

## Section 1A: $\mathbb{R}^n$ and $\mathbb{C}^n$

### 1. Complex Numbers

Before we can study vector spaces, we need to understand the scalars we'll use. The real numbers  $\mathbb{R}$  are familiar, but linear algebra becomes more powerful when we also work with complex numbers  $\mathbb{C}$ .

#### 1.1 Definition: Complex Numbers, $\mathbb{C}$

- A **complex number** is an ordered pair  $(a, b)$  where  $a, b \in \mathbb{R}$ , written as  $a + bi$ .
- The set of all complex numbers is denoted by  $\mathbb{C}$ :

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

- **Addition** and **multiplication** on  $\mathbb{C}$  are defined by:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

where  $a, b, c, d \in \mathbb{R}$ .

**Intuition:** Think of  $\mathbb{C}$  as a 2D plane where the horizontal axis represents real numbers and the vertical axis represents imaginary numbers. If  $a \in \mathbb{R}$ , we identify  $a + 0i$  with the real number  $a$ , so  $\mathbb{R} \subset \mathbb{C}$ . We write  $0 + bi$  as just  $bi$ , and  $0 + 1i$  as just  $i$ .

**Why complex numbers?** Even when working with real matrices, eigenvalues (Chapter 5) often require complex numbers. The completeness of  $\mathbb{C}$  makes linear algebra more elegant—every polynomial has roots in  $\mathbb{C}$ .

**Why this multiplication formula?** We define  $i$  as a symbol satisfying  $i^2 = -1$ . This is consistent and creates an algebraically closed field. Using the usual rules of arithmetic:

$$\begin{aligned} (a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

#### 1.2 Example: Complex Arithmetic

Compute  $(2 + 3i)(4 + 5i)$ .

Using the distributive and commutative properties:

$$\begin{aligned} (2 + 3i)(4 + 5i) &= 2 \cdot (4 + 5i) + (3i)(4 + 5i) \\ &= 2 \cdot 4 + 2 \cdot 5i + 3i \cdot 4 + (3i)(5i) \\ &= 8 + 10i + 12i - 15 \\ &= \boxed{-7 + 22i} \end{aligned}$$

#### 1.3 Properties of Complex Arithmetic

For all  $\alpha, \beta, \lambda \in \mathbb{C}$ :

**commutativity:**  $\alpha + \beta = \beta + \alpha$  and  $\alpha\beta = \beta\alpha$

**associativity:**  $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$  and  $(\alpha\beta)\lambda = \alpha(\beta\lambda)$

**identities:**  $\lambda + 0 = \lambda$  and  $\lambda \cdot 1 = \lambda$

**additive inverse:** For every  $\alpha \in \mathbb{C}$ , there exists a unique  $\beta \in \mathbb{C}$  such that  $\alpha + \beta = 0$

**multiplicative inverse:** For every  $\alpha \in \mathbb{C}$  with  $\alpha \neq 0$ , there exists a unique  $\beta \in \mathbb{C}$  such that  $\alpha\beta = 1$

**distributive property:**  $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$

#### 1.4 Example: Commutativity of Complex Multiplication

To show  $\alpha\beta = \beta\alpha$  for all  $\alpha, \beta \in \mathbb{C}$ , suppose  $\alpha = a + bi$  and  $\beta = c + di$  where  $a, b, c, d \in \mathbb{R}$ .

**LHS:**

$$\alpha\beta = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

**RHS:**

$$\beta\alpha = (c + di)(a + bi) = (ca - db) + (cb + da)i$$

Since real multiplication is commutative ( $ac = ca$ ,  $bd = db$ , etc.), we have  $\alpha\beta = \beta\alpha$ .  $\square$

#### 1.5 Definition: $-\alpha$ , Subtraction, $1/\alpha$ , Division

Suppose  $\alpha, \beta \in \mathbb{C}$ .

- Let  $-\alpha$  denote the **additive inverse** of  $\alpha$ : the unique complex number such that  $\alpha + (-\alpha) = 0$ .
- **Subtraction** on  $\mathbb{C}$  is defined by  $\beta - \alpha = \beta + (-\alpha)$ .
- For  $\alpha \neq 0$ , let  $1/\alpha$  and  $\frac{1}{\alpha}$  denote the **multiplicative inverse** of  $\alpha$ : the unique complex number such that  $\alpha(1/\alpha) = 1$ .
- For  $\alpha \neq 0$ , **division** by  $\alpha$  is defined by  $\beta/\alpha = \beta(1/\alpha)$ .

