Section 1B: Definition of Vector Space

1. Addition and Scalar Multiplication

In Section 1A, we defined addition and scalar multiplication on $\mathbb{F}^n$. Now we abstract these operations to define vector spaces in general.

1.19 Definition: Addition, Scalar Multiplication

- An **addition** on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
- A **scalar multiplication** on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and each $v \in V$.

Key point: Addition takes two elements of V and produces an element of V. Scalar multiplication takes a scalar from $\mathbb{F}$ and an element of V, producing an element of V. These are *functions*: every input has exactly one output.

Notation: We will also use juxtaposition for scalar multiplication: λv means the same as $\lambda \cdot v$.

2. Definition of Vector Space

The following definition is the central definition of linear algebra.

1.20 Definition: Vector Space
A **vector space** is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:
commutativity:

$$u + v = v + u \quad \text{for all } u, v \in V$$

associativity:

$$(u + v) + w = u + (v + w) \text{ and } (ab)v = a(bv)$$

for all $u, v, w \in V$ and all $a, b \in \mathbb{F}$
additive identity: there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$
additive inverse: for every $v \in V$, there exists $w \in V$ such that $v + w = 0$
multiplicative identity:

$$1v = v \quad \text{for all } v \in V$$

distributive properties:

$$a(u + v) = au + av \text{ and } (a + b)v = av + bv$$

for all $a, b \in \mathbb{F}$ and all $u, v \in V$

1.21 Definition: Vector, Point
Elements of a vector space are called **vectors** or **points**.

Terminology: The elements of $\mathbb{F}$ are called **scalars**. The word "scalar" is used because elements of $\mathbb{F}$ scale vectors via scalar multiplication.

1.22 Definition: Real Vector Space, Complex Vector Space

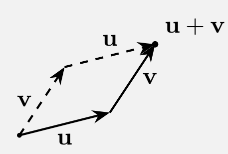
- A vector space over $\mathbb{R}$ is called a **real vector space**.
- A vector space over $\mathbb{C}$ is called a **complex vector space**.

Note: The simplest vector space is $\{0\}$, which contains only the additive identity.

Mnemonic for the 8 axioms: "CAAIMD^2"

- **C**ommutativity of addition
- **A**ssociativity of addition
- **A**ssociativity of scalar multiplication
- Additive **I**dentity
- Additive inverse ("**I**nverse")
- **M**ultiplicative identity
- **D**istributive (scalar over vector sum)
- **D**istributive (scalar sum over vector)

Why These Axioms? (Geometric Intuition)
These axioms aren't arbitrary; each captures an essential geometric property:
Commutativity & Associativity: The parallelogram rule for adding vectors works regardless of which vector you draw first. Moving "3 steps east then 2 north" reaches the same point as "2 north then 3 east."



Additive Identity & Inverse: We need a "do nothing" operation (0) and a way to "undo" addition ($-v$). Geometrically: the origin exists, and every direction has an opposite.
Multiplicative Identity: Scaling by 1 leaves a vector unchanged. This anchors our notion of "unit scale."
Distributive Laws: Scaling respects the structure of addition. Stretching $\mathbf{u} + \mathbf{v}$ by factor λ is the same

as stretching each separately, then adding.

3. Examples of Vector Spaces

The set $\mathbb{F}^n$ with the addition and scalar multiplication defined in Section 1A is a vector space over $\mathbb{F}$. We verified commutativity in Section 1A; the other vector space properties follow similarly by working coordinate-by-coordinate.

1.23 Example: $\mathbb{F}^\infty$
Define $\mathbb{F}^\infty$ as the set of all sequences of elements of $\mathbb{F}$:

$$\mathbb{F}^\infty = \{(x_1, x_2, x_3, \ldots) : x_k \in \mathbb{F} \text{ for } k = 1, 2, \ldots\}$$

Addition and scalar multiplication are defined coordinate-wise:

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots)$$
$$\lambda(x_1, x_2, \ldots) = (\lambda x_1, \lambda x_2, \ldots)$$

With these operations, $\mathbb{F}^\infty$ is a vector space over $\mathbb{F}$.

