

1 Suppose $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W , the matrix of T has at least $\dim \text{range } T$ nonzero entries.

SOLUTION Suppose v_1, \dots, v_n is a basis of V and w_1, \dots, w_m is a basis of W . Thus $\dim V = n$.

Suppose p of the columns of $\mathcal{M}(T)$ contain only 0's. Thus $Tv_k = 0$ for p different choices of k . Because v_1, \dots, v_n is a linearly independent list, this implies that $\dim \text{null } T \geq p$.

The fundamental theorem of linear maps (3.21) implies that

$$\begin{aligned} n - p &= \dim \text{range } T + \dim \text{null } T - p \\ &\geq \dim \text{range } T. \end{aligned}$$

The definition of p implies that the number of columns of $\mathcal{M}(T)$ that have at least one nonzero entry is $n - p$. The inequality above implies that there are at least $\dim \text{range } T$ such columns. Hence the matrix of T has at least $\dim \text{range } T$ nonzero entries.

2 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \text{range } T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1.

SOLUTION First suppose there exist a basis v_1, \dots, v_n of V and a basis w_1, \dots, w_m of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1. Thus

$$Tv_k = w_1 + \cdots + w_m$$

for each $k = 1, \dots, n$. Hence

$$\text{range } T = \text{span}(w_1 + \cdots + w_m)$$

and thus $\dim \text{range } T = 1$ (note that $w_1 + \cdots + w_m \neq 0$ because w_1, \dots, w_m is linearly independent).

Conversely, suppose now that $\dim \text{range } T = 1$. Let $n = \dim V$. By the fundamental theorem of linear maps (3.21), we have $\dim \text{null } T = n - 1$. Let u_1, \dots, u_{n-1} be a basis of $\text{null } T$. Extend to a basis u_1, \dots, u_{n-1}, u_n of V . Note that $u_n \notin \text{null } T$ and thus $Tu_n \neq 0$. Let $w = Tu_n$.

It is easy to verify that $u_1 + u_n, \dots, u_{n-1} + u_n, u_n$ is a basis of V , which we will relabel as v_1, \dots, v_n . Clearly $Tv_k = w$ for each $k = 1, \dots, n$.

Extend w to a basis w, w_2, \dots, w_m of W . It is easy to verify that

$$w - w_2, w_2 - w_3, \dots, w_{m-1} - w_m, w_m$$

is a basis of W . Furthermore,

$$w = (w - w_2) + (w_2 - w_3) + \cdots + (w_{m-1} - w_m) + w_m.$$

Thus the matrix of T with respect to the basis v_1, \dots, v_n of V and the basis $w - w_2, w_2 - w_3, \dots, w_{m-1} - w_m, w_m$ of W consists of all 1's.

3 Suppose v_1, \dots, v_n is a basis of V and w_1, \dots, w_m is a basis of W .

(a) Show that if $S, T \in \mathcal{L}(V, W)$, then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

(b) Show that if $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$, then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

This exercise asks you to verify 3.35 and 3.38.

SOLUTION

(a) Suppose $S, T \in \mathcal{L}(V, W)$. Let $A = \mathcal{M}(S)$ and $C = \mathcal{M}(T)$. Thus if $k = 1, \dots, n$, then

$$Sv_k = \sum_{j=1}^m A_{j,k} w_j \quad \text{and} \quad Tv_k = \sum_{j=1}^m C_{j,k} w_j.$$

Hence

$$(S + T)v_k = \sum_{j=1}^m (A_{j,k} + C_{j,k}) w_j.$$

The equation above implies that the entry in row j , column k , of $\mathcal{M}(S + T)$ equals $A_{j,k} + C_{j,k}$. Thus $\mathcal{M}(S + T) = A + C = \mathcal{M}(S) + \mathcal{M}(T)$.

(b) Suppose $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$. Let $A = \mathcal{M}(T)$. Thus if $k = 1, \dots, n$, then

$$Tv_k = \sum_{j=1}^m A_{j,k} w_j.$$

Hence

$$(\lambda T)v_k = \sum_{j=1}^m \lambda A_{j,k} w_j.$$

The equation above implies that the entry in row j , column k , of $\mathcal{M}(\lambda T)$ equals $\lambda A_{j,k}$. Thus $\mathcal{M}(\lambda T) = \lambda A = \lambda \mathcal{M}(T)$.

