Upper Division Tutoring Program Topic Review

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Section I: Definitions and Theorems Review

Eigenvalues and Eigenvectors

Throughout, vector spaces are over a field F.

Definition 1 (invariant subspace)**.** Fix a linear map $T: V \to V$ of vector spaces. Then a subspace $U \subseteq V$ is *invariant* to T if and only if $Tu \in U$ for each $u \in U$.

Definition 2 (eigenvalue, eigenvector)**.** Fix a linear map $T: V \to V$ of vector spaces. A scalar λ is an *eigenvalue* of T if and only if there exists a nonzero vector $v \in V$ such that $Tv = \lambda v$. In this situation, v is called an *eigenvector* with eigenvalue λ.

Proposition 1. Fix a linear map $T: V \to V$ of vector spaces. Given a scalar λ, the following are equivalent.

1. λ is an eigenvalue of T.
2. $T - \lambda I$ is not invertible.
3. $T - \lambda I$ is not injective.
4. $T - \lambda I$ is not surjective.

Definition 3 (eigenspace)**.** Fix an eigenvalue λ of a linear map $T: V \to V$ of vector spaces. Then we define the *eigenspace* of λ to be

$$E(\lambda, T) \coloneqq \{v \in V : Tv = \lambda v\}.$$

Proposition 2. Fix a linear map $T: V \to V$ of vector spaces with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. Then

$$E(\lambda_1, T) + E(\lambda_2, T) + \cdots + E(\lambda_m, T)$$

is a direct sum.

Definition 4 (diagonalizable)**.** A linear map $T: V \to V$ is *diagonalizable* if and only if there is a basis of V for which the associated matrix is diagonal, meaning that the only nonzero terms of the matrix lie on the central diagonal

$$\begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{bmatrix}.$$

Proposition 3. Fix a linear map $T: V \to V$ of vector spaces. Then the following are equivalent.

1. T is diagonalizable.
2. T has a basis of eigenvectors.
3. Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues for T. Then $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.
4. Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues for T. Then $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

Definition 5 (upper-triangular)**.** A matrix M is *upper-triangular* if and only if all entries strictly below the central diagonal are zero. In other words, the matrix has the form

$$\begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{bmatrix}.$$

Proposition 4. Let V be a vector space over $\mathbb{C}$. Then any linear map $T\colon V \to V$ has a basis $\{v_1, \ldots, v_m\}$ such that the matrix associated to T is upper-triangular.

Proposition 5. Let $T\colon V \to V$ be a linear map. Suppose that we have a basis $\{v_1, \ldots, v_m\}$ such that the associated matrix of T is upper-triangular. Then the eigenvalues are precisely the diagonal entries of the associated matrix.

Definition 6 (minimal polynomial)**.** Fix a linear map $T\colon V \to V$ of vector spaces. Then the *minimal polynomial* p of T is the unique monic polynomial of minimal degree such that $p(T)$ is the zero operator.

Proposition 6. Fix a linear map $T\colon V \to V$ of vector spaces with minimal polynomial p. If λ is an eigenvalue for T, then $p(\lambda) = 0$.

Remark. In fact, the converse of the above proposition is also true if V is finite-dimensional: if λ is a root of the minimal polynomial p, then λ is an eigenvalue of T.

Definition 7 (Generalized Eigenvector)**.** Let $T\colon V \to V$ be a linear map with eigenvalue λ. A (nonzero) vector $v \in V$ is a generalized eigenvector of T corresponding to λ if $(T - \lambda I)^k v = 0$ for some positive integer k.

Definition 8 (Generalized Eigenspace)**.** Let $T\colon V \to V$ be a linear map and $\lambda \in \mathbf{F}$. The generalized eigenspace of T corresponding to λ is $G(\lambda, T) = \{v \in V : (T - \lambda I)^k v = 0 \text{ for some positive integer k}\}$.

Proposition 7. Suppose $\mathbf{F} = \mathbf{C}$ and $T\colon V \to V$ is a linear map with distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. Then $V = G(\lambda_1, T) \oplus \ldots G(\lambda_m, T)$.

Section II: Practice Exercises

Exercises are organized thematically, not by difficulty.

1. (Axler 5.B.10) Let p be a polynomial and $T\colon V \to V$ be a linear map. Given an eigenvector $v \in V$ with eigenvalue λ, show that $p(T)v = p(\lambda)v$.

2. Let V denote the $\mathbb{R}$-vector space space spanned by the functions $\{1, \sin x, \cos x, \sin 2x, \cos 2x\}$, and let $d\colon V \to V$ denote the differentiation operator. What are the eigenvalues of d? Is d diagonalizable?

3. (a) Define the linear map $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ by $T(x, y) := (y, -x)$. What are the eigenvalues of T? Find a basis of $\mathbb{C}^2$ such that the matrix associated to T is upper-triangular.

   (b) Define the linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) := (y, -x)$. Show that there is no basis of $\mathbb{R}^2$ such that the matrix associated to T is upper-triangular.

4. (Axler 6.B.18) Fix an invertible linear map $T\colon V \to V$.

   (a) If $p(x)$ is the minimal polynomial of T, show that $x^{\deg p} p(1/x)$ is the minimal polynomial of T^{-1}.

   (b) For each nonzero scalar λ, show that $E(\lambda, T) = E\left(\lambda^{-1}, T^{-1}\right)$.

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5. Fix a finite-dimensional vector space V. Fix a linear map $T\colon V \to V$ with nonzero eigenvalues $\lambda_1, \ldots, \lambda_m$.
   (a) Show that $E(\lambda_i, T) \subseteq \operatorname{range} T$ for each eigenvalue λ_i.
   (b) Use (a) to conclude that
   $$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim \operatorname{range} T.$$
6. Find the bases for the generalized eigenspace(s) of the following matrix:
   $$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 4 & -4 & 7 \end{bmatrix}$$
7. (Axler 8.B.4) Suppose $\dim V \geq 2$ and $T\colon V \to V$ is a linear map such that null $T^{\dim V-2} \neq$ null $T^{\dim V-1}$. Prove that T has at most two distinct eigenvalues.