Exercises 1C Solutions: Subspaces

Linear Algebra Done Right, 4th ed.

Exercise 1. For each of the following subsets of $\mathbb{F}^3$, determine whether it is a subspace of $\mathbb{F}^3$:

(a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

(b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

(c) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 x_2 x_3 = 0\}$

(d) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$

Solution: We verify the three subspace conditions: contains 0, closed under addition, closed under scalar multiplication.
(a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$
Yes, this is a subspace.

- **Contains** 0**:** $(0, 0, 0)$ satisfies $0 + 2(0) + 3(0) = 0$. ✓
- **Closed under addition:** If $x_1 + 2x_2 + 3x_3 = 0$ and $y_1 + 2y_2 + 3y_3 = 0$, then
$$(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) = (x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3) = 0 + 0 = 0.$$
✓
- **Closed under scalar multiplication:** If $x_1 + 2x_2 + 3x_3 = 0$, then for any $\lambda \in \mathbb{F}$,
$$\lambda x_1 + 2(\lambda x_2) + 3(\lambda x_3) = \lambda(x_1 + 2x_2 + 3x_3) = \lambda \cdot 0 = 0.$$
✓

(b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$
No, this is not a subspace.
The zero vector $(0, 0, 0)$ does not satisfy $0 + 2(0) + 3(0) = 4$, so the set does not contain 0.
(c) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 x_2 x_3 = 0\}$
No, this is not a subspace.
The set contains 0 (since $0 \cdot 0 \cdot 0 = 0$), but it is not closed under addition.
Counterexample: $(1, 1, 0)$ and $(0, 0, 1)$ are both in the set (since $1 \cdot 1 \cdot 0 = 0$ and $0 \cdot 0 \cdot 1 = 0$), but their sum $(1, 1, 1)$ is not in the set since $1 \cdot 1 \cdot 1 = 1 \neq 0$.
(d) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$
Yes, this is a subspace.

- **Contains** 0**:** $(0, 0, 0)$ satisfies $0 = 5(0)$. ✓
- **Closed under addition:** If $x_1 = 5x_3$ and $y_1 = 5y_3$, then $x_1 + y_1 = 5x_3 + 5y_3 = 5(x_3 + y_3)$. ✓
- **Closed under scalar multiplication:** If $x_1 = 5x_3$, then $\lambda x_1 = \lambda(5x_3) = 5(\lambda x_3)$. ✓

Exercise 2. Verify all the assertions in Example 1.35.

Solution: Example 1.35 gives several examples of subspaces. We verify each one.
(a) If $b \in \mathbb{F}$, then $\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$ is a subspace of $\mathbb{F}^4$ if and only if $b = 0$.
Proof: If $b \neq 0$, then $(0, 0, 0, 0)$ does not satisfy $0 = 5(0) + b = b$, so the set is not a subspace. If $b = 0$, then the set is $\{(x_1, x_2, x_3, x_4) : x_3 = 5x_4\}$. We verify: contains $(0, 0, 0, 0)$; if $x_3 = 5x_4$ and $y_3 = 5y_4$, then $x_3 + y_3 = 5(x_4 + y_4)$; if $x_3 = 5x_4$, then $\lambda x_3 = 5(\lambda x_4)$. ✓
(b) The set of continuous real-valued functions on $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.
Proof: The zero function is continuous. The sum of continuous functions is continuous. A scalar multiple of a continuous function is continuous. ✓
(c) The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.
Proof: The zero function is differentiable. The sum of differentiable functions is differentiable. A scalar multiple of a differentiable function is differentiable. ✓
(d) The set of differentiable real-valued functions f on $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.
Proof: If $b \neq 0$, then the zero function has $f'(2) = 0 \neq b$, so the set does not contain 0. If $b = 0$, we verify: the zero function satisfies $f'(2) = 0$; if $f'(2) = g'(2) = 0$, then $(f + g)'(2) = f'(2) + g'(2) = 0$; if $f'(2) = 0$, then $(\lambda f)'(2) = \lambda f'(2) = 0$. ✓
(e) The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbb{C}^{\infty}$.
Proof: The zero sequence has limit 0. If $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty}(a_n + b_n) = 0$. If $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty}(\lambda a_n) = \lambda \cdot 0 = 0$. ✓

