Exercises 1B Solutions: Definition of Vector Space

Linear Algebra Done Right, 4th ed.

Definition 1.19 (Addition, Scalar Multiplication).
- An **addition** on V assigns $u + v \in V$ to each pair $u, v \in V$.
- A **scalar multiplication** on V assigns $\lambda v \in V$ to each $\lambda \in \mathbb{F}$, $v \in V$.

Definition 1.20 (Vector Space). A **vector space** over $\mathbb{F}$ is a set V with addition and scalar multiplication satisfying:

(C) Commutativity:	$u + v = v + u$
(A1) Associativity (add):	$(u + v) + w = u + (v + w)$
(A2) Associativity (scalar):	$(ab)v = a(bv)$
(I) Additive identity:	$\exists\, 0 \in V$ such that $v + 0 = v$
(Inv) Additive inverse:	$\forall v \in V$, $\exists w \in V$ such that $v + w = 0$
(M) Multiplicative identity:	$1v = v$
(D1) Distributivity:	$a(u + v) = au + av$
(D2) Distributivity:	$(a + b)v = av + bv$

Definition 1.21. Elements of a vector space are called **vectors** or **points**.

Definition 1.22. A vector space over $\mathbb{R}$ is a **real vector space**; over $\mathbb{C}$ is a **complex vector space**.

Notation 1.24. $\mathbb{F}^S$ denotes all functions $f : S \to \mathbb{F}$, with pointwise operations: $(f + g)(x) = f(x) + g(x)$ and $(\lambda f)(x) = \lambda f(x)$.

Notation 1.28. $-v$ denotes the additive inverse of v; $\quad w - v := w + (-v)$.

Key Results:
- **1.26** The additive identity is unique.
- **1.27** Every element has a unique additive inverse (denoted $-v$).
- **1.30** $0v = 0$ for all $v \in V$ (scalar zero $\to$ vector zero).
- **1.31** $a0 = 0$ for all $a \in \mathbb{F}$ (any scalar $\times$ zero vector $= 0$).
- **1.32** $(-1)v = -v$ (negation = scalar multiplication by -1).

Exercise 1. Prove that $-(-v) = v$ for every $v \in V$.

Reading the Question

Step 1: Notice the objects. The statement involves:
- $v \in V$ (a vector)
- $-v$ (additive inverse)
- equality of vectors

No coordinates, no scalars, no calculations—only **addition**, **zero**, and **inverse**.

Step 2: Recognize defined symbols. Is "$-v$" something you can manipulate algebraically? **No.** In LADR, $-v$ is *defined* as the unique vector satisfying $v + (-v) = 0$. When you see inverse, identity, or zero, your brain should trigger: **unpack the definition**.

Step 3: Ask "what does it mean?" Before proving, translate:

- $-v$ = the unique vector that adds to v to give zero
- $-(-v)$ = the unique vector that adds to $(-v)$ to give zero

So the problem becomes: show that v is the thing that cancels $(-v)$.

Step 4: Spot the word "prove." The problem says *prove*, not compute, find, or simplify. This means: no formulas, no coordinates—use **axioms + logic**.

Step 5: Recognize the uniqueness pattern. Whenever you see inverse, identity, zero, or cancellation, think: "There is a **uniqueness theorem** I can exploit."

Intuition: What Does "Inverse of an Inverse" Mean?

Before diving into the proof, pause and *think*:

Q1: What is $-v$ conceptually? It's the vector that "undoes" v—adding them returns you to zero, the neutral element. Think of $-v$ as the **reversal** of v.

Q2: What happens when you reverse a reversal? If you turn around, then turn around again, you face your original direction. If you undo an undo, you're back where you started.

Q3: Why should $-(-v) = v$? Because v is the thing that undoes $-v$. The definition of $-(-v)$ is "the unique vector that cancels $-v$"—and v does exactly that!

Pattern recognition: This is an example of an **involution**—an operation that is its own inverse. Negation applied twice returns to the original. You'll see this pattern throughout mathematics:

- Numbers: $-(-5) = 5$
- Logic: $\neg(\neg P) = P$
- Functions: $(f^{-1})^{-1} = f$
- Geometry: reflecting twice across a line returns to original position

The proof below makes this intuition rigorous using only the axioms.

Why other proof styles don't work here:

- Contradiction? Nothing to contradict
- Induction? No integer structure
- Computation? No coordinates given
- Examples? Must prove for *all* v

The proof template:

1. State the definition of $-(-v)$
2. Show that v satisfies that definition
3. Invoke uniqueness (Lemma 1.25)

Axioms used: commutativity, existence of additive inverse, uniqueness of additive inverse.

Solution: We use the uniqueness of the additive inverse.

Proof Strategy: We cannot "compute" $-(-v)$ directly—there's no formula. Instead, we use the **defining property** of $-(-v)$: it is the unique vector w such that $(-v)+w=0$. If we can show that v has this property, then v must equal $-(-v)$.

Step 1: State what we need to show.
By definition, $-(-v)$ is characterized by:

$$(-v)+(-(-v))=0.$$

Our goal: verify that v satisfies this same equation, i.e., that $(-v)+v=0$.

