Exercises 1A Solutions: $\mathbb{R}^n$ and $\mathbb{C}^n$

Linear Algebra Done Right, 4th ed.

Exercise 1. Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Solution: Let $\alpha = a + bi$ and $\beta = c + di$ where $a, b, c, d \in \mathbb{R}$.
Computing $\alpha + \beta$:

$$\alpha + \beta = (a + bi) + (c + di) = (a + c) + (b + d)i.$$

Computing $\beta + \alpha$:

$$\beta + \alpha = (c + di) + (a + bi) = (c + a) + (d + b)i.$$

Since addition of real numbers is commutative ($a + c = c + a$ and $b + d = d + b$):

$$\alpha + \beta = (a + c) + (b + d)i = (c + a) + (d + b)i = \beta + \alpha.$$

Therefore $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$. □

Exercise 2. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Solution: Let $\alpha = a + bi$, $\beta = c + di$, and $\lambda = e + fi$ where $a, b, c, d, e, f \in \mathbb{R}$.
Computing the left-hand side:

$$\begin{aligned}(\alpha + \beta) + \lambda &= \big((a + c) + (b + d)i\big) + (e + fi) \\ &= \big((a + c) + e\big) + \big((b + d) + f\big)i.\end{aligned}$$

Computing the right-hand side:

$$\begin{aligned}\alpha + (\beta + \lambda) &= (a + bi) + \big((c + e) + (d + f)i\big) \\ &= \big(a + (c + e)\big) + \big(b + (d + f)\big)i.\end{aligned}$$

Since addition of real numbers is associative:

$$(a + c) + e = a + (c + e) \quad \text{and} \quad (b + d) + f = b + (d + f).$$

Therefore $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$. □

Exercise 3. Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.

Solution: Let $\alpha = a + bi$, $\beta = c + di$, and $\lambda = e + fi$ where $a, b, c, d, e, f \in \mathbb{R}$.
First, compute $\alpha\beta$:

$$\alpha\beta = (a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Then compute $(\alpha\beta)\lambda$:

$$\begin{aligned}(\alpha\beta)\lambda &= \big((ac-bd)+(ad+bc)i\big)(e+fi)\\ &= \big((ac-bd)e-(ad+bc)f\big)+\big((ac-bd)f+(ad+bc)e\big)i\\ &= (ace-bde-adf-bcf)+(acf-bdf+ade+bce)i.\end{aligned}$$

Now compute $\beta\lambda$:

$$\beta\lambda = (c+di)(e+fi) = (ce-df)+(cf+de)i.$$

Then compute $\alpha(\beta\lambda)$:

$$\begin{aligned}\alpha(\beta\lambda) &= (a+bi)\big((ce-df)+(cf+de)i\big)\\ &= \big(a(ce-df)-b(cf+de)\big)+\big(a(cf+de)+b(ce-df)\big)i\\ &= (ace-adf-bcf-bde)+(acf+ade+bce-bdf)i.\end{aligned}$$

Comparing terms, we see that LHS = RHS. Therefore $(\alpha\beta)\lambda = \alpha(\beta\lambda)$. $\square$

Exercise 4. Show that $\lambda(\alpha+\beta)=\lambda\alpha+\lambda\beta$ for all $\lambda,\alpha,\beta\in\mathbb{C}$.

Solution: Let $\lambda=a+bi$, $\alpha=c+di$, and $\beta=e+fi$ where $a,b,c,d,e,f\in\mathbb{R}$.
Computing the left-hand side:

$$\begin{aligned}\lambda(\alpha+\beta)&=(a+bi)\big((c+e)+(d+f)i\big)\\&=a(c+e)-b(d+f)+\big(a(d+f)+b(c+e)\big)i\\&=(ac+ae-bd-bf)+(ad+af+bc+be)i.\end{aligned}$$

Computing the right-hand side:

$$\begin{aligned}\lambda\alpha&=(a+bi)(c+di)=(ac-bd)+(ad+bc)i\\\lambda\beta&=(a+bi)(e+fi)=(ae-bf)+(af+be)i\end{aligned}$$

