1 Suppose $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries.

SOLUTION Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Thus $\dim V = n$.

Suppose p of the columns of $\mathcal{M}(T)$ contain only 0's. Thus $Tv_k = 0$ for p different choices of k. Because $v_1, \ldots, v_n$ is a linear independent list, this implies that $\dim \operatorname{null} T \geq p$.

The fundamental theorem of linear maps (3.21) implies that

$$\begin{aligned} n - p &= \dim \operatorname{range} T + \dim \operatorname{null} T - p \\ &\geq \dim \operatorname{range} T. \end{aligned}$$

The definition of p implies that the number of columns of $\mathcal{M}(T)$ that have at least one nonzero entry is $n - p$. The inequality above implies that there are at least $\dim \operatorname{range} T$ such columns. Hence the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries.

2 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1.

SOLUTION First suppose there exist a basis $v_1, \ldots, v_n$ of V and a basis $w_1, \ldots, w_m$ of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1. Thus

$$Tv_k = w_1 + \cdots + w_m$$

for each $k = 1, \ldots, n$. Hence

$$\operatorname{range} T = \operatorname{span}(w_1 + \cdots + w_m)$$

and thus $\dim \operatorname{range} T = 1$ (note that $w_1 + \cdots + w_m \neq 0$ because $w_1, \ldots, w_m$ is linearly independent).

Conversely, suppose now that $\dim \operatorname{range} T = 1$. Let $n = \dim V$. By the fundamental theorem of linear maps (3.21), we have $\dim \operatorname{null} T = n - 1$. Let $u_1, \ldots, u_{n-1}$ be a basis of $\operatorname{null} T$. Extend to a basis $u_1, \ldots, u_{n-1}, u_n$ of V. Note that $u_n \notin \operatorname{null} T$ and thus $Tu_n \neq 0$. Let $w = Tu_n$.

It is easy to verify that $u_1 + u_n, \ldots, u_{n-1} + u_n, u_n$ is a basis of V, which we will relabel as $v_1, \ldots, v_n$. Clearly $Tv_k = w$ for each $k = 1, \ldots, n$.

Extend w to a basis $w, w_2, \ldots, w_m$ of W. It is easy to verify that

$$w - w_2, w_2 - w_3, \ldots, w_{m-1} - w_m, w_m$$

is a basis of W. Furthermore,

$$w = (w - w_2) + (w_2 - w_3) + \cdots + (w_{m-1} - w_m) + w_m.$$

Thus the matrix of T with respect to the basis $v_1, \ldots, v_n$ of V and the basis $w - w_2, w_2 - w_3, \ldots, w_{m-1} - w_m, w_m$ of W consists of all 1's.

3 Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W.

(a) Show that if $S, T \in \mathcal{L}(V, W)$, then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

(b) Show that if $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$, then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

This exercise asks you to verify 3.35 and 3.38.

SOLUTION

(a) Suppose $S, T \in \mathcal{L}(V, W)$. Let $A = \mathcal{M}(S)$ and $C = \mathcal{M}(T)$. Thus if $k = 1, \ldots, n$, then

$$Sv_k = \sum_{j=1}^{m} A_{j,k} w_j \quad \text{and} \quad Tv_k = \sum_{j=1}^{m} C_{j,k} w_j.$$

Hence

$$(S + T)v_k = \sum_{j=1}^{m} (A_{j,k} + C_{j,k}) w_j.$$

The equation above implies that the entry in row j, column k, of $\mathcal{M}(S + T)$ equals $A_{j,k} + C_{j,k}$. Thus $\mathcal{M}(S + T) = A + C = \mathcal{M}(S) + \mathcal{M}(T)$.

(b) Suppose $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$. Let $A = \mathcal{M}(T)$. Thus if $k = 1, \ldots, n$, then

$$Tv_k = \sum_{j=1}^{m} A_{j,k} w_j.$$

Hence

$$(\lambda T)v_k = \sum_{j=1}^{m} \lambda A_{j,k} w_j.$$

The equation above implies that the entry in row j, column k, of $\mathcal{M}(\lambda T)$ equals $\lambda A_{j,k}$. Thus $\mathcal{M}(\lambda T) = \lambda A = \lambda \mathcal{M}(T)$.

4 Suppose that $D \in \mathcal{L}(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbf{R})$ and a basis of $\mathcal{P}_2(\mathbf{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Compare with Example 3.33. The next exercise generalizes this exercise.