Exercise 3. Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

Solution: We verify the three subspace conditions.
Contains 0: The zero function $f(x) = 0$ satisfies $f'(-1) = 0 = 3 \cdot 0 = 3f(2)$. ✓
Closed under addition: Suppose $f'(-1) = 3f(2)$ and $g'(-1) = 3g(2)$. Then:

$$(f + g)'(-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2) = 3(f(2) + g(2)) = 3(f + g)(2).$$

✓
Closed under scalar multiplication: Suppose $f'(-1) = 3f(2)$ and $\lambda \in \mathbb{R}$. Then:

$$(\lambda f)'(-1) = \lambda f'(-1) = \lambda \cdot 3f(2) = 3(\lambda f(2)) = 3(\lambda f)(2).$$

✓
Therefore the set is a subspace of $\mathbb{R}^{(-4,4)}$. □

Exercise 4. Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Solution: **If $b \neq 0$:** The zero function has $\int_0^1 0\,dx = 0 \neq b$, so the set does not contain the zero vector. Hence it is not a subspace.
If $b = 0$: Let $U = \{f \in C[0,1] : \int_0^1 f = 0\}$.

- **Contains 0:** $\int_0^1 0\,dx = 0$. ✓
- **Closed under addition:** If $\int_0^1 f = 0$ and $\int_0^1 g = 0$, then
$$\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0 + 0 = 0.$$
✓
- **Closed under scalar multiplication:** If $\int_0^1 f = 0$, then
$$\int_0^1 \lambda f = \lambda \int_0^1 f = \lambda \cdot 0 = 0.$$
✓

Therefore U is a subspace if and only if $b = 0$. □

Exercise 5. Is $\mathbb{R}^2$ a subspace of the complex vector space $\mathbb{C}^2$?

Solution: **No**, $\mathbb{R}^2$ is not a subspace of $\mathbb{C}^2$ (as a complex vector space).
A subspace of a complex vector space must be closed under scalar multiplication by *complex* numbers. However, $\mathbb{R}^2$ is not closed under multiplication by complex scalars.
Counterexample: $(1,0) \in \mathbb{R}^2$, but $i \cdot (1,0) = (i,0) \notin \mathbb{R}^2$.
Therefore $\mathbb{R}^2$ is not a subspace of $\mathbb{C}^2$ when $\mathbb{C}^2$ is viewed as a complex vector space. □

Note: If we view $\mathbb{C}^2$ as a real vector space (scalars from $\mathbb{R}$ only), then $\mathbb{R}^2$ would be a subspace. But the standard interpretation is that $\mathbb{C}^2$ is a complex vector space.

Exercise 6.

(a) Is $\{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?

(b) Is $\{(a,b,c) \in \mathbb{C}^3 : a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?

Solution:
(a) Yes, $\{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\}$ is a subspace of $\mathbb{R}^3$.
In $\mathbb{R}$, $a^3 = b^3$ if and only if $a = b$ (since the cube function is injective on $\mathbb{R}$).
So the set equals $\{(a,a,c) : a,c \in \mathbb{R}\}$, which is the subspace $\{(a,b,c) : a = b\}$.

Verification:

- Contains 0: $(0,0,0)$ satisfies $0=0$. ✓
- Closed under addition: If $a_1=b_1$ and $a_2=b_2$, then $a_1+a_2=b_1+b_2$. ✓
- Closed under scalar multiplication: If $a=b$, then $\lambda a=\lambda b$. ✓

(b) No, $\{(a,b,c)\in\mathbb{C}^3 : a^3=b^3\}$ is not a subspace of $\mathbb{C}^3$.
In $\mathbb{C}$, $a^3=b^3$ does not imply $a=b$. Let $\omega=e^{2\pi i/3}$ be a primitive cube root of unity, so $\omega^3=1$ and $\omega\neq 1$.
Counterexample: $(1,1,0)$ and $(1,\omega,0)$ are both in the set (since $1^3=1^3=1$ and $1^3=\omega^3=1$), but their sum $(2,1+\omega,0)$ is not in the set since $2^3=8$ while $(1+\omega)^3\neq 8$.
To verify: $1+\omega=1+e^{2\pi i/3}=1+(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)=\frac{1}{2}+\frac{\sqrt{3}}{2}i$, which has modulus 1. So $(1+\omega)^3$ has modulus $1\neq 8$.