Adding:

$$\begin{aligned}\lambda\alpha+\lambda\beta&=\big((ac-bd)+(ae-bf)\big)+\big((ad+bc)+(af+be)\big)i\\&=(ac+ae-bd-bf)+(ad+af+bc+be)i.\end{aligned}$$

Comparing, LHS = RHS. Therefore $\lambda(\alpha+\beta)=\lambda\alpha+\lambda\beta$. $\square$

Exercise 5. Show that for every $\alpha\in\mathbb{C}$, there exists a unique $\beta\in\mathbb{C}$ such that $\alpha+\beta=0$.

Solution: Let $\alpha=a+bi$ where $a,b\in\mathbb{R}$.
Existence: Define $\beta=-a+(-b)i=-a-bi$. Then:

$$\alpha+\beta=(a+bi)+(-a-bi)=(a-a)+(b-b)i=0+0i=0.$$

So such a β exists.
Uniqueness: Suppose β_1 and β_2 both satisfy $\alpha+\beta_1=0$ and $\alpha+\beta_2=0$.
Then:

$$\begin{aligned}\beta_1&=\beta_1+0=\beta_1+(\alpha+\beta_2)\\&=(\beta_1+\alpha)+\beta_2=(\alpha+\beta_1)+\beta_2\\&=0+\beta_2=\beta_2.\end{aligned}$$

Therefore β is unique. $\square$

Exercise 6. Show that for every $\alpha\in\mathbb{C}$ with $\alpha\neq 0$, there exists a unique $\beta\in\mathbb{C}$ such that $\alpha\beta=1$.

Solution: Let $\alpha = a + bi$ where $a, b \in \mathbb{R}$ and a and b are not both zero.
Existence: Define $\beta = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$.
Note that $a^2 + b^2 \neq 0$ since $\alpha \neq 0$, so β is well-defined. Then:

$$\begin{aligned}
\alpha\beta &= (a+bi)\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\right) \\
&= \frac{a^2}{a^2+b^2} - \frac{ab}{a^2+b^2}i + \frac{ab}{a^2+b^2}i - \frac{b^2}{a^2+b^2}i^2 \\
&= \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1.
\end{aligned}$$

Uniqueness: Suppose β_1 and β_2 both satisfy $\alpha\beta_1 = 1$ and $\alpha\beta_2 = 1$.
Then:

$$\beta_1 = \beta_1 \cdot 1 = \beta_1(\alpha\beta_2) = (\beta_1\alpha)\beta_2 = (\alpha\beta_1)\beta_2 = 1 \cdot \beta_2 = \beta_2.$$

Therefore β is unique. $\square$

Exercise 7. Show that

$$\frac{-1+\sqrt{3}\,i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Solution: Let $\omega = \frac{-1+\sqrt{3}\,i}{2}$. We compute ω^3 step by step.
First, compute ω^2:

$$\begin{aligned}\omega^2 &= \left(\frac{-1+\sqrt{3}\,i}{2}\right)^2 = \frac{(-1+\sqrt{3}\,i)^2}{4} \\ &= \frac{1-2\sqrt{3}\,i+3i^2}{4} = \frac{1-2\sqrt{3}\,i-3}{4} \\ &= \frac{-2-2\sqrt{3}\,i}{4} = \frac{-1-\sqrt{3}\,i}{2}.\end{aligned}$$

Now compute $\omega^3 = \omega^2 \cdot \omega$:

$$\begin{aligned}\omega^3 &= \frac{-1-\sqrt{3}\,i}{2} \cdot \frac{-1+\sqrt{3}\,i}{2} \\ &= \frac{(-1-\sqrt{3}\,i)(-1+\sqrt{3}\,i)}{4} \\ &= \frac{(-1)^2-(\sqrt{3}\,i)^2}{4} = \frac{1-3i^2}{4} = \frac{1+3}{4} = 1.\end{aligned}$$

Therefore $\omega^3 = 1$, so ω is a cube root of 1. $\square$

Exercise 8. Find two distinct square roots of i.

