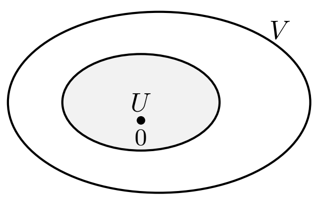
Section 1C: Subspaces

1. Introduction to Subspaces

Often we want to find vector spaces that "live inside" a larger vector space. For example, lines and planes through the origin in $\mathbb{R}^3$ turn out to be vector spaces in their own right. The key idea is that a subset can inherit the vector space structure from its parent.

1.33 Definition: Subspace
A subset U of V is called a **subspace** of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Key point: A subspace uses the *same* operations as the parent space. We don't invent new addition or scalar multiplication—we just restrict to a subset.



A subspace U sits inside V and contains the origin.

Note: The additive identity of a subspace must be the same as the additive identity of the larger space. If 0_U is the additive identity of U and 0 is the additive identity of V, then for any $v \in U$:

$$0_U = 0_U + 0 = 0 + 0_U = 0$$

So subspaces always contain the zero vector of V.

2. The Subspace Test

Checking all 8 vector space axioms would be tedious. Fortunately, most axioms are inherited automatically. The following result gives us a simpler test.

1.34 Conditions for a Subspace
A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

additive identity:

$$0 \in U$$

closed under addition:

$$u, w \in U \implies u + w \in U$$

closed under scalar multiplication:

$$a \in \mathbb{F} \text{ and } u \in U \implies au \in U$$

Proof:

($\Rightarrow$) *Subspace $\implies$ three conditions.* **(Easy direction.)**

If U is a subspace, it is already a vector space. So it automatically has a zero vector, is closed under addition, and is closed under scalar multiplication. By uniqueness of the additive identity in V (Result 1.26), the zero vector of U must be 0. So $0 \in U$.

($\Leftarrow$) *Three conditions $\implies$ subspace.* **(Important direction.)**

Suppose U satisfies the three conditions. We verify every vector space axiom:

- **Step 1: Additive identity.** $0 \in U$ by assumption.
- **Step 2: Closure under addition.** Closure under addition means that if $u, w \in U$, then $u + w \in U$. Addition stays inside U.
- **Step 3: Closure under scalar multiplication.** Closure under scalar multiplication means that if $a \in \mathbb{F}$ and $u \in U$, then $au \in U$. Scalar multiplication stays inside U.
- **Step 4: Additive inverses.** For any $u \in U$, we need $-u \in U$. Observe that

  $$-u = (-1)u.$$

  Since $-1 \in \mathbb{F}$ and U is closed under scalar multiplication, $(-1)u \in U$. So additive inverses exist in U.
- **Step 5: Remaining axioms.** Commutativity, associativity, distributivity, and $1u = u$ are properties of *elements*, not of the set. Since $U \subseteq V$, every element of U is an element of V, and these equations already hold in V. So they hold in U automatically.

All vector space axioms hold in U, so U is a subspace of V. $\square$

Big picture: This proof explains why the Subspace Test works. It reduces checking ∼8 vector space axioms down to just 3 simple checks: zero, closed under $+$, closed under scalar $\times$. That is the test you will use constantly in Math 110.
Why not reduce further? The ∼5 axioms we skip (commutativity, associativity, distributivity, etc.) are **equations about elements**. If $u, w \in U \subseteq V$, then $u + w = w + u$ is already true because it is true in V.

You cannot "break" commutativity by restricting to a subset. These are *element properties*, and they are inherited automatically.
The 3 checks we keep are different — they ask whether the **output** of an operation lands back in U. That is a property of the *set*, and V cannot guarantee it for you:

- **Zero vector:** Without $0 \in U$, the set could be empty (and the empty set is not a vector space). This is the existence check.
- **Closed under $+$:** V guarantees $u + w$ exists *somewhere* in V, but not that it stays in U. For example, $U = \{(1,0), (0,1)\} \subset \mathbb{R}^2$ has $(1,0) + (0,1) = (1,1) \notin U$.
- **Closed under scalar $\times$:** Similarly, λu exists in V but might leave U. For example, $U = \{(1,0)\}$ has $2 \cdot (1,0) = (2,0) \notin U$.

The fundamental distinction: element properties are inherited; set properties are not.

Alternative condition: Instead of checking $0 \in U$, you can check that U is **nonempty**. Here's why:
If $U \neq \emptyset$, pick any $u \in U$. By closure under scalar multiplication:

$$0 \cdot u = 0 \in U$$

So "U is nonempty" $\Leftrightarrow$ "$0 \in U$" when the other conditions hold.