4 Suppose that $D \in \mathcal{L}(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbf{R})$ and a basis of $\mathcal{P}_2(\mathbf{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Compare with Example 3.33. The next exercise generalizes this exercise.

SOLUTION Take $x^3, x^2, x, 1$ as a basis of $\mathcal{P}_3(\mathbf{R})$ and take $3x^2, 2x, 1$ as a basis of $\mathcal{P}_2(\mathbf{R})$. Then the matrix of D with respect to these bases is the matrix above.

5 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row k , column k , equal 1 if $1 \leq k \leq \dim \text{range } T$.

SOLUTION Let u_1, \dots, u_m be a basis of $\text{null } T$. Extend u_1, \dots, u_m to a basis $u_1, \dots, u_m, v_1, \dots, v_n$ of V . Then Tv_1, \dots, Tv_n is a basis of $\text{range } T$, as was proved in the proof of 3.21. Thus $n = \dim \text{range } T$.

Because Tv_1, \dots, Tv_n is a basis of the subspace $\text{range } T$, this list is linearly independent in W . Extend the linearly independent list Tv_1, \dots, Tv_n to a basis $Tv_1, \dots, Tv_n, w_1, \dots, w_p$ of W .

With respect to the basis $v_1, \dots, v_n, u_1, \dots, u_m$ of V (note that the v 's now come before the u 's) and the basis $Tv_1, \dots, Tv_n, w_1, \dots, w_p$ of W , the matrix of T has the desired form.

[Here some of m , n , or p might be 0.]

6 Suppose v_1, \dots, v_m is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis w_1, \dots, w_n of W such that all entries in the first column of $\mathcal{M}(T)$ [with respect to the bases v_1, \dots, v_m and w_1, \dots, w_n] are 0 except for possibly a 1 in the first row, first column.

In this exercise, unlike Exercise 5, you are given the basis of V instead of being able to choose a basis of V .

SOLUTION If $Tv_1 = 0$, then the first column of $\mathcal{M}(T)$ will consist of all 0's for every choice of basis of W (using, of course, v_1, \dots, v_m as the basis of V).

Thus suppose $Tv_1 \neq 0$. Let $w_1 = Tv_1$. Extend the list w_1 to a basis w_1, \dots, w_n of W . Then with respect to the bases v_1, \dots, v_m and w_1, \dots, w_n , the first column of $\mathcal{M}(T)$ consists of all 0's except for a 1 in the first row, first column.

7 Suppose w_1, \dots, w_n is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis v_1, \dots, v_m of V such that all entries in the first row of $\mathcal{M}(T)$ [with respect to the bases v_1, \dots, v_m and w_1, \dots, w_n] are 0 except for possibly a 1 in the first row, first column.

In this exercise, unlike Exercise 5, you are given the basis of W instead of being able to choose a basis of W .

SOLUTION If $\text{range } T \subseteq \text{span}(w_2, \dots, w_n)$, then the first row will consist of all 0's for every choice of basis of V (using, of course, w_1, \dots, w_n as the basis of W).

Thus suppose $\text{range } T \not\subseteq \text{span}(w_2, \dots, w_n)$. Let $u_1 \in V$ be such that $Tu_1 \notin \text{span}(w_2, \dots, w_n)$. Because w_1, \dots, w_n is a basis of W , we can write

$$Tu_1 = c_1w_1 + \cdots + c_nw_n$$

for some $c_1, \dots, c_n \in \mathbf{F}$. Because $Tu_1 \notin \text{span}(w_2, \dots, w_n)$, we know that $c_1 \neq 0$. Thus replacing u_1 with $\frac{1}{c_1}u_1$, we can assume that $c_1 = 1$. In other words,

$$Tu_1 = w_1 + c_2w_2 + \cdots + c_nw_n.$$

Extend u_1 to a basis u_1, \dots, u_m of V . For each $k \in \{2, \dots, m\}$, we can write

$$Tu_k = a_{1,k}w_1 + \cdots + a_{n,k}w_n.$$

Thus

$$T(u_k - a_{1,k}u_1) = (a_{2,k} - a_{1,k}c_2)w_2 + \cdots + (a_{n,k} - a_{1,k}c_n)w_n.$$

Thus with respect to the basis

$$u_1, u_2 - a_{1,2}u_1, u_3 - a_{1,3}u_1, \dots, u_m - a_{1,m}u_1$$

of V and w_1, \dots, w_n of W , we see that the first row of $\mathcal{M}(T)$ consists of all 0's except for a 1 in row 1, column 1.

