1 Suppose $b, c \in \mathbf{R}$. Define $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b,\, 6x + cxyz).$$

Show that T is linear if and only if $b = c = 0$.

SOLUTION First suppose $b = c = 0$. Then

$$T(x, y, z) = (2x - 4y + 3z,\, 6x),$$

which easily implies that T is linear.

Conversely, now suppose that T is linear. Note that

$$T(0, 0, 0) = (b, 0).$$

Now 3.10 implies that $b = 0$.

Note that

$$T(1, 1, 1) = (1, 6 + c) \quad \text{and} \quad T(2, 2, 2) = (2, 12 + 8c).$$

The equation $T(2, 2, 2) = 2T(1, 1, 1)$ implies that $12 + 8c = 12 + 2c$, which implies that $c = 0$.

2 Suppose $b, c \in \mathbf{R}$. Define $T\colon \mathcal{P}(\mathbf{R}) \to \mathbf{R}^2$ by

$$Tp = \bigl(3p(4) + 5p'(6) + bp(1)p(2),\ \int_{-1}^{2} x^3 p(x)\,dx + c\sin p(0)\bigr).$$

Show that T is linear if and only if $b = c = 0$.

SOLUTION First suppose $b = c = 0$. Then

$$Tp = \bigl(3p(4) + 5p'(6),\ \int_{-1}^{2} x^3 p(x)\,dx\bigr),$$

which easily implies that T is linear.

Conversely, now suppose that T is linear. Note that

$$T1 = \bigl(3 + b,\ \tfrac{15}{4} + c\sin 1\bigr) \quad \text{and} \quad T2 = \bigl(6 + 4b,\ \tfrac{15}{2} + c\sin 2\bigr)$$

The equation $T2 = 2T1$ implies that $6 + 4b = 6 + 2b$ and $\frac{15}{2} + c\sin 2 = \frac{15}{2} + 2c\sin 1$, which implies that $b = 0$ and $c = 0$.

3 Suppose that $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbf{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}\,x_n,\ \ldots,\ A_{m,1}x_1 + \cdots + A_{m,n}\,x_n)$$

for every $(x_1, \ldots, x_n) \in \mathbf{F}^n$.

This exercise shows that the linear map T has the form promised in the second to last item of Example 3.3.

SOLUTION Let $e_1, \ldots, e_n$ denote the standard basis of $\mathbf{F}^n$. Furthermore, let $f_1, \ldots, f_m$ denote the standard basis of $\mathbf{F}^m$. Then for $k = 1, \ldots, n$, there are numbers $A_{j,k}$ such that

$$Te_k = \sum_{j=1}^{m} A_{j,k} f_j.$$

Thus

$$\begin{aligned} T(x_1, \ldots, x_n) &= x_1 Te_1 + \cdots + x_n Te_n \\ &= x_1 \sum_{j=1}^{m} A_{j,1} f_j + \cdots + x_n \sum_{j=1}^{m} A_{j,n} f_j \\ &= \sum_{j=1}^{m} (A_{j,1}x_1 + \cdots + A_{j,n}x_n) f_j. \end{aligned}$$

Thus the j^{th}-coordinate of $T(x_1, \ldots, x_n)$ is $A_{j,1}x_1 + \cdots + A_{j,n}x_n$, as desired.

4 Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_m$ is a list of vectors in V such that $Tv_1, \dots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \dots, v_m$ is linearly independent.

SOLUTION Suppose $c_1, \dots, c_n \in \mathbf{F}$ are such that

$$c_1 v_1 + \cdots + c_n v_n = 0.$$

Applying T to both sides of the equation above, we have

$$c_1 T v_1 + \cdots + c_n T v_n = 0.$$

Because $Tv_1, \dots, Tv_m$ is linearly independent, this implies that

$$c_1 = \cdots = c_n = 0.$$

Thus $v_1, \dots, v_m$ is linearly independent.

5 Prove that $\mathcal{L}(V,W)$ is a vector space, as was asserted in 3.6.

SOLUTION Suppose $S,T \in \mathcal{L}(V,W)$. Then

$$(S+T)(v) = Sv + Tv = Tv + Sv = (T+S)v$$

for every $v \in V$. Thus $S+T = T+S$. In other words, addition is commutative on $\mathcal{L}(V,W)$.

Suppose $R,S,T \in \mathcal{L}(V,W)$. Then

$$\begin{aligned}((R+S)+T)(v) &= (R+S)v + Tv\\ &= (Rv+Sv)+Tv\\ &= Rv+(Sv+Tv)\\ &= Rv+(S+T)v\\ &= (R+(S+T))(v)\end{aligned}$$

for every $v \in V$. Thus $(R+S)+T = R+(S+T)$. In other words, addition is associative on $\mathcal{L}(V,W)$.

Suppose $a,b \in \mathbf{F}$ and $T \in \mathcal{L}(V,W)$. Then

$$\begin{aligned}((ab)T)(v) &= (ab)(Tv)\\ &= a(bTv)\\ &= (a(bT))(v)\end{aligned}$$

for every $v \in V$. Thus $(ab)T = a(bT)$.

The linear map $0 \in \mathcal{L}(V,W)$ defined by $0v = 0$ (where the 0 on the right is the additive identity for W) is clearly an additive identity for $\mathcal{L}(V,W)$.

For $T \in \mathcal{L}(V,W)$, define $-T \in \mathcal{L}(V,W)$ by

$$(-T)(v) = -(Tv)$$

for all $v \in V$. It is easy to verify that $-T$ is an additive inverse of T.

Suppose $T \in \mathcal{L}(V,W)$. Then

$$(1T)(v) = 1(Tv) = Tv$$

for every $v \in V$. Thus $1T = T$.

Suppose $a \in \mathbf{F}$ and $S,T \in \mathcal{L}(V,W)$. Then

$$\begin{aligned}(a(S+T))(v) &= a((S+T)(v))\\ &= a(Sv+Tv)\\ &= a(Sv)+a(Tv)\\ &= (aS)(v)+(aT)(v)\\ &= (aS+aT)(v)\end{aligned}$$

for every $v \in V$. Thus $a(S+T) = aS+aT$.

Suppose $a,b \in \mathbf{F}$ and $T \in \mathcal{L}(V,W)$. Then

$$\begin{aligned}((a+b)T)(v) &= (a+b)(Tv)\\ &= a(Tv)+b(Tv)\\ &= (aT)(v)+(bT)(v)\\ &= (aT+bT)(v)\end{aligned}$$

for every $v \in V$. Thus $(a + b)T = aT + bT$.

We have now verified that $\mathcal{L}(V, W)$ is a vector space with the operations of addition and scalar multiplication that were defined on $\mathcal{L}(V, W)$.

6 Prove that multiplication of linear maps has the associative, identity, and distributive properties asserted in 3.8.

SOLUTION Suppose U, V, W, X are vector spaces over $\mathbf{F}$ and

$$T_3 \in \mathcal{L}(U, V), \quad T_2 \in \mathcal{L}(V, W), \quad T_1 \in \mathcal{L}(W, X).$$

Thus $(T_1T_2)T_3$ and $T_1(T_2T_3)$ are both elements of $\mathcal{L}(U, X)$.

Now

$$\begin{aligned}((T_1T_2)T_3)(u) &= (T_1T_2)(T_3u)\\ &= T_1(T_2(T_3u))\\ &= T_1((T_2T_3)(u))\\ &= (T_1(T_2T_3))(u)\end{aligned}$$

for every $u \in U$. Thus $(T_1T_2)T_3 = T_1(T_2T_3)$.

Suppose $T \in \mathcal{L}(V, W)$. Then

$$(TI)v = T(Iv) = Tv$$

for every $v \in V$. Thus $TI = T$. Similarly,

$$(IT)(v) = I(Tv) = Tv$$

for every $v \in V$. Thus $IT = T$. Hence $TI = IT = T$.

Suppose U, V, W are vector spaces over $\mathbf{F}$. Suppose also that $T \in \mathcal{L}(U, V)$ and $S_1, S_2 \in \mathcal{L}(V, W)$. Then

$$\begin{aligned}((S_1 + S_2)T)(u) &= (S_1 + S_2)(Tu)\\ &= S_1(Tu) + S_2(Tu)\\ &= (S_1T)(u) + (S_2T)(u)\\ &= (S_1T + S_2T)(u)\end{aligned}$$

for every $u \in U$. Thus $(S_1 + S_2)T = S_1T + S_2T$.

Suppose U, V, W are vector spaces over $\mathbf{F}$. Suppose also that $T_1, T_2 \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$. Then

$$\begin{aligned}(S(T_1 + T_2))(u) &= S((T_1 + T_2)u)\\ &= S(T_1u + T_2u)\\ &= S(T_1u) + S(T_2u)\\ &= (ST_1)(u) + (ST_2)(u)\\ &= (ST_1 + ST_2)(u)\end{aligned}$$

for every $u \in U$. Thus $S(T_1 + T_2) = ST_1 + ST_2$.

7 Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

SOLUTION Suppose $\dim V = 1$ and $T \in \mathcal{L}(V)$. Let u be a nonzero vector in V. Then every vector in V is a scalar multiple of u. In particular, $Tu = \lambda u$ for some $\lambda \in \mathbf{F}$.

Now consider a typical vector $v \in V$. There exists $b \in \mathbf{F}$ such that $v = bu$. Thus

$$\begin{aligned} Tv &= T(bu) \\ &= bT(u) \\ &= b(\lambda u) \\ &= \lambda(bu) \\ &= \lambda v. \end{aligned}$$

8 Give an example of a function $\varphi\colon \mathbf{R}^2 \to \mathbf{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but φ is not linear.

This exercise and the next exercise show that neither homogeneity nor additivity alone is enough to imply that a function is a linear map.

SOLUTION Define $\varphi\colon \mathbf{R}^2 \to \mathbf{R}$ by

$$\varphi(x, y) = (x^3 + y^3)^{1/3}.$$

Then $\varphi(av) = a\varphi(v)$ for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$. However, φ is not linear, because $\varphi(1, 0) = 1$ and $\varphi(0, 1) = 1$ but

$$\begin{aligned} \varphi((1,0) + (0,1)) &= \varphi(1,1) \\ &= 2^{1/3} \\ &\neq \varphi(1,0) + \varphi(0,1). \end{aligned}$$

Of course there are also many other examples.

