dual cone* of K.

Proof. Let $\vec{y}, \vec{z} \in K^\star$ and let $\alpha, \beta \geq 0$. Then, for any $\vec{x} \in K$, we have

$$(\alpha\vec{y} + \beta\vec{z})^\top \vec{x} = \underbrace{\alpha}_{\geq 0}\underbrace{(\vec{y}^\top \vec{x})}_{\geq 0} + \underbrace{\beta}_{\geq 0}\underbrace{(\vec{z}^\top \vec{x})}_{\geq 0} \geq 0. \tag{5.57}$$

In particular, this holds for $\beta = 0$ (so that $K^\star$ is a cone), and $\alpha \in [0, 1]$ and $\beta = 1 - \alpha$ (so that $K^\star$ is convex). Thus, $K^\star$ is a convex cone.

Now we want to show that K contains its limits. Let $(\vec{y}_k)_{k=1}^\infty$ be a sequence in $K^\star$ that converges to some $\vec{y} \in \mathbb{R}^n$. We want to show that $\vec{y} \in K^\star$. Indeed, for any $\vec{x} \in K$, we want to show that $\vec{y}^\top \vec{x} \geq 0$. But this is true because we have

$$\vec{y}^\top \vec{x} = \left(\lim_{k\to\infty} \vec{y}_k\right)^\top \vec{x} = \lim_{k\to\infty} \underbrace{\vec{y}_k^\top \vec{x}}_{\geq 0} \geq 0. \tag{5.58}$$

Since $\vec{x}$ was arbitrary, we have $\vec{y} \in K^\star$. Thus $K^\star$ contains its limits and is a closed cone. □

A geometric interpretation of the dual cone is that $K^\star$ is the intersection of the half-spaces $\mathcal{H}_{\vec{x}} \doteq \{\vec{y} \in \mathbb{R}^n \mid \vec{y}^\top \vec{x} \geq 0\}$ defined by each vector $\vec{x}$ in K.

Below, we provide some examples of cones and their dual cones. The reader is encouraged to verify the following statements.

Example 114.

(a) The set $\mathbb{R}^n_+ \doteq \{\vec{x} \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i \in \{1, \ldots, n\}\}$ is a convex cone, and its dual cone in $\mathbb{R}^n$ is itself.

(b) Let $S \doteq \{\vec{x} \in \mathbb{R}^2 \mid x_1 = 0 \text{ or } x_2 = 0\}$. Then S is a cone but is not a convex cone, and the dual cone of S is $\{\vec{0}\}$, the singleton set comprised of the 2-dimensional zero vector.

(c) Let $S \subseteq \mathbb{R}^n$ be a subspace. Then S is a convex cone, and the orthogonal complement $S^\perp$ of S is the dual cone of S.

Two proper cones with interesting properties that are widely used in convex optimization are the cone of symmetric positive semi-definite matrices and the second-order cone. The propositions below explore their properties.

Proposition 115

Let $\mathbb{S}^n$ be the vector space of $n \times n$ real-valued symmetric matrices equipped with the Frobenius inner product:

$$\langle A,\ B\rangle_F \doteq \operatorname{tr}(AB) = \sum_{i=1}^n \sum_{j=1}^n A_{ij}B_{ij}, \qquad \text{for any } A, B \in \mathbb{S}^n. \tag{5.59}$$

and the Frobenius norm $\|\cdot\|_F$. Let $\mathbb{S}^n_+$ denote the set of all $n \times n$ positive semidefinite matrices.

(a) $\mathbb{S}^n_+$ is a proper cone in $\mathbb{S}^n$;

(b) The dual cone of $\mathbb{S}^n_+$ in $\mathbb{S}^n$ is itself.

To start with, this is an instance of the earlier discussion: not every application of cones will be with reference to $\mathbb{R}^n$, but could be with reference to another inner product space. Here it is the vector space $\mathbb{S}^n$ with the appropriate inner product and norm. The intuition for why we can do this is that $\mathbb{S}^n$ is a subspace of $\mathbb{R}^{n\times n}$, the space of $n \times n$ matrices. But by stacking up the entries in an $n \times n$ matrix we get an n^2-dimensional vector, i.e., an element of $\mathbb{R}^{n^2}$. Indeed, the Frobenius norm and inner product on matrices are exactly the ℓ^2 inner product and norm applied to the "unrolled" matrices in $\mathbb{R}^{n^2}$. Thus, one can informally view $\mathbb{S}^n$ as a subspace of $\mathbb{R}^{n^2}$ (though remember that it is a vector space in its own right, so that we can define things like interiors and dual cones with respect to it instead of its "parent" space $\mathbb{R}^{n^2}$), so the same proof techniques and intuitions carry over.