8 Suppose A is an m -by- n matrix and B is an n -by- p matrix. Prove that

$$(AB)_{j,\cdot} = A_{j,\cdot} B$$

for each $1 \leq j \leq m$. In other words, show that row j of AB equals (row j of A) times B .

This exercise gives the row version of 3.48.

SOLUTION Note that both $(AB)_{j,\cdot}$ and $A_{j,\cdot} B$ are 1-by- p matrices.

By the definition of matrix multiplication, the entry in column k of $(AB)_{j,\cdot}$ is

$$\sum_{r=1}^n A_{j,r} B_{r,k}.$$

Again by the definition of matrix multiplication, the entry in column k of $A_{j,\cdot} B$ is also equal to

$$\sum_{r=1}^n A_{j,r} B_{r,k}.$$

Thus $(AB)_{j,\cdot} = A_{j,\cdot} B$.

9 Suppose $a = (a_1 \ \cdots \ a_n)$ is a 1-by- n matrix and B is an n -by- p matrix. Prove that

$$aB = a_1B_{1,\cdot} + \cdots + a_nB_{n,\cdot}.$$

In other words, show that aB is a linear combination of the rows of B , with the scalars that multiply the rows coming from a .

This exercise gives the row version of 3.50.

SOLUTION Note that both aB and $a_1B_{1,\cdot} + \cdots + a_nB_{n,\cdot}$ are 1-by- p matrices.

By the definition of matrix multiplication, the entry in column k of aB is

$$\sum_{r=1}^n a_r B_{r,k}.$$

For $r = 1, \dots, n$, the entry in column k of $a_r B_{r,\cdot}$ is $a_r B_{r,k}$. Thus the entry in column k of $a_1B_{1,\cdot} + \cdots + a_nB_{n,\cdot}$ is

$$\sum_{r=1}^n a_r B_{r,k}.$$

Thus $aB = a_1B_{1,\cdot} + \cdots + a_nB_{n,\cdot}$.

10 Give an example of 2-by-2 matrices A and B such that $AB \neq BA$.

SOLUTION Almost any randomly chosen 2-by-2 matrices will work.

11 Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A , B , C , D , E , and F are matrices whose sizes are such that $A(B + C)$ and $(D + E)F$ make sense. Explain why $AB + AC$ and $DF + EF$ both make sense and prove that

$$A(B + C) = AB + AC \quad \text{and} \quad (D + E)F = DF + EF.$$

SOLUTION Because $A(B + C)$ makes sense, B and C have the same size. Furthermore, the number of columns of A (let's call this number n) equals the number of rows of B and C . All this means that $AB + AC$ makes sense.

To prove that $A(B + C) = AB + AC$, just use the definition of matrix addition, the definition of matrix multiplication, and the usual distributive property for elements of \mathbf{F} . Specifically, let $A_{j,k}$, $b_{j,k}$, and $c_{j,k}$ denote the entries in row j , column k of A , B , and C , respectively. The entry in row j , column k of $B + C$ is $b_{j,k} + c_{j,k}$. Thus the entry in row j , column k of $A(B + C)$ is

$$\sum_{r=1}^n A_{j,r}(b_{r,k} + c_{r,k}),$$

which equals

$$\sum_{r=1}^n A_{j,r}b_{r,k} + \sum_{r=1}^n A_{j,r}c_{r,k},$$

which equals the entry in row j , column k of $AB + AC$, as desired.

12 Prove that matrix multiplication is associative. In other words, suppose A , B , and C are matrices whose sizes are such that $(AB)C$ makes sense. Explain why $A(BC)$ makes sense and prove that

$$(AB)C = A(BC).$$

Try to find a clean proof that illustrates the following quote from Emil Artin: "It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."

SOLUTION This exercise can be done by a brute force calculation, in the style of the solution to Exercise 11. Here is a solution that uses only the associativity of the product of linear maps (which is easy to verify because composition of functions is clearly associative) and the nice property that the matrix of the product of two linear maps equals the product of the matrices of the two linear maps (see 3.43).