COMMENT This exercise shows that homogeneity alone is not enough to imply that a function is a linear map. Additivity alone is also not enough to imply that a function is a linear map, although the proof of this involves advanced tools that are beyond the scope of this book.

9 Give an example of a function $\varphi\colon \mathbf{C} \to \mathbf{C}$ such that

$$\varphi(w + z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbf{C}$ but φ is not linear. (Here $\mathbf{C}$ is thought of as a complex vector space.)

There also exists a function $\varphi\colon \mathbf{R} \to \mathbf{R}$ such that φ satisfies the additivity condition above but φ is not linear. However, showing the existence of such a function involves considerably more advanced tools.

SOLUTION Define $\varphi\colon \mathbf{C} \to \mathbf{C}$ by

$$\varphi(x + yi) = x - iy$$

for all $x, y \in \mathbf{R}$. Then φ is additive by φ is not homogeneous with respect to complex scalars. For example,

$$\varphi(ii) = \varphi(-1) = -1 \neq 1 = i(-i) = i\varphi(i)$$

and thus $\varphi(ii) \neq i\varphi(i)$.

10 Prove or give a counterexample: If $q \in \mathcal{P}(\mathbf{R})$ and $T\colon \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is defined by $Tp = q \circ p$, then T is a linear map.

The function T defined here differs from the function T defined in the last bullet point of 3.3 by the order of the functions in the compositions.

SOLUTION To construct a counterexample, take $q(x) = x^2$, $p_1(x) = 1$, and $p_2(x) = x$. Then

$$(Tp_1)(x) = 1 \quad \text{and} \quad (Tp_2)(x) = x^2$$

for all $x \in \mathbf{R}$ but

$$(T(p_1 + p_2))(x) = (1 + x)^2 = 1 + 2x + x^2.$$

Thus $T(p_1 + p_2) \neq Tp_1 + Tp_2$, which implies that T is not linear.

11 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

SOLUTION First $T = aI$ for some $a \in \mathbf{F}$. Let $S \in \mathcal{L}(V)$. Then

$$\begin{aligned} ST &= S(aI) \\ &= aS \\ &= (aI)S \\ &= TS. \end{aligned}$$

To prove the implication in the other direction, suppose now that $ST = TS$ for all $S \in \mathcal{L}(V)$. We begin by proving that v, Tv is linearly dependent for every $v \in V$. To do this, fix $v \in V$, and suppose v, Tv is linearly independent. Then v, Tv can be extended to a basis $v, Tv, u_1, \dots, u_n$ of V. Define $S \in \mathcal{L}(V)$ by

$$S(av + bTv + c_1 u_1 + \cdots + c_n u_n) = bv.$$

Thus $S(Tv) = v$ and $Sv = 0$. Thus the equation $S(Tv) = T(Sv)$ becomes the equation $v = 0$, a contradiction because v, Tv was assumed to be linearly independent. This contradiction shows that v, Tv is linearly dependent for every $v \in V$. This implies that for each $v \in V \setminus \{0\}$, there exists $a_v \in \mathbf{F}$ such that

$$Tv = a_v v.$$

To show that T is a scalar multiple of the identity, we must show that a_v is independent of v. To do this, suppose $v, w \in V \setminus \{0\}$. We want to show that $a_v = a_w$. First consider the case where v, w is linearly dependent. Then there exists $b \in \mathbf{F}$ such that $w = bv$. We have

$$\begin{aligned} a_w w &= Tw \\ &= T(bv) \\ &= bTv \\ &= b(a_v v) \\ &= a_v(bv) \\ &= a_v w, \end{aligned}$$

which shows that $a_v = a_w$, as desired.

Finally, consider the case where v, w is linearly independent. We have

$$\begin{aligned} a_{v+w}(v + w) &= T(v + w) \\ &= Tv + Tw \\ &= a_v v + a_w w, \end{aligned}$$

which implies that

$$(a_{v+w} - a_v)v + (a_{v+w} - a_w)w = 0.$$

Because v, w is linearly independent, this implies that $a_{v+w} = a_v$ and $a_{v+w} = a_w$, so again we have $a_v = a_w$, as desired.

12 Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means $Su \neq 0$ for some $u \in U$). Define $T \colon V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is not a linear map on V.

SOLUTION Let $u \in U$ be such that $Su \neq 0$. Let $w \in V$ be such that $w \notin U$. Then $u + w \notin U$ [because $u + w \in U$ would imply that $(u + w) + (-u)$, which equals w, is in U]. Thus

$$T(u + w) = 0.$$

However,

$$Tu + Tw = Tu + 0 = Tu = Su \neq 0.$$

Thus $T(u + w) \neq Tu + Tw$. Hence T is not linear.

13 Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

The result in this exercise is used in the proof of 3.125.

SOLUTION Suppose U is a subspace of V and $S \in \mathcal{L}(U, W)$. Choose a basis $u_1, \ldots, u_m$ of U. Then $u_1, \ldots, u_m$ is a linearly independent list of vectors in V, and so can be extended to a basis $u_1, \ldots, u_m, v_1, \ldots, v_n$ of V (by 2.32). Define $T \in \mathcal{L}(V, W)$ by

$$T(a_1 u_1 + \ldots a_m u_m + b_1 v_1 + \cdots + b_n v_n) = a_1 S u_1 + \cdots + a_m S u_m.$$

Then $Tu = Su$ for all $u \in U$.

14 Suppose V is finite-dimensional with $\dim V > 0$, and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

SOLUTION Suppose W is infinite-dimensional. Let $v_1, \dots, v_m$ be a basis of V.

Because W is infinite-dimensional, there exists a sequence $w_1, w_2, \dots$ such that $w_1, \dots, w_n$ is linearly independent for each positive integer n.

For each positive integer n, use the linear map lemma (3.4) to define a linear map $T_n \in \mathcal{L}(V, W)$ such that

$$T_n(v_k) = w_n$$

for all $k = 1, \dots, m$.

Suppose n is a positive integer and $c_1, \dots, c_n \in \mathbf{F}$ are such that

$$c_1 T_1 + \cdots + c_n T_n = 0.$$

Applying both sides of the equation above to the vector v_1 gives

$$c_1 w_1 + \cdots + c_n w_n = 0.$$

Because $w_1, \dots, w_n$ is linearly independent, the equation above implies that $c_1 = \cdots = c_n = 0$. Thus $T_1, \dots, T_n$ is a linearly independent list in $\mathcal{L}(V, W)$.

Because $\mathcal{L}(V, W)$ has a linearly independent list of length n for every positive integer n, we conclude that $\mathcal{L}(V, W)$ is infinite-dimensional.

15 Suppose $v_1, \ldots, v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, \ldots, w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$.

SOLUTION If $v_1 = 0$, let w_1 be a nonzero vector in W. Then there does not exist a linear map $T \in \mathcal{L}(V, W)$ such that $Tv_1 = w_1$ (by 3.10).

If $v_1 \neq 0$, then by the linear dependence lemma (2.19) there exists $j \in \{2, \ldots, m\}$ and $c_1, \ldots, c_{j-1} \in \mathbf{F}$ such that

$$v_j = c_1 v_1 + \cdots + c_{j-1} v_{j-1}.$$

Let $w_k = 0$ for all $k = 1, \ldots, j-1$ and let w_j equal any nonzero vector in W. Then the equation above and 3.10 imply that there does not exist a linear map $T \in \mathcal{L}(V, W)$ such that $Tv_k = w_k$ for all $k = 1, \ldots, j$.

16 Suppose V is finite-dimensional with $\dim V > 1$. Prove that there exist $S, T \in \mathcal{L}(V)$ such that $ST \neq TS$.

SOLUTION Let $v_1, \ldots, v_n$ be a basis of V. Use the linear map lemma (3.4) to define $S, T \in \mathcal{L}(V)$ such that

$$Sv_k = \begin{cases} v_2 & \text{if } k = 1, \\ 0 & \text{if } k \neq 1, \end{cases} \quad \text{and} \quad Tv_k = \begin{cases} v_1 & \text{if } k = 2, \\ 0 & \text{if } k \neq 2. \end{cases}$$

Then

$$(ST)(v_1) = S(Tv_1) = S0 = 0$$

but

$$(TS)(v_1) = T(Sv_1) = Tv_2 = v_1.$$

Thus $ST \neq TS$.

17 Suppose V is finite-dimensional. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.

A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called a ***two-sided ideal*** *of $\mathcal{L}(V)$ if $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$ for all $E \in \mathcal{E}$ and all $T \in \mathcal{L}(V)$.*

SOLUTION Suppose $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$ and $\mathcal{E} \neq \{0\}$. We will prove that $\mathcal{E} = \mathcal{L}(V)$.

Let $T \in \mathcal{E}$ be such that $T \neq 0$. Let $w \in V$ be such that $Tw \neq 0$. Let $v_1, \ldots, v_n$ be a basis of V (thus $n = \dim V$).

For $1 \leq k \leq n$, let $S_k \in \mathcal{L}(V)$ be the linear map (whose existence is guaranteed by the linear map lemma 3.4) such that

$$S_k v_j = \begin{cases} w & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Because $Tw \neq 0$, the list Tw of length one can be extended to a basis of V. Thus for each $1 \leq k \leq n$, there exists $R_k \in \mathcal{L}(V)$ such that

$$R_k(Tw) = v_k,$$

where again we have used the linear map lemma (3.4).

Now for each $1 \leq k \leq n$, we have

$$R_k T S_k v_j = \begin{cases} v_j & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Hence

$$\Big(\sum_{k=1}^{n} R_k T S_k\Big) v_j = v_j.$$

for every $j = 1, \ldots, n$. This implies that

$$\sum_{k=1}^{n} R_k T S_k = I,$$

where I is the identity operator on V.

Because $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$, each $R_k T S_k \in \mathcal{E}$. Thus the equation above implies that $I \in \mathcal{E}$.

