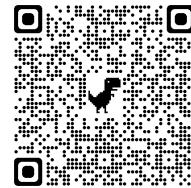


## Upper Division Tutoring Program Topic Review

Facilitators: Matthew Corbo, [corbo@berkeley.edu](mailto:corbo@berkeley.edu)  
Kyle Fenimore, [kyle.fenimore@berkeley.edu](mailto:kyle.fenimore@berkeley.edu)  
Drop-In: Check [Drop-In Schedule](#)



## Vector Spaces, Linear Maps, and Duality

### Section I: Definitions and Theorems Review

#### Vector Spaces

**Definition 1** (vector space). Fix a field  $F$ . Then a *vector space*  $V$  over  $F$  is a set equipped with an addition  $+$  and scalar multiplication  $\cdot$  satisfying the following conditions.

- Addition commutes and associates:  $u + v = v + u$  and  $u + (v + w) = (u + v) + w$  for any  $u, v, w \in V$ .
- Additive identity: there is a vector  $0 \in V$  such that  $v + 0 = 0 + v = v$  for any  $v \in V$ .
- Additive inverse: for any  $v \in V$ , there is a vector  $-v \in V$  such that  $v + (-v) = (-v) + v = 0$ .
- Multiplicative identity: for any  $v \in V$ , we have  $1v = v$ .
- Distribution: for any  $a, b \in F$  and  $v, w \in V$ , we have  $(a + b)(v + w) = av + aw + bv + bw$ .

**Remark.** One can show that the identity  $0 \in V$  and inverse  $-v \in V$  of  $v \in V$  are unique. As such, we may say “the identity” or “the inverse of  $v$ .”

**Definition 2** (subspace). Fix a vector space  $V$  over a field  $F$ . A subset  $W \subseteq V$  is a *subspace* if and only if it satisfies the following.

- Zero:  $0 \in W$ .
- Addition: for  $v, w \in W$ , we have  $v + w \in W$ .
- Scalar multiplication: for  $a \in F$  and  $v \in W$ , we have  $av \in W$ .

**Definition 3** (sum, direct sum). Fix a vector space  $V$  over a field  $F$ . Given some subsets  $U_1, \dots, U_n$ , we define the *sum* as

$$U_1 + \dots + U_n := \{u_1 + \dots + u_n : u_1 \in U_1, \dots, u_n \in U_n\}.$$

This is a *direct sum* if it satisfies the following property: for each  $v \in U_1 + \dots + U_n$ , there are unique vectors  $u_1 \in U_1, \dots, u_n \in U_n$  such that  $v = u_1 + \dots + u_n$ . In this case, we write  $U_1 \oplus \dots \oplus U_n$ .

**Definition 4** (span). Fix a vector space  $V$  over a field  $F$ . Given a set of vectors  $\{v_1, \dots, v_n\}$ , the *span* of these vectors is given by

$$\text{span}(\{v_1, \dots, v_n\}) := \{a_1v_1 + \dots + a_nv_n : a_1, \dots, a_n \in F\}.$$

In other words, the span is the set of linear combinations of the vectors in  $\{v_1, \dots, v_n\}$ .

**Definition 5** (linearly independent). Fix a vector space  $V$  over a field  $F$ . Then a set of vectors  $\{v_1, \dots, v_m\}$  in  $V$  is *linearly independent* if and only if

$$a_1v_1 + \dots + a_nv_n = 0$$

for  $a_1, \dots, a_n \in F$  implies that  $a_1 = \dots = a_n = 0$ . Otherwise, we say that these vectors are *linearly dependent*.

**Definition 6** (basis, dimension). Fix a vector space  $V$  over a field  $F$ . A *basis* of a set of vectors  $B$  which is linearly independent and spans  $V$ . The *dimension*  $\dim V$  of  $V$  is the size of the set  $B$ .

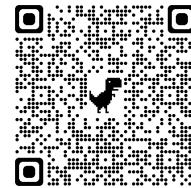
**Proposition 1.** Fix a vector space  $V$  over a field  $F$  and a subset of vectors  $\{v_1, \dots, v_n\}$ . Then this set is a basis if and only if it satisfies the following: for any vector  $v \in V$ , there are scalars  $a_1, \dots, a_n \in F$  such that

$$v = a_1v_1 + \dots + a_nv_n.$$

**Theorem 1.** Every vector space has a (possibly infinite) basis.

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## Linear Maps

**Definition 7** (linear map). Fix vector spaces  $V$  and  $W$  over a field  $F$ . Then a *linear map* is a function  $T: V \rightarrow W$  satisfying the following.