SOLUTION Take $x^3, x^2, x, 1$ as a basis of $\mathcal{P}_3(\mathbf{R})$ and take $3x^2, 2x, 1$ as a basis of $\mathcal{P}_2(\mathbf{R})$. Then the matrix of D with respect to these bases is the matrix above.

5 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row k, column k, equal 1 if $1 \leq k \leq \dim \operatorname{range} T$.

SOLUTION Let $u_1, \ldots, u_m$ be a basis of $\operatorname{null} T$. Extend $u_1, \ldots, u_m$ to a basis $u_1, \ldots, u_m, v_1, \ldots, v_n$ of V. Then $Tv_1, \ldots, Tv_n$ is a basis of $\operatorname{range} T$, as was proved in the proof of 3.21. Thus $n = \dim \operatorname{range} T$.

Because $Tv_1, \ldots, Tv_n$ is a basis of the subspace $\operatorname{range} T$, this list is linearly independent in W. Extend the linearly independent list $Tv_1, \ldots, Tv_n$ to a basis $Tv_1, \ldots, Tv_n, w_1, \ldots, w_p$ of W.

With respect to the basis $v_1, \ldots, v_n, u_1, \ldots, u_m$ of V (note that the v's now come before the u's) and the basis $Tv_1, \ldots, Tv_n, w_1, \ldots, w_p$ of W, the matrix of T has the desired form.

[Here some of m, n, or p might be 0.]

6 Suppose $v_1, \ldots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \ldots, w_n$ of W such that all entries in the first column of $\mathcal{M}(T)$ [with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$] are 0 except for possibly a 1 in the first row, first column.

In this exercise, unlike Exercise 5, you are given the basis of V instead of being able to choose a basis of V.

SOLUTION If $Tv_1 = 0$, then the first column of $\mathcal{M}(T)$ will consist of all 0's for every choice of basis of W (using, of course, $v_1, \ldots, v_m$ as the basis of V).

Thus suppose $Tv_1 \neq 0$. Let $w_1 = Tv_1$. Extend the list w_1 to a basis $w_1, \ldots, w_n$ of W. Then with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$, the first column of $\mathcal{M}(T)$ consists of all 0's except for a 1 in the first row, first column.

7 Suppose $w_1, \ldots, w_n$ is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $v_1, \ldots, v_m$ of V such that all entries in the first row of $\mathcal{M}(T)$ [with respect to the bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$] are 0 except for possibly a 1 in the first row, first column.

In this exercise, unlike Exercise 5, you are given the basis of W instead of being able to choose a basis of W.

SOLUTION If range $T \subseteq \operatorname{span}(w_2, \ldots, w_n)$, then the first row of $\mathcal{M}(T)$ will consist of all 0's for every choice of basis of V (using, of course, $w_1, \ldots, w_n$ as the basis of W).

Thus suppose range $T \not\subseteq \operatorname{span}(w_2, \ldots, w_n)$. Let $u_1 \in V$ be such that $Tu_1 \notin \operatorname{span}(w_2, \ldots, w_n)$. Because $w_1, \ldots, w_n$ is a basis of W, we can write

$$Tu_1 = c_1 w_1 + \cdots + c_n w_n$$

for some $c_1, \ldots, c_n \in \mathbf{F}$. Because $Tu_1 \notin \operatorname{span}(w_2, \ldots, w_n)$, we know that $c_1 \neq 0$. Thus replacing u_1 with $\frac{1}{c_1} u_1$, we can assume that $c_1 = 1$. In other words,

$$Tu_1 = w_1 + c_2 w_2 + \cdots + c_n w_n.$$

Extend u_1 to a basis $u_1, \ldots, u_m$ of V. For each $k \in \{2, \ldots, m\}$, we can write

$$Tu_k = a_{1,k} w_1 + \cdots + a_{n,k} w_n.$$

Thus

$$T(u_k - a_{1,k} u_1) = (a_{2,k} - a_{1,k} c_2) w_2 + \cdots + (a_{n,k} - a_{1,k} c_n) w_n.$$

Thus with respect to the basis

$$u_1, u_2 - a_{1,2} u_1, u_3 - a_{1,3} u_1, \ldots, u_m - a_{1,m} u_1$$

of V and $w_1, \ldots, w_n$ of W, we see that the first row of $\mathcal{M}(T)$ consists of all 0's except for a 1 in row 1, column 1.

8 Suppose A is an m-by-n matrix and B is an n-by-p matrix. Prove that

$$(AB)_{j,\cdot} = A_{j,\cdot}\, B$$

for each $1 \leq j \leq m$. In other words, show that row j of AB equals (row j of A) times B.

