Section 1A: $\mathbb{R}^n$ and $\mathbb{C}^n$

1. Complex Numbers

Before we can study vector spaces, we need to understand the scalars we'll use. The real numbers $\mathbb{R}$ are familiar, but linear algebra becomes more powerful when we also work with complex numbers $\mathbb{C}$.

1.1 Definition: Complex Numbers, $\mathbb{C}$

- A **complex number** is an ordered pair (a, b) where $a, b \in \mathbb{R}$, written as $a + bi$.
- The set of all complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

- **Addition** and **multiplication** on $\mathbb{C}$ are defined by:

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i \\ (a + bi)(c + di) &= (ac - bd) + (ad + bc)i\end{aligned}$$

  where $a, b, c, d \in \mathbb{R}$.

Intuition: Think of $\mathbb{C}$ as a 2D plane where the horizontal axis represents real numbers and the vertical axis represents imaginary numbers. If $a \in \mathbb{R}$, we identify $a + 0i$ with the real number a, so $\mathbb{R} \subset \mathbb{C}$. We write $0 + bi$ as just bi, and $0 + 1i$ as just i.

Why complex numbers? Even when working with real matrices, eigenvalues (Chapter 5) often require complex numbers. The completeness of $\mathbb{C}$ makes linear algebra more elegant—every polynomial has roots in $\mathbb{C}$.

Why this multiplication formula? We *define* i as a symbol satisfying $i^2 = -1$. This is consistent and creates an algebraically closed field. Using the usual rules of arithmetic:

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

1.2 Example: Complex Arithmetic
Compute $(2 + 3i)(4 + 5i)$.
Using the distributive and commutative properties:

$$\begin{aligned}(2 + 3i)(4 + 5i) &= 2 \cdot (4 + 5i) + (3i)(4 + 5i) \\ &= 2 \cdot 4 + 2 \cdot 5i + 3i \cdot 4 + (3i)(5i) \\ &= 8 + 10i + 12i - 15 \\ &= \boxed{-7 + 22i}\end{aligned}$$

1.3 Properties of Complex Arithmetic
For all $\alpha, \beta, \lambda \in \mathbb{C}$:
commutativity: $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$
associativity: $(\alpha+\beta)+\lambda = \alpha+(\beta+\lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$
identities: $\lambda + 0 = \lambda$ and $\lambda \cdot 1 = \lambda$
additive inverse: For every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$
multiplicative inverse: For every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$
distributive property: $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$

1.4 Example: Commutativity of Complex Multiplication
To show $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$, suppose $\alpha = a+bi$ and $\beta = c + di$ where $a, b, c, d \in \mathbb{R}$.
LHS:

$$\alpha\beta = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

RHS:

$$\beta\alpha = (c + di)(a + bi) = (ca - db) + (cb + da)i$$

Since real multiplication is commutative ($ac = ca$, $bd = db$, etc.), we have $\alpha\beta = \beta\alpha$. □

1.5 Definition: $-\alpha$, Subtraction, $1/\alpha$, Division
Suppose $\alpha, \beta \in \mathbb{C}$.

- Let $-\alpha$ denote the **additive inverse** of α: the unique complex number such that $\alpha + (-\alpha) = 0$.
- **Subtraction** on $\mathbb{C}$ is defined by $\beta-\alpha = \beta+(-\alpha)$.
- For $\alpha \neq 0$, let $1/\alpha$ and $\frac{1}{\alpha}$ denote the **multiplicative inverse** of α: the unique complex number such that $\alpha(1/\alpha) = 1$.
- For $\alpha \neq 0$, **division** by α is defined by $\beta/\alpha = \beta(1/\alpha)$.

Computing $1/\alpha$: For $\alpha = a + bi \neq 0$, multiply by the conjugate:

$$\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

Key insight: Why multiply by the conjugate? Because $(a+bi)(a-bi) = a^2+b^2$ is always real and positive. Multiplying by the conjugate eliminates i from the denominator.

Example: Complex Division. Compute $\dfrac{2 + 3i}{4 + 5i}$.