- Addition: for  $v, v' \in V$ , we have  $T(v + v') = T(v) + T(v')$ .
- Scalar multiplication: for  $v \in V$  and  $a \in F$ , we have  $T(av) = aT(v)$ .

We denote the vector space of linear transformations  $V \rightarrow W$  by  $\mathcal{L}(V, W)$

**Example 1.** Given a vector space  $V$ , define the function  $\text{id}_V: V \rightarrow V$  by  $\text{id}_V(v) := v$  for each  $v \in V$ . Then  $\text{id}_V$  is a linear map.

**Remark.** If  $V$  and  $W$  are finite-dimensional, then  $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$ .

**Proposition 2.** Fix a finite-dimensional vector space  $V$  with a basis  $\{v_1, \dots, v_n\}$ . Given a vector space  $W$  and some vectors  $\{w_1, \dots, w_n\}$ , there is a unique linear transformation  $T: V \rightarrow W$  such that

$$T(v_1) = w_1, \quad T(v_2) = w_2, \quad \dots, \quad T(v_n) = w_n.$$

**Definition 8** (null space, range). Fix a linear map  $T: V \rightarrow W$  of vector spaces.

- The *null space* of  $T$  is  $\text{null } T := \{v \in V : T(v) = 0\}$ . It is a subspace of  $V$ .
- The *range* of  $T$  is  $\text{range } T := \{T(v) : v \in V\}$ . It is a subspace of  $W$ .

**Definition 9** (injective, surjective, bijective). Fix a linear map  $T: V \rightarrow W$  of vector spaces.

- We say  $T$  is *injective* if and only if  $T(v) = T(v')$  implies  $v = v'$  for any  $v, v' \in V$ .
- We say  $T$  is *surjective* if and only if any vector  $w \in W$  has some vector  $v \in V$  such that  $T(v) = w$ .
- We say  $T$  is *bijective* or an *isomorphism* if and only if  $T$  is both injective and surjective.

**Remark.** One can show that  $T$  is injective if and only if  $\text{null } T = \{0\}$ , and  $T$  is surjective if and only if  $\text{range } T = W$ .

**Theorem 2.** Fix a linear map  $T: V \rightarrow W$  of vector spaces. Then

$$\dim V = \dim \text{null } T + \dim \text{range } T.$$

**Definition 10** (matrix). Fix a linear map  $T: V \rightarrow W$  of finite-dimensional vector spaces  $V$  and  $W$  with bases  $\{v_1, \dots, v_m\}$  and  $\{w_1, \dots, w_n\}$  respectively. For each  $v_i$ , find scalars  $a_{i1}, \dots, a_{in}$  such that

$$Tv_i = \sum_{j=1}^n a_{ij}w_j.$$

Then the matrix  $\{a_{ij}\}$  is the *matrix associated to  $T$* .

## Duality

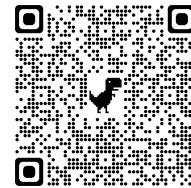
**Definition 11** (linear functional). Let  $V$  be a vector space,  $\phi$  is a linear functional on  $V$  if  $\phi \in \mathcal{L}(V, \mathbb{F})$ .

**Definition 12** (dual space). Fix a vector space  $V$ , its dual space  $V'$  is the space of all linear functionals on  $V$ . We write  $V' := \mathcal{L}(V, \mathbb{F})$ .

**Proposition 3.** Fix a vector space  $V$ . Then  $\dim V = \dim V'$ .

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**Definition 13** (dual operator). Fix a linear operator  $T: V \rightarrow W$ , then its dual operator  $T': W' \rightarrow V'$  is defined by  $T'(\phi) := \phi \circ T$  for any  $\phi \in W'$ .

**Definition 14** (dual basis). Let  $v_1, \dots, v_n$  be a basis of a finite-dimensional vector space  $V$ . Then the dual basis of this list is given by  $\varphi_1, \dots, \varphi_n \in V'$ , where each  $\varphi_j \in V'$  has

$$\varphi_j(v_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

**Proposition 4.** The dual basis defined as above is indeed a basis of  $V'$ .

**Proposition 5.** The matrix of the dual operator  $T'$  is the transpose of the matrix of  $T$ .

**Definition 15.** Let  $V$  be a vector space and  $U \subset V$  be a subspace. Then, the annihilator of  $U$ , denoted by  $U^0$ , is the set of all  $\phi \in V'$  such that  $\phi(u) = 0$  for all  $u \in U$ .