This exercise gives the row version of 3.48.

SOLUTION Note that both $(AB)_{j,\cdot}$ and $A_{j,\cdot}\, B$ are 1-by-p matrices.

By the definition of matrix multiplication, the entry in column k of $(AB)_{j,\cdot}$ is

$$\sum_{r=1}^{n} A_{j,r} B_{r,k}.$$

Again by the definition of matrix multiplication, the entry in column k of $A_{j,\cdot}\, B$ is also equal to

$$\sum_{r=1}^{n} A_{j,r} B_{r,k}.$$

Thus $(AB)_{j,\cdot} = A_{j,\cdot}\, B$.

9 Suppose $a = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$ is a 1-by-n matrix and B is an n-by-p matrix. Prove that

$$aB = a_1 B_{1,\cdot} + \cdots + a_n B_{n,\cdot}\,.$$

In other words, show that aB is a linear combination of the rows of B, with the scalars that multiply the rows coming from a.

This exercise gives the row version of 3.50.

SOLUTION Note that both aB and $a_1 B_{1,\cdot} + \cdots + a_n B_{n,\cdot}$ are 1-by-p matrices.

By the definition of matrix multiplication, the entry in column k of aB is

$$\sum_{r=1}^{n} a_r B_{r,k}.$$

For $r = 1, \ldots, n$, the entry in column k of $a_r B_{r,\cdot}$ is $a_r B_{r,k}$. Thus the entry in column k of $a_1 B_{1,\cdot} + \cdots + a_n B_{n,\cdot}$ is

$$\sum_{r=1}^{n} a_r B_{r,k}.$$

Thus $aB = a_1 B_{1,\cdot} + \cdots + a_n B_{n,\cdot}\,.$

10 Give an example of 2-by-2 matrices A and B such that $AB \neq BA$.

SOLUTION Almost any randomly chosen 2-by-2 matrices will work.

11 Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, C, D, E, and F are matrices whose sizes are such that $A(B+C)$ and $(D+E)F$ make sense. Explain why $AB + AC$ and $DF + EF$ both make sense and prove that

$$A(B+C) = AB + AC \quad \text{and} \quad (D+E)F = DF + EF.$$

SOLUTION Because $A(B+C)$ makes sense, B and C have the same size. Furthermore, the number of columns of A (let's call this number n) equals the number of rows of B and C. All this means that $AB + AC$ makes sense.

To prove that $A(B+C) = AB+AC$, just use the definition of matrix addition, the definition of matrix multiplication, and the usual distributive property for elements of $\mathbf{F}$. Specifically, let $A_{j,k}$, $b_{j,k}$, and $c_{j,k}$ denote the entries in row j, column k of A, B, and C, respectively. The entry in row j, column k of $B + C$ is $b_{j,k} + c_{j,k}$. Thus the entry in row j, column k of $A(B+C)$ is

$$\sum_{r=1}^{n} A_{j,r}(b_{r,k} + c_{r,k}),$$

which equals

$$\sum_{r=1}^{n} A_{j,r}b_{r,k} + \sum_{r=1}^{n} A_{j,r}c_{r,k},$$

which equals the entry in row j, column k of $AB + AC$, as desired.

12 Prove that matrix multiplication is associative. In other words, suppose A, B, and C are matrices whose sizes are such that $(AB)C$ makes sense. Explain why $A(BC)$ makes sense and prove that

$$(AB)C = A(BC).$$

Try to find a clean proof that illustrates the following quote from Emil Artin: "It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."

SOLUTION This exercise can be done by a brute force calculation, in the style of the solution to Exercise 11. Here is a solution that uses only the associativity of the product of linear maps (which is easy to verify because composition of functions is clearly associative) and the nice property that the matrix of the product of two linear maps equals the product of the matrices of the two linear maps (see 3.43).

Suppose A is an m-by-n matrix, B is an n-by-p matrix, and C is a p-by-q matrix; the sizes much match up like this in order for $(AB)C$ to make sense. Let $R \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$, $S \in \mathcal{L}(\mathbf{F}^p, \mathbf{F}^n)$, $T \in \mathcal{L}(\mathbf{F}^q, \mathbf{F}^p)$ be such that, with respect to the standard bases, $\mathcal{M}(R) = A$, $\mathcal{M}(S) = B$, $\mathcal{M}(T) = C$; 3.71 ensures that such linear maps exist. Now

$$\begin{aligned}
(AB)C &= (\mathcal{M}(R)\mathcal{M}(S))\mathcal{M}(T)\\
&= \mathcal{M}(RS)\mathcal{M}(T)\\
&= \mathcal{M}((RS)T)\\
&= \mathcal{M}(R(ST))\\
&= \mathcal{M}(R)\mathcal{M}(ST)\\
&= \mathcal{M}(R)(\mathcal{M}(S)\mathcal{M}(T))\\
&= A(BC).
\end{aligned}$$

