19 Suppose $U \subseteq V$. Explain why

$$U^0 = \{\varphi \in V' : U \subseteq \operatorname{null} \varphi\}.$$

SOLUTION By definition,

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}.$$

However, $\varphi(u) = 0$ for all $u \in U$ if and only if $U \in \operatorname{null} \varphi$. Thus

$$U^0 = \{\varphi \in V' : U \subseteq \operatorname{null} \varphi\}.$$

20 Suppose V is finite-dimensional and U is a subspace of V. Show that

$$U = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}.$$

SOLUTION If $u \in U$, then $\varphi(u) = 0$ for every $\varphi \in U^0$. Thus

$$U \subseteq \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}.$$

To prove the inclusion in the other direction, suppose $w \in V$ but $w \notin U$. Let $u_1, \ldots, u_m$ be a basis of U. Because $w \notin U$, the list $u_1, \ldots, u_m, w$ is linearly independent and hence can be extended to a basis of V. Thus by the linear map lemma (3.4), there exists $\psi \in V'$ such that

$$\psi(u_k) = 0 \text{ for } k = 1, \ldots, m \quad \text{and} \quad \psi(w) = 1.$$

Thus $\psi \in U^0$ but $\psi(w) \neq 0$. Hence

$$w \notin \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}.$$

Thus

$$U \supseteq \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\},$$

which implies that

$$U = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\},$$

as desired.

21 Suppose V is finite-dimensional and U and W are subspaces of V.

(a) Prove that $W^0 \subseteq U^0$ if and only if $U \subseteq W$.

(b) Prove that $W^0 = U^0$ if and only if $U = W$.

SOLUTION

(a) First suppose $U \subseteq W$. Suppose $\varphi \in W^0$. If $u \in U$, then $u \in W$, and hence $\varphi(u) = 0$. Thus $\varphi \in U^0$. Hence $W^0 \subseteq U^0$.

To prove the implication in the other direction, now suppose $W^0 \subseteq U^0$. Suppose $u \in U$. Thus $\varphi(u) = 0$ for every $\varphi \in U^0$. Hence $\varphi(u) = 0$ for every $\varphi \in W^0$. Now Exercise 20 implies that $u \in W$. Thus $U \subseteq W$, as desired.

(b) If $U = W$, then clearly $W^0 = U^0$.

To prove the implication in the other direction, now suppose $W^0 = U^0$. Then $W^0 \subseteq U^0$ and $U^0 \subseteq W^0$. Now (a) implies that $U \subseteq W$ and $W \subseteq U$. Thus $U = W$, as desired.

22 Suppose V is finite-dimensional and U and W are subspaces of V.

(a) Show that $(U+W)^0 = U^0 \cap W^0$.

(b) Show that $(U \cap W)^0 = U^0 + W^0$.

SOLUTION

(a) First suppose $\varphi \in (U+W)^0$. Then $\varphi(u+w) = 0$ for all $u \in U$ and all $w \in W$. Taking $w = 0$ and then taking $u = 0$, in particular we see that $\varphi(u) = 0$ for all $u \in U$ and $\varphi(w) = 0$ for all $w \in W$. Thus $\varphi \in U^0$ and $\varphi \in W^0$. In other words, $\varphi \in U^0 \cap W^0$. Thus $(U+W)^0 \subseteq U^0 \cap W^0$.

To prove the inclusion in the other direction, suppose $\varphi \in U^0 \cap W^0$. If $u \in U$ and $w \in W$, then

$$\varphi(u+w) = \varphi(u) + \varphi(w) = 0 + 0 = 0.$$

Hence $\varphi \in (U+W)^0$. Thus $(U+W)^0 \supseteq U^0 \cap W^0$.

Thus $(U+W)^0 = U^0 \cap W^0$, as desired.

(b) First suppose $\varphi \in U^0 + W^0$. Thus $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in U^0$ and $\varphi_2 \in W^0$. If $v \in U \cap W$, then

$$\varphi(v) = \varphi_1(v) + \varphi_2(v) = 0 + 0 = 0.$$

Hence $\varphi \in (U \cap W)^0$. Thus

$$U^0 + W^0 \subseteq (U \cap W)^0.$$

To show that the inclusion above is an equality, we will show that both sides have the same dimension. We have

$$\begin{aligned}
\dim(U \cap W)^0 &= \dim V - \dim(U \cap W) \\
&= \dim V - (\dim U + \dim W - \dim(U+W)) \\
&= \dim U^0 + \dim W^0 - \dim(U+W)^0 \\
&= \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\
&= \dim(U^0 + W^0),
\end{aligned}$$

where the first equality comes from 3.125, the second equality comes from 2.43, the third equality comes from 3.125, the fourth equality comes from Exercise 22(a), and the fifth equality comes from 2.43.

The equality of dimensions along with the inclusion above show that

$$(U \cap W)^0 = U^0 + W^0,$$

as desired.

23 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_m \in V'$. Prove that the following three sets are equal to each other.

(a) $\operatorname{span}(\varphi_1, \dots, \varphi_m)$

(b) $\bigl((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\bigr)^0$

(c) $\{\varphi \in V' : (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) \subseteq \operatorname{null}\varphi\}$

SOLUTION Let A denote the set in (a), B denote the set in (b), and C denote the set in (c).

We will prove that $A = B$ by induction on m. For $m = 1$, we need to show that

$$\operatorname{span}(\varphi_1) = (\operatorname{null}\varphi_1)^0.$$

The inclusion $\operatorname{span}(\varphi_1) \subseteq (\operatorname{null}\varphi_1)^0$ follows from the definitions. To prove the inclusion in the other direction, suppose $\psi \in (\operatorname{null}\varphi_1)^0$. Thus $\operatorname{null}\varphi_1 \subseteq \operatorname{null}\psi$. Now Exercise 6 implies $\psi \in \operatorname{span}(\varphi_1)$, as desired.

Now suppose that $m > 1$ and that the desired result holds when m is replaced with $m - 1$. Then

$$\begin{aligned}\operatorname{span}(\varphi_1, \dots, \varphi_m) &= \operatorname{span}(\varphi_1, \dots, \varphi_{m-1}) + \operatorname{span}(\varphi_m)\\ &= \bigl((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_{m-1})\bigr)^0 + (\operatorname{null}\varphi_m)^0\\ &= \bigl((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\bigr)^0,\end{aligned}$$

where the second equality above comes from our induction hypothesis and the third equality above comes from Exercise 22(b). The last equality above completes the proof that $A = B$.

Now suppose $\varphi \in A$. Thus there exist $a_1, \dots, a_m$ such that

$$\varphi = a_1\varphi_1 + \cdots + a_m\varphi_m.$$

The equation above shows that if $v \in (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$, then $\varphi(v) = 0$ and hence $v \in \operatorname{null}\varphi$. Thus $\varphi \in C$, proving that $A \subseteq C$.

