1/20/2026

Lec 1 Lecture

Modules : Resources

Syllabus ✓

Proof. 1. Formal System = alphabet, grammar rules. $\begin{cases} \text{Alphabet, sentences, axioms. (initial seed)} \\ \text{(RI) Rules of Inferences.} \to \text{Theorems.} \end{cases}$

"Proof of S_n"

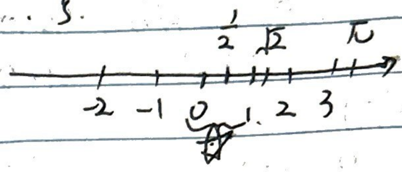
$S_1 \dots S_n$: S_i = either axiom / obtained from the preceding sentences. by Rules of Inference.

S_n : $\boxed{\text{Theorem}}$.

Syntax

Notions: 1. Set : Examples: ① Natural #. $\{0, 1, \dots\}$. ($\mathbb{N}$)

② Real Numbers : ($\mathbb{R}$)



ZFC ("choice") : $\begin{cases} A, B, C \dots \Rightarrow \text{Sets.} \\ a, b, c \dots \Rightarrow \text{elements} \end{cases}$

"$a \notin A$" does a belong.

"$a \in A$" "a belongs to A"

"$\forall$" = "for any",

"$\exists$" = "there exists"

Ex: Sentence in Peano Arithmetic:

Logical: "$\forall a \ \exists l \ (a = 2 \cdot l \ \vee \ a = 2 \cdot l + 1)$"

multiplication   addition

Formal System of Fields

1. A field is a set F equipped with 2 operations (binary) : $\begin{cases} + \text{ addition} \\ \cdot \text{ Multiplication} \end{cases}$

$a \to \boxed{+} \to a+b$, $b \to$ $a \to \boxed{\cdot} \to a \cdot b$, $b \to$

2. We also have functions (maps)

$A \xrightarrow{f} B$ (A, B - sets)

$a \mapsto f(B)$

Ex: $A = \mathbb{R}$, $B = \mathbb{R}$. $\mathbb{R} \xrightarrow{f} \mathbb{R}$, $x \mapsto x^2$ $\boxed{f(x) = x^2}$

$\mathbb{R}^2 \to \mathbb{R}$
$(x, y) \mapsto x^2 + y^2$
$\boxed{f(x,y) = x^2 + y^2}$

Fields

ZFC

Sets.

Cartesian Product

1. $A \times B = \{(a,b) \mid a \in A,\ b \in B\}$.

$\begin{cases} \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x \in \mathbb{R},\ y \in \mathbb{R}\}. \\ \mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) \mid x \in \mathbb{R},\ y \in \mathbb{R},\ z \in \mathbb{R}\}. \end{cases}$

Back to 2

Operations: 1. Addition: $F \times F \xrightarrow{+,\cdot} F$

$(a,b) \longmapsto a+b$

Additional Axioms. (Page 3 SIA)

1. (F1+) $\forall a \forall b.\ (a+b = b+a)$

(F1•) $a \cdot b = b \cdot a$

2. (F2+) $(a+b)+c = a+(b+c)$

(F2•) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

3. (F3+) $\exists 0\ \forall a: a+0 = a$

4. (F3•) $\exists 1\ \forall a: a \cdot 1 = a$

5. (F4) inverses

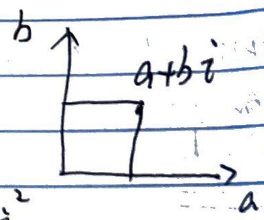
6. (F5) distributive law: $a \cdot (b+c) = a \cdot b + a \cdot c$

☆ $\mathbb{R}$ is a model of this formal system. (as well as $\mathbb{C}$)

① $F = \mathbb{R}$. ② $F = \mathbb{C}$ - complex $\sqrt{-1} \notin \mathbb{R}$.