Worked Example: Is $U = \{(x, 2x) : x \in \mathbb{R}\}$ a subspace of $\mathbb{R}^2$?

Geometrically, U is the line $y = 2x$, which passes through the origin.

Check the three conditions:

1. **Zero vector:** Setting $x = 0$ gives $(0,0) \in U$. ✓
2. **Closed under addition:** Take $(a, 2a), (b, 2b) \in U$. Then
$$(a,2a)+(b,2b) = (a+b,\ 2a+2b) = \underbrace{(a+b,\ 2(a+b))}_{\text{has the form } (x,2x)} \in U. \quad ✓$$
3. **Closed under scalar multiplication:** Take $\lambda \in \mathbb{R}$ and $(a, 2a) \in U$. Then
$$\lambda(a,2a) = (\lambda a,\ 2\lambda a) = \underbrace{(\lambda a,\ 2(\lambda a))}_{\text{has the form } (x,2x)} \in U. \quad ✓$$

Conclusion: U is a subspace of $\mathbb{R}^2$.

Remark: Any line through the origin in $\mathbb{R}^2$ is a subspace. A line that does *not* pass through the origin, such as $\{(x, 2x+1) : x \in \mathbb{R}\}$, fails the zero vector condition and is *not* a subspace.

3. Examples of Subspaces

1.35(a) Example: $\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$
Let $b \in \mathbb{F}$ and define

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}.$$

We will show that U is a subspace of $\mathbb{F}^4$ if and only if $b = 0$.

Step 0: $U \subseteq V$: Every element of U is a 4-tuple in $\mathbb{F}^4$, so $U \subseteq \mathbb{F}^4$. By the subspace test, U is a subspace of $\mathbb{F}^4$ if and only if it contains the zero vector and is closed under addition and scalar multiplication.

Direction 1 ($b \neq 0$): U is not a subspace.

Step 1: Check the zero vector. The zero vector in $\mathbb{F}^4$ is $\mathbf{0} = (0, 0, 0, 0)$. We need $x_3 = 5x_4 + b$, so plugging in gives

$$0 = 5 \cdot 0 + b = b.$$

Thus $\mathbf{0} \in U$ if and only if $b = 0$. If $b \neq 0$, then $\mathbf{0} \notin U$, so U is *not* a subspace. ×

Direction 2 ($b = 0$): U is a subspace.

Now $U = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4\}$. We verify the three subspace conditions.

Step 2: Contains $\mathbf{0}$. The vector $(0, 0, 0, 0)$ satisfies $0 = 5 \cdot 0$, so $\mathbf{0} \in U$. ✓

Step 3: Closed under addition. Let $u = (x_1, x_2, x_3, x_4)$ and $w = (y_1, y_2, y_3, y_4)$ be elements of U, so $x_3 = 5x_4$ and $y_3 = 5y_4$. Then

$$u + w = (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3,\ x_4 + y_4).$$

The third component of $u + w$ is $x_3 + y_3 = 5x_4 + 5y_4 = 5(x_4 + y_4)$, which equals 5 times the fourth component. Hence $u + w \in U$. ✓

Step 4: Closed under scalar multiplication. Let $a \in \mathbb{F}$ and $u = (x_1, x_2, x_3, x_4) \in U$, so $x_3 = 5x_4$. Then

$$au = (ax_1,\ ax_2,\ ax_3,\ ax_4).$$

The third component of au is $ax_3 = a(5x_4) = 5(ax_4)$, which equals 5 times the fourth component. Hence $au \in U$. ✓

Conclusion: All three conditions are satisfied when $b = 0$, so U is a subspace of $\mathbb{F}^4$ if and only if $b = 0$.

Lesson: Subspaces must pass through the origin. A constraint like $x_3 = 5x_4 + b$ with $b \neq 0$ defines an "affine subspace" (a shifted subspace), not a true subspace.

1.35(b) Example: Continuous Functions
Let $V = \mathbb{R}^{[0,1]}$ be the vector space of all real-valued functions on $[0, 1]$, and let

$$U = C([0, 1]) = \{\text{continuous real-valued functions on } [0, 1]\}.$$

We will show that U is a subspace of V.

Step 0: Every continuous function $f\colon [0, 1] \to \mathbb{R}$ is certainly a function $[0, 1] \to \mathbb{R}$, so $U \subseteq V$. By the subspace test, it suffices to verify three conditions.