If $T \in \mathcal{L}(V)$ then $TI = T$ and hence $T \in \mathcal{L}(V)$ [because $I \in \mathcal{E}$ and $\mathcal{E}$ is a two-sided ideal of $\mathcal{L}(V)$]. Hence we conclude that $\mathcal{E} = \mathcal{L}(V)$, as desired.

1 Give an example of a linear map T with $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

SOLUTION Define $T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^2)$ by

$$T(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2).$$

Then $\operatorname{null} T = \{(0, 0, x_3, x_4, x_5) : x_3, x_4, x_5 \in \mathbf{R}\}$, which has dimension three, and $\operatorname{range} T = \mathbf{R}^2$, which has dimension two.

2 Suppose $S, T \in \mathcal{L}(V)$ are such that $\operatorname{range} S \subseteq \operatorname{null} T$. Prove that $(ST)^2 = 0$.

SOLUTION Suppose $v \in V$. Then

$$(ST)^2 v = ST(S(Tv)).$$

Now $S(Tv) \in \operatorname{range} S \subseteq \operatorname{null} T$. Thus $T(S(Tv)) = 0$. Hence the equation above shows that $(ST)^2 v = 0$. Thus $(ST)^2 = 0$.

3 Suppose $v_1, \ldots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$

(a) What property of T corresponds to $v_1, \ldots, v_m$ spanning V?

(b) What property of T corresponds to the list $v_1, \ldots, v_m$ being linearly independent?

SOLUTION

(a) The list $v_1, \ldots, v_m$ spans V if and only if T is surjective.

(b) The list $v_1, \ldots, v_m$ is linearly independent if and only if T is injective.

4 Show that $\{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \operatorname{null} T > 2\}$ is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

SOLUTION Define $S, T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$ by

$$S(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, 0, 0)$$

and

$$T(x_1, x_2, x_3, x_4, x_5) = (0, 0, x_3, x_4).$$

Then

$$\operatorname{null} S = \{(0, 0, x_3, x_4, x_5) : x_3, x_4, x_5 \in \mathbf{R}\}$$

and

$$\operatorname{null} T = \{(x_1, x_2, 0, 0, x_5) : x_1, x_2, x_5 \in \mathbf{R}\}.$$

Thus $\dim \operatorname{null} S = \dim \operatorname{null} T = 3$. However,

$$(S + T)(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, x_4).$$

Thus $\operatorname{null}(S + T) = \{(0, 0, 0, 0, x_5) : x_5 \in \mathbf{R}\}$, which has dimension one. Thus $\{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \operatorname{null} T > 2\}$ is not closed under addition and hence is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

5 Give an example of $T \in \mathcal{L}(\mathbf{R}^4)$ such that $\operatorname{range} T = \operatorname{null} T$.

SOLUTION Define $T \in \mathcal{L}(\mathbf{R}^4, \mathbf{R}^4)$ by

$$T(x_1, x_2, x_3, x_4) = (x_3, x_4, 0, 0).$$

Then $\operatorname{range} T = \operatorname{null} T = \{(x_3, x_4, 0, 0) \in \mathbf{R}^4 : x_3, x_4 \in \mathbf{R}\}$.

6 Prove that there does not exist $T \in \mathcal{L}(\mathbf{R}^5)$ such that $\operatorname{range} T = \operatorname{null} T$.

SOLUTION Suppose $T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^5)$. By the fundamental theorem of linear maps (3.21), we have

$$5 = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

If $\operatorname{range} T = \operatorname{null} T$, then the right side of the equation above would be an even number, which is a contradiction to the value on the left side. Hence $\operatorname{range} T \neq \operatorname{null} T$.

7 Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

SOLUTION Let $v_1, \ldots, v_m$ be a basis of V and let $w_1, \ldots, w_n$ be a basis of W. Thus $2 \leq m \leq n$.

Using the linear map lemma (3.4), let $S, T \in \mathcal{L}(V, W)$ be such that

$$Sv_1 = 0 \quad \text{and} \quad Sv_k = w_k \text{ for } k = 2, \ldots, m$$

and

$$Tv_1 = w_1 \quad \text{and} \quad Tv_k = 0 \text{ for } k = 2, \ldots, m.$$

Then neither S nor T is injective because $Sv_1 = Tv_2 = 0$. However, $(S + T)v_k = w_k$ for each $k = 1, \ldots, m$, and thus $S + T$ is injective (as is easy to see). Hence the set of linear maps from V to W that are not injective is not closed under addition and thus is not a subspace of $\mathcal{L}(V, W)$.

8 Suppose V and W are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that $\{T \in \mathcal{L}(V,W) : T \text{ is not surjective}\}$ is not a subspace of $\mathcal{L}(V,W)$.

SOLUTION Let $v_1, \ldots, v_m$ be a basis of V and let $w_1, \ldots, w_n$ be a basis of W. Thus $m \geq n \geq 2$.

Using the linear map lemma (3.4), let $S, T \in \mathcal{L}(V,W)$ be such that

$$Sv_k = w_k \text{ for } k = 2, \ldots, n \quad \text{and} \quad Sv_k = 0 \text{ for } k = 1, n+1, n+2, \ldots, m$$

and

$$Tv_1 = w_1 \quad \text{and} \quad Tv_k = 0 \text{ for } k = 2, \ldots, m.$$

Then neither S nor T is surjective because $w_1 \notin \operatorname{range} S$ and $w_2 \notin \operatorname{range} T$. However, $(S+T)v_k = w_k$ for each $k = 1, \ldots, n$, and thus $S + T$ is surjective (as is easy to see). Hence the set of linear maps from V to W that are not surjective is not closed under addition and thus is not a subspace of $\mathcal{L}(V,W)$.

9 Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

SOLUTION To show that $Tv_1, \ldots, Tv_n$ is linearly independent, suppose that $a_1, \ldots, a_n \in \mathbf{F}$ are such that

$$a_1 T v_1 + \cdots + a_n T v_n = 0.$$

Because T is a linear map, this equation can be rewritten as

$$T(a_1 v_1 + \cdots + a_n v_n) = 0.$$

Because T is injective, this implies that

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

Because $v_1, \ldots, v_n$ is linearly independent, the equation above implies that $a_1 = \cdots = a_n = 0$. Thus $Tv_1, \ldots, Tv_n$ is linearly independent.

10 Suppose $v_1, \dots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Show that $Tv_1, \dots, Tv_n$ spans $\operatorname{range} T$.

SOLUTION Let $w \in \operatorname{range} T$. Thus there exists $v \in V$ such that $Tv = w$. Because $v_1, \dots, v_n$ spans V, there exist $a_1, \dots, a_n \in \mathbf{F}$ such that

$$v = a_1 v_1 + \cdots + a_n v_n.$$

Applying T to both sides of this equation, we get

$$Tv = a_1 T v_1 + \cdots + a_n T v_n.$$

Because $Tv = w$, the equation above implies that $w \in \operatorname{span}(Tv_1, \dots, Tv_n)$. Because w was an arbitrary vector in $\operatorname{range} T$, this implies that $Tv_1, \dots, Tv_n$ spans $\operatorname{range} T$.

11 Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that

$$U \cap \operatorname{null} T = \{0\} \quad \text{and} \quad \operatorname{range} T = \{Tu : u \in U\}.$$

SOLUTION There exists a subspace U of V such that

$$V = \operatorname{null} T \oplus U;$$

this follows from 2.33 (with $\operatorname{null} T$ playing the role of U and U playing the role of W).

From the definition of direct sum, we have $U \cap \operatorname{null} T = \{0\}$.

Clearly $\operatorname{range} T \supseteq \{Tu : u \in U\}$. To prove the inclusion in the other direction, suppose $v \in V$. Then there exist $w \in \operatorname{null} T$ and $x \in U$ such that

$$v = w + x.$$

Applying T to both sides of this equation, we have $Tv = Tw + Tx = Tx$. Thus $Tv \in \{Tu : u \in U\}$. Because v was an arbitrary vector in V (and thus Tv is an arbitrary vector in $\operatorname{range} T$), this implies that

$$\operatorname{range} T \subseteq \{Tu : u \in U\}.$$

Thus $\operatorname{range} T = \{Tu : u \in U\}$, as desired.

12 Suppose T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

SOLUTION The hypothesis implies that $(5, 1, 0, 0)$, $(0, 0, 7, 1)$ is a basis of $\operatorname{null} T$. Hence $\dim \operatorname{null} T = 2$. From the fundamental theorem of linear maps we have

$$\begin{aligned} \dim \operatorname{range} T &= \dim \mathbf{F}^4 - \dim \operatorname{null} T \\ &= 4 - 2 \\ &= 2. \end{aligned}$$

Because $\operatorname{range} T$ is a two-dimensional subspace of $\mathbf{F}^2$, we have $\operatorname{range} T = \mathbf{F}^2$. In other words, T is surjective.

13 Suppose U is a three-dimensional subspace of $\mathbf{R}^8$ and that T is a linear map from $\mathbf{R}^8$ to $\mathbf{R}^5$ such that $\operatorname{null} T = U$. Prove that T is surjective.

SOLUTION From the fundamental theorem of linear maps (3.21), we have

$$\begin{aligned} 8 &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= \dim U + \dim \operatorname{range} T \\ &= 3 + \dim \operatorname{range} T. \end{aligned}$$

Thus $\dim \operatorname{range} T = 5$. Hence $\operatorname{range} T = \mathbf{R}^5$. Thus T is surjective.

14 Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$.

SOLUTION Suppose U is the subspace of $\mathbf{F}^5$ displayed above. Then

$$(3, 1, 0, 0, 0), \quad (0, 0, 1, 1, 1)$$

is a basis of U. Hence $\dim U = 2$.

If $T \in \mathcal{L}(\mathbf{F}^5, \mathbf{F}^2)$ then from the fundamental theorem of linear maps we have

$$\begin{aligned} \dim \operatorname{null} T &= \dim \mathbf{F}^5 - \dim \operatorname{range} T \\ &= 5 - \dim \operatorname{range} T \\ &\geq 3 \\ &> \dim U, \end{aligned}$$

where the first inequality holds because $\operatorname{range} T \subseteq \mathbf{F}^2$. The inequality above shows that if $T \in \mathcal{L}(\mathbf{F}^5, \mathbf{F}^2)$, then $\operatorname{null} T \neq U$, as desired.

