Exercises 1A Solutions: $\mathbb{R}^n$ and $\mathbb{C}^n$

Linear Algebra Done Right, 4th ed.

Exercise 1. Suppose a and b are real numbers, not both 0. Find real numbers c and d such that

$$\frac{1}{a+bi} = c + di.$$

Solution. Multiply the numerator and denominator by the complex conjugate of the denominator:

$$\frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2}.$$

Since a and b are not both 0, we have $a^2 + b^2 \neq 0$. Separating into real and imaginary parts:

$$\frac{1}{a+bi} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i.$$

Therefore:

$$\boxed{c = \frac{a}{a^2+b^2}, \quad d = \frac{-b}{a^2+b^2}}$$

□

Exercise 2. Show that

$$\frac{-1+\sqrt{3}\,i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Solution. Let $\omega = \frac{-1+\sqrt{3}\,i}{2}$. We compute ω^3 step by step.

First, compute ω^2:

$$\begin{aligned}
\omega^2 = \left(\frac{-1+\sqrt{3}\,i}{2}\right)^2 &= \frac{(-1+\sqrt{3}\,i)^2}{4} \\
&= \frac{1-2\sqrt{3}\,i+3i^2}{4} = \frac{1-2\sqrt{3}\,i-3}{4} \\
&= \frac{-2-2\sqrt{3}\,i}{4} = \frac{-1-\sqrt{3}\,i}{2}.
\end{aligned}$$

Now compute $\omega^3 = \omega^2 \cdot \omega$:

$$\begin{aligned}
\omega^3 &= \frac{-1-\sqrt{3}\,i}{2} \cdot \frac{-1+\sqrt{3}\,i}{2} \\
&= \frac{(-1-\sqrt{3}\,i)(-1+\sqrt{3}\,i)}{4} \\
&= \frac{(-1)^2-(\sqrt{3}\,i)^2}{4} = \frac{1-3i^2}{4} = \frac{1+3}{4} = 1.
\end{aligned}$$

Therefore $\omega^3 = 1$, so ω is a cube root of 1. □

□

Exercise 3. Find two distinct square roots of i.

Solution. We seek $z = a + bi$ such that $z^2 = i$, where $a, b \in \mathbb{R}$.

Expanding z^2:

$$z^2 = (a + bi)^2 = a^2 + 2abi + b^2 i^2 = (a^2 - b^2) + 2abi.$$

Setting $z^2 = i = 0 + 1 \cdot i$, we equate real and imaginary parts:

$$\begin{aligned} a^2 - b^2 &= 0 \\ 2ab &= 1 \end{aligned}$$

From the first equation, $a^2 = b^2$, so $a = \pm b$.

If $a = b$: From $2ab = 1$, we get $2a^2 = 1$, so $a = \pm\frac{1}{\sqrt{2}}$. This gives $a = b = \frac{1}{\sqrt{2}}$ or $a = b = -\frac{1}{\sqrt{2}}$.

If $a = -b$: From $2ab = 1$, we get $-2a^2 = 1$, which has no real solutions.

Therefore the two square roots of i are:

$$\boxed{z_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i = \frac{\sqrt{2}}{2}(1 + i), \quad z_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i = -\frac{\sqrt{2}}{2}(1 + i)}$$

□

Exercise 4. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Solution. Let $\alpha = a + bi$ and $\beta = c + di$ where $a, b, c, d \in \mathbb{R}$.

Computing $\alpha + \beta$:

$$\alpha + \beta = (a + bi) + (c + di) = (a + c) + (b + d)i.$$

Computing $\beta + \alpha$:

$$\beta + \alpha = (c + di) + (a + bi) = (c + a) + (d + b)i.$$

Since addition of real numbers is commutative ($a + c = c + a$ and $b + d = d + b$):

$$\alpha + \beta = (a + c) + (b + d)i = (c + a) + (d + b)i = \beta + \alpha.$$

Therefore $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$. $\square$ $\square$

Exercise 5. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Solution. Let $\alpha = a + bi$, $\beta = c + di$, and $\lambda = e + fi$ where $a, b, c, d, e, f \in \mathbb{R}$.

Computing the left-hand side:

$$\begin{aligned}(\alpha + \beta) + \lambda &= \big((a + c) + (b + d)i\big) + (e + fi) \\ &= \big((a + c) + e\big) + \big((b + d) + f\big)i.\end{aligned}$$

Computing the right-hand side:

$$\begin{aligned}\alpha + (\beta + \lambda) &= (a + bi) + \big((c + e) + (d + f)i\big) \\ &= \big(a + (c + e)\big) + \big(b + (d + f)\big)i.\end{aligned}$$

Since addition of real numbers is associative:

$$(a + c) + e = a + (c + e) \quad \text{and} \quad (b + d) + f = b + (d + f).$$

Therefore $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$. $\square$ $\square$

Exercise 6. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Solution. Let $\alpha = a + bi$, $\beta = c + di$, and $\lambda = e + fi$ where $a, b, c, d, e, f \in \mathbb{R}$.

