1 Find a list of four distinct vectors in $\mathbf{F}^3$ whose span equals

$$\{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}.$$

SOLUTION Let $U = \{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}$.

The list $(1, -1, 0), (0, -1, 1), (0, 0, 0), (2, -2, 0)$ consists of four distinct vectors, each of which is in U. Thus the span of these four vectors is contained in U.

Conversely, if $(x, y, z) \in U$, then $y = -x - z$ and thus

$$(x, y, z) = x(1, -1, 0) + z(0, -1, 1) + 0(0, 0, 0) + 0(2, -2, 0).$$

Thus $(x, y, z) \in \operatorname{span}\big((1, -1, 0), (0, -1, 1), (0, 0, 0), (2, -2, 0)\big)$. This shows that U is contained in the span of our list of four vectors.

Hence U equals the span of our list of four vectors.

2 Prove or give a counterexample: If v_1, v_2, v_3, v_4 spans V, then the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

SOLUTION The statement above is true. To prove it, let $v \in V$. To show that $v \in \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$, we need to find $a_1, a_2, a_3, a_4 \in \mathbf{F}$ such that

$$v = a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4.$$

Rearranging the equation above, we see that we need to find $a_1, a_2, a_3, a_4 \in \mathbf{F}$ such that

$$v = a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4.$$

Because v_1, v_2, v_3, v_4 spans V, there exist $b_1, b_2, b_3, b_4 \in \mathbf{F}$ such that

$$v = b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4.$$

Comparing the last two equations, we see that the first of these two equations will be satisfied if we choose a_1 to equal b_1 and then choose a_2 to equal $b_2 + a_1$ and then choose a_3 to equal $b_3 + a_2$, and then choose a_4 to equal $b_4 + a_3$.

3 Suppose $v_1, \dots, v_m$ is a list of vectors in V. For $k \in \{1, \dots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

SOLUTION Suppose $k \in \{1, \dots, m\}$. If $k = 1$, then $v_k = w_k$. If $k > 1$, then

$$v_k = w_k - w_{k-1}.$$

Thus we see that $v_k \in \operatorname{span}(w_1, \dots, w_m)$.

The paragraph above implies that $\operatorname{span}(v_1, \dots, v_m) \subseteq \operatorname{span}(w_1, \dots, w_m)$. To prove the inclusion in the other direction, note that for each $k \in \{1, \dots, m\}$ we have

$$w_k \in \operatorname{span}(v_1, \dots, v_k) \subseteq \operatorname{span}(v_1, \dots, v_m).$$

Thus $\operatorname{span}(w_1, \dots, w_m) \subseteq \operatorname{span}(v_1, \dots, v_m)$. Hence

$$\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m),$$

as desired.

4 (a) Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.

(b) Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.

SOLUTION

(a) Suppose $v \in V$.

If $v = 0$, then the list v of length one is not linearly independent because $1v = 0$.

Conversely, if $v \neq 0$, then the list v of length one is linearly independent because the only scalar $a \in \mathbf{F}$ such that $av = 0$ is $a = 0$.

(b) Suppose $v_1, v_2 \in V$.

Suppose there is a scalar $a \in \mathbf{F}$ such that $v_1 = av_2$ or $v_2 = av_1$. Then $1v_1 - av_2 = 0$ or $av_1 - 1v_2 = 0$. Thus the list v_1, v_2 of length two is linearly dependent.

Conversely, suppose the list v_1, v_2 of length two is linearly dependent. Then there exist scalars $a_1, a_2 \in \mathbf{F}$, not both 0, such that $a_1v_1 + a_2v_2 = 0$. If $a_1 \neq 0$ then $v_1 = -\frac{a_2}{a_1}v_2$. If $a_2 \neq 0$ then $v_2 = -\frac{a_1}{a_2}v_1$.

5 Find a number t such that

$$(3,1,4),\ (2,-3,5),\ (5,9,t)$$

is not linearly independent in $\mathbf{R}^3$.

SOLUTION We begin by looking just at the first two coordinates of each vector above. To write $(5,9)$ as a linear combination of $(3,1)$, $(2,-3)$, we must find $a, b \in \mathbf{R}$ such that

$$a(3,1) + b(2,-3) = (5,9),$$

which is equivalent to the system of equations

$$\begin{aligned} 3a + 2b &= 5 \\ a - 3b &= 9. \end{aligned}$$

Solving for a, b, we get $a = 3$, $b = -2$.

Thus to choose t so that $(3,1,4)$, $(2,-3,5)$, $(5,9,t)$ is linearly dependent, we need

$$3(3,1,4) - 2(2,-3,5) = (5,9,t),$$

which implies that $t = 2$.

6 Show that the list $(2, 3, 1), (1, -1, 2), (7, 3, c)$ is linearly dependent in $\mathbf{F}^3$ if and only if $c = 8$.

SOLUTION The equation in the first bullet point in Example 2.18 shows that $(2, 3, 1), (1, -1, 2), (7, 3, 8)$ is linearly dependent.

Conversely, suppose $(2, 3, 1), (1, -1, 2), (7, 3, c)$ is linearly dependent. The first vector in this list is not the 0 vector, and the second vector in this list is not a scalar multiple of the first vector. Thus by the linear dependence lemma (2.19), the vector $(7, 3, c)$ is in the span of $(2, 3, 1), (1, -1, 2)$.

The list $(2, 3), (1, -1)$ is linearly independent (neither vector is a scalar multiple of the other) and thus $(7, 3)$ can be written as a linear combination of $(2, 3), (1, -1)$ in at most one way. From the first bullet point in 2.18, we see that this one way of writing $(7, 3)$ as a linear combination of $(2, 3), (1, -1)$ is

$$2(2, 3) + 3(1, -1) = (7, 3).$$

Thus the only possible way to write $(7, 3, c)$ as a linear combination of the vectors $(2, 3, 1), (1, -1, 2)$ is

$$2(2, 3, 1) + 3(1, -1, 2) = (7, 3, c).$$

The equation above implies that the list $(2, 3, 1), (1, -1, 2), (7, 3, c)$ is linearly dependent only when $c = 8$.

