Section 1A: $\mathbb{R}^n$ and $\mathbb{C}^n$

1. Complex Numbers

Before we can study vector spaces, we need to understand the scalars we'll use. The real numbers $\mathbb{R}$ are familiar, but linear algebra becomes more powerful when we also work with complex numbers $\mathbb{C}$.

1.1 Definition: Complex Numbers, $\mathbb{C}$

- A **complex number** is an ordered pair (a, b) where $a, b \in \mathbb{R}$, written as $a + bi$.
- The set of all complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

- **Addition** and **multiplication** on $\mathbb{C}$ are defined by:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

where $a, b, c, d \in \mathbb{R}$.

Intuition: Think of $\mathbb{C}$ as a 2D plane where the horizontal axis represents real numbers and the vertical axis represents imaginary numbers. If $a \in \mathbb{R}$, we identify $a+0i$ with the real number a, so $\mathbb{R} \subset \mathbb{C}$. We write $0+bi$ as just bi, and $0 + 1i$ as just i.

Why complex numbers? Even when working with real matrices, eigenvalues (Chapter 5) often require complex numbers. The completeness of $\mathbb{C}$ makes linear algebra more elegant—every polynomial has roots in $\mathbb{C}$.

Why this multiplication formula? We *define* i as a symbol satisfying $i^2 = -1$. This is consistent and creates an algebraically closed field. Using the usual rules of arithmetic:

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

1.2 Example: Complex Arithmetic
Compute $(2 + 3i)(4 + 5i)$.
Using the distributive and commutative properties:

$$\begin{aligned}(2 + 3i)(4 + 5i) &= 2 \cdot (4 + 5i) + (3i)(4 + 5i) \\ &= 2 \cdot 4 + 2 \cdot 5i + 3i \cdot 4 + (3i)(5i) \\ &= 8 + 10i + 12i - 15 \\ &= \boxed{-7 + 22i}\end{aligned}$$

1.3 Properties of Complex Arithmetic
For all $\alpha, \beta, \lambda \in \mathbb{C}$:
commutativity: $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$
associativity: $(\alpha+\beta)+\lambda = \alpha+(\beta+\lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$
identities: $\lambda + 0 = \lambda$ and $\lambda \cdot 1 = \lambda$
additive inverse: For every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$
multiplicative inverse: For every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$
distributive property: $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$

1.4 Example: Commutativity of Complex Multiplication
To show $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$, suppose $\alpha = a+bi$ and $\beta = c + di$ where $a, b, c, d \in \mathbb{R}$.
LHS:

$$\alpha\beta = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

RHS:

$$\beta\alpha = (c + di)(a + bi) = (ca - db) + (cb + da)i$$

Since real multiplication is commutative ($ac = ca$, $bd = db$, etc.), we have $\alpha\beta = \beta\alpha$. □

1.5 Definition: $-\alpha$, Subtraction, $1/\alpha$, Division
Suppose $\alpha, \beta \in \mathbb{C}$.

- Let $-\alpha$ denote the **additive inverse** of α: the unique complex number such that $\alpha + (-\alpha) = 0$.
- **Subtraction** on $\mathbb{C}$ is defined by $\beta-\alpha = \beta+(-\alpha)$.
- For $\alpha \neq 0$, let $1/\alpha$ and $\frac{1}{\alpha}$ denote the **multiplicative inverse** of α: the unique complex number such that $\alpha(1/\alpha) = 1$.
- For $\alpha \neq 0$, **division** by α is defined by $\beta/\alpha = \beta(1/\alpha)$.

Computing $1/\alpha$: For $\alpha = a + bi \neq 0$, multiply by the conjugate:

$$\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

Key insight: Why multiply by the conjugate? Because $(a+bi)(a-bi) = a^2+b^2$ is always real and positive. Multiplying by the conjugate eliminates i from the denominator.

Example: Complex Division. Compute $\dfrac{2 + 3i}{4 + 5i}$.

Multiply by the conjugate of the denominator:

$$\frac{2+3i}{4+5i} \cdot \frac{4-5i}{4-5i} = \frac{(2+3i)(4-5i)}{(4+5i)(4-5i)}$$

Denominator: $(4+5i)(4-5i) = 16+25 = 41$

Numerator: $(2+3i)(4-5i) = 8-10i+12i+15 = 23+2i$

Answer: $\frac{2+3i}{4+5i} = \boxed{\frac{23}{41} + \frac{2}{41}i}$

1.6 Notation: $\mathbb{F}$

Throughout this book, $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$.