**Computing  $1/\alpha$ :** For  $\alpha = a + bi \neq 0$ , multiply by the conjugate:

$$\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

**Key insight:** Why multiply by the conjugate? Because  $(a+bi)(a-bi) = a^2+b^2$  is always real and positive. Multiplying by the conjugate eliminates  $i$  from the denominator.

**Example: Complex Division.** Compute  $\frac{2 + 3i}{4 + 5i}$ .

Multiply by the conjugate of the denominator:

$$\frac{2+3i}{4+5i} \cdot \frac{4-5i}{4-5i} = \frac{(2+3i)(4-5i)}{(4+5i)(4-5i)}$$

Denominator:  $(4+5i)(4-5i) = 16 + 25 = 41$

Numerator:  $(2+3i)(4-5i) = 8 - 10i + 12i + 15 = 23 + 2i$

**Answer:**  $\frac{2+3i}{4+5i} = \boxed{\frac{23}{41} + \frac{2}{41}i}$

### 1.6 Notation: $\mathbb{F}$

Throughout this book,  $\mathbb{F}$  stands for either  $\mathbb{R}$  or  $\mathbb{C}$ .

**Why use  $\mathbb{F}$ ?** The letter  $\mathbb{F}$  reminds us of “field.” Both  $\mathbb{R}$  and  $\mathbb{C}$  are fields: sets with addition and multiplication satisfying the properties in 1.3. Using  $\mathbb{F}$  lets us state theorems once and have them apply to both  $\mathbb{R}$  and  $\mathbb{C}$ .

Elements of  $\mathbb{F}$  are called **scalars**.

**Powers of scalars:** For  $\alpha \in \mathbb{F}$  and a positive integer  $m$ :

$$\alpha^m = \underbrace{\alpha \cdot \alpha \cdots \alpha}_{m \text{ times}}$$

This definition implies:

- $(\alpha^m)^n = \alpha^{mn}$
- $(\alpha\beta)^m = \alpha^m\beta^m$

## 2. Lists

To generalize  $\mathbb{R}^2$  and  $\mathbb{R}^3$  to higher dimensions, we first need to discuss the concept of lists.

### 1.7 Example: $\mathbb{R}^2$ and $\mathbb{R}^3$

- $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$  (the plane)
- $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$  (3D space)

### 1.8 Definition: List, Length

Suppose  $n$  is a nonnegative integer. A **list of length  $n$**  is an ordered collection of  $n$  elements (which might be numbers, other lists, or more abstract objects) separated by commas and surrounded by parentheses.

A list of length  $n$  is also called an  **$n$ -tuple**.

**Key point:** Two lists are equal if and only if they have the same length and the same elements in the same order.

### 1.9 Example: Lists versus Sets

- Lists  $(3, 5)$  and  $(5, 3)$  are **not equal**, but sets  $\{3, 5\} = \{5, 3\}$
- Lists  $(4, 4)$  and  $(4, 4, 4)$  are **not equal**, but sets  $\{4, 4\} = \{4, 4, 4\} = \{4\}$

Key difference: order and repetition matter in lists, not in sets.

**Why lists?** Linear algebra needs ordered data. The coordinates  $(1, 2, 3)$  represent a different point than  $(3, 2, 1)$ . Order encodes meaning—the first coordinate might be position, the second velocity, the third acceleration.

## 3. $\mathbb{F}^n$

To define the higher-dimensional analogues of  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , we simply replace  $\mathbb{R}$  with  $\mathbb{F}$  (which equals  $\mathbb{R}$  or  $\mathbb{C}$ ) and replace the 2 or 3 with an arbitrary positive integer.

### 1.10 Notation: $n$

Fix a positive integer  $n$  for the rest of this chapter.

### 1.11 Definition: $\mathbb{F}^n$

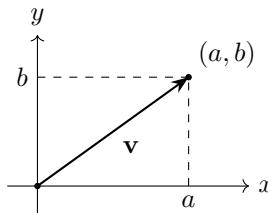
$\mathbb{F}^n$  is the set of all lists of length  $n$  of elements of  $\mathbb{F}$ :

$$\mathbb{F}^n = \{(x_1, x_2, \dots, x_n) : x_k \in \mathbb{F} \text{ for } k = 1, \dots, n\}$$

For  $(x_1, \dots, x_n) \in \mathbb{F}^n$  and  $k \in \{1, \dots, n\}$ , we say  $x_k$  is the  $k^{\text{th}}$  **coordinate** of  $(x_1, \dots, x_n)$ .

