Inner Products

Linear Algebra Done Right, 4th ed.

Exercise 1. Consider the space $V = \mathcal{P}_n(\mathbb{C})$. Show that

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$$

defines an inner product on V.

Exercise 2. Consider a fixed set of points $x_1, \ldots, x_n \in \mathbb{C}$. For two functions $f, g \in \mathcal{P}_n(\mathbb{C})$, is

$$\langle f, g\rangle = \sum_{i=1}^{n} f(x_i)\overline{g(x_i)}$$

an inner product? For which values of k is this an inner product on $\mathcal{P}_k(\mathbb{C})$?

Exercise 3. (Axler 6.A.22) Show that if $u, v \in V$, then

$$\|u+v\| \cdot \|u-v\| \leq \|u\|^2 + \|v\|^2.$$

Exercise 4. (Axler 6.A.23) Suppose $v_1, \ldots, v_m \in V$ are such that $\|v_k\| \leq 1$ for each $k = 1, \ldots, m$. Show that there exists $a_1, \ldots, a_m \in \{1, -1\}$ such that

$$\|a_1 v_1 + \cdots + a_m v_m\| \leq \sqrt{m}.$$

Exercise 5. For $n = 1, 2, 3$, use the Gram-Schmidt algorithm to orthonormalize the basis $1, x, \ldots, x^n$ of $\mathcal{P}_n(\mathbb{C})$ equipped with the inner product of Exercise 1.

Exercise 6. (Axler 6.B.6a) Suppose $e_1, e_2, \ldots, e_n$ is an orthonormal basis of V. Prove that if $v_1, v_2, \ldots, v_n$ are vectors in V such that

$$\|e_k - v_k\| < \frac{1}{\sqrt{n}}$$

for each k, then $v_1, v_2, \ldots, v_n$ is a basis of V.

Exercise 7. (Axler 6.B.18) Suppose $u_1, u_2, \ldots, u_m$ is a linearly independent list of vectors in V. Show that there exists $v \in V$ such that $\langle u_i, v \rangle = 1$ for all $i = 1, \ldots, m$.

Exercise 8. Prove that every (pairwise) orthogonal list of non-zero vectors is linearly independent.

Exercise 9. (Axler 6.C.10) Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V. Prove that U and $U^{\perp}$ are both invariant under T if and only if $P_U T = T P_U$.

Exercise 10. (Axler 6.C.12) Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(0) = 0$, $p'(0) = 0$, and

$$\int_0^1 |2 + 3x - p(x)|^2 \, dx$$

is as small as possible.

Answer: ____

Exercise 11. (Axler 6.C.9) Suppose V is finite-dimensional. Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in null(P) is orthogonal to every vector in range(P). Prove that there exists a subspace U of V such that $P = P_U$.