Why use $\mathbb{F}$? The letter $\mathbb{F}$ reminds us of "field." Both $\mathbb{R}$ and $\mathbb{C}$ are fields: sets with addition and multiplication satisfying the properties in 1.3. Using $\mathbb{F}$ lets us state theorems once and have them apply to both $\mathbb{R}$ and $\mathbb{C}$.

Elements of $\mathbb{F}$ are called **scalars**.

Powers of scalars: For $\alpha \in \mathbb{F}$ and a positive integer m:

$$\alpha^m = \underbrace{\alpha \cdot \alpha \cdots \alpha}_{m \text{ times}}$$

This definition implies:

- $(\alpha^m)^n = \alpha^{mn}$
- $(\alpha\beta)^m = \alpha^m \beta^m$

2. Lists

To generalize $\mathbb{R}^2$ and $\mathbb{R}^3$ to higher dimensions, we first need to discuss the concept of lists.

1.7 Example: $\mathbb{R}^2$ and $\mathbb{R}^3$

- $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ (the plane)
- $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ (3D space)

1.8 Definition: List, Length

Suppose n is a nonnegative integer. A **list** of **length** n is an ordered collection of n elements (which might be numbers, other lists, or more abstract objects) separated by commas and surrounded by parentheses.

A list of length n is also called an n**-tuple**.

Key point: Two lists are equal if and only if they have the same length and the same elements in the same order.

1.9 Example: Lists versus Sets

- Lists $(3, 5)$ and $(5, 3)$ are **not equal**, but sets $\{3, 5\} = \{5, 3\}$
- Lists $(4, 4)$ and $(4, 4, 4)$ are **not equal**, but sets $\{4, 4\} = \{4, 4, 4\} = \{4\}$

Key difference: order and repetition matter in lists, not in sets.

Why lists? Linear algebra needs ordered data. The coordinates $(1, 2, 3)$ represent a different point than $(3, 2, 1)$. Order encodes meaning—the first coordinate might be position, the second velocity, the third acceleration.

3. $\mathbb{F}^n$

To define the higher-dimensional analogues of $\mathbb{R}^2$ and $\mathbb{R}^3$, we simply replace $\mathbb{R}$ with $\mathbb{F}$ (which equals $\mathbb{R}$ or $\mathbb{C}$) and replace the 2 or 3 with an arbitrary positive integer.

1.10 Notation: n

Fix a positive integer n for the rest of this chapter.

1.11 Definition: $\mathbb{F}^n$

$\mathbb{F}^n$ is the set of all lists of length n of elements of $\mathbb{F}$:

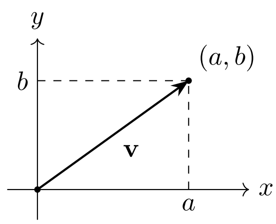
$$\mathbb{F}^n = \{(x_1, x_2, \ldots, x_n) : x_k \in \mathbb{F} \text{ for } k = 1, \ldots, n\}$$

For $(x_1, \ldots, x_n) \in \mathbb{F}^n$ and $k \in \{1, \ldots, n\}$, we say x_k is the k^{th} **coordinate** of $(x_1, \ldots, x_n)$.

Example 3.1 (Elements of $\mathbb{F}^n$).

- $(2, -1, 5) \in \mathbb{R}^3$ is a list with 3 real coordinates.
- $(1+i, 2, -3i) \in \mathbb{C}^3$ is a list with 3 complex coordinates.
- $(7, -2) \in \mathbb{R}^2$ corresponds to a point in the plane.
- $\mathbb{F}^1$ can be identified with $\mathbb{F}$.

Intuition: Think of $\mathbb{R}^2$ as the plane and $\mathbb{R}^3$ as 3-dimensional space. For $n > 3$, we lose geometric visualization but the algebra works identically.



Elements of $\mathbb{R}^2$ can be thought of as points or as vectors.