13 Suppose A is an n-by-n matrix and $1 \leq j, k \leq n$. Show that the entry in row j, column k, of A^3 (which is defined to mean AAA) is

$$\sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p} A_{p,r} A_{r,k}.$$

SOLUTION By the definition of matrix multiplication, we have

$$\begin{aligned}
(A^3)_{j,k} &= (AA^2)_{j,k} \\
&= \sum_{p=1}^{n} A_{j,p} (A^2)_{p,k} \\
&= \sum_{p=1}^{n} A_{j,p} (AA)_{p,k} \\
&= \sum_{p=1}^{n} A_{j,p} \sum_{r=1}^{n} A_{p,r} A_{r,k} \\
&= \sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p} A_{p,r} A_{r,k}.
\end{aligned}$$

14 Suppose m and n are positive integers. Prove that the function $A \mapsto A^{t}$ is a linear map from $\mathbf{F}^{m,n}$ to $\mathbf{F}^{n,m}$.

SOLUTION The linearity of the map that takes A to A^{t} follows from the definitions.

15 Prove that if A is an m-by-n matrix and C is an n-by-p matrix, then

$$(AC)^{\mathrm{t}} = C^{\mathrm{t}} A^{\mathrm{t}}.$$

This exercise shows that the transpose of the product of two matrices is the product of the transposes in the opposite order.

SOLUTION Suppose $1 \leq k \leq p$ and $1 \leq j \leq m$. Then

$$\begin{aligned} \big((AC)^{\mathrm{t}}\big)_{k,j} &= (AC)_{j,k} \\ &= \sum_{r=1}^{n} A_{j,r} C_{r,k} \\ &= \sum_{r=1}^{n} (C^{\mathrm{t}})_{k,r} (A^{\mathrm{t}})_{r,j} \\ &= (C^{\mathrm{t}} A^{\mathrm{t}})_{k,j}. \end{aligned}$$

Thus $(AC)^{\mathrm{t}} = C^{\mathrm{t}} A^{\mathrm{t}}$, as desired.

16 Suppose A is an m-by-n matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, \ldots, c_m) \in \mathbf{F}^m$ and $(d_1, \ldots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \ldots, m$ and every $k = 1, \ldots, n$.

SOLUTION First suppose there exist $(c_1, \ldots, c_m) \in \mathbf{F}^m$ and $(d_1, \ldots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \ldots, m$ and every $k = 1, \ldots, n$. Let

$$d = \begin{pmatrix} d_1 & d_2 & \ldots & d_n \end{pmatrix} \in \mathbf{F}^{1,n}.$$

Then row j of A equals $c_j d$ for each $j = 1, \ldots, m$. Thus each row of A is in $\operatorname{span}(d)$. Thus the row rank of A is less than or equal to 1. However, the rank of A does not equal 0 (because $A \neq 0$), and hence the rank of A is 1.

To prove the other direction, suppose the rank of A is 1. Let

$$d = \begin{pmatrix} d_1 & d_2 & \ldots & d_n \end{pmatrix} \in \mathbf{F}^{1,n}$$

be a row of A that is not identically 0 (such a row must exist because $A \neq 0$). Because A has row rank 1, for each $j = 1, \ldots, m$ there exists $c_j \in \mathbf{F}$ such that row j of A equals $c_j d$. Hence $A_{j,k} = c_j d_k$ for every $j = 1, \ldots, m$ and every $k = 1, \ldots, n$, as desired.

17 Suppose $T \in \mathcal{L}(V)$, and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Prove that the following are equivalent.

(a) T is injective.

(b) The columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$.

(c) The columns of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$.

(d) The rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$.

(e) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{1,n}$.

Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$.

SOLUTION First suppose (a) holds, so T is injective. Define $S\colon V \to \mathbf{F}^{n,1}$ by

$$Sv = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix},$$

where $a_1, \ldots, a_n \in \mathbf{F}$ are such that $v = a_1 v_1 + \cdots + a_n v_n$. Then S is an injective linear map from V to $\mathbf{F}^{n,1}$. Thus ST is also an injective linear map from V to $\mathbf{F}^{n,1}$. Hence $STu_1, \ldots, STu_n$ is a linearly independent list in $\mathbf{F}^{n,1}$ (by Exercise 9 in Section 3B). Because $STu_1, \ldots, STu_n$ is the list of columns of $\mathcal{M}(T)$, we conclude that the columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$, which shows that (a) implies (b).