- Example 3.1** (Elements of  $\mathbb{F}^n$ ). •  $(2, -1, 5) \in \mathbb{R}^3$  is a list with 3 real coordinates.
- $(1+i, 2, -3i) \in \mathbb{C}^3$  is a list with 3 complex coordinates.
  - $(7, -2) \in \mathbb{R}^2$  corresponds to a point in the plane.
  - $\mathbb{F}^1$  can be identified with  $\mathbb{F}$ .

**Intuition:** Think of  $\mathbb{R}^2$  as the plane and  $\mathbb{R}^3$  as 3-dimensional space. For  $n > 3$ , we lose geometric visualization but the algebra works identically.



Elements of  $\mathbb{R}^2$  can be thought of as points or as vectors.

### 1.12 Example: $\mathbb{C}^4$

$$\mathbb{C}^4 = \{(z_1, z_2, z_3, z_4) : z_1, z_2, z_3, z_4 \in \mathbb{C}\}$$

### 1.13 Definition: Addition in $\mathbb{F}^n$

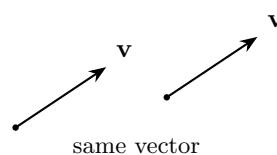
Addition in  $\mathbb{F}^n$  is defined by adding corresponding coordinates:

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

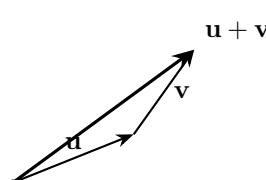
**Example 3.2** (Vector Addition). In  $\mathbb{R}^3$ :

$$(1, 2, 3) + (10, 20, 30) = (1+10, 2+20, 3+30) = (11, 22, 33)$$

**Geometric intuition:** In  $\mathbb{R}^2$  and  $\mathbb{R}^3$ , addition corresponds to the parallelogram rule: place the tail of the second vector at the head of the first.



A vector—same length and direction = same vector.



The sum of two vectors (tip-to-tail method).

### 1.14 Commutativity of Addition in $\mathbb{F}^n$

If  $x, y \in \mathbb{F}^n$ , then  $x + y = y + x$ .

**Proof:** Suppose  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$ . Then:

$$\begin{aligned} x + y &= (x_1, \dots, x_n) + (y_1, \dots, y_n) \\ &= (x_1 + y_1, \dots, x_n + y_n) \\ &= (y_1 + x_1, \dots, y_n + x_n) \\ &= (y_1, \dots, y_n) + (x_1, \dots, x_n) \\ &= y + x \end{aligned}$$

where the third equality uses commutativity of addition in  $\mathbb{F}$ .  $\square$

### 1.15 Notation: 0

Let 0 denote the list of length  $n$  whose coordinates are all 0:

$$0 = (0, \dots, 0)$$

**Geometric intuition:** The zero vector 0 represents the origin. Adding 0 to any vector leaves it unchanged—you’re adding “no displacement.”

### 1.16 Example: The Zero Vector Notation

When we write  $x + 0 = x$  for  $x \in \mathbb{F}^n$ , the symbol 0 means the **zero vector**:

$$0 = (0, 0, \dots, 0) \quad (n \text{ zeros})$$

**Why?** Addition in  $\mathbb{F}^n$  is only defined for two vectors. Since  $x$  is a vector, the 0 in “ $x + 0$ ” must also be a vector—not the number zero.

**Example in  $\mathbb{R}^3$ :**

$$(1, 2, 3) + 0 = (1, 2, 3) + (0, 0, 0) = (1, 2, 3) \checkmark$$

**Key point:** The symbol “0” means different things depending on context:

- In  $\mathbb{F}$  (scalars): 0 is the number zero
- In  $\mathbb{F}^n$  (vectors): 0 is the zero vector  $(0, \dots, 0)$

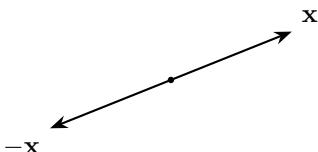
### 1.17 Definition: Additive Inverse in $\mathbb{F}^n$

For  $x \in \mathbb{F}^n$ , the **additive inverse** of  $x$ , denoted  $-x$ , is the vector  $-x \in \mathbb{F}^n$  such that  $x + (-x) = 0$ .