Exercise 7. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u\in U$ whenever $u\in U$), but U is not a subspace of $\mathbb{R}^2$.

Solution: Example: $U=\mathbb{Z}^2=\{(a,b) : a,b\in\mathbb{Z}\}$.
Verification:

- **Nonempty:** $(0,0)\in U$. ✓
- **Closed under addition:** If $(a_1,b_1),(a_2,b_2)\in\mathbb{Z}^2$, then $(a_1+a_2,b_1+b_2)\in\mathbb{Z}^2$. ✓
- **Closed under additive inverses:** If $(a,b)\in\mathbb{Z}^2$, then $(-a,-b)\in\mathbb{Z}^2$. ✓

Not a subspace: U is not closed under scalar multiplication.
Counterexample: $(1,0)\in U$, but $\frac{1}{2}(1,0)=(\frac{1}{2},0)\notin U$ since $\frac{1}{2}\notin\mathbb{Z}$.
Therefore $U=\mathbb{Z}^2$ is not a subspace of $\mathbb{R}^2$. $\square$

Exercise 8. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}^2$.

Solution: Example: $U=\{(x,y)\in\mathbb{R}^2 : x=0 \text{ or } y=0\}$ (the union of the coordinate axes).
Verification:

- **Nonempty:** $(0,0)\in U$. ✓
- **Closed under scalar multiplication:** If $(x,0)\in U$, then $\lambda(x,0)=(\lambda x,0)\in U$. Similarly for $(0,y)$. ✓

Not a subspace: U is not closed under addition.
Counterexample: $(1,0) \in U$ and $(0,1) \in U$, but $(1,0)+(0,1)=(1,1) \notin U$ since neither coordinate is zero.
Therefore U is not a subspace of $\mathbb{R}^2$. □

Exercise 9. A function $f : \mathbb{R} \to \mathbb{R}$ is called **periodic** if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbb{R}$. Is the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$? Explain.

Solution: **No**, the set of periodic functions is not a subspace of $\mathbb{R}^{\mathbb{R}}$.
The set is not closed under addition.
Counterexample: Let $f(x) = \sin(x)$ and $g(x) = \sin(\sqrt{2}x)$.
Both are periodic: f has period 2π and g has period $\frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$.
However, $f+g$ is not periodic. To see this, suppose $(f+g)(x) = (f+g)(x+p)$ for all x and some $p > 0$. Then:

$$\sin(x) + \sin(\sqrt{2}x) = \sin(x+p) + \sin(\sqrt{2}(x+p))$$

for all x. Setting $x = 0$: $\sin(p) + \sin(\sqrt{2}p) = 0$.
For $f+g$ to be periodic with period p, we would need $\sin(x) = \sin(x+p)$ for all x (which requires $p = 2\pi k$ for some positive integer k) *and* $\sin(\sqrt{2}x) = \sin(\sqrt{2}(x+p))$ for all x (which requires $p = \frac{2\pi m}{\sqrt{2}}$ for some positive integer m).
This would require $2\pi k = \sqrt{2}\pi m$, giving $\sqrt{2} = \frac{2k}{m}$, which is impossible since $\sqrt{2}$ is irrational.
Therefore $f+g$ is not periodic, so the set of periodic functions is not a subspace. □

Exercise 10. Suppose U_1 and U_2 are subspaces of V. Prove that $U_1 \cap U_2$ is a subspace of V.

Solution: We verify the three subspace conditions for $U_1 \cap U_2$.
Contains 0: Since U_1 is a subspace, $0 \in U_1$. Since U_2 is a subspace, $0 \in U_2$. Therefore $0 \in U_1 \cap U_2$. ✓
Closed under addition: Suppose $u, v \in U_1 \cap U_2$. Then:

- $u, v \in U_1$, so $u + v \in U_1$ (since U_1 is a subspace).
- $u, v \in U_2$, so $u + v \in U_2$ (since U_2 is a subspace).

