19 Suppose $U \subseteq V$. Explain why

$$U^0 = \{\varphi \in V' : U \subseteq \operatorname{null} \varphi\}.$$

20 Suppose V is finite-dimensional and U is a subspace of V. Show that

$$U = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}.$$

21 Suppose V is finite-dimensional and U and W are subspaces of V.

(a) Prove that $W^0 \subseteq U^0$ if and only if $U \subseteq W$.

(b) Prove that $W^0 = U^0$ if and only if $U = W$.

22 Suppose V is finite-dimensional and U and W are subspaces of V.

(a) Show that $(U + W)^0 = U^0 \cap W^0$.

(b) Show that $(U \cap W)^0 = U^0 + W^0$.

23 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_m \in V'$. Prove that the following three sets are equal to each other.

(a) $\operatorname{span}(\varphi_1, \dots, \varphi_m)$

(b) $\big((\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)\big)^0$

(c) $\{\varphi \in V' : (\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m) \subseteq \operatorname{null} \varphi\}$

24 Suppose V is finite-dimensional and $v_1, \ldots, v_m \in V$. Define a linear map $\Gamma \colon V' \to \mathbf{F}^m$ by $\Gamma(\varphi) = \bigl(\varphi(v_1), \ldots, \varphi(v_m)\bigr)$.

(a) Prove that $v_1, \ldots, v_m$ spans V if and only if Γ is injective.

(b) Prove that $v_1, \ldots, v_m$ is linearly independent if and only if Γ is surjective.

25 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_m \in V'$. Define a linear map $\Gamma\colon V \to \mathbf{F}^m$ by $\Gamma(v) = \bigl(\varphi_1(v), \dots, \varphi_m(v)\bigr)$.

(a) Prove that $\varphi_1, \dots, \varphi_m$ spans V' if and only if Γ is injective.

(b) Prove that $\varphi_1, \dots, \varphi_m$ is linearly independent if and only if Γ is surjective.

26 Suppose V is finite-dimensional and Ω is a subspace of V'. Prove that

$$\Omega = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in \Omega\}^0.$$

27 Suppose $T \in \mathcal{L}\big(\mathcal{P}_5(\mathbf{R})\big)$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$, where φ is the linear functional on $\mathcal{P}_5(\mathbf{R})$ defined by $\varphi(p) = p(8)$. Prove that

$$\operatorname{range} T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}.$$

28 Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_m$ is a linearly independent list in V'. Prove that

$$\dim\bigl((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\bigr) = (\dim V) - m.$$

29 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$.

(a) Prove that if $\varphi \in W'$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$, then $\operatorname{range} T = \operatorname{null} \varphi$.

(b) Prove that if $\psi \in V'$ and $\operatorname{range} T' = \operatorname{span}(\psi)$, then $\operatorname{null} T = \operatorname{null} \psi$.

30 Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_n$ is a basis of V'. Show that there exists a basis of V whose dual basis is $\varphi_1, \ldots, \varphi_n$.

31 Suppose U is a subspace of V. Let $i\colon U \to V$ be the inclusion map defined by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.

(a) Show that $\operatorname{null} i' = U^0$.

(b) Prove that if V is finite-dimensional, then $\operatorname{range} i' = U'$.

(c) Prove that if V is finite-dimensional, then $\tilde{i}'$ is an isomorphism from V'/U^0 onto U'.

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.

32 The *double dual space* of V, denoted by V'', is defined to be the dual space of V'. In other words, $V'' = (V')'$. Define $\Lambda\colon V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for each $v \in V$ and each $\varphi \in V'$.

(a) Show that Λ is a linear map from V to V''.

(b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.

(c) Show that if V is finite-dimensional, then Λ is an isomorphism from V onto V''.

Suppose V is finite-dimensional. Then V and V' are isomorphic, but finding an isomorphism from V onto V' generally requires choosing a basis of V. In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

33 Suppose U is a subspace of V. Let $\pi\colon V \to V/U$ be the usual quotient map. Thus $\pi' \in \mathcal{L}\big((V/U)', V'\big)$.

(a) Show that π' is injective.

(b) Show that $\operatorname{range} \pi' = U^0$.

(c) Conclude that π' is an isomorphism from $(V/U)'$ onto U^0.

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space. In fact, there is no assumption here that any of these vector spaces are finite-dimensional.