Exercises 1B Solutions: Definition of Vector Space

Linear Algebra Done Right, 4th ed.

Exercise 1. Prove that $-(-v) = v$ for every $v \in V$.

Solution: By definition, $-(-v)$ is the additive inverse of $(-v)$, meaning it is the unique element satisfying $(-v) + (-(-v)) = 0$.
We claim that v satisfies this property. Indeed:

$$(-v) + v = v + (-v) = 0$$

where the first equality uses commutativity of addition.
Since v is an element such that $(-v) + v = 0$, and the additive inverse of $(-v)$ is unique, we conclude that $-(-v) = v$. $\square$

Exercise 2. Suppose $a \in \mathbb{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Solution: Suppose $a \neq 0$. We will show that $v = 0$.
Since $a \neq 0$ and $\mathbb{F}$ is a field, a has a multiplicative inverse $a^{-1} \in \mathbb{F}$.
From $av = 0$, multiply both sides by a^{-1}:

$$a^{-1}(av) = a^{-1} \cdot 0.$$

The left side simplifies using associativity of scalar multiplication:

$$a^{-1}(av) = (a^{-1}a)v = 1 \cdot v = v.$$

For the right side, we show $a^{-1} \cdot 0_V = 0_V$: Since $0_V = 0_V + 0_V$, we have

$$a^{-1} \cdot 0_V = a^{-1}(0_V + 0_V) = a^{-1} \cdot 0_V + a^{-1} \cdot 0_V.$$

Adding $-(a^{-1} \cdot 0_V)$ to both sides gives $0_V = a^{-1} \cdot 0_V$.
Therefore $v = 0$.
We have shown: if $a \neq 0$, then $v = 0$. Equivalently, $a = 0$ or $v = 0$. $\square$

Exercise 3. Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution: We prove both existence and uniqueness.
Existence: Define $x = \frac{1}{3}(w + (-v)) = \frac{1}{3}(w - v)$.

We verify that this x satisfies $v + 3x = w$:

$$\begin{aligned} v + 3x &= v + 3 \cdot \frac{1}{3}(w - v) \\ &= v + 1 \cdot (w - v) \\ &= v + (w - v) \\ &= v + w + (-v) \\ &= w + v + (-v) \\ &= w + 0 = w. \end{aligned}$$

Uniqueness: Suppose x_1 and x_2 both satisfy the equation:

$$v + 3x_1 = w \quad \text{and} \quad v + 3x_2 = w.$$

Then $v + 3x_1 = v + 3x_2$.
Adding $(-v)$ to both sides:

$$3x_1 = 3x_2.$$

Multiplying both sides by $\frac{1}{3}$:

$$x_1 = x_2.$$

Therefore there exists a unique $x \in V$ such that $v + 3x = w$:

$$\boxed{x = \frac{1}{3}(w - v)}$$

□

Exercise 4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

Solution: The empty set fails the **additive identity** axiom.
The additive identity axiom requires the existence of an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.
Since the empty set contains no elements, there is no element that could serve as the additive identity. Even though the condition "$v + 0 = v$ for all $v \in V$" is vacuously true (there are no v to check), the requirement that such an element *exists* fails.
All other axioms are vacuously satisfied because they are statements about elements of V, and the empty set has no elements to violate them. □

Why not additive inverse? The additive inverse axiom says: "for every $v \in V$, there exists $w \in V$ such that $v + w = 0$." For the empty set, there are no elements v to consider, so this statement is vacuously true. The key difference is that additive identity requires something to *exist*, while additive inverse only makes claims about elements that are already there.

Exercise 5. Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

Solution: We must show that if V satisfies all vector space axioms except the additive inverse axiom, but does satisfy $0v = 0$ for all $v \in V$, then the additive inverse axiom holds.
We need to show: for every $v \in V$, there exists $w \in V$ such that $v + w = 0$.
Given $v \in V$, consider $w = (-1)v$.
We claim $v + w = 0$:

$$\begin{aligned}
v + (-1)v &= 1 \cdot v + (-1)v && \text{(multiplicative identity axiom)}\\
&= (1 + (-1))v && \text{(distributivity over scalar addition)}\\
&= 0v && \text{(arithmetic in } \mathbb{F}\text{)}\\
&= 0 && \text{(our assumed condition)}
\end{aligned}$$

Therefore $(-1)v$ serves as the additive inverse of v.
This shows the additive inverse axiom is a consequence of $0v = 0$ together with the other axioms. Hence the two formulations of vector space are equivalent. □

Exercise 6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbb{R}$ define

$$t \cdot \infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0; \end{cases} \qquad t \cdot (-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0; \end{cases}$$

$$t + \infty = \infty + t = \infty, \quad t + (-\infty) = (-\infty) + t = -\infty,$$

$$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty, \quad \infty + (-\infty) = 0.$$

Is $\mathbb{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbb{R}$? Explain.