Step 2: Verify that v satisfies the defining property.
We compute:

$$(-v)+v=v+(-v)\quad\text{(commutativity of addition)}$$

$$=0\quad\text{(definition of additive inverse: }-v\text{ is the inverse of }v\text{)}$$

So v satisfies $(-v)+v=0$—exactly the defining property of $-(-v)$.

Step 3: Apply uniqueness.
Lemma 1.25 (uniqueness of additive inverse) states: for any vector u, there is exactly **one** vector w satisfying $u+w=0$.
Applying this with $u=-v$:

- By definition, $-(-v)$ is the unique solution to $(-v)+w=0$.
- We just showed v satisfies $(-v)+v=0$.
- By uniqueness, $v=-(-v)$.

Conclusion:

$$\boxed{-(-v)=v}$$

Reflection: Notice we never "canceled" anything or used subtraction rules. We only used: (1) the definition of additive inverse, (2) commutativity, and (3) uniqueness. This is a template for many proofs in abstract algebra. □

Exercise 2. Suppose $a\in\mathbb{F}$, $v\in V$, and $av=0$. Prove that $a=0$ or $v=0$.

Reading the Question
Step 0: Spot the "or" in the conclusion. The statement asks you to prove:

$$a=0\quad\text{or}\quad v=0$$

A standard logical equivalence: $(P \text{ or } Q)\Leftrightarrow(\neg P\Rightarrow Q)$. So the problem is *really* asking: **if** $a\neq 0$**, then** $v=0$. This is why the solution assumes $a\neq 0$ and derives $v=0$.
Step 1: Notice "$a\in\mathbb{F}$" (field). What is the key property of a field? Every nonzero element has a **multiplicative inverse**. So assuming $a\neq 0$ gives you a^{-1}—the engine of the proof.
Step 2: The equation $av=0$ is a vector equation. You want to "cancel" the scalar a. Cancellation works when inverses exist—exactly why we assumed $a\neq 0$.

Step 3: No coordinates given. This means: use **axioms only**, not computation.

Clue-to-Strategy Map

Clue in problem	**What it hints**
"$a \in \mathbb{F}$"	Use multiplicative inverse
"$av = 0$"	Cancellation argument
"or" in conclusion	Use contrapositive
No coordinates	Axioms only

Intuition: Why Can't Scaling Produce Zero from Nothing?
Before the proof, let's build intuition through questions:
Q1: What does scalar multiplication do geometrically? Multiplying a vector by a **scales** it—stretching if $|a| > 1$, shrinking if $|a| < 1$, and flipping direction if $a < 0$.
Q2: Can scaling ever "annihilate" a vector? If $v \neq 0$, then v points somewhere in space. Scaling changes *how far* it points, but not *that* it points somewhere (unless the scalar is zero). You can't shrink a nonzero vector into nothing with a nonzero scale factor.
Q3: What if $av = 0$? Something collapsed to zero. Either:

- The scalar $a = 0$ (you scaled by nothing), or
- The vector $v = 0$ (there was nothing to scale).

There's no third option. This is the "no zero divisors" property.
Q4: Why does this require a field? In $\mathbb{Z}/6\mathbb{Z}$ (integers mod 6), we have $2 \cdot 3 = 0$ even though $2 \neq 0$ and $3 \neq 0$. Fields exclude such "zero divisors" by requiring every nonzero element to have an inverse. The proof exploits this: if $a \neq 0$, we can "undo" the scaling by multiplying by a^{-1}.
Analogy: Think of a as a "lens" and v as "light." If the output is darkness (0), either the lens blocked everything ($a = 0$) or there was no light to begin with ($v = 0$). A clear lens ($a \neq 0$) cannot create darkness from light.
Looking ahead: This is a load-bearing result for linear independence, bases, and solving linear equations.

Solution: We prove the contrapositive: if $a \neq 0$, then $v = 0$.

Proof Strategy: The statement "$a = 0$ or $v = 0$" is logically equivalent to "if $a \neq 0$, then $v = 0$." This reformulation tells us exactly what to assume ($a \neq 0$) and what to prove ($v = 0$). The key insight: if $a \neq 0$, then a has a multiplicative inverse a^{-1} in the field $\mathbb{F}$. We can "undo" the scalar multiplication by applying a^{-1}.

Step 1: Setup—use the field structure.
Suppose $a \neq 0$. Since $\mathbb{F}$ is a field, every nonzero element has a multiplicative inverse. Therefore, $a^{-1} \in \mathbb{F}$ exists and satisfies $a^{-1} \cdot a = 1$.

Step 2: Apply the inverse to both sides.

Starting from the given equation $av = 0$, multiply both sides on the left by a^{-1}:

$$a^{-1}(av) = a^{-1} \cdot 0_V.$$

Why is this valid? Scalar multiplication is a function from $\mathbb{F} \times V \to V$. We're applying the same scalar to both sides of a vector equation.

Step 3: Simplify the left side.
Using the *associativity of scalar multiplication* (axiom):

$$a^{-1}(av) = (a^{-1} \cdot a)v = 1 \cdot v.$$

Then using the *multiplicative identity axiom*:

$$1 \cdot v = v.$$

So the left side simplifies to v.

