Section 1B: Definition of Vector Space

1. Addition and Scalar Multiplication

In Section 1A, we defined addition and scalar multiplication on $\mathbb{F}^n$. Now we abstract these operations to define vector spaces in general.

1.19 Definition: Addition, Scalar Multiplication

- An **addition** on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
- A **scalar multiplication** on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and each $v \in V$.

Key point: Addition takes two elements of V and produces an element of V. Scalar multiplication takes a scalar from $\mathbb{F}$ and an element of V, producing an element of V. These are *functions*: every input has exactly one output.

Notation: We will also use juxtaposition for scalar multiplication: λv means the same as $\lambda \cdot v$.

2. Definition of Vector Space

The following definition is the central definition of linear algebra.

1.20 Definition: Vector Space
A **vector space** is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:
commutativity:

$$u + v = v + u \quad \text{for all } u, v \in V$$

associativity:

$$(u + v) + w = u + (v + w) \text{ and } (ab)v = a(bv)$$

for all $u, v, w \in V$ and all $a, b \in \mathbb{F}$
additive identity: there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$
additive inverse: for every $v \in V$, there exists $w \in V$ such that $v + w = 0$
multiplicative identity:

$$1v = v \quad \text{for all } v \in V$$

distributive properties:

$$a(u + v) = au + av \text{ and } (a + b)v = av + bv$$

for all $a, b \in \mathbb{F}$ and all $u, v \in V$

1.21 Definition: Vector, Point
Elements of a vector space are called **vectors** or **points**.

Terminology: The elements of $\mathbb{F}$ are called **scalars**. The word "scalar" is used because elements of $\mathbb{F}$ scale vectors via scalar multiplication.

1.22 Definition: Real Vector Space, Complex Vector Space

- A vector space over $\mathbb{R}$ is called a **real vector space**.
- A vector space over $\mathbb{C}$ is called a **complex vector space**.

Note: The simplest vector space is $\{0\}$, which contains only the additive identity.

Mnemonic for the 8 axioms: "CAAIMD^2"

- **C**ommutativity of addition
- **A**ssociativity of addition
- **A**ssociativity of scalar multiplication
- Additive **I**dentity
- Additive inverse ("**I**nverse")
- **M**ultiplicative identity
- **D**istributive (scalar over vector sum)
- **D**istributive (scalar sum over vector)

Why these axioms? They capture the essential properties of $\mathbb{F}^n$ that make linear algebra work. The axioms ensure we can:

- Rearrange sums (commutativity, associativity)
- Solve equations like $v + x = w$ (additive inverse)
- Scale vectors predictably (distributive laws)

3. Examples of Vector Spaces

The set $\mathbb{F}^n$ with the addition and scalar multiplication defined in Section 1A is a vector space over $\mathbb{F}$. We verified commutativity in Section 1A; the other vector space properties follow similarly by working coordinate-by-coordinate.

1.23 Example: $\mathbb{F}^\infty$
Define $\mathbb{F}^\infty$ as the set of all sequences of elements of $\mathbb{F}$:

$$\mathbb{F}^\infty = \{(x_1, x_2, x_3, \ldots) : x_k \in \mathbb{F} \text{ for } k = 1, 2, \ldots\}$$

Addition and scalar multiplication are defined coordinate-wise:

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (x_1 + y_1, x_2 + y_2, \ldots)$$
$$\lambda(x_1, x_2, \ldots) = (\lambda x_1, \lambda x_2, \ldots)$$

With these operations, $\mathbb{F}^\infty$ is a vector space over $\mathbb{F}$.

Intuition: $\mathbb{F}^\infty$ is like $\mathbb{F}^n$ but with infinitely many coordinates. The verification is identical; each axiom is checked coordinate-wise.

1.24 Notation: $\mathbb{F}^S$
If S is a nonempty set, then $\mathbb{F}^S$ denotes the set of all functions from S to $\mathbb{F}$.
For $f, g \in \mathbb{F}^S$, the **sum** $f + g \in \mathbb{F}^S$ is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

for all $x \in S$.
For $\lambda \in \mathbb{F}$ and $f \in \mathbb{F}^S$, the **product** $\lambda f \in \mathbb{F}^S$ is the function defined by

$$(\lambda f)(x) = \lambda f(x)$$

for all $x \in S$.

1.25 Example: $\mathbb{F}^S$ **is a Vector Space**
If S is a nonempty set, then $\mathbb{F}^S$ (with the operations of addition and scalar multiplication as defined in 1.24) is a vector space over $\mathbb{F}$.
Details:

- The additive identity of $\mathbb{F}^S$ is the function $0 : S \to \mathbb{F}$ defined by $0(x) = 0$ for all $x \in S$.
- For $f \in \mathbb{F}^S$, the additive inverse of f is the function $-f : S \to \mathbb{F}$ defined by $(-f)(x) = -f(x)$ for all $x \in S$.

Key insight: The "vectors" in $\mathbb{F}^S$ are *functions*. Addition means adding function values pointwise.

Unifying perspective: $\mathbb{F}^n$ is a special case of $\mathbb{F}^S$ where $S = \{1, 2, \ldots, n\}$. A list $(x_1, \ldots, x_n)$ is the same as the function $f : \{1, \ldots, n\} \to \mathbb{F}$ defined by $f(k) = x_k$.
Similarly, $\mathbb{F}^\infty$ is $\mathbb{F}^S$ where $S = \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

4. Uniqueness Results

The axioms guarantee existence of additive identities and inverses, but we should verify they are unique.