Step 1: Check the zero vector. In $V = \mathbb{R}^{[0,1]}$, the zero vector is the *zero function*:

$$f_0(x) = 0 \quad \text{for all } x \in [0, 1].$$

This function is continuous on $[0, 1]$, so $f_0 \in U$. ✓

Step 2: Check closure under addition. Let $f, g \in U$, so f and g are continuous on $[0, 1]$. Their sum $f + g$ is defined by

$$(f + g)(x) = f(x) + g(x).$$

By a standard fact from calculus, the sum of continuous functions is continuous. So $f + g$ is continuous on $[0, 1]$, hence $f + g \in U$. ✓

Step 3: Check closure under scalar multiplication. Let $a \in \mathbb{R}$ and $f \in U$, so f is continuous on $[0, 1]$. The scalar multiple af is defined by

$$(af)(x) = a \cdot f(x).$$

By a standard fact from calculus, a constant times a continuous function is continuous. So af is continuous on $[0, 1]$, hence $af \in U$. ✓

Conclusion: Since U contains the zero function and is closed under addition and scalar multiplication, $C([0, 1])$ is a subspace of $\mathbb{R}^{[0,1]}$.

1.35(c) Example: Differentiable Functions
The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.

Step 0: Identify the spaces. Let $V = \mathbb{R}^{\mathbb{R}}$ be the vector space of all functions $\mathbb{R} \to \mathbb{R}$, and let

$$U = \{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is differentiable on } \mathbb{R}\}.$$

Every differentiable function $\mathbb{R} \to \mathbb{R}$ is certainly a function $\mathbb{R} \to \mathbb{R}$, so $U \subseteq V$. By the subspace test, it suffices to verify three conditions.

Step 1: Check the zero vector. In $V = \mathbb{R}^{\mathbb{R}}$, the zero vector is the *zero function*:

$$f_0(x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

This function is differentiable everywhere, with $f_0'(x) = 0$ for all $x \in \mathbb{R}$. So $f_0 \in U$. ✓

Step 2: Check closure under addition. Let $f, g \in U$, so f and g are differentiable on $\mathbb{R}$. Their sum $f + g$ is defined by

$$(f + g)(x) = f(x) + g(x).$$

By the *sum rule* from calculus, $(f+g)'(x) = f'(x)+g'(x)$. Since both f' and g' exist for all $x \in \mathbb{R}$, the derivative of $f + g$ exists everywhere. So $f + g$ is differentiable, hence $f + g \in U$. ✓

Step 3: Check closure under scalar multiplication. Let $a \in \mathbb{R}$ and $f \in U$, so f is differentiable on $\mathbb{R}$. The scalar multiple af is defined by

$$(af)(x) = a \cdot f(x).$$

By the *constant multiple rule* from calculus, $(af)'(x) = a \cdot f'(x)$. Since f' exists for all $x \in \mathbb{R}$, the derivative of af exists everywhere. So af is differentiable, hence $af \in U$. ✓

Conclusion: Since U contains the zero function and is closed under addition and scalar multiplication, the set of differentiable functions is a subspace of $\mathbb{R}^{\mathbb{R}}$.

Chain of subspaces: Each level of smoothness gives a subspace of the next:

$$\{\text{polynomials}\} \subseteq C^\infty \subseteq \cdots \subseteq C^1 \subseteq C^0 \subseteq \mathbb{R}^{\mathbb{R}}$$

where C^0 = continuous functions, C^1 = once-differentiable, ..., C^∞ = infinitely differentiable (smooth). Each inclusion holds because every differentiable function is continuous, every twice-differentiable function is differentiable, and so on.

Big-picture reason: Differentiability is preserved under linear combinations: if f and g are differentiable and $a, b \in \mathbb{R}$, then $af+bg$ is differentiable. Any property preserved under linear combinations gives a subspace.

1.35(d) Example: $\{f \in \mathbb{R}^{(0,3)} : f$ differentiable and $f'(2) = b\}$

Let $b \in \mathbb{R}$ and define

$$U_b = \{f \in \mathbb{R}^{(0,3)} : f \text{ is differentiable on } (0,3) \text{ and } f'(2) = b\}.$$

Let $V = \mathbb{R}^{(0,3)}$ be the vector space of all functions $(0,3) \to \mathbb{R}$. We will show that U_b is a subspace of V if and only if $b = 0$.

Step 0: $U_b \subseteq V$: Every differentiable function $(0,3) \to \mathbb{R}$ is certainly a function $(0,3) \to \mathbb{R}$, so $U_b \subseteq V$.

Direction 1 ($b \neq 0$): U_b is not a subspace.

