

Section 1A: \mathbb{R}^n and \mathbb{C}^n

1. Complex Numbers

Before we can study vector spaces, we need to understand the scalars we'll use. The real numbers \mathbb{R} are familiar, but linear algebra becomes more powerful when we also work with complex numbers \mathbb{C} .

1.1 Definition: Complex Numbers, \mathbb{C}

- A **complex number** is an ordered pair (a, b) where $a, b \in \mathbb{R}$, written as $a + bi$.
- The set of all complex numbers is denoted by \mathbb{C} :

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

- **Addition** and **multiplication** on \mathbb{C} are defined by:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

where $a, b, c, d \in \mathbb{R}$.

Intuition: Think of \mathbb{C} as a 2D plane where the horizontal axis represents real numbers and the vertical axis represents imaginary numbers. If $a \in \mathbb{R}$, we identify $a + 0i$ with the real number a , so $\mathbb{R} \subset \mathbb{C}$. We write $0 + bi$ as just bi , and $0 + 1i$ as just i .

Why complex numbers? Even when working with real matrices, eigenvalues (Chapter 5) often require complex numbers. The completeness of \mathbb{C} makes linear algebra more elegant—every polynomial has roots in \mathbb{C} .

Why this multiplication formula? We define i as a symbol satisfying $i^2 = -1$. This is consistent and creates an algebraically closed field. Using the usual rules of arithmetic:

$$\begin{aligned} (a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

1.2 Example: Complex Arithmetic

Compute $(2 + 3i)(4 + 5i)$.

Using the distributive and commutative properties:

$$\begin{aligned} (2 + 3i)(4 + 5i) &= 2 \cdot (4 + 5i) + (3i)(4 + 5i) \\ &= 2 \cdot 4 + 2 \cdot 5i + 3i \cdot 4 + (3i)(5i) \\ &= 8 + 10i + 12i - 15 \\ &= \boxed{-7 + 22i} \end{aligned}$$

1.3 Properties of Complex Arithmetic

For all $\alpha, \beta, \lambda \in \mathbb{C}$:

commutativity: $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$

associativity: $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$

identities: $\lambda + 0 = \lambda$ and $\lambda \cdot 1 = \lambda$

additive inverse: For every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$

multiplicative inverse: For every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$

distributive property: $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$

1.4 Example: Commutativity of Complex Multiplication

To show $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$, suppose $\alpha = a + bi$ and $\beta = c + di$ where $a, b, c, d \in \mathbb{R}$.

LHS:

$$\alpha\beta = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

RHS:

$$\beta\alpha = (c + di)(a + bi) = (ca - db) + (cb + da)i$$

Since real multiplication is commutative ($ac = ca$, $bd = db$, etc.), we have $\alpha\beta = \beta\alpha$. \square

1.5 Definition: $-\alpha$, Subtraction, $1/\alpha$, Division

Suppose $\alpha, \beta \in \mathbb{C}$.

- Let $-\alpha$ denote the **additive inverse** of α : the unique complex number such that $\alpha + (-\alpha) = 0$.
- **Subtraction** on \mathbb{C} is defined by $\beta - \alpha = \beta + (-\alpha)$.
- For $\alpha \neq 0$, let $1/\alpha$ and $\frac{1}{\alpha}$ denote the **multiplicative inverse** of α : the unique complex number such that $\alpha(1/\alpha) = 1$.
- For $\alpha \neq 0$, **division** by α is defined by $\beta/\alpha = \beta(1/\alpha)$.

Computing $1/\alpha$: For $\alpha = a + bi \neq 0$, multiply by the conjugate:

$$\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

Key insight: Why multiply by the conjugate? Because $(a + bi)(a - bi) = a^2 + b^2$ is always real and positive. Multiplying by the conjugate eliminates i from the denominator.

Example: Complex Division. Compute $\frac{2 + 3i}{4 + 5i}$.

Multiply by the conjugate of the denominator:

$$\frac{2+3i}{4+5i} \cdot \frac{4-5i}{4-5i} = \frac{(2+3i)(4-5i)}{(4+5i)(4-5i)}$$

Denominator: $(4+5i)(4-5i) = 16+25 = 41$

Numerator: $(2+3i)(4-5i) = 8-10i+12i+15 = 23+2i$

Answer: $\frac{2+3i}{4+5i} = \boxed{\frac{23}{41} + \frac{2}{41}i}$

1.6 Notation: \mathbb{F}

Throughout this book, \mathbb{F} stands for either \mathbb{R} or \mathbb{C} .

Why use \mathbb{F} ? The letter \mathbb{F} reminds us of “field.” Both \mathbb{R} and \mathbb{C} are fields: sets with addition and multiplication satisfying the properties in 1.3. Using \mathbb{F} lets us state theorems once and have them apply to both \mathbb{R} and \mathbb{C} .

Elements of \mathbb{F} are called **scalars**.