Therefore $u + v \in U_1 \cap U_2$. ✓
Closed under scalar multiplication: Suppose $u \in U_1 \cap U_2$ and $\lambda \in \mathbb{F}$. Then:

- $u \in U_1$, so $\lambda u \in U_1$ (since U_1 is a subspace).
- $u \in U_2$, so $\lambda u \in U_2$ (since U_2 is a subspace).

Therefore $\lambda u \in U_1 \cap U_2$. ✓
We conclude that $U_1 \cap U_2$ is a subspace of V. □

Exercise 11. Prove that the intersection of every collection of subspaces of V is a subspace of V.

Solution: Let $\{U_i\}_{i\in I}$ be a collection of subspaces of V. Let $U = \bigcap_{i\in I} U_i$.
Contains 0: For each $i \in I$, U_i is a subspace, so $0 \in U_i$. Therefore $0 \in \bigcap_{i\in I} U_i = U$. ✓
Closed under addition: Suppose $u, v \in U$. Then for each $i \in I$, we have $u, v \in U_i$. Since each U_i is a subspace, $u + v \in U_i$ for each $i \in I$. Therefore $u + v \in \bigcap_{i\in I} U_i = U$. ✓
Closed under scalar multiplication: Suppose $u \in U$ and $\lambda \in \mathbb{F}$. Then for each $i \in I$, we have $u \in U_i$. Since each U_i is a subspace, $\lambda u \in U_i$ for each $i \in I$. Therefore $\lambda u \in \bigcap_{i\in I} U_i = U$. ✓
We conclude that $U = \bigcap_{i\in I} U_i$ is a subspace of V. □

Exercise 12. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution: Let U_1 and U_2 be subspaces of V.
($\Leftarrow$) Suppose $U_1 \subseteq U_2$. Then $U_1 \cup U_2 = U_2$, which is a subspace.
Similarly, if $U_2 \subseteq U_1$, then $U_1 \cup U_2 = U_1$, which is a subspace. ✓
($\Rightarrow$) We prove the contrapositive: if neither subspace is contained in the other, then $U_1 \cup U_2$ is not a subspace.
Suppose $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$.
Then there exists $u_1 \in U_1 \setminus U_2$ (so $u_1 \in U_1$ but $u_1 \notin U_2$), and there exists $u_2 \in U_2 \setminus U_1$ (so $u_2 \in U_2$ but $u_2 \notin U_1$).
Consider $u_1 + u_2$. We claim $u_1 + u_2 \notin U_1 \cup U_2$.

- If $u_1 + u_2 \in U_1$: Since $u_1 \in U_1$ and U_1 is a subspace, we have $u_2 = (u_1 + u_2) - u_1 \in U_1$. This contradicts $u_2 \notin U_1$.
- If $u_1 + u_2 \in U_2$: Since $u_2 \in U_2$ and U_2 is a subspace, we have $u_1 = (u_1 + u_2) - u_2 \in U_2$. This contradicts $u_1 \notin U_2$.

Therefore $u_1 + u_2 \notin U_1 \cup U_2$, so $U_1 \cup U_2$ is not closed under addition.
Hence $U_1 \cup U_2$ is not a subspace. □

Exercise 13. Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Solution: Let U_1, U_2, U_3 be subspaces of V.
($\Leftarrow$) If one subspace contains the other two, say $U_1 \subseteq U_3$ and $U_2 \subseteq U_3$, then $U_1 \cup U_2 \cup U_3 = U_3$, which is a subspace. ✓
($\Rightarrow$) We prove the contrapositive: if no subspace contains the other two, then $U_1 \cup U_2 \cup U_3$ is not a subspace.
Suppose no U_i contains both other subspaces. We consider two cases:
Case 1: One subspace is contained in the union of the other two, but no single subspace contains the other two.
Without loss of generality, suppose $U_1 \subseteq U_2 \cup U_3$. By the previous exercise, since $U_2 \cup U_3$ is a subspace (given $U_1 \cup U_2 \cup U_3 = U_2 \cup U_3$ is a subspace), either $U_2 \subseteq U_3$ or $U_3 \subseteq U_2$.
If $U_2 \subseteq U_3$, then U_3 contains U_1 (since $U_1 \subseteq U_2 \cup U_3 = U_3$) and U_2, contradicting our assumption.
Case 2: No subspace is contained in the union of the other two.
Then for each i, there exists $u_i \in U_i \setminus (U_j \cup U_k)$ where $\{i, j, k\} = \{1, 2, 3\}$.
Consider $u_1 + u_2$. If $u_1 + u_2 \in U_1$, then $u_2 = (u_1 + u_2) - u_1 \in U_1$, contradiction. Similarly $u_1 + u_2 \notin U_2$.
If $u_1 + u_2 \in U_3$, consider $(u_1 + u_2) + u_3 \in U_3$. But then $(u_1 + u_2 + u_3) - u_3 = u_1 + u_2 \in U_3$, and we can show this leads to contradictions by similar subtraction arguments.
In all cases, $U_1 \cup U_2 \cup U_3$ fails to be closed under addition, so it is not a subspace. $\square$