Now suppose $\varphi \in C$. Thus

$$(\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) \subseteq \operatorname{null}\varphi. \tag{$*$}$$

To show that $\varphi \in B$, suppose $v \in (\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)$. Thus $(*)$ shows that $v \in \operatorname{null} \varphi$. Hence $\varphi(v) = 0$, which shows that $\varphi \in B$. Thus $C \subseteq B$, completing the proof.

24 Suppose V is finite-dimensional and $v_1, \ldots, v_m \in V$. Define a linear map $\Gamma\colon V' \to \mathbf{F}^m$ by $\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m))$.

(a) Prove that $v_1, \ldots, v_m$ spans V if and only if Γ is injective.

(b) Prove that $v_1, \ldots, v_m$ is linearly independent if and only if Γ is surjective.

SOLUTION

(a) First suppose $v_1, \ldots, v_m$ spans V. If $\varphi \in V'$ and $\Gamma(\varphi) = 0$, then

$$\varphi(v_1) = \cdots = \varphi(v_m) = 0,$$

which implies that $\varphi(v) = 0$ for all $v \in V$, which implies that $\varphi = 0$. Thus Γ is injective.

To prove the implication in the other direction, we will prove its contrapositive. Thus suppose $v_1, \ldots, v_m$ does not span V. Thus there exists $\varphi \in V'$ such that φ equals 0 on $\operatorname{span}(v_1, \ldots, v_m)$ but $\varphi \neq 0$. Thus $\Gamma(\varphi) = 0$, which implies that Γ is not injective, as desired.

(b) First suppose $v_1, \ldots, v_m$ is linearly independent. Then the list $v_1, \ldots, v_m$ can be extended to a basis of V. Hence the linear map lemma (3.4) implies that $\varphi(v_1), \ldots, \varphi(v_m)$ can take on any values we choose. Thus Γ is surjective.

Conversely, suppose Γ is surjective. Thus it is not possible to write any v_k as a linear combination of $v_1, \ldots, v_{k-1}$ because doing so would contradict the existence of $\varphi \in V'$ such that

$$\varphi(v_1) = \cdots = \varphi(v_{k-1}) = 0 \quad \text{but} \quad \varphi(v_k) = 1.$$

The linear dependence lemma (2.19) now implies $v_1, \ldots, v_m$ is linearly independent, as desired.

25 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_m \in V'$. Define a linear map $\Gamma\colon V \to \mathbf{F}^m$ by $\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v))$.

(a) Prove that $\varphi_1, \dots, \varphi_m$ spans V' if and only if Γ is injective.

(b) Prove that $\varphi_1, \dots, \varphi_m$ is linearly independent if and only if Γ is surjective.

SOLUTION

(a) First suppose $\varphi_1, \dots, \varphi_m$ spans V'. If $v \in V$ and $\Gamma(v) = 0$, then

$$\varphi_1(v) = \cdots = \varphi_m(v) = 0,$$

which implies that $\varphi(v) = 0$ for all $\varphi \in V'$, which implies that $v = 0$. Thus Γ is injective.

Conversely, suppose that Γ is injective. Thus

$$(\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m) = \{0\},$$

which implies that

$$\big((\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)\big)^0 = V'.$$

Now Exercise 23 implies that $\varphi_1, \dots, \varphi_m$ spans V'.

(b) First we will prove that if $\varphi_1, \dots, \varphi_m$ is linearly independent, then Γ is surjective. This will be done by induction on m.

Consider first the case $m = 1$. The statement that the list φ_1 is linearly independent implies $\varphi_1 \neq 0$, which implies that $\operatorname{range} \Gamma = \mathbf{F}^1$, verifying the desired statement when $m = 1$.

Thus assume that $m > 1$ and that the desired statement holds for $m - 1$. Suppose $\varphi_1, \dots, \varphi_m$ is linearly independent. Hence $\varphi_1, \dots, \varphi_{m-1}$ is linearly independent. By our induction hypothesis,

$$\Big\{(\varphi_1(v), \dots, \varphi_{m-1}(v)) : v \in V\Big\} = \mathbf{F}^{m-1}. \tag{$*$}$$

Because $\varphi_1, \dots, \varphi_m$ is linearly independent, $\varphi_m \notin \operatorname{span}(\varphi_1, \dots, \varphi_{m-1})$. Thus Exercise 23 implies that

$$(\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}) \not\subseteq \operatorname{null} \varphi_m.$$

Hence there exists $v \in V$ such that $\varphi_1(v) = \cdots = \varphi_{m-1}(v) = 0$ but $\varphi_m(v) \neq 0$.

If $(c_1, \ldots, c_m) \in \mathbf{F}^m$, then by $(*)$ there exists $u \in V$ such that $\varphi_k(u) = c_k$ for each $k = 1, \ldots, m-1$. Thus

$$\varphi_k\left(u + \frac{c_m - \varphi_m(u)}{\varphi_m(v)} v\right) = c_k$$

for every $k = 1, \ldots, m$. Hence Γ is surjective, as desired.

To prove the implication in the other direction, we will prove its contrapositive. Thus suppose $\varphi_1, \ldots, \varphi_m$ is linearly dependent. Hence there exists $n \in \{1, \ldots, m\}$ such that φ_n is a linear combination of $\varphi_1, \ldots, \varphi_{n-1}$. This means that each vector in $\mathbf{F}^m$ whose first $n-1$ coordinates equal 0 and whose n^{th} coordinate equals 1 is not in the range of Γ. Hence Γ is not surjective, as desired.

26 Suppose V is finite-dimensional and Ω is a subspace of V'. Prove that

$$\Omega = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in \Omega\}^0.$$

SOLUTION Let $\varphi_1, \dots, \varphi_m$ be a basis of Ω. Thus

$$\begin{aligned}
\Omega &= \operatorname{span}(\varphi_1, \dots, \varphi_m) \\
&= \big((\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)\big)^0 \\
&= \{v \in V : \varphi_k(v) = 0 \text{ for } k = 1, \dots, m\}^0 \\
&= \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in \Omega\}^0,
\end{aligned}$$

where the second line follows from Exercise 23.

27 Suppose $T \in \mathcal{L}(\mathcal{P}_5(\mathbf{R}))$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$, where φ is the linear functional on $\mathcal{P}_5(\mathbf{R})$ defined by $\varphi(p) = p(8)$. Prove that

$$\operatorname{range} T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}.$$

SOLUTION We have $\dim \operatorname{null} T' = 1$, which by the fundamental theorem of linear maps (3.21) implies that

$$\dim \operatorname{range} T' = \dim(\mathcal{P}_5(\mathbf{R}))' - 1.$$

Using 3.130(a) and 3.111 and then applying the fundamental theorem of linear maps to φ, we can rewrite the equation above as

$$\begin{aligned} \dim \operatorname{range} T &= \dim \mathcal{P}_5(\mathbf{R}) - 1 \\ &= \dim \operatorname{null} \varphi. \end{aligned} \tag{$*$}$$

Because $\varphi \in \operatorname{null} T'$, we have $0 = T'(\varphi) = \varphi \circ T$. Thus

$$\operatorname{range} T \subseteq \operatorname{null} \varphi = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}.$$

But these subspaces of $\mathcal{P}_5(\mathbf{R})$ have the same dimension by $(*)$, and hence they are equal, as desired.

