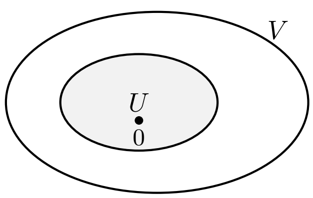
Section 1C: Subspaces

1. Introduction to Subspaces

Often we want to find vector spaces that "live inside" a larger vector space. For example, lines and planes through the origin in $\mathbb{R}^3$ turn out to be vector spaces in their own right. The key idea is that a subset can inherit the vector space structure from its parent.

1.33 Definition: Subspace
A subset U of V is called a **subspace** of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Key point: A subspace uses the *same* operations as the parent space. We don't invent new addition or scalar multiplication—we just restrict to a subset.



A subspace U sits inside V and contains the origin.

Note: The additive identity of a subspace must be the same as the additive identity of the larger space. If 0_U is the additive identity of U and 0 is the additive identity of V, then for any $v \in U$:

$$0_U = 0_U + 0 = 0 + 0_U = 0$$

So subspaces always contain the zero vector of V.

2. The Subspace Test

Checking all 8 vector space axioms would be tedious. Fortunately, most axioms are inherited automatically. The following result gives us a simpler test.

1.34 Conditions for a Subspace
A subset U of V is a subspace of V if and only if U satisfies the following three conditions:
additive identity:

$$0 \in U$$

closed under addition:

$$u, w \in U \implies u + w \in U$$

closed under scalar multiplication:

$$a \in \mathbb{F} \text{ and } u \in U \implies au \in U$$

Proof sketch:

($\Rightarrow$) If U is a subspace, it's a vector space, so it has an additive identity. Since the additive identity is unique in V (by 1.26), it must be 0. Closure under addition and scalar multiplication follow because these are operations on U.

($\Leftarrow$) Suppose U satisfies the three conditions. We verify U is a vector space:

- The three conditions give us $0 \in U$, closure under $+$, and closure under scalar multiplication.
- For $u \in U$: $-u = (-1)u \in U$ by closure under scalar multiplication. So additive inverses exist in U.
- Commutativity, associativity, distributivity, and the multiplicative identity hold because they hold in V and $U \subseteq V$.

□

Alternative condition: Instead of checking $0 \in U$, you can check that U is **nonempty**. Here's why:
If $U \neq \emptyset$, pick any $u \in U$. By closure under scalar multiplication:

$$0 \cdot u = 0 \in U$$

So "U is nonempty" $\Leftrightarrow$ "$0 \in U$" when the other conditions hold.

3. Examples of Subspaces

1.35(a) Example: $\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$
For which values of $b \in \mathbb{F}$ is this set a subspace of $\mathbb{F}^4$?
Answer: Only when $b = 0$.
If $b = 0$: The set is $\{(x_1, x_2, x_3, x_4) : x_3 = 5x_4\}$.

- $0 = (0, 0, 0, 0)$ satisfies $0 = 5 \cdot 0$. ✓
- Closed under $+$: If $x_3 = 5x_4$ and $y_3 = 5y_4$, then $x_3 + y_3 = 5(x_4 + y_4)$. ✓
- Closed under scalar mult: If $x_3 = 5x_4$, then $\lambda x_3 = 5(\lambda x_4)$. ✓

If $b \neq 0$: The zero vector $(0, 0, 0, 0)$ does not satisfy $0 = 5 \cdot 0 + b = b$. So $0 \notin U$, and U is not a subspace.

Lesson: Subspaces must pass through the origin. A constraint like $x_3 = 5x_4 + b$ with $b \neq 0$ defines an "affine subspace" (a shifted subspace), not a true subspace.