15 Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

SOLUTION Suppose T is a linear map from V into some vector space such that $\operatorname{null} T$ and $\operatorname{range} T$ are both finite-dimensional. Thus there exist vectors $u_1, \dots, u_m \in V$ and $w_1, \dots, w_n \in \operatorname{range} T$ such that $u_1, \dots, u_m$ spans $\operatorname{null} T$ and $w_1, \dots, w_n$ spans $\operatorname{range} T$. Because each $w_k \in \operatorname{range} T$, there exists $v_k \in V$ such that $w_k = Tv_k$.

Suppose $v \in V$. Then $Tv \in \operatorname{range} T$, so there exist $b_1, \dots, b_n \in \mathbf{F}$ such that

$$Tv = b_1 w_1 + \cdots + b_n w_n.$$

$$Tv = b_1 Tv_1 + \cdots + b_n Tv_n.$$

The equation above implies that $T(v - b_1 v_1 - \cdots - b_n v_n) = 0$. In other words, $v - b_1 v_1 - \cdots - b_n v_n \in \operatorname{null} T$. Thus there exist $a_1, \dots, a_m \in \mathbf{F}$ such that

$$v - b_1 v_1 - \cdots - b_n v_n = a_1 u_1 + \cdots + a_m u_m.$$

The equation above can be rewritten as

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n.$$

The equation above shows that every vector $v \in V$ is a linear combination of $u_1, \dots, u_m, v_1, \dots, v_n$. In other words, $u_1, \dots, u_m, v_1, \dots, v_n$ spans V. Thus V is finite-dimensional.

COMMENT The hypothesis of the fundamental theorem of linear maps is that V is finite-dimensional (which is what we are trying to prove in this exercise), so the fundamental theorem of linear maps cannot be used in this exercise.

16 Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

SOLUTION First suppose that there exists an injective linear map T from V to W. Then by the fundamental theorem of linear maps (3.21), we have

$$\begin{aligned}\dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ &= \dim\{0\} + \dim \operatorname{range} T \\ &= \dim \operatorname{range} T \\ &\leq \dim W.\end{aligned}$$

Conversely, suppose $\dim V \leq \dim W$. Let $v_1, \ldots, v_m$ be a basis of V and let $w_1, \ldots, w_n$ be a basis of W. Thus $m \leq n$. Use the linear map lemma (3.4) to define a linear map $T \in \mathcal{L}(V, W)$ such that

$$Tv_k = w_k \text{ for } k = 1, \ldots, m.$$

Suppose $v \in V$ and $Tv = 0$. There exist $c_1, \ldots, c_m \in \mathbf{F}$ such that

$$v = c_1 v_1 + \cdots + c_m v_m.$$

Thus

$$\begin{aligned}0 &= Tv \\ &= c_1 T v_1 + \cdots + c_m T v_m \\ &= c_1 w_1 + \cdots + c_m w_m.\end{aligned}$$

Because $w_1, \ldots, w_m$ is linearly independent, we have $c_1 = \cdots = c_m = 0$. Thus $v = 0$. Thus T is injective, as desired.

17 Suppose V and W are both finite-dimensional. Prove that there exists a surjective linear map from V onto W if and only if $\dim V \geq \dim W$.

SOLUTION First suppose there exists a surjective linear map T from V onto W. Then

$$\begin{aligned}\dim W &= \dim \operatorname{range} T \\ &= \dim V - \dim \operatorname{null} T \\ &\leq \dim V,\end{aligned}$$

where the second equality comes from the fundamental theorem of linear maps.

To prove the other direction, now suppose $\dim W \leq \dim V$. Let $w_1, \dots, w_m$ be a basis of W and let $v_1, \dots, v_n$ be a basis of V. For scalars $a_1, \dots, a_n \in \mathbf{F}$ define $T(a_1 v_1 + \cdots + a_n v_n)$ by

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_m w_m.$$

Because $\dim W \leq \dim V$, we have $m \leq n$ and so a_m on the right side of the equation above makes sense. Clearly T is a surjective linear map from V onto W.

18 Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

SOLUTION First suppose there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$. Then

$$\begin{aligned}\dim U &= \dim \operatorname{null} T \\ &= \dim V - \dim \operatorname{range} T \\ &\geq \dim V - \dim W,\end{aligned}$$

where the second equality comes from the fundamental theorem of linear maps.

To prove the other direction, now suppose $\dim U \geq \dim V - \dim W$. Let $u_1, \dots, u_m$ be a basis of U. Extend to a basis $u_1, \dots, u_m, v_1, \dots, v_n$ of V. Let $w_1, \dots, w_p$ be a basis of W. For $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F}$ define

$$T(a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n) = b_1 w_1 + \cdots + b_n w_n.$$

Because $\dim W \geq \dim V - \dim U$, we have $p \geq n$ and so w_n on the right side of the equation above makes sense. Clearly $T \in \mathcal{L}(V, W)$ and $\operatorname{null} T = U$.

19 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity operator on V.

SOLUTION First suppose T is injective. Define $S_1 \colon \operatorname{range} T \to V$ by

$$S_1(Tv) = v;$$

because T is injective, each element of $\operatorname{range} T$ can be represented in the form Tv in only one way, so T is well defined. As can easily be checked, S_1 is a linear map on $\operatorname{range} T$. By Exercise 13 in Section 3A, S_1 can be extended to a linear map S from W to V. If $v \in V$, then $(ST)v = S(Tv) = S_1(Tv) = v$. Thus ST is the identity operator on V.

To prove the implication in the other direction, now suppose there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity operator on V. If $u, v \in V$ are such that $Tu = Tv$, then

$$u = (ST)(u) = S(Tu) = S(Tv) = (ST)v = v.$$

Hence $u = v$. Thus T is injective, as desired.

20 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity operator on W.

SOLUTION First suppose T is surjective. Let $w_1, \dots, w_m$ be a basis of W. Because T is surjective, for each k there exists $v_k \in V$ such that $w_k = Tv_k$. Define $S \in \mathcal{L}(W, V)$ by

$$S(a_1 w_1 + \cdots + a_m w_m) = a_1 v_1 + \cdots + a_m v_m.$$

Then

$$\begin{aligned}(TS)(a_1 w_1 + \cdots + a_m w_m) &= T(a_1 v_1 + \cdots + a_m v_m)\\ &= a_1 T v_1 + \cdots + a_m T v_m\\ &= a_1 w_1 + \cdots + a_m w_m.\end{aligned}$$

Thus TS is the identity operator on W.

To prove the implication in the other direction, now suppose there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity operator on W. If $w \in W$, then $w = T(Sw)$, and hence $w \in \operatorname{range} T$. Thus $\operatorname{range} T = W$. In other words, T is surjective, as desired.

21 Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and U is a subspace of W. Prove that $\{v \in V : Tv \in U\}$ is a subspace of V and

$$\dim\{v \in V : Tv \in U\} = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T).$$

SOLUTION Let

$$E = \{v \in V : Tv \in U\}.$$

If $v \in E$ and $\lambda \in \mathbf{F}$, then $T(\lambda v) = \lambda Tv \in U$, which means $\lambda v \in E$. Also, if $v, w \in E$ then $Tv \in U$ and $Tw \in U$ and hence $T(u + w) = Tu + Tw \in U$, which means that $v + w \in E$. Thus E is a subspace of V.

Note that $\operatorname{null} T \subseteq E$. Let $S = T|_E$. Then

$$\operatorname{null} S = \operatorname{null} T \quad \text{and} \quad \operatorname{range} S = U \cap \operatorname{range} T.$$

Thus the fundamental theorem of linear maps, as applied to S, shows that

$$\dim E = \dim \operatorname{null} S + \dim \operatorname{range} S = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T),$$

as desired.

22 Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T.$$

SOLUTION Define a linear map $R\colon \operatorname{null} ST \to V$ by $Ru = Tu$. If $u \in \operatorname{null} ST$, then $S(Tu) = 0$, which means that $Tu \in \operatorname{null} S$. In other words, $\operatorname{range} R \subseteq \operatorname{null} S$. Now

$$\begin{aligned}
\dim \operatorname{null} ST &= \dim \operatorname{null} R + \dim \operatorname{range} R \\
&\leq \dim \operatorname{null} R + \dim \operatorname{null} S \\
&\leq \dim \operatorname{null} T + \dim \operatorname{null} S,
\end{aligned}$$

where the first line follows from the fundamental theorem of linear maps (applied to R), the second line holds because $\operatorname{range} R \subseteq \operatorname{null} S$, and the third line holds because of the inclusion $\operatorname{null} R \subseteq \operatorname{null} T$.

23 Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{range} ST \leq \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}.$$

SOLUTION First note that $\operatorname{range} ST \subseteq \operatorname{range} S$. Thus

$$\dim \operatorname{range} ST \leq \dim \operatorname{range} S.$$

Next, note that

$$\dim \operatorname{range} ST = \dim S|_{\operatorname{range} T} \leq \dim \operatorname{range} T,$$

where the inequality above follows from the fundamental theorem of linear maps (3.21).

Combining the two displayed inequalities above gives the desired result.

24

(a) Suppose $\dim V = 5$ and $S, T \in \mathcal{L}(V)$ are such that $ST = 0$. Prove that $\dim \operatorname{range} TS \leq 2$.

(b) Give an example of $S, T \in \mathcal{L}(\mathbf{F}^5)$ with $ST = 0$ and $\dim \operatorname{range} TS = 2$.

SOLUTION

(a) The equation $ST = 0$ implies that $\operatorname{range} T \subseteq \operatorname{null} S$. Thus

$$\dim \operatorname{range} T \leq \dim \operatorname{null} S = \dim V - \dim \operatorname{range} S = 5 - \dim \operatorname{range} S.$$

Hence

$$\dim \operatorname{range} S + \dim \operatorname{range} T \leq 5,$$

which implies that at least one of $\dim \operatorname{range} S$ and $\dim \operatorname{range} T$ is less than or equal to 2. Thus Exercise 23 implies that $\dim \operatorname{range} TS \leq 2$.

