Vector Spaces, Linear Maps, and Duality

Linear Algebra Done Right, 4th ed.

Exercise 1. Given a matrix

$$A := \begin{bmatrix} a_{11} & \cdots & a_{15} \\ \vdots & \ddots & \vdots \\ a_{51} & \cdots & a_{55} \end{bmatrix} \in \mathbb{R}^{5\times 5},$$

we say that A is *symmetric* if and only if $a_{ij} = a_{ji}$ for each i and j, and we say that A is *skew-symmetric* if and only if $a_{ij} = -a_{ji}$.

(a) Let Y and K be the set of symmetric and skew-symmetric matrices in $\mathbb{R}^{5\times 5}$, respectively. Show that Y and K are subspaces of $\mathbb{R}^{5\times 5}$.

(b) Compute $\dim Y$ and $\dim K$.

Answer: ______________________________

(c) Show that $\mathbb{R}^{5\times 5}$ is the direct sum of the subspaces Y and K.

Exercise 2. (Axler 2.C.17) Fix a finite-dimensional vector space V over a field $\mathbb{F}$, and fix subspaces $U_1, \ldots, U_n$ of V.

(a) Show that

$$\dim(U_1 + \cdots + U_n) \leq \dim U_1 + \cdots + \dim U_n.$$

(b) In fact, show that equality holds in (a) if and only if the sum $U_1 + \cdots + U_n$ is direct.

Exercise 3. Fix a linear transformation $T \colon V \to W$ of vector spaces over a field $\mathbb{F}$. For any $w \in W$, suppose there is a vector $v_0 \in V$ such that $T(v_0) = w$. Then show that

$$\{v \in V : T(v) = w\} = \{v + v_0 : v \in \operatorname{null} T\}.$$

Exercise 4. (Axler 3.B.28) Suppose $p \in \mathcal{P}(\mathbb{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbb{R})$ such that

$$5q'' + 3q' = p.$$

Exercise 5. Let I be the linear map $I \colon \mathcal{P}_5(\mathbb{R}) \to \mathcal{P}_6(\mathbb{R})$ by

$$I(f) := \int_0^x f(t)\, dt.$$

(a) Convince yourself that I is a linear map.

(b) Find bases of $\mathcal{P}_5(\mathbb{R})$ and $\mathcal{P}_6(\mathbb{R})$.

Answer: ____________________

(c) Use the bases found in (b) in order to write dual bases for $\mathcal{P}_5(\mathbb{R})'$ and $\mathcal{P}_6(\mathbb{R})'$.

(d) Use the dual bases found in (c) in order to write $I' \colon \mathcal{P}_6(\mathbb{R})' \to \mathcal{P}_5(\mathbb{R})'$ as a matrix.

Exercise 6. Let V be a 2-dimensional vector space, and let $\varphi, \psi \in V'$. Show that φ and ψ are linearly independent if and only if

$$\operatorname{null}(\varphi) \cap \operatorname{null}(\psi) = \{0\}.$$

Exercise 7. Fix a finite-dimensional vector space V. Call a linear map $T\colon V \to V$ a *projection* if and only if $T \circ T = T$.

(a) Give an example of a projection $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which is neither the zero nor the identity operators.

Answer: ______________________

(b) For any projection T, show that T fixes $\operatorname{range} T$.

(c) For any projection T, show that $\operatorname{range} T \cap \operatorname{null} T = \{0\}$. Conclude that $V = \operatorname{range} T \oplus \operatorname{null} T$.

(d) We say that a linear transformation $T \in \mathcal{L}(V)$ is *diagonal* if and only if there is a basis $\{v_1, \ldots, v_n\}$ of V and constants $\lambda_1, \ldots, \lambda_n$ such that $Tv_i = \lambda_i v_i$ for each i. Show that T can be written as a linear combination of projections.

Exercise 8. (Axler 3.F.23) Let V be a finite-dimensional vector space, and let U and W be subspaces of V.

(a) Show that $(U+W)^0 = U^0 \cap W^0$.

(b) Show that $(U \cap W)^0 = U^0 + W^0$.