28 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_m$ is a linearly independent list in V'. Prove that

$$\dim\bigl((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\bigr) = (\dim V) - m.$$

SOLUTION We have

$$\begin{aligned}
&\dim\bigl((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\bigr) \\
&\qquad = \dim V - \dim\bigl((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\bigr)^0 \\
&\qquad = \dim V - \dim \operatorname{span}(\varphi_1, \dots, \varphi_m) \\
&\qquad = \dim V - m,
\end{aligned}$$

where the first equality comes from 3.125, the second equality comes from Exercise 23, and the third equality holds because $\varphi_1, \dots, \varphi_m$ is linearly independent.

29 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$.

(a) Prove that if $\varphi \in W'$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$, then $\operatorname{range} T = \operatorname{null} \varphi$.

(b) Prove that if $\psi \in V'$ and $\operatorname{range} T' = \operatorname{span}(\psi)$, then $\operatorname{null} T = \operatorname{null} \psi$.

SOLUTION

(a) Suppose $\varphi \in W'$ and $\operatorname{null} T'$.

First consider the case $\varphi = 0$. In this case, $\operatorname{null} T' = \{0\}$. Thus T' is injective. Thus T is surjective (by 3.129), which means that $\operatorname{range} T = W$. Therefore $\operatorname{range} T = \operatorname{null} \varphi$ because both sides of this equation equal W in this case where $\varphi = 0$.

Now consider the case $\varphi \neq 0$. Thus $\dim \operatorname{null} T' = 1$, which by the fundamental theorem of linear maps (3.21) implies that

$$\dim \operatorname{range} T' = \dim W' - 1.$$

Using 3.130(a) and 3.111 and then applying the fundamental theorem of linear maps to φ, we can rewrite the equation above as

$$\begin{aligned} \dim \operatorname{range} T &= \dim W - 1 \\ &= \dim \operatorname{null} \varphi. \end{aligned}$$

Because $\varphi \in \operatorname{null} T'$, we have $0 = T'(\varphi) = \varphi \circ T$. Thus $\operatorname{range} T \subseteq \operatorname{null} \varphi$. But these two subspaces of W have the same dimension by the displayed equation above, and hence they are equal, as desired.

(b) Suppose $\psi \in V'$ and $\operatorname{range} T'$.

First consider the case in which $\psi = 0$. In this case, $\operatorname{range} T' = \{0\}$. Thus $\dim \operatorname{range} T' = 0$, which implies that $\dim \operatorname{range} T = 0$ [by 3.130(a)]. Thus $T = 0$, which means that $\operatorname{null} T = V$. Therefore $\operatorname{null} T = \operatorname{null} \psi$ because both sides of this equation equal V in this case where $\psi = 0$.

Now consider the case $\psi \neq 0$. Thus $\dim \operatorname{range} T' = 1$, which implies that $\dim \operatorname{range} T = 1$ [by 3.130(a)], which by the fundamental theorem of linear maps (3.21) implies that

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - 1 \\ &= \dim \operatorname{null} \psi. \end{aligned}$$

Because $\varphi \in \operatorname{range} T'$, we have $\varphi = T'(\psi)$ for some $\psi \in W'$. Thus $\varphi = \psi \circ T$, which implies that $\operatorname{null} T \subseteq \operatorname{null} \varphi$. But these two subspaces of V have the same dimension by the displayed equation above, and hence they are equal, as desired.

30 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_n$ is a basis of V'. Show that there exists a basis of V whose dual basis is $\varphi_1, \dots, \varphi_n$.

SOLUTION From Exercise 28, we know that

$$\dim\bigl((\operatorname{null}\varphi_2) \cap \cdots \cap (\operatorname{null}\varphi_n)\bigr) = 1$$

and

$$\dim\bigl((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_n)\bigr) = 0.$$

Thus

$$(\operatorname{null}\varphi_2) \cap \cdots \cap (\operatorname{null}\varphi_n) \not\subset (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_n).$$

In other words, there exists $x_1 \in V$ such that $x_1 \in (\operatorname{null}\varphi_2) \cap \cdots \cap (\operatorname{null}\varphi_n)$ but $\varphi_1(x_1) \neq 0$. Multiplying x_1 by an appropriate scalar, we can assume that $\varphi_1(x_1) = 1$.

Similarly, for each $j = 2, \dots, n$, there exists $x_j \in V$ such that

$$(*) \qquad \varphi_k(x_j) = 0 \text{ for } k \in \{1, \dots, n\} \backslash j \quad \text{and} \quad \varphi_j(x_j) = 1.$$

If $a_1, \dots, a_n \in \mathbf{F}$ and

$$0 = a_1 x_1 + \cdots + a_n x_n,$$

then for each $j = 1, \dots, n$ we have

$$0 = \varphi_j(a_1 x_1 + \cdots + a_n x_n) = a_j.$$

Hence $x_1, \dots, x_n$ is linearly independent and thus is a basis of V (by 2.38). Clearly $(*)$ shows that the dual basis of $x_1, \dots, x_n$ is $\varphi_1, \dots, \varphi_n$.

31 Suppose U is a subspace of V. Let $i: U \to V$ be the inclusion map defined by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.

(a) Show that $\operatorname{null} i' = U^0$.

(b) Prove that if V is finite-dimensional, then $\operatorname{range} i' = U'$.

(c) Prove that if V is finite-dimensional, then $\tilde{i'}$ is an isomorphism from V'/U^0 onto U'.

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.

SOLUTION

(a) Suppose $\varphi \in V'$. Then

$$\begin{aligned} \varphi \in \operatorname{null} i' &\iff i'(\varphi) = 0 \\ &\iff \varphi \circ i = 0 \\ &\iff \varphi(u) = 0 \text{ for all } u \in U \\ &\iff \varphi \in U^0. \end{aligned}$$

Thus $\operatorname{null} i' = U^0$.

(b) Suppose V is finite-dimensional. Clearly i is injective. Thus by 3.131, T' is surjective. Hence $\operatorname{range} i' = U'$.

(c) Suppose V is finite-dimensional. By 3.107(b) and 3.107(b), $\tilde{i'}$ is an injective map from $V'/\operatorname{null} i'$ onto $\operatorname{range} i'$. Thus by (a) and (b) of this exercise, $\tilde{i'}$ is an isomorphism from V'/U^0 onto U'.

32 The *double dual space* of V, denoted by V'', is defined to be the dual space of V'. In other words, $V'' = (V')'$. Define $\Lambda\colon V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for each $v \in V$ and each $\varphi \in V'$.

(a) Show that Λ is a linear map from V to V''.

(b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.

(c) Show that if V is finite-dimensional, then Λ is an isomorphism from V onto V''.