1.12 Example: $\mathbb{C}^4$
$\mathbb{C}^4 = \{(z_1, z_2, z_3, z_4) : z_1, z_2, z_3, z_4 \in \mathbb{C}\}$

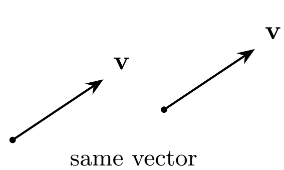
1.13 Definition: Addition in $\mathbb{F}^n$
Addition in $\mathbb{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

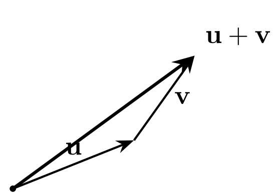
Example 3.2 (Vector Addition). In $\mathbb{R}^3$:

$$(1, 2, 3)+(10, 20, 30) = (1+10, 2+20, 3+30) = \boxed{(11, 22, 33)}$$

Geometric intuition: In $\mathbb{R}^2$ and $\mathbb{R}^3$, addition corresponds to the parallelogram rule: place the tail of the second vector at the head of the first.



A vector—same length and direction = same vector.



The sum of two vectors (tip-to-tail method).

1.14 Commutativity of Addition in $\mathbb{F}^n$
If $x, y \in \mathbb{F}^n$, then $x + y = y + x$.

Proof: Suppose $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then:

$$\begin{aligned} x + y &= (x_1, \ldots, x_n) + (y_1, \ldots, y_n) \\ &= (x_1 + y_1, \ldots, x_n + y_n) \\ &= (y_1 + x_1, \ldots, y_n + x_n) \\ &= (y_1, \ldots, y_n) + (x_1, \ldots, x_n) \\ &= y + x \end{aligned}$$

where the third equality uses commutativity of addition in $\mathbb{F}$. ☐

1.15 Notation: 0
Let 0 denote the list of length n whose coordinates are all 0:

$$0 = (0, \ldots, 0)$$

Geometric intuition: The zero vector 0 represents the origin. Adding 0 to any vector leaves it unchanged—you're adding "no displacement."

1.16 Example: The Zero Vector Notation
When we write $x + 0 = x$ for $x \in \mathbb{F}^n$, the symbol 0 means the **zero vector**:

$$0 = (0, 0, \ldots, 0) \quad (n \text{ zeros})$$

Why? Addition in $\mathbb{F}^n$ is only defined for two vectors. Since x is a vector, the 0 in "$x + 0$" must also be a vector—not the number zero.
Example in $\mathbb{R}^3$:

$$(1, 2, 3) + 0 = (1, 2, 3) + (0, 0, 0) = (1, 2, 3)✓$$

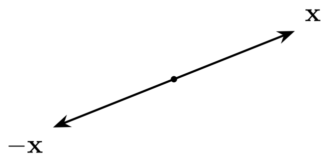
Key point: The symbol "0" means different things depending on context:

- In $\mathbb{F}$ (scalars): 0 is the number zero
- In $\mathbb{F}^n$ (vectors): 0 is the zero vector $(0, \ldots, 0)$

1.17 Definition: Additive Inverse in $\mathbb{F}^n$
For $x \in \mathbb{F}^n$, the **additive inverse** of x, denoted $-x$, is the vector $-x \in \mathbb{F}^n$ such that $x + (-x) = 0$.
If $x = (x_1, \ldots, x_n)$, then $-x = (-x_1, \ldots, -x_n)$.

Note: Subtraction is defined by $x - y = x + (-y)$.

Geometric intuition: In $\mathbb{R}^2$, $-x$ is the vector with the same length as x but pointing in the opposite direction.



A vector and its additive inverse.

Why not coordinate-wise multiplication? We *could* define multiplication of two vectors by multiplying corresponding coordinates, but this is not useful for linear algebra. Instead, **scalar multiplication** (multiplying a vector by a number) is central to our subject.

1.18 Definition: Scalar Multiplication in $\mathbb{F}^n$
The product of a number $\lambda \in \mathbb{F}$ and a vector in $\mathbb{F}^n$ is computed by multiplying each coordinate by λ:

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Example 3.3 (Scalar Multiplication in $\mathbb{R}^n$)**.** Here $\lambda = -2 \in \mathbb{R}$.

$$-2(1, 2, 3) = (-2 \cdot 1, -2 \cdot 2, -2 \cdot 3) = \boxed{(-2, -4, -6)}$$

Example 3.4 (Scalar Multiplication in $\mathbb{C}^n$)**.** Here $\lambda = 1 + i \in \mathbb{C}$.