Step 1: Check the zero vector. In $V = \mathbb{R}^{(0,3)}$, the zero vector is the zero function $f_0(x) = 0$ for all $x \in (0,3)$. Its derivative is $f_0'(x) = 0$ for all x, so $f_0'(2) = 0$. For $f_0 \in U_b$ we need $f_0'(2) = b$, i.e. $0 = b$. If $b \neq 0$, then $f_0 \notin U_b$, so U_b does not contain the zero vector and is *not* a subspace. ×

Direction 2 ($b = 0$): U_0 is a subspace.

Now $U_0 = \{f \in \mathbb{R}^{(0,3)} : f \text{ is differentiable on } (0,3) \text{ and } f'(2) = 0\}$. We verify the three subspace conditions.

Step 2: Contains the zero vector. The zero function $f_0(x) = 0$ has $f_0'(2) = 0$, so $f_0 \in U_0$. ✓

Step 3: Closed under addition. Let $f, g \in U_0$, so f and g are differentiable on $(0,3)$ with $f'(2) = 0$ and $g'(2) = 0$. By the *sum rule* from calculus,

$$(f + g)'(2) = f'(2) + g'(2) = 0 + 0 = 0.$$

So $f+g$ is differentiable on $(0,3)$ with $(f+g)'(2) = 0$, hence $f + g \in U_0$. ✓

Step 4: Closed under scalar multiplication. Let $a \in \mathbb{R}$ and $f \in U_0$, so f is differentiable on $(0,3)$ with $f'(2) = 0$. By the *constant multiple rule* from calculus,

$$(af)'(2) = a \cdot f'(2) = a \cdot 0 = 0.$$

So af is differentiable on $(0,3)$ with $(af)'(2) = 0$, hence $af \in U_0$. ✓

Conclusion: All three conditions are satisfied when $b = 0$, so U_0 is a subspace of V. Combined with Direction 1: U_b is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.

Trick insight: The condition $f'(2) = b$ is a *linear* condition on the function f only when $b = 0$; when $b \neq 0$ it is an affine shift, so the zero function fails. This is the function-space analogue of Example (a), where $x_3 = 5x_4 + b$ is a subspace condition only when $b = 0$.

Proof tricks used:

- **"For which b?" $\Rightarrow$ iff proof.** When the problem asks "for which values of a parameter is this a subspace?", structure the proof as two directions: one showing failure, the other showing all three conditions hold.
- **Zero vector test kills $b \neq 0$ immediately.** Before checking closure properties, always test whether the zero vector belongs to the set. If it doesn't, you're done — the set is not a subspace. This is the fastest way to rule out non-subspaces.
- **Derivative rules do the heavy lifting.** The *sum rule* $(f+g)' = f' + g'$ gives closure under addition; the *constant multiple rule* $(af)' = a \cdot f'$ gives closure under scalar multiplication. In function-space subspace proofs, calculus rules translate directly into subspace conditions.
- **Evaluating at a point preserves linearity.** The key computation $f'(2) + g'(2) = 0 + 0 = 0$ works because differentiation is linear *and* evaluation at a point is linear. The condition "$f'(2) = 0$" composes two linear operations: differentiate, then evaluate at 2.
- **Homogeneous vs. non-homogeneous.** A condition of the form $L(f) = 0$ (where L is a linear operation) always gives a subspace. A condition $L(f) = b$ with $b \neq 0$ never does, because $L(f_0) = L(0) = 0 \neq b$. This pattern appears whenever a parameter b shifts the constraint away from zero.

1.35(e) Example: Sequences Converging to 0
Let $V = \mathbb{C}^\infty$ be the vector space of all complex sequences

$$(a_1, a_2, a_3, \dots), \quad a_n \in \mathbb{C}$$

with addition and scalar multiplication defined term-by-term. Let

$$U = \{(a_n) \in \mathbb{C}^\infty : \lim_{n\to\infty} a_n = 0\}.$$

Claim: U is a subspace of V.

Step 0: $U \subseteq V$: Every sequence with limit 0 is still a complex sequence, so $U \subseteq \mathbb{C}^\infty$.