7 (a) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{R}$, then the list $1 + i, 1 - i$ is linearly independent.

(b) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{C}$, then the list $1 + i, 1 - i$ is linearly dependent.

SOLUTION

(a) Think of $\mathbf{C}$ as a vector space over $\mathbf{R}$. Suppose $a, b \in \mathbf{R}$ and

$$a(1+i) + b(1-i) = 0.$$

By looking at the real and imaginary parts of the left side of the equation above, we see that $a + b = 0$ and $a - b = 0$, which implies $a = b = 0$. Hence the list $(1 + i, 1 - i)$ is linearly independent.

(b) Think of $\mathbf{C}$ as a vector space over $\mathbf{C}$. Then

$$1 - i = a(1+i),$$

where $a = \frac{1-i}{1+i}$. Thus the list $(1 + i, 1 - i)$ is linearly dependent.

8 Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

SOLUTION To prove that the list displayed above is linearly independent, suppose $a_1, a_2, a_3, a_4 \in \mathbf{F}$ are such that

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = 0.$$

Rearranging terms, the equation above can be rewritten as

$$a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0.$$

Because v_1, v_2, v_3, v_4 is linearly independent, the equation above implies that

$$\begin{aligned} a_1 &= 0 \\ a_2 - a_1 &= 0 \\ a_3 - a_2 &= 0 \\ a_4 - a_3 &= 0. \end{aligned}$$

The first equation above tells us that $a_1 = 0$. That information, combined with the second equation, tells us that $a_2 = 0$. That information, combined with the third equation, tells us that $a_3 = 0$. That information, combined with the fourth equation, tells us that $a_4 = 0.$. Thus $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is linearly independent.

9 Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, \ldots, v_m$$

is linearly independent.

SOLUTION Suppose $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V. Suppose $a_1, a_2, \ldots, a_m \in \mathbf{F}$ are such that

$$a_1(5v_1 - 4v_2) + a_2 v_2 + \cdots + a_m v_m = 0.$$

Then

$$5a_1 v_1 + (a_2 - 4a_1)v_2 + a_3 v_3 + \cdots + a_m v_m = 0.$$

Because $v_1, v_2, \ldots, v_m$ is linearly independent, we have

$$a_1 = a_2 - 4a_1 = a_3 = \cdots = a_m = 0.$$

Thus $a_1 = a_2 = a_3 = \cdots = a_m = 0$. Hence $5v_1 - 4v_2, v_2, v_3, \ldots, v_m$ is linearly independent.

10 Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$ is linearly independent.

SOLUTION Suppose $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$. Suppose $a_1, a_2, \ldots, a_m \in \mathbf{F}$ are such that

$$a_1 \lambda v_1 + \cdots + a_m \lambda v_m = 0.$$

Because $v_1, v_2, \ldots, v_m$ is linearly independent, we have

$$a_1 \lambda = \cdots = a_m \lambda = 0.$$

Because $\lambda \neq 0$, we have $a_1 = \cdots = a_m = 0$. Hence $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$ is linearly independent.

11 Prove or give a counterexample: If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in V, then the list $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

SOLUTION The statement above is not true. For example, take $v_1, \ldots, v_m$ to be any linearly independent lists of vectors in V, and then let

$$w_1 = -v_1, \ldots, w_m = -v_m.$$

The list $v_1 + w_1, \ldots, v_m + w_m$ will then consist of all 0's and thus will not be linearly independent.

12 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.

SOLUTION Suppose $v_1 + w, \ldots, v_m + w$ is linearly dependent. Then there exist scalars $a_1, \ldots, a_m$, not all 0, such that

$$a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0.$$

Rearranging this equation, we have

$$a_1 v_1 + \cdots + a_m v_m = -(a_1 + \cdots + a_m)w.$$

If $a_1 + \cdots + a_m$ were 0, then the equation above would contradict the linear independence of $v_1, \ldots, v_m$. Thus $a_1 + \cdots + a_m \neq 0$. Hence we can divide both sides of the equation above by $-(a_1 + \cdots + a_m)$, showing that $w \in \operatorname{span}(v_1, \ldots, v_m)$.

13 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that

$$v_1, \ldots, v_m, w \text{ is linearly independent} \iff w \notin \operatorname{span}(v_1, \ldots, v_m).$$

SOLUTION First suppose $v_1, \ldots, v_m, w$ is linearly independent. Then no vector in that list is a linear combination of the other vectors in the list. In particular, $w \notin \operatorname{span}(v_1, \ldots, v_m)$.

Conversely, suppose $w \notin \operatorname{span}(v_1, \ldots, v_m)$. Because $v_1, \ldots, v_m$ is linearly independent, no v_k is in the span of $v_1, \ldots, v_{k-1}$. Thus the linear dependence lemma (2.19) implies that $v_1, \ldots, v_m, w$ is linearly independent.

14 Suppose $v_1, \dots, v_m$ is a list of vectors in V. For $k \in \{1, \dots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that the list $v_1, \dots, v_m$ is linearly independent if and only if the list $w_1, \dots, w_m$ is linearly independent.

SOLUTION First suppose $v_1, \dots, v_m$ is linearly independent. Suppose $c_1, \dots, c_m \in \mathbf{F}$ and

$$c_1 w_1 + \cdots + c_m w_m = 0.$$

In the equation above, replace each w_k with $v_1 + \cdots + v_k$ and rewrite the left side of the equation above as a linear combination of $v_1, \dots, v_m$. Because $v_1, \dots, v_m$ is linearly independent, the coefficient of each v_k equals 0. The coefficient of v_m (after replacing each w_k with $v_1 + \cdots + v_k$ in the equation above) is c_m. Thus $c_m = 0$.

Now that $c_m = 0$, the coefficient of v_{m-1} is c_{m-1}. Thus $c_{m-1} = 0$. Continuing in this fashion, we see that $c_m = c_{m-1} = \cdots = c_1 = 0$. Thus $w_1, \dots, w_m$ is linearly independent.