Solution: We seek $z = a + bi$ such that $z^2 = i$, where $a, b \in \mathbb{R}$.
Expanding z^2:

$$z^2 = (a+bi)^2 = a^2 + 2abi + b^2 i^2 = (a^2 - b^2) + 2abi.$$

Setting $z^2 = i = 0 + 1 \cdot i$, we equate real and imaginary parts:

$$\begin{aligned}a^2 - b^2 &= 0 \\ 2ab &= 1\end{aligned}$$

From the first equation, $a^2 = b^2$, so $a = \pm b$.
If $a = b$: From $2ab = 1$, we get $2a^2 = 1$, so $a = \pm\frac{1}{\sqrt{2}}$. This gives $a = b = \frac{1}{\sqrt{2}}$ or $a = b = -\frac{1}{\sqrt{2}}$.
If $a = -b$: From $2ab = 1$, we get $-2a^2 = 1$, which has no real solutions.

Therefore the two square roots of i are:

$$\boxed{z_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = \frac{\sqrt{2}}{2}(1+i), \quad z_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i = -\frac{\sqrt{2}}{2}(1+i)}$$

Intuition: The phrase "two distinct square roots of i" simply means: find two *different* complex numbers whose square equals i.

Why do two exist? If $z^2 = i$, then $(-z)^2 = z^2 = i$ as well. So square roots come in $\pm$ pairs. Unless $z = 0$, these two are different.

Why say "distinct"? To emphasize we want two different answers, not the same number listed twice.

Why does "no real solutions" matter for complex numbers? The solution writes $z = a + bi$ where $a, b \in \mathbb{R}$. This is crucial: even though z is complex, the components a and b are *required to be real* by the definition of complex number notation. When $a = -b$, we get $-2a^2 = 1$, which has no solution in $\mathbb{R}$. Since a must be real, this case is impossible.

Geometric view: Squaring a complex number doubles its angle. The angles $\frac{\pi}{4}$ and $\frac{5\pi}{4}$ both double to $\frac{\pi}{2}$ (the angle of i). This is why every nonzero complex number has exactly two square roots.

Exercise 9. Find $x \in \mathbb{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Solution: Isolate $2x$ by subtracting $(4, -3, 1, 7)$ from both sides:

$$2x = (5, 9, -6, 8) - (4, -3, 1, 7) = (5 - 4, 9 - (-3), -6 - 1, 8 - 7) = (1, 12, -7, 1).$$

Divide by 2:

$$x = \frac{1}{2}(1, 12, -7, 1) = \left(\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2}\right).$$

Therefore:

$$\boxed{x = \left(\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2}\right)}$$

Exercise 10. Explain why there does not exist $\lambda \in \mathbb{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Solution: If such a λ exists, then each component equation must hold simultaneously.
From the first component: $\lambda(2 - 3i) = 12 - 5i$.
Solving for λ:

$$\lambda = \frac{12 - 5i}{2 - 3i} = \frac{(12 - 5i)(2 + 3i)}{(2 - 3i)(2 + 3i)} = \frac{24 + 36i - 10i - 15i^2}{4 + 9} = \frac{24 + 26i + 15}{13} = \frac{39 + 26i}{13} = 3+2i.$$

Now check the second component with $\lambda = 3 + 2i$:

$$(3 + 2i)(5 + 4i) = 15 + 12i + 10i + 8i^2 = 15 + 22i - 8 = 7 + 22i. \quad \checkmark$$

Check the third component with $\lambda = 3 + 2i$:

$$(3 + 2i)(-6 + 7i) = -18 + 21i - 12i + 14i^2 = -18 + 9i - 14 = -32 + 9i.$$

But we need $-32 - 9i$, not $-32 + 9i$.
Since the imaginary parts differ ($9i \neq -9i$), there is no single $\lambda \in \mathbb{C}$ that satisfies all three component equations simultaneously. $\square$

Exercise 11. Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{F}^n$.