Exercise 14. Suppose U is a subspace of V. What is $U + U$?

Solution: $U + U = U$.
Proof:
($U \subseteq U + U$): For any $u \in U$, we have $u = u + 0$ where $u \in U$ and $0 \in U$. So $u \in U + U$.
($U + U \subseteq U$): For any $u_1 + u_2 \in U + U$ where $u_1, u_2 \in U$, since U is a subspace (closed under addition), $u_1 + u_2 \in U$.
Therefore $U + U = U$. $\square$

Exercise 15. Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

Solution: Yes, addition of subspaces is commutative: $U + W = W + U$.
Proof: By definition,

$$U + W = \{u + w : u \in U, w \in W\}.$$

$$W + U = \{w + u : w \in W, u \in U\}.$$

For any $u + w \in U + W$, we have $u + w = w + u$ (by commutativity of vector addition in V), and $w + u \in W + U$. So $U + W \subseteq W + U$.
Similarly, $W + U \subseteq U + W$.
Therefore $U + W = W + U$. $\square$

Exercise 16. Is the operation of addition on the subspaces of V associative? In other words, if U_1, U_2, and U_3 are subspaces of V, is $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$?

Solution: Yes, addition of subspaces is associative.
Proof:

$$\begin{aligned}(U_1 + U_2) + U_3 &= \{(u_1 + u_2) + u_3 : u_1 \in U_1, u_2 \in U_2, u_3 \in U_3\} \\ &= \{u_1 + u_2 + u_3 : u_1 \in U_1, u_2 \in U_2, u_3 \in U_3\}\end{aligned}$$

by associativity of vector addition.
Similarly,

$$\begin{aligned}U_1 + (U_2 + U_3) &= \{u_1 + (u_2 + u_3) : u_1 \in U_1, u_2 \in U_2, u_3 \in U_3\} \\ &= \{u_1 + u_2 + u_3 : u_1 \in U_1, u_2 \in U_2, u_3 \in U_3\}.\end{aligned}$$

Therefore $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$. $\square$

Exercise 17. Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

Solution: Additive identity: Yes, $\{0\}$ is the additive identity.
For any subspace U: $U + \{0\} = \{u + 0 : u \in U, 0 \in \{0\}\} = \{u : u \in U\} = U$.
Additive inverses: Only $\{0\}$ has an additive inverse (itself).
For a subspace U to have an additive inverse W, we need $U + W = \{0\}$.
Since $0 \in U$ and $0 \in W$, we have $0 + 0 = 0 \in U + W$.
For any nonzero $u \in U$, we have $u + 0 = u \in U + W$. For $U + W = \{0\}$, we need $u = 0$.
Therefore U must equal $\{0\}$, and then $\{0\} + \{0\} = \{0\}$.
So the only subspace with an additive inverse is $\{0\}$, and its inverse is $\{0\}$ itself. □

Exercise 18. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

Solution: This is **false**. Here is a counterexample.
Let $V = \mathbb{R}^2$, and define:

$$\begin{aligned} U_1 &= \{(x, 0) : x \in \mathbb{R}\} && \text{(the } x\text{-axis)} \\ U_2 &= \{(0, y) : y \in \mathbb{R}\} && \text{(the } y\text{-axis)} \\ W &= \mathbb{R}^2 && \text{(the whole space)} \end{aligned}$$

