Lec 3

Orthogonal Decomposition of a Space

Vector space X, S subspace of X

$\vec{x} \in X \quad \vec{x} = \vec{s} + \vec{s}^{\perp}$ "perp"

$S^{\perp} = \{\vec{y} \mid \langle y, \vec{s} \rangle = 0, \ \forall \vec{s} \in S\}$

Fundamental Subspaces Associated with matrix A

$\text{Null}(A), \underbrace{\text{Range}(A)}_{\text{col space}}, \text{Null}(A^T), \text{Range}(A^T)$

1. $\text{Null}(A) \oplus \text{Range}(A^T) = \mathbb{R}^n$
2. $\text{Null}(A^T) \oplus \text{Range}(A) = \mathbb{R}^n$

Recall

Null A: $\vec{x}$ s.t. $A\vec{x} = \vec{0}$

Range A $\vec{y}$ s.t. $\exists \vec{x}$ s.t. $\vec{y} = A\vec{x}$

Set Equality: $N(A) = R(A^T)^{\perp}$

$N(A) \subseteq R(A^T)^{\perp}$

$\forall \vec{x} \in N(A) \quad \vec{x} \in R(A^T)^{\perp}$

$A\vec{x} = 0$

We will show $\vec{w} \in R(A^T)$, $\langle \vec{w}, \vec{x} \rangle = 0$

$w = A^T\vec{z}$

Consider $\langle \vec{w}, \vec{x} \rangle$

Reminder: $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T\vec{y} = \vec{y}^T x$

$= \langle A^T\vec{z}, \vec{x} \rangle = \vec{x}^T A^T \vec{z}$

$= (A\vec{x})^T \vec{z}$

$= 0\vec{z} = 0$

$R(A^T)^{\perp} \subseteq N(A)$

$\forall \vec{w}$ s.t. $\vec{w} = A^T\vec{y}$, $\langle \vec{w}, \vec{x} \rangle = 0$

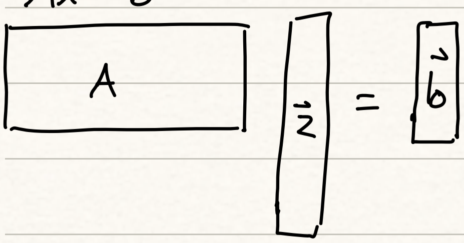
$\langle A^T\vec{y}, \vec{x} \rangle = 0$

$\vec{y}^T A\vec{x} = 0$ if $\vec{y}^T\vec{z} = 0 \;\; \forall \vec{y}$, $\vec{z} = 0$ $\because$ only $\vec{0}$ orthogonal to all vectors

$\therefore A\vec{x} = 0$

Minimum Norm Problem

$A\vec{x} = b$



$A \in \mathbb{R}^{m,n}$, full row rank

$m < n$ underdetermined

$A\vec{x} = \vec{b}$

$\vec{x} = x_1 + x_2 + \dots + x_n$

$A\vec{x} = A\vec{x}_1 + A\vec{x}_2 + \dots + Ax_n$

$\vec{x} = \underbrace{\vec{x}_1}_{\in N(A)} + \vec{x}_2$

$A\vec{x} = Ax^2$

$\vec{x} = \vec{y} + \vec{z}$

$y \in N(A)$, $\vec{z} \in R(A^T)$

$\|x\|_2^2 = \underbrace{\|y\|_2^2}_{0} + \|z\|_2^2$

Want $\vec{x} \in R(A^T)$

$\vec{x} = A^T\vec{w}$ because $\vec{x} \in R(A^T)$

$A\vec{x} = \vec{b}$

$\underbrace{AA^T}_{\text{square + full row rank}} w = \vec{b}$ $\quad\therefore \vec{w} = (AA^T)^{-1}\vec{b}$

$x = A^T(AA^T)^{-1}\vec{b}$

Symmetric Matrices

$A^T = A$, $a_{ij} = a_{ji}$ for all i,j

- Always diagonalizable

$A = P \Lambda P^{-1}$