Multiply by the conjugate of the denominator:

$$\frac{2+3i}{4+5i}\cdot\frac{4-5i}{4-5i}=\frac{(2+3i)(4-5i)}{(4+5i)(4-5i)}$$

Denominator: $(4+5i)(4-5i)=16+25=41$

Numerator: $(2+3i)(4-5i)=8-10i+12i+15=23+2i$

Answer: $\frac{2+3i}{4+5i}=\boxed{\frac{23}{41}+\frac{2}{41}i}$

1.6 Notation: $\mathbb{F}$

Throughout this book, $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$.

Why use $\mathbb{F}$? The letter $\mathbb{F}$ reminds us of "field." Both $\mathbb{R}$ and $\mathbb{C}$ are fields: sets with addition and multiplication satisfying the properties in 1.3. Using $\mathbb{F}$ lets us state theorems once and have them apply to both $\mathbb{R}$ and $\mathbb{C}$.

Elements of $\mathbb{F}$ are called **scalars**.

Powers of scalars: For $\alpha \in \mathbb{F}$ and a positive integer m:

$$\alpha^m = \underbrace{\alpha\cdot\alpha\cdots\alpha}_{m\text{ times}}$$

This definition implies:

- $(\alpha^m)^n = \alpha^{mn}$
- $(\alpha\beta)^m = \alpha^m\beta^m$

2. Lists

To generalize $\mathbb{R}^2$ and $\mathbb{R}^3$ to higher dimensions, we first need to discuss the concept of lists.

1.7 Example: $\mathbb{R}^2$ and $\mathbb{R}^3$

- $\mathbb{R}^2 = \{(x,y) : x,y\in\mathbb{R}\}$ (the plane)
- $\mathbb{R}^3 = \{(x,y,z) : x,y,z\in\mathbb{R}\}$ (3D space)

1.8 Definition: List, Length

Suppose n is a nonnegative integer. A **list** of **length** n is an ordered collection of n elements (which might be numbers, other lists, or more abstract objects) separated by commas and surrounded by parentheses.

A list of length n is also called an **n-tuple**.

Key point: Two lists are equal if and only if they have the same length and the same elements in the same order.

1.9 Example: Lists versus Sets

- Lists $(3,5)$ and $(5,3)$ are **not equal**, but sets $\{3,5\}=\{5,3\}$
- Lists $(4,4)$ and $(4,4,4)$ are **not equal**, but sets $\{4,4\}=\{4,4,4\}=\{4\}$

Key difference: order and repetition matter in lists, not in sets.

Why lists? Linear algebra needs ordered data. The coordinates $(1,2,3)$ represent a different point than $(3,2,1)$. Order encodes meaning—the first coordinate might be position, the second velocity, the third acceleration.

3. $\mathbb{F}^n$

To define the higher-dimensional analogues of $\mathbb{R}^2$ and $\mathbb{R}^3$, we simply replace $\mathbb{R}$ with $\mathbb{F}$ (which equals $\mathbb{R}$ or $\mathbb{C}$) and replace the 2 or 3 with an arbitrary positive integer.

1.10 Notation: n

Fix a positive integer n for the rest of this chapter.

1.11 Definition: $\mathbb{F}^n$

$\mathbb{F}^n$ is the set of all lists of length n of elements of $\mathbb{F}$:

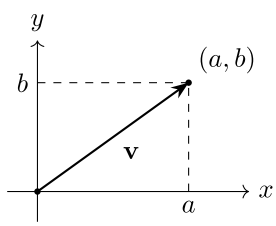
$$\mathbb{F}^n = \{(x_1,x_2,\ldots,x_n) : x_k\in\mathbb{F}\text{ for }k=1,\ldots,n\}$$

For $(x_1,\ldots,x_n)\in\mathbb{F}^n$ and $k\in\{1,\ldots,n\}$, we say x_k is the k^{th} **coordinate** of $(x_1,\ldots,x_n)$.

Example 3.1 (Elements of $\mathbb{F}^n$).