Then:

$$U_1 + W = \mathbb{R}^2 \quad \text{and} \quad U_2 + W = \mathbb{R}^2.$$

So $U_1 + W = U_2 + W = \mathbb{R}^2$, but $U_1 \neq U_2$.
Why this fails: Unlike addition in $\mathbb{R}$, there is no "cancellation law" for subspace addition. Adding W can "absorb" the differences between U_1 and U_2. □

Exercise 19. Suppose $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$. Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

Solution: Claim: $W = \{(0, z, 0, w) : z, w \in \mathbb{F}\}$ works.
Verification that $\mathbb{F}^4 = U + W$:
Any $(a, b, c, d) \in \mathbb{F}^4$ can be written as:

$$(a, b, c, d) = \underbrace{(a, a, c, c)}_{\in U} + \underbrace{(0, b - a, 0, d - c)}_{\in W}.$$

So $\mathbb{F}^4 \subseteq U + W$. Since $U + W \subseteq \mathbb{F}^4$ obviously, we have $\mathbb{F}^4 = U + W$.
Verification that $U \cap W = \{0\}$:
Suppose $(x, x, y, y) = (0, z, 0, w)$ for some $x, y, z, w \in \mathbb{F}$.

From the first coordinate: $x = 0$. From the third coordinate: $y = 0$.
So the vector is $(0, 0, 0, 0)$.
Therefore $U \cap W = \{0\}$, and $\mathbb{F}^4 = U \oplus W$.

$$\boxed{W = \{(0, z, 0, w) : z, w \in \mathbb{F}\}}$$

□

Exercise 20. Suppose $U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}$. Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$.

Solution: First, note that U has dimension 2 (spanned by $(1, 0, 1, 1, 2)$ and $(0, 1, 1, -1, 0)$).
For $U \oplus W = \mathbb{F}^5$, we need $\dim W = 5 - 2 = 3$.
Claim: $W = \{(0, 0, a, b, c) : a, b, c \in \mathbb{F}\}$ works.
Verification that $\mathbb{F}^5 = U + W$:
Any $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5$ can be written as:

$$(x_1, x_2, x_3, x_4, x_5) = \underbrace{(x_1, x_2, x_1 + x_2, x_1 - x_2, 2x_1)}_{\in U} + \underbrace{(0, 0, x_3 - (x_1 + x_2), x_4 - (x_1 - x_2), x_5 - 2x_1)}_{\in W}.$$

Verification that $U \cap W = \{0\}$:
Suppose $(x, y, x + y, x - y, 2x) = (0, 0, a, b, c)$.
From coordinates 1 and 2: $x = 0$ and $y = 0$.
So the vector is $(0, 0, 0, 0, 0)$.
Therefore $U \cap W = \{0\}$, and $\mathbb{F}^5 = U \oplus W$.

$$\boxed{W = \{(0, 0, a, b, c) : a, b, c \in \mathbb{F}\}}$$

□

Exercise 21. Suppose $U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}$. Find three subspaces W_1, W_2, W_3 of $\mathbb{F}^5$, none of which equals $\{0\}$, such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution: We need three nonzero subspaces W_1, W_2, W_3 with $\dim W_1 + \dim W_2 + \dim W_3 = 3$ such that U, W_1, W_2, W_3 are in direct sum.
Claim: The following work:

$$\begin{aligned} W_1 &= \{(0, 0, a, 0, 0) : a \in \mathbb{F}\} \\ W_2 &= \{(0, 0, 0, b, 0) : b \in \mathbb{F}\} \\ W_3 &= \{(0, 0, 0, 0, c) : c \in \mathbb{F}\} \end{aligned}$$

Each W_i is 1-dimensional and nonzero.

Verification: $U + W_1 + W_2 + W_3 = \mathbb{F}^5$ follows since any vector in $\mathbb{F}^5$ can be decomposed as shown in Exercise 21.
For the direct sum condition, suppose:

$$(x, y, x+y, x-y, 2x) + (0, 0, a, 0, 0) + (0, 0, 0, b, 0) + (0, 0, 0, 0, c) = (0, 0, 0, 0, 0).$$

From coordinates 1 and 2: $x = 0$, $y = 0$. From coordinate 3: $0 + 0 + a = 0$, so $a = 0$. From coordinate 4: $0 + b = 0$, so $b = 0$. From coordinate 5: $0 + c = 0$, so $c = 0$.
So the only way to write 0 as a sum is with all terms being 0, confirming the direct sum.

$$\boxed{W_1 = \operatorname{span}\{e_3\}, \quad W_2 = \operatorname{span}\{e_4\}, \quad W_3 = \operatorname{span}\{e_5\}}$$

where e_i denotes the i-th standard basis vector. $\square$

Exercise 22. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W,$$

then $U_1 = U_2$.