First, compute $\alpha\beta$:

$$\alpha\beta = (a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Then compute $(\alpha\beta)\lambda$:

$$\begin{aligned}(\alpha\beta)\lambda &= \big((ac - bd) + (ad + bc)i\big)(e + fi) \\ &= \big((ac - bd)e - (ad + bc)f\big) + \big((ac - bd)f + (ad + bc)e\big)i \\ &= (ace - bde - adf - bcf) + (acf - bdf + ade + bce)i.\end{aligned}$$

Now compute $\beta\lambda$:

$$\beta\lambda = (c + di)(e + fi) = (ce - df) + (cf + de)i.$$

Then compute $\alpha(\beta\lambda)$:

$$\begin{aligned}\alpha(\beta\lambda) &= (a + bi)\big((ce - df) + (cf + de)i\big) \\ &= \big(a(ce - df) - b(cf + de)\big) + \big(a(cf + de) + b(ce - df)\big)i \\ &= (ace - adf - bcf - bde) + (acf + ade + bce - bdf)i.\end{aligned}$$

Comparing terms, we see that LHS = RHS. Therefore $(\alpha\beta)\lambda = \alpha(\beta\lambda)$. $\square$ $\square$

Exercise 7. Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

Solution. Let $\alpha = a + bi$ where $a, b \in \mathbb{R}$.
Existence: Define $\beta = -a + (-b)i = -a - bi$. Then:

$$\alpha + \beta = (a + bi) + (-a - bi) = (a - a) + (b - b)i = 0 + 0i = 0.$$

So such a β exists.
Uniqueness: Suppose β_1 and β_2 both satisfy $\alpha + \beta_1 = 0$ and $\alpha + \beta_2 = 0$.
Then:

$$\begin{aligned}\beta_1 &= \beta_1 + 0 = \beta_1 + (\alpha + \beta_2)\\ &= (\beta_1 + \alpha) + \beta_2 = (\alpha + \beta_1) + \beta_2\\ &= 0 + \beta_2 = \beta_2.\end{aligned}$$

Therefore β is unique. □ □

Exercise 8. Show that for every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

Solution. Let $\alpha = a + bi$ where $a, b \in \mathbb{R}$ and a and b are not both zero.
Existence: Define $\beta = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$.
Note that $a^2 + b^2 \neq 0$ since $\alpha \neq 0$, so β is well-defined. Then:

$$\begin{aligned}\alpha\beta &= (a + bi)\left(\frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i\right)\\ &= \frac{a^2}{a^2 + b^2} - \frac{ab}{a^2 + b^2} i + \frac{ab}{a^2 + b^2} i - \frac{b^2}{a^2 + b^2} i^2\\ &= \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1.\end{aligned}$$

Uniqueness: Suppose β_1 and β_2 both satisfy $\alpha\beta_1 = 1$ and $\alpha\beta_2 = 1$.
Then:

$$\beta_1 = \beta_1 \cdot 1 = \beta_1(\alpha\beta_2) = (\beta_1\alpha)\beta_2 = (\alpha\beta_1)\beta_2 = 1 \cdot \beta_2 = \beta_2.$$

Therefore β is unique. □ □

Exercise 9. Find $x \in \mathbb{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Solution. Isolate $2x$ by subtracting $(4, -3, 1, 7)$ from both sides:

$$2x = (5, 9, -6, 8) - (4, -3, 1, 7) = (5 - 4, 9 - (-3), -6 - 1, 8 - 7) = (1, 12, -7, 1).$$

Divide by 2:

$$x = \frac{1}{2}(1, 12, -7, 1) = \left(\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2}\right).$$

Therefore:

$$\boxed{x = \left(\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2}\right)}$$

□

Exercise 10. Explain why there does not exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Solution. If such a λ exists, then each component equation must hold simultaneously.

From the first component: $\lambda(2 - 3i) = 12 - 5i$.

Solving for λ:

$$\lambda = \frac{12 - 5i}{2 - 3i} = \frac{(12 - 5i)(2 + 3i)}{(2 - 3i)(2 + 3i)} = \frac{24 + 36i - 10i - 15i^2}{4 + 9} = \frac{24 + 26i + 15}{13} = \frac{39 + 26i}{13} = 3 + 2i.$$

Now check the second component with $\lambda = 3 + 2i$:

$$(3 + 2i)(5 + 4i) = 15 + 12i + 10i + 8i^2 = 15 + 22i - 8 = 7 + 22i. \quad \checkmark$$

Check the third component with $\lambda = 3 + 2i$:

$$(3 + 2i)(-6 + 7i) = -18 + 21i - 12i + 14i^2 = -18 + 9i - 14 = -32 + 9i.$$

But we need $-32 - 9i$, not $-32 + 9i$.

