Problem Set X: Topic Name

Chapter Y Exercises

Instructions: Show all work. Justify each step.

Part I: Computation

Problem 1. Let $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(a) Find the eigenvalues of $\mathbf{A}$.

(b) Find a basis for each eigenspace.

(c) Is $\mathbf{A}$ diagonalizable? If so, find matrices $\mathbf{P}$ and $\mathbf{D}$.

Solution. **(a)** The characteristic polynomial is:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3).$$

So $\boxed{\lambda_1 = 1, \lambda_2 = 3}$.

(b) For $\lambda = 1$: Solve $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0}$.

$$E_1 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

For $\lambda = 3$: Solve $(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \mathbf{0}$.

$$E_3 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

(c) Yes, $\mathbf{A}$ is diagonalizable since we have 2 linearly independent eigenvectors.

$$\boxed{\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}}$$

□

Part II: Proofs

Problem 2. Prove that if $T : V \to V$ is a linear operator and $T^2 = T$, then $V = \ker(T) \oplus \operatorname{im}(T)$.

Hint: For any $\mathbf{v} \in V$, write $\mathbf{v} = (\mathbf{v} - T(\mathbf{v})) + T(\mathbf{v})$.

End of Problem Set