Suppose A is an m -by- n matrix, B is an n -by- p matrix, and C is a p -by- q matrix; the sizes much match up like this in order for $(AB)C$ to make sense. Let $R \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$, $S \in \mathcal{L}(\mathbf{F}^p, \mathbf{F}^n)$, $T \in \mathcal{L}(\mathbf{F}^q, \mathbf{F}^p)$ be such that, with respect to the standard bases, $\mathcal{M}(R) = A$, $\mathcal{M}(S) = B$, $\mathcal{M}(T) = C$; 3.71 ensures that such linear maps exist. Now

$$\begin{aligned} (AB)C &= (\mathcal{M}(R)\mathcal{M}(S))\mathcal{M}(T) \\ &= \mathcal{M}(RS)\mathcal{M}(T) \\ &= \mathcal{M}((RS)T) \\ &= \mathcal{M}(R(ST)) \\ &= \mathcal{M}(R)\mathcal{M}(ST) \\ &= \mathcal{M}(R)(\mathcal{M}(S)\mathcal{M}(T)) \\ &= A(BC). \end{aligned}$$

13 Suppose A is an n -by- n matrix and $1 \leq j, k \leq n$. Show that the entry in row j , column k , of A^3 (which is defined to mean AAA) is

$$\sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}.$$

SOLUTION By the definition of matrix multiplication, we have

$$\begin{aligned}(A^3)_{j,k} &= (AA^2)_{j,k} \\&= \sum_{p=1}^n A_{j,p} (A^2)_{p,k} \\&= \sum_{p=1}^n A_{j,p} (AA)_{p,k} \\&= \sum_{p=1}^n A_{j,p} \sum_{r=1}^n A_{p,r} A_{r,k} \\&= \sum_{p=1}^n \sum_{r=1}^n A_{j,p} A_{p,r} A_{r,k}.\end{aligned}$$

14 Suppose m and n are positive integers. Prove that the function $A \mapsto A^t$ is a linear map from $\mathbf{F}^{m,n}$ to $\mathbf{F}^{n,m}$.

SOLUTION The linearity of the map that takes A to A^t follows from the definitions.

15 Prove that if A is an m -by- n matrix and C is an n -by- p matrix, then

$$(AC)^t = C^t A^t.$$

This exercise shows that the transpose of the product of two matrices is the product of the transposes in the opposite order.

SOLUTION Suppose $1 \leq k \leq p$ and $1 \leq j \leq m$. Then

$$\begin{aligned} ((AC)^t)_{k,j} &= (AC)_{j,k} \\ &= \sum_{r=1}^n A_{j,r} C_{r,k} \\ &= \sum_{r=1}^n (C^t)_{k,r} (A^t)_{r,j} \\ &= (C^t A^t)_{k,j}. \end{aligned}$$

Thus $(AC)^t = C^t A^t$, as desired.

16 Suppose A is an m -by- n matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, \dots, c_m) \in \mathbf{F}^m$ and $(d_1, \dots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$.

SOLUTION First suppose there exist $(c_1, \dots, c_m) \in \mathbf{F}^m$ and $(d_1, \dots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$. Let

$$d = \begin{pmatrix} d_1 & d_2 & \dots & d_n \end{pmatrix} \in \mathbf{F}^{1,n}.$$

Then row j of A equals $c_j d$ for each $j = 1, \dots, m$. Thus each row of A is in $\text{span}(d)$. Thus the row rank of A is less than or equal to 1. However, the rank of A does not equal 0 (because $A \neq 0$), and hence the rank of A is 1.

To prove the other direction, suppose the rank of A is 1. Let

$$d = \begin{pmatrix} d_1 & d_2 & \dots & d_n \end{pmatrix} \in \mathbf{F}^{1,n}$$

be a row of A that is not identically 0 (such a row must exist because $A \neq 0$). Because A has row rank 1, for each $j = 1, \dots, m$ there exists $c_j \in \mathbf{F}$ such that row j of A equals $c_j d$. Hence $A_{j,k} = c_j d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$, as desired.

17 Suppose $T \in \mathcal{L}(V)$, and u_1, \dots, u_n and v_1, \dots, v_n are bases of V . Prove that the following are equivalent.

- (a) T is injective.
- (b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$.
- (c) The columns of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$.
- (d) The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$.
- (e) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{1,n}$.

Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$.