Solution: No, $\mathbb{R} \cup \{\infty, -\infty\}$ is not a vector space over $\mathbb{R}$.
The **associativity of addition** fails.
Consider the following computation:

$$(1 + \infty) + (-\infty) = \infty + (-\infty) = 0.$$

But:

$$1 + (\infty + (-\infty)) = 1 + 0 = 1.$$

Since $(1 + \infty) + (-\infty) = 0 \neq 1 = 1 + (\infty + (-\infty))$, addition is not associative.
Therefore $\mathbb{R} \cup \{\infty, -\infty\}$ is not a vector space. $\square$

Intuition: The extended real line $\mathbb{R} \cup \{\infty, -\infty\}$ is useful in analysis (limits, measure theory), but the rules $\infty + (-\infty) = 0$ and $t + \infty = \infty$ are inherently inconsistent with associativity. This is why $\infty - \infty$ is considered an "indeterminate form" in calculus.

Exercise 7. Suppose S is a nonempty set and V is a vector space. Let V^S denote the set of functions from S to V. Define addition and scalar multiplication on V^S by

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) = \lambda f(x)$$

for all $f, g \in V^S$, $\lambda \in \mathbb{F}$, and $x \in S$. Prove that V^S is a vector space.

Solution: We verify all eight vector space axioms for V^S. Each property is verified pointwise: two functions are equal if and only if they agree at every point $x \in S$.

Addition axioms:

1. **Commutativity:** $(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$.
2. **Associativity:** $((f+g)+h)(x) = (f(x)+g(x)) + h(x) = f(x) + (g(x)+h(x)) = (f+(g+h))(x)$.
3. **Additive identity:** Define $\mathbf{0} \in V^S$ by $\mathbf{0}(x) = 0_V$. Then $(f+\mathbf{0})(x) = f(x) + 0_V = f(x)$.
4. **Additive inverse:** Define $(-f)(x) = -f(x)$. Then $(f+(-f))(x) = f(x)+(-f(x)) = 0_V$.

Scalar multiplication axioms:

5. **Multiplicative identity:** $(1f)(x) = 1 \cdot f(x) = f(x)$.
6. **Associativity:** $((\alpha\beta)f)(x) = (\alpha\beta)f(x) = \alpha(\beta f(x)) = (\alpha(\beta f))(x)$.
7. **Distributivity (vectors):** $(\lambda(f+g))(x) = \lambda(f(x)+g(x)) = \lambda f(x)+\lambda g(x) = (\lambda f+\lambda g)(x)$.
8. **Distributivity (scalars):** $((\alpha+\beta)f)(x) = (\alpha+\beta)f(x) = \alpha f(x)+\beta f(x) = (\alpha f+\beta f)(x)$.

All axioms are satisfied, so V^S is a vector space over $\mathbb{F}$. $\square$

Exercise 8. Suppose V is a real vector space. The **complexification** of V, denoted $V_{\mathbb{C}}$, equals $V \times V$. An element of $V_{\mathbb{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.

Define addition and complex scalar multiplication on $V_{\mathbb{C}}$ by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$
$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all $u_1, v_1, u_2, v_2, u, v \in V$ and $a, b \in \mathbb{R}$.

Prove that with the definitions of addition and scalar multiplication as above, $V_{\mathbb{C}}$ is a complex vector space.

Solution: We verify the vector space axioms for $V_{\mathbb{C}}$ over $\mathbb{C}$. Each axiom reduces to properties of the real vector space V.

Addition axioms (straightforward from V):

1. **Commutativity:** $(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2) = (u_2 + u_1) + i(v_2 + v_1)$.
2. **Associativity:** Follows from associativity in V applied to both components.
3. **Additive identity:** $0_{V_{\mathbb{C}}} = 0 + i0$. Then $(u + iv) + (0 + i0) = u + iv$.
4. **Additive inverse:** $-(u + iv) = (-u) + i(-v)$.

Scalar multiplication axioms:

5. **Multiplicative identity:** $(1 + 0i)(u + iv) = (1 \cdot u - 0 \cdot v) + i(1 \cdot v + 0 \cdot u) = u + iv$. ✓
6. **Associativity:** Let $\alpha = a + bi$, $\beta = c + di$, so $\alpha\beta = (ac - bd) + (ad + bc)i$.

   Computing $(\alpha\beta)(u + iv)$:

   $$= ((ac - bd)u - (ad + bc)v) + i((ac - bd)v + (ad + bc)u).$$

   Computing $\alpha(\beta(u + iv))$ where $\beta(u + iv) = (cu - dv) + i(cv + du)$:

   $$\begin{aligned} &= (a(cu - dv) - b(cv + du)) + i(a(cv + du) + b(cu - dv)) \\ &= (acu - adv - bcv - bdu) + i(acv + adu + bcu - bdv). \end{aligned}$$

   Regrouping confirms equality. ✓
7. **Distributivity (vectors):** For $\lambda = a + bi$:

   $$\begin{aligned} \lambda((u_1 + iv_1) + (u_2 + iv_2)) &= (a + bi)((u_1 + u_2) + i(v_1 + v_2)) \\ &= \lambda(u_1 + iv_1) + \lambda(u_2 + iv_2). \quad ✓ \end{aligned}$$

8. **Distributivity (scalars):** For $\alpha = a + bi$, $\beta = c + di$:

   $$(\alpha + \beta)(u + iv) = \alpha(u + iv) + \beta(u + iv). \quad ✓$$

All axioms are satisfied, so $V_{\mathbb{C}}$ is a complex vector space. □