1.35(b) Example: Continuous Functions
The set of continuous real-valued functions on $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$ (the space of all functions from $[0, 1]$ to $\mathbb{R}$).
Verification:

- The zero function $0(x) = 0$ is continuous. ✓
- Sum of continuous functions is continuous. ✓
- Scalar multiple of a continuous function is continuous. ✓

1.35(c) Example: Differentiable Functions
The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.
Verification:

- The zero function is differentiable (with derivative 0). ✓
- Sum of differentiable functions is differentiable. ✓
- Scalar multiple of a differentiable function is differentiable. ✓

Note: We also have a chain of subspaces:

$$\{\text{polynomials}\} \subseteq \{C^\infty\} \subseteq \{C^1\} \subseteq \{C^0\} \subseteq \mathbb{R}^{\mathbb{R}}$$

1.35(d) Example: $\{f \in \mathbb{R}^{\mathbb{R}} :$ f differentiable and $f'(2) = b\}$
For which values of $b \in \mathbb{R}$ is this a subspace?
Answer: Only when $b = 0$.
If $b = 0$: This is the set of differentiable functions whose derivative vanishes at 2.

- The zero function has $0'(2) = 0$. ✓
- If $f'(2) = 0$ and $g'(2) = 0$, then $(f + g)'(2) = 0$. ✓
- If $f'(2) = 0$, then $(\lambda f)'(2) = \lambda \cdot 0 = 0$. ✓

If $b \neq 0$: The zero function has $0'(2) = 0 \neq b$, so $0 \notin U$.

1.35(e) Example: Sequences Converging to 0
The set

$$\{(a_1, a_2, \ldots) \in \mathbb{C}^\infty : \lim_{n\to\infty} a_n = 0\}$$

is a subspace of $\mathbb{C}^\infty$.
Verification:

- The zero sequence $(0, 0, \ldots)$ has limit 0. ✓
- If $\lim a_n = 0$ and $\lim b_n = 0$, then $\lim(a_n + b_n) = 0$. ✓
- If $\lim a_n = 0$, then $\lim(\lambda a_n) = \lambda \cdot 0 = 0$. ✓

Extreme subspaces: Every vector space V has two "trivial" subspaces:

- The **smallest subspace**: $\{0\}$ (just the zero vector)
- The **largest subspace**: V itself

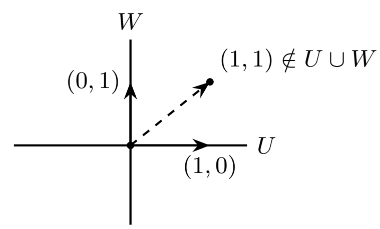
Why the empty set is not a subspace: $\emptyset$ fails the condition $0 \in U$. Every subspace must contain the zero vector.

4. Sums of Subspaces

Given subspaces $U_1, \ldots, U_m$ of V, we want to build new subspaces from them. The natural candidate—the union—usually fails.

Why not unions? If U and W are subspaces of V, then $U \cup W$ is *usually not* a subspace. For example, in $\mathbb{R}^2$:

- Let $U = \{(x, 0) : x \in \mathbb{R}\}$ (the x-axis)
- Let $W = \{(0, y) : y \in \mathbb{R}\}$ (the y-axis)
- Then $(1, 0) \in U$ and $(0, 1) \in W$
- But $(1, 0) + (0, 1) = (1, 1) \notin U \cup W$



Union fails closure under +

The union is not closed under addition! We need something else.

1.36 Definition: Sum of Subspaces
Suppose $U_1, \ldots, U_m$ are subspaces of V. The **sum** of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$:

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_j \in U_j\}$$

Key insight: The sum $U_1 + \cdots + U_m$ is the *smallest* subspace of V containing all of $U_1, \ldots, U_m$.

Verification that the sum is a subspace:

- $0 = 0 + \cdots + 0 \in U_1 + \cdots + U_m$ ✓
- Closed under addition: $(u_1 + \cdots + u_m) + (v_1 + \cdots + v_m) = (u_1 + v_1) + \cdots + (u_m + v_m)$, and each $u_j + v_j \in U_j$. ✓
- Closed under scalar mult: $\lambda(u_1 + \cdots + u_m) = \lambda u_1 + \cdots + \lambda u_m$, and each $\lambda u_j \in U_j$. ✓

1.37 Example: Sum of Subspaces of $\mathbb{F}^3$
Let $U = \{(x, 0, 0) \in \mathbb{F}^3 : x \in \mathbb{F}\}$ (the x-axis).
Let $W = \{(0, y, 0) \in \mathbb{F}^3 : y \in \mathbb{F}\}$ (the y-axis).