Now suppose (b) holds, so the columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$. Then, because we have a linearly independent list of length n in a vector space of dimension n, the list of columns of $\mathcal{M}(T)$ spans $\mathbf{F}^{n,1}$ (by 2.38). Thus (b) implies (c).

Now suppose (c) holds, so the columns of $\mathcal{M}(T)$ span $\mathbf{F}^{n,1}$. Thus the column rank of $\mathcal{M}(T)$ equals n, which implies that the row rank of $\mathcal{M}(T)$ equals n (by 3.57). Hence the rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$, proving that (c) implies (d).

Now suppose (d) holds, so the rows of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$. Then, because we have a spanning list of length n in a vector space of dimension n, the list of rows of $\mathcal{M}(T)$ is linearly independent in $\mathbf{F}^{1,n}$ (by 2.42). Thus (d) implies (e).

Now suppose (e) holds, so the rows of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{1,n}$. Then, because we have a linearly independent list of length n in a vector space of dimension n, this list also spans $\mathbf{F}^{1,n}$ (by 2.38). Thus the row rank of $\mathcal{M}(T)$ equals n, which implies that the column rank of $\mathcal{M}(T)$ equals n (by 3.57). Hence the columns of $\mathcal{M}(T)$ span $\mathbf{F}^{1,n}$. Thus the columns of $\mathcal{M}(T)$ are linearly independent in $\mathbf{F}^{n,1}$ (by 2.42).

To prove that T is injective, suppose $u \in V$and $Tu = 0$. There exist $b_1, \ldots, b_n \in \mathbf{F}$ such that

$$u = b_1 u_1 + \cdots + b_n u_n.$$

Thus

$$0 = Tu = b_1 T u_1 + \cdots + b_n T u_n.$$

Applying the linear map S defined in the first paragraph of this solution to both sides of the equation above, we have

$$0 = STu = b_1 STu_1 + \cdots + b_n STu_n.$$

However, $STu_1, \ldots, STu_n$ are the columns of $\mathcal{M}(T)$, which are linearly independent by the paragraph above. Thus the equation above implies that $b_1 = \cdots = b_n = 0$. This implies that $u = 0$. Hence T is injective, completing the proof that (e) implies (a).

1 Explain why each linear functional is surjective or is the zero map.

SOLUTION A linear functional is a linear map into $\mathbf{F}$. The range of a linear map is a subspace of the target space. The only subspaces of $\mathbf{F}$ are $\mathbf{F}$ and $\{0\}$. Thus each linear functional is surjective (if the range equals $\mathbf{F}$) or is the zero map (if the range equals $\{0\}$).

2 Give three distinct examples of linear functionals on $\mathbf{R}^{[0,1]}$.

SOLUTION For $t \in [0,1]$, define $\varphi_t \colon \mathbf{R}^{[0,1]} \to \mathbf{R}$ by

$$\varphi_t(f) = f(t)$$

for each $f \in \mathbf{R}^{[0,1]}$.

For each $t \in [0,1]$, φ_t is a linear functional on $\mathbf{R}^{[0,1]}$. Thus three distinct linear functionals on $\mathbf{R}^{[0,1]}$ are φ_0, $\varphi_{\frac{1}{2}}$, and φ_1.

3 Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

SOLUTION Because $v \neq 0$, we can extend v to a basis $v, v_2, \ldots, v_n$ of V (by 2.32). Now the linear map lemma (3.4) implies that there exists $\varphi \in V'$ such that $\varphi(v) = 1$ [and $\varphi(v_k)$ equals whatever we want for each $k = 2, \ldots, n$].

4 Suppose V is finite-dimensional and U is a subspace of V such that $U \neq V$. Prove that there exists $\varphi \in V'$ such that $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$.

SOLUTION Let $u_1, \ldots, u_m$ be a basis of U. Extend this list to a basis

$$u_1, \ldots, u_m, v_1, \ldots, v_n$$

of V (using 2.32). Now the linear map lemma (3.4) implies that there exists $\varphi \in V'$ such that

$$\varphi(u_k) = 0 \text{ for } k = 1, \ldots, m$$

and

$$\varphi(v_k) = 1 \text{ for } k = 1, \ldots, n.$$

Thus $\varphi(u) = 0$ for every $u \in U$ but $\varphi \neq 0$, as desired.