Powers of scalars: For $\alpha \in \mathbb{F}$ and a positive integer m :

$$\alpha^m = \underbrace{\alpha \cdot \alpha \cdots \alpha}_{m \text{ times}}$$

This definition implies:

- $(\alpha^m)^n = \alpha^{mn}$
- $(\alpha\beta)^m = \alpha^m\beta^m$

2. Lists

To generalize \mathbb{R}^2 and \mathbb{R}^3 to higher dimensions, we first need to discuss the concept of lists.

1.7 Example: \mathbb{R}^2 and \mathbb{R}^3

- $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ (the plane)
- $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ (3D space)

1.8 Definition: List, Length

Suppose n is a nonnegative integer. A **list** of **length** n is an ordered collection of n elements (which might be numbers, other lists, or more abstract objects) separated by commas and surrounded by parentheses.

A list of length n is also called an **n -tuple**.

Key point: Two lists are equal if and only if they have the same length and the same elements in the same order.

1.9 Example: Lists versus Sets

- Lists $(3, 5)$ and $(5, 3)$ are **not equal**, but sets $\{3, 5\} = \{5, 3\}$
- Lists $(4, 4)$ and $(4, 4, 4)$ are **not equal**, but sets $\{4, 4\} = \{4, 4, 4\} = \{4\}$

Key difference: order and repetition matter in lists, not in sets.

Why lists? Linear algebra needs ordered data. The coordinates $(1, 2, 3)$ represent a different point than $(3, 2, 1)$. Order encodes meaning—the first coordinate might be position, the second velocity, the third acceleration.

3. \mathbb{F}^n

To define the higher-dimensional analogues of \mathbb{R}^2 and \mathbb{R}^3 , we simply replace \mathbb{R} with \mathbb{F} (which equals \mathbb{R} or \mathbb{C}) and replace the 2 or 3 with an arbitrary positive integer.

1.10 Notation: n

Fix a positive integer n for the rest of this chapter.

1.11 Definition: \mathbb{F}^n

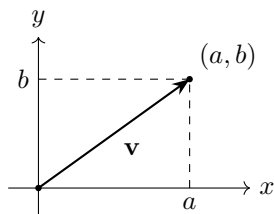
\mathbb{F}^n is the set of all lists of length n of elements of \mathbb{F} :

$$\mathbb{F}^n = \{(x_1, x_2, \dots, x_n) : x_k \in \mathbb{F} \text{ for } k = 1, \dots, n\}$$

For $(x_1, \dots, x_n) \in \mathbb{F}^n$ and $k \in \{1, \dots, n\}$, we say x_k is the k^{th} **coordinate** of (x_1, \dots, x_n) .

- Example 3.1** (Elements of \mathbb{F}^n). • $(2, -1, 5) \in \mathbb{R}^3$ is a list with 3 real coordinates.
- $(1 + i, 2, -3i) \in \mathbb{C}^3$ is a list with 3 complex coordinates.
 - $(7, -2) \in \mathbb{R}^2$ corresponds to a point in the plane.
 - \mathbb{F}^1 can be identified with \mathbb{F} .

Intuition: Think of \mathbb{R}^2 as the plane and \mathbb{R}^3 as 3-dimensional space. For $n > 3$, we lose geometric visualization but the algebra works identically.



Elements of \mathbb{R}^2 can be thought of as points or as vectors.

1.12 Example: \mathbb{C}^4

$$\mathbb{C}^4 = \{(z_1, z_2, z_3, z_4) : z_1, z_2, z_3, z_4 \in \mathbb{C}\}$$

1.13 Definition: Addition in \mathbb{F}^n

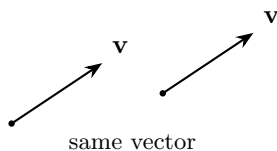
Addition in \mathbb{F}^n is defined by adding corresponding coordinates:

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

Example 3.2 (Vector Addition). In \mathbb{R}^3 :

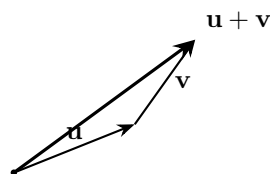
$$(1, 2, 3) + (10, 20, 30) = (1+10, 2+20, 3+30) = (11, 22, 33)$$

Geometric intuition: In \mathbb{R}^2 and \mathbb{R}^3 , addition corresponds to the parallelogram rule: place the tail of the second vector at the head of the first.



same vector

A vector—same length and direction = same vector.



The sum of two vectors (tip-to-tail method).

1.14 Commutativity of Addition in \mathbb{F}^n

If $x, y \in \mathbb{F}^n$, then $x + y = y + x$.

Proof: Suppose $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Then:

$$\begin{aligned} x + y &= (x_1, \dots, x_n) + (y_1, \dots, y_n) \\ &= (x_1 + y_1, \dots, x_n + y_n) \\ &= (y_1 + x_1, \dots, y_n + x_n) \\ &= (y_1, \dots, y_n) + (x_1, \dots, x_n) \\ &= y + x \end{aligned}$$

where the third equality uses commutativity of addition in \mathbb{F} . \square

1.15 Notation: 0

Let 0 denote the list of length n whose coordinates are all 0:

$$0 = (0, \dots, 0)$$

Geometric intuition: The zero vector 0 represents the origin. Adding 0 to any vector leaves it unchanged—you're adding “no displacement.”