$$\begin{aligned}(1+i)(2, -i) &= ((1+i) \cdot 2, (1+i)(-i)) \\ &= (2 + 2i, -i - i^2) \\ &= (2 + 2i, -i + 1) \\ &= \boxed{(2 + 2i, 1 - i)}\end{aligned}$$

Geometric intuition in $\mathbb{R}^2$:

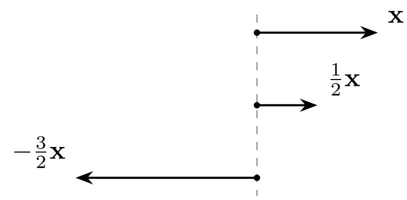
- If $\lambda > 0$: λx points in the same direction as x, with length λ times the length of x.
- If $\lambda > 1$: stretches (longer). If $0 < \lambda < 1$: shrinks (shorter).
- If $\lambda < 0$: λx points in the opposite direction, with length $|\lambda|$ times the length of x.

Direction Preservation Property
All scalar multiples of a nonzero vector x lie on a single line through the origin.
Proof: Let $x = (x_1, \ldots, x_n) \neq 0$. The set of all scalar multiples is:

$$\{\lambda x : \lambda \in \mathbb{F}\} = \{(\lambda x_1, \ldots, \lambda x_n) : \lambda \in \mathbb{F}\}$$

This is precisely the parametric equation of the line through the origin with direction vector x. As λ varies over $\mathbb{F}$, we trace out every point on this line. □

Key insight: This property is fundamental to linear maps (Chapter 3)—they preserve these lines through the origin.



Scalar multiplication: scaling and reversing vectors.

Scalar multiplication vs dot product: Scalar multiplication takes a scalar and a vector, producing a **vector**. The dot product (Chapter 6) takes two vectors and produces a **scalar**. These are different operations.
Generalization: The dot product in $\mathbb{R}^n$ generalizes to the **inner product** in Chapter 6, which works for both $\mathbb{R}^n$ and $\mathbb{C}^n$ (and more abstract vector spaces).

4. Digression on Fields

A **field** is a set containing at least two distinct elements called 0 and 1, along with operations of addition and multiplication satisfying all properties listed in 1.3.

- $\mathbb{R}$ and $\mathbb{C}$ are fields.
- The set of rational numbers $\mathbb{Q}$ is a field.
- The set $\{0, 1\}$ with usual addition and multiplication (except $1 + 1 = 0$) is a field.

Note: This book deals only with $\mathbb{R}$ and $\mathbb{C}$. However, many definitions, theorems, and proofs work for arbitrary fields. If you prefer, think of $\mathbb{F}$ as denoting an arbitrary field (except in Chapters 6–7 on inner products, where $\mathbb{F} = \mathbb{C}$ is sometimes required).

Exercise Reference

Key Formulas

Complex Inverse:

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$$

Addition in $\mathbb{C}$: $(a+bi)+(c+di) = (a+c)+(b+d)i$
Multiplication in $\mathbb{C}$: $(a+bi)(c+di) = (ac-bd)+(ad+bc)i$
Addition in $\mathbb{F}^n$: $(x_1,\ldots,x_n)+(y_1,\ldots,y_n) = (x_1+y_1,\ldots,x_n+y_n)$
Scalar Multiplication: $\lambda(x_1,\ldots,x_n) = (\lambda x_1,\ldots,\lambda x_n)$

Common Problem Types

Computing Complex Inverses/Roots
Multiply numerator and denominator by the conjugate. To verify a root, substitute and compute the power directly.

Proving Field Properties in $\mathbb{C}$
Write complex numbers as $\alpha = a+bi$, $\beta = c+di$. Expand both sides using definitions; compare real and imaginary parts.

Proving Uniqueness (of inverses, identities)
Assume two objects satisfy the defining property. Use that property to show they must be equal.

Solving Vector Equations
Isolate the unknown using additive inverses and scalar multiplication. Work component-wise.

Showing No Scalar λ Exists
If $\lambda x = y$, then each component ratio y_j/x_j must equal λ. Check if all ratios are consistent.

Proving $\mathbb{F}^n$ Properties
Write vectors as $x = (x_1,\ldots,x_n)$. Apply definitions of addition/scalar multiplication, then use field axioms on each component.