Since the imaginary parts differ ($9i \neq -9i$), there is no single $\lambda \in \mathbb{C}$ that satisfies all three component equations simultaneously. □ □

Exercise 11. Show that $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbb{F}^n$.

Solution. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, and $z = (z_1, \ldots, z_n)$ where $x_j, y_j, z_j \in \mathbb{F}$.
Computing the left-hand side:

$$\begin{aligned}(x+y)+z &= (x_1+y_1, \ldots, x_n+y_n) + (z_1, \ldots, z_n) \\ &= ((x_1+y_1)+z_1, \ldots, (x_n+y_n)+z_n).\end{aligned}$$

Computing the right-hand side:

$$\begin{aligned}x+(y+z) &= (x_1, \ldots, x_n) + (y_1+z_1, \ldots, y_n+z_n) \\ &= (x_1+(y_1+z_1), \ldots, x_n+(y_n+z_n)).\end{aligned}$$

Since addition in $\mathbb{F}$ is associative, for each j:

$$(x_j+y_j)+z_j = x_j+(y_j+z_j).$$

Therefore $(x+y)+z = x+(y+z)$. □ □

Exercise 12. Show that $(ab)x = a(bx)$ for all $x \in \mathbb{F}^n$ and all $a, b \in \mathbb{F}$.

Solution. Let $x = (x_1, \ldots, x_n)$ where $x_j \in \mathbb{F}$.
Computing the left-hand side:

$$(ab)x = ((ab)x_1, \ldots, (ab)x_n).$$

Computing the right-hand side:

$$a(bx) = a(bx_1, \ldots, bx_n) = (a(bx_1), \ldots, a(bx_n)).$$

Since multiplication in $\mathbb{F}$ is associative, for each j:

$$(ab)x_j = a(bx_j).$$

Therefore $(ab)x = a(bx)$. □ □

Exercise 13. Show that $1x = x$ for all $x \in \mathbb{F}^n$.

Solution. Let $x = (x_1, \ldots, x_n)$ where $x_j \in \mathbb{F}$.

By the definition of scalar multiplication:

$$1x = (1 \cdot x_1, \ldots, 1 \cdot x_n).$$

Since 1 is the multiplicative identity in $\mathbb{F}$, we have $1 \cdot x_j = x_j$ for each j.

Therefore:

$$1x = (x_1, \ldots, x_n) = x. \quad \square$$

$\square$

Exercise 14. Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{F}$ and all $x, y \in \mathbb{F}^n$.

Solution. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ where $x_j, y_j \in \mathbb{F}$.

Computing the left-hand side:

$$\begin{aligned}\lambda(x + y) &= \lambda(x_1 + y_1, \ldots, x_n + y_n) \\ &= (\lambda(x_1 + y_1), \ldots, \lambda(x_n + y_n)).\end{aligned}$$

Computing the right-hand side:

$$\begin{aligned}\lambda x + \lambda y &= (\lambda x_1, \ldots, \lambda x_n) + (\lambda y_1, \ldots, \lambda y_n) \\ &= (\lambda x_1 + \lambda y_1, \ldots, \lambda x_n + \lambda y_n).\end{aligned}$$

Since multiplication distributes over addition in $\mathbb{F}$, for each j:

$$\lambda(x_j + y_j) = \lambda x_j + \lambda y_j.$$

Therefore $\lambda(x + y) = \lambda x + \lambda y$. $\square$ $\square$

Exercise 15. Show that $(a + b)x = ax + bx$ for all $a, b \in \mathbb{F}$ and all $x \in \mathbb{F}^n$.

Solution. Let $x = (x_1, \ldots, x_n)$ where $x_j \in \mathbb{F}$.

Computing the left-hand side:

$$(a + b)x = ((a + b)x_1, \ldots, (a + b)x_n).$$

Computing the right-hand side:

$$\begin{aligned}ax + bx &= (ax_1, \ldots, ax_n) + (bx_1, \ldots, bx_n) \\ &= (ax_1 + bx_1, \ldots, ax_n + bx_n).\end{aligned}$$

Since multiplication distributes over addition in $\mathbb{F}$, for each j:

$$(a + b)x_j = ax_j + bx_j.$$

Therefore $(a + b)x = ax + bx$. $\square$ $\square$