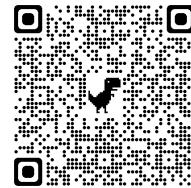


Upper Division Tutoring Program Topic Review

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Section I: Definitions and Theorems Review

Eigenvalues and Eigenvectors

Throughout, vector spaces are over a field F .

Definition 1 (invariant subspace). Fix a linear map $T: V \rightarrow V$ of vector spaces. Then a subspace $U \subseteq V$ is *invariant* to T if and only if $Tu \in U$ for each $u \in U$.

Definition 2 (eigenvalue, eigenvector). Fix a linear map $T: V \rightarrow V$ of vector spaces. A scalar λ is an *eigenvalue* of T if and only if there exists a nonzero vector $v \in V$ such that $Tv = \lambda v$. In this situation, v is called an *eigenvector* with eigenvalue λ .

Proposition 1. Fix a linear map $T: V \rightarrow V$ of vector spaces. Given a scalar λ , the following are equivalent.

1. λ is an eigenvalue of T .
2. $T - \lambda I$ is not invertible.
3. $T - \lambda I$ is not injective.
4. $T - \lambda I$ is not surjective.

Definition 3 (eigenspace). Fix an eigenvalue λ of a linear map $T: V \rightarrow V$ of vector spaces. Then we define the *eigenspace* of λ to be

$$E(\lambda, T) := \{v \in V : Tv = \lambda v\}.$$

Proposition 2. Fix a linear map $T: V \rightarrow V$ of vector spaces with distinct eigenvalues $\lambda_1, \dots, \lambda_m$. Then

$$E(\lambda_1, T) + E(\lambda_2, T) + \cdots + E(\lambda_m, T)$$

is a direct sum.

Definition 4 (diagonalizable). A linear map $T: V \rightarrow V$ is *diagonalizable* if and only if there is a basis of V for which the associated matrix is diagonal, meaning that the only nonzero terms of the matrix lie on the central diagonal

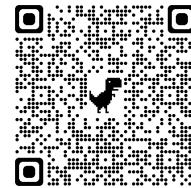
$$\begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{bmatrix}.$$

Proposition 3. Fix a linear map $T: V \rightarrow V$ of vector spaces. Then the following are equivalent.

1. T is diagonalizable.
2. T has a basis of eigenvectors.
3. Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues for T . Then $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.
4. Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues for T . Then $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

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Definition 5 (upper-triangular). A matrix M is *upper-triangular* if and only if all entries strictly below the central diagonal are zero. In other words, the matrix has the form

$$\begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{bmatrix}.$$

Proposition 4. Let V be a vector space over \mathbb{C} . Then any linear map $T: V \rightarrow V$ has a basis $\{v_1, \dots, v_m\}$ such that the matrix associated to T is upper-triangular.

Proposition 5. Let $T: V \rightarrow V$ be a linear map. Suppose that we have a basis $\{v_1, \dots, v_m\}$ such that the associated matrix of T is upper-triangular. Then the eigenvalues are precisely the diagonal entries of the associated matrix.

Definition 6 (minimal polynomial). Fix a linear map $T: V \rightarrow V$ of vector spaces. Then the *minimal polynomial* p of T is the unique monic polynomial of minimal degree such that $p(T)$ is the zero operator.

Proposition 6. Fix a linear map $T: V \rightarrow V$ of vector spaces with minimal polynomial p . If λ is an eigenvalue for T , then $p(\lambda) = 0$.

Remark. In fact, the converse of the above proposition is also true if V is finite-dimensional: if λ is a root of the minimal polynomial p , then λ is an eigenvalue of T .

Definition 7 (Generalized Eigenvector). Let $T: V \rightarrow V$ be a linear map with eigenvalue λ . A (nonzero) vector $v \in V$ is a generalized eigenvector of T corresponding to λ if $(T - \lambda I)^k v = 0$ for some positive integer k .

Definition 8 (Generalized Eigenspace). Let $T: V \rightarrow V$ be a linear map and $\lambda \in \mathbb{F}$. The generalized eigenspace of T corresponding to λ is $G(\lambda, T) = \{v \in V : (T - \lambda I)^k v = 0 \text{ for some positive integer } k\}$.

Proposition 7. Suppose $\mathbf{F} = \mathbf{C}$ and $T: V \rightarrow V$ is a linear map with distinct eigenvalues $\lambda_1, \dots, \lambda_m$. Then $V = G(\lambda_1, T) \oplus \dots \oplus G(\lambda_m, T)$.

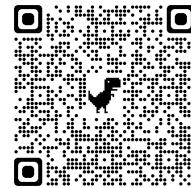
Section II: Practice Exercises

Exercises are organized thematically, not by difficulty.

1. (Axler 5.B.10) Let p be a polynomial and $T: V \rightarrow V$ be a linear map. Given an eigenvector $v \in V$ with eigenvalue λ , show that $p(T)v = p(\lambda)v$.
2. Let V denote the \mathbb{R} -vector space spanned by the functions $\{1, \sin x, \cos x, \sin 2x, \cos 2x\}$, and let $d: V \rightarrow V$ denote the differentiation operator. What are the eigenvalues of d ? Is d diagonalizable?
3. (a) Define the linear map $T: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by $T(x, y) := (y, -x)$. What are the eigenvalues of T ? Find a basis of \mathbb{C}^2 such that the matrix associated to T is upper-triangular.
 (b) Define the linear map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T(x, y) := (y, -x)$. Show that there is no basis of \mathbb{R}^2 such that the matrix associated to T is upper-triangular.
4. (Axler 6.B.18) Fix an invertible linear map $T: V \rightarrow V$.
 - (a) If $p(x)$ is the minimal polynomial of T , show that $x^{\deg p} p(1/x)$ is the minimal polynomial of T^{-1} .
 - (b) For each nonzero scalar λ , show that $E(\lambda, T) = E(\lambda^{-1}, T^{-1})$.

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5. Fix a finite-dimensional vector space V . Fix a linear map $T: V \rightarrow V$ with nonzero eigenvalues $\lambda_1, \dots, \lambda_m$.

- Show that $E(\lambda_i, T) \subseteq \text{range } T$ for each eigenvalue λ_i .
- Use (a) to conclude that

$$\dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T) \leq \dim \text{range } T.$$

6. Find the bases for the generalized eigenspace(s) of the following matrix:

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 4 & -4 & 7 \end{bmatrix}$$

7. (Axler 8.B.4) Suppose $\dim V \geq 2$ and $T: V \rightarrow V$ is a linear map such that $\text{null } T^{\dim V - 2} \neq \text{null } T^{\dim V - 1}$. Prove that T has at most two distinct eigenvalues.