To prove the implication in the other direction, now suppose $w_1, \dots, w_m$ is linearly independent. Suppose $a_1, \dots, a_m \in \mathbf{F}$ and

$$a_1 v_1 + \cdots + a_m v_m = 0.$$

In the equation above, replace v_1 with w_1 and replace each v_k, for $k > 1$, with $w_k - w_{k-1}$ and rewrite the left side of the equation above as a linear combination of $w_1, \dots, w_m$. Because $w_1, \dots, w_m$ is linearly independent, the coefficient of each w_k equals 0. The coefficient of w_m (after the replacements just described) is a_m. Thus $a_m = 0$.

Now that $a_m = 0$, the coefficient of w_{m-1} is a_{m-1}. Thus $a_{m-1} = 0$. Continuing in this fashion, we see that $a_m = a_{m-1} = \cdots = a_1 = 0$. Thus $v_1, \dots, v_m$ is linearly independent.

15 Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbf{F})$.

SOLUTION The list $1, z, z^2, z^3, z^4$ spans $\mathcal{P}_4(\mathbf{F})$. This list has length five. Thus no list of length six is linearly independent in $\mathcal{P}_4(\mathbf{F})$ (by 2.22).

16 Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbf{F})$.

SOLUTION The list $1, z, z^2, z^3, z^4$ is linearly independent in $\mathcal{P}_4(\mathbf{F})$. This list has length five. Thus no list of length four spans $\mathcal{P}_4(\mathbf{F})$ (by 2.22).

17 Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

SOLUTION First suppose V is infinite-dimensional. Choose v_1 to be any nonzero vector in V. Choose $v_2, v_3, \ldots$ by the following inductive process: suppose $v_1, \ldots, v_{m-1}$ have been chosen; choose any vector $v_m \in V$ such that $v_m \notin \operatorname{span}(v_1, \ldots, v_{m-1})$—because V is not finite-dimensional, $\operatorname{span}(v_1, \ldots, v_{m-1})$ cannot equal V so choosing v_m in this fashion is possible. The linear dependence lemma (2.19) implies that $v_1, \ldots, v_m$ is linearly independent for every positive integer m, as desired.

Conversely, suppose there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m. The existence of a spanning list in V would contradict 2.22. Thus V is infinite-dimensional.

18 Prove that $\mathbf{F}^\infty$ is infinite-dimensional.

SOLUTION For each positive integer m, let e_m be the element of $\mathbf{F}^\infty$ whose m^{th} coordinate equals 1 and whose other coordinates equal 0:

$$e_m = (0, \ldots, 0, \underset{\substack{\uparrow \\ m^{\text{th}} \text{ coordinate}}}{1}, 0, \ldots).$$

Then $e_1, \ldots, e_m$ is a linearly independent list of vectors in $\mathbf{F}^\infty$, as is easy to verify. Exercise 17 now implies that $\mathbf{F}^\infty$ is infinite-dimensional.

19 Prove that the real vector space of all continuous real-valued functions on the interval $[0,1]$ is infinite-dimensional.

SOLUTION Let V denote the real vector space of all continuous real-valued functions on the interval $[0,1]$. For each positive integer m, the list $1, x, \dots, x^m$ is linearly independent in V (because if $a_0, \dots, a_m \in \mathbf{R}$ are such that

$$a_0 + a_1 x + \cdots + a_m x^m = 0$$

for every $x \in [0,1]$, then the polynomial above has infinitely many zeros and hence all its coefficients equal 0). The existence of a spanning list in V would contradict 2.22. Thus V is infinite-dimensional.

20 Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \ldots, m\}$. Prove that $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

SOLUTION Because $p_k(2) = 0$ for each k, the constant polynomial 1 is not in $\operatorname{span}(p_0, \ldots, p_m)$. Hence if the list $p_0, p_1, \ldots, p_m$ was linearly independent, then the list $p_0, p_1, \ldots, p_m, 1$ would also be linearly independent. But this is impossible in $\mathcal{P}_m(\mathbf{F})$ because this list has length $m + 2$, which is larger than the length of the spanning list $1, z, \ldots, z^m$.

1 Find all vector spaces that have exactly one basis.

SOLUTION If $v_1, \ldots, v_n$ is a basis of V, then so is $\lambda v_1, \ldots, \lambda v_n$ for each $\lambda \in \mathbf{F}$. Thus the only vector space having exactly one basis is the vector space $\{0\}$, which has the empty list $(\,)$ as its unique basis.

2 Verify all assertions in Example 2.27.

SOLUTION

(a) The list $(1,0,\dots,0), (0,1,0,\dots,0), \dots, (0,\dots,0,1)$ is linearly independent and it spans $\mathbf{F}^n$; thus it is a basis of $\mathbf{F}^n$.

(b) The list $(1,2), (3,5)$ is linearly independent because neither vector in the list is a scalar multiple of the other.

The $(1,2), (3,5)$ spans $\mathbf{F}^2$ because

$$(-5x+3y)(1,2) + (2x-y)(3,5) = (x,y)$$

for all $(x,y) \in \mathbf{F}^2$.

Because the list $(1,2), (3,5)$ is linearly independent and it spans $\mathbf{F}^2$, it is a basis of $\mathbf{F}^2$.

(c) The list $(1,2,-4), (7,-5,6)$ is linearly independent in $\mathbf{F}^3$ because neither vector in the list is a scalar multiple of the other.

The list $(1,2,-4), (7,-5,6)$ does not span $\mathbf{F}^3$ because $(8,-3,0)$ is not in the span of $(1,2,-4), (7,-5,6)$, as can be seen by trying to solve the equations

$$\begin{aligned} a + 7b &= 8 \\ 2a - 5b &= -3 \\ -4a + 6b &= 0. \end{aligned}$$

Because the list $(1,2,-4), (7,-5,6)$ does not span $\mathbf{F}^3$, it is not a basis of $\mathbf{F}^3$.

(d) The list $(1,2), (3,5), (4,13)$ spans $\mathbf{F}^2$ because the list $(1,2), (3,5)$ spans $\mathbf{F}^2$ [by (b)] and adjoining additional vectors to a spanning list clearly gives a spanning list.

