

- 1** Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.

- 5** Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

- 6** Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

7 Show that

$$\frac{-1 + \sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

- 8 Find two distinct square roots of i .

10 Explain why there does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

- 1** Prove that $-(-v) = v$ for every $v \in V$.

2 Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

- 3** Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

4 The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

5 Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V .

The phrase a “condition can be replaced” in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.

7 Suppose S is a nonempty set. Let V^S denote the set of functions from S to V . Define a natural addition and scalar multiplication on V^S , and show that V^S is a vector space with these definitions.

8 Suppose V is a real vector space.

- The *complexification* of V , denoted by $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v) , where $u, v \in V$, but we write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

for all $u_1, v_1, u_2, v_2 \in V$.

- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbf{C}}$ is a complex vector space.

Think of V as a subset of $V_{\mathbf{C}}$ by identifying $u \in V$ with $u + i0$. The construction of $V_{\mathbf{C}}$ from V can then be thought of as generalizing the construction of \mathbf{C}^n from \mathbf{R}^n .

1 For each of the following subsets of \mathbf{F}^3 , determine whether it is a subspace of \mathbf{F}^3 .

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1x_2x_3 = 0\}$

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$

3 Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$.

6

- (a) Is $\{(a, b, c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of \mathbf{R}^3 ?
- (b) Is $\{(a, b, c) \in \mathbf{C}^3 : a^3 = b^3\}$ a subspace of \mathbf{C}^3 ?

7 Prove or give a counterexample: If U is a nonempty subset of \mathbf{R}^2 such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of \mathbf{R}^2 .

8 Give an example of a nonempty subset U of \mathbf{R}^2 such that U is closed under scalar multiplication, but U is not a subspace of \mathbf{R}^2 .

9 A function $f: \mathbf{R} \rightarrow \mathbf{R}$ is called *periodic* if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from \mathbf{R} to \mathbf{R} a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

10 Suppose V_1 and V_2 are subspaces of V . Prove that the intersection $V_1 \cap V_2$ is a subspace of V .

- 11** Prove that the intersection of every collection of subspaces of V is a subspace of V .

12 Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

13 Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

This exercise is surprisingly harder than Exercise 12, possibly because this exercise is not true if we replace \mathbf{F} with a field containing only two elements.

14 Suppose

$$U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\} \quad \text{and} \quad W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}.$$

Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.

15 Suppose U is a subspace of V . What is $U + U$?

19 Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V_1 + U = V_2 + U,$$

then $V_1 = V_2$.

20 Suppose

$$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}.$$

Find a subspace W of \mathbf{F}^4 such that $\mathbf{F}^4 = U \oplus W$.

21 Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find a subspace W of \mathbf{F}^5 such that $\mathbf{F}^5 = U \oplus W$.

22 Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find three subspaces W_1, W_2, W_3 of \mathbf{F}^5 , none of which equals $\{0\}$, such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

23 Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V = V_1 \oplus U \quad \text{and} \quad V = V_2 \oplus U,$$

then $V_1 = V_2$.

Hint: When trying to discover whether a conjecture in linear algebra is true or false, it is often useful to start by experimenting in \mathbf{F}^2 .

24 A function $f: \mathbf{R} \rightarrow \mathbf{R}$ is called *even* if

$$f(-x) = f(x)$$

for all $x \in \mathbf{R}$. A function $f: \mathbf{R} \rightarrow \mathbf{R}$ is called *odd* if

$$f(-x) = -f(x)$$

for all $x \in \mathbf{R}$. Let V_e denote the set of real-valued even functions on \mathbf{R} and let V_o denote the set of real-valued odd functions on \mathbf{R} . Show that $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$.