# Vector Norms

- $\|\vec{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2}$ (2 norm / euclidean norm)

A function $f: X \to \mathbb{R}$ is a norm if

1. $\|\vec{x}\| \geq 0 \quad \forall \vec{x} \in X$ and $\|\vec{x}\| = 0$ iff $\vec{x} = 0$
2. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \quad \forall x, y \in X$
3. $\|\alpha \vec{x}\| = |\alpha| \, \|\vec{x}\|$, $\forall \alpha \in \mathbb{R}, \vec{x} \in X$

## $\ell_p$-norm

$$\|\vec{x}\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

$p=2$: Euclidean norm

$p=1$: $\|\vec{x}\|_1 = \sum |x_i|$

$p=\infty$: $\|\vec{x}\|_\infty = \max_{i=1,2,\ldots n} |x_i|$

## Cauchy Schwartz

$|\langle \vec{x}^T \vec{y} \rangle| \leq \|\vec{x}\|_2 \|\vec{y}\|_2$ (proof in Axler)

## Generalization: Holder's Inequality

$\vec{x}, \vec{y} \in \mathbb{R}^n \quad p, q \in \mathbb{R}, \; p, q \geq 1$

$\frac{1}{p} + \frac{1}{q} = 1$

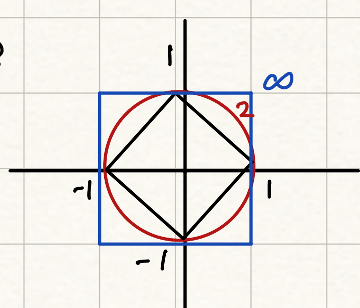
$|\langle \vec{x}^T \vec{y} \rangle| \leq \|\vec{x}\|_p \|\vec{y}\|_q$

## Norm Ball

$B_p = \{ \vec{x} \in \mathbb{R}^n \mid \|\vec{x}\|_p \leq 1 \}$

$\vec{x} \in \mathbb{R}^2$

$p = 2$

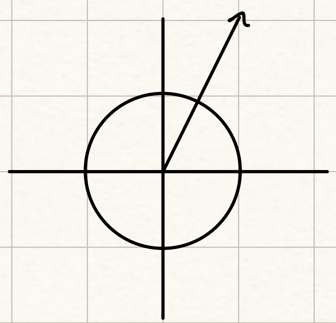



1 norm: $|x_1| + |x_2| \leq 1$

## Optimization Problem

$$\max_{\text{st } \|\vec{x}\|_p \le 1} \vec{x}^T\vec{y}$$

1. $p=2$, $y=\begin{bmatrix}1\\2\end{bmatrix}$



$$\text{argmax} = \vec{x}^* = \frac{\vec{y}}{\|\vec{y}\|}$$

2. $p=1$ $\quad \max_{\|\vec{x}\|_1=1} \vec{x}^T\vec{y}$

$$\vec{x}^T\vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

Constraint: $|x_1| + |x_2| + \cdots + |x_n| \le 1$

Choose $x_1 = 1 \Rightarrow x_2 = \cdots = x_n = 0$

$\vec{x}^T\vec{y} = y_1$

If $x_i = (\text{sign})_i$ where $i = \text{argmax}|y_i|$, then $\vec{x}^T\vec{y} = \max|y_i|$

$$\vec{x}^T\vec{y} = \sum_{i=1}^n x_iy_i \le \sum_{i=1}^n |x_i||y_i| \le \sum_{i=1}^n |x_i||y_{max}|$$

$$= |y_{max}|\left(\sum_{i=1}^n |x_i|\right) \le |y_{max}| \cdot 1 = \|\vec{y}\|_\infty$$

3. $\max \vec{x}^T\vec{y}$
s.t. $\|\vec{x}\|_\infty \le 1$

$x_i \in [-1, 1]$

$$\vec{y} = \begin{bmatrix}-2\\3\end{bmatrix} \rightarrow \vec{x} = \begin{bmatrix}\text{sign}(-2)\\\text{sign}(3)\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$$

$$\vec{x}^T\vec{y} = \sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{n} |x_i y_i| = \sum_{i=1} |x_i||y_i| \le \sum_{i=1}^{n} |y_i| = \|\vec{y}\|_1$$

## Gram Schmidt / QR

$\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m\} \rightarrow$ generate an orthonormal basis

Choose $\vec{q}_1 = \dfrac{\vec{q}_1}{\|\vec{q}_1\|_2}$

Project $\vec{a}_2$ onto $\vec{q}_1$: $\langle \vec{a}_2, \vec{q}_1 \rangle \vec{q}_1$

Residual: $\vec{q}_2 - \langle \vec{q}_2, \vec{q}_1 \rangle \vec{q}_1 = \vec{S}_2$

$\vec{q}_2 = \dfrac{\vec{S}_2}{\|S_2\|_2}$

Project $\vec{a}_3$ onto $\vec{q}_1, \vec{q}_2$

Residual $\vec{a}_3 - \langle a_3, \vec{q}_2 \rangle \vec{q}_2 - \langle \vec{a}_3, \vec{q}_1 \rangle \vec{q}_1 = \vec{S}_3$

$$A = [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m]$$
$$= \underbrace{[\vec{q}_1, \vec{q}_2, \dots, \vec{q}_3]}_{Q} \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ 0 & Y_{22} & Y_{23} \\ 0 & 0 & Y_{33} \end{bmatrix}$$