Step 1: Check $\mathbf{0} \in U$. The zero vector in $\mathbb{C}^\infty$ is the zero sequence:

$$\mathbf{0} = (0, 0, 0, \dots).$$

Its limit is $\lim_{n\to\infty} 0 = 0$, so $\mathbf{0} \in U$. ✓

Step 2: Closed under addition. Take any $(a_n), (b_n) \in U$, so

$$\lim_{n\to\infty} a_n = 0, \qquad \lim_{n\to\infty} b_n = 0.$$

Consider their sum $(a_n + b_n)$. By the limit sum law:

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = 0 + 0 = 0.$$

So $(a_n + b_n) \in U$. ✓

Step 3: Closed under scalar multiplication. Take $(a_n) \in U$ and $c \in \mathbb{C}$, so $\lim a_n = 0$. By the limit scalar law:

$$\lim_{n\to\infty} (c\, a_n) = c \cdot \lim_{n\to\infty} a_n = c \cdot 0 = 0.$$

So $(c\, a_n) \in U$. ✓

Conclusion: Since U contains the zero sequence and is closed under addition and scalar multiplication, U is a subspace of $\mathbb{C}^\infty$.

Trick insight: Anything defined by "limit equals 0" tends to be a subspace because limits respect linear combinations:

$$\lim(c\, a_n + d\, b_n) = c \lim a_n + d \lim b_n.$$

Compare: the set of sequences with limit 1 is *not* a subspace — the zero sequence has limit $0 \neq 1$.

Pattern: Across all these examples, constraints of the form "something $= 0$" (homogeneous) tend to give subspaces, because the zero vector satisfies them and the arithmetic of 0 plays nicely with addition and scaling. Constraints of the form "something $= b$" with $b \neq 0$ (non-homogeneous) do *not* give subspaces, because they exclude the zero vector. This pattern appears in both finite-dimensional settings (Example (a): $x_3 = 5x_4 + b$) and function spaces (Example (d): $f'(2) = b$).

Extreme subspaces: Every vector space V has two "trivial" subspaces:

- The **smallest subspace**: $\{0\}$ (just the zero vector)
- The **largest subspace**: V itself

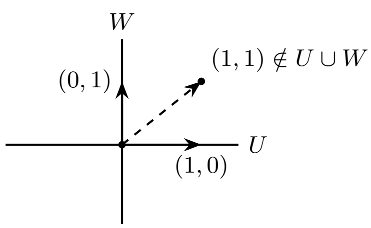
Why the empty set is not a subspace: $\emptyset$ fails the condition $0 \in U$. Every subspace must contain the zero vector.

4. Sums of Subspaces

Given subspaces $U_1, \ldots, U_m$ of V, we want to build new subspaces from them. The natural candidate—the union—usually fails.

Why not unions? If U and W are subspaces of V, then $U \cup W$ is *usually not* a subspace. For example, in $\mathbb{R}^2$:

- Let $U = \{(x, 0) : x \in \mathbb{R}\}$ (the x-axis)
- Let $W = \{(0, y) : y \in \mathbb{R}\}$ (the y-axis)
- Then $(1, 0) \in U$ and $(0, 1) \in W$
- But $(1, 0) + (0, 1) = (1, 1) \notin U \cup W$



Union fails closure under +

The union is not closed under addition! We need something else.

1.36 Definition: Sum of Subspaces
Suppose $U_1, \ldots, U_m$ are subspaces of V. The **sum** of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$:

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_j \in U_j\}$$

Notation: What does "|" or ":" mean?
The definition above uses **set-builder notation**. In

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_j \in U_j\},$$

the vertical bar "|" means **"such that."** You read it as:

> The set of all vectors of the form $u_1 + \cdots + u_m$ **such that** each $u_j \in U_j$.

In general, {expression | condition} means "all expressions where the condition holds." For example,

$$\{x \in \mathbb{R} \mid x > 0\} \quad = \quad \text{all real numbers such that } x > 0.$$

So in the sum-of-subspaces definition:

1. Pick one vector u_1 from U_1,
2. Pick one vector u_2 from U_2,
3. ...
4. Add them together,
5. Collect all possible such sums.

Remark: You will also see a colon in place of the bar: $\{x \in V : \text{something about } x\}$. Both "|" and ":" mean the same thing — "such that."

Key insight: The sum $U_1 + \cdots + U_m$ is the *smallest* subspace of V containing all of $U_1, \ldots, U_m$.