Because the list $(1,2), (3,5)$ of length two spans $\mathbf{F}^2$, no list of length larger than 2 is linearly independent in $\mathbf{F}^2$ (by 2.22). Thus the $(1,2), (3,5), (4,13)$ is not linearly independent and hence it is not a basis of $\mathbf{F}^2$.

(e) The list $(1,1,0), (0,0,1)$ is linearly independent because neither vector in this list of length two is a scalar multiple of the other.

For all $x, y \in \mathbf{F}$, we have

$$(x, x, y) = x(1, 1, 0) + y(0, 0, 1).$$

Thus $(1, 1, 0), (0, 0, 1)$ spans $\{(x, x, y) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}$.

Because the list $(1, 1, 0), (0, 0, 1)$ is linearly independent and spans the subspace $\{(x, x, y) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}$, this list of length two is a basis of $\{(x, x, y) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}$.

(f) The list $(1, -1, 0), (1, 0, -1)$ is linearly independent because neither vector in this list of length two is a scalar multiple of the other.

For all $x, y, z \in \mathbf{F}$ with $x + y + z = 0$, we have

$$(x, y, z) = -y(1, -1, 0) - z(1, 0, -1).$$

Thus $(1, -1, 0), (1, 0, -1)$ spans $\{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}$.

Because the list $(1, -1, 0), (1, 0, -1)$ is linearly independent and spans the subspace $\{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}$, this list is a basis of $\{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}$.

(g) The list $1, z, \ldots, z^m$ is linearly independent in $\mathcal{P}_M(\mathbf{F})$ and spans $\mathcal{P}_M(\mathbf{F})$ and hence is a basis of $\mathcal{P}_m(\mathbf{F})$.

3 (a) Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathbf{R}^5$.

(c) Find a subspace W of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$.

SOLUTION

(a) Note that

$$U = \{(3x_2, x_2, 7x_4, x_4, x_5) : x_2, x_4, x_5 \in \mathbf{R}\}.$$

From this representation of U, we see easily that

$$(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$$

is a basis of U.

Of course there are also other possible choices of bases of U.

(b) The list

$$(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1), (1, 0, 0, 0, 0), (0, 0, 1, 0, 0)$$

is a basis of $\mathbf{R}^5$.

(c) Let

$$W = \{((x, 0, y, 0, 0)) : x, y \in \mathbf{R}\}.$$

4 (a) Let U be the subspace of $\mathbf{C}^5$ defined by

$$U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbf{C}^5 : 6z_1 = z_2 \text{ and } z_3 + 2z_4 + 3z_5 = 0\}.$$

Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathbf{C}^5$.

(c) Find a subspace W of $\mathbf{C}^5$ such that $\mathbf{C}^5 = U \oplus W$.

SOLUTION

(a) The list $(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1)$ consists of vectors in U. It is easy to see from the definition of linear independence that this list is linearly independent.

If $(z_1, z_2, z_3, z_4, z_5) \in U$, then

$$\begin{aligned}(z_1, z_2, z_3, z_4, z_5)\\ = z_1(1, 6, 0, 0, 0) + z_4(0, 0, -2, 1, 0) + z_5(0, 0, -3, 0, 1).\end{aligned}$$

Thus the list $(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1)$ spans U.

Because $(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1)$ is linearly independent and spans U, this list is a basis of U.

(b) The list

$$\begin{aligned}(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1),\\ (1, 0, 0, 0, 0), (0, 0, 1, 0, 0)\end{aligned}$$

is a basis of $\mathbf{C}^5$, as is easy to verify.

(c) Using the idea of the proof of 2.33 and the answer above to (b), we see that taking

$$W = \{(\alpha, 0, \beta, 0, 0) : \alpha, \beta \in \mathbf{C}\}$$

gives a subspace W such that $\mathbf{C}^5 = U \oplus W$.

5 Suppose V is finite-dimensional and U, W are subspaces of V such that $V = U + W$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.

SOLUTION Because V is finite-dimensional, the subspaces U and W are also finite-dimensional (by 2.25). Thus there exists a list of vectors in U that spans U and there exists a list of vectors in W that spans W. Put these two lists together to get a list of vectors in $U \cup W$ that spans V (becasue $V = U + W$). By 2.30, this list can be reduced to a basis of V, thus producing a basis of V consisting of vectors in $U \cup W$.

6 Prove or give a counterexample: If p_0, p_1, p_2, p_3 is a list in $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2, then p_0, p_1, p_2, p_3 is not a basis of $\mathcal{P}_3(\mathbf{F})$.

SOLUTION To construct a counterexample, define $p_0, p_1, p_2, p_3 \in \mathcal{P}_3(\mathbf{F})$ by

$$\begin{aligned} p_0(z) &= 1, \\ p_1(z) &= z, \\ p_2(z) &= z^2 + z^3, \\ p_3(z) &= z^3. \end{aligned}$$

None of the polynomials p_0, p_1, p_2, p_3 has degree 2, but p_0, p_1, p_2, p_3 is a basis of $\mathcal{P}_3(\mathbf{F})$, as is easy to verify.

7 Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

SOLUTION To prove that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is linearly independent, suppose $a, b, c, d \in \mathbf{F}$ and

$$a(v_1 + v_2) + b(v_2 + v_3) + c(v_3 + v_4) + dv_4 = 0.$$

Then

$$av_1 + (a + b)v_2 + (b + c)v_3 + (c + d)v_4 = 0.$$

Because v_1, v_2, v_3, v_4 is linearly independent, this implies that

$$a = a + b = b + c = c + d = 0,$$

which implies that $a = b = c = d = 0$. Thus $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is linearly independent.

To prove that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ spans V, suppose $u \in V$. Because v_1, v_2, v_3, v_4 is a basis of V, there exist $a, b, c, d \in \mathbf{F}$ such that

$$u = av_1 + bv_2 + cv_3 + dv_4.$$

Thus

$$u = a(v_1 + v_2) + (b - a)(v_2 + v_3) + (c - b + a)(v_3 + v_4) + (d - c + b - a)v_4.$$

Hence $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ spans V.