Step 4: Simplify the right side (sub-lemma).
We need: $a^{-1} \cdot 0_V = 0_V$ for any scalar a^{-1}.
Proof of sub-lemma: The zero vector satisfies $0_V + 0_V = 0_V$ (additive identity). Therefore:

$$\begin{aligned} a^{-1} \cdot 0_V &= a^{-1} \cdot (0_V + 0_V) && \text{(property of } 0_V\text{)} \\ &= a^{-1} \cdot 0_V + a^{-1} \cdot 0_V && \text{(distributivity)} \end{aligned}$$

Now we have $a^{-1} \cdot 0_V = a^{-1} \cdot 0_V + a^{-1} \cdot 0_V$. Adding $-(a^{-1} \cdot 0_V)$ to both sides:

$$0_V = a^{-1} \cdot 0_V.$$

Step 5: Combine results.
From Steps 3 and 4:

$$v = a^{-1}(av) = a^{-1} \cdot 0_V = 0_V.$$

Conclusion:
We have shown: if $a \neq 0$, then $v = 0$.
By contrapositive equivalence, this proves: $av = 0 \Rightarrow a = 0$ or $v = 0$.

$$\boxed{a = 0 \text{ or } v = 0}$$

Reflection: The proof has two engines: (1) the field structure gives us a^{-1}, and (2) the vector space axioms let us "cancel" and simplify. This result is foundational—it's why we can solve $av = b$ uniquely when $a \neq 0$. □

Exercise 3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Reading the Question

Step 1: Notice the structure. The statement involves:

- Two given vectors $v, w \in V$
- An unknown vector $x \in V$ to find
- A linear equation $v + 3x = w$
- Two claims: existence and uniqueness

Step 2: Recognize the pattern. This is a **linear equation in one vector variable**. In ordinary algebra, solving $a + 3x = b$ gives $x = \frac{1}{3}(b - a)$. The same algebraic manipulation works in vector spaces—but we must justify each step using axioms.

Step 3: The word "explain." This signals: show your reasoning clearly. For existence-uniqueness problems, you need two parts:

- **Existence:** Construct an x and verify it works
- **Uniqueness:** Show any two solutions must be equal

Step 4: Why does uniqueness require proof? In some algebraic structures, equations can have multiple solutions or none. Here, the field structure (specifically, $3 \neq 0$ has inverse $\frac{1}{3}$) guarantees exactly one solution.

Intuition: Solving Equations in Vector Spaces

Before the proof, build intuition through questions:

Q1: What does $v + 3x = w$ mean geometrically? Starting at v, we want to find a vector x such that adding $3x$ (three copies of x) lands us at w. We need $3x$ to "bridge the gap" from v to w.

Q2: What is that gap? The displacement from v to w is $w - v$. So we need $3x = w - v$, meaning x is one-third of that displacement.

Q3: Why is there exactly one solution?

- **Existence:** We can always compute $\frac{1}{3}(w - v)$ because $\frac{1}{3} \in \mathbb{F}$ exists (fields have multiplicative inverses for nonzero elements).
- **Uniqueness:** If two different x's worked, their difference would satisfy $3(x_1 - x_2) = 0$. By Exercise 2 (no zero divisors), this forces $x_1 - x_2 = 0$.

Q4: What makes this different from solving equations in $\mathbb{Z}$? Over integers, $3x = 6$ has solution $x = 2$, but $3x = 7$ has no integer solution. In a vector space over a field, we can always "divide" by nonzero scalars.

Pattern recognition: This is the prototype for solving $\alpha x = \beta$ in any algebraic structure with inverses. The existence of α^{-1} is the key.

Solution: We prove both existence and uniqueness.

Proof Strategy: The equation $v + 3x = w$ is linear in x. To solve it:

1. Isolate x algebraically to guess the answer
2. Verify the guess satisfies the equation (existence)
3. Show any two solutions must coincide (uniqueness)

Part 1: Existence—construct a solution.

Finding the candidate: Rearranging $v + 3x = w$ suggests:

$$3x = w - v = w + (-v), \quad \text{so} \quad x = \tfrac{1}{3}(w - v).$$

This is our candidate. Now we verify it actually works.
Verification: Substitute $x = \frac{1}{3}(w - v)$ into $v + 3x$:

$$\begin{aligned} v + 3x &= v + 3 \cdot \tfrac{1}{3}(w - v) = v + (w - v) \\ &= v + w + (-v) = w + (v + (-v)) \\ &= w + 0 = w. \quad \checkmark \end{aligned}$$

So $x = \frac{1}{3}(w - v)$ is indeed a solution. ✓

Part 2: Uniqueness—show the solution is the only one.
Setup: Suppose x_1 and x_2 both satisfy the equation:

$$v + 3x_1 = w \quad \text{and} \quad v + 3x_2 = w.$$

Step 1: Since both equal w, we have:

$$v + 3x_1 = v + 3x_2.$$

Step 2: Add $(-v)$ to both sides (using existence of additive inverse):

$$(-v) + (v + 3x_1) = (-v) + (v + 3x_2).$$

Step 3: Apply associativity:

$$((-v) + v) + 3x_1 = ((-v) + v) + 3x_2.$$

Step 4: Simplify using $(-v) + v = 0$ and $0 + 3x_i = 3x_i$:

$$3x_1 = 3x_2.$$

Step 5: Multiply both sides by $\frac{1}{3}$ (the multiplicative inverse of 3 in $\mathbb{F}$):

$$\tfrac{1}{3}(3x_1) = \tfrac{1}{3}(3x_2).$$

Step 6: Apply associativity of scalar multiplication:

$$(\tfrac{1}{3} \cdot 3)x_1 = (\tfrac{1}{3} \cdot 3)x_2 \implies 1 \cdot x_1 = 1 \cdot x_2 \implies x_1 = x_2.$$