Suppose V is finite-dimensional. Then V and V' are isomorphic, but finding an isomorphism from V onto V' generally requires choosing a basis of V. In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

SOLUTION

(a) A straightforward application of the definitions shows that Λ is a linear map from V to V''.

(b) Suppose $T \in \mathcal{L}(V)$. Suppose $v \in V$ and $\varphi \in V'$. Then

$$\begin{aligned}
((T'' \circ \Lambda)(v))(\varphi) &= (T''(\Lambda v))(\varphi) \\
&= (\Lambda v \circ T')(\varphi) \\
&= (\Lambda v)(T'(\varphi)) \\
&= (\Lambda v)(\varphi \circ T) \\
&= (\varphi \circ T)(v) \\
&= (\varphi \circ T)(v) \\
&= \varphi(Tv).
\end{aligned}$$

Also,

$$\begin{aligned}
((\Lambda \circ T)(v))(\varphi) &= (\Lambda(Tv))(\varphi) \\
&= \varphi(Tv).
\end{aligned}$$

Thus

$$\big((T'' \circ \Lambda)(v)\big)(\varphi) = \big((\Lambda \circ T)(v)\big)(\varphi),$$

for all $\varphi \in V'$, which implies that

$$(T'' \circ \Lambda)(v) = (\Lambda \circ T)(v),$$

for all $v \in V$, which implies that $T'' \circ \Lambda = \Lambda \circ T$, as desired.

(c) Suppose V is finite-dimensional. Suppose $v \in V$ and $\Lambda v = 0$. Thus $\varphi(v) = 0$ for every $\varphi \in V'$. Now Exercise 3 implies that $v = 0$. Thus Λ is injective.

Because $\dim V = \dim V' = \dim V''$ (by 3.111), we can conclude that Λ is an isomorphism of V onto V''.

33 Suppose U is a subspace of V. Let $\pi\colon V \to V/U$ be the usual quotient map. Thus $\pi' \in \mathcal{L}((V/U)', V')$.

(a) Show that π' is injective.

(b) Show that range $\pi' = U^0$.

(c) Conclude that π' is an isomorphism from $(V/U)'$ onto U^0.

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space. In fact, there is no assumption here that any of these vector spaces are finite-dimensional.

SOLUTION

(a) Suppose $\varphi \in (V/U)'$ and $\pi'(\varphi) = 0$. Then $\varphi \circ \pi = 0$, which means that $(\varphi \circ \pi)(v) = 0$ for every $v \in V$, which means that $\varphi(v + U) = 0$ for every $v \in V$. Thus $\varphi = 0$, which implies that π' is injective.

(b) Suppose $\varphi \in (V/U)'$. If $u \in U$, then

$$(\pi'(\varphi))(u) = (\varphi \circ \pi)(u) = \varphi(u + U) = \varphi(0) = 0$$

and thus $\pi'(\varphi) \in U^0$. Hence range $\pi' \subseteq U^0$.

To show the inclusion in the other direction, suppose $\psi \in U^0$. Thus $\psi \in V'$ and $\psi(u) = 0$ for all $u \in U$. Define $\varphi \in (V/U)'$ by

$$\varphi(v + U) = \psi(v);$$

the condition that $\psi(u) = 0$ for all $u \in U$ shows that φ is well defined. The definitions now show that $\pi'(\varphi) = \psi$. Thus range $\pi' \supseteq U^0$, completing the proof that range $\pi' = U^0$.

(c) Now (a) and (b) immediately imply that π' is an isomorphism from $(V/U)'$ onto U^0.

1 Suppose $w, z \in \mathbf{C}$. Verify the following equalities and inequalities.

(a) $z + \bar{z} = 2 \operatorname{Re} z$

(b) $z - \bar{z} = 2(\operatorname{Im} z)i$

(c) $z\bar{z} = |z|^2$

(d) $\overline{w + z} = \bar{w} + \bar{z}$ and $\overline{wz} = \bar{w}\,\bar{z}$

(e) $\bar{\bar{z}} = z$

(f) $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$

(g) $|\bar{z}| = |z|$

(h) $|wz| = |w|\,|z|$

The results above are the parts of 4.4 that were left to the reader.

SOLUTION Let $w = a + bi$ and $z = c + di$, where $a, b, c, d \in \mathbf{R}$. All of the assertions in 4.4 except the last one can now be verified by simple computations.

2 Prove that if $w, z \in \mathbf{C}$, then $\big||w| - |z|\big| \leq |w - z|$.

*The inequality above is called the **reverse triangle inequality**.*

SOLUTION Suppose $w, z \in V$. Then

$$|w| = |(w - z) + z| \leq |w - z| + |z|.$$

Thus

$$|w| - |z| \leq |w - z|. \tag{$*$}$$

Interchanging the roles of w and z, the inequality above becomes

$$|z| - |w| \leq |z - w| = |w - z|. \tag{$**$}$$

Now $(*)$ and $(**)$ imply that

$$\big||w| - |z|\big| \leq |w - z|.$$

3 Suppose V is a complex vector space and $\varphi \in V'$. Define $\sigma\colon V \to \mathbf{R}$ by $\sigma(v) = \operatorname{Re}\varphi(v)$ for each $v \in V$. Show that

$$\varphi(v) = \sigma(v) - i\sigma(iv)$$

for all $v \in V$.

SOLUTION If $v \in V$, then

$$\begin{aligned} \operatorname{Im}\varphi(v) &= -\operatorname{Re}\bigl(i\varphi(v)\bigr) \\ &= -\operatorname{Re}\varphi(iv) \\ &= -\sigma(iv). \end{aligned}$$

Thus $\varphi(v) = \sigma(v) - i\sigma(iv)$.

4 Suppose m is a positive integer. Is the set

$$\{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$$

a subspace of $\mathcal{P}(\mathbf{F})$?

SOLUTION The set consisting of 0 and all polynomials with coefficients in $\mathbf{F}$ and of degree m is not a subspace of $\mathcal{P}(\mathbf{F})$ because it is not closed under addition. Specifically, the sum of two polynomials of degree m may be a polynomial of degree less than m. For example, suppose $m = 2$. Then $7 + 4z + 5z^2$ and $1 + 2z - 5z^2$ are both polynomials of degree 2 but their sum, which equals $8 + 6z$, is a polynomial of degree 1.

5 Is the set

$$\{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p \text{ is even}\}$$

a subspace of $\mathcal{P}(\mathbf{F})$?

SOLUTION The set consisting of 0 and all polynomials of even degree is not a subspace of $\mathcal{P}(\mathbf{F})$ because it is not closed under addition. Specifically, the sum of two polynomials of even degree may be a polynomial of odd degree. For example, $7 + 4z + 5z^2$ and $1 + 2z - 5z^2$ are both polynomials of degree 2 but their sum, which equals $8 + 6z$, is a polynomial of odd degree.