Because $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is linearly independent and spans V, it is a basis of V.

8 Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

SOLUTION The statement above is false. For example, take $V = \mathbf{R}^4$, let v_1, v_2, v_3, v_4 be the standard basis of $\mathbf{R}^4$, and let

$$U = \{(a, b, c, c) : a, b, c \in \mathbf{R}\}.$$

Then $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$. However, v_1, v_2 is not a basis of U, because $(0, 0, 1, 1) \in U$ but $(0, 0, 1, 1) \notin \operatorname{span}(v_1, v_2)$.

9 Suppose $v_1, \dots, v_m$ is a list of vectors in V. For $k \in \{1, \dots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $v_1, \dots, v_m$ is a basis of V if and only if $w_1, \dots, w_m$ is a basis of V.

SOLUTION First suppose $v_1, \dots, v_m$ is a basis of V. Thus $v_1, \dots, v_m$ is linearly independent and spans V. By Exercises 14 and 3 in Section 2A, $w_1, \dots, w_m$ is linearly independent and spans V. Thus $w_1, \dots, w_m$ is a basis of V.

Now suppose $w_1, \dots, w_m$ is a basis of V. Thus $w_1, \dots, w_m$ is linearly independent and spans V. By Exercises 14 and 3 in Section 2A, $v_1, \dots, v_m$ is linearly independent and spans V. Thus $v_1, \dots, v_m$ is a basis of V.

10 Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V.

SOLUTION First suppose $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$ and

$$a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n = 0.$$

Because $a_1 u_1 + \cdots + a_m u_m \in U$ and $b_1 w_1 + \cdots + b_n w_n \in W$, the equation above and 1.45 imply that

$$a_1 u_1 + \cdots + a_m u_m = 0 \quad \text{and} \quad b_1 w_1 + \cdots + b_n w_n = 0.$$

Because $u_1, \ldots, u_m$ is linearly independent and $w_1, \ldots, w_n$ is linearly independent, this implies that

$$a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0.$$

Thus $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent.

Suppose $v \in V$. Then there exist $u \in U$ and $w \in W$ such that $v = u + w$. Because $u_1, \ldots, u_m$ spans U and $w_1, \ldots, w_n$ spans W, there exist $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$ such that

$$u = a_1 u_1 + \cdots + a_m u_m \quad \text{and} \quad w = b_1 w_1 + \cdots + b_n w_n.$$

Thus

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n,$$

which shows that $u_1, \ldots, u_m, w_1, \ldots, w_n$ spans V.

Because $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent and spans V, this list is a basis of V.

11 Suppose V is a real vector space. Show that if $v_1, \dots, v_n$ is a basis of V (as a real vector space), then $v_1, \dots, v_n$ is also a basis of the complexification $V_{\mathbf{C}}$ (as a complex vector space).

See Exercise 8 in Section 1B for the definition of the complexification $V_{\mathbf{C}}$.

SOLUTION Suppose that $v_1, \dots, v_n$ is a basis of the real vector space V. Then $\operatorname{span}(v_1, \dots, v_n)$ in the complex vector space $V_{\mathbf{C}}$ contains all the vectors

$$v_1, \dots, v_n, iv_1, \dots, iv_n.$$

Thus $v_1, \dots, v_n$ spans the complex vector space $V_{\mathbf{C}}$.

To show that $v_1, \dots, v_n$ is linearly independent in the complex vector space $V_{\mathbf{C}}$, suppose $\lambda_1, \dots, \lambda_n \in \mathbf{C}$ and

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = 0.$$

Then the equation above and our definitions imply that

$$(\operatorname{Re}\lambda_1)v_1 + \cdots + (\operatorname{Re}\lambda_n)v_n = 0 \quad \text{and} \quad (\operatorname{Im}\lambda_1)v_1 + \cdots + (\operatorname{Im}\lambda_n)v_n = 0.$$

Because $v_1, \dots, v_n$ is linearly independent in V, the equations above imply that

$$\operatorname{Re}\lambda_1 = \cdots = \operatorname{Re}\lambda_n = 0 \quad \text{and} \quad \operatorname{Im}\lambda_1 = \cdots = \operatorname{Im}\lambda_n = 0.$$

Thus $\lambda_1 = \cdots = \lambda_n = 0$. Hence $v_1, \dots, v_n$ is linearly independent in $V_{\mathbf{C}}$ and thus is a basis of $V_{\mathbf{C}}$.

1 Show that the subspaces of $\mathbf{R}^2$ are precisely $\{0\}$, all lines in $\mathbf{R}^2$ containing the origin, and $\mathbf{R}^2$.

SOLUTION The set $\{0\}$, the set $\mathbf{R}^2$, and all lines in $\mathbf{R}^2$ through the origin are all subspaces of $\mathbf{R}^2$.

To show that there are no other subspaces of $\mathbf{R}^2$, suppose U is a subspace of $\mathbf{R}^2$. Then by 2.37, $\dim U$ equals 0, 1, or 2.

If $\dim U = 0$, then $U = \{0\}$.

If $\dim U = 1$, then there is a basis of U consisting on one vector v, and U equals all scalar multiples of v; thus U is a line through the origin.

If $\dim U = 2$, then $U = \mathbf{R}^2$.

2 Show that the subspaces of $\mathbf{R}^3$ are precisely $\{0\}$, all lines in $\mathbf{R}^3$ containing the origin, all planes in $\mathbf{R}^3$ containing the origin, and $\mathbf{R}^3$.

SOLUTION The set $\{0\}$, the set $\mathbf{R}^3$, all lines in $\mathbf{R}^2$ through the origin, and all planes in $\mathbf{R}^3$ through the origin are all subspaces of $\mathbf{R}^2$.

To show that there are no other subspaces of $\mathbf{R}^3$, suppose U is a subspace of $\mathbf{R}^3$. Then by 2.37, $\dim U$ equals 0, 1, 2, or 3.

If $\dim U = 0$, then $U = \{0\}$.