Verification that the sum is a subspace:

- $0 = 0 + \cdots + 0 \in U_1 + \cdots + U_m$ ✓
- Closed under addition: $(u_1 + \cdots + u_m) + (v_1 + \cdots + v_m) = (u_1 + v_1) + \cdots + (u_m + v_m)$, and each $u_j + v_j \in U_j$. ✓
- Closed under scalar mult: $\lambda(u_1 + \cdots + u_m) = \lambda u_1 + \cdots + \lambda u_m$, and each $\lambda u_j \in U_j$. ✓

1.37 Example: Sum of Subspaces of $\mathbb{F}^3$
Let $U = \{(x, 0, 0) \in \mathbb{F}^3 : x \in \mathbb{F}\}$ (the x-axis).
Let $W = \{(0, y, 0) \in \mathbb{F}^3 : y \in \mathbb{F}\}$ (the y-axis).
Then:

$$U + W = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$$

This is the xy-plane! A general element of $U + W$ is:

$$(x, 0, 0) + (0, y, 0) = (x, y, 0)$$

1.38 Example: Sum of Subspaces of $\mathbb{F}^4$
Let $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$.
Let $W = \{(x, x, x, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$.
Claim: $U + W = \{(x, x, y, z) \in \mathbb{F}^4 : x, y, z \in \mathbb{F}\}$.
Proof of $\subseteq$: Take $u \in U$ and $w \in W$:

$$\begin{aligned} u &= (a, a, b, b) \quad \text{for some } a, b \in \mathbb{F} \\ w &= (c, c, c, d) \quad \text{for some } c, d \in \mathbb{F} \\ u + w &= (a + c, a + c, b + c, b + d) \end{aligned}$$

Notice that the first two coordinates are equal. So $u + w$ has the form (x, x, y, z).
Proof of $\supseteq$: **(1.39)** Given $(x, x, y, z) \in \mathbb{F}^4$, we want to write it as $u + w$ with $u \in U$ and $w \in W$. Choose:

$$u = (x, x, y, y) \in U$$
$$w = (0, 0, 0, z - y) \in W \quad (\text{since } (0, 0, 0, z - y) \text{ has form } (t, t, t, s) \text{ w}$$

Then $u + w = (x, x, y, y) + (0, 0, 0, z - y) = (x, x, y, z)$. ✓

5. Sum is Smallest Containing Subspace

1.40 Sum of Subspaces is the Smallest Containing Subspace
Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

What "smallest" means: A subspace W is the smallest subspace containing $U_1, \ldots, U_m$ if:

1. $U_j \subseteq W$ for all $j = 1, \ldots, m$
2. If S is any subspace containing all U_j, then $W \subseteq S$

Proof:
First, we show that each $U_j \subseteq U_1 + \cdots + U_m$:
For any $u \in U_j$, we can write:

$$u = 0 + \cdots + 0 + u + 0 + \cdots + 0$$

where u is in the j-th position. Since $0 \in U_k$ for all $k \neq j$, this shows $u \in U_1 + \cdots + U_m$.

Second, we show that $U_1 + \cdots + U_m$ is contained in any subspace S that contains all U_j:

Suppose S is a subspace of V containing each U_j. Take any element $u_1 + \cdots + u_m \in U_1 + \cdots + U_m$.

Since $u_j \in U_j \subseteq S$ for each j, and S is closed under addition, we have:

$$u_1 + \cdots + u_m \in S$$

Therefore $U_1 + \cdots + U_m \subseteq S$. □

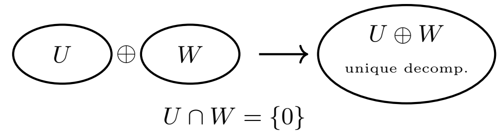
6. Direct Sums

Given a sum $U_1 + \cdots + U_m$, we know every element can be written as $u_1 + \cdots + u_m$ with $u_j \in U_j$. But can an element be written in *multiple* ways? When the representation is always **unique**, the sum is called a **direct sum**.

1.41 Definition: Direct Sum
Suppose $U_1, \ldots, U_m$ are subspaces of V.

- The sum $U_1 + \cdots + U_m$ is called a **direct sum** if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where each $u_j \in U_j$.
- If $U_1 + \cdots + U_m$ is a direct sum, then we write $U_1 \oplus \cdots \oplus U_m$ instead of $U_1 + \cdots + U_m$.

Notation: The symbol $\oplus$ ("direct sum") signals that every element has a *unique* decomposition into summands from each subspace.



Direct sum: unique decomposition when subspaces meet only at 0.

