

# Problem Set X: Topic Name

## Chapter Y Exercises

**Instructions:** Show all work. Justify each step.

### Part I: Computation

**Problem 1.** Let  $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ .

- (a) Find the eigenvalues of  $\mathbf{A}$ .
- (b) Find a basis for each eigenspace.
- (c) Is  $\mathbf{A}$  diagonalizable? If so, find matrices  $\mathbf{P}$  and  $\mathbf{D}$ .

*Solution.* (a) The characteristic polynomial is:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = (2 - \lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3).$$

So  $\lambda_1 = 1, \lambda_2 = 3$ .

(b) For  $\lambda = 1$ : Solve  $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0}$ .

$$E_1 = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

For  $\lambda = 3$ : Solve  $(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \mathbf{0}$ .

$$E_3 = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

(c) Yes,  $\mathbf{A}$  is diagonalizable since we have 2 linearly independent eigenvectors.

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

□

### Part II: Proofs

**Problem 2.** Prove that if  $T : V \rightarrow V$  is a linear operator and  $T^2 = T$ , then  $V = \ker(T) \oplus \text{im}(T)$ .

**Hint:** For any  $\mathbf{v} \in V$ , write  $\mathbf{v} = (\mathbf{v} - T(\mathbf{v})) + T(\mathbf{v})$ .

*End of Problem Set*