If $\dim U = 1$, then there is a basis of U consisting on one vector v, and U equals all scalar multiples of v; thus U is a line through the origin.

If $\dim U = 2$, then there is a basis v_1, v_2 of the subspace U, and hence U equals $\operatorname{span}(v_1, v_2)$; thus U is a plane through the origin.

If $\dim U = 3$, then $U = \mathbf{R}^3$.

3 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(6) = 0\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

SOLUTION

(a) A basis of U is

$$x - 6, (x-6)^2, (x-6)^3, (x-6)^4.$$

To verify that the list above is indeed a basis of U, first note that the list above is linearly independent (using the same reasoning as was used in Example 2.41 to show that the list in that example is linearly independent). Then note that the linearly independent list above has length four, and thus $\dim U \geq 4$. However, $\dim \mathcal{P}_4(\mathbf{F}) = 5$, which implies that $\dim U = 4$ or $\dim U = 5$. Because U is a proper subspace of $\mathcal{P}_4(\mathbf{F})$, this implies that $\dim U = 4$. Hence the list above is a basis of U.

(b) The constant function 1 clearly is not in U. Thus

$$x - 6, (x-6)^2, (x-6)^3, (x-6)^4, 1$$

is a linearly independent list in $\mathcal{P}_4(\mathbf{F})$ of length five. By 2.38, the list above is a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Using the idea of the proof of 2.33 and the answer above to (b), we see that taking W to be the subspace of $\mathcal{P}_4(\mathbf{F})$ consisting of the constant functions gives a subspace W such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

4 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{R}) : p''(6) = 0\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{R})$ such that $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

SOLUTION

(a) A basis of U is

$$1, x - 6, (x - 6)^3, (x - 6)^4.$$

Each polynomial in the list above is clearly in U. To verify that the list above is indeed a basis of U, first note that the list above is linearly independent (using the same reasoning as was used in Example 2.41 to show that the list in that example is linearly independent). Then note that the linearly independent list above has length four, and thus $\dim U \geq 4$. However, $\dim \mathcal{P}_4(\mathbf{R}) = 5$, which implies that $\dim U = 4$ or $\dim U = 5$. Because U is a proper subspace of $\mathcal{P}_4(\mathbf{R})$, this implies that $\dim U = 4$. Hence the list above is a basis of U.

(b) The polynomial $(x - 2)^2$ clearly is not in U. Thus

$$1, x - 6, (x - 6)^3, (x - 6)^4, (x - 6)^2$$

is a linearly independent list in $\mathcal{P}_4(\mathbf{R})$ of length five. By 2.38, the list above is a basis of $\mathcal{P}_4(\mathbf{R})$.

(c) Using the idea of the proof of 2.33 and the answer above to (b), we see that taking W to be the subspace of $\mathcal{P}_4(\mathbf{R})$ consisting of the constant multiples of $(x - 6)^2$ gives a subspace W such that $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

5 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5)\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

SOLUTION

(a) A basis of U is

$$1, (x-2)(x-5), (x-2)^2(x-5), (x-2)^3(x-5).$$

Each polynomial in the list above is clearly in U. To verify that the list above is indeed a basis of U, first note that the list above is linearly independent (using the same reasoning as was used in Example 2.41 to show that the list in that example is linearly independent). Then note that the linearly independent list above has length four, and thus $\dim U \geq 4$. However, $\dim \mathcal{P}_4(\mathbf{F}) = 5$, which implies that $\dim U = 4$ or $\dim U = 5$. Because U is a proper subspace of $\mathcal{P}_4(\mathbf{F})$, this implies that $\dim U = 4$. Hence the list above is a basis of U.

(b) The polynomial x clearly is not in U. Thus

$$1, (x-2)(x-5), (x-2)^2(x-5), (x-2)^3(x-5), x$$

is a linearly independent list in $\mathcal{P}_4(\mathbf{F})$ of length five. By 2.38, the list above is a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Using the idea of the proof of 2.33 and the answer above to (b), we see that taking W to be the subspace of $\mathcal{P}_4(\mathbf{F})$ consisting of the constant multiples of x gives a subspace W such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

6 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

SOLUTION

(a) A basis of U is

$$1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6).$$

Each polynomial in the list above is clearly in U. To verify that the list above is indeed a basis of U, first note that the list above is linearly independent (using the same reasoning as was used in Example 2.41 to show that the list in that example is linearly independent). Then note that the linearly independent list above has length three and thus $\dim U \geq 3$. However, U is a proper subspace of $\{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5)\}$, which from the solution to Exercise 5 has dimension four. This implies that $\dim U = 3$. Hence the list above is a basis of U.

(b) The list

$$1, (x-2)(x-5)(x-6), (x-2)^2(x-5)(x-6), x, x^2$$

is a linearly independent list in $\mathcal{P}_4(\mathbf{F})$ of length five. By 2.38, the list above is a basis of $\mathcal{P}_4(\mathbf{F})$.

(c) Using the idea of the proof of 2.33 and the answer above to (b), we see that taking $W = \operatorname{span}(x, x^2)$ gives a subspace W such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

7 (a) Let $U = \left\{p \in \mathcal{P}_4(\mathbf{R}) : \int_{-1}^{1} p = 0\right\}$. Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{R})$.

(c) Find a subspace W of $\mathcal{P}_4(\mathbf{R})$ such that $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

SOLUTION

(a) A basis of U is

$$x, x^2 - \tfrac{1}{3}, x^3, x^4 - \tfrac{1}{5}.$$

Simple calculus shows that each polynomial in the list above is in U. To verify that the list above is indeed a basis of U, first note that the list above is linearly independent (using the same reasoning as was used in Example 2.41 to show that the list in that example is linearly independent). Then note that the linearly independent list above has length four, and thus $\dim U \geq 4$. However, $\dim \mathcal{P}_4(\mathbf{R}) = 5$, which implies that $\dim U = 4$ or $\dim U = 5$. Because U is a proper subspace of $\mathcal{P}_4(\mathbf{R})$, this implies that $\dim U = 4$. Hence the list above is a basis of U.