Then:

$$U + W = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$$

This is the xy-plane! A general element of $U + W$ is:

$$(x, 0, 0) + (0, y, 0) = (x, y, 0)$$

1.38 Example: Sum of Subspaces of $\mathbb{F}^4$
Let $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$.
Let $W = \{(x, x, x, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$.
Claim: $U + W = \{(x, x, y, z) \in \mathbb{F}^4 : x, y, z \in \mathbb{F}\}$.
Proof of $\subseteq$: Take $u \in U$ and $w \in W$:

$$\begin{aligned} u &= (a, a, b, b) \quad \text{for some } a, b \in \mathbb{F} \\ w &= (c, c, c, d) \quad \text{for some } c, d \in \mathbb{F} \\ u + w &= (a + c, a + c, b + c, b + d) \end{aligned}$$

Notice that the first two coordinates are equal. So $u + w$ has the form (x, x, y, z).
Proof of $\supseteq$: (1.39) Given $(x, x, y, z) \in \mathbb{F}^4$, we want to write it as $u + w$ with $u \in U$ and $w \in W$.
Choose:

$$\begin{aligned} &u = (x, x, y, y) \in U \\ &w = (0, 0, 0, z - y) \in W \quad \text{(since } (0, 0, 0, z - y) \text{ has form } (t, t, t, s) \text{ with } t = 0) \end{aligned}$$

Then $u+w = (x, x, y, y)+(0, 0, 0, z-y) = (x, x, y, z)$. ✓

5. Sum is Smallest Containing Subspace

1.40 Sum of Subspaces is the Smallest Containing Subspace
Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

What "smallest" means: A subspace W is the smallest subspace containing $U_1, \ldots, U_m$ if:

1. $U_j \subseteq W$ for all $j = 1, \ldots, m$
2. If S is any subspace containing all U_j, then $W \subseteq S$

Proof:
First, we show that each $U_j \subseteq U_1 + \cdots + U_m$:
For any $u \in U_j$, we can write:

$$u = 0 + \cdots + 0 + u + 0 + \cdots + 0$$

where u is in the j-th position. Since $0 \in U_k$ for all $k \neq j$, this shows $u \in U_1 + \cdots + U_m$.

Second, we show that $U_1 + \cdots + U_m$ is contained in any subspace S that contains all U_j:

Suppose S is a subspace of V containing each U_j. Take any element $u_1 + \cdots + u_m \in U_1 + \cdots + U_m$.

Since $u_j \in U_j \subseteq S$ for each j, and S is closed under addition, we have:

$$u_1 + \cdots + u_m \in S$$

Therefore $U_1 + \cdots + U_m \subseteq S$. □

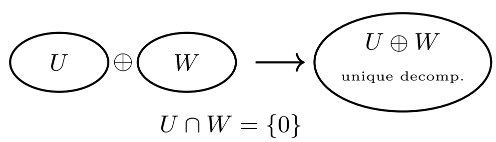
6. Direct Sums

Given a sum $U_1 + \cdots + U_m$, we know every element can be written as $u_1 + \cdots + u_m$ with $u_j \in U_j$. But can an element be written in *multiple* ways? When the representation is always **unique**, the sum is called a **direct sum**.

1.41 Definition: Direct Sum
Suppose $U_1, \ldots, U_m$ are subspaces of V.

- The sum $U_1 + \cdots + U_m$ is called a **direct sum** if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where each $u_j \in U_j$.
- If $U_1 + \cdots + U_m$ is a direct sum, then we write $U_1 \oplus \cdots \oplus U_m$ instead of $U_1 + \cdots + U_m$.

Notation: The symbol $\oplus$ ("direct sum") signals that every element has a *unique* decomposition into summands from each subspace.



Direct sum: unique decomposition when subspaces meet only at 0.

