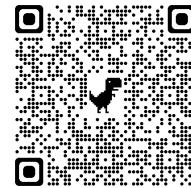


## Upper Division Tutoring Program Topic Review

Facilitators: Cici Wang, [yijin.wang@berkeley.edu](mailto:yijin.wang@berkeley.edu)  
Sambhabi Bose, [sbose812@berkeley.edu](mailto:sbose812@berkeley.edu) Please sign-in!  
Drop-In: Check [Drop-In Schedule](#)



## Math 110 Midterm Exam Review

### Section I: Definitions and Theorems Review

**Definition 1** (Linear Map). Let  $V, W$  be vector spaces over a field  $\mathbb{F}$ . A linear map  $T : V \rightarrow W$  is a map that satisfies the following properties:

- For all  $v, w \in V$ ,  $T(v) + T(w) = T(v + w)$ , and
- for all  $\lambda \in \mathbb{F}$ ,  $\lambda T(v) = T(\lambda v)$ .

**Definition 2** (Isomorphism Between Vector Spaces). An isomorphism between two vector spaces is an invertible linear map. Moreover, two finite-dimensional vector spaces over  $\mathbb{F}$  are isomorphic if and only if they have the same dimension.

**Theorem 1.** A linear map is invertible if and only if it is injective and surjective.

**Theorem 2.** Suppose  $V$  and  $W$  are finite-dimensional. Then  $\mathcal{L}(V, W)$  is finite-dimensional and  $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$ .

**Definition 3.** A linear functional on  $V$  is a linear map from  $V$  to  $\mathbb{F}$ .

**Definition 4** (Dual Space). The dual space of  $V$ , otherwise known as  $V'$ , is the vector space of all linear functionals on  $V$ .

**Definition 5** (Dual Basis). If  $v_1, \dots, v_n$  is a basis of  $V$ , then the dual basis of  $v_1, \dots, v_n$  is a list of linear functionals  $\phi_j \in V'$  such that

$$\phi_j(v_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

**Definition 6** (Eigenvalue). Suppose  $T \in \mathcal{L}(V)$  is an operator. A number  $\lambda \in \mathbb{F}$  is an eigenvalue of  $T$  if there exists  $v \in V$  such that  $Tv = \lambda v$ .

**Theorem 3.** Every operator on a finite-dimensional non-zero complex vector space has an eigenvalue.

**Theorem 4** (Conditions Equivalent to Diagonalizability). Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Let  $\lambda_1, \dots, \lambda_m$  denote the distinct eigenvalues of  $T$ . Then the following are equivalent:

1.  $T$  is diagonalizable.
2.  $V$  has a basis consisting of eigenvectors of  $T$ .
3.  $V = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$ .
4.  $\dim V = \dim E(\lambda_1, T) + \dots + \dim E(\lambda_m, T)$ .

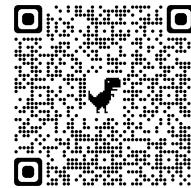
**Theorem 5.** Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$  has  $\dim V$  distinct eigenvalues. Then  $T$  is diagonalizable.

### Section II: Exercises

1. Given a finite-dimensional vector space  $V$  with basis  $v_1, \dots, v_n$ .
  - (a) Prove that if two linear maps  $S$  and  $T$  have  $S(v_1) = T(v_1), \dots, S(v_n) = T(v_n)$  then  $S = T$
  - (b) Prove that for a linear map  $T$ , that  $T(v) = \varphi_1(v)T(v_1) + \dots + \varphi_n(v)T(v_n)$  with  $\varphi_1, \dots, \varphi_n$  the dual basis of  $v_1, \dots, v_n$ .

## Upper Division Tutoring Program Topic Review

Facilitators: Cici Wang, [yijin.wang@berkeley.edu](mailto:yijin.wang@berkeley.edu)  
Sambhabi Bose, [sbose812@berkeley.edu](mailto:sbose812@berkeley.edu) Please sign-in!  
Drop-In: Check [Drop-In Schedule](#)



2. Let  $V$  denote the  $\mathbb{R}$ -vector space spanned by the functions  $\{1, \sin x, \cos x, \sin 2x, \cos 2x\}$ , and let  $d: V \rightarrow V$  denote the differentiation operator. What are the eigenvalues of  $d$ ? Is  $d$  diagonalizable?
3. (a) Define the linear map  $T: \mathbb{C}^2 \rightarrow \mathbb{C}^2$  by  $T(x, y) := (y, -x)$ . What are the eigenvalues of  $T$ ? Find a basis of  $\mathbb{C}^2$  such that the matrix associated to  $T$  is upper-triangular.  
(b) Define the linear map  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by  $T(x, y) := (y, -x)$ . Show that there is no basis of  $\mathbb{R}^2$  such that the matrix associated to  $T$  is upper-triangular.
4. (Spring 2024 Frenkel Mock Midterm Problem 3) Prove that a vector space  $V$  over a field  $\mathbb{F}$  is isomorphic to  $\mathbb{F}^n$  (where  $n$  is a positive integer) if and only if  $\dim(V) = n$ .