Exercises 1B: Definition of Vector Space

Linear Algebra Done Right, 4th ed.

Exercise 1. Prove that $-(-v) = v$ for every $v \in V$.

Exercise 2. Suppose $a \in \mathbb{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Exercise 3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Exercise 4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Answer: ______________________________

Exercise 5. Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \quad \text{for all } v \in V.$$

The 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

Hint: The point here is that if the expression "for every $v \in V$, there exists $w \in V$ such that $v + w = 0$" in the definition of vector space is replaced with "$0v = 0$ for all $v \in V$," then the other conditions in the definition of vector space imply that for each $v \in V$, there exists an additive inverse of v.

Exercise 6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty, -\infty\}$ as follows: the usual operations on $\mathbb{R}$ are unchanged; for $t \in \mathbb{R}$ define

$$\begin{aligned} t + \infty = \infty + t &= \infty, \\ t + (-\infty) = (-\infty) + t &= -\infty, \\ \infty + \infty &= \infty, \\ (-\infty) + (-\infty) &= -\infty, \\ \infty + (-\infty) &= 0; \end{aligned}$$

for $t \in \mathbb{R}$ with $t > 0$ define

$$t \cdot \infty = \infty, \qquad t \cdot (-\infty) = -\infty;$$

for $t \in \mathbb{R}$ with $t < 0$ define

$$t \cdot \infty = -\infty, \qquad t \cdot (-\infty) = \infty;$$

and define

$$0 \cdot \infty = 0, \qquad 0 \cdot (-\infty) = 0.$$

Is $\mathbb{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbb{R}$? Explain.

Exercise 7. Suppose S is a nonempty set and V is a vector space. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Exercise 8. Suppose V is a real vector space. The **complexification** of V, denoted $V_{\mathbb{C}}$, equals $V \times V$. An element of $V_{\mathbb{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we will write this as $u + iv$.

Define addition on $V_{\mathbb{C}}$ by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

for $u_1, v_1, u_2, v_2 \in V$.

Define complex scalar multiplication on $V_{\mathbb{C}}$ by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for $a, b \in \mathbb{R}$ and $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbb{C}}$ is a complex vector space.