1.42 Example: $\mathbb{F}^3$ as a Direct Sum
Let $U = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$ (the xy-plane).
Let $W = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\}$ (the z-axis).
Claim: $\mathbb{F}^3 = U \oplus W$.
Proof: Take any $(x, y, z) \in \mathbb{F}^3$. We can write:

$$(x, y, z) = (x, y, 0) + (0, 0, z)$$

where $(x, y, 0) \in U$ and $(0, 0, z) \in W$. This shows $\mathbb{F}^3 = U + W$.
Is this decomposition unique? Suppose $(x, y, z) = (a, b, 0) + (0, 0, c)$ for some $a, b, c \in \mathbb{F}$.
Then $(x, y, z) = (a, b, c)$, which forces $a = x$, $b = y$, $c = z$.
So the decomposition is unique, and $\mathbb{F}^3 = U \oplus W$. ✓

1.43 Example: $\mathbb{F}^n$ as Direct Sum of Coordinate Axes
For $j = 1, \ldots, n$, define:

$$U_j = \{(0, \ldots, 0, x, 0, \ldots, 0) \in \mathbb{F}^n : x \in \mathbb{F}\}$$

where x is in the j-th coordinate (all other coordinates are 0).
Then:

$$\mathbb{F}^n = U_1 \oplus U_2 \oplus \cdots \oplus U_n$$

Why? Any $(x_1, \ldots, x_n) \in \mathbb{F}^n$ can be written uniquely as:

$$(x_1, 0, \ldots, 0) + (0, x_2, 0, \ldots, 0) + \cdots + (0, \ldots, 0, x_n)$$

The coordinates completely determine each summand, so the decomposition is unique.

7. Testing for Direct Sums

How can we tell if a sum is direct? The following result gives a simple test.

1.44 Example: A Sum That is NOT Direct

Consider $\mathbb{F}^3$ and define:

$$U_1 = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\} \quad \text{(the } xy\text{-plane)}$$
$$U_2 = \{(0, y, z) \in \mathbb{F}^3 : y, z \in \mathbb{F}\} \quad \text{(the } yz\text{-plane)}$$
$$U_3 = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\} \quad \text{(the } z\text{-axis)}$$

Then $U_1+U_2+U_3 = \mathbb{F}^3$, but this sum is **not direct**. **Why not?** The zero vector $0 \in \mathbb{F}^3$ can be written as a sum of elements from U_1, U_2, U_3 in *multiple* ways:

$$0 = 0 + 0 + 0 \quad \text{(the obvious way)}$$
$$0 = (0, 1, 0) + (0, -1, 0) + 0 \quad \text{(another way!)}$$

Since $(0, 1, 0) \in U_1$, $(0, -1, 0) \in U_2$, and $0 \in U_3$, the second decomposition is valid. The representation of 0 is not unique, so this is not a direct sum.

The previous example suggests a key insight: to check if a sum is direct, we only need to check if 0 has a unique representation.

1.45 Condition for a Direct Sum
Suppose $U_1, \dots, U_m$ are subspaces of V. Then $U_1 + \dots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \dots + u_m$, where each $u_j \in U_j$, is by taking each $u_j = 0$.

Proof:

($\Rightarrow$) If the sum is direct, then every element has a unique representation. Since $0 = 0 + \dots + 0$ is one representation of 0, it must be the only one.

($\Leftarrow$) Suppose the only way to write 0 is $0 + \dots + 0$. We show the representation of any element is unique.

Take $v \in U_1 + \dots + U_m$ and suppose:

$$v = u_1 + \dots + u_m = w_1 + \dots + w_m$$

where $u_j, w_j \in U_j$ for each j.

Subtracting:

$$0 = (u_1 - w_1) + \dots + (u_m - w_m)$$

Since $u_j - w_j \in U_j$ (each U_j is a subspace), and the only way to write 0 is with all summands equal to 0:

$$u_j - w_j = 0 \quad \text{for all } j$$

Therefore $u_j = w_j$ for all j, proving uniqueness. $\square$

For the case of *two* subspaces, there's an even simpler criterion:

1.46 Direct Sum of Two Subspaces
Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof:

($\Rightarrow$) Suppose $U + W$ is a direct sum. Take any $v \in U \cap W$.

Then $v \in U$ and $v \in W$. Also $-v \in W$ (since W is a subspace).

We can write:

$$0 = v + (-v)$$

where $v \in U$ and $-v \in W$. By 1.45, the only way to write 0 is $0 + 0$, so $v = 0$.

Therefore $U \cap W = \{0\}$.

($\Leftarrow$) Suppose $U \cap W = \{0\}$. We show the only way to write $0 = u + w$ with $u \in U$, $w \in W$ is $u = w = 0$.

If $0 = u+w$, then $u = -w$. Since $u \in U$ and $-w \in W$ (and $u = -w$), we have $u \in U \cap W = \{0\}$.