1.42 Example: $\mathbb{F}^3$ as a Direct Sum
Let $U = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$ (the xy-plane).
Let $W = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\}$ (the z-axis).
Claim: $\mathbb{F}^3 = U \oplus W$.
Proof: Take any $(x, y, z) \in \mathbb{F}^3$. We can write:

$$(x, y, z) = (x, y, 0) + (0, 0, z)$$

where $(x, y, 0) \in U$ and $(0, 0, z) \in W$. This shows $\mathbb{F}^3 = U + W$.
Is this decomposition unique? Suppose $(x, y, z) = (a, b, 0) + (0, 0, c)$ for some $a, b, c \in \mathbb{F}$.
Then $(x, y, z) = (a, b, c)$, which forces $a = x$, $b = y$, $c = z$.
So the decomposition is unique, and $\mathbb{F}^3 = U \oplus W$. ✓

1.43 Example: $\mathbb{F}^n$ as Direct Sum of Coordinate Axes
For $j = 1, \ldots, n$, define:

$$U_j = \{(0, \ldots, 0, x, 0, \ldots, 0) \in \mathbb{F}^n : x \in \mathbb{F}\}$$

where x is in the j-th coordinate (all other coordinates are 0).
Then:

$$\mathbb{F}^n = U_1 \oplus U_2 \oplus \cdots \oplus U_n$$

Why? Any $(x_1, \ldots, x_n) \in \mathbb{F}^n$ can be written uniquely as:

$$(x_1, 0, \ldots, 0) + (0, x_2, 0, \ldots, 0) + \cdots + (0, \ldots, 0, x_n)$$

The coordinates completely determine each summand, so the decomposition is unique.

7. Testing for Direct Sums

How can we tell if a sum is direct? The following result gives a simple test.

1.44 Example: A Sum That is NOT Direct
Consider $\mathbb{F}^3$ and define:

$$U_1 = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\} \quad \text{(the } xy\text{-plane)}$$
$$U_2 = \{(0, y, z) \in \mathbb{F}^3 : y, z \in \mathbb{F}\} \quad \text{(the } yz\text{-plane)}$$
$$U_3 = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\} \quad \text{(the } z\text{-axis)}$$

Then $U_1 + U_2 + U_3 = \mathbb{F}^3$, but this sum is **not direct**.
Why not? The zero vector $0 \in \mathbb{F}^3$ can be written as a sum of elements from U_1, U_2, U_3 in *multiple* ways:

$$0 = 0 + 0 + 0 \quad \text{(the obvious way)}$$
$$0 = (0, 1, 0) + (0, -1, 0) + 0 \quad \text{(another way!)}$$

Since $(0, 1, 0) \in U_1$, $(0, -1, 0) \in U_2$, and $0 \in U_3$, the second decomposition is valid. The representation of 0 is not unique, so this is not a direct sum.

The previous example suggests a key insight: to check if a sum is direct, we only need to check if 0 has a unique representation.

1.45 Condition for a Direct Sum
Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each $u_j \in U_j$, is by taking each $u_j = 0$.

Proof:
($\Rightarrow$) If the sum is direct, then every element has a unique representation. Since $0 = 0 + \cdots + 0$ is one representation of 0, it must be the only one.

($\Leftarrow$) Suppose the only way to write 0 is $0 + \cdots + 0$. We show the representation of any element is unique.

Take $v \in U_1 + \cdots + U_m$ and suppose:

$$v = u_1 + \cdots + u_m = w_1 + \cdots + w_m$$

where $u_j, w_j \in U_j$ for each j.

Subtracting:

$$0 = (u_1 - w_1) + \cdots + (u_m - w_m)$$

Since $u_j - w_j \in U_j$ (each U_j is a subspace), and the only way to write 0 is with all summands equal to 0:

$$u_j - w_j = 0 \quad \text{for all } j$$

Therefore $u_j = w_j$ for all j, proving uniqueness. □

For the case of *two* subspaces, there's an even simpler criterion:

1.46 Direct Sum of Two Subspaces
Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof:
($\Rightarrow$) Suppose $U + W$ is a direct sum. Take any $v \in U \cap W$.

Then $v \in U$ and $v \in W$. Also $-v \in W$ (since W is a subspace).

We can write:

$$0 = v + (-v)$$

where $v \in U$ and $-v \in W$. By 1.45, the only way to write 0 is $0 + 0$, so $v = 0$.

Therefore $U \cap W = \{0\}$.

($\Leftarrow$) Suppose $U \cap W = \{0\}$. We show the only way to write $0 = u + w$ with $u \in U$, $w \in W$ is $u = w = 0$.