5 Suppose $T \in \mathcal{L}(V, W)$ and $w_1, \dots, w_m$ is a basis of range T. Hence for each $v \in V$, there exist unique numbers $\varphi_1(v), \dots, \varphi_m(v)$ such that

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m,$$

thus defining functions $\varphi_1, \dots, \varphi_m$ from V to $\mathbf{F}$. Show that each of the functions $\varphi_1, \dots, \varphi_m$ is a linear functional on V.

SOLUTION Suppose $u, v \in V$. Then

$$\begin{aligned}
&\varphi_1(u+v)w_1 + \cdots + \varphi_m(u+v)w_m \\
&\qquad = T(u+v) \\
&\qquad = Tu + Tv \\
&\qquad = \big(\varphi_1(u)w_1 + \cdots + \varphi_m(u)w_m\big) + \big(\varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m\big) \\
&\qquad = \big(\varphi_1(u) + \varphi_1(v)\big)w_1 + \cdots + \big(\varphi_w(u) + \varphi_m(v)\big)w_m.
\end{aligned}$$

Because $w_1, \dots, w_m$ is linearly independent, the equation above shows that

$$\varphi_1(u+v) = \varphi_1(u) + \varphi_1(v),\ \dots,\ \varphi_m(u+v) = \varphi_m(u) + \varphi_m(v).$$

Similarly, suppose now that $\lambda \in \mathbf{F}$ and $v \in V$. Then

$$\begin{aligned}
\varphi_1(\lambda v)w_1 + \cdots + \varphi_m(\lambda v)w_m &= T(\lambda v) \\
&= \lambda Tv \\
&= \lambda\big(\varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m\big) \\
&= \lambda\varphi_1(v)w_1 + \cdots + \lambda\varphi_m(v)w_m.
\end{aligned}$$

Because $w_1, \dots, w_m$ is linearly independent, the equation above shows that

$$\varphi_1(\lambda v) = \lambda\varphi_1(v),\ \dots,\ \varphi_m(\lambda v) = \lambda\varphi_m(v),$$

completing the proof that each of the functions $\varphi_1, \dots, \varphi_m$ is a linear functional on V.

6 Suppose $\varphi, \beta \in V'$. Prove that $\operatorname{null}\varphi \subseteq \operatorname{null}\beta$ if and only if there exists $c \in \mathbf{F}$ such that $\beta = c\varphi$.

SOLUTION If $\beta = c\varphi$ for some $c \in \mathbf{F}$, then clearly $\operatorname{null}\varphi \subseteq \operatorname{null}\beta$.

The implication in the other direction follows from Exercise 25 in Section 3B (with $W = \mathbf{F}$) and from Exercise 7 in Section 3A.

7 Suppose that $V_1, \dots, V_m$ are vector spaces. Prove that $(V_1 \times \cdots \times V_m)'$ and ${V_1}' \times \cdots \times {V_m}'$ are isomorphic vector spaces.

SOLUTION For $k = 1, \dots, m$ and $\varphi \in (V_1 \times \cdots \times V_m)'$, define $\varphi_k \in {V_k}'$ by

$$\varphi_k(v) = \varphi(0, \dots, 0, v, 0, \dots, 0),$$

for each $v \in V_k$, where the v on the right side above appears in the k^{th} slot.

Now define $\Gamma\colon (V_1 \times \cdots \times V_m)' \to {V_1}' \times \cdots \times {V_m}'$ by

$$\Gamma\varphi = (\varphi_1, \dots, \varphi_n).$$

It is easy to verify that Γ is a linear map and that Γ is injective and surjective. Thus $(V_1 \times \cdots \times V_m)'$ and ${V_1}' \times \cdots \times {V_m}'$ are isomorphic vector spaces.

8 Suppose $v_1, \ldots, v_n$ is a basis of V and $\varphi_1, \ldots, \varphi_n$ is the dual basis of V'. Define $\Gamma\colon V \to \mathbf{F}^n$ and $\Lambda\colon \mathbf{F}^n \to V$ by

$$\Gamma(v) = (\varphi_1(v), \ldots, \varphi_n(v)) \quad \text{and} \quad \Lambda(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n.$$

Explain why Γ and Λ are inverses of each other.