Conclusion:
There exists a unique $x \in V$ such that $v + 3x = w$, namely:

$$\boxed{x = \tfrac{1}{3}(w - v)}$$

Reflection: This proof uses two key features of vector spaces over fields:
1. **Additive structure:** We can "move v to the other side" using $(-v)$.
2. **Scalar inverses:** We can "divide by 3" using $\frac{1}{3} \in \mathbb{F}$.

The same template solves any equation $v + ax = w$ when $a \neq 0$: the unique solution is $x = \frac{1}{a}(w - v)$. □

Exercise 4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Reading the Question

Step 1: Understand what's being asked. The problem says the empty set fails **exactly one** axiom. Your job:

- Identify which axiom fails
- Explain why all others are satisfied

Step 2: Recall the axioms. Definition 1.20 lists eight axioms. Each has a specific logical structure—some begin with "there exists," others begin with "for all."

Step 3: What does "empty" mean logically?

- If $V = \varnothing$, then V contains **no elements**
- Any statement "for all $v \in V$, ..." is **vacuously true** (there's nothing to check)
- Any statement "there exists $v \in V$ such that ..." is **false** (there's nothing to find)

Intuition: Why Does the Empty Set Fail?

Q1: What's the difference between "for all" and "there exists"?

- "For all $v \in V$, $P(v)$" is true when $V = \varnothing$ (no counterexamples exist)
- "There exists $v \in V$ such that $P(v)$" is false when $V = \varnothing$ (no witnesses exist)

Q2: Which axiom requires something to exist? The additive identity axiom:

$$\exists\, 0 \in V \text{ such that } \forall v \in V,\ v + 0 = v$$

The red $\exists$ is the problem—it demands that V **contain** an element.

Q3: Why not the additive inverse axiom? It says:

$$\forall\, v \in V,\ \exists w \in V \text{ such that } v + w = 0$$

The blue $\forall$ comes **first**. When $V = \varnothing$, there are no v's to check, so this is vacuously true.

Key Insight: The order of quantifiers matters. "$\exists \ldots \forall$" fails on empty sets; "$\forall \ldots \exists$" is vacuously true.

Claim. The empty set $\varnothing$ is not a vector space. Among the axioms in Definition 1.20, it fails **exactly one**: the additive identity axiom.

Proof.

Recall the **additive identity axiom** for a vector space V:

$$\exists\, 0 \in V \text{ such that } \forall v \in V,\ v + 0 = v. \tag{AI}$$

This axiom has two logically distinct components:

1. **Existence:** there exists an element $0 \in V$;
2. **Universal property:** for all $v \in V$, $v + 0 = v$.

Now suppose $V = \varnothing$.

- Since $\varnothing$ contains no elements, the existential statement $\exists\, 0 \in V$ is **false**.
- Therefore, axiom (AI) fails.

Hence, the empty set does **not** satisfy the additive identity axiom and is not a vector space.

Why all other axioms are satisfied.
All remaining vector space axioms have the logical form

$$\forall v, w \in V,\ P(v,w) \quad \text{or} \quad \forall v \in V,\ \exists w \in V \text{ such that } Q(v,w),$$

where the universal quantifier ranges over elements of V.
Since $V = \varnothing$, there are **no elements** v or w to check. Therefore, every such universally quantified statement is **vacuously true**.
Note: The additive inverse axiom begins with $\forall v \in V$, so it's vacuously true on $\varnothing$ (see hintbox discussion of quantifier order).

Conclusion.
The empty set satisfies all vector space axioms **except** the additive identity axiom. Consequently, the empty set is not a vector space.

$$\boxed{\varnothing \text{ is not a vector space because it lacks an additive identity.}}$$

Conceptual one-liner: A vector space must *contain* an additive identity; the empty set contains nothing. □

Exercise 5. Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

Reading the Question
Step 1: Clarify the two zeros. This problem uses the same symbol "0" for two different objects:

- Left side: 0 is the **scalar zero** in $\mathbb{F}$
- Right side: 0 is the **vector zero** 0_V in V

The condition $0v = 0$ says: scaling any vector by the scalar zero yields the zero vector.
Step 2: Understand "replaced." The problem asks: can we swap out one axiom for another equivalent condition? We need to show that under the other axioms, "$0v = 0$ for all v" implies "every v has an additive inverse."
Step 3: Identify the candidate inverse. What vector could serve as $-v$? The natural guess is $(-1)v$—scaling by -1. We must show $v + (-1)v = 0_V$.

