Exercises 1C: Subspaces

Linear Algebra Done Right, 4th ed.

Exercise 1. Prove that the intersection of every collection of subspaces of V is a subspace of V.

Exercise 2. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Exercise 3. Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Hint: This exercise is more complicated than the previous exercise; there is no quick argument. First prove this in the special case where $\mathbb{F}$ has only two elements, and then prove it when $\mathbb{F}$ has more than two elements.

Exercise 4. Suppose U is the subspace of $\mathbb{F}^3$ defined by

$$U = \{(x, -x, 2x) \in \mathbb{F}^3 : x \in \mathbb{F}\}.$$

Also suppose W is the subspace of $\mathbb{F}^3$ defined by

$$W = \{(x, x, 2x) \in \mathbb{F}^3 : x \in \mathbb{F}\}.$$

Describe $U + W$ both symbolically and non-symbolically.

Answer: ______________________________

Exercise 5. Suppose U is a subspace of V. What is $U + U$?

Answer: ______________________________

Exercise 6. Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

Exercise 7. Is the operation of addition on the subspaces of V associative? In other words, if V_1, V_2, V_3 are subspaces of V, is

$$(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)?$$

Exercise 8. Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

Exercise 9. Suppose V_1, V_2, and U are subspaces of V. Prove or give a counterexample: If $V_1 + U = V_2 + U$, then $V_1 = V_2$.

Exercise 10. Suppose V_1, V_2, and U are subspaces of V. Prove or give a counterexample: If $V = V_1 \oplus U$ and $V = V_2 \oplus U$, then $V_1 = V_2$.

Hint: Think about subspaces of $\mathbb{F}^2$.

Exercise 11. Suppose

$$U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

Answer: ____

Exercise 12. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$.

Answer: ____________________

Exercise 13. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$

Find three nonzero subspaces W_1, W_2, W_3 of $\mathbb{F}^5$, none of which equals $\{0\}$, such that

$$\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3.$$

Exercise 14. A function $f : \mathbb{R} \to \mathbb{R}$ is called **even** if

$$f(-x) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

A function $f : \mathbb{R} \to \mathbb{R}$ is called **odd** if

$$f(-x) = -f(x) \quad \text{for all } x \in \mathbb{R}.$$

Let V_e denote the set of real-valued even functions on $\mathbb{R}$, and let V_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that

$$\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o.$$