1 Suppose $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries.

2 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1.

3 Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W.

(a) Show that if $S, T \in \mathcal{L}(V, W)$, then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$.

(b) Show that if $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$, then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

This exercise asks you to verify 3.35 and 3.38.

4 Suppose that $D \in \mathcal{L}(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbf{R})$ and a basis of $\mathcal{P}_2(\mathbf{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Compare with Example 3.33. The next exercise generalizes this exercise.

5 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row k, column k, equal 1 if $1 \leq k \leq \dim \operatorname{range} T$.

6 Suppose $v_1, \dots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \dots, w_n$ of W such that all entries in the first column of $\mathcal{M}(T)$ [with respect to the bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$] are 0 except for possibly a 1 in the first row, first column.

In this exercise, unlike Exercise 5, you are given the basis of V instead of being able to choose a basis of V.

7 Suppose $w_1, \ldots, w_n$ is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $v_1, \ldots, v_m$ of V such that all entries in the first row of $\mathcal{M}(T)$ [with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$] are 0 except for possibly a 1 in the first row, first column.

In this exercise, unlike Exercise 5, you are given the basis of W instead of being able to choose a basis of W.

8 Suppose A is an m-by-n matrix and B is an n-by-p matrix. Prove that

$$(AB)_{j,\cdot} = A_{j,\cdot}\, B$$

for each $1 \leq j \leq m$. In other words, show that row j of AB equals (row j of A) times B.

This exercise gives the row version of 3.48.

9 Suppose $a = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$ is a 1-by-n matrix and B is an n-by-p matrix. Prove that

$$aB = a_1 B_{1,\cdot} + \cdots + a_n B_{n,\cdot}.$$

In other words, show that aB is a linear combination of the rows of B, with the scalars that multiply the rows coming from a.

This exercise gives the row version of 3.50.

10 Give an example of 2-by-2 matrices A and B such that $AB \neq BA$.

11 Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, C, D, E, and F are matrices whose sizes are such that $A(B+C)$ and $(D+E)F$ make sense. Explain why $AB+AC$ and $DF+EF$ both make sense and prove that

$$A(B+C) = AB + AC \quad \text{and} \quad (D+E)F = DF + EF.$$

12 Prove that matrix multiplication is associative. In other words, suppose A, B, and C are matrices whose sizes are such that $(AB)C$ makes sense. Explain why $A(BC)$ makes sense and prove that

$$(AB)C = A(BC).$$

Try to find a clean proof that illustrates the following quote from Emil Artin: "It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."

13 Suppose A is an n-by-n matrix and $1 \le j, k \le n$. Show that the entry in row j, column k, of A^3 (which is defined to mean AAA) is

$$\sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p}\, A_{p,r}\, A_{r,k}.$$

14 Suppose m and n are positive integers. Prove that the function $A \mapsto A^{\mathrm{t}}$ is a linear map from $\mathbf{F}^{m,n}$ to $\mathbf{F}^{n,m}$.

15 Prove that if A is an m-by-n matrix and C is an n-by-p matrix, then

$$(AC)^{\mathrm{t}} = C^{\mathrm{t}} A^{\mathrm{t}}.$$

This exercise shows that the transpose of the product of two matrices is the product of the transposes in the opposite order.

16 Suppose A is an m-by-n matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, \ldots, c_m) \in \mathbf{F}^m$ and $(d_1, \ldots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \ldots, m$ and every $k = 1, \ldots, n$.

17 Suppose $T \in \mathcal{L}(V)$, and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Prove that the following are equivalent.

(a) T is injective.

(b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$.

(c) The columns of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$.

(d) The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$.

(e) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{1,n}$.

Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$.

1 Explain why each linear functional is surjective or is the zero map.

2 Give three distinct examples of linear functionals on $\mathbf{R}^{[0,1]}$.

3 Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

4 Suppose V is finite-dimensional and U is a subspace of V such that $U \neq V$. Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

5 Suppose $T \in \mathcal{L}(V, W)$ and $w_1, \dots, w_m$ is a basis of $\operatorname{range} T$. Hence for each $v \in V$, there exist unique numbers $\varphi_1(v), \dots, \varphi_m(v)$ such that

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m,$$

thus defining functions $\varphi_1, \dots, \varphi_m$ from V to $\mathbf{F}$. Show that each of the functions $\varphi_1, \dots, \varphi_m$ is a linear functional on V.

6 Suppose $\varphi, \beta \in V'$. Prove that $\operatorname{null} \varphi \subseteq \operatorname{null} \beta$ if and only if there exists $c \in \mathbf{F}$ such that $\beta = c\varphi$.

7 Suppose that $V_1, \dots, V_m$ are vector spaces. Prove that $(V_1 \times \cdots \times V_m)'$ and $V_1' \times \cdots \times V_m'$ are isomorphic vector spaces.

8 Suppose $v_1, \dots, v_n$ is a basis of V and $\varphi_1, \dots, \varphi_n$ is the dual basis of V'. Define $\Gamma\colon V \to \mathbf{F}^n$ and $\Lambda\colon \mathbf{F}^n \to V$ by

$$\Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v)) \quad \text{and} \quad \Lambda(a_1, \dots, a_n) = a_1 v_1 + \cdots + a_n v_n.$$

Explain why Γ and Λ are inverses of each other.

9 Suppose m is a positive integer. Show that the dual basis of the basis $1, x, \ldots, x^m$ of $\mathcal{P}_m(\mathbf{R})$ is $\varphi_0, \varphi_1, \ldots, \varphi_m$, where

$$\varphi_k(p) = \frac{p^{(k)}(0)}{k!}.$$

Here $p^{(k)}$ *denotes the* k^{th} *derivative of* p*, with the understanding that the* 0^{th} *derivative of* p *is* p*.*

10 Suppose m is a positive integer.

(a) Show that $1, x-5, \ldots, (x-5)^m$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

(b) What is the dual basis of the basis in (a)?

11 Suppose $v_1, \dots, v_n$ is a basis of V and $\varphi_1, \dots, \varphi_n$ is the corresponding dual basis of V'. Suppose $\psi \in V'$. Prove that

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n.$$

12 Suppose $S, T \in \mathcal{L}(V, W)$.

(a) Prove that $(S + T)' = S' + T'$.

(b) Prove that $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbf{F}$.

This exercise asks you to verify (a) and (b) in 3.120.

13 Show that the dual map of the identity operator on V is the identity operator on V'.

14 Define $T: \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T(x, y, z) = (4x + 5y + 6z,\ 7x + 8y + 9z).$$

Suppose φ_1, φ_2 denotes the dual basis of the standard basis of $\mathbf{R}^2$ and ψ_1, ψ_2, ψ_3 denotes the dual basis of the standard basis of $\mathbf{R}^3$.

(a) Describe the linear functionals $T'(\varphi_1)$ and $T'(\varphi_2)$.

(b) Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.

15 Define $T\colon \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by

$$(Tp)(x) = x^2 p(x) + p''(x)$$

for each $x \in \mathbf{R}$.

(a) Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = p'(4)$. Describe the linear functional $T'(\varphi)$ on $\mathcal{P}(\mathbf{R})$.

(b) Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = \int_0^1 p$. Evaluate $(T'(\varphi))(x^3)$.

16 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that

$$T' = 0 \iff T = 0.$$

17 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is invertible if and only if $T' \in \mathcal{L}(W', V')$ is invertible.

18 Suppose V and W are finite-dimensional. Prove that the map that takes $T \in \mathcal{L}(V, W)$ to $T' \in \mathcal{L}(W', V')$ is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.