Intuition: $\mathbb{F}^\infty$ is like $\mathbb{F}^n$ but with infinitely many coordinates. The verification is identical; each axiom is checked coordinate-wise.

The Abstraction Ladder: From $\mathbb{F}^n$ to $\mathbb{F}^S$
We've now seen two vector spaces with similar structure:

- $\mathbb{F}^n$: lists indexed by $\{1, 2, \ldots, n\}$
- $\mathbb{F}^\infty$: sequences indexed by $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$

What do these have in common? In both cases, we have a set of "indices" ($\{1, \ldots, n\}$ or $\mathbb{Z}^+$), and a vector assigns a scalar to each index. This leads to a powerful abstraction: what if we index by *any* set S?

1.24 Notation: $\mathbb{F}^S$
If S is a nonempty set, then $\mathbb{F}^S$ denotes the set of all functions from S to $\mathbb{F}$.
For $f, g \in \mathbb{F}^S$, the **sum** $f + g \in \mathbb{F}^S$ is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

for all $x \in S$.
For $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, the **product** $\lambda f \in \mathbb{F}^S$ is the function defined by

$$(\lambda f)(x) = \lambda f(x)$$

for all $x \in S$.

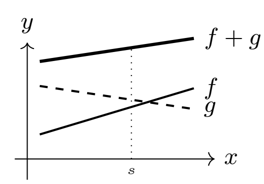
1.25 Example: $\mathbb{F}^S$ is a Vector Space
If S is a nonempty set, then $\mathbb{F}^S$ (with the operations of addition and scalar multiplication as defined in 1.24) is a vector space over $\mathbb{F}$.
Details:

- The additive identity of $\mathbb{F}^S$ is the function $0 : S \to \mathbb{F}$ defined by $0(x) = 0$ for all $x \in S$.
- For $f \in \mathbb{F}^S$, the additive inverse of f is the function $-f : S \to \mathbb{F}$ defined by $(-f)(x) = -f(x)$ for all $x \in S$.

Key insight: The "vectors" in $\mathbb{F}^S$ are *functions*. Addition means adding function values pointwise.

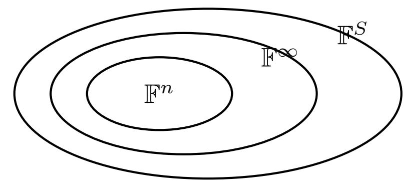
Why functions are vectors: This may seem abstract, but it's incredibly powerful. A function $f : S \to \mathbb{F}$ assigns a number to each point in S—just like a tuple $(x_1, \ldots, x_n)$ assigns a number to each index. The key realization is that *any* way of assigning numbers to a collection of labels can be viewed as a vector. This viewpoint unifies:

- Polynomials (coefficients indexed by degree)
- Signals (amplitude indexed by time)
- Images (pixel values indexed by position)



Pointwise addition: $(f + g)(s) = f(s) + g(s)$

Unifying perspective: $\mathbb{F}^n$ is a special case of $\mathbb{F}^S$ where $S = \{1, 2, \ldots, n\}$. A list $(x_1, \ldots, x_n)$ is the same as the function $f : \{1, \ldots, n\} \to \mathbb{F}$ defined by $f(k) = x_k$.
Similarly, $\mathbb{F}^\infty$ is $\mathbb{F}^S$ where $S = \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.



Vector space generalization hierarchy

4. Uniqueness Results

Geometric intuition: The axioms guarantee that additive identities and inverses *exist*, but why should they be *unique*? Geometrically, if there were two different "origins" in our space, we'd have two incompatible reference points—vectors would have ambiguous positions. Similarly, if a vector had multiple additive inverses, the notion of "opposite direction" would be ill-defined. Uniqueness ensures our geometric intuition is well-founded.

Proof Technique: Uniqueness Arguments
To prove an object is unique:

1. Assume two objects satisfy the defining property
2. Show they must be equal using that property

This "assume two, show equal" pattern appears throughout mathematics.

The axioms guarantee existence of additive identities and inverses, but we should verify they are unique.