1. General Complex number: $a + b\sqrt{-1} \in \mathbb{C}$

$= a + bi$



2. $(a_1 + b_1 i) + (a_2 + b_2 i) = a_1 + a_2 + (b_1 + b_2) \cdot i$

3. $(a_1 + b_1 i) \cdot (a_2 + b_2 i) = a_1 a_2 + a_1 b_2 i + b_1 \cdot a_2 \cdot i + b_1 \cdot b_2 \cdot i^2$

Addition in $\mathbb{R}^2$:

$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2,\ b_1 + b_2)$.

$\mathbb{R}^2$ $u \to \boxed{+} \to u+v$
$\mathbb{R}^2$ $v \to$

Scalar multiplication

$\mathbb{R}: \alpha \to \boxed{\cdot} \to (\alpha \cdot a_2,\ \alpha \cdot b_2)$
$\mathbb{R}^2: v \to$

Formal System of Vector Spaces.

Let F be a field

1. A vector space over F is a set V .... 2 operations:

$+$, $\cdot$ scalar multiplication.

$$V \ni a \rightarrow \quad V \ni b \rightarrow \boxed{+} \rightarrow a+b \in V \qquad V \times V \xrightarrow{+} V$$

$$F \times V \rightarrow V$$

$$F \ni \alpha \rightarrow \quad V \ni a \rightarrow \boxed{\cdot} \rightarrow \alpha \cdot a$$

(V1) $u+v = v+u$

(V2) Associativity

(V3) $\exists \underline{0} \in V,\ \exists v \in V,\ v + \underline{0} = v$ $\quad (0 \in F,\ \underline{0} \in V)$

Theorem: Suppose 0 and $0'$ satisfy (V3), then $0 = 0'$

Example of Proofs:

S1: $v+0=v$ (axiom V3 for 0)

S2: $0'+0=0'$ (RI = substitution $v=0'$)

S3: $v+0'=v$ (V3 for $0'$)

S4: $0+0'=0$ (RI = Subs. $v=0$)

S5: $u+v=v+u$

S6: $0'+0=0+0'$

S7: $0+0'=0'$

$\boxed{\text{S8: } 0=0'}$

1/22/2026

Lec 2 -Lecture

PS1

Field 1. a field ($F=\mathbb{R}$ or $\mathbb{C}$). fixed.

Def 2. A vector space over F is a set V with 2 binary operations.

$$V\times V \xrightarrow{+} V \quad (u,v)\mapsto u+v \in V$$
$$F\times V \xrightarrow{\cdot} V \quad (\alpha\in F, v\in V)\mapsto \alpha\cdot v\in V$$

- $v\in V$ call v a "vector"
- $\alpha\in F$ call α a "scalar" (number)
- These 2 operations follow axioms:

① Ax(1) ~~commutivity~~ commutivity : $\forall u\in V, \forall v\in V, u+v=v+u$

② Ax(2): associtivity

③ Ax(3): $\exists 0_V\in V, \forall v, v+0_V=v$

~~④ $\forall 0\in V, \exists V, V+0=V$~~

④ $\forall v\in V, \exists w\in V, v+w=0_V$ ... "negative of v"

Remark: $0_F\in F$, $0_V\in V$

Theorem 1: (Uniqueness of $0_V\in V$): Suppose $0\in V$ and $0'\in V$ both satisfy Axiom 3, ($\forall v, v+0=v$ and $\forall v, v+0'=v$) Then, $0=0'$

Proof ("Accelerated"): ① $0'=0'+0$ (Ax(3) for 0 with $v=0'$)

Note: "$A=B \iff B=A$)
2. $A=B=C \Rightarrow A=C$ } R.I.

② By Ax(1): $0'=0'+0=0+0'=0$ (Ax(3) for $0'$ with $v=0$)

③ Thus, $0'=0$. $\square$

Theorem 2: Uniqueness of Additive Inverse: Let $v\in V$; suppose $v+w=0$ and $v+w'=0$. Then $w=w'$.