If $0 = u + w$, then $u = -w$. Since $u \in U$ and $-w \in W$ (and $u = -w$), we have $u \in U \cap W = \{0\}$.

So $u = 0$, which means $w = -u = 0$. By 1.45, the sum is direct. □

Warning: Pairwise Intersection is Not Enough
For three or more subspaces, having $U_i \cap U_j = \{0\}$ for all pairs $i \neq j$ does **NOT** guarantee a direct sum.
Example: In 1.44, notice that:

- $U_1 \cap U_2 = \{(0, y, 0) : y \in \mathbb{F}\}$ (the y-axis) $\neq \{0\}$

But even if we modify the example to have pairwise intersections equal to $\{0\}$, the sum might still not be direct. The condition in 1.45 (uniqueness of the zero decomposition) is the correct test for $m \geq 3$

subspaces.

Strategy: How to Check if U is a Subspace

The Subspace Checklist:
To verify that $U \subseteq V$ is a subspace, check three things:

1. **Zero vector:** Show $0 \in U$.
   (Alternatively, show $U \neq \emptyset$.)
2. **Closed under addition:** Take *arbitrary* $u, w \in U$. Show that $u + w \in U$.
   (Use the definition of U to verify the sum satisfies the defining property.)
3. **Closed under scalar multiplication:** Take *arbitrary* $a \in \mathbb{F}$ and $u \in U$. Show that $au \in U$.

Common mistakes to avoid:

- Don't use specific vectors; use *arbitrary* elements.
- Don't forget to check $0 \in U$ (or nonemptiness).
- Remember: the operations come from V, not something new.

Quick tests for NON-subspaces:
A subset $U \subseteq V$ is **NOT** a subspace if any of these hold:

- $0 \notin U$ (e.g., $\{x : x_1 = 1\}$)
- Not closed under $+$ (find $u, w \in U$ with $u+w \notin U$)
- Not closed under scalar mult (find $a \in \mathbb{F}$, $u \in U$ with $au \notin U$)

Rule of thumb: Constraints of the form "$= b$" with $b \neq 0$ usually fail the subspace test because 0 won't satisfy the constraint.

Key Results Summary

Definitions:

- **Subspace** (1.33): A subset that is itself a vector space with the inherited operations
- **Sum of subspaces** (1.36): $U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_j \in U_j\}$
- **Direct sum** (1.41): A sum where each element has a *unique* representation; written $U_1 \oplus \cdots \oplus U_m$

The Subspace Test (1.34):
U is a subspace of V $\Leftrightarrow$

1. $0 \in U$
2. $u, w \in U \Rightarrow u + w \in U$
3. $a \in \mathbb{F}, u \in U \Rightarrow au \in U$

Direct Sum Tests:

- (1.45) $U_1 + \cdots + U_m$ is direct $\Leftrightarrow$ the only way to write $0 = u_1 + \cdots + u_m$ is with all $u_j = 0$

- (1.46) For two subspaces: $U + W$ is direct $\Leftrightarrow$ $U \cap W = \{0\}$

Key Facts:

- Every subspace contains 0
- $\{0\}$ and V are always subspaces of V
- The empty set is never a subspace
- Sums of subspaces are subspaces
- Unions of subspaces are usually NOT subspaces
- (1.40) $U_1 + \cdots + U_m$ is the smallest subspace containing all U_j

Common Problem Types:

Determine if U is a subspace
Use the three-condition test. Check $0 \in U$, closure under $+$, closure under scalar mult.

For which b is U a subspace?
Usually $b = 0$. Check whether 0 satisfies the defining condition.

Describe $U + W$
Write a general element as $u + w$ where $u \in U$, $w \in W$. Simplify to find the pattern.

Prove $U + W =$ some set S
Show $U + W \subseteq S$ (every sum has the right form) and $S \subseteq U + W$ (every element of S can be written as a sum).

Prove a sum is direct
For two subspaces: show $U \cap W = \{0\}$.
For multiple subspaces: show $u_1 + \cdots + u_m = 0$ implies all $u_j = 0$.

Show a sum is NOT direct
Find a nonzero way to write $0 = u_1 + \cdots + u_m$ with $u_j \in U_j$.

Relevant Exercises

Practice these problems from LADR to reinforce the material:

- Section 1C: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24