Proof.

(a) To show that $\mathbb{S}^n_+$ is a convex cone, let $A, B \in \mathbb{S}^n_+$ and $\alpha, \beta \geq 0$ be given. We wish to show that $\alpha A + \beta B \in \mathbb{S}^n_+$, which will confirm that $\mathbb{S}^n_+$ is a convex cone. Indeed, $\alpha A + \beta B$ is symmetric as the linear combination of two symmetric matrices. To show that it is positive semidefinite, let $\vec{v} \in \mathbb{R}^n$ be arbitrarily given. Then we have

$$\vec{v}^\top(\alpha A + \beta B)\vec{v} = \underbrace{\alpha}_{\geq 0}\underbrace{\vec{v}^\top A\vec{v}}_{\geq 0} + \underbrace{\beta}_{\geq 0}\underbrace{\vec{v}^\top B\vec{v}}_{\geq 0} \geq 0. \tag{5.60}$$

Since $\vec{v}$ was arbitrarily given, $\alpha A + \beta B$ is positive semidefinite. Thus $\alpha A + \beta B \in \mathbb{S}^n_+$. This holds for $\beta = 0$, so $\mathbb{S}^n_+$ is a cone, and also for $\alpha \in [0, 1]$ and $\beta = 1 - \alpha$, so $\mathbb{S}^n_+$ is convex.

We now show that $\mathbb{S}^n_+$ is pointed, i.e., it contains no lines through the origin. Let $A \in \mathbb{S}^n_+$ be a nonzero matrix. Then $-A \notin \mathbb{S}^n_+$, because there exists $\vec{v} \in \mathbb{R}^n$ such that $\vec{v}^\top A\vec{v} > 0$, at which point $\vec{v}^\top(-A)\vec{v} = -\vec{v}^\top A\vec{v} < 0$, so $-A$ is not positive semidefinite. Thus $-A \notin \mathbb{S}^n_+$, so that for any $A \in \mathbb{S}^n_+$ there exists $\alpha \in \mathbb{R}$ such that $\alpha A \notin \mathbb{S}^n_+$. Thus $\mathbb{S}^n_+$ contains no lines through the origin and is pointed.

We now show that $\mathbb{S}^n_+$ is solid, i.e., has nonempty interior. We show that the open ball in $\mathbb{S}^n$ defined by

$$\mathcal{B} \doteq \left\{A \in \mathbb{S}^n \,\middle|\, \|A - I\|_F < \frac{1}{2}\right\} \tag{5.61}$$

is contained in $\mathbb{S}^n_+$. Indeed, let $A \in \mathcal{B}$. By definition of $\mathcal{B}$, we have that A is symmetric. Moreover, for each $\vec{v} \in \mathbb{R}^n$ we have

$$\begin{aligned}
\vec{v}^\top A\vec{v} &= \vec{v}^\top((A - I) + I)\vec{v} && (5.62)\\
&= \vec{v}^\top(A - I)\vec{v} + \vec{v}^\top\vec{v} && (5.63)\\
&= \vec{v}^\top(A - I)\vec{v} + \|\vec{v}\|_2^2 && (5.64)\\
&\geq -\|A - I\|_2\|\vec{v}\|_2^2 + \|\vec{v}\|_2^2 && (5.65)\\
&\geq -\|A - I\|_F\|\vec{v}\|_2^2 + \|\vec{v}\|_2^2 && (5.66)\\
&> -\frac{1}{2}\|\vec{v}\|_2^2 + \|\vec{v}\|_2^2 && (5.67)\\
&= \frac{1}{2}\|\vec{v}\|_2^2. && (5.68)
\end{aligned}$$

For $\vec{v}$ nonzero, we have $\frac{1}{2}\|\vec{v}\|_2^2$, and so $\vec{v}^\top A\vec{v} > 0$. Thus, $A \in \mathbb{S}^n_+$.

Finally, we need to show that $\mathbb{S}^n_+$ is a closed cone. Let $(A_k)_{k=1}^\infty$ be a sequence in $\mathbb{S}^n_+$ that converges to some $A \in \mathbb{S}^n$. We want to show that $A \in \mathbb{S}^n_+$. As the limit of symmetric matrices, A is symmetric. Now for any

$\vec{v} \in \mathbb{R}^n$ we have

$$\vec{v}^\top A\vec{v} = \vec{v}^\top \left(\lim_{k\to\infty} A_k\right)\vec{v} = \lim_{k\to\infty} \underbrace{\vec{v}^\top A_k\vec{v}}_{\geq 0} \geq 0. \tag{5.69}$$

Thus $\vec{v}^\top A\vec{v} \geq 0$ for all $\vec{v} \in \mathbb{R}^n$, so that $A \in \mathbb{S}^n_+$. Thus $\mathbb{S}^n_+$ contains its limits and is a closed cone.