(b) Define $S, T \in \mathcal{L}(\mathbf{F}^5)$ by

$$S(z_1, z_2, z_3, z_4, z_5) = (z_3, z_4, z_5, 0, 0)$$

and

$$T(z_1, z_2, z_3, z_4, z_5) = (z_1, z_2, 0, 0, 0).$$

Then $ST = 0$ and

$$TS(z_1, z_2, z_3, z_4, z_5) = (z_3, z_4, 0, 0, 0).$$

Thus $\dim \operatorname{range} TS = 2$, as desired.

25 Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} S \subseteq \operatorname{null} T$ if and only if there exists $E \in \mathcal{L}(W)$ such that $T = ES$.

SOLUTION First suppose there exists $E \in \mathcal{L}(W)$ such that $T = ES$. Suppose $v \in \operatorname{null} S$. Thus

$$Tv = (ES)v = E(Sv) = E(0) = 0.$$

Thus $v \in \operatorname{null} T$. Hence $\operatorname{null} S \subseteq \operatorname{null} T$, as desired.

Now suppose that $\operatorname{null} S \subseteq \operatorname{null} T$. Define $E_1 \colon \operatorname{range} S \to W$ by

$$E_1(Sv) = Tv$$

for each $v \in V$. To show that this definition makes sense, we must show that if $u, v \in V$ and $Su = Sv$, then $Tu = Tv$. But this is true, because if $Su = Sv$ then $u - v \in \operatorname{null} S \subseteq \operatorname{null} T$ and hence $Tu = Tv$.

Now that E_1 is well defined, it is easy to verify that E_1 is a linear map from $\operatorname{range} S$ to W. Thus E_1 can be extended to a linear map E from W to W (see Exercise 13 in Section 3A). The displayed equation above shows that $T = ES$, as desired.

26 Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} S \subseteq \operatorname{range} T$ if and only if there exists $E \in \mathcal{L}(V)$ such that $S = TE$.

SOLUTION First suppose there exists $E \in \mathcal{L}(V)$ such that $S = TE$. Suppose $w \in \operatorname{range} S$. Hence there exists $v \in V$ such that $w = Sv$. Thus

$$w = Sv = (TE)v = T(Ev).$$

Thus $w \in \operatorname{range} T$. Hence $\operatorname{range} S \subseteq \operatorname{range} T$, as desired.

Now suppose that $\operatorname{range} S \subseteq \operatorname{range} T$. Let $v_1, \ldots, v_n$ be a basis of V. For each $k = 1, \ldots, n$, we have $Sv_k \in \operatorname{range} S \subseteq \operatorname{range} T$, and hence there exists $u_k \in V$ such that

$$Sv_k = Tu_k.$$

Use the linear map lemma (3.4) to define a linear map $E \in \mathcal{L}(V)$ such that

$$Ev_k = u_k$$

for each $k = 1, \ldots, n$. Then

$$Sv_k = Tu_k = TEv_k$$

for each $k = 1, \ldots, n$. Because S and TE are linear maps on V that agree on a basis of V, we have $S = TE$, as desired.

27 Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.

SOLUTION First suppose $u \in \operatorname{null} P \cap \operatorname{range} P$. Then $Pu = 0$, and there exists $w \in V$ such that $u = Pw$. Applying P to both sides of the last equation, we have $Pu = P^2 w = Pw$. But $Pu = 0$, so this implies that $Pw = 0$. Because $u = Pw$, this implies that $u = 0$. Because u was an arbitrary vector in $\operatorname{null} P \cap \operatorname{range} P$, this implies that $\operatorname{null} P \cap \operatorname{range} P = \{0\}$.

Now suppose $v \in V$. Then obviously

$$v = (v - Pv) + Pv.$$

Note that $P(v - Pv) = Pv - P^2 v = 0$, so $(v - Pv) \in \operatorname{null} P$. Clearly $Pv \in \operatorname{range} P$. Thus the equation above shows that $v \in \operatorname{null} P + \operatorname{range} P$. Because v was an arbitrary vector in V, this implies that $V = \operatorname{null} P + \operatorname{range} P$.

We have shown that $\operatorname{null} P \cap \operatorname{range} P = \{0\}$ and $V = \operatorname{null} P + \operatorname{range} P$. Thus $V = \operatorname{null} P \oplus \operatorname{range} P$ (by 1.46).

28 Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is such that $\deg Dp = (\deg p) - 1$ for every nonconstant polynomial $p \in \mathcal{P}(\mathbf{R})$. Prove that D is surjective.

The notation D is used above to remind you of the differentiation map that sends a polynomial p to p'.

SOLUTION Let m be a positive integer. Let $V = \operatorname{span}(x, x^2, \dots, x^m)$ in $\mathcal{P}(\mathbf{R})$. Note that $\dim V = m$.

Let $T = D|_V$. Our hypothesis on D implies that D is a linear map from V into $\mathcal{P}_{m-1}(\mathbf{R})$. Furthermore, our hypothesis on D implies that $\operatorname{null} D = \{0\}$. The fundamental theorem of linear maps (3.21) thus implies that

$$m = \dim V = \dim \operatorname{range} T.$$

Because $\operatorname{range} T$ has dimension m and $\operatorname{range} T$ is a subspace of the m-dimensional vector space $\mathcal{P}_{m-1}(\mathbf{R})$, we conclude that $\operatorname{range} T = \mathcal{P}_{m-1}(\mathbf{R})$.

Thus $\operatorname{range} D$ contains $\mathcal{P}_n(\mathbf{R})$ for every nonnegative integer n. This implies that $\operatorname{range} D$ contains $\mathcal{P}(\mathbf{R})$. Hence D is surjective.

29 Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.
This exercise can be done without linear algebra, but it's more fun to do it using linear algebra.

SOLUTION Define $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ by

$$Dq = 5q'' + 3q'.$$

Then D satisfies the hypotheses of Exercise 28. Thus by Exercise 28, D is surjective. Hence there exists a polynomial $q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.

30 Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$ and $\varphi \neq 0$. Suppose $u \in V$ is not in $\operatorname{null} \varphi$. Prove that

$$V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}.$$

SOLUTION If $a \in \mathbf{F}$ and $au \in \operatorname{null} \varphi$, then $0 = \varphi(au) = a\varphi(u)$, which implies that $a = 0$ (because $\varphi(u) \neq 0$). Thus

$$\operatorname{null} \varphi \cap \{au : a \in \mathbf{F}\} = \{0\}.$$

If $v \in V$, then

$$v = \Bigl(v - \frac{\varphi(v)}{\varphi(u)} u\Bigr) + \frac{\varphi(v)}{\varphi(u)} u.$$

Note that $\varphi\bigl(v - \frac{\varphi(v)}{\varphi(u)} u\bigr) = \varphi(v) - \frac{\varphi(v)}{\varphi(u)} \varphi(u) = 0$. Thus the equation above expresses an arbitrary vector $v \in V$ as the sum of a vector in $\operatorname{null} \varphi$ and a scalar multiple of u. Hence $V = \operatorname{null} \varphi + \{au : a \in \mathbf{F}\}$. Using 1.46, we conclude that $V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}$.

31 Suppose V is finite-dimensional, X is a subspace of V, and Y is a finite-dimensional subspace of W. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = X$ and $\operatorname{range} T = Y$ if and only if $\dim X + \dim Y = \dim V$.

SOLUTION First suppose there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = X$ and $\operatorname{range} T = Y$. Then $\dim X + \dim Y = \dim V$ by the fundamental theorem of linear maps.

To prove the implication in the other direction, now suppose

$$\dim X + \dim Y = \dim V.$$

Let $x_1, \ldots, x_m$ be a basis of X, and extend to a basis $x_1, \ldots, x_m, v_1, \ldots, v_n$ of V. Let $(y_1, \ldots, y_n)$ be a basis of Y. Our hypothesis implies that $n = \dim Y$.

Let $T \in \mathcal{L}(V, W)$ be the linear map from V to W such that

$$Tx_j = 0 \text{ for } j = 1, \ldots, m \quad \text{and} \quad Tv_k = y_k \text{ for } k = 1, \ldots, n.$$

Then $\operatorname{null} T = X$ and $\operatorname{range} T = Y$.

32 Suppose V is finite-dimensional with $\dim V > 1$. Show that if $\varphi\colon \mathcal{L}(V) \to \mathbf{F}$ is a linear map such that $\varphi(ST) = \varphi(S)\varphi(T)$ for all $S, T \in \mathcal{L}(V)$, then $\varphi = 0$.

Hint: The description of the two-sided ideals of $\mathcal{L}(V)$ given by Exercise 17 in Section 3A might be useful.

SOLUTION Suppose $\varphi\colon \mathcal{L}(V) \to \mathbf{F}$ is a linear map such that

$$\varphi(ST) = \varphi(S)\varphi(T)$$

for all $S, T \in \mathcal{L}(V)$. Then it is straightforward to verify that $\operatorname{null}\varphi$ is a two-sided ideal of $\mathcal{L}(V)$. By Exercise 17 in Section 3A, this implies that

$$\operatorname{null}\varphi = \{0\} \quad \text{or} \quad \operatorname{null}\varphi = \mathcal{L}(V).$$

Suppose $v_1, \ldots, v_n$ is a basis of V, where $n \geq 2$. Then there exist $S, T \in \mathcal{L}(V)$ such that

$$Sv_1 = v_1 \quad \text{and} \quad Sv_2 = 0$$

and

$$Tv_1 = 0 \quad \text{and} \quad Tv_2 = v_2,$$

where we have used the linear map lemma (3.4). Neither S nor T is a scalar multiple of the other, and thus S, T is a linearly independent list of length two in $\mathcal{L}(V)$. Hence

$$\dim \mathcal{L}(V) \geq 2.$$

The inequality above implies that φ is not injective (see 3.22). Thus $\operatorname{null}\varphi \neq \{0\}$, which implies that $\operatorname{null}\varphi = \mathcal{L}(V)$. Hence $\varphi = 0$.

33 Suppose that V and W are real vector spaces and $T \in \mathcal{L}(V, W)$. Define $T_{\mathbf{C}} \colon V_{\mathbf{C}} \to W_{\mathbf{C}}$ by

$$T_{\mathbf{C}}(u + iv) = Tu + iTv$$

for all $u, v \in V$.