1.26 Unique Additive Identity
A vector space has a unique additive identity.

Proof: Suppose 0 and $0'$ are both additive identities for V. Then:

$$0' = 0' + 0 = 0 + 0' = 0$$

The first equality holds because 0 is an additive identity. The second uses commutativity. The third holds because $0'$ is an additive identity. □

1.27 Unique Additive Inverse
Every element in a vector space has a unique additive inverse.

Proof: Suppose $v \in V$ and both w and w' satisfy $v + w = 0$ and $v + w' = 0$. Then:

$$\begin{aligned} w' &= w' + 0 \\ &= w' + (v + w) \\ &= (w' + v) + w \\ &= (v + w') + w \\ &= 0 + w \\ &= w \end{aligned}$$

Thus $w = w'$. □

1.28 Notation: $-v$, $w - v$
Let $v, w \in V$.

- $-v$ denotes the additive inverse of v.
- $w - v$ is defined by $w - v = w + (-v)$.

1.29 Notation: V
For the rest of this book, V denotes a vector space over $\mathbb{F}$.

5. Properties of Vector Spaces

The following properties are not axioms; they are *consequences* of the axioms.

1.30 $0v = 0$
For every v in a vector space, $0v = 0$.

Proof: For $v \in V$:

$$\begin{aligned} 0v &= (0 + 0)v \\ &= 0v + 0v \end{aligned}$$

Adding the additive inverse of $0v$ to both sides:

$$0 = 0v$$

□

Notation warning: In "$0v = 0$", the left 0 is the *scalar* zero, and the right 0 is the *vector* zero.

1.31 $a0 = 0$
For every $a \in \mathbb{F}$, $a0 = 0$.

Proof: For $a \in \mathbb{F}$:

$$\begin{aligned} a0 &= a(0 + 0) \\ &= a0 + a0 \end{aligned}$$

Adding $-(a0)$ to both sides gives $0 = a0$. □

1.32 $(-1)v = -v$
For every v in a vector space, $(-1)v = -v$.

Proof: For $v \in V$:

$$\begin{aligned} v + (-1)v &= 1v + (-1)v \\ &= (1 + (-1))v \\ &= 0v \\ &= 0 \end{aligned}$$

This shows $(-1)v$ is an additive inverse of v. By uniqueness (1.27), $(-1)v = -v$. □

Key insight: We can now write $-v$ as $(-1)v$. Subtraction is just a special case of scalar multiplication!

Summary of key results:

- $0v = 0$ (scalar zero times any vector is the zero vector)
- $a0 = 0$ (any scalar times the zero vector is the zero vector)
- $(-1)v = -v$ (the additive inverse is scalar multiplication by -1)

These three results connect the scalar and vector versions of "zero" and "negative."

Non-Examples and Pitfalls

Understanding what is *not* a vector space is as important as knowing the definition.

Non-Example: The Empty Set
The empty set $\emptyset$ is not a vector space because it fails the additive identity axiom: there is no element $0 \in \emptyset$.
Key point: Every vector space must contain at least one element (the zero vector).

Non-Example: $\mathbb{R}^2$ with "wrong" scalar multiplication
Consider $\mathbb{R}^2$ with standard addition but scalar multiplication defined by:

$$\lambda(x, y) = (\lambda x, 0)$$

This is **not** a vector space. Check the multiplicative identity:

$$1(x, y) = (1 \cdot x, 0) = (x, 0) \neq (x, y)$$

unless $y = 0$. The axiom $1v = v$ fails.

Non-Example: Positive Reals
Consider $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ with standard addition and scalar multiplication. This is **not** a vector space because:

- The additive identity would need to be 0, but $0 \notin \mathbb{R}^+$.
- For $v = 1 \in \mathbb{R}^+$, the additive inverse $-1 \notin \mathbb{R}^+$.

Note: With different operations (multiplication as "addition" and exponentiation as "scalar multiplication"), $\mathbb{R}^+$ *can* be made into a vector space!

Verifying Vector Space Axioms

Strategy for proving V is a vector space:

1. Define addition on V (check it maps $V \times V \to V$).
2. Define scalar multiplication (check it maps $\mathbb{F} \times V \to V$).
3. Verify all 8 axioms.

Common shortcuts:

- If V inherits operations from a known vector space (like $\mathbb{F}^n$ or $\mathbb{F}^S$), most axioms follow automatically.
- Commutativity and associativity often follow from the corresponding properties in $\mathbb{F}$.

Strategy for proving V is NOT a vector space: Find ONE axiom that fails and provide a specific counterexample.
Common failures:

- No additive identity ($0 \notin V$)
- No additive inverses ($-v \notin V$ for some v)
- Not closed under addition ($u + v \notin V$)
- Not closed under scalar multiplication ($\lambda v \notin V$)
- Multiplicative identity fails ($1v \neq v$)

Key Takeaways

1. **Vector space axioms (CAAIMD^2):** 8 properties defining addition and scalar multiplication
2. **Uniqueness:** Additive identity and inverses are unique
3. **Key consequences:** $0v = 0$, $a0 = 0$, $(-1)v = -v$
4. **Verification strategy:** Check all 8 axioms (or find one that fails for non-examples)

Relevant Exercises

Practice these problems from LADR to reinforce the material:

- Section 1B: 1, 2, 3, 4, 5, 6

Common Problem Types:

Prove a property from axioms
Start with one side. Apply axioms step-by-step. Justify each step by naming the axiom used.

Prove uniqueness
Assume two objects satisfy the definition. Show they must be equal using the defining property.

Verify a set is a vector space
Define operations clearly. Verify closure. Check all 8 axioms (use shortcuts when operations are inherited).

Show a set is NOT a vector space
Find one axiom that fails. Give a specific counterexample with explicit values.