- $(2,-1,5) \in \mathbb{R}^3$ is a list with 3 real coordinates.
- $(1+i, 2, -3i) \in \mathbb{C}^3$ is a list with 3 complex coordinates.
- $(7,-2) \in \mathbb{R}^2$ corresponds to a point in the plane.
- $\mathbb{F}^1$ can be identified with $\mathbb{F}$.

Intuition: Think of $\mathbb{R}^2$ as the plane and $\mathbb{R}^3$ as 3-dimensional space. For $n > 3$, we lose geometric visualization but the algebra works identically.



Elements of $\mathbb{R}^2$ can be thought of as points or as vectors.

1.12 Example: $\mathbb{C}^4$
$\mathbb{C}^4 = \{(z_1, z_2, z_3, z_4) : z_1, z_2, z_3, z_4 \in \mathbb{C}\}$

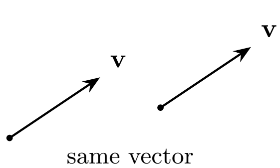
1.13 Definition: Addition in $\mathbb{F}^n$
Addition in $\mathbb{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

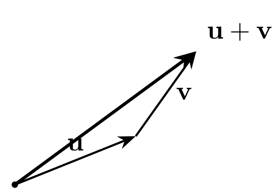
Example 3.2 (Vector Addition). In $\mathbb{R}^3$:

$$(1,2,3)+(10,20,30) = (1+10, 2+20, 3+30) = \boxed{(11,22,33)}$$

Geometric intuition: In $\mathbb{R}^2$ and $\mathbb{R}^3$, addition corresponds to the parallelogram rule: place the tail of the second vector at the head of the first.



A vector—same length and direction = same vector.



The sum of two vectors (tip-to-tail method).

1.14 Commutativity of Addition in $\mathbb{F}^n$
If $x, y \in \mathbb{F}^n$, then $x + y = y + x$.

Proof: Suppose $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then:

$$\begin{aligned} x + y &= (x_1, \ldots, x_n) + (y_1, \ldots, y_n) \\ &= (x_1 + y_1, \ldots, x_n + y_n) \\ &= (y_1 + x_1, \ldots, y_n + x_n) \\ &= (y_1, \ldots, y_n) + (x_1, \ldots, x_n) \\ &= y + x \end{aligned}$$

where the third equality uses commutativity of addition in $\mathbb{F}$. $\square$

1.15 Notation: 0
Let 0 denote the list of length n whose coordinates are all 0:

$$0 = (0, \ldots, 0)$$

Geometric intuition: The zero vector 0 represents the origin. Adding 0 to any vector leaves it unchanged—you're adding "no displacement."

1.16 Example: The Zero Vector Notation
When we write $x + 0 = x$ for $x \in \mathbb{F}^n$, the symbol 0 means the **zero vector**:

$$0 = (0, 0, \ldots, 0) \quad (n \text{ zeros})$$

Why? Addition in $\mathbb{F}^n$ is only defined for two vectors. Since x is a vector, the 0 in "$x + 0$" must also be a vector—not the number zero.
Example in $\mathbb{R}^3$:

$$(1,2,3) + 0 = (1,2,3) + (0,0,0) = (1,2,3)\checkmark$$

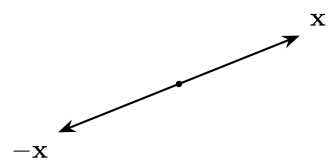
Key point: The symbol "0" means different things depending on context:

- In $\mathbb{F}$ (scalars): 0 is the number zero
- In $\mathbb{F}^n$ (vectors): 0 is the zero vector $(0, \ldots, 0)$

1.17 Definition: Additive Inverse in $\mathbb{F}^n$
For $x \in \mathbb{F}^n$, the **additive inverse** of x, denoted $-x$, is the vector $-x \in \mathbb{F}^n$ such that $x + (-x) = 0$.
If $x = (x_1, \ldots, x_n)$, then $-x = (-x_1, \ldots, -x_n)$.

Note: Subtraction is defined by $x - y = x + (-y)$.

Geometric intuition: In $\mathbb{R}^2$, $-x$ is the vector with the same length as x but pointing in the opposite direction.



A vector and its additive inverse.