(a) Show that $T_{\mathbf{C}}$ is a (complex) linear map from $V_{\mathbf{C}}$ to $W_{\mathbf{C}}$.

(b) Show that $T_{\mathbf{C}}$ is injective if and only if T is injective.

(c) Show that $\operatorname{range} T_{\mathbf{C}} = W_{\mathbf{C}}$ if and only if $\operatorname{range} T = W$.

See Exercise 8 in Section 1B for the definition of the complexification $V_{\mathbf{C}}$. The linear map $T_{\mathbf{C}}$ is called the ***complexification*** *of the linear map T.*

SOLUTION

(a) Suppose $u_1, v_1, u_2, v_2 \in V$. Then

$$\begin{aligned} T_{\mathbf{C}}((u_1 + iv_1) + (u_2 + iv_2)) &= T_{\mathbf{C}}((u_1 + u_2) + i(v_1 + v_2)) \\ &= T(u_1 + u_2) + i(T(v_1 + v_2)) \\ &= (Tu_1 + Tu_2) + i(Tv_1 + Tv_2) \\ &= (Tu_1 + iTv_1) + (Tu_2 + iTv_2) \\ &= T_{\mathbf{C}}(u_1 + iv_1) + T_{\mathbf{C}}(u_2 + iv_2). \end{aligned}$$

Thus $T_{\mathbf{C}}$ is additive.

Suppose now that $a, b \in \mathbf{R}$ and $u, v \in V$. Then

$$\begin{aligned} T_{\mathbf{C}}((a + bi)(u + iv)) &= T_{\mathbf{C}}((au - bv) + i(av + bu)) \\ &= T(au - bv) + iT(av + bu) \\ &= (aTu - bTv) + i(aTv + bTu) \\ &= (a + bi)(Tu + iTv) \\ &= (a + bi)T_{\mathbf{C}}(u + iv). \end{aligned}$$

Thus $T_{\mathbf{C}}$ is homogeneous.

(b) First suppose $T_{\mathbf{C}}$ is injective. If $v \in V$ and $Tv = 0$, then $T_{\mathbf{C}}(v) = Tv = 0$, which implies that $v = 0$. Hence T is injective.

To prove the implication in the other direction, now suppose T is injective. If $u, v \in V$ and $T_{\mathbf{C}}(u + iv) = 0$, then $Tu = 0$ and $Tv = 0$, which implies that $u = 0$ and $v = 0$, which implies that $u + iv = 0$. Hence $T_{\mathbf{C}}$ is injective.

(c) First suppose $\operatorname{range} T_{\mathbf{C}} = W_{\mathbf{C}}$. If $w \in W$, then there exist $u, v \in V$ such that $T_{\mathbf{C}}(u + iv) = w$, which implies that $Tu = w$, which implies that $w \in \operatorname{range} T$. Thus $\operatorname{range} T = W$.

To prove the implication in the other direction, now suppose

$$\operatorname{range} T = W.$$

If $w_1, w_2 \in W$, then there exist $v_1, v_2 \in V$ such that $Tv_1 = w_1$ and $Tv_2 = w_2$, which implies that $T_{\mathbf{C}}(v_1 + iv_2) = w_1 + iw_2$, which implies that $w_1 + iw_2 \in \operatorname{range} T_{\mathbf{C}}$. Thus $\operatorname{range} T_{\mathbf{C}} = W_{\mathbf{C}}$.

1 Suppose $T \in \mathcal{L}(V, W)$ is invertible. Show that T^{-1} is invertible and

$$(T^{-1})^{-1} = T.$$

SOLUTION The desired result follows from the equations

$$TT^{-1} = I \quad \text{and} \quad T^{-1}T = I,$$

which show that T is an inverse of T^{-1}.

2 Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

SOLUTION Note that

$$(ST)(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = SIS^{-1} = SS^{-1} = I$$

and

$$(T^{-1}S^{-1})(ST) = T^{-1}(S^{-1}S)T = T^{-1}IT = T^{-1}T = I.$$

Thus ST is invertible and $(ST)^{-1} = T^{-1}S^{-1}$.

3 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent.

(a) T is invertible.

(b) $Tv_1, \dots, Tv_n$ is a basis of V for every basis $v_1, \dots, v_n$ of V.

(c) $Tv_1, \dots, Tv_n$ is a basis of V for some basis $v_1, \dots, v_n$ of V.

SOLUTION First suppose (a) holds, so T is invertible. Suppose $v_1, \dots, v_n$ is a basis of V. To show that $Tv_1, \dots, Tv_n$ is linearly independent, suppose $c_1, \dots, c_n \in \mathbf{F}$ are such that

$$c_1 Tv_1 + \cdots + c_n Tv_n = 0.$$

Then

$$T(c_1 v_1 + \cdots + c_n v_n) = 0.$$

Because T is invertible, the equation above implies that

$$c_1 v_1 + \cdots + c_n v_n = 0.$$

Because $v_1, \dots, v_n$ is a basis of V, the equation above implies that $c_1 = \cdots = c_n = 0$. Thus the list $Tv_1, \dots, Tv_n$ is linearly independent and hence is a basis of V, completing the proof that (a) implies (b).

Clearly (b) implies (c).

Now suppose (c) holds. Thus there exists a basis $v_1, \dots, v_n$ of V such that $Tv_1, \dots, Tv_n$ is a basis of V. Suppose $v \in V$ is such that $Tv = 0$. There exist $c_1, \dots, c_n \in \mathbf{F}$ such that

$$v = c_1 v_1 + \cdots + c_n v_n.$$

Because $Tv = 0$, the equation above implies that

$$0 = c_1 Tv_1 + \cdots + c_n Tv_n.$$

Because $Tv_1, \dots, Tv_n$ is a basis of V, the equation above implies that $c_1 = \cdots = c_n = 0$. Hence $v = 0$. This shows that T is injective and thus is invertible, completing the proof that (c) implies (a).

4 Suppose V is finite-dimensional and $\dim V > 1$. Prove that the set of noninvertible linear maps from V to itself is not a subspace of $\mathcal{L}(V)$.

SOLUTION Let $n = \dim V$ and let $v_1, \dots, v_n$ be a basis of V. Define $S, T \in \mathcal{L}(V)$ by

$$S(a_1 v_1 + \cdots + a_n v_n) = a_1 v_1$$

and

$$T(a_1 v_1 + \cdots + a_n v_n) = a_2 v_2 + \cdots + a_n v_n.$$

Then S is not injective because $Sv_2 = 0$ (this is where we use the hypothesis that $\dim V > 1$), and T is not injective because $Tv_1 = 0$. Thus both S and T are not invertible. However, $S + T$ equals I, which is invertible. Thus the set of noninvertible linear maps from V to itself is not closed under addition, and hence it is not a subspace of $\mathcal{L}(V)$.

COMMENT If $\dim V \leq 1$, then the set of noninvertible linear maps from V to itself equals $\{0\}$, which is a subspace of $\mathcal{L}(V)$.

5 Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible linear map T from V to itself such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

SOLUTION First suppose there exists an invertible linear map T from V to itself such that $Tu = Su$ for every $u \in U$. Because T is injective, it is clear that S is injective.

Conversely, now suppose S is injective. Let $u_1, \ldots, u_m$ be a basis of U. Extend $u_1, \ldots, u_m$ to a basis $u_1, \ldots, u_m, v_1, \ldots, v_n$ of V.

By Exercise 9 in Section 3B, the list $Su_1, \ldots, Su_m$ is linearly independent. Thus we can extend to a basis $Su_1, \ldots, Su_m, w_1, \ldots, w_n$ of V.

Use the linear map lemma (3.4) to define $T \in \mathcal{L}(V)$ by

$$Tu_k = Su_k \text{ for } k = 1, \ldots, m \quad \text{and} \quad Tv_k = w_k \text{ for } k = 1, \ldots, n.$$

Then $Tu = Su$ for every $u \in U$ (because T and S agree on a basis of U) and T is invertible (because the range of T includes $Su_1, \ldots, Su_m, w_1, \ldots, w_n$, we see that T is surjective and hence invertible).

6 Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} S = \operatorname{null} T$ if and only if there exists an invertible $E \in \mathcal{L}(W)$ such that $S = ET$.

SOLUTION First suppose that there exists an invertible $E \in \mathcal{L}(W)$ such that $S = ET$. Let $v \in V$. Then

$$\begin{aligned} v \in \operatorname{null} S &\iff Sv = 0 \\ &\iff E(Tv) = 0 \\ &\iff Tv = 0 \\ &\iff v \in \operatorname{null} T, \end{aligned}$$

where the second equivalence holds because E is invertible (and hence injective). Thus $\operatorname{null} S = \operatorname{null} T$, as desired.

Now suppose that $\operatorname{null} S = \operatorname{null} T$. Define $E_1 \colon \operatorname{range} T \to W$ by

$$E_1(Tv) = Sv$$

for each $v \in V$. To show that this definition makes sense, we must show that if $u, v \in V$ and $Tu = Tv$, then $Su = Sv$. But this is true, because if $Tu = Tv$ then $u - v \in \operatorname{null} T = \operatorname{null} S$ and hence $Su = Sv$.

Now that E_1 is well defined, it is easy to verify that E_1 is a linear map from $\operatorname{range} T$ to W. Suppose $w \in \operatorname{range} T$ and $E_1 w = 0$. There exists $v \in V$ such that $w = Tv$. Now

$$0 = E_1 w = E_1(Tv) = Sv,$$

and thus $v \in \operatorname{null} S = \operatorname{null} T$. Hence $w = Tv = 0$. Thus E_1 is injective.

By Exercise 5, we can extend E_1 to an invertible linear map E from W to itself.

The definition of E_1 shows that $S = ET$.

7 Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} S = \operatorname{range} T$ if and only if there exists an invertible $E \in \mathcal{L}(V)$ such that $S = TE$.