So $u = 0$, which means $w = -u = 0$. By 1.45, the sum is direct. $\square$

Warning: Pairwise Intersection is Not Enough
For three or more subspaces, having $U_i \cap U_j = \{0\}$ for all pairs $i \neq j$ does **NOT** guarantee a direct sum.
Example: In 1.44, notice that:

- $U_1 \cap U_2 = \{(0, y, 0) : y \in \mathbb{F}\}$ (the y-axis) $\neq \{0\}$

But even if we modify the example to have pairwise intersections equal to $\{0\}$, the sum might still not be direct. The condition in 1.45 (uniqueness of the zero decomposition) is the correct test for $m \geq 3$ subspaces.

Strategy: How to Check if U is a Subspace

The Subspace Checklist:
To verify that $U \subseteq V$ is a subspace, check three things:

1. **Zero vector:** Show $0 \in U$.
   (Alternatively, show $U \neq \emptyset$.)
2. **Closed under addition:** Take *arbitrary* $u, w \in U$. Show that $u + w \in U$.
   (Use the definition of U to verify the sum satisfies the defining property.)
3. **Closed under scalar multiplication:** Take *arbitrary* $a \in \mathbb{F}$ and $u \in U$. Show that $au \in U$.

Common mistakes to avoid:

- Don't use specific vectors; use *arbitrary* elements.
- Don't forget to check $0 \in U$ (or nonemptiness).
- Remember: the operations come from V, not something new.

Quick tests for NON-subspaces:
A subset $U \subseteq V$ is **NOT** a subspace if any of these hold:

- $0 \notin U$ (e.g., $\{x : x_1 = 1\}$)
- Not closed under $+$ (find $u, w \in U$ with $u+w \notin U$)
- Not closed under scalar mult (find $a \in \mathbb{F}$, $u \in U$ with $au \notin U$)

Rule of thumb: Constraints of the form "$= b$" with $b \neq 0$ usually fail the subspace test because 0 won't satisfy the constraint.

Key Results Summary

Definitions:

- **Subspace** (1.33): A subset that is itself a vector space with the inherited operations
- **Sum of subspaces** (1.36): $U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_j \in U_j\}$
- **Direct sum** (1.41): A sum where each element has a *unique* representation; written $U_1 \oplus \cdots \oplus U_m$

The Subspace Test (1.34):
U is a subspace of $V \Leftrightarrow$

1. $0 \in U$
2. $u, w \in U \Rightarrow u + w \in U$
3. $a \in \mathbb{F}, u \in U \Rightarrow au \in U$

Direct Sum Tests:

- (1.45) $U_1 + \cdots + U_m$ is direct $\Leftrightarrow$ the only way to write $0 = u_1 + \cdots + u_m$ is with all $u_j = 0$
- (1.46) For two subspaces: $U + W$ is direct $\Leftrightarrow$ $U \cap W = \{0\}$

Key Facts:

- Every subspace contains 0
- $\{0\}$ and V are always subspaces of V
- The empty set is never a subspace
- Sums of subspaces are subspaces
- Unions of subspaces are usually NOT subspaces
- (1.40) $U_1 + \cdots + U_m$ is the smallest subspace containing all U_j

Common Problem Types:

Determine if U is a subspace
Use the three-condition test. Check $0 \in U$, closure under $+$, closure under scalar mult.

For which b is U a subspace?
Usually $b = 0$. Check whether 0 satisfies the defining condition.

Describe $U + W$
Write a general element as $u + w$ where $u \in U$, $w \in W$. Simplify to find the pattern.

Prove $U + W =$ some set S
Show $U + W \subseteq S$ (every sum has the right form) and $S \subseteq U + W$ (every element of S can be written as a sum).

Prove a sum is direct
For two subspaces: show $U \cap W = \{0\}$.
For multiple subspaces: show $u_1 + \cdots + u_m = 0$ implies all $u_j = 0$.

Show a sum is NOT direct
Find a nonzero way to write $0 = u_1 + \cdots + u_m$ with $u_j \in U_j$.

Key Takeaways

1. **Subspace test (1.34):** Check three conditions: $0 \in U$, closed under $+$, closed under scalar $\times$
2. **Homogeneous constraints:** "Something $=$ 0" gives subspaces; "something $= b$" with $b \neq 0$ does not
3. **Sums vs unions:** Unions of subspaces usually fail; use sums $U_1 + \cdots + U_m$ instead
4. **Direct sum tests:** For two subspaces, $U \cap W = \{0\}$; for multiple, check uniqueness of 0 decomposition

Relevant Exercises

Practice these problems from LADR to reinforce the material:

- Section 1C: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24