SOLUTION First suppose (a) holds, so T is injective. Define $S: V \rightarrow \mathbf{F}^{n,1}$ by

$$Sv = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix},$$

where $a_1, \dots, a_n \in \mathbf{F}$ are such that $v = a_1v_1 + \dots + a_nv_n$. Then S is an injective linear map from V to $\mathbf{F}^{n,1}$. Thus ST is also an injective linear map from V to $\mathbf{F}^{n,1}$. Hence STu_1, \dots, STu_n is a linearly independent list in $\mathbf{F}^{n,1}$ (by Exercise 9 in Section 3B). Because STu_1, \dots, STu_n is the list of columns of $\mathcal{M}(T)$, we conclude that the columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$, which shows that (a) implies (b).

Now suppose (b) holds, so the columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$. Then, because we have a linearly independent list of length n in a vector space of dimension n , the list of columns of $\mathcal{M}(T)$ spans $\mathbf{F}^{n,1}$ (by 2.38). Thus (b) implies (c).

Now suppose (c) holds, so the columns of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$. Thus the column rank of $\mathcal{M}(T)$ equals n , which implies that the row rank of $\mathcal{M}(T)$ equals n (by 3.57). Hence the rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$. Thus (c) implies (d).

Now suppose (d) holds, so the rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$. Then, because we have a spanning list of length n in a vector space of dimension n , the list of rows of $\mathcal{M}(T)$ is linearly independent in $\mathbf{F}^{1,n}$ (by 2.42). Thus (d) implies (e).

Now suppose (e) holds, so the rows of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{1,n}$. Then, because we have a linearly independent list of length n in a vector space of dimension n , this list also spans $\mathbf{F}^{1,n}$ (by 2.38). Thus the row rank of $\mathcal{M}(T)$ equals n , which implies that the column rank of $\mathcal{M}(T)$ equals n (by 3.57). Hence the columns of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$. Thus the columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$ (by 2.42).

To prove that T is injective, suppose $u \in V$ and $Tu = 0$. There exist $b_1, \dots, b_n \in \mathbf{F}$ such that

$$u = b_1u_1 + \dots + b_nu_n.$$

Thus

$$0 = Tu = b_1Tu_1 + \dots + b_nTu_n.$$

Applying the linear map S defined in the first paragraph of this solution to both sides of the equation above, we have

$$0 = STu = b_1STu_1 + \dots + b_nSTu_n.$$

However, STu_1, \dots, STu_n are the columns of $\mathcal{M}(T)$, which are linearly independent by the paragraph above. Thus the equation above implies that $b_1 = \dots = b_n = 0$. This implies that $u = 0$. Hence T is injective, completing the proof that (e) implies (a).

1 Explain why each linear functional is surjective or is the zero map.

SOLUTION A linear functional is a linear map into \mathbf{F} . The range of a linear map is a subspace of the target space. The only subspaces of \mathbf{F} are \mathbf{F} and $\{0\}$. Thus each linear functional is surjective (if the range equals \mathbf{F}) or is the zero map (if the range equals $\{0\}$).

2 Give three distinct examples of linear functionals on $\mathbf{R}^{[0,1]}$.

SOLUTION For $t \in [0, 1]$, define $\varphi_t: \mathbf{R}^{[0,1]} \rightarrow \mathbf{R}$ by

$$\varphi_t(f) = f(t)$$

for each $f \in \mathbf{R}^{[0,1]}$.

For each $t \in [0, 1]$, φ_t is a linear functional on $\mathbf{R}^{[0,1]}$. Thus three distinct linear functionals on $\mathbf{R}^{[0,1]}$ are φ_0 , $\varphi_{1/2}$, and φ_1 .

3 Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

SOLUTION Because $v \neq 0$, we can extend v to a basis v, v_2, \dots, v_n of V (by 2.32). Now the linear map lemma (3.4) implies that there exists $\varphi \in V'$ such that $\varphi(v) = 1$ [and $\varphi(v_k)$ equals whatever we want for each $k = 2, \dots, n$].

4 Suppose V is finite-dimensional and U is a subspace of V such that $U \neq V$. Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

SOLUTION Let u_1, \dots, u_m be a basis of U . Extend this list to a basis

$$u_1, \dots, u_m, v_1, \dots, v_n$$

of V (using 2.32). Now the linear map lemma (3.4) implies that there exists $\varphi \in V'$ such that

$$\varphi(u_k) = 0 \text{ for } k = 1, \dots, m$$

and

$$\varphi(v_k) = 1 \text{ for } k = 1, \dots, n.$$

Thus $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$, as desired.