SOLUTION First suppose there exists an invertible $E \in \mathcal{L}(V)$ such that $S = TE$. If $w \in \operatorname{range} S$, then there exists $v \in V$ such that $w = Sv$. Thus

$$w = Sv = (TE)(v) = T(Ev),$$

and hence $w \in \operatorname{range} T$. Thus $\operatorname{range} S \subseteq \operatorname{range} T$.

We have $T = SE^{-1}$. Thus by the same reasoning as in the paragraph above, we conclude that $\operatorname{range} T \subseteq \operatorname{range} S$.

Combining the conclusions of the two paragraphs above shows that

$$\operatorname{range} S = \operatorname{range} T,$$

as desired.

To prove the other direction, now suppose that $\operatorname{range} S = \operatorname{range} T$. The fundamental theorem of linear maps (3.21) implies that

$$\dim \operatorname{null} S = \dim \operatorname{null} T.$$

Let $v_1, \dots, v_m$ be a basis of $\operatorname{null} S$, and let $u_1, \dots, u_m$ be a basis of $\operatorname{null} T$.

Extend $v_1, \dots, v_m$ to a basis $v_1, \dots, v_m, \dots v_n$ of V, and extend $u_1, \dots, u_m$ to a basis $u_1, \dots, u_m, \dots, u_n$ of V. For $m < k \le n$, we have $Sv_k \in \operatorname{range} S = \operatorname{range} T$, and thus there exists $u_k \in V$ such that $Sv_k = Tu_k$.

Note that if $1 \le k \le m$, then we also have $Sv_k = Tu_k$ because both sides equal 0.

Now use the linear map lemma (3.4) to define a linear map E from V to V such that $Ev_k = u_k$ for each $k = 1, \dots, n$. If $1 \le k \le n$, then

$$Sv_k = Tu_k = T(Ev_k) = (TE)(v_k).$$

Thus $S = TE$ because these two linear maps agree on a basis.

To prove that E is invertible, suppose $v \in V$ and $Ev = 0$. Then $Sv = TEv = 0$, and hence $v \in \operatorname{null} S = \operatorname{span}(v_1, \dots, v_m)$, and we can write

$$v = a_1 v_1 + \cdots + a_m v_m$$

for some $a_1, \dots, a_m \in \mathbf{F}$. Applying E to both sides of the equation above gives

$$0 = a_1 u_1 + \cdots + a_m u_m.$$

Because $u_1, \dots, u_m$ is linearly independent, this implies $a_1 = \cdots = a_m = 0$. Hence $v = 0$. Thus E is injective.

Now 3.63 implies that E is invertible, as desired.

8 Suppose V and W are finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that there exist invertible $E_1 \in \mathcal{L}(V)$ and $E_2 \in \mathcal{L}(W)$ such that $S = E_2 T E_1$ if and only if $\dim \operatorname{null} S = \dim \operatorname{null} T$.

SOLUTION First suppose that there exist invertible $E_2 \in \mathcal{L}(W)$ and $E_1 \in \mathcal{L}(V)$ such that $S = E_2 T E_1$. Suppose $v \in V$. Then

$$\begin{aligned} v \in \operatorname{null} S &\iff Sv = 0 \\ &\iff E_2 T E_1 v = 0 \\ &\iff T E_1 v = 0 \\ &\iff E_1 v \in \operatorname{null} T \\ &\iff v \in \operatorname{range}(E_1^{-1}|_{\operatorname{null} T}) \end{aligned}$$

Thus $\operatorname{null} S = \operatorname{range}(E_1^{-1}|_{\operatorname{null} T})$. Hence

$$\dim \operatorname{null} S = \dim \operatorname{range}(E_1^{-1}|_{\operatorname{null} T}) = \dim \operatorname{null} T,$$

where the last equality follows from the fundamental theorem of linear maps (3.21). Thus $\dim \operatorname{null} S = \dim \operatorname{null} T$, as desired.

To prove the other direction, now suppose $\dim \operatorname{null} S = \dim \operatorname{null} T$. Let $v_1, \dots, v_m$ be a basis of $\operatorname{null} S$, and let $u_1, \dots, u_m$ be a basis of $\operatorname{null} T$.

Extend $v_1, \dots, v_m$ to a basis $v_1, \dots, v_m, \dots, v_n$ of V, and extend $u_1, \dots, u_m$ to a basis $u_1, \dots, u_m, \dots, u_n$ of V. Use the linear map lemma (3.4) to define $E_1 \in \mathcal{L}(V)$ such that $E_1 v_k = u_k$ for all $k = 1, \dots, n$. Because $v_1, \dots, v_n$ and $u_1, \dots, u_n$ are both bases of V, it is easy to see that E_1 is an invertible linear map from V to V.

Suppose $v \in V$. Then

$$\begin{aligned} v \in \operatorname{null} S &\iff E_1 v \in \operatorname{null} T \\ &\iff T(E_1 v) = 0 \\ &\iff v \in \operatorname{null} T E_1. \end{aligned}$$

Thus $\operatorname{null} S = \operatorname{null} T E_1$. Applying Exercise 6 to the linear maps S and $T E_1$, we conclude that there exists an invertible $E_2 \in \mathcal{L}(W)$ such that $S = E_2 T E_1$.

9 Suppose V is finite-dimensional and $T: V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W.

Here $T|_U$ means the function T restricted to U. Thus $T|_U$ is the function whose domain is U, with $T|_U(u) = Tu$ for every $u \in U$.

SOLUTION Let $w_1, \ldots, w_m$ be a basis of W. Because T is surjective, for each $k = 1, \ldots, m$, there exists $v_k \in V$ such that $Tv_k = w_k$.

Let $U = \operatorname{span}(v_1, \ldots, v_m)$. Thus $\dim U \leq m$.

Now $\operatorname{range} T|_U = W$ (because $w_k \in \operatorname{range} T|_U$ for each $k = 1, \ldots, m$). The fundamental theorem of linear maps (3.21) thus tells us that

$$m \geq \dim U = \dim \operatorname{null} T|_U + m,$$

which implies that $\dim \operatorname{null} T|_U = 0$, which implies that $T|_U$ is injective. Thus $T|_U$ is an isomorphism of U onto W.

10 Suppose V and W are finite-dimensional and U is a subspace of V. Let

$$\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\}.$$

(a) Show that $\mathcal{E}$ is a subspace of $\mathcal{L}(V, W)$.

(b) Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$, and $\dim U$.

Hint: Define $\Phi\colon \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ *by* $\Phi(T) = T|_U$. *What is* $\operatorname{null} \Phi$*? What is* $\operatorname{range} \Phi$*?*

SOLUTION Define $\Phi\colon \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ by $\Phi(T) = T|_U$.

(a) Because Φ is a linear map and $\operatorname{null} \Phi = \mathcal{E}$, we can conclude that $\mathcal{E}$ is a subspace of $\mathcal{L}(V, W)$.

(b) Exercise 13 in Section 3A implies that $\operatorname{range} \Phi = \mathcal{L}(U, W)$. Thus the fundamental theorem of linear maps shows that

$$\begin{aligned}(\dim V)(\dim W) &= \dim \mathcal{L}(V, W)\\ &= \dim \operatorname{null} \Phi + \dim \operatorname{range} \Phi\\ &= \dim \mathcal{E} + \dim \mathcal{L}(U, W)\\ &= \dim \mathcal{E} + (\dim U)(\dim W).\end{aligned}$$

Thus

$$\dim \mathcal{E} = (\dim V - \dim U)(\dim W).$$

11 Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that

$$ST \text{ is invertible} \iff S \text{ and } T \text{ are invertible.}$$

SOLUTION First suppose ST is invertible. Thus there exists $R \in \mathcal{L}(V)$ such that $R(ST) = (ST)R = I$. If $v \in V$ is such that $Tv = 0$, then

$$\begin{aligned} v &= Iv \\ &= R(ST)v \\ &= 0. \end{aligned}$$

Because v was an arbitrary vector in $\operatorname{null} T$, this shows that $\operatorname{null} T = \{0\}$. Thus T is injective (by 3.15). Hence T is invertible (by 3.65), as desired.

If $u \in V$, then

$$\begin{aligned} u &= Iu \\ &= (ST)Ru \\ &= S(TRu), \end{aligned}$$

which shows that $u \in \operatorname{range} S$. Because u was an arbitrary vector in V, this implies that $\operatorname{range} S = V$. Thus S is surjective. Hence S is invertible (by 3.65), as desired.

To prove the implication in the other direction, now suppose both S and T are invertible. Then

$$\begin{aligned} (ST)(T^{-1}S^{-1}) &= S(TT^{-1})S^{-1} \\ &= SS^{-1} \\ &= I \end{aligned}$$

and

$$\begin{aligned} (T^{-1}S^{-1})(ST) &= T^{-1}(S^{-1}S)T \\ &= T^{-1}T \\ &= I. \end{aligned}$$

Thus $T^{-1}S^{-1}$ satisfies the properties required for an inverse of ST. Thus ST is invertible and $(ST)^{-1} = T^{-1}S^{-1}$.

12 Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

SOLUTION The equation $(ST)U = I$ implies (by 3.68) that $U(ST) = I$, which we can write as

$$(US)T = I.$$

Now 3.68 implies that

$$T(US) = I.$$

The two displayed equations above imply that T is invertible and $T^{-1} = US$.

13 Show that the result in Exercise 12 can fail without the hypothesis that V is finite-dimensional.

SOLUTION Let $V = \mathbf{F}^\infty$ and let S be the identity operator on $\mathbf{F}^\infty$.

Let T be the backward shift on $\mathbf{F}^\infty$:

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots),$$

and let U be the forward shift on $\mathbf{F}^\infty$:

$$U(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots).$$

Then $STU = I$, but T is not invertible.

14 Prove or give a counterexample: If V is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective, then S is injective.

SOLUTION Because RST is surjective, $R(ST)$ is invertible (by 3.65). Exercise 11 now implies that ST is invertible. Another use of Exercise 11 implies that S is invertible. Thus S is injective.

15 Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_m$ is a list in V such that $Tv_1, \ldots, Tv_m$ spans V. Prove that $v_1, \ldots, v_m$ spans V.

SOLUTION Because the list $Tv_1, \ldots, Tv_m$ spans V, the vector space V is finite-dimensional and the linear map T is surjective. Thus T is injective (by 3.65).