Intuition: Why $(-1)v$ Should Be the Inverse
Q1: What does $(-1)v$ mean geometrically? Scaling by -1 flips the direction of v. Adding v and

its flip should return to the origin.
Q2: How does $0v = 0$ help? We need to show $v + (-1)v = 0$. Using distributivity:

$$v + (-1)v = 1 \cdot v + (-1)v = (1 + (-1))v = 0v$$

If we know $0v = 0_V$, we're done!
Q3: Why is distributivity the key link? Distributivity connects scalar addition $(1 + (-1) = 0)$ to vector structure. Without it, we couldn't "factor out" the v.

Proof template:

1. State the goal: show additive inverse exists for each v
2. Propose candidate: $w = (-1)v$
3. Verify: compute $v + (-1)v$ using distributivity
4. Apply assumed condition $0v = 0_V$
5. Conclude $(-1)v$ is the additive inverse

Solution: The additive inverse axiom follows from $0v = 0$ using distributivity.

Claim. If V satisfies all vector space axioms except the additive inverse axiom, but satisfies $0v = 0_V$ for all $v \in V$, then every vector has an additive inverse.

Proof Strategy: For each $v \in V$, we must exhibit a vector w such that $v + w = 0_V$. The candidate is $w = (-1)v$. We verify this works using distributivity to "factor out" v, then apply the assumed condition.

Step 1: Identify the candidate inverse.
Given any $v \in V$, consider $w = (-1)v$.
This is well-defined since $-1 \in \mathbb{F}$ and scalar multiplication is defined.

Step 2: Verify $v + (-1)v = 0_V$.

$$\begin{aligned}
v + (-1)v &= 1 \cdot v + (-1)v && \text{(multiplicative identity: } 1 \cdot v = v) \\
&= (1 + (-1))v && \text{(distributivity over scalar addition)} \\
&= 0 \cdot v && \text{(arithmetic in } \mathbb{F}\text{: } 1 + (-1) = 0) \\
&= 0_V && \text{(assumed condition: } 0v = 0_V)
\end{aligned}$$

Step 3: Conclude additive inverses exist.
For every $v \in V$, we have found $w = (-1)v \in V$ such that $v + w = 0_V$.
This is precisely the additive inverse axiom.

Conclusion:

$$\boxed{\text{The condition } 0v = 0 \text{ can replace the additive inverse axiom}}$$

The two formulations of vector space are equivalent: under the other axioms, each implies the other.

Reflection: The distributive law is the bridge between scalar arithmetic and vector structure. It lets us transfer the equation $1 + (-1) = 0$ in $\mathbb{F}$ to the equation $v + (-1)v = 0_V$ in V.

Converse: Standard axiom implies $0v = 0_V$.
In any vector space (with the standard additive inverse axiom), we have:

$$\begin{aligned} 0v &= (0+0)v && \text{(arithmetic: } 0+0=0\text{)} \\ &= 0v + 0v && \text{(distributivity over scalar addition)} \end{aligned}$$

Adding $-(0v)$ to both sides:

$$0_V = 0v + 0v + (-(0v)) = 0v + 0_V = 0v.$$

Thus the two formulations are equivalent. $\square$

Exercise 6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$t \cdot \infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0; \end{cases} \qquad t \cdot (-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0; \end{cases}$$

$$t + \infty = \infty + t = \infty, \quad t + (-\infty) = (-\infty) + t = -\infty,$$

$$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty, \quad \infty + (-\infty) = 0.$$

Is $\mathbb{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbb{R}$? Explain.

Reading the Question

Step 1: Understand the verification strategy. To show something is **not** a vector space, we need to find **one** axiom that fails. We don't need to check all eight—just find a counterexample to one.

Step 2: Identify suspicious operations. Look at the definition $\infty + (-\infty) = 0$. This is unusual—it defines a specific value for something that is typically "indeterminate." Whenever you see a forced definition for a problematic case, suspect trouble.

Step 3: Test associativity. The axiom $(a + b) + c = a + (b + c)$ requires all groupings to give the same result. The elements 1, ∞, and $-\infty$ interact in potentially inconsistent ways.

Intuition: Why Extended Reals Fail

Q1: What goes wrong with ∞? The symbol ∞ "absorbs" finite additions: $1 + \infty = \infty$. But when ∞ meets $-\infty$, they "cancel" to 0. These two behaviors clash.

Q2: Why is this problematic for associativity? Consider $(1 + \infty) + (-\infty)$:

- First grouping: $1 + \infty = \infty$, then $\infty + (-\infty) = 0$
- Second grouping: $\infty + (-\infty) = 0$, then $1 + 0 = 1$

The finite 1 gets "lost" in the first grouping but "survives" in the second.

Q3: Why is $\infty - \infty$ an indeterminate form in calculus? Precisely because of this ambiguity—the "answer" depends on how you approach it. This is a feature in analysis but a bug for algebra.

Proof template:

1. Choose elements that exploit the $\infty + (-\infty) = 0$ rule
2. Compute LHS: $(a + b) + c$
3. Compute RHS: $a + (b + c)$
4. Show LHS $\neq$ RHS

Solution: No, $\mathbb{R} \cup \{\infty, -\infty\}$ is not a vector space over $\mathbb{R}$.

Claim. The set $\mathbb{R} \cup \{\infty, -\infty\}$ with the given operations fails the associativity of addition axiom.

