Upper Division Tutoring Program Topic Review

Facilitators: Matthew Corbo, corbo@berkeley.edu
Kyle Fenimore, kyle.fenimore@berkeley.edu

Drop-In: Check Drop-In Schedule

Vector Spaces, Linear Maps, and Duality

Section I: Definitions and Theorems Review

Vector Spaces

Definition 1 (vector space)**.** Fix a field F. Then a *vector space* V over F is a set equipped with an addition $+$ and scalar multiplication $\cdot$ satisfying the following conditions.

- Addition commutes and associates: $u+v=v+u$ and $u+(v+w)=(u+v)+w$ for any $u,v,w \in V$.
- Additive identity: there is a vector $0 \in V$ such that $v+0=0+v=v$ for any $v \in V$.
- Additive inverse: for any $v \in V$, there is a vector $-v \in V$ such that $v+(-v)=(-v)+v=0$.
- Multiplicative identity: for any $v \in V$, we have $1v=v$.
- Distribution: for any $a,b \in F$ and $v,w \in V$, we have $(a+b)(v+w)=av+aw+bv+bw$.

Remark. One can show that the identity $0 \in V$ and inverse $-v \in V$ of $v \in V$ are unique. As such, we may say "the identity" or "the inverse of v."

Definition 2 (subspace)**.** Fix a vector space V over a field F. A subset $W \subseteq V$ is a *subspace* if and only if it satisfies the following.

- Zero: $0 \in V$.
- Addition: for $v,w \in W$, we have $v+w \in W$.
- Scalar multiplication: for $a \in F$ and $v \in W$, we have $av \in W$.

Definition 3 (sum, direct sum)**.** Fix a vector space V over a field F. Given some subsets $U_1,\ldots,U_n$, we define the *sum* as

$$U_1+\cdots+U_n \coloneqq \{u_1+\cdots+u_n : u_1 \in U_1,\ldots,u_n \in U_n\}.$$

This is a *direct sum* if it satisfies the following property: for each $v \in U_1+\cdots+U_n$, there are unique vectors $u_1 \in U_1,\ldots,u_n \in U_n$ such that $v=u_1+\cdots+u_n$. In this case, we write $U_1 \oplus \cdots \oplus U_n$.

Definition 4 (span)**.** Fix a vector space V over a field F. Given a set of vectors $\{v_1,\ldots,v_n\}$, the *span* of these vectors is given by

$$\operatorname{span}(\{v_1,\ldots,v_n\}) \coloneqq \{a_1v_1+\cdots+a_nv_n : a_1,\ldots,a_n \in F\}.$$

In other words, the span is the set of linear combinations of the vectors in $\{v_1,\ldots,v_n\}$.

Definition 5 (linearly independent)**.** Fix a vector space V over a field F. Then a set of vectors $\{v_1,\ldots,v_m\}$ in V is *linearly independent* if and only if

$$a_1v_1+\cdots+a_nv_n=0$$

for $a_1,\ldots,a_n \in F$ implies that $a_1=\cdots=a_n=0$. Otherwise, we say that these vectors are *linearly dependent*.

Definition 6 (basis, dimension)**.** Fix a vector space V over a field F. A *basis* of a set of vectors B which is linearly independent and spans V. The *dimension* $\dim V$ of V is the size of the set B.

Proposition 1. Fix a vector space V over a field F and a subset of vectors $\{v_1,\ldots,v_n\}$. Then this set is a basis if and only if it satisfies the following: for any vector $v \in V$, there are scalars $a_1,\ldots,a_n \in F$ such that

$$v=a_1v_1+\cdots+a_nv_n.$$

Theorem 1. Every vector space has a (possibly infinite) basis.

Upper Division Tutoring Program Topic Review

Facilitators: Matthew Corbo, corbo@berkeley.edu
Kyle Fenimore, kyle.fenimore@berkeley.edu
Drop-In: Check Drop-In Schedule

Linear Maps

Definition 7 (linear map)**.** Fix vector spaces V and W over a field F. Then a *linear map* is a function $T\colon V \to W$ satisfying the following.