We have thus proved that $\mathbb{S}^n_+$ is a convex, pointed, solid, and closed cone, so it is a proper cone.

(b) We now show that the dual cone of $\mathbb{S}^n_+$ in $\mathbb{S}^n$ is itself. That is, defining the dual cone as $(\mathbb{S}^n_+)^\star = \{A \in \mathbb{S}^n \mid \langle A,\ B\rangle_F \geq 0 \text{ for all } B \in \mathbb{S}^n_+\}$, we want to show that $(\mathbb{S}^n_+)^\star = \mathbb{S}^n_+$. To do this, we show that $(\mathbb{S}^n_+)^\star \subseteq \mathbb{S}^n_+$ and that $(\mathbb{S}^n_+)^\star \supseteq \mathbb{S}^n_+$.

We first show that $(\mathbb{S}^n_+)^\star \subseteq \mathbb{S}^n_+$. Fix $A \in (\mathbb{S}^n_+)^\star$, and let $\vec{v} \in \mathbb{R}^n$ be given arbitrarily. Then, since $\vec{v}\vec{v}^\top \in \mathbb{S}^n_+$, we have

$$\begin{aligned}
\vec{v}^\top A\vec{v} &= \operatorname{tr}\left(\vec{v}^\top A\vec{v}\right) && (5.70)\\
&= \operatorname{tr}\left(A\vec{v}\vec{v}^\top\right) && (5.71)\\
&= \left\langle A,\ \vec{v}\vec{v}^\top\right\rangle_F && (5.72)\\
&\geq 0, && (5.73)
\end{aligned}$$

where in the first line we use the fact that the trace of a scalar is a scalar, in the second line we use the cyclic trace inequality,and the last inequality is justified because $\vec{v}\vec{v}^\top \in \mathbb{S}^n_+$ and $A \in (\mathbb{S}^n_+)^\star$. Since $\vec{v}$ was selected arbitrarily, $\vec{v}^\top A\vec{v} \geq 0$ for all $\vec{v} \in \mathbb{R}^n$. This (along with the fact that A is symmetric) proves that $A \in \mathbb{S}^n_+$. Since A was selected arbitrarily, $(\mathbb{S}^n_+)^\star \subseteq \mathbb{S}^n_+$.

Now we show that $\mathbb{S}^n_+ \subseteq (\mathbb{S}^n_+)^\star$. Let $B \in \mathbb{S}^n_+$. We aim to show that $B \in (\mathbb{S}^n_+)^\star$, i.e., that $\langle B,\ C\rangle_F \geq 0$ for any $C \in \mathbb{S}^n_+$. By the spectral theorem, we may diagonalize $C = \sum_{i=1}^n \lambda_i\vec{v}_i\vec{v}_i^\top$, where $\lambda_i \geq 0$ are the eigenvalues of C and $\vec{v}_i \in \mathbb{R}^n$ are orthonormal eigenvectors of C. Thus we have

$$\begin{aligned}
\langle B,\ C\rangle_F &= \operatorname{tr}(BC) && (5.74)\\
&= \operatorname{tr}\left(B\left(\sum_{i=1}^n \lambda_i\vec{v}_i\vec{v}_i^\top\right)\right) && (5.75)\\
&= \operatorname{tr}\left(\sum_{i=1}^n \lambda_i B\vec{v}_i\vec{v}_i^\top\right) && (5.76)\\
&= \sum_{i=1}^n \lambda_i \operatorname{tr}\left(B\vec{v}_i\vec{v}_i^\top\right) && (5.77)\\
&= \sum_{i=1}^n \lambda_i \operatorname{tr}\left(\vec{v}_i^\top B\vec{v}_i\right) && (5.78)\\
&= \sum_{i=1}^n \underbrace{\lambda_i}_{\geq 0}\underbrace{\vec{v}_i^\top B\vec{v}_i}_{\geq 0} && (5.79)\\
&\geq 0. && (5.80)
\end{aligned}$$

Thus we have $\langle B,\ C\rangle_F \geq 0$. Since $C \in \mathbb{S}^n_+$ were arbitrary, we have $B \in (\mathbb{S}^n_+)^\star$. Since B were arbitrary, we have $\mathbb{S}^n_+ \subseteq (\mathbb{S}^n_+)^\star$.