1.26 Unique Additive Identity
A vector space has a unique additive identity.

Proof: Suppose 0 and $0'$ are both additive identities for V. Then:

$$0' = 0' + 0 = 0 + 0' = 0$$

The first equality holds because 0 is an additive identity. The second uses commutativity. The third holds because $0'$ is an additive identity. $\square$

1.27 Unique Additive Inverse
Every element in a vector space has a unique additive inverse.

Proof: Suppose $v \in V$ and both w and w' satisfy $v + w = 0$ and $v + w' = 0$. Then:

$$\begin{aligned} w' &= w' + 0 \\ &= w' + (v + w) \\ &= (w' + v) + w \\ &= (v + w') + w \\ &= 0 + w \\ &= w \end{aligned}$$

Thus $w = w'$. $\square$

1.28 Notation: $-v$, $w - v$
Let $v, w \in V$.

- $-v$ denotes the additive inverse of v.
- $w - v$ is defined by $w - v = w + (-v)$.

1.29 Notation: V
For the rest of this book, V denotes a vector space over $\mathbb{F}$.

5. Properties of Vector Spaces

The following properties are not axioms; they are *consequences* of the axioms.

1.30 $0v = 0$
For every v in a vector space, $0v = 0$.

Proof: For $v \in V$:

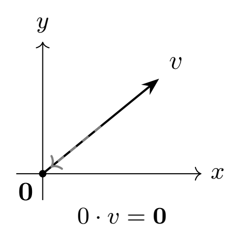
$$\begin{aligned} 0v &= (0 + 0)v \\ &= 0v + 0v \end{aligned}$$

Adding the additive inverse of $0v$ to both sides:

$$0 = 0v$$

$\square$

Notation warning: In "$0v = 0$", the left 0 is the *scalar* zero, and the right 0 is the *vector* zero.



Scalar zero "collapses" any vector to the origin.

1.31 $a0 = 0$
For every $a \in \mathbb{F}$, $a0 = 0$.

Proof: For $a \in \mathbb{F}$:

$$\begin{aligned} a0 &= a(0 + 0) \\ &= a0 + a0 \end{aligned}$$

Adding $-(a0)$ to both sides gives $0 = a0$. $\square$

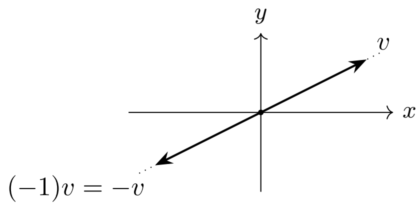
1.32 $(-1)v = -v$
For every v in a vector space, $(-1)v = -v$.

Proof: For $v \in V$:

$$\begin{aligned} v + (-1)v &= 1v + (-1)v \\ &= (1 + (-1))v \\ &= 0v \\ &= 0 \end{aligned}$$

This shows $(-1)v$ is an additive inverse of v. By uniqueness (1.27), $(-1)v = -v$. □

Key insight: We can now write $-v$ as $(-1)v$. Subtraction is just a special case of scalar multiplication!



Scalar multiplication by -1 reverses direction.

Summary of key results:

- $0v = 0$ (scalar zero times any vector is the zero vector)
- $a0 = 0$ (any scalar times the zero vector is the zero vector)
- $(-1)v = -v$ (the additive inverse is scalar multiplication by -1)

These three results connect the scalar and vector versions of "zero" and "negative."

Non-Examples and Pitfalls

Understanding what is *not* a vector space is as important as knowing the definition. For each non-example, we identify the **specific axiom that fails**.

Non-Example	Axiom Violated
$\emptyset$	Additive identity
$\mathbb{R}^2$, wrong $\cdot$	Multiplicative identity
$\mathbb{R}^+$	Add. identity & inverse

Quick reference: which axiom each non-example violates

Non-Example: The Empty Set
The empty set $\emptyset$ is not a vector space.
Axiom violated: *Additive identity* — there is no element $0 \in \emptyset$.
Key point: Every vector space must contain at least one element (the zero vector).