1.16 Example: The Zero Vector Notation

When we write $x + 0 = x$ for $x \in \mathbb{F}^n$, the symbol 0 means the **zero vector**:

$$0 = (0, 0, \dots, 0) \quad (n \text{ zeros})$$

Why? Addition in \mathbb{F}^n is only defined for two vectors. Since x is a vector, the 0 in “ $x + 0$ ” must also be a vector—not the number zero.

Example in \mathbb{R}^3 :

$$(1, 2, 3) + 0 = (1, 2, 3) + (0, 0, 0) = (1, 2, 3) \checkmark$$

Key point: The symbol “0” means different things depending on context:

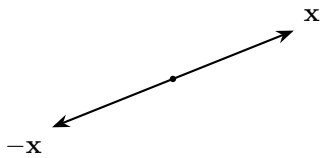
- In \mathbb{F} (scalars): 0 is the number zero
- In \mathbb{F}^n (vectors): 0 is the zero vector $(0, \dots, 0)$

1.17 Definition: Additive Inverse in \mathbb{F}^n

For $x \in \mathbb{F}^n$, the **additive inverse** of x , denoted $-x$, is the vector $-x \in \mathbb{F}^n$ such that $x + (-x) = 0$. If $x = (x_1, \dots, x_n)$, then $-x = (-x_1, \dots, -x_n)$.

Note: Subtraction is defined by $x - y = x + (-y)$.

Geometric intuition: In \mathbb{R}^2 , $-x$ is the vector with the same length as x but pointing in the opposite direction.



A vector and its additive inverse.

Why not coordinate-wise multiplication? We *could* define multiplication of two vectors by multiplying corresponding coordinates, but this is not useful for linear algebra. Instead, **scalar multiplication** (multiplying a vector by a number) is central to our subject.

1.18 Definition: Scalar Multiplication in \mathbb{F}^n

The product of a number $\lambda \in \mathbb{F}$ and a vector in \mathbb{F}^n is computed by multiplying each coordinate by λ :

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

Example 3.3 (Scalar Multiplication in \mathbb{R}^n).

$$-2(1, 2, 3) = (-2 \cdot 1, -2 \cdot 2, -2 \cdot 3) = \boxed{(-2, -4, -6)}$$

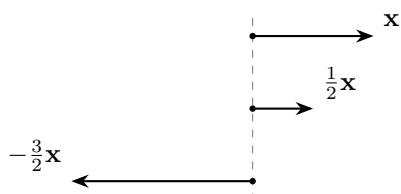
Example 3.4 (Scalar Multiplication in \mathbb{C}^n).

$$\begin{aligned} (1+i)(2, -i) &= ((1+i) \cdot 2, (1+i)(-i)) \\ &= (2+2i, -i-i^2) \\ &= (2+2i, -i+1) \\ &= \boxed{(2+2i, 1-i)} \end{aligned}$$

Geometric intuition in \mathbb{R}^2 :

- If $\lambda > 0$: λx points in the same direction as x , with length λ times the length of x .
- If $\lambda > 1$: stretches (longer). If $0 < \lambda < 1$: shrinks (shorter).
- If $\lambda < 0$: λx points in the opposite direction, with length $|\lambda|$ times the length of x .

Note: All scalar multiples of x lie on a single line through the origin. This “direction preservation” is fundamental to linear maps (Chapter 3).



Scalar multiplication: scaling and reversing vectors.

Scalar multiplication vs dot product: Scalar multiplication takes a scalar and a vector, producing a **vector**. The dot product (Chapter 6) takes two vectors and produces a **scalar**. These are different operations.

4. Digression on Fields

A **field** is a set containing at least two distinct elements called 0 and 1, along with operations of addition and multiplication satisfying all properties listed in 1.3.

- \mathbb{R} and \mathbb{C} are fields.
- The set of rational numbers \mathbb{Q} is a field.
- The set $\{0, 1\}$ with usual addition and multiplication (except $1 + 1 = 0$) is a field.

Note: This book deals only with \mathbb{R} and \mathbb{C} . However, many definitions, theorems, and proofs work for arbitrary fields. If you prefer, think of \mathbb{F} as denoting an arbitrary field (except in Chapters 6–7 on inner products, where $\mathbb{F} = \mathbb{C}$ is sometimes required).

Key Takeaways

1. **Complex numbers** $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ with $i^2 = -1$.
2. **Arithmetic:** $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.
3. **Field properties** (1.3): commutativity, associativity, identities, inverses, distributivity.
4. **\mathbb{F} notation:** \mathbb{F} denotes either \mathbb{R} or \mathbb{C} .
5. \mathbb{F}^n is the set of all n -tuples (x_1, \dots, x_n) with $x_j \in \mathbb{F}$.
6. **Addition** is coordinate-wise.
7. **Scalar multiplication:** $\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$.
8. **Zero vector** $0 = (0, \dots, 0)$ is the additive identity.