Lec 4

Spectral Theorem

A is a symmetric matrix $\in \mathbb{R}^n \equiv S^n$

1. $\lambda_i \in \mathbb{R}$, all eigenvalues real
2. Eigenspaces associated with distinct eigenvalues are orthogonal
3. $\phi_i = \text{Null}(A - \lambda_i I)$

Let μ_i be the multiplicity of λ_i

$\dim(\phi_i) = \mu_i$

dim of eigenspace not necessarily equal to multiplicity of eigenvalue

Proof (of 3)

Lemma: $A \in S^n$ Let $(\lambda, \vec{v})$ be an eigenpair of A

Then, $\exists$ an orthonormal matrix U s.t. $U^T A U = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \boxed{B} & \\ 0 & & \end{bmatrix}$, $B \in S^{n-1}$

Proof: Consider

$$U = \begin{bmatrix} \vec{v} & U_1 \end{bmatrix} \qquad U^T A U = \begin{bmatrix} \vec{v}^T \\ U_1^T \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \vec{v} & U_1 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{v}^T \\ U_1^T \end{bmatrix} \begin{bmatrix} A\vec{v} & AU_1 \end{bmatrix}$$

$$= \begin{bmatrix} \overset{=\lambda}{\vec{v}^T A \vec{v}} & \vec{v}^T A U_1 \\ U_1^T A \vec{v} & U_1^T A U_1 \end{bmatrix} \qquad \vec{v}^T A U_1 = \vec{v}^T A^T U_1 = \lambda \vec{v} U_1 = \vec{0}$$

$$= \begin{bmatrix} \lambda & 0 \cdots 0 \\ 0 & \\ \vdots & U_1^T A U_1 \\ 0 & \end{bmatrix}$$

Principal Component Analysis

$\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \in \mathbb{R}^p$ (zero-mean)

Uncover an underlying low dimensional structure.