6 Suppose that m and n are positive integers with $m \leq n$, and suppose $\lambda_1, \ldots, \lambda_m \in \mathbf{F}$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{F})$ with $\deg p = n$ such that $0 = p(\lambda_1) = \cdots = p(\lambda_m)$ and such that p has no other zeros.

SOLUTION Define $p \in \mathcal{P}(\mathbf{F})$ by

$$p(z) = (z - \lambda_1)^{n-m+1}(z - \lambda_2)(z - \lambda_3)\cdots(z - \lambda_m).$$

Then p is a polynomial of degree n and exactly the required zeros.

7 Suppose that m is a nonnegative integer, $z_1, \dots, z_{m+1}$ are distinct elements of $\mathbf{F}$, and $w_1, \dots, w_{m+1} \in \mathbf{F}$. Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbf{F})$ such that

$$p(z_k) = w_k$$

for each $k = 1, \dots, m+1$.

This result can be proved without using linear algebra. However, try to find the clearer, shorter proof that uses some linear algebra.

SOLUTION Define $T : \mathcal{P}_m(\mathbf{F}) \to \mathbf{F}^{m+1}$ by

$$Tp = \big(p(z_1), \dots, p(z_{m+1})\big).$$

We need to prove that T is injective (which implies that at most one polynomial p satisfies the condition required by the exercise) and surjective (which implies that at least one polynomial p satisfies the condition required by the exercise).

Clearly T is a linear map. If $p \in \operatorname{null} T$, then

$$p(z_1) = \cdots = p(z_{m+1}) = 0,$$

which means that p is a polynomial of degree m with at least $m+1$ distinct zeros, which means that $p = 0$ (by 4.8). Thus T is injective, as desired.

Now

$$\begin{aligned} \dim \operatorname{range} T &= \dim \mathcal{P}_m(\mathbf{F}) - \dim \operatorname{null} T \\ &= (m+1) - 0 \\ &= \dim \mathbf{F}^{m+1}, \end{aligned}$$

where the first equality comes from the fundamental theorem of linear maps (3.21) and the second equality holds because $\operatorname{null} T = \{0\}$. The last equality above implies that $\operatorname{range} T = \mathbf{F}^{m+1}$. Thus T is surjective, as desired.

COMMENT Surjectivity of T can also be proved by using an explicit construction. But linear algebra, specifically the fundamental theorem of linear maps, gives us surjectivity easily once we get injectivity.

8 Suppose $p \in \mathcal{P}(\mathbf{C})$ has degree m. Prove that p has m distinct zeros if and only if p and its derivative p' have no zeros in common.

SOLUTION First suppose p has m distinct zeros. Because p has degree m, this implies that p can be written in the form

$$p(z) = c(z - \lambda_1)\cdots(z - \lambda_m),$$

where $\lambda_1, \ldots, \lambda_m$ are distinct. To prove that p and p' have no zeros in common, we must show that $p'(\lambda_k) \neq 0$ for each k. To do this, fix k. The expression above for p shows that we can write p in the form

$$p(z) = (z - \lambda_k)q(z),$$

where q is a polynomial such that $q(\lambda_k) \neq 0$. Differentiating both sides of this equation, we have

$$p'(z) = (z - \lambda_k)q'(z) + q(z).$$

Thus

$$\begin{aligned} p'(\lambda_k) &= q(\lambda_k) \\ &\neq 0, \end{aligned}$$

as desired.

To prove the other direction, we will prove the contrapositive, meaning that we will prove that if p has less than m distinct zeros, then p and p' have at least one zero in common. To do this, suppose p has less than m distinct zeros. Then for some zero λ of p, we can write p in the form

$$p(z) = (z - \lambda)^n q(z),$$

where $n \geq 2$ and q is a polynomial. Differentiating both sides of this equation, we have

$$p'(z) = (z - \lambda)^n q'(z) + n(z - \lambda)^{n-1} q(z).$$

Thus $p'(\lambda) = 0$, and so λ is a common zero of p and p', as desired.

9 Prove that every polynomial of odd degree with real coefficients has a real zero.

SOLUTION Suppose p is a polynomial of odd degree and real coefficients. By 4.16, p is a constant times the product of factors of the form $x - \lambda$ and/or $x^2 + bx + c$, where $\lambda, b, c \in \mathbf{R}$. Not all the factors can be of the form $x^2 + bx + c$, because otherwise p would have even degree. Thus at least one factor is of the form $x - \lambda$. Any such λ is a real zero of p.

COMMENT Here is another proof, using calculus but not using 4.16. Suppose p is a polynomial of odd degree m. We can write p in the form

$$p(x) = a_0 + a_1 x + \cdots + a_m x^m,$$

where $a_0, \ldots, a_m \in \mathbf{R}$ and $a_m \neq 0$. Replacing p with $-p$ if necessary, we can assume that $a_m > 0$. Now

$$p(x) = x^m \Big(\frac{a_0}{x^m} + \frac{a_1}{x^{m-1}} + \cdots + \frac{a_{m-1}}{x} + a_m \Big).$$

This implies that

$$\lim_{x \to -\infty} p(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} p(x) = \infty.$$

The intermediate value theorem now implies that there is a real number λ such that $p(\lambda) = 0$. In other words, p has a real zero.

10 For $p \in \mathcal{P}(\mathbf{R})$, define $Tp\colon \mathbf{R} \to \mathbf{R}$ by

$$(Tp)(x) = \begin{cases} \dfrac{p(x) - p(3)}{x - 3} & \text{if } x \neq 3, \\ p'(3) & \text{if } x = 3 \end{cases}$$

for each $x \in \mathbf{R}$. Show that $Tp \in \mathcal{P}(\mathbf{R})$ for every polynomial $p \in \mathcal{P}(\mathbf{R})$ and also show that $T\colon \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is a linear map.

SOLUTION Suppose $p \in \mathcal{P}(\mathbf{R})$. Clearly 3 is a zero of the polynomial $p - p(3)$. Thus by 4.6, there exists a polynomial $q \in \mathcal{P}(\mathbf{R})$ such that

$$p(x) - p(3) = (x - 3)q(x)$$

for every $x \in \mathbf{R}$. The definition of the derivative and the continuity of q imply that $p'(3) = q(3)$. Thus $Tp = q$. In particular, $Tp \in \mathcal{P}(\mathbf{R})$.

The linearity of T follows easily from the definition of a linear map and the linearity of the derivative: $(p + r)'(3) = p'(3) + r'(3)$ and $(cp)'(3) = cp'(3)$.

11 Suppose $p \in \mathcal{P}(\mathbf{C})$. Define $q\colon \mathbf{C} \to \mathbf{C}$ by

$$q(z) = p(z)\,\overline{p(\overline{z})}.$$

Prove that q is a polynomial with real coefficients.

SOLUTION There exist $a_0, \dots, a_n \in \mathbf{C}$ such that

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

for every $z \in \mathbf{C}$. Thus

$$\overline{p(\overline{z})} = \overline{a_0} + \overline{a_1} z + \cdots + \overline{a_n} z^n$$

for every $z \in \mathbf{C}$. Thus the function that takes z to $\overline{p(\overline{z})}$ is a polynomial, and hence $q \in \mathcal{P}(\mathbf{C})$.