Proof: ① By Ax(3) $w=w+0=w+(v+w') \overset{\text{By Ax(2)}}{=} (w+v)+w'=0+w' \overset{Ax(1)}{=} w'+0 \overset{Ax(3)}{=} w'$. $\square$

(Set Theory)

Subspace 1. Let S be a set. Then we say that set T is a $\boxed{\text{Subset}}$ of S if T consists of some of the elements of S (and nothing more). Notation $T \subset S \Rightarrow \forall a \in T,\ a \in S$

(Note: $a \in S \neq T \subset S$)

Example: $\{\text{Even integers}\} \subset \{\text{All integers}\}$.

Union / Intersection

$$\begin{cases} T_1 \subset S \rightsquigarrow T_1 \cap T_2 \subset S = \{a \in S \mid a \in T_1 \ \& \ a \in T_2\}. & \text{(intersect)} \\ T_2 \subset S \rightsquigarrow T_1 \cup T_2 \text{ (union)} = \{a \in S \mid a \in T_1 \text{ or } a \in T_2\}. \end{cases}$$

Ex: T_1 = even numbers, $T_2 = \{n = 3 \cdot m\}$.

Then: $T_1 \cap T_2 = \{6 \cdot m\}$.

- Suppose V is a vector space over F ($\mathbb{R}$ or $\mathbb{C}$)
  - More correctly, $(V, +, \cdot) \equiv$ v. sp. over F.
- V is a set, Let U be a subset.
- In general, U is not going to be a vector space.
- Suppose further, U is closed under addition (+) & scalar multiplication ($\cdot$)

This means (

(1) $u_1, u_2 \in U$. View them as elements of V.
Then, $u_1 + u_2 \in V$. closed under "+" if $u_1 + u_2 \in U \ \forall u_1, u_2 \in U$

(2) closed under "$\cdot$" means: $\forall u \in U$ ($\in V$), $\forall \alpha \in F$, $\alpha \cdot u$ ($\in V$) is in U

Examples for Vector Space over $\mathbb{R}$:

① $\mathbb{R}^n$ ($n = 1, 2, 3, \ldots$) $= \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \middle| \ a_i \in \mathbb{R},\ \forall i = 1, \ldots, n \right\}$

"such that"

② $\mathbb{C}$ version:

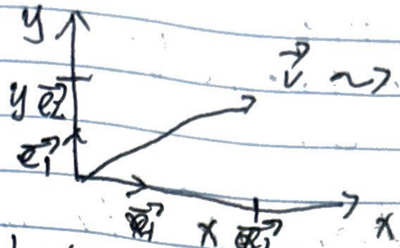
$$\mathbb{C}^n = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \middle| \ a_i \in \mathbb{C},\ \forall i = 1, \ldots, n \right\}.$$

Remark: ① $\boxed{\mathbb{R} \subset \mathbb{C}}$ $\{a + b\sqrt{-1}\} \subset \{a + b\sqrt{-1} \mid a, b \in \mathbb{R}\}$.

② $\mathbb{C}^n$ can also be viewed as a vector sp. over $\mathbb{R}$ ($\underset{\text{isomorphic}}{\simeq} \mathbb{R}^n$)

(continued)

Example 2



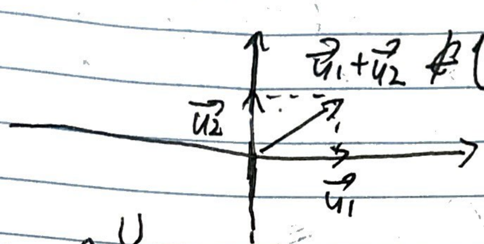
$\vec{v} \rightsquigarrow \binom{x}{y} \in \mathbb{R}^2$

- Let $U = \{\vec{e_1}\} \subset V = \mathbb{R}^2 = \binom{1}{0}$

e.g. $2 \cdot \vec{e_1} \in V$, but $\notin U$

Ex 1: - $U \in \{\alpha \cdot \vec{e_1} \mid \alpha \in \mathbb{R}\}$.

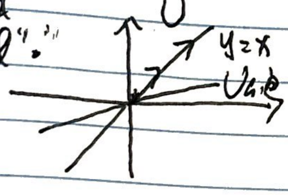
Ex2: -



$\vec{u_1} + \vec{u_2} \notin U \Rightarrow U$ is not closed under "+"

$U = \{x\text{-axis}\} \cup \{y\text{-axis}\}$.