Solution: Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, and $z = (z_1, \dots, z_n)$ where $x_j, y_j, z_j \in \mathbb{F}$.
Computing the left-hand side:

$$\begin{aligned}(x + y) + z &= (x_1 + y_1, \dots, x_n + y_n) + (z_1, \dots, z_n) \\ &= ((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n).\end{aligned}$$

Computing the right-hand side:

$$\begin{aligned}x + (y + z) &= (x_1, \dots, x_n) + (y_1 + z_1, \dots, y_n + z_n) \\ &= (x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n)).\end{aligned}$$

Since addition in $\mathbb{F}$ is associative, for each j:

$$(x_j + y_j) + z_j = x_j + (y_j + z_j).$$

Therefore $(x + y) + z = x + (y + z)$. □

Exercise 12. Show that $(ab)x = a(bx)$ for all $x \in \mathbb{F}^n$ and all $a, b \in \mathbb{F}$.

Solution: Let $x = (x_1, \dots, x_n)$ where $x_j \in \mathbb{F}$.
Computing the left-hand side:

$$(ab)x = ((ab)x_1, \dots, (ab)x_n).$$

Computing the right-hand side:

$$a(bx) = a(bx_1, \dots, bx_n) = (a(bx_1), \dots, a(bx_n)).$$

Since multiplication in $\mathbb{F}$ is associative, for each j:

$$(ab)x_j = a(bx_j).$$

Therefore $(ab)x = a(bx)$. □

Exercise 13. Show that $1x = x$ for all $x \in \mathbb{F}^n$.

Solution: Let $x = (x_1, \ldots, x_n)$ where $x_j \in \mathbb{F}$.
By the definition of scalar multiplication:

$$1x = (1 \cdot x_1, \ldots, 1 \cdot x_n).$$

Since 1 is the multiplicative identity in $\mathbb{F}$, we have $1 \cdot x_j = x_j$ for each j.
Therefore:

$$1x = (x_1, \ldots, x_n) = x. \quad \square$$

Exercise 14. Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbb{F}$ and all $x, y \in \mathbb{F}^n$.

Solution: Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ where $x_j, y_j \in \mathbb{F}$.
Computing the left-hand side:

$$\begin{aligned} \lambda(x + y) &= \lambda(x_1 + y_1, \ldots, x_n + y_n) \\ &= (\lambda(x_1 + y_1), \ldots, \lambda(x_n + y_n)). \end{aligned}$$

Computing the right-hand side:

$$\begin{aligned} \lambda x + \lambda y &= (\lambda x_1, \ldots, \lambda x_n) + (\lambda y_1, \ldots, \lambda y_n) \\ &= (\lambda x_1 + \lambda y_1, \ldots, \lambda x_n + \lambda y_n). \end{aligned}$$

Since multiplication distributes over addition in $\mathbb{F}$, for each j:

$$\lambda(x_j + y_j) = \lambda x_j + \lambda y_j.$$

Therefore $\lambda(x + y) = \lambda x + \lambda y$. $\square$

Exercise 15. Show that $(a + b)x = ax + bx$ for all $a, b \in \mathbb{F}$ and all $x \in \mathbb{F}^n$.

Solution: Let $x = (x_1, \ldots, x_n)$ where $x_j \in \mathbb{F}$.
Computing the left-hand side:

$$(a + b)x = ((a + b)x_1, \ldots, (a + b)x_n).$$

Computing the right-hand side:

$$\begin{aligned} ax + bx &= (ax_1, \ldots, ax_n) + (bx_1, \ldots, bx_n) \\ &= (ax_1 + bx_1, \ldots, ax_n + bx_n). \end{aligned}$$

Since multiplication distributes over addition in $\mathbb{F}$, for each j:

$$(a + b)x_j = ax_j + bx_j.$$

Therefore $(a+b)x = ax + bx$. $\square$