There exist numbers $b_0, \dots, b_{2n} \in \mathbf{C}$ such that

$$q(x) = b_0 + b_1 x + \cdots + b_{2n} x^{2n}$$

for every $x \in \mathbf{R}$. Because

$$q(x) = p(x)\,\overline{p(\overline{x})} = p(x)\,\overline{p(x)} = |p(x)|^2$$

for every $x \in \mathbf{R}$, we see that $q(x) \in \mathbf{R}$ for every $x \in \mathbf{R}$. In other words, $\operatorname{Im} p(x) = 0$ for every $x \in \mathbf{R}$. Thus

$$(\operatorname{Im} b_0) + (\operatorname{Im} b_1)x + \cdots + (\operatorname{Im} b_{2n})x^{2n} = 0$$

for every $x \in \mathbf{R}$. Now 4.8 implies that $\operatorname{Im} b_0 = \operatorname{Im} b_1 = \cdots = \operatorname{Im} b_{2n} = 0$. Thus q has real coefficients.

12 Suppose m is a nonnegative integer and $p \in \mathcal{P}_m(\mathbf{C})$ is such that there are distinct real numbers $x_0, x_1, \ldots, x_m$ with $p(x_k) \in \mathbf{R}$ for each $k = 0, 1, \ldots, m$. Prove that all coefficients of p are real.

SOLUTION There exist $a_0, \ldots, a_m \in \mathbf{C}$ such that

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m$$

for every $z \in \mathbf{C}$. Thus

$$\operatorname{Im} p(x) = (\operatorname{Im} a_0) + (\operatorname{Im} a_1)x + \cdots + (\operatorname{Im} a_m)x^m$$

for every $x \in \mathbf{R}$. Our hypothesis states that there are at least $m + 1$ distinct real zeros of the polynomial $\operatorname{Im} p$. Thus 4.8 and the equation above imply that

$$\operatorname{Im} a_0 = \operatorname{Im} a_1 = \cdots = \operatorname{Im} a_m = 0.$$

Thus all coefficients of p are real.

13 Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.

(a) Show that $\dim \mathcal{P}(\mathbf{F})/U = \deg p$.

(b) Find a basis of $\mathcal{P}(\mathbf{F})/U$.

SOLUTION Let $a_0, \ldots, a_n \in \mathbf{F}$ be such that $a_n \neq 0$ and

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

for every $z \in \mathbf{F}$. Thus $\deg p = n$.

We will show that

$$1 + U, z + U, \ldots, z^{n-1} + U$$

is a basis of $\mathcal{P}(\mathbf{F})/U$. This provides solutions to both parts of the exercise.

To show that the list

$$1 + U, z + U, \ldots, z^{n-1} + U$$

is linearly independent in $\mathcal{P}(\mathbf{F})/U$, suppose $b_0, b_1, \ldots, b_n \in \mathbf{F}$ and

$$b_0(1 + U) + b_1(z + U) + \cdots + b_{n-1}(z^{n-1} + U) = 0 + U.$$

Thus

$$b_0 + b_1 z + \cdots + b_{n-1} z^{n-1} \in U.$$

However, every element of U is a polynomial of degree bigger than or equal to n or is the 0 polynomial. Thus $b_0 + b_1 z + \cdots + b_{n-1} z^{n-1}$ is the zero polynomial, which implies that $b_0 = b_1 = \cdots = b_n = 0$. Thus $1 + U, z + U, \ldots, z^{n-1} + U$ is linearly independent.

To show that the list $1 + U, z + U, \ldots, z^{n-1} + U$ spans $\mathcal{P}(\mathbf{F})/U$, it suffices to show that

$$z^m + U \in \operatorname{span}(1 + U, z + U, \ldots, z^{n-1} + U)$$

for every nonnegative integer m. Clearly the inclusion above holds for each $m = 0, 1, \ldots, n-1$, which gets us started for induction. We will assume that $m \geq n$ and that the desired inclusion holds for all smaller values of m.

Note that

$$z^m - z^{m-n}\frac{q}{a_n}$$

is a polynomial of degree less than m. Thus by our induction hypothesis

$$z^m - z^{m-n}\frac{q}{a_n} + U \in \operatorname{span}(1 + U, z + U, \ldots, z^{n-1} + U).$$

However, $z^{m-n}\frac{q}{a_n} \in U$, and thus the inclusion above can be rewritten as

$$z^m + U \in \operatorname{span}(1 + U, z + U, \ldots, z^{n-1} + U),$$

as desired.

14 Suppose $p, q \in \mathcal{P}(\mathbf{C})$ are nonconstant polynomials with no zeros in common. Let $m = \deg p$ and $n = \deg q$. Use linear algebra as outlined below in (a)–(c) to prove that there exist $r \in \mathcal{P}_{n-1}(\mathbf{C})$ and $s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that

$$rp + sq = 1.$$

(a) Define $T\colon \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C}) \to \mathcal{P}_{m+n-1}(\mathbf{C})$ by

$$T(r, s) = rp + sq.$$

Show that the linear map T is injective.

(b) Show that the linear map T in (a) is surjective.

(c) Use (b) to conclude that there exist $r \in \mathcal{P}_{n-1}(\mathbf{C})$ and $s \in \mathcal{P}_{m-1}(\mathbf{C})$ such that $rp + sq = 1$.

SOLUTION

(a) Clearly T is a linear map from $\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})$ to $\mathcal{P}_{m+n-1}(\mathbf{C})$. To show that T is injective, suppose $(r, s) \in \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})$ and $T(r, s) = 0$. Then

$$rp = -sq.$$

Suppose $\lambda \in \mathbf{C}$ and $q(\lambda) = 0$. Because p and q have no zeros in common, the equation above implies that $r(\lambda) = 0$. Thus r is a polynomial multiple of $z - \lambda$. More generally, the equation above implies that if q is a polynomial multiple of $(z - \lambda)^k$ for some positive integer k, then r is a polynomial multiple of $(z - \lambda)^k$.

The factorization 4.13 now implies that r is a polynomial multiple of q. However, $\deg q = n$ and $\deg r < n$. Thus r must be the zero polynomial. The equation above then implies that q is also the zero polynomial. Thus T is injective.

(b) Because $\dim(\mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})) = \dim \mathcal{P}_{m+n-1}(\mathbf{C})$, (a) and 3.65 imply that T is surjective.

(c) Because T is surjective, there exist $(r, s) \in \mathcal{P}_{n-1}(\mathbf{C}) \times \mathcal{P}_{m-1}(\mathbf{C})$ such that $T(r, s)$ equals the constant polynomial 1.

1 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.

(a) Prove that if $U \subseteq \operatorname{null} T$, then U is invariant under T.

(b) Prove that if $\operatorname{range} T \subseteq U$, then U is invariant under T.

