Upper Division Tutoring Program Topic Review
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Math 110 Midterm Exam Review

Section I: Definitions and Theorems Review

Definition 1 (Linear Map)**.** Let V, W be vector spaces over a field $\mathbb{F}$. A linear map $T : V \to W$ is a map that satisfies the following properties:

- For all $v, w \in V$, $T(v) + T(w) = T(v + w)$, and
- for all $\lambda \in \mathbb{F}, \lambda T(v) = T(\lambda v)$.

Definition 2 (Isomorphism Between Vector Spaces)**.** An isomorphism between two vector spaces is an invertible linear map. Moreover, two finite-dimensional vector spaces over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.

Theorem 1. A linear map is invertible if and only if it is injective and surjective.

Theorem 2. Suppose V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finite-dimensional and $\text{dim}\mathcal{L}(V, W) = (\text{dim}V)(\text{dim}W)$.

Definition 3. A linear functional on V is a linear map from V to $\mathbb{F}$.

Definition 4 (Dual Space)**.** The dual space of V, otherwise known as V', is the vector space of all linear functionals on V.

Definition 5 (Dual Basis)**.** If $v_1, \ldots, v_n$ is a basis of V, then the dual basis of $v_1, \ldots, v_n$ is a list of linear functionals $\phi_j \in V'$ such that

$$\phi_j(v_k) = \begin{cases} 1 \text{ if } k = j, \\ 0 \text{ if } k \neq j. \end{cases}$$

Definition 6 (Eigenvalue)**.** Suppose $T \in \mathcal{L}(V)$ is an operator. A number $\lambda \in \mathbb{F}$ is an eigenvalue of T is there exists $v \in V$ such that $Tv = \lambda v$.

Theorem 3. Every operator on a finite-dimensional non-zero complex vector space has an eigenvalue.

Theorem 4 (Conditions Equivalent to Diagonalizability)**.** Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent:

1. T is diagonalizable.
2. V has a basis consisting of eigenvectors of T.
3. $V = E(\lambda_1, T) \oplus \ldots \oplus E(\lambda_m, T)$.
4. $\text{dim}V = \text{dim}E(\lambda_1, T) + \ldots + \text{dim}E(\lambda_m, T)$.

Theorem 5. Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ has $\text{dim}V$ distinct eigenvalues. Then T is diagonalizable.

Section II: Exercises

1. Given a finite-dimensional vector space V with basis $v_1, \ldots, v_n$.
   (a) Prove that if two linear maps S and T have $S(v_1) = T(v_1), \ldots, S(v_n) = T(v_n)$ then $S = T$
   (b) Prove that for a linear map T, that $T(v) = \varphi_1(v)T(v_1) + \cdots + \varphi_n(v)T(v_n)$ with $\varphi_1, \ldots, \varphi_n$ the dual basis of $v_1, \ldots, v_n$.

2. Let V denote the $\mathbb{R}$-vector space space spanned by the functions $\{1, \sin x, \cos x, \sin 2x, \cos 2x\}$, and let $d \colon V \to V$ denote the differentiation operator. What are the eigenvalues of d? Is d diagonalizable?

3. (a) Define the linear map $T \colon \mathbb{C}^2 \to \mathbb{C}^2$ by $T(x, y) \coloneqq (y, -x)$. What are the eigenvalues of T? Find a basis of $\mathbb{C}^2$ such that the matrix associated to T is upper-triangular.

   (b) Define the linear map $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) \coloneqq (y, -x)$. Show that there is no basis of $\mathbb{R}^2$ such that the matrix associated to T is upper-triangular.

4. (Spring 2024 Frenkel Mock Midterm Problem 3) Prove that a vector space V over a field $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$ (where n is a positive integer) if and only if $\dim(V) = n$.