Ex 3: closed "+" & "·"



$\vec{u} \in U \iff x - y = 0$

" $\binom{x}{y}$

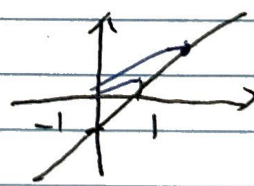
- More generally: $\forall \{\binom{x}{y} \mid a \cdot x + b \cdot y = 0\}$ for some fixed $a, b \in \mathbb{R}$

(homogeneous eq.)

- But consider an inhomogeneous example:

$x - y = 1$ ← not equal to 0.

or, $y = x - 1$



Let $U' = \{x - y = 1\}$

$x, y \in \mathbb{R}$

① addi on "+".

② mult "·": $0 \in \mathbb{R}$. $\underset{\mathbb{R}}{0} \binom{x}{y} = \binom{0}{0}$ ← $0 \cdot \mathbb{R}^2$

$\Rightarrow U'$ is not closed under "·" or "+"

Ex of closed: - Let V be a v.s.p. over F, $+$, $\cdot$
$U \subset V$ a subset closed under $+$, $\cdot$ in V
then obtain $+$ and $\cdot$ on U.

~~Def: U is a subspace~~

Def: $U \subset V$ (subset) called a subspace if it's closed under "$+$ and $\cdot$", and with respect to $+$, and $\cdot$, U is a vector space over the same field. F.

Short Cut! Theorem 3: $U \subset V$ is a subspace $\boxed{\text{iff}}$:
(a) U is closed under "$+$"
(b) U is closed under "$\cdot$"
(c) $0_V \in U$.

for verifying this!

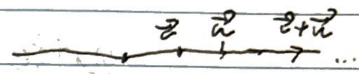
Proof for both directions.

Remark: $U \subset V$ means $U = V$ for "~~st~~ ...

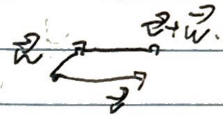
Discussion 1 Vector Space, Subspace 1/23/2026

Q1: What is a field? Examples? Non-examples? (All with Proofs)

Q2: What is a vector space? Examples? Non-examples?

A2: ① Geometric Examples: a) line, with an fixed origin 

$+ \checkmark \quad \cdot \checkmark$

b) plane, fixed origin 

over $\mathbb{R}$

$+$:

$\cdot$: $\checkmark$

c) solid space, fixed origin: $+ \checkmark$, $\cdot \checkmark$

② Constructive Examples: Let F be a field.

- $F \checkmark$
- $F^2 = \{(x,y) \mid x, y \in F\} \checkmark$

$\boxed{\text{• vector operations are defined coordinate-wise.}}$

$$\begin{cases} + : (x_1,y_1)+(x_2,y_2) = (x_1+x_2, y_1+y_2) \\ \cdot : \lambda(x,y) = (\lambda x, \lambda y) \end{cases}$$

Special: ③ Functional Examples:

- All polynomials with real coefficients with vector space.
- $\mathbb{R} \to \mathbb{R}$

other domains work.
- All functions including discontinuous ones.

Q3: What does it mean to say one vector space is a subspace of another vector space? Examples vs Non-examples.

A3 ① Geometric Examples: a) line: 

U: {0}. line itself.

{1} is not!

{0,1,2...} is not! (no inverse!!!)

b) solid:

c) Plane: {origin},

- V, vs, $U_1, \dots, U_n$ sub
- Def $U_1 + \dots + U_n := \{u_1 + \dots + u_n : u_1 \in U_1, \dots, u_n \in U_n\}$ is a subspace of V

Write $U_1 + \dots + U_n = U_1 \oplus \dots \oplus U_n$

if $u_1 + \dots + u_n = u_1' + \dots + u_n'$
$u_1 = u_1' \dots u_n' = u_n'$

Thm 1.31; 1.32