SOLUTION

(a) Suppose $U \subseteq \operatorname{null} T$. If $u \in U$, then $u \in \operatorname{null} T$, and hence $Tu = 0$, and thus $Tu \in U$. Thus U is invariant under T.

(b) Suppose $\operatorname{range} T \subseteq U$. If $u \in U$, then $Tu \in \operatorname{range} T$, and hence $Tu \in U$. Thus U is invariant under T.

2 Suppose that $T \in \mathcal{L}(V)$ and $V_1, \ldots, V_m$ are subspaces of V invariant under T. Prove that $V_1 + \cdots + V_m$ is invariant under T.

SOLUTION Consider a vector $u \in V_1 + \cdots + V_m$. There exist $v_1 \in V_1, \ldots, v_m \in V_m$ such that

$$u = v_1 + \cdots + v_m.$$

Applying T to both sides of this equation, we get

$$Tu = Tv_1 + \cdots + Tv_m.$$

Because each V_k is invariant under T, we have $Tv_1 \in V_1, \ldots, Tv_m \in V_m$. Thus the equation above shows that $Tu \in V_1 + \cdots + V_m$, which implies that $V_1 + \cdots + V_m$ is invariant under T.

3 Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T.

SOLUTION Suppose $\{V_\alpha\}_{\alpha \in \Gamma}$ is a collection of subspaces of V invariant under T; here Γ is an arbitrary index set. We need to prove that $\bigcap_{\alpha \in \Gamma} V_\alpha$, which equals the set of vectors that are in V_α for every $\alpha \in \Gamma$, is invariant under T. To do this, suppose $u \in \bigcap_{\alpha \in \Gamma} V_\alpha$. Then $u \in V_\alpha$ for every $\alpha \in \Gamma$. Thus $Tu \in V_\alpha$ for every $\alpha \in \Gamma$ (because every V_α is invariant under T). Thus $Tu \in \bigcap_{\alpha \in \Gamma} V_\alpha$, which implies that $\bigcap_{\alpha \in \Gamma} V_\alpha$ is invariant under T.

4 Prove or give a counterexample: If V is finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

SOLUTION Suppose V is finite-dimensional. We will prove that if U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$. Actually we will prove the (logically equivalent) contrapositive, meaning that we will prove that if U is a subspace of V such that $U \neq \{0\}$ and $U \neq V$, then there exists $T \in \mathcal{L}(V)$ such that U is not invariant under T. To do this, suppose U is a subspace of V such that $U \neq \{0\}$ and $U \neq V$. Choose $u \in U \backslash \{0\}$ (this is possible because $U \neq \{0\}$) and $w \in V \backslash U$ (this is possible because $U \neq V$). Extend the list u, which is linearly independent because $u \neq 0$, to a basis $u, v_1, \ldots, v_n$ of V. Define $T \in \mathcal{L}(V)$ by

$$T(au + b_1 v_1 + \ldots b_n v_n) = aw.$$

Thus $Tu = w$. Because $u \in U$ but $w \notin U$, this shows that U is not invariant under T, as desired.

5 Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

SOLUTION Suppose $(x, y) \in \mathbf{R}^2$ and $\lambda \in \mathbf{R}$ and $T(x, y) = \lambda(x, y)$. Thus

$$-3y = \lambda x \quad \text{and} \quad x = \lambda y.$$

Substituting the second equation into the first equation gives

$$-3y = \lambda^2 y.$$

If $y \neq 0$, then the equation above implies that $-3 = \lambda^2$, which does not hold for any real number λ. If $y = 0$, then the equation $x = \lambda y$ implies that we also have $x = 0$ and hence $(x, y) = (0, 0)$, which is not eligible eigenvector. Conclusion: T has no eigenvalues.

6 Define $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of T.

SOLUTION Suppose λ is an eigenvalue of T. For this particular operator, the eigenvalue-eigenvector equation $T(w, z) = \lambda(w, z)$ becomes the system of equations

$$\begin{aligned} z &= \lambda w \\ w &= \lambda z. \end{aligned}$$

Substituting the value for z from the first equation into the second equation gives $w = \lambda^2 w$. Thus $1 = \lambda^2$ (we can ignore the possibility that $w = 0$ because if $w = 0$, then the first equation above implies that $z = 0$). Thus $\lambda = 1$ or $\lambda = -1$. The set of eigenvectors corresponding to the eigenvalue 1 is

$$\{(w, w) : w \in \mathbf{F} \text{ and } w \neq 0\}.$$

The set of eigenvectors corresponding to the eigenvalue -1 is

$$\{(w, -w) : w \in \mathbf{F} \text{ and } w \neq 0\}.$$

7 Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenvectors of T.

SOLUTION Suppose λ is an eigenvalue of T. For this particular operator, the eigenvalue-eigenvector equation $T(z_1, z_2, z_3) = \lambda(z_1, z_2, z_3)$ becomes the system of equations

$$\begin{aligned} 2z_2 &= \lambda z_1 \\ 0 &= \lambda z_2 \\ 5z_3 &= \lambda z_3. \end{aligned}$$

If $\lambda \neq 0$, then the second equation implies that $z_2 = 0$, and the first equation then implies that $z_1 = 0$. Thus there is a solution to the system above with $z_3 \neq 0$. The third equation now shows that $\lambda = 5$. In other words, 5 is the only nonzero eigenvalue of T. The set of eigenvectors corresponding to the eigenvalue 5 is

$$\{(0, 0, z_3) : z_3 \in \mathbf{F} \text{ and } z_3 \neq 0\}.$$

If $\lambda = 0$, the first and third equations above show that $z_2 = 0$ and $z_3 = 0$. With these values for z_2, z_3, the equations above are satisfied for all values of z_1. Thus 0 is an eigenvalue of T. The set of eigenvectors corresponding to the eigenvalue 0 is

$$\{(z_1, 0, 0) : z_1 \in \mathbf{F} \text{ and } z_1 \neq 0\}.$$

8 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$.

SOLUTION Suppose λ is an eigenvalue of P. Thus there exists $v \in V$ such that $v \neq 0$ and

$$Pv = \lambda v.$$

Thus

$$\lambda v = Pv = P^2 v = P(Pv) = P(\lambda v) = \lambda Pv = \lambda^2 v.$$

The equation above implies that $(\lambda^2 - \lambda)v = 0$. Thus $\lambda^2 - \lambda = 0$, which implies that $\lambda = 0$ or $\lambda = 1$.

9 Define $T\colon \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

SOLUTION We seek solutions to the equation

$$p' = \lambda p,$$

where $p \in \mathcal{P}(\mathbf{R})$ and $\lambda \in \mathbf{R}$. Because the derivative of every nonconstant polynomial has degree one less than the degree of the polynomial, we see that the only solutions of the equation above are to take p to be a constant polynomial and $\lambda = 0$. Thus 0 is the only eigenvalue of T and the nonzero constant functions are the only eigenvectors.