5 Suppose $T \in \mathcal{L}(V, W)$ and w_1, \dots, w_m is a basis of $\text{range } T$. Hence for each $v \in V$, there exist unique numbers $\varphi_1(v), \dots, \varphi_m(v)$ such that

$$Tv = \varphi_1(v)w_1 + \dots + \varphi_m(v)w_m,$$

thus defining functions $\varphi_1, \dots, \varphi_m$ from V to \mathbf{F} . Show that each of the functions $\varphi_1, \dots, \varphi_m$ is a linear functional on V .

SOLUTION Suppose $u, v \in V$. Then

$$\begin{aligned} \varphi_1(u+v)w_1 + \dots + \varphi_m(u+v)w_m &= T(u+v) \\ &= Tu + Tv \\ &= (\varphi_1(u)w_1 + \dots + \varphi_m(u)w_m) + (\varphi_1(v)w_1 + \dots + \varphi_m(v)w_m) \\ &= (\varphi_1(u) + \varphi_1(v))w_1 + \dots + (\varphi_m(u) + \varphi_m(v))w_m. \end{aligned}$$

Because w_1, \dots, w_m is linearly independent, the equation above shows that

$$\varphi_1(u+v) = \varphi_1(u) + \varphi_1(v), \dots, \varphi_m(u+v) = \varphi_m(u) + \varphi_m(v).$$

Similarly, suppose now that $\lambda \in \mathbf{F}$ and $v \in V$. Then

$$\begin{aligned} \varphi_1(\lambda v)w_1 + \dots + \varphi_m(\lambda v)w_m &= T(\lambda v) \\ &= \lambda Tv \\ &= \lambda(\varphi_1(v)w_1 + \dots + \varphi_m(v)w_m) \\ &= \lambda\varphi_1(v)w_1 + \dots + \lambda\varphi_m(v)w_m. \end{aligned}$$

Because w_1, \dots, w_m is linearly independent, the equation above shows that

$$\varphi_1(\lambda v) = \lambda\varphi_1(v), \dots, \varphi_m(\lambda v) = \lambda\varphi_m(v),$$

completing the proof that each of the functions $\varphi_1, \dots, \varphi_m$ is a linear functional on V .

6 Suppose $\varphi, \beta \in V'$. Prove that $\text{null } \varphi \subseteq \text{null } \beta$ if and only if there exists $c \in \mathbf{F}$ such that $\beta = c\varphi$.

SOLUTION If $\beta = c\varphi$ for some $c \in \mathbf{F}$, then clearly $\text{null } \varphi \subseteq \text{null } \beta$.

The implication in the other direction follows from Exercise 25 in Section 3B (with $W = \mathbf{F}$) and from Exercise 7 in Section 3A.

7 Suppose that V_1, \dots, V_m are vector spaces. Prove that $(V_1 \times \cdots \times V_m)'$ and $V_1' \times \cdots \times V_m'$ are isomorphic vector spaces.

SOLUTION For $k = 1, \dots, m$ and $\varphi \in (V_1 \times \cdots \times V_m)'$, define $\varphi_k \in V_k'$ by

$$\varphi_k(v) = \varphi(0, \dots, 0, v, 0, \dots, 0),$$

for each $v \in V_k$, where the v on the right side above appears in the k^{th} slot.

Now define $\Gamma: (V_1 \times \cdots \times V_m)' \rightarrow V_1' \times \cdots \times V_m'$ by

$$\Gamma\varphi = (\varphi_1, \dots, \varphi_m).$$

It is easy to verify that Γ is a linear map and that Γ is injective and surjective. Thus $(V_1 \times \cdots \times V_m)'$ and $V_1' \times \cdots \times V_m'$ are isomorphic vector spaces.

8 Suppose v_1, \dots, v_n is a basis of V and $\varphi_1, \dots, \varphi_n$ is the dual basis of V' . Define $\Gamma: V \rightarrow \mathbf{F}^n$ and $\Lambda: \mathbf{F}^n \rightarrow V$ by

$$\Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v)) \quad \text{and} \quad \Lambda(a_1, \dots, a_n) = a_1 v_1 + \dots + a_n v_n.$$

Explain why Γ and Λ are inverses of each other.