Why not coordinate-wise multiplication? We *could* define multiplication of two vectors by multiplying corresponding coordinates, but this is not useful for linear algebra. Instead, **scalar multiplication** (multiplying a vector by a number) is central to our subject.

1.18 Definition: Scalar Multiplication in $\mathbb{F}^n$
The product of a number $\lambda \in \mathbb{F}$ and a vector in $\mathbb{F}^n$ is computed by multiplying each coordinate by λ:

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Example 3.3 (Scalar Multiplication in $\mathbb{R}^n$)**.** Here $\lambda = -2 \in \mathbb{R}$.

$$-2(1, 2, 3) = (-2 \cdot 1, -2 \cdot 2, -2 \cdot 3) = \boxed{(-2, -4, -6)}$$

Example 3.4 (Scalar Multiplication in $\mathbb{C}^n$)**.** Here $\lambda = 1 + i \in \mathbb{C}$.

$$\begin{aligned}(1+i)(2, -i) &= ((1+i) \cdot 2, (1+i)(-i)) \\ &= (2 + 2i, -i - i^2) \\ &= (2 + 2i, -i + 1) \\ &= \boxed{(2 + 2i, 1 - i)}\end{aligned}$$

Geometric intuition in $\mathbb{R}^2$:

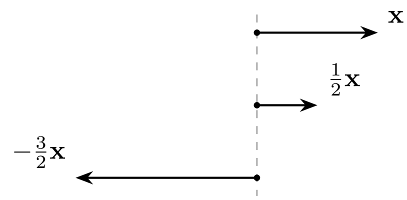
- If $\lambda > 0$: λx points in the same direction as x, with length λ times the length of x.
- If $\lambda > 1$: stretches (longer). If $0 < \lambda < 1$: shrinks (shorter).
- If $\lambda < 0$: λx points in the opposite direction, with length $|\lambda|$ times the length of x.

Direction Preservation Property
All scalar multiples of a nonzero vector x lie on a single line through the origin.
Proof: Let $x = (x_1, \ldots, x_n) \neq 0$. The set of all scalar multiples is:

$$\{\lambda x : \lambda \in \mathbb{F}\} = \{(\lambda x_1, \ldots, \lambda x_n) : \lambda \in \mathbb{F}\}$$

This is precisely the parametric equation of the line through the origin with direction vector x. As λ varies over $\mathbb{F}$, we trace out every point on this line. □

Key insight: This property is fundamental to linear maps (Chapter 3)—they preserve these lines through the origin.



Scalar multiplication: scaling and reversing vectors.

Scalar multiplication vs dot product: Scalar multiplication takes a scalar and a vector, producing a **vector**. The dot product (Chapter 6) takes two vectors and produces a **scalar**. These are different operations.

4. Digression on Fields

A **field** is a set containing at least two distinct elements called 0 and 1, along with operations of addition and multiplication satisfying all properties listed in 1.3.

- $\mathbb{R}$ and $\mathbb{C}$ are fields.
- The set of rational numbers $\mathbb{Q}$ is a field.
- The set $\{0, 1\}$ with usual addition and multiplication (except $1 + 1 = 0$) is a field.

Note: This book deals only with $\mathbb{R}$ and $\mathbb{C}$. However, many definitions, theorems, and proofs work for arbitrary fields. If you prefer, think of $\mathbb{F}$ as denoting an arbitrary field (except in Chapters 6–7 on inner products, where $\mathbb{F} = \mathbb{C}$ is sometimes required).

Key Takeaways

1. **Complex numbers** $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ with $i^2 = -1$.

2. **Arithmetic:** $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$.
3. **Field properties** (1.3): commutativity, associativity, identities, inverses, distributivity.
4. $\mathbb{F}$ **notation:** $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.
5. $\mathbb{F}^n$ is the set of all n-tuples $(x_1, \ldots, x_n)$ with $x_j \in \mathbb{F}$.
6. **Addition** is coordinate-wise.
7. **Scalar multiplication:** $\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$.
8. **Zero vector** $0 = (0, \ldots, 0)$ is the additive identity.