1 Suppose $b, c \in \mathbf{R}$. Define $T\colon \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b,\ 6x + cxyz).$$

Show that T is linear if and only if $b = c = 0$.

2 Suppose $b, c \in \mathbf{R}$. Define $T\colon \mathcal{P}(\mathbf{R}) \to \mathbf{R}^2$ by

$$Tp = \bigl(3p(4) + 5p'(6) + bp(1)p(2),\ \int_{-1}^{2} x^3 p(x)\,dx + c\sin p(0)\bigr).$$

Show that T is linear if and only if $b = c = 0$.

3 Suppose that $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbf{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}\,x_n,\ \ldots,\ A_{m,1}x_1 + \cdots + A_{m,n}\,x_n)$$

for every $(x_1, \ldots, x_n) \in \mathbf{F}^n$.

This exercise shows that the linear map T has the form promised in the second to last item of Example 3.3.

4 Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $Tv_1, \ldots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

5 Prove that $\mathcal{L}(V, W)$ is a vector space, as was asserted in 3.6.

6 Prove that multiplication of linear maps has the associative, identity, and distributive properties asserted in 3.8.

7 Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

8 Give an example of a function $\varphi\colon \mathbf{R}^2 \to \mathbf{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but φ is not linear.

This exercise and the next exercise show that neither homogeneity nor additivity alone is enough to imply that a function is a linear map.

9 Give an example of a function $\varphi\colon \mathbf{C} \to \mathbf{C}$ such that

$$\varphi(w + z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbf{C}$ but φ is not linear. (Here $\mathbf{C}$ is thought of as a complex vector space.)

There also exists a function $\varphi\colon \mathbf{R} \to \mathbf{R}$ such that φ satisfies the additivity condition above but φ is not linear. However, showing the existence of such a function involves considerably more advanced tools.

10 Prove or give a counterexample: If $q \in \mathcal{P}(\mathbf{R})$ and $T\colon \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is defined by $Tp = q \circ p$, then T is a linear map.

The function T defined here differs from the function T defined in the last bullet point of 3.3 by the order of the functions in the compositions.

11 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

12 Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means $Su \neq 0$ for some $u \in U$). Define $T: V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is not a linear map on V.

13 Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

The result in this exercise is used in the proof of 3.125.

14 Suppose V is finite-dimensional with $\dim V > 0$, and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

15 Suppose $v_1, \ldots, v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, \ldots, w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$.

16 Suppose V is finite-dimensional with $\dim V > 1$. Prove that there exist $S, T \in \mathcal{L}(V)$ such that $ST \neq TS$.

17 Suppose V is finite-dimensional. Show that the only two-sided ideals of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$.

*A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called a **two-sided ideal** of $\mathcal{L}(V)$ if $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$ for all $E \in \mathcal{E}$ and all $T \in \mathcal{L}(V)$.*

1 Give an example of a linear map T with $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

2 Suppose $S, T \in \mathcal{L}(V)$ are such that $\operatorname{range} S \subseteq \operatorname{null} T$. Prove that $(ST)^2 = 0$.

3 Suppose $v_1, \ldots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$

(a) What property of T corresponds to $v_1, \ldots, v_m$ spanning V?

(b) What property of T corresponds to the list $v_1, \ldots, v_m$ being linearly independent?

4 Show that $\{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \operatorname{null} T > 2\}$ is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

5 Give an example of $T \in \mathcal{L}(\mathbf{R}^4)$ such that $\operatorname{range} T = \operatorname{null} T$.

6 Prove that there does not exist $T \in \mathcal{L}(\mathbf{R}^5)$ such that $\operatorname{range} T = \operatorname{null} T$.

7 Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

8 Suppose V and W are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not surjective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

9 Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

10 Suppose $v_1, \ldots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Show that $Tv_1, \ldots, Tv_n$ spans range T.

11 Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that

$$U \cap \operatorname{null} T = \{0\} \quad \text{and} \quad \operatorname{range} T = \{Tu : u \in U\}.$$

12 Suppose T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

13 Suppose U is a three-dimensional subspace of $\mathbf{R}^8$ and that T is a linear map from $\mathbf{R}^8$ to $\mathbf{R}^5$ such that $\operatorname{null} T = U$. Prove that T is surjective.

