Eigenvalues and Eigenvectors

Linear Algebra Done Right, 4th ed.

Exercise 1. (Axler 5.B.10) Let p be a polynomial and $T\colon V \to V$ be a linear map. Given an eigenvector $v \in V$ with eigenvalue λ, show that $p(T)v = p(\lambda)v$.

Exercise 2. Let V denote the $\mathbb{R}$-vector space spanned by the functions $\{1, \sin x, \cos x, \sin 2x, \cos 2x\}$, and let $d\colon V \to V$ denote the differentiation operator. What are the eigenvalues of d? Is d diagonalizable?

Answer: ____________________

Exercise 3.

(a) Define the linear map $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ by $T(x,y) := (y,-x)$. What are the eigenvalues of T? Find a basis of $\mathbb{C}^2$ such that the matrix associated to T is upper-triangular.

Answer: ____________________

(b) Define the linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x,y) := (y,-x)$. Show that there is no basis of $\mathbb{R}^2$ such that the matrix associated to T is upper-triangular.

Exercise 4. (Axler 5.B.18) Fix an invertible linear map $T\colon V \to V$.

(a) If $p(x)$ is the minimal polynomial of T, show that $x^{\deg p}p(1/x)$ is the minimal polynomial of T^{-1}.

(b) For each nonzero scalar λ, show that $E(\lambda, T) = E(\lambda^{-1}, T^{-1})$.

Exercise 5. Fix a finite-dimensional vector space V. Fix a linear map $T\colon V \to V$ with nonzero eigenvalues $\lambda_1, \ldots, \lambda_m$.

(a) Show that $E(\lambda_i, T) \subseteq \operatorname{range} T$ for each eigenvalue λ_i.

(b) Use (a) to conclude that

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim \operatorname{range} T.$$

Exercise 6. Find the bases for the generalized eigenspace(s) of the following matrix:

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 4 & -4 & 7 \end{bmatrix}$$

Answer: ____

Exercise 7. (Axler 8.B.4) Suppose $\dim V \geq 2$ and $T\colon V \to V$ is a linear map such that $\operatorname{null} T^{\dim V-2} \neq \operatorname{null} T^{\dim V-1}$. Prove that T has at most two distinct eigenvalues.