(b) The constant polynomial 1 clearly is not in U. Thus

$$x, x^2 - \tfrac{1}{3}, x^3, x^4 - \tfrac{1}{5}, 1$$

is a linearly independent list in $\mathcal{P}_4(\mathbf{R})$ of length five. By 2.38, the list above is a basis of $\mathcal{P}_4(\mathbf{R})$.

(c) Using the idea of the proof of 2.33 and the answer above to (b), we see that taking W to be the subspace of $\mathcal{P}_4(\mathbf{R})$ consisting of the constant polynomials gives a subspace W such that $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

8 Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

SOLUTION We have

$$v_k - v_m = (v_k + w) - (v_m + w) \in \operatorname{span}(v_1 + w, \ldots, v_m + w)$$

for $k = 1, 2, \ldots, m - 1$. Because $v_1, \ldots, v_m$ is linearly independent, it is easy to see that

$$v_1 - v_m, v_2 - v_m, \ldots, v_{m-1} - v_m$$

is also linearly independent. Thus we have a linearly independent list of length $m - 1$ in $\operatorname{span}(v_1 + w, \ldots, v_m + w)$. Hence

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

9 Suppose m is a positive integer and $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_k has degree k. Prove that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

SOLUTION The same reasoning as used in Example 2.41 shows that $p_0, p_1, \ldots, p_m$ is linearly independent. This list has length $m + 1$, which equals the dimension of $\mathcal{P}_m(\mathbf{F})$. Thus by 2.38, this list is a basis of $\mathcal{P}_m(\mathbf{F})$.

10 Suppose m is a positive integer. For $0 \le k \le m$, let

$$p_k(x) = x^k(1-x)^{m-k}.$$

Show that $p_0, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

The basis in this exercise leads to what are called ***Bernstein polynomials****. You can do a web search to learn how Bernstein polynomials are used to approximate continuous functions on* $[0,1]$.

SOLUTION To show that $p_0, \ldots, p_m$ is a linearly independent list, suppose that $a_0, \ldots, a_m \in \mathbf{F}$ are such that

$$a_0 p_0 + \cdots + a_m p_m = 0.$$

The only polynomial in the list $p_0, \ldots, p_m$ that has a nonzero constant term is p_0; thus the equation above implies that $a_0 = 0$. Hence the equation above can now be rewritten as

$$a_1 p_1 + \cdots + a_m p_m = 0.$$

The only polynomial in the list $p_1, \ldots, p_m$ that has a nonzero x term is p_1; thus the equation above implies that $a_1 = 0$. Hence the equation above can now be rewritten as

$$a_2 p_2 + \cdots + a_m p_m = 0.$$

Continuing in this fashion, we see that $a_0 = \cdots = a_m = 0$. Thus the list $p_0, \ldots, p_m$ is linearly independent in $\mathcal{P}_m(\mathbf{F})$.

Because $\dim \mathcal{P}_m(\mathbf{F}) = m + 1$ and $p_0, \ldots, p_m$ is a linearly independent list of length $m + 1$, we conclude that $p_0, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$ (by 2.38).

11 Suppose U and W are both four-dimensional subspaces of $\mathbf{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

SOLUTION From 2.43, we have

$$\begin{aligned}\dim(U \cap W) &= \dim U + \dim W - \dim(U + W)\\ &= 8 - \dim(U + W)\\ &\geq 8 - 6\\ &= 2.\end{aligned}$$

Because $\dim(U \cap W) \geq 2$, there exists a linearly independent list v_1, v_2 of vectors in $U \cap W$. Thus neither v_1 nor v_2 is a scalar multiple of the other vector.

12 Suppose that U and W are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $\mathbf{R}^8 = U \oplus W$.

SOLUTION We know (from 2.43) that

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Because $\dim(U + W) = 8$, $\dim U = 3$, and $\dim W = 5$, this implies that $\dim(U \cap W) = 0$. Thus $U \cap W = \{0\}$. Now 1.46 implies that $\mathbf{R}^8 = U \oplus W$.

13 Suppose U and W are both five-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

SOLUTION Using 2.43 we have

$$\begin{aligned} 9 &\geq \dim(U+W) \\ &= \dim U + \dim W - \dim(U \cap W) \\ &= 10 - \dim(U \cap W). \end{aligned}$$

Thus $\dim(U \cap W) \geq 1$. In particular, $U \cap W \neq \{0\}$.

14 Suppose V is a ten-dimensional vector space and V_1, V_2, V_3 are subspaces of V with $\dim V_1 = \dim V_2 = \dim V_3 = 7$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

SOLUTION Using 2.43, we have

$$\begin{aligned}\dim(V_1 \cap V_2) &= \dim V_1 + \dim V_2 - \dim(V_1 + V_2)\\ &= 14 - \dim(V_1 + V_2)\\ &\geq 4.\end{aligned}$$

Using 2.43 and the inequality above, we have

$$\begin{aligned}\dim(V_1 \cap V_2 \cap V_3) &= \dim((V_1 \cap V_2) \cap V_3)\\ &= \dim(V_1 \cap V_2) + \dim V_3 - \dim((V_1 \cap V_2) + V_3)\\ &\geq 4 + 7 - \dim((V_1 \cap V_2) + V_3)\\ &\geq 1.\end{aligned}$$

Thus $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

15 Suppose V is finite-dimensional and V_1, V_2, V_3 are subspaces of V with $\dim V_1 + \dim V_2 + \dim V_3 > 2\dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

SOLUTION Using 2.43, we have

$$\begin{aligned}\dim(V_1 \cap V_2) &= \dim V_1 + \dim V_2 - \dim(V_1 + V_2)\\ &> 2\dim V - \dim V_3 - \dim(V_1 + V_2).\end{aligned}$$

Using 2.43 and the inequality above, we have

$$\begin{aligned}\dim(V_1 \cap V_2 \cap V_3) &= \dim\big((V_1 \cap V_2) \cap V_3\big)\\ &= \dim(V_1 \cap V_2) + \dim V_3 - \dim\big((V_1 \cap V_2) + V_3\big)\\ &> 2\dim V - \dim(V_1 + V_2) - \dim\big((V_1 \cap V_2) + V_3\big)\\ &\geq 0.\end{aligned}$$

Thus $\dim(V_1 \cap V_2 \cap V_3) > 0$; hence $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

16 Suppose V is finite-dimensional and U is a subspace of V with $U \neq V$. Let $n = \dim V$ and $m = \dim U$. Prove that there exist $n - m$ subspaces of V, each of dimension $n - 1$, whose intersection equals U.