14 Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$.

15 Suppose there exists a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

16 Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

17 Suppose V and W are both finite-dimensional. Prove that there exists a surjective linear map from V onto W if and only if $\dim V \geq \dim W$.

18 Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

19 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity operator on V.

20 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity operator on W.

21 Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and U is a subspace of W. Prove that $\{v \in V : Tv \in U\}$ is a subspace of V and

$$\dim\{v \in V : Tv \in U\} = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T).$$

22 Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T.$$

23 Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{range} ST \leq \min\{\dim \operatorname{range} S, \dim \operatorname{range} T\}.$$

24

(a) Suppose $\dim V = 5$ and $S, T \in \mathcal{L}(V)$ are such that $ST = 0$. Prove that $\dim \operatorname{range} TS \leq 2$.

(b) Give an example of $S, T \in \mathcal{L}(\mathbf{F}^5)$ with $ST = 0$ and $\dim \operatorname{range} TS = 2$.

25 Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} S \subseteq \operatorname{null} T$ if and only if there exists $E \in \mathcal{L}(W)$ such that $T = ES$.

26 Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} S \subseteq \operatorname{range} T$ if and only if there exists $E \in \mathcal{L}(V)$ such that $S = TE$.

27 Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.

28 Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is such that $\deg Dp = (\deg p) - 1$ for every nonconstant polynomial $p \in \mathcal{P}(\mathbf{R})$. Prove that D is surjective.

The notation D is used above to remind you of the differentiation map that sends a polynomial p to p'.

29 Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.
This exercise can be done without linear algebra, but it's more fun to do it using linear algebra.

30 Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$ and $\varphi \neq 0$. Suppose $u \in V$ is not in $\operatorname{null} \varphi$. Prove that

$$V = \operatorname{null} \varphi \oplus \{au : a \in \mathbf{F}\}.$$

31 Suppose V is finite-dimensional, X is a subspace of V, and Y is a finite-dimensional subspace of W. Prove that there exists $T \in \mathcal{L}(V,W)$ such that $\operatorname{null} T = X$ and $\operatorname{range} T = Y$ if and only if $\dim X + \dim Y = \dim V$.

32 Suppose V is finite-dimensional with $\dim V > 1$. Show that if $\varphi\colon \mathcal{L}(V) \to \mathbf{F}$ is a linear map such that $\varphi(ST) = \varphi(S)\varphi(T)$ for all $S, T \in \mathcal{L}(V)$, then $\varphi = 0$.

Hint: The description of the two-sided ideals of $\mathcal{L}(V)$ given by Exercise 17 in Section 3A might be useful.

33 Suppose that V and W are real vector spaces and $T \in \mathcal{L}(V, W)$. Define $T_{\mathbf{C}} \colon V_{\mathbf{C}} \to W_{\mathbf{C}}$ by

$$T_{\mathbf{C}}(u + iv) = Tu + iTv$$

for all $u, v \in V$.

(a) Show that $T_{\mathbf{C}}$ is a (complex) linear map from $V_{\mathbf{C}}$ to $W_{\mathbf{C}}$.

(b) Show that $T_{\mathbf{C}}$ is injective if and only if T is injective.

(c) Show that $\operatorname{range} T_{\mathbf{C}} = W_{\mathbf{C}}$ if and only if $\operatorname{range} T = W$.

See Exercise 8 in Section 1B for the definition of the complexification $V_{\mathbf{C}}$. The linear map $T_{\mathbf{C}}$ is called the ***complexification*** *of the linear map T.*

1 Suppose $T \in \mathcal{L}(V, W)$ is invertible. Show that T^{-1} is invertible and

$$(T^{-1})^{-1} = T.$$

2 Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

3 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent.

(a) T is invertible.

(b) $Tv_1, \ldots, Tv_n$ is a basis of V for every basis $v_1, \ldots, v_n$ of V.