Suppose $v \in V$. Because $Tv \in V$ and $Tv_1, \ldots, Tv_m$ spans V, there exist $a_1, \ldots, a_m \in \mathbf{F}$ such that

$$Tv = a_1 Tv_1 + \cdots + a_m Tv_m = T(a_1 v_1 + \cdots + a_m v_m).$$

Because T is injective, the equation above implies that

$$v = a_1 v_1 + \cdots + a_m v_m.$$

Thus $v_1, \ldots, v_m$ spans V.

16 Prove that every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then there exists an m-by-n matrix A such that $Tx = Ax$ for every $x \in \mathbf{F}^{n,1}$.

SOLUTION The vector spaces $\mathbf{F}^{n,1}$ and $\mathbf{F}^{m,1}$ have standard bases (consisting of matrices that have 0 in all entries except for a 1 in one entry). Let A be the matrix of T with respect to these bases. Note that if $x \in \mathbf{F}^{n,1}$, then $\mathcal{M}(x) = x$ and $\mathcal{M}(Tx) = Tx$. Thus

$$\begin{aligned} Tx &= \mathcal{M}(Tx) \\ &= \mathcal{M}(T)\mathcal{M}(x) \\ &= Ax, \end{aligned}$$

where the second equality comes from 3.76.

17 Suppose V is finite-dimensional and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by

$$\mathcal{A}(T) = ST$$

for $T \in \mathcal{L}(V)$.

(a) Show that $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.

(b) Show that $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

SOLUTION

(a) Note that

$$\operatorname{null} \mathcal{A} = \{T \in \mathcal{L}(V) : \operatorname{range} T \subseteq \operatorname{null} S\}.$$

In other words, $\operatorname{null} \mathcal{A}$ is the vector space of linear maps from V to $\operatorname{null} S$. By 3.72, the dimension of this vector space is $(\dim V)(\dim \operatorname{null} S)$. Thus $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.

(b) We have

$$\begin{aligned}
\dim \operatorname{range} \mathcal{A} &= \dim \mathcal{L}(V) - \dim \operatorname{null} \mathcal{A} \\
&= (\dim V)^2 - (\dim V)(\dim \operatorname{null} S) \\
&= (\dim V)(\dim V - \dim \operatorname{null} S) \\
&= (\dim V)(\dim \operatorname{range} S),
\end{aligned}$$

where the first and last equalities come from the fundamental theorem of linear maps and the second equality comes from 3.72 and (a).

18 Show that V and $\mathcal{L}(\mathbf{F}, V)$ are isomorphic vector spaces.

SOLUTION Define $T \colon V \to \mathcal{L}(\mathbf{F}, V)$ by

$$(Tv)(\lambda) = \lambda v$$

for each $v \in V$ and each $\lambda \in \mathbf{F}$. The easy verifications that Tv is a linear map from $\mathbf{F}$ to V and that T is a linear map from V to $\mathcal{L}(\mathbf{F}, V)$ follow from the definition of a vector space.

Suppose $v \in V$ and $Tv = 0$. Thus $0 = (Tv)(1) = v$. Hence T is injective by 3.15.

To show that T is surjective, suppose $S \in \mathcal{L}(\mathbf{F}, V)$. Let $v = S(1)$. If $\lambda \in \mathbf{F}$, then

$$\begin{aligned}(Tv)(\lambda) &= \lambda v \\ &= \lambda S(1) \\ &= S(\lambda).\end{aligned}$$

Thus $S = Tv$. Hence T is surjective.

Because T is injective and surjective, we conclude that V and $\mathcal{L}(\mathbf{F}, V)$ are isomorphic vector spaces, as desired.

19 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has the same matrix with respect to every basis of V if and only if T is a scalar multiple of the identity operator.

SOLUTION First suppose T has the same matrix with respect to every basis of V. We begin by proving that v, Tv is linearly dependent for every $v \in V$. To do this, fix $v \in V$, and suppose v, Tv is linearly independent. Then v, Tv can be extended to a basis $v, Tv, u_1, \ldots, u_n$ of V. The first column of the matrix of T with respect to this basis is

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Clearly $2v, Tv, u_1, \ldots, u_n$ is also a basis of V. The first column of the matrix of T with respect to this basis is

$$\begin{pmatrix} 0 \\ 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus T has different matrices with respect to the two bases we have considered. This contradiction shows that v, Tv is linearly dependent for every $v \in V$. This implies that for every nonzero vector in V is an eigenvector of T. This implies that T is a scalar multiple of the identity operator (by Exercise 26 in Chapter 5).

To prove the implication in the other direction, note that if $\lambda \in \mathbf{F}$ and $T = \lambda I$, then the matrix of T with respect to every basis of V is the diagonal matrix with only λ on the diagonal.

20 Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{R})$ such that

$$q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$$

for all $x \in \mathbf{R}$.

SOLUTION Let $m = \deg q$. Define $T \in \mathcal{L}(\mathcal{P}_m(\mathbf{R}))$ by

$$Tp = (x^2 + x)p''(x) + 2xp'(x) + p(3)$$

for all $x \in \mathbf{R}$. The linear map T indeed maps $\mathcal{P}_m(\mathbf{R})$ to itself because

$$\deg(x^2 + x)p''(x) + 2xp'(x) + p(3) = \deg p$$

for every polynomial $p \in \mathcal{P}(\mathbf{R})$.

The equation above also implies that if $Tp = 0$ then $p = 0$. Thus T is injective. Hence T is surjective (by 3.65). Thus there exists $p \in \mathcal{P}_m(\mathbf{R})$ such that $Tp = q$.

21 Suppose n is a positive integer and $A_{j,k} \in \mathbf{F}$ for all $j, k = 1, \ldots, n$. Prove that the following are equivalent (note that in both parts below, the number of equations equals the number of variables).

(a) The trivial solution $x_1 = \cdots = x_n = 0$ is the only solution to the homogeneous system of equations
$$\sum_{k=1}^{n} A_{1,k}\, x_k = 0 \qquad \cdots \qquad \sum_{k=1}^{n} A_{n,k}\, x_k = 0.$$

(b) For every $c_1, \ldots, c_n \in \mathbf{F}$, there exists a solution to the system of equations
$$\sum_{k=1}^{n} A_{1,k}\, x_k = c_1 \qquad \cdots \qquad \sum_{k=1}^{n} A_{n,k}\, x_k = c_n.$$

SOLUTION Define $T \in \mathcal{L}(\mathbf{F}^n)$ by
$$T(x_1, \ldots, x_n) = \Big(\sum_{k=1}^{n} A_{1,k}\, x_k, \ \ldots, \ \sum_{k=1}^{n} A_{n,k}\, x_k\Big).$$

Then (a) above is the assertion that T is injective, and (b) above is the assertion that T is surjective. By 3.65, these two assertions are equivalent.

22 Suppose $T \in \mathcal{L}(V)$ and $v_1, \dots, v_n$ is a basis of V. Prove that

$$\mathcal{M}(T, (v_1, \dots, v_n)) \text{ is invertible} \iff T \text{ is invertible.}$$

SOLUTION First suppose $\mathcal{M}(T)$ is an invertible matrix (because the only basis in sight is $v_1, \dots, v_n$, we can leave the basis out of the notation). Thus there exists an n-by-n matrix B such that

$$\mathcal{M}(T)B = B\mathcal{M}(T) = I.$$

There exists an operator $S \in \mathcal{L}(V)$ such that $\mathcal{M}(S) = B$ (see 3.71). Thus the equations above become

$$\mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I,$$

which we can rewrite as

$$\mathcal{M}(TS) = \mathcal{M}(ST) = \mathcal{M}(I),$$

which implies that

$$TS = ST = I.$$

Thus T is invertible, as desired, with inverse S.

To prove the implication in the other direction, suppose now that T is invertible. Thus there exists $S \in \mathcal{L}(V)$ such that

$$TS = ST = I.$$

This implies that

$$\mathcal{M}(TS) = \mathcal{M}(ST) = \mathcal{M}(I),$$

which implies that

$$\mathcal{M}(T)\mathcal{M}(S) = \mathcal{M}(S)\mathcal{M}(T) = I.$$

Thus $\mathcal{M}(T)$ is invertible, as desired, with inverse $\mathcal{M}(S)$.

23 Suppose that $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Let $T \in \mathcal{L}(V)$ be such that $Tv_k = u_k$ for each $k = 1, \ldots, n$. Prove that

$$\mathcal{M}(T, (v_1, \ldots, v_n)) = \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)).$$

SOLUTION Fix k. Write

$$u_k = A_1 v_1 + \cdots + A_n v_n,$$

where $A_1, \ldots, A_n \in \mathbf{F}$.

Because $Tv_k = u_k$, the k^{th} column of the matrix $\mathcal{M}(T, (v_1, \ldots, v_n))$ consists of the numbers $A_1, \ldots, A_n$.

Because $Iu_k = u_k$, the k^{th} column of

$$\mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$$

also consists of the numbers $A_1, \ldots, A_n$.

Because $\mathcal{M}(T, (v_1, \ldots, v_n))$ and $\mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$ have the same columns, these two matrices are equal.

24 Suppose A and B are square matrices of the same size and $AB = I$. Prove that $BA = I$.

SOLUTION Suppose A and B are n-by-n matrices and $AB = I$. There exist $S, T \in \mathcal{L}(\mathbf{F}^n)$ such that

$$\mathcal{M}(S) = A \quad \text{and} \quad \mathcal{M}(T) = B;$$

here we are using the standard basis of $\mathbf{F}^n$ (the existence of $S, T \in \mathcal{L}(\mathbf{F}^n)$ satisfying the equations above follows from 3.71). Because $AB = I$, we have $\mathcal{M}(S)\mathcal{M}(T) = I$, which implies that $\mathcal{M}(ST) = \mathcal{M}(I)$, which implies that $ST = I$, which implies that $TS = I$ (by 3.68). Thus

$$\begin{aligned} BA &= \mathcal{M}(T)\mathcal{M}(S) \\ &= \mathcal{M}(TS) \\ &= \mathcal{M}(I) \\ &= I. \end{aligned}$$