SOLUTION First suppose $a_1, \ldots, a_n \in \mathbf{F}^n$. Then

$$\begin{aligned}\Gamma(\Lambda(a_1, \ldots, a_n)) &= \Gamma(a_1 v_1 + \cdots + a_n v_n)\\ &= (\varphi_1(a_1 v_1 + \cdots + a_n v_n), \ldots, \varphi_n(a_1 v_1 + \cdots + a_n v_n))\\ &= (a_1, \ldots, a_n).\end{aligned}$$

Also,

$$\begin{aligned}\Lambda(\Gamma(v)) &= \Lambda(\varphi_1(v), \ldots, \varphi_n(v))\\ &= \varphi_1(v) v_1 + \cdots \varphi_n(v) v_n\\ &= v,\end{aligned}$$

where the last line follows from 3.114.

9 Suppose m is a positive integer. Show that the dual basis of the basis $1, x, \dots, x^m$ of $\mathcal{P}_m(\mathbf{R})$ is $\varphi_0, \varphi_1, \dots, \varphi_m$, where

$$\varphi_k(p) = \frac{p^{(k)}(0)}{k!}.$$

Here $p^{(k)}$ denotes the k^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.

SOLUTION Note that

$$\varphi_k(x^j) = \frac{(x^j)^{(k)}(0)}{k!} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Thus $\varphi_0, \varphi_1, \dots, \varphi_m$ is indeed the dual basis of $1, x, \dots, x^m$.

10 Suppose m is a positive integer.

(a) Show that $1, x-5, \dots, (x-5)^m$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

(b) What is the dual basis of the basis in (a)?

SOLUTION

(a) Define $\varphi_0, \varphi_1, \dots, \varphi_m \in (\mathcal{P}_m(\mathbf{R}))'$ by

$$\varphi_j(p) = \frac{p^{(j)}(5)}{j!}.$$

Suppose $a_0, a_1, \dots, a_m \in \mathbf{R}$ and

$$a_0 + a_1(x-5) + \cdots + a_m(x-5)^m = 0.$$

Then for each $j = 0, 1, \dots, m$, we have

$$a_j = \varphi_j(a_0 + a_1(x-5) + \cdots + a_m(x-5)^m) = \varphi_j(0) = 0.$$

Thus $a_0 = a_1 = \cdots = a_m = 0$. Hence $1, x-5, \dots, (x-5)^m$ is linearly independent list in $\mathcal{P}_m(\mathbf{R})$ of length $m+1$, which equals the dimension of $\mathcal{P}_m(\mathbf{R})$. Thus $1, x-5, \dots, (x-5)^m$ is a basis of $\mathcal{P}_m(\mathbf{R})$ (by 2.38).

(b) Let $\varphi_0, \varphi_1, \dots, \varphi_m \in (\mathcal{P}_m(\mathbf{R}))'$ be defined as in (a). Then

$$\varphi_j((x-5)^k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Thus $\varphi_0, \varphi_1, \dots, \varphi_m$ is the dual basis of $1, x-5, \dots, (x-5)^m$.

11 Suppose $v_1, ..., v_n$ is a basis of V and $\varphi_1, ..., \varphi_n$ is the corresponding dual basis of V'. Suppose $\psi \in V'$. Prove that

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n.$$

SOLUTION We have

$$\big(\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n\big)(v_k) = \psi(v_k)$$

for each $k = 1, ..., n$. Because the linear functionals $\psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n$ and ψ agree on a basis, they are equal by the uniqueness part of the linear map lemma (3.4).

12 Suppose $S, T \in \mathcal{L}(V, W)$.

(a) Prove that $(S+T)' = S' + T'$.

(b) Prove that $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbf{F}$.

This exercise asks you to verify (a) and (b) in 3.120.

SOLUTION

(a) We have

$$\begin{aligned}(S+T)'(\varphi) &= \varphi \circ (S+T)\\ &= \varphi \circ S + \varphi \circ T\\ &= S'(\varphi) + T'(\varphi).\end{aligned}$$

Thus $(S+T)' = S' + T'$, as desired.

(b) Suppose $\lambda \in \mathbf{F}$. Then

$$\begin{aligned}(\lambda T)'(\varphi) &= \varphi \circ (\lambda T)\\ &= \lambda(\varphi \circ T)\\ &= \lambda T'(\varphi).\end{aligned}$$

Thus $(\lambda T)' = \lambda T'$, as desired.

13 Show that the dual map of the identity operator on V is the identity operator on V'.

SOLUTION Let I denote the identity operator on V. For $\varphi \in V'$ we have

$$I'(\varphi) = \varphi \circ I = \varphi.$$

Thus I' is the identity operator on V'.