(c) $Tv_1, \ldots, Tv_n$ is a basis of V for some basis $v_1, \ldots, v_n$ of V.

4 Suppose V is finite-dimensional and $\dim V > 1$. Prove that the set of noninvertible linear maps from V to itself is not a subspace of $\mathcal{L}(V)$.

5 Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible linear map T from V to itself such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

6 Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{null} S = \operatorname{null} T$ if and only if there exists an invertible $E \in \mathcal{L}(W)$ such that $S = ET$.

7 Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\operatorname{range} S = \operatorname{range} T$ if and only if there exists an invertible $E \in \mathcal{L}(V)$ such that $S = TE$.

8 Suppose V and W are finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that there exist invertible $E_1 \in \mathcal{L}(V)$ and $E_2 \in \mathcal{L}(W)$ such that $S = E_2 T E_1$ if and only if $\dim \operatorname{null} S = \dim \operatorname{null} T$.

9 Suppose V is finite-dimensional and $T: V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W.

Here $T|_U$ means the function T restricted to U. Thus $T|_U$ is the function whose domain is U, with $T|_U(u) = Tu$ for every $u \in U$.

10 Suppose V and W are finite-dimensional and U is a subspace of V. Let

$$\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\}.$$

(a) Show that $\mathcal{E}$ is a subspace of $\mathcal{L}(V, W)$.

(b) Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$, and $\dim U$.

Hint: Define $\Phi\colon \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ *by* $\Phi(T) = T|_U$. *What is* $\operatorname{null} \Phi$*? What is* $\operatorname{range} \Phi$*?*

11 Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that

$$ST \text{ is invertible} \iff S \text{ and } T \text{ are invertible.}$$

12 Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

13 Show that the result in Exercise 12 can fail without the hypothesis that V is finite-dimensional.

14 Prove or give a counterexample: If V is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective, then S is injective.

15 Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_m$ is a list in V such that $Tv_1, \ldots, Tv_m$ spans V. Prove that $v_1, \ldots, v_m$ spans V.

16 Prove that every linear map from $\mathbf{F}^{n,1}$ to $\mathbf{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbf{F}^{n,1}, \mathbf{F}^{m,1})$, then there exists an m-by-n matrix A such that $Tx = Ax$ for every $x \in \mathbf{F}^{n,1}$.

17 Suppose V is finite-dimensional and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by

$$\mathcal{A}(T) = ST$$

for $T \in \mathcal{L}(V)$.

(a) Show that $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.

(b) Show that $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

18 Show that V and $\mathcal{L}(\mathbf{F}, V)$ are isomorphic vector spaces.

19 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has the same matrix with respect to every basis of V if and only if T is a scalar multiple of the identity operator.

20 Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{R})$ such that

$$q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$$

for all $x \in \mathbf{R}$.

21 Suppose n is a positive integer and $A_{j,k} \in \mathbf{F}$ for all $j, k = 1, \ldots, n$. Prove that the following are equivalent (note that in both parts below, the number of equations equals the number of variables).

(a) The trivial solution $x_1 = \cdots = x_n = 0$ is the only solution to the homogeneous system of equations

$$\sum_{k=1}^{n} A_{1,k}\, x_k = 0 \qquad \cdots \qquad \sum_{k=1}^{n} A_{n,k}\, x_k = 0.$$

(b) For every $c_1, \ldots, c_n \in \mathbf{F}$, there exists a solution to the system of equations

$$\sum_{k=1}^{n} A_{1,k}\, x_k = c_1 \qquad \cdots \qquad \sum_{k=1}^{n} A_{n,k}\, x_k = c_n.$$

22 Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Prove that

$$\mathcal{M}(T, (v_1, \ldots, v_n)) \text{ is invertible} \iff T \text{ is invertible.}$$

23 Suppose that $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are bases of V. Let $T \in \mathcal{L}(V)$ be such that $Tv_k = u_k$ for each $k = 1, \dots, n$. Prove that

$$\mathcal{M}(T, (v_1, \dots, v_n)) = \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n)).$$

24 Suppose A and B are square matrices of the same size and $AB = I$. Prove that $BA = I$.