Exercises 1C: Subspaces

Linear Algebra Done Right, 4th ed.

Exercise 1. For each of the following subsets of $\mathbb{F}^3$, determine whether it is a subspace of $\mathbb{F}^3$.

(a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

(b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

(c) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 x_2 x_3 = 0\}$

(d) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$

Exercise 2. Verify all assertions about subspaces in Example 1.35.

Exercise 3. Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

Exercise 4. Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Exercise 5. Is $\mathbb{R}^2$ a subspace of the complex vector space $\mathbb{C}^2$?

Exercise 6.

(a) Is $\{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?

(b) Is $\{(a,b,c) \in \mathbb{C}^3 : a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?

Exercise 7. Prove or give a counterexample: If U is a nonempty subset of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbb{R}^2$.

Exercise 8. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

Exercise 9. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called **periodic** if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbb{R}$. Is the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$? Explain.

Exercise 10. Suppose V_1 and V_2 are subspaces of V. Prove that the intersection $V_1 \cap V_2$ is a subspace of V.

Exercise 11. Prove that the intersection of every collection of subspaces of V is a subspace of V.

Exercise 12. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Exercise 13. Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

This exercise is surprisingly harder than Exercise 12, possibly because this exercise is not true if we replace $\mathbb{F}$ with a field containing only two elements.

Exercise 14. Suppose

$$U = \{(x, -x, 2x) \in \mathbb{F}^3 : x \in \mathbb{F}\} \quad \text{and} \quad W = \{(x, x, 2x) \in \mathbb{F}^3 : x \in \mathbb{F}\}.$$

Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.

Answer: ____

Exercise 15. Suppose U is a subspace of V. What is $U + U$?

Answer: ____

Exercise 16. Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

Exercise 17. Is the operation of addition on the subspaces of V associative? In other words, if V_1, V_2, V_3 are subspaces of V, is

$$(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)?$$

Exercise 18. Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

Exercise 19. Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V_1 + U = V_2 + U,$$

then $V_1 = V_2$.

Exercise 20. Suppose

$$U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

Answer: ______________________________

Exercise 21. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$.

Answer: ______________________________

Exercise 22. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbb{F}^5$, none of which equals $\{0\}$, such that

$$\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3.$$

Exercise 23. Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V = V_1 \oplus U \quad \text{and} \quad V = V_2 \oplus U,$$

then $V_1 = V_2$.

Hint: When trying to discover whether a conjecture in linear algebra is true or false, it is often useful to start by experimenting in $\mathbb{F}^2$.

Exercise 24. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called **even** if

$$f(-x) = f(x)$$

for all $x \in \mathbb{R}$. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called **odd** if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$. Let V_{e} denote the set of real-valued even functions on $\mathbb{R}$ and let V_{o} denote the set of real-valued odd functions on $\mathbb{R}$. Show that

$$\mathbb{R}^{\mathbb{R}} = V_{\mathrm{e}} \oplus V_{\mathrm{o}}.$$