SOLUTION First suppose $a_1, \dots, a_n \in \mathbf{F}^n$. Then

$$\begin{aligned} \Gamma(\Lambda(a_1, \dots, a_n)) &= \Gamma(a_1 v_1 + \dots + a_n v_n) \\ &= (\varphi_1(a_1 v_1 + \dots + a_n v_n), \dots, \varphi_n(a_1 v_1 + \dots + a_n v_n)) \\ &= (a_1, \dots, a_n). \end{aligned}$$

Also,

$$\begin{aligned} \Lambda(\Gamma(v)) &= \Lambda(\varphi_1(v), \dots, \varphi_n(v)) \\ &= \varphi_1(v) v_1 + \dots + \varphi_n(v) v_n \\ &= v, \end{aligned}$$

where the last line follows from 3.114.

9 Suppose m is a positive integer. Show that the dual basis of the basis $1, x, \dots, x^m$ of $\mathcal{P}_m(\mathbf{R})$ is $\varphi_0, \varphi_1, \dots, \varphi_m$, where

$$\varphi_k(p) = \frac{p^{(k)}(0)}{k!}.$$

Here $p^{(k)}$ denotes the k^{th} derivative of p , with the understanding that the 0^{th} derivative of p is p .

SOLUTION Note that

$$\varphi_k(x^j) = \frac{(x^j)^{(k)}(0)}{k!} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Thus $\varphi_0, \varphi_1, \dots, \varphi_m$ is indeed the dual basis of $1, x, \dots, x^m$.

10 Suppose m is a positive integer.

- (a) Show that $1, x - 5, \dots, (x - 5)^m$ is a basis of $\mathcal{P}_m(\mathbf{R})$.
- (b) What is the dual basis of the basis in (a)?

SOLUTION

- (a) Define $\varphi_0, \varphi_1, \dots, \varphi_m \in (\mathcal{P}_m(\mathbf{R}))'$ by

$$\varphi_j(p) = \frac{p^{(j)}(5)}{j!}.$$

Suppose $a_0, a_1, \dots, a_m \in \mathbf{R}$ and

$$a_0 + a_1(x - 5) + \dots + a_m(x - 5)^m = 0.$$

Then for each $j = 0, 1, \dots, m$, we have

$$a_j = \varphi_j(a_0 + a_1(x - 5) + \dots + a_m(x - 5)^m) = \varphi_j(0) = 0.$$

Thus $a_0 = a_1 = \dots = a_m = 0$. Hence $1, x - 5, \dots, (x - 5)^m$ is a linearly independent list in $\mathcal{P}_m(\mathbf{R})$ of length $m + 1$, which equals the dimension of $\mathcal{P}_m(\mathbf{R})$. Thus $1, x - 5, \dots, (x - 5)^m$ is a basis of $\mathcal{P}_m(\mathbf{R})$ (by 2.38).

- (b) Let $\varphi_0, \varphi_1, \dots, \varphi_m \in (\mathcal{P}_m(\mathbf{R}))'$ be defined as in (a). Then

$$\varphi_j((x - 5)^k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Thus $\varphi_0, \varphi_1, \dots, \varphi_m$ is the dual basis of $1, x - 5, \dots, (x - 5)^m$.

11 Suppose v_1, \dots, v_n is a basis of V and $\varphi_1, \dots, \varphi_n$ is the corresponding dual basis of V' . Suppose $\psi \in V'$. Prove that

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n.$$

SOLUTION We have

$$(\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n)(v_k) = \psi(v_k)$$

for each $k = 1, \dots, n$. Because the linear functionals $\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$ and ψ agree on a basis, they are equal by the uniqueness part of the linear map lemma (3.4).

12 Suppose $S, T \in \mathcal{L}(V, W)$.

- (a) Prove that $(S + T)' = S' + T'$.
- (b) Prove that $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbf{F}$.

This exercise asks you to verify (a) and (b) in 3.120.

SOLUTION

- (a) We have

$$\begin{aligned}(S + T)'(\varphi) &= \varphi \circ (S + T) \\ &= \varphi \circ S + \varphi \circ T \\ &= S'(\varphi) + T'(\varphi).\end{aligned}$$

Thus $(S + T)' = S' + T'$, as desired.

- (b) Suppose $\lambda \in \mathbf{F}$. Then

$$\begin{aligned}(\lambda T)'(\varphi) &= \varphi \circ (\lambda T) \\ &= \lambda(\varphi \circ T) \\ &= \lambda T'(\varphi).\end{aligned}$$

Thus $(\lambda T)' = \lambda T'$, as desired.

13 Show that the dual map of the identity operator on V is the identity operator on V' .

SOLUTION Let I denote the identity operator on V . For $\varphi \in V'$ we have

$$I'(\varphi) = \varphi \circ I = \varphi.$$

Thus I' is the identity operator on V' .