- Addition: for $v, v' \in V$, we have $T(v+v') = T(v) + T(v')$.
- Scalar multiplication: for $v \in V$ and $a \in F$, we have $T(cv) = cT(v)$.

We denote the vector space of linear transformations $V \to W$ by $\mathcal{L}(V, W)$

Example 1. Given a vector space V, define the function $\mathrm{id}_V : V \to V$ by $\mathrm{id}_V(v) \coloneqq v$ for each $v \in V$. Then id_V is a linear map.

Remark. If V and W are finite-dimensional, then $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$.

Proposition 2. Fix a finite-dimensional vector space V with a basis $\{v_1, \dots, v_n\}$. Given a vector space W and some vectors $\{w_1, \dots, w_n\}$, there is a unique linear transformation $T\colon V \to W$ such that

$$T(v_1) = w_1, \qquad T(v_2 = w_2), \qquad \dots, \qquad T(v_n) = w_n.$$

Definition 8 (null space, range)**.** Fix a linear map $T\colon V \to W$ of vector spaces.

- The *null space* of T is $\operatorname{null} T \coloneqq \{v \in V : T(v) = 0\}$. It is a subspace of V.
- The *range* of T is $\operatorname{range} T \coloneqq \{T(v) : v \in V\}$. It is a subspace of W.

Definition 9 (injective, surjective, bijective)**.** Fix a linear map $T\colon V \to W$ of vector spaces.

- We say T is *injective* if and only if $T(v) = T(v')$ implies $v = v'$ for any $v, v' \in V$.
- We say T is *surjective* if and only if any vector $w \in W$ has some vector $v \in V$ such that $T(v) = w$.
- We say T is *bijective* or an *isomorphism* if and only if T is both injective and surjective.

Remark. One can show that T is injective if and only if $\operatorname{null} T = \{0\}$, and T is surjective if and only if $\operatorname{range} T = W$.

Theorem 2. Fix a linear map $T\colon V \to W$ of vector spaces. Then

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Definition 10 (matrix)**.** Fix a linear map $T\colon V \to W$ of finite-dimensional vector spaces V and W with bases $\{v_1, \dots, v_m\}$ and $\{w_1, \dots, w_m\}$ respectively. For each v_i, find scalars $a_{i1}, \dots, a_{in}$ such that

$$Tv_i = \sum_{j=1}^{n} a_{ij} w_j.$$

Then the matrix $\{a_{ij}\}$ is the *matrix associated to* T.

Duality

Definition 11 (linear functional)**.** Let V be a vector space, ϕ is a linear functional on V if $\phi \in \mathcal{L}(V, \mathbb{F})$.

Definition 12 (dual space)**.** Fix a vector space V, its dual space V' is the space of all linear functionals on V. We write $V' \coloneqq \mathcal{L}(V, \mathbb{F})$.

Proposition 3. Fix a vector space V. Then $\dim V = \dim V'$.

Facilitators: Matthew Corbo, corbo@berkeley.edu
Kyle Fenimore, kyle.fenimore@berkeley.edu

Definition 13 (dual operator)**.** Fix a linear operator $T\colon V \to W$, then its dual operator $T'\colon W' \to V'$ is defined by $T'(\phi) := \phi \circ T$ for any $\phi \in W'$.

Definition 14 (dual basis)**.** Let $v_1, \dots, v_n$ be a basis of a finite-dimensional vector space V. Then the dual basis of this list is given by $\varphi_1, \dots, \varphi_n \in V'$, where each $\varphi_j \in V'$ has

$$\varphi_j(v_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Proposition 4. The dual basis defined as above is indeed a basis of V'.

Proposition 5. The matrix of the dual operator T' is the transpose of the matrix of T.

Definition 15. Let V be a vector space and $U \subset V$ be a subspace. Then, the annihilator of U, denoted by U^0, is the set of all $\phi \in V'$ such that $\phi(u) = 0$ for all $u \in U$.

Remark. The annihilator U^0 is a subspace of V'.