Proof Strategy: To disprove associativity, we find one counterexample: specific elements

a, b, c such that $(a+b)+c \neq a+(b+c)$. The key insight is that ∞ "absorbs" finite numbers, but $\infty + (-\infty) = 0$ creates information loss.

Step 1: Choose elements.
Let $a = 1$, $b = \infty$, and $c = -\infty$.

Step 2: Compute LHS = $(a+b)+c$.

$$\begin{aligned} (1+\infty)+(-\infty) &= \infty + (-\infty) && \text{(since } t + \infty = \infty \text{ for } t \in \mathbb{R}) \\ &= 0 && \text{(given definition)} \end{aligned}$$

Step 3: Compute RHS = $a+(b+c)$.

$$\begin{aligned} 1+(\infty+(-\infty)) &= 1+0 && \text{(given definition)} \\ &= 1 && \text{(standard addition in } \mathbb{R}) \end{aligned}$$

Step 4: Compare LHS and RHS.

$$\text{LHS} = 0 \neq 1 = \text{RHS}$$

Conclusion:
Since $(1+\infty)+(-\infty) \neq 1+(\infty+(-\infty))$, addition is **not associative**.

$$\boxed{\mathbb{R} \cup \{\infty, -\infty\} \text{ is not a vector space (associativity fails)}}$$

Reflection: The extended real line is useful in analysis for limits and measure theory, but the rule $\infty + (-\infty) = 0$ forces "indeterminate form" behavior that violates associativity. This is why $\infty - \infty$ remains undefined in rigorous calculus. □

Exercise 7. Suppose S is a nonempty set and V is a vector space. Let V^S denote the set of functions from S to V. Define addition and scalar multiplication on V^S by

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) = \lambda f(x)$$

for all $f, g \in V^S$, $\lambda \in \mathbb{F}$, and $x \in S$. Prove that V^S is a vector space.

Reading the Question

Step 1: Understand "pointwise" operations. The definitions

$$(f+g)(x) = f(x) + g(x), \qquad (\lambda f)(x) = \lambda f(x)$$

mean we perform operations **at each point** $x \in S$ separately, using the operations in V.

Step 2: Recognize the proof pattern. Every axiom in V^S reduces to the corresponding axiom in V. The template is:

1. Write the definition of the operation in V^S
2. Apply the axiom in V at each point x
3. Translate back to the definition in V^S

Step 3: Understand function equality. Two functions $f, g \in V^S$ are equal if and only if $f(x) = g(x)$ for all $x \in S$. This is why "pointwise" proofs work.

Intuition: Function Spaces as "Component-wise" Structures

Q1: What is V^S concretely? Think of $S = \{1, 2, 3\}$. Then $f \in V^S$ is determined by three values: $f(1), f(2), f(3) \in V$. This is essentially $V^3 = V \times V \times V$.

Q2: What if $S = \mathbb{R}$? Then V^S is the space of all functions $\mathbb{R} \to V$. For $V = \mathbb{R}$, this includes polynomials, continuous functions, even discontinuous monsters.

Q3: Why does the proof work uniformly? Because we never use specific properties of S—only that V is a vector space. The pointwise operations "lift" the vector space structure from V to V^S.

Pattern recognition: This construction appears everywhere: function spaces $C([0,1])$, sequence spaces ℓ^p, and spaces of matrices (which are functions $\{1, \ldots, m\} \times \{1, \ldots, n\} \to \mathbb{F}$).

Proof template for each axiom:

1. Let $x \in S$ be arbitrary
2. Compute both sides using the pointwise definition
3. Apply the corresponding axiom in V
4. Conclude equality holds for all x, hence functions are equal

Solution: We verify that V^S is a vector space over $\mathbb{F}$.

Proof Strategy: Each axiom is verified **pointwise**: we show both sides of each equation agree at every $x \in S$. Since functions are equal if and only if they agree everywhere, this proves the axiom in V^S. The key is that each step reduces to the corresponding axiom in V.

Concrete example: Let $S = \{1, 2\}$ and $V = \mathbb{R}$. Then $V^S \cong \mathbb{R}^2$: a function $f \in V^S$ corresponds to the pair $(f(1), f(2))$. The operations become:

$$(f+g) \leftrightarrow (f(1)+g(1),\ f(2)+g(2)), \quad (\lambda f) \leftrightarrow (\lambda f(1),\ \lambda f(2))$$

This is exactly $\mathbb{R}^2$ with component-wise operations. The general proof below works for *any* S and V.

Part 1: Addition Axioms

Step 1: Commutativity of addition.
Let $f, g \in V^S$ and $x \in S$ be arbitrary.

$$\begin{aligned}
(f+g)(x) &= f(x) + g(x) && \text{(definition of addition in } V^S\text{)} \\
&= g(x) + f(x) && \text{(commutativity of addition in } V\text{)} \\
&= (g+f)(x) && \text{(definition of addition in } V^S\text{)}
\end{aligned}$$

Since this holds for all $x \in S$, we have $f+g = g+f$.