Thus, $(\mathbb{S}^n_+)^\star = \mathbb{S}^n_+$.

□

The next example of a cone will be useful when discussing the eponymous *second-order cone programs* (SOCPs).

Definition 116 (Second Order Cone)
The *second-order cone* in $\mathbb{R}^{n+1}$ is the set:

$$K \doteq \{(\vec{x}, t) \in \mathbb{R}^n \times \mathbb{R} \mid \|\vec{x}\|_2 \leq t\}. \tag{5.81}$$

Proposition 117
Let K be the second-order cone in $\mathbb{R}^{n+1}$.

(a) K is a proper cone.

(b) The dual cone of K in $\mathbb{R}^{n+1}$ is itself.

Proof.

(a) We first show that K is a convex cone. Let $(\vec{x}_1, t_1), (\vec{x}_2, t_2) \in K$ and let $\alpha_1, \alpha_2 \geq 0$. Then

$$\begin{aligned} \|\alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2\|_2 &\leq \alpha_1 \|\vec{x}_1\|_2 + \alpha_2 \|\vec{x}_2\|_2 && (5.82) \\ &\leq \alpha_1 t_1 + \alpha_2 t_2 && (5.83) \end{aligned}$$

where the first inequality is by triangle inequality and the second is by definition of a second-order cone. This holds for $\alpha_2 = 0$, showing that K is indeed a cone, and $\alpha_1 \in [0, 1]$ and $\alpha_2 = 1 - \alpha_1$, showing that K is convex. Thus K is a convex cone.

We show that K is pointed, i.e., contains no lines through the origin. Indeed, let $(\vec{x}, t) \in K$ be nonzero. Then either $\vec{x}$ is nonzero or t is nonzero (or both); in the first case, $\|\vec{x}\|_2 > 0$ so since $t \geq \|\vec{x}\|_2$, we have $t > 0$ as well. Thus $t > 0$ in all cases. Thus we certainly *do not* have $\|-\vec{x}\|_2 \leq -t$(in fact, norms can never be negative) so that $-(\vec{x}, t) = (-\vec{x}, -t) \notin K$. Thus for any $(\vec{x}, t) \in K$ there exists $\alpha \in \mathbb{R}$ such that $\alpha(\vec{x}, t) \notin K$, so K is pointed.

We now show that K is solid, i.e., has nonempty interior. We claim that the open ball in $\mathbb{R}^{n+1}$ of radius 1 centered at $(\vec{0}, 2)$, where $\vec{0}$ is the n-dimensional zero vector, is contained in K. Formally, define

$$\mathcal{B} \doteq \{(\vec{x}, t) \in \mathbb{R}^{n+1} \mid \left\|(\vec{x}, t) - (\vec{0}, 2)\right\|_2 < 1\}. \tag{5.84}$$

Let $(\vec{x}, t) \in \mathcal{B}$; we show that $(\vec{x}, t) \in K$. Indeed, we have

$$\begin{aligned} &\left\|(\vec{x}, t) - (\vec{0}, 2)\right\|_2 < 1 && (5.85) \\ \implies &\left\|(\vec{x}, t) - (\vec{0}, 2)\right\|_2^2 < 1^2 = 1 && (5.86) \\ \implies &\|\vec{x}\|_2^2 + (t-2)^2 < 1, && (5.87) \end{aligned}$$

which implies that $\|\vec{x}\|_2^2 < 1$ and $(t-2)^2 < 1$, namely $t \in (1, 3)$. Thus $\|\vec{x}\|_2^2 < 1 < t$, so $\|\vec{x}\|_2 < 1 < t$, so $\|\vec{x}\|_2 \leq t$, so $(\vec{x}, t) \in K$ as desired. Since $(\vec{x}, t) \in \mathcal{B}$ were arbitrarily chosen, $\mathcal{B} \subseteq K$ and K is solid.

We now show that K is closed, i.e., contains its limits. Let $((\vec{x}_k, t_k))_{k=1}^{\infty}$ be a sequence in K that converges to some $(\vec{x}, t) \in \mathbb{R}^{n+1}$. We want to show that $(\vec{x}, t) \in K$. Indeed, we have that $(\vec{x}, t) \in K$ if and only if $t - \|\vec{x}\|_2 \geq 0$. We have

$$t - \|\vec{x}\|_2 = \lim_{k \to \infty} t_k - \left\|\lim_{k \to \infty} \vec{x}_k\right\|_2 \tag{5.88}$$