# Midterm Review

*Linear Algebra Done Right, 4th ed.*

**Exercise 1.** Given a finite-dimensional vector space $V$ with basis $v_1, \ldots, v_n$.

(a) Prove that if two linear maps $S$ and $T$ have $S(v_1) = T(v_1), \ldots, S(v_n) = T(v_n)$, then $S = T$.

(b) Prove that for a linear map $T$, we have $T(v) = \varphi_1(v)T(v_1) + \cdots + \varphi_n(v)T(v_n)$ with $\varphi_1, \ldots, \varphi_n$ the dual basis of $v_1, \ldots, v_n$.

**Exercise 2.** Let $V$ denote the $\mathbb{R}$-vector space spanned by the functions $\{1, \sin x, \cos x, \sin 2x, \cos 2x\}$, and let $d\colon V \to V$ denote the differentiation operator. What are the eigenvalues of $d$? Is $d$ diagonalizable?

**Answer:** ______________________________

**Exercise 3.**

(a) Define the linear map $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ by $T(x, y) := (y, -x)$. What are the eigenvalues of $T$? Find a basis of $\mathbb{C}^2$ such that the matrix associated to $T$ is upper-triangular.

**Answer:** ______________________________

(b) Define the linear map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) := (y, -x)$. Show that there is no basis of $\mathbb{R}^2$ such that the matrix associated to $T$ is upper-triangular.

**Exercise 4.** (Spring 2024 Frenkel Mock Midterm Problem 3) Prove that a vector space $V$ over a field $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$ (where $n$ is a positive integer) if and only if $\dim(V) = n$.