Step 2: Associativity of addition.
Let $f, g, h \in V^S$ and $x \in S$.

$$\begin{aligned}
((f+g)+h)(x) &= (f+g)(x) + h(x) && \text{(definition)} \\
&= (f(x)+g(x)) + h(x) && \text{(definition)} \\
&= f(x) + (g(x)+h(x)) && \text{(associativity in } V\text{)} \\
&= f(x) + (g+h)(x) && \text{(definition)} \\
&= (f+(g+h))(x) && \text{(definition)}
\end{aligned}$$

Hence $(f+g)+h = f+(g+h)$.

Step 3: Additive identity.
Define $\mathbf{0} \in V^S$ by $\mathbf{0}(x) = 0_V$ for all $x \in S$ (the constant zero function).
For any $f \in V^S$ and $x \in S$:

$$\begin{aligned}
(f+\mathbf{0})(x) &= f(x) + \mathbf{0}(x) && \text{(definition)} \\
&= f(x) + 0_V && \text{(definition of } \mathbf{0}\text{)} \\
&= f(x) && \text{(additive identity in } V\text{)}
\end{aligned}$$

Hence $f + \mathbf{0} = f$, so $\mathbf{0}$ is the additive identity in V^S.

Step 4: Additive inverse.
Given $f \in V^S$, define $(-f) \in V^S$ by $(-f)(x) = -f(x)$ for all $x \in S$.
For any $x \in S$:

$$\begin{aligned}
(f+(-f))(x) &= f(x) + (-f)(x) && \text{(definition)} \\
&= f(x) + (-f(x)) && \text{(definition of } -f\text{)} \\
&= 0_V && \text{(additive inverse in } V\text{)} \\
&= \mathbf{0}(x) && \text{(definition of } \mathbf{0}\text{)}
\end{aligned}$$

Hence $f + (-f) = \mathbf{0}$, so $(-f)$ is the additive inverse of f.

Part 2: Scalar Multiplication Axioms

The remaining axioms follow the same pointwise pattern. We show the key step for each:

Step 5 (Multiplicative identity): $(1 \cdot f)(x) = 1 \cdot f(x) = f(x)$, so $1 \cdot f = f$.

Step 6 (Associativity): $((\alpha\beta)f)(x) = (\alpha\beta)f(x) = \alpha(\beta f(x)) = (\alpha(\beta f))(x)$.

Step 7 (Dist. over vectors): $(\lambda(f+g))(x) = \lambda(f(x)+g(x)) = \lambda f(x)+\lambda g(x) = (\lambda f+\lambda g)(x)$.

Step 8 (Dist. over scalars): $((\alpha+\beta)f)(x) = (\alpha+\beta)f(x) = \alpha f(x)+\beta f(x) = (\alpha f+\beta f)(x)$.

Conclusion:

$$\boxed{V^S \text{ is a vector space over } \mathbb{F}}$$

Reflection: Function spaces are a fundamental construction in mathematics. The proof shows that pointwise operations "lift" the vector space structure from V to V^S. This same pattern gives us spaces of continuous functions, integrable functions, and infinite-dimensional spaces central to analysis. □

Exercise 8. Suppose V is a real vector space. The **complexification** of V, denoted $V_{\mathbb{C}}$, equals $V \times V$. An element of $V_{\mathbb{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.

Define addition and complex scalar multiplication on $V_{\mathbb{C}}$ by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all $u_1, v_1, u_2, v_2, u, v \in V$ and $a, b \in \mathbb{R}$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbb{C}}$ is a complex vector space.

Reading the Question

Step 1: Parse the notation. The notation $u + iv$ is suggestive but potentially confusing:

- This is **not** actual addition—$u, v \in V$ and $i \notin V$
- Think of $u + iv$ as an **ordered pair** $(u, v) \in V \times V$
- The "i" is a formal symbol marking the second component

Step 2: Understand the operations. The definitions mimic complex arithmetic:

- Addition: component-wise, just like $(a, b) + (c, d) = (a + c, b + d)$
- Scalar mult.: $(a + bi)(u + iv) = (au - bv) + i(av + bu)$ uses the rule $i^2 = -1$

Compare to $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$ in $\mathbb{C}$.

Step 3: Proof strategy overview. We must verify 8 axioms. The addition axioms are straightforward (reduce to V). The scalar multiplication axioms require more care, especially associativity.

Intuition: Why Complexify?

Q1: What is $V_{\mathbb{C}}$ for a concrete V? If $V = \mathbb{R}^n$, then $V_{\mathbb{C}} \cong \mathbb{C}^n$. An element $(x_1, \ldots, x_n) + i(y_1, \ldots, y_n)$ corresponds to $(x_1 + iy_1, \ldots, x_n + iy_n) \in \mathbb{C}^n$.

Q2: Why do we need complexification? Real vector spaces may lack eigenvalues. For example, the rotation matrix $0 - 1$
10 has no real eigenvalues, but has complex eigenvalues $\pm i$. Complexification gives us access to all eigenvalues.

Q3: Why does the scalar multiplication formula work? Expanding $(a + bi)(u + iv)$ formally and using $i^2 = -1$:

$$au + aiv + biu + bi^2v = au + i(av) + i(bu) - bv = (au - bv) + i(av + bu)$$

Pattern recognition: Complexification is a **base change**—extending scalars from $\mathbb{R}$ to $\mathbb{C}$. This construction generalizes to other field extensions in algebra.