14 Define $T: \mathbf{R}^3 \rightarrow \mathbf{R}^2$ by

$$T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z).$$

Suppose φ_1, φ_2 denotes the dual basis of the standard basis of \mathbf{R}^2 and ψ_1, ψ_2, ψ_3 denotes the dual basis of the standard basis of \mathbf{R}^3 .

- (a) Describe the linear functionals $T'(\varphi_1)$ and $T'(\varphi_2)$.
- (b) Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3 .

SOLUTION

- (a) Note that $\varphi_1(a, b) = a$ and $\varphi_2(a, b) = b$ for all $(a, b) \in \mathbf{R}^2$. Recall that $T'(\varphi_1) = \varphi_1 \circ T$ and $T'(\varphi_2) = \varphi_2 \circ T$. Thus

$$(T'(\varphi_1))(x, y, z) = 4x + 5y + 6z$$

and

$$(T'(\varphi_2))(x, y, z) = 7x + 8y + 9z.$$

- (b) We have $\psi_1(x, y, z) = x$, $\psi_2(x, y, z) = y$, and $\psi_3(x, y, z) = z$. Using (a), we thus see that

$$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$$

and

$$T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3.$$

15 Define $T: \mathcal{P}(\mathbf{R}) \rightarrow \mathcal{P}(\mathbf{R})$ by

$$(Tp)(x) = x^2p(x) + p''(x)$$

for each $x \in \mathbf{R}$.

- (a) Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = p'(4)$. Describe the linear functional $T'(\varphi)$ on $\mathcal{P}(\mathbf{R})$.
 (b) Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = \int_0^1 p$. Evaluate $(T'(\varphi))(x^3)$.

SOLUTION

- (a) Suppose $p \in \mathcal{P}(\mathbf{R})$. Then

$$\begin{aligned} (T'(\varphi))(p) &= (\varphi \circ T)(p) \\ &= \varphi(Tp) \\ &= \varphi(x^2p + p'') \\ &= (x^2p + p'')'(4) \\ &= (2xp + x^2p' + p''')(4) \\ &= 8p(4) + 16p'(4) + p'''(4). \end{aligned}$$

- (b) We have

$$\begin{aligned} (T'(\varphi))(x^3) &= (\varphi \circ T)(x^3) \\ &= \varphi(Tx^3) \\ &= \varphi(x^5 + 6x) \\ &= \int_0^1 (x^5 + 6x) dx \\ &= \frac{19}{6}. \end{aligned}$$

16 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that

$$T' = 0 \iff T = 0.$$

SOLUTION First suppose $T = 0$. If $\varphi \in W'$, then $T'(\varphi) = \varphi \circ T = 0$, and thus $T' = 0$.

To prove the other direction, now suppose $T' = 0$. Thus

$$0 = T'(\varphi) = \varphi \circ T$$

for every $\varphi \in W'$. Thus

$$0 = \varphi(Tv)$$

for every $v \in V$ and every $\varphi \in W'$. Exercise 3 now implies that $Tv = 0$ for every $v \in V$. Thus $T = 0$, as desired.

17 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is invertible if and only if $T' \in \mathcal{L}(W', V')$ is invertible.

SOLUTION A linear map is invertible if and only if it is injective and surjective. Thus the desired result follows from 3.129 and 3.131.

18 Suppose V and W are finite-dimensional. Prove that the map that takes $T \in \mathcal{L}(V, W)$ to $T' \in \mathcal{L}(W', V')$ is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

SOLUTION Define $\Gamma: \mathcal{L}(V, W) \rightarrow \mathcal{L}(W', V')$ by $\Gamma(T) = T'$. By 3.120, Γ is a linear map.

Exercise 16 implies that Γ is injective. Note that

$$\begin{aligned}\dim \mathcal{L}(V, W) &= (\dim V)(\dim W) \\ &= (\dim V')(\dim W') \\ &= (\dim W')(\dim V') \\ &= \dim \mathcal{L}(W', V'),\end{aligned}$$

where the first and last equalities above come from 3.72 and the second equality above comes from 3.111.

The fundamental theorem of linear maps (3.21) and the equation above now imply that $\dim \text{range } \Gamma = \dim(\mathcal{L}(W', V'))$. Thus Γ is surjective and hence is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$, as desired.