SOLUTION Let $u_1, \ldots, u_m$ be a basis of U. Extend to a basis

$$u_1, \ldots, u_m, v_1, \ldots, v_{n-m}$$

of V. For each $k = 1, \ldots, n - m$, let V_k denote the span of the list obtained by deleting v_k from the list above. Thus each V_k has dimension $n - 1$, and

$$\bigcap_{k=1}^{n-m} V_k = U.$$

17 Suppose that $V_1, \ldots, V_m$ are finite-dimensional subspaces of V. Prove that $V_1 + \cdots + V_m$ is finite-dimensional and

$$\dim(V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m.$$

The inequality above is an equality if and only if $V_1 + \cdots + V_m$ is a direct sum, as will be shown in 3.94.

SOLUTION For each $k = 1, \ldots m$, choose a basis of V_k. Put these bases together to form a single list of vectors in V. Clearly this list spans $V_1 + \cdots + V_m$, and thus $V_1 + \cdots + V_m$ is finite-dimensional. Furthermore, the dimension of $V_1 + \cdots + V_m$ is less than or equal to the number of vectors in this list (by 2.30), which equals $\dim V_1 + \cdots + \dim V_m$. In other words,

$$\dim(V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m.$$

18 Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist one-dimensional subspaces $V_1, \ldots, V_n$ of V such that

$$V = V_1 \oplus \cdots \oplus V_n.$$

SOLUTION Let $v_1, \ldots, v_n$ be a basis of V. For each k, let V_k equal $\operatorname{span}(v_k)$; in other words, $V_k = \{av_k : a \in \mathbf{F}\}$. Because $v_1, \ldots, v_n$ is a basis of V, each vector in V can be written uniquely in the form

$$a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in \mathbf{F}$ (see 2.28). By definition of direct sum, this means that $V = V_1 \oplus \cdots \oplus V_n$.

19 Explain why you might guess, motivated by analogy with the formula for the number of elements in the union of three finite sets, that if V_1, V_2, V_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(V_1 + V_2 + V_3)&\\ &= \dim V_1 + \dim V_2 + \dim V_3\\ &\quad - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3)\\ &\quad + \dim(V_1 \cap V_2 \cap V_3).\end{aligned}$$

Then either prove the formula above or give a counterexample.

SOLUTION Recall that for S a finite set, $\#(S)$ denote the numbers of elements of S.

If S_1, S_2, S_3 are finite sets, then

$$\begin{aligned}\#(S_1 \cup S_2 \cup S_3)&\\ &= \#(S_1) + \#(S_2) + \#(S_3)\\ &\quad - \#(S_1 \cap S_2) - \#(S_1 \cap S_3) - \#(S_2 \cap S_3)\\ &\quad + \#(S_1 \cap S_2 \cap S_3).\end{aligned}$$

The formula above holds because to count the number of elements of $S_1 \cup S_2 \cup S_3$, first we add up the number of elements in the three sets. But we have double-counted the number of elements that lie in two or more of the sets, so we subtract the number of elements that lie in the pairwise intersections. However, for elements that lie in all three sets, we have counted them three times and then subtracted them three times. Thus we add the number of elements that lie in all three sets.

The formula above suggests that if V_1, V_2, V_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(V_1 + V_2 + V_3)&\\ &= \dim V_1 + \dim V_2 + \dim V_3\\ &\quad - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3)\\ &\quad + \dim(V_1 \cap V_2 \cap V_3).\end{aligned}$$

However, the formula above does not hold in general. To give a counterexample, let $V = \mathbf{R}^2$, and let

$$\begin{aligned} V_1 &= \{(x,0) : x \in \mathbf{R}\}, \\ V_2 &= \{(0,y) : y \in \mathbf{R}\}, \\ V_3 &= \{(x,x) : x \in \mathbf{R}\}. \end{aligned}$$

Then $V_1 + V_2 + V_3 = \mathbf{R}^2$, so $\dim(V_1 + V_2 + V_3) = 2$. However,

$$\dim V_1 = \dim V_2 = \dim V_3 = 1$$

and

$$\dim(V_1 \cap V_2) = \dim(V_1 \cap V_3) = \dim(V_2 \cap V_3) = \dim(V_1 \cap V_2 \cap V_3) = 0.$$

Thus in this case our guess would reduce to the formula $2 = 3$, which is false.

20 Prove that if V_1, V_2, and V_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}&\dim(V_1 + V_2 + V_3)\\ &\quad = \dim V_1 + \dim V_2 + \dim V_3\\ &\qquad - \frac{\dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3)}{3}\\ &\qquad - \frac{\dim((V_1+V_2)\cap V_3) + \dim((V_1+V_3)\cap V_2) + \dim((V_2+V_3)\cap V_1)}{3}.\end{aligned}$$

The formula above may seem strange because the right side does not look like an integer.

SOLUTION Using the formula for the dimension of the sum of two subspaces, we have

$$\begin{aligned}&\dim(V_1 + V_2 + V_3)\\ &= \dim((V_1 + V_2) + V_3)\\ &= \dim(V_1 + V_2) + \dim V_3 - \dim((V_1 + V_2) \cap V_3)\\ &= \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim((V_1 + V_2) \cap V_3),\end{aligned}$$

with two similar formulas for $\dim(V_1 + V_2 + V_3)$ based on the groupings

$$\dim((V_1 + V_3) + V_2) \quad \text{and} \quad \dim(V_1 + (V_2 + V_3)).$$

Adding together these three formulas and then dividing by 3 gives the desired result.