If  $x = (x_1, \dots, x_n)$ , then  $-x = (-x_1, \dots, -x_n)$ .

**Note:** Subtraction is defined by  $x - y = x + (-y)$ .

**Geometric intuition:** In  $\mathbb{R}^2$ ,  $-x$  is the vector with the same length as  $x$  but pointing in the opposite direction.



*A vector and its additive inverse.*

**Why not coordinate-wise multiplication?** We could define multiplication of two vectors by multiplying corresponding coordinates, but this is not useful for linear algebra. Instead, **scalar multiplication** (multiplying a vector by a number) is central to our subject.

### 1.18 Definition: Scalar Multiplication in $\mathbb{F}^n$

The product of a number  $\lambda \in \mathbb{F}$  and a vector in  $\mathbb{F}^n$  is computed by multiplying each coordinate by  $\lambda$ :

$$\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$$

**Example 3.3** (Scalar Multiplication in  $\mathbb{R}^n$ ).

$$-2(1, 2, 3) = (-2 \cdot 1, -2 \cdot 2, -2 \cdot 3) = \boxed{(-2, -4, -6)}$$

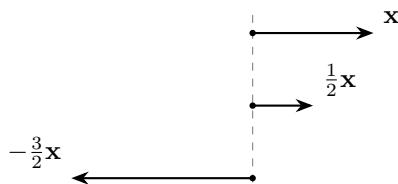
**Example 3.4** (Scalar Multiplication in  $\mathbb{C}^n$ ).

$$\begin{aligned} (1+i)(2, -i) &= ((1+i) \cdot 2, (1+i)(-i)) \\ &= (2+2i, -i-i^2) \\ &= (2+2i, -i+1) \\ &= \boxed{(2+2i, 1-i)} \end{aligned}$$

### Geometric intuition in $\mathbb{R}^2$ :

- If  $\lambda > 0$ :  $\lambda x$  points in the same direction as  $x$ , with length  $\lambda$  times the length of  $x$ .
- If  $\lambda > 1$ : stretches (longer). If  $0 < \lambda < 1$ : shrinks (shorter).
- If  $\lambda < 0$ :  $\lambda x$  points in the opposite direction, with length  $|\lambda|$  times the length of  $x$ .

**Note:** All scalar multiples of  $x$  lie on a single line through the origin. This “direction preservation” is fundamental to linear maps (Chapter 3).



*Scalar multiplication: scaling and reversing vectors.*

**Scalar multiplication vs dot product:** Scalar multiplication takes a scalar and a vector, producing a **vector**. The dot product (Chapter 6) takes two vectors and produces a **scalar**. These are different operations.

## 4. Digression on Fields

A **field** is a set containing at least two distinct elements called 0 and 1, along with operations of addition and multiplication satisfying all properties listed in 1.3.

- $\mathbb{R}$  and  $\mathbb{C}$  are fields.
- The set of rational numbers  $\mathbb{Q}$  is a field.
- The set  $\{0, 1\}$  with usual addition and multiplication (except  $1 + 1 = 0$ ) is a field.

**Note:** This book deals only with  $\mathbb{R}$  and  $\mathbb{C}$ . However, many definitions, theorems, and proofs work for arbitrary fields. If you prefer, think of  $\mathbb{F}$  as denoting an arbitrary field (except in Chapters 6–7 on inner products, where  $\mathbb{F} = \mathbb{C}$  is sometimes required).

### Key Takeaways

1. **Complex numbers**  $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$  with  $i^2 = -1$ .
2. **Arithmetic:**  $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$ .
3. **Field properties** (1.3): commutativity, associativity, identities, inverses, distributivity.
4. **F notation:**  $\mathbb{F}$  denotes either  $\mathbb{R}$  or  $\mathbb{C}$ .
5.  $\mathbb{F}^n$  is the set of all  $n$ -tuples  $(x_1, \dots, x_n)$  with  $x_j \in \mathbb{F}$ .
6. **Addition** is coordinate-wise.
7. **Scalar multiplication:**  $\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$ .
8. **Zero vector**  $0 = (0, \dots, 0)$  is the additive identity.