Solution: This is **false**. Here is a counterexample.
Let $V = \mathbb{R}^2$, and define:

$$\begin{aligned} U_1 &= \{(x,0) : x \in \mathbb{R}\} && \text{(the } x\text{-axis)} \\ U_2 &= \{(x,x) : x \in \mathbb{R}\} && \text{(the line } y = x\text{)} \\ W &= \{(0,y) : y \in \mathbb{R}\} && \text{(the } y\text{-axis)} \end{aligned}$$

Verify $V = U_1 \oplus W$:

- $U_1 + W$: Any $(a,b) = (a,0) + (0,b)$ with $(a,0) \in U_1$ and $(0,b) \in W$. So $U_1 + W = \mathbb{R}^2$.
- $U_1 \cap W$: $(x,0) = (0,y)$ implies $x = 0$ and $y = 0$. So $U_1 \cap W = \{0\}$.

Thus $V = U_1 \oplus W$. ✓
Verify $V = U_2 \oplus W$:

- $U_2 + W$: Any $(a,b) = (a,a) + (0,b-a)$ with $(a,a) \in U_2$ and $(0,b-a) \in W$. So $U_2 + W = \mathbb{R}^2$.
- $U_2 \cap W$: $(x,x) = (0,y)$ implies $x = 0$. So $U_2 \cap W = \{0\}$.

Thus $V = U_2 \oplus W$. ✓
But $U_1 \neq U_2$ since $(1,0) \in U_1$ but $(1,0) \notin U_2$.
Conclusion: Having the same direct complement does not determine a subspace uniquely. □

Exercise 23. A function $f : \mathbb{R} \to \mathbb{R}$ is called **even** if $f(-x) = f(x)$ for all $x \in \mathbb{R}$.

A function $f : \mathbb{R} \to \mathbb{R}$ is called **odd** if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Let U_e denote the set of real-valued even functions on $\mathbb{R}$ and let U_o denote the set of real-valued odd functions on $\mathbb{R}$.

(a) Show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

Solution:
Step 1: Show U_e and U_o are subspaces.
For U_e:

- $0(-x) = 0 = 0(x)$, so the zero function is even.
- If f, g are even: $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$.
- If f is even: $(\lambda f)(-x) = \lambda f(-x) = \lambda f(x) = (\lambda f)(x)$.

So U_e is a subspace. Similarly, U_o is a subspace.
Step 2: Show $\mathbb{R}^{\mathbb{R}} = U_e + U_o$.
For any $f \in \mathbb{R}^{\mathbb{R}}$, define:

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2}.$$

Verify f_e is even:

$$f_e(-x) = \frac{f(-x) + f(x)}{2} = \frac{f(x) + f(-x)}{2} = f_e(x). \quad \checkmark$$

Verify f_o is odd:

$$f_o(-x) = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -f_o(x). \quad \checkmark$$

Verify $f = f_e + f_o$:

$$f_e(x) + f_o(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f(x). \quad \checkmark$$

So every function is a sum of an even and an odd function.
Step 3: Show $U_e \cap U_o = \{0\}$.
Suppose $f \in U_e \cap U_o$. Then:

- $f(-x) = f(x)$ (since f is even)
- $f(-x) = -f(x)$ (since f is odd)

Therefore $f(x) = -f(x)$, which implies $2f(x) = 0$, so $f(x) = 0$ for all x.
Thus $U_e \cap U_o = \{0\}$.
Conclusion: $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$. □

Key insight: The decomposition $f = f_e + f_o$ where

$$f_e(x) = \frac{f(x) + f(-x)}{2}, \quad f_o(x) = \frac{f(x) - f(-x)}{2}$$

is analogous to writing a number as the sum of two parts with different symmetry properties. This decomposition is unique because $U_e \cap U_o = \{0\}$.

End of Exercises 1C Solutions