Non-Example: $\mathbb{R}^2$ with "wrong" scalar multiplication
Consider $\mathbb{R}^2$ with standard addition but scalar multiplication defined by:

$$\lambda(x, y) = (\lambda x, 0)$$

Axiom violated: *Multiplicative identity* ($1v = v$).
Check: $1(x, y) = (1 \cdot x, 0) = (x, 0) \neq (x, y)$ unless $y = 0$.
The operation "forgets" the second coordinate, so scaling by 1 doesn't preserve vectors.

Non-Example: Positive Reals
Consider $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ with standard addition and scalar multiplication.
Axioms violated:

- *Additive identity*: $0 \notin \mathbb{R}^+$ (zero is not positive)
- *Additive inverse*: For $v = 1 \in \mathbb{R}^+$, we need -1, but $-1 \notin \mathbb{R}^+$

Twist: With different operations (multiplication as "addition" and exponentiation as "scalar multiplication"), $\mathbb{R}^+$ *can* be made into a vector space!

Verifying Vector Space Axioms

Strategy for proving V is a vector space:

1. Define addition on V (check it maps $V \times V \to V$).
2. Define scalar multiplication (check it maps $\mathbb{F} \times V \to V$).
3. Verify all 8 axioms.

Common shortcuts:

- If V inherits operations from a known vector space (like $\mathbb{F}^n$ or $\mathbb{F}^S$), most axioms follow automatically.
- Commutativity and associativity often follow from the corresponding properties in $\mathbb{F}$.

Strategy for proving V is NOT a vector space: Find ONE axiom that fails and provide a specific counterexample.

Common failures:

- No additive identity ($0 \notin V$)
- No additive inverses ($-v \notin V$ for some v)
- Not closed under addition ($u + v \notin V$)
- Not closed under scalar multiplication ($\lambda v \notin V$)
- Multiplicative identity fails ($1v \neq v$)

Worked Example: Verifying $\mathbb{F}^2$ is a Vector Space

Let's verify that $\mathbb{F}^2 = \{(x, y) : x, y \in \mathbb{F}\}$ with coordinate-wise operations is a vector space.

1. Commutativity of addition: Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$.

$$u+v = (u_1+v_1, u_2+v_2) = (v_1+u_1, v_2+u_2) = v+u \checkmark$$

(using commutativity in $\mathbb{F}$)

2. Additive identity: The vector $0 = (0, 0)$ satisfies:

$$(x, y) + (0, 0) = (x + 0, y + 0) = (x, y) \checkmark$$

3. Multiplicative identity: For any $(x, y) \in \mathbb{F}^2$:

$$1(x, y) = (1 \cdot x, 1 \cdot y) = (x, y) \checkmark$$

4. Distributive (scalar over sum): Let $a \in \mathbb{F}$, $u, v \in \mathbb{F}^2$:

$$a(u+v) = a(u_1+v_1, u_2+v_2) = (a(u_1+v_1), a(u_2+v_2))$$

$$= (au_1 + av_1, au_2 + av_2) = au + av \checkmark$$

The remaining axioms (associativity of addition, associativity of scalar multiplication, additive inverses, other distributive law) follow similarly from properties of $\mathbb{F}$.

Key Takeaways

1. **Vector space axioms (CAAIMD^2):** 8 properties defining addition and scalar multiplication
2. **Uniqueness:** Additive identity and inverses are unique
3. **Key consequences:** $0v = 0$, $a0 = 0$, $(-1)v = -v$
4. **Verification strategy:** Check all 8 axioms (or find one that fails for non-examples)

Relevant Exercises

Practice these problems from LADR to reinforce the material:

- Section 1B: 1, 2, 3, 4, 5, 6

Common Problem Types:

Prove a property from axioms
Start with one side. Apply axioms step-by-step. Justify each step by naming the axiom used.

Prove uniqueness
Assume two objects satisfy the definition. Show they must be equal using the defining property.

Verify a set is a vector space
Define operations clearly. Verify closure. Check all 8 axioms (use shortcuts when operations are inherited).

Show a set is NOT a vector space
Find one axiom that fails. Give a specific counterexample with explicit values.