Proof template:

1. Addition axioms: reduce to component-wise properties in V
2. Scalar mult. axioms: expand using the formula, regroup, use V axioms
3. Associativity requires careful algebra with real/imaginary parts

Solution: We verify that $V_{\mathbb{C}}$ is a vector space over $\mathbb{C}$.

Proof Strategy: The addition axioms follow directly from component-wise operations in V. The scalar multiplication axioms require expanding the definition and using properties of V. The trickiest is associativity of scalar multiplication, which requires tracking real and imaginary parts carefully.

Part 1: Addition Axioms (component-wise, inherited from V)

- **Commutativity:** $(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2) = (u_2 + u_1) + i(v_2 + v_1)$ by commutativity in V.
- **Associativity:** Follows from associativity of $+$ in each component.
- **Identity:** $0_{V_{\mathbb{C}}} = 0_V + i \cdot 0_V$ satisfies $(u + iv) + 0_{V_{\mathbb{C}}} = u + iv$.
- **Inverse:** $-(u + iv) = (-u) + i(-v)$ satisfies $(u + iv) + (-(u + iv)) = 0_{V_{\mathbb{C}}}$.

Part 2: Scalar Multiplication Axioms

Step 5: Multiplicative identity.
The identity in $\mathbb{C}$ is $1 = 1 + 0i$.

$$\begin{aligned}
(1 + 0i)(u + iv) &= (1 \cdot u - 0 \cdot v) + i(1 \cdot v + 0 \cdot u) && \text{(definition)} \\
&= (u - 0_V) + i(v + 0_V) && \text{(scalar mult. in } V\text{)} \\
&= u + iv && \text{(additive identity in } V\text{)}
\end{aligned}$$

Step 6: Associativity of scalar multiplication.
Let $\alpha = a + bi$, $\beta = c + di \in \mathbb{C}$, and $u + iv \in V_{\mathbb{C}}$.
First, compute $\alpha\beta$ in $\mathbb{C}$:

$$\alpha\beta = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Compute LHS: $(\alpha\beta)(u + iv)$.
Let $p = ac - bd$ and $q = ad + bc$, so $\alpha\beta = p + qi$.

$$\begin{aligned}
(\alpha\beta)(u + iv) &= (p + qi)(u + iv) \\
&= (pu - qv) + i(pv + qu) && \text{(definition)} \\
&= ((ac - bd)u - (ad + bc)v) + i((ac - bd)v + (ad + bc)u)
\end{aligned}$$

Compute RHS: $\alpha(\beta(u + iv))$.
First, compute $\beta(u + iv)$:

$$\begin{aligned}
\beta(u + iv) &= (c + di)(u + iv) \\
&= (cu - dv) + i(cv + du) && \text{(definition)}
\end{aligned}$$

Let $w_1 = cu - dv$ and $w_2 = cv + du$. Now compute $\alpha(w_1 + iw_2)$:

$$\begin{aligned}
\alpha(w_1 + iw_2) &= (a + bi)(w_1 + iw_2) \\
&= (aw_1 - bw_2) + i(aw_2 + bw_1) && \text{(definition)}
\end{aligned}$$

Expand the real part:

$$\begin{aligned} aw_1 - bw_2 &= a(cu - dv) - b(cv + du) \\ &= acu - adv - bcv - bdu \\ &= (ac - bd)u + (-ad - bc)v \\ &= (ac - bd)u - (ad + bc)v \end{aligned}$$

Expand the imaginary part:

$$\begin{aligned} aw_2 + bw_1 &= a(cv + du) + b(cu - dv) \\ &= acv + adu + bcu - bdv \\ &= (ac - bd)v + (ad + bc)u \end{aligned}$$

Therefore:

$$\alpha(\beta(u + iv)) = ((ac - bd)u - (ad + bc)v) + i((ac - bd)v + (ad + bc)u)$$

Compare: LHS = RHS. Hence $(\alpha\beta)(u + iv) = \alpha(\beta(u + iv))$.

Verification summary:

Component	**Value (both sides)**
Real part	$(ac - bd)u - (ad + bc)v$
Imaginary part	$(ac - bd)v + (ad + bc)u$

Since both components match, $(\alpha\beta)(u + iv) = \alpha(\beta(u + iv))$. ✓

Step 7: Distributivity over vector addition.
Expanding $\lambda((u_1 + iv_1) + (u_2 + iv_2))$ and regrouping by components:

$$= ((au_1 - bv_1) + (au_2 - bv_2)) + i((av_1 + bu_1) + (av_2 + bu_2)) = \lambda(u_1 + iv_1) + \lambda(u_2 + iv_2). \checkmark$$

Step 8: Distributivity over scalar addition.
Similarly, $(\alpha + \beta)(u + iv) = ((a + c)u - (b + d)v) + i((a + c)v + (b + d)u)$ regroups to $\alpha(u + iv) + \beta(u + iv)$. ✓

Conclusion:

$$\boxed{V_{\mathbb{C}} \text{ is a complex vector space}}$$

Reflection: Complexification extends a real vector space to allow complex scalars. The construction is essential for spectral theory: every linear operator on a finite-dimensional complex vector space has eigenvalues (Fundamental Theorem of Algebra), but real operators may not. Complexification gives us access to the full spectrum. □