14 Define $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T(x, y, z) = (4x + 5y + 6z, 7x + 8y + 9z).$$

Suppose φ_1, φ_2 denotes the dual basis of the standard basis of $\mathbf{R}^2$ and ψ_1, ψ_2, ψ_3 denotes the dual basis of the standard basis of $\mathbf{R}^3$.

(a) Describe the linear functionals $T'(\varphi_1)$ and $T'(\varphi_2)$.

(b) Write $T'(\varphi_1)$ and $T'(\varphi_2)$ as linear combinations of ψ_1, ψ_2, ψ_3.

SOLUTION

(a) Note that $\varphi_1(a, b) = a$ and $\varphi_2(a, b) = b$ for all $(a, b) \in \mathbf{R}^2$.
Recall that $T'(\varphi_1) = \varphi_1 \circ T$ and $T'(\varphi_2) = \varphi_2 \circ T$. Thus

$$(T'(\varphi_1))(x, y, z) = 4x + 5y + 6z$$

and

$$(T'(\varphi_2))(x, y, z) = 7x + 8y + 9z.$$

(b) We have $\psi_1(x, y, z) = x$, $\psi_2(x, y, z) = y$, and $\psi_3(x, y, z) = z$. Using (a), we thus see that

$$T'(\varphi_1) = 4\psi_1 + 5\psi_2 + 6\psi_3$$

and

$$T'(\varphi_2) = 7\psi_1 + 8\psi_2 + 9\psi_3.$$

15 Define $T: \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by

$$(Tp)(x) = x^2 p(x) + p''(x)$$

for each $x \in \mathbf{R}$.

(a) Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = p'(4)$. Describe the linear functional $T'(\varphi)$ on $\mathcal{P}(\mathbf{R})$.

(b) Suppose $\varphi \in \mathcal{P}(\mathbf{R})'$ is defined by $\varphi(p) = \int_0^1 p$. Evaluate $(T'(\varphi))(x^3)$.

SOLUTION

(a) Suppose $p \in \mathcal{P}(\mathbf{R})$. Then

$$\begin{aligned}(T'(\varphi))(p) &= (\varphi \circ T)(p)\\ &= \varphi(Tp)\\ &= \varphi(x^2 p + p'')\\ &= (x^2 p + p'')'(4)\\ &= (2xp + x^2 p' + p''')(4)\\ &= 8p(4) + 16p'(4) + p'''(4).\end{aligned}$$

(b) We have

$$\begin{aligned}(T'(\varphi))(x^3) &= (\varphi \circ T)(x^3)\\ &= \varphi(Tx^3)\\ &= \varphi(x^5 + 6x)\\ &= \int_0^1 (x^5 + 6x)\,dx\\ &= \frac{19}{6}.\end{aligned}$$

16 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that

$$T' = 0 \iff T = 0.$$

SOLUTION First suppose $T = 0$. If $\varphi \in W'$, then $T'(\varphi) = \varphi \circ T = 0$, and thus $T' = 0$.

To prove the other direction, now suppose $T' = 0$. Thus

$$0 = T'(\varphi) = \varphi \circ T$$

for every $\varphi \in W'$. Thus

$$0 = \varphi(Tv)$$

for every $v \in V$ and every $\varphi \in W'$. Exercise 3 now implies that $Tv = 0$ for every $v \in V$. Thus $T = 0$, as desired.

17 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is invertible if and only if $T' \in \mathcal{L}(W', V')$ is invertible.

SOLUTION A linear map is invertible if and only if it is injective and surjective. Thus the desired result follows from 3.129 and 3.131.

18 Suppose V and W are finite-dimensional. Prove that the map that takes $T \in \mathcal{L}(V, W)$ to $T' \in \mathcal{L}(W', V')$ is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$.

SOLUTION Define $\Gamma\colon \mathcal{L}(V, W) \to \mathcal{L}(W', V')$ by $\Gamma(T) = T'$. By 3.120, Γ is a linear map.

Exercise 16 implies that Γ is injective. Note that

$$\begin{aligned}\dim \mathcal{L}(V, W) &= (\dim V)(\dim W)\\ &= (\dim V')(\dim W')\\ &= (\dim W')(\dim V')\\ &= \dim \mathcal{L}(W', V'),\end{aligned}$$

where the first and last equalities above come from 3.72 and the second equality above comes from 3.111.

The fundamental theorem of linear maps (3.21) and the equation above now imply that $\dim \operatorname{range} \Gamma = \dim(\mathcal{L}(W', V')$. Thus Γ is surjective and hence is an isomorphism of $\mathcal{L}(V, W)$ onto $\mathcal{L}(W', V')$, as desired.