**Remark.** The annihilator  $U^0$  is a subspace of  $V'$ .

### Section II: Practice Exercises

Exercises are organized thematically, not by difficulty.

1. Given a matrix

$$A := \begin{bmatrix} a_{11} & \cdots & a_{15} \\ \vdots & \ddots & \vdots \\ a_{51} & \cdots & a_{55} \end{bmatrix} \in \mathbb{R}^{5 \times 5},$$

we say that  $A$  is *symmetric* if and only if  $a_{ij} = a_{ji}$  for each  $i$  and  $j$ , and we say that  $A$  is *skew-symmetric* if and only if  $a_{ij} = -a_{ji}$ .

- (a) Let  $Y$  and  $K$  be the set of symmetric and skew-symmetric matrices in  $\mathbb{R}^{5 \times 5}$ , respectively. Show that  $Y$  and  $K$  are subspaces of  $\mathbb{R}^{5 \times 5}$ .
- (b) Compute  $\dim Y$  and  $\dim K$ .
- (c) Show that  $\mathbb{R}^{5 \times 5}$  is the direct sum of the subspaces  $Y$  and  $K$ .

2. (Axler 2.C.17) Fix a finite-dimensional vector space  $V$  over a field  $F$ , and fix subspaces  $U_1, \dots, U_n$  of  $V$ .

- (a) Show that

$$\dim(U_1 + \cdots + U_n) \leq \dim U_1 + \cdots + \dim U_n.$$

- (b) In fact, show that equality holds in (a) if and only if the sum  $U_1 + \cdots + U_n$  is direct.

3. Fix a linear transformation  $T: V \rightarrow W$  of vector spaces over a field  $F$ . For any  $w \in W$ , suppose there is a vector  $v_0 \in V$  such that  $f(v_0) = w$ . Then show that

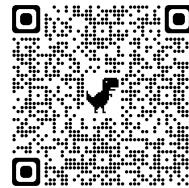
$$\{v \in V : f(v) = w\} = \{v + v_0 : v \in \text{null } T\}.$$

4. (Axler 3.B.28) Suppose  $p \in \mathcal{P}(\mathbb{R})$ . Prove that there exists a polynomial  $q \in \mathcal{P}(\mathbb{R})$  such that

$$5q'' + 3q' = p.$$

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5. Let  $I$  be the linear map  $I: \mathcal{P}_5(\mathbb{R}) \rightarrow \mathcal{P}_6(\mathbb{R})$  by

$$I(f) := \int_0^x f(t) dt.$$

- (a) Convince yourself that  $I$  is a linear map.
- (b) Find bases of  $\mathcal{P}_5(\mathbb{R})$  and  $\mathcal{P}_6(\mathbb{R})$ .
- (c) Use the bases found in (b) in order to write dual bases for  $\mathcal{P}_5(\mathbb{R})'$  and  $\mathcal{P}_6(\mathbb{R})'$ .
- (d) Use the dual bases found in (c) in order to write  $I': \mathcal{P}_6(\mathbb{R})' \rightarrow \mathcal{P}_5(\mathbb{R})'$  as a matrix.

6. Let  $V$  be a 2-dimensional vector space, and let  $\varphi, \psi \in V'$ . Show that  $\varphi$  and  $\psi$  are linearly independent if and only if

$$\text{null}(\varphi) \cap \text{null}(\psi) = \{0\}.$$

7. Fix a finite-dimensional vector space  $V$ . Call a linear map  $T: V \rightarrow V$  a *projection* if and only if  $T \circ T = T$ .

- (a) Give an example of a projection  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  which is neither the zero nor the identity operators.
- (b) For any projection  $T$ , show that  $T$  fixes range  $T$ .
- (c) For any projection  $T$ , show that  $\text{range } T \cap \text{null } T = \{0\}$ . Conclude that  $V = \text{range } T \oplus \text{null } T$ .
- (d) We say that a linear transformation  $T \in \mathcal{L}(V)$  is *diagonal* if and only if there is a basis  $\{v_1, \dots, v_n\}$  of  $V$  and constants  $\lambda_1, \dots, \lambda_n$  such that  $Tv_i = \lambda_i v_i$  for each  $i$ . Show that  $T$  can be written as a linear combination of projections.

8. (Axler 3.F.23) Let  $V$  be a finite-dimensional vector space, and let  $U$  and  $W$  be subspaces of  $V$ .

- (a) Show that  $(U + W)^0 = U^0 \cap W^0$ .
- (b) Show that  $(U \cap W)^0 = U^0 + W^0$ .