Section II: Practice Exercises

Exercises are organized thematically, not by difficulty.

1. Given a matrix
$$A := \begin{bmatrix} a_{11} & \cdots & a_{15} \\ \vdots & \ddots & \vdots \\ a_{51} & \cdots & a_{55} \end{bmatrix} \in \mathbb{R}^{5\times 5},$$
we say that A is *symmetric* if and only if $a_{ij} = a_{ji}$ for each i and j, and we say that A is *skew-symmetric* if and only if $a_{ij} = -a_{ji}$.
   (a) Let Y and K be the set of symmetric and skew-symmetric matrices in $\mathbb{R}^{5\times 5}$, respectively. Show that Y and K are subspaces of $\mathbb{R}^{5\times 5}$.
   (b) Compute $\dim Y$ and $\dim K$.
   (c) Show that $\mathbb{R}^{5\times 5}$ is the direct sum of the subspaces Y and K.

2. (Axler 2.C.17) Fix a finite-dimensional vector space V over a field F, and fix subspaces $U_1, \dots, U_n$ of V.
   (a) Show that
$$\dim(U_1 + \cdots + U_n) \leq \dim U_1 + \cdots + \dim U_n.$$
   (b) In fact, show that equality holds in (a) if and only if the sum $U_1 + \cdots + U_n$ is direct.

3. Fix a linear transformation $T\colon V \to W$ of vector spaces over a field F. For any $w \in W$, suppose there is a vector $v_0 \in V$ such that $f(v_0) = w$. Then show that
$$\{v \in V : f(v) = w\} = \{v + v_0 : v \in \operatorname{null} T\}.$$

4. (Axler 3.B.28) Suppose $p \in \mathcal{P}(\mathbb{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbb{R})$ such that
$$5q'' + 3q' = p.$$

Facilitators: Matthew Corbo, corbo@berkeley.edu
Kyle Fenimore, kyle.fenimore@berkeley.edu
Drop-In: Check Drop-In Schedule

5. Let I be the linear map $I\colon \mathcal{P}_5(\mathbb{R}) \to \mathcal{P}_6(\mathbb{R})$ by
$$I(f) := \int_0^x f(t)\,dt.$$
   (a) Convince yourself that I is a linear map.
   (b) Find bases of $\mathcal{P}_5(\mathbb{R})$ and $\mathcal{P}_6(\mathbb{R})$.
   (c) Use the bases found in (b) in order to write dual bases for $\mathcal{P}_5(\mathbb{R})'$ and $\mathcal{P}_6(\mathbb{R})'$.
   (d) Use the dual bases found in (c) in order to write $I'\colon \mathcal{P}_6(\mathbb{R})' \to \mathcal{P}_5(\mathbb{R})'$ as a matrix.

6. Let V be a 2-dimensional vector space, and let $\varphi, \psi \in V'$. Show that φ and ψ are linearly independent if and only if
$$\operatorname{null}(\varphi) \cap \operatorname{null}(\psi) = \{0\}.$$

7. Fix a finite-dimensional vector space V. Call a linear map $T\colon V \to V$ a *projection* if and only if $T \circ T = T$.
   (a) Give an example of a projection $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which is neither the zero nor the identity operators.
   (b) For any projection T, show that T fixes $\operatorname{range} T$.
   (c) For any projection T, show that $\operatorname{range} T \cap \operatorname{null} T = \{0\}$. Conclude that $V = \operatorname{range} T \oplus \operatorname{null} T$.
   (d) We say that a linear transformation $T \in \mathcal{L}(V)$ is *diagonal* if and only if there is a basis $\{v_1, \ldots, v_n\}$ of V and constants $\lambda_1, \ldots, \lambda_n$ such that $Tv_i = \lambda_i v_i$ for each i. Show that T can be written as a linear combination of projections.

8. (Axler 3.F.23) Let V be a finite-dimensional vector space, and let U and W be subspaces of V.
   (a) Show that $(U + W)^0 = U^0 \cap W^0$.
   (b) Show that $(U \cap W)^0 = U^0 + W^0$.