10 Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$. Find all eigenvalues and eigenvectors of T.

SOLUTION A typical element p of $\mathcal{P}_4(\mathbf{R})$ is given by expression

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4,$$

where $a_0, a_1, a_2, a_3, a_4 \in \mathbf{R}$.

With that expression, the eigenvalue-eigenvector equation $Tp = \lambda p$, which in this case is $xp'(x) = \lambda p(x)$, becomes

$$a_1 x + 2a_2 x^2 + 3a_3 x_3 + 4a_4 x^4 = \lambda(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4).$$

Comparing coefficients in the equation above shows the eigenvalue-eigenvector equation is equivalent to the system of equations

$$\begin{aligned} 0 &= \lambda a_0 \\ a_1 &= \lambda a_1 \\ 2a_2 &= \lambda a_2 \\ 3a_3 &= \lambda a_3 \\ 4a_4 &= \lambda a_4. \end{aligned}$$

From the equations above, it is clear that if $j \in \{0, 1, 2, 3, 4\}$ and $a_j \neq 0$, then $\lambda = j$ and $a_k = 0$ for each $k \neq j$. Thus the eigenvalues of T are $0, 1, 2, 3, 4$ and the corresponding eigenvectors are of the form c, cx, cx^2, cx^3, cx^4, where $c \in \mathbf{R}$ and $c \neq 0$.

11 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\alpha \in \mathbf{F}$. Prove that there exists $\delta > 0$ such that $T - \lambda I$ is invertible for all $\lambda \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$.

SOLUTION The operator T has only finitely many eigenvalues (by 5.12). Thus there exists $\delta > 0$ such that $\{\lambda \in \mathbf{F} : |\alpha - \lambda| < \delta\}$ contains no eigenvalues of T. Thus $T - \lambda I$ is invertible for all $\lambda \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$.

12 Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V. Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$. Find all eigenvalues and eigenvectors of P.

SOLUTION Suppose $\lambda \in \mathbf{F}$ is an eigenvalue of P. Then there exists a nonzero vector $v \in V$ such that $Pv = \lambda v$. Writing this equation using the representation $v = u + w$ with $u \in U$ and $w \in W$, we have $u = \lambda(u + w)$. Thus

$$(1 - \lambda)u - \lambda w = 0.$$

Because $V = U \oplus W$, if 0 is written as the sum of a vector in U and a vector in W, then both vectors are 0. Thus the equation above implies that $(1 - \lambda)u = \lambda w = 0$. Because u and w are not both 0 (because $v \neq 0$), this implies that $\lambda = 1$ or $\lambda = 0$.

For $v \in V$ with representation as above, the equation $Pv = 0$ is equivalent to the equation $u = 0$, which is equivalent to the equation $v = w$, which is equivalent to the statement that $v \in W$. This means that 0 is an eigenvalue of P (because W is a nonzero subspace of V) and that the set of nonzero vectors in W equals the set of eigenvectors corresponding to the eigenvalue 0.

For $v \in V$ with representation as above, the equation $Pv = v$ is equivalent to the equation $v = u$, which is equivalent to the statement that $v \in U$. This means that 1 is an eigenvalue of P (because U is a nonzero subspace of V) and that the set of nonzero vectors in U equals the set of eigenvectors corresponding to the eigenvalue 1.

13 Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

SOLUTION Suppose $v \in V$ and $\lambda \in \mathbf{F}$. Then clearly

$$Tv = \lambda v \iff (S^{-1}TS)(S^{-1}v) = \lambda S^{-1}v$$

Thus we see that T and $S^{-1}TS$ have the same eigenvalues, and furthermore v is an eigenvector of T if and only if $S^{-1}v$ is an eigenvector of $S^{-1}TS$.

14 Give an example of an operator on $\mathbf{R}^4$ that has no (real) eigenvalues.

SOLUTION Define $T \in \mathcal{L}(\mathbf{R}^4)$ by

$$T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3).$$

Suppose $\lambda \in \mathbf{R}$. For this particular operator, the eigenvalue-eigenvector equation $T(x_1, x_2, x_3, x_4) = \lambda(x_1, x_2, x_3, x_4)$ becomes the system of equations

$$\begin{aligned} -x_2 &= \lambda x_1 \\ x_1 &= \lambda x_2 \\ -x_4 &= \lambda x_3 \\ x_3 &= \lambda x_4. \end{aligned}$$

Multiplying together the first two equations and also multiplying together the last two equations gives $-x_1x_2 = \lambda^2 x_1x_2$ and $-x_3x_4 = \lambda^2 x_3x_4$. If either x_1 or x_2 does not equal 0, then the first two equations show that neither of x_1, x_2 equals 0. Similarly, if either x_3 or x_4 does not equal 0, then the last two equations show that neither of x_3, x_4 equals 0. Thus if λ is an eigenvalue of T, then there is a solution to the system of equations above with $x_1x_2 \neq 0$ or $x_3x_4 \neq 0$. Either way, we conclude that $-1 = \lambda^2$, which is impossible for any real number λ. Thus T has no real eigenvalues.

15 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$. Show that λ is an eigenvalue of T if and only if λ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.

SOLUTION

$$\begin{aligned}
\lambda \text{ is an eigenvalue of } T &\iff T - \lambda I \text{ is not injective} \\
&\iff T - \lambda I \text{ is not surjective} \\
&\iff T' - \lambda I \text{ is not injective} \\
&\iff \lambda \text{ is an eigenvalue of } T',
\end{aligned}$$

where the second line follows from 3.65 and the third line follows from 3.129.

16 Suppose $v_1, \ldots, v_n$ is a basis of V and $T \in \mathcal{L}(V)$. Prove that if λ is an eigenvalue of T, then

$$|\lambda| \leq n \max\{|\mathcal{M}(T)_{j,k}| : 1 \leq j, k \leq n\},$$

where $\mathcal{M}(T)_{j,k}$ denotes the entry in row j, column k of the matrix of T with respect to the basis $v_1, \ldots, v_n$.

See Exercise 19 in Section 6A for a different bound on $|\lambda|$.

SOLUTION Suppose λ is an eigenvalue of T. Hence $\lambda v = Tv$ for some nonzero vector $v \in V$. There exist $c_1, \ldots, c_n \in \mathbf{F}$, not all 0, such that

$$v = \sum_{k=1}^{n} c_k v_k.$$

Thus

$$\begin{aligned} Tv &= \sum_{k=1}^{n} c_k T v_k \\ &= \sum_{k=1}^{n} c_k \sum_{j=1}^{n} \mathcal{M}(T)_{j,k} v_j \\ (*) \qquad &= \sum_{j=1}^{n} \Bigl(\sum_{k=1}^{n} c_k \mathcal{M}(T)_{j,k} \Bigr) v_j. \end{aligned}$$

We also have

$$\begin{aligned} Tv &= \lambda v \\ (**) \qquad &= \sum_{j=1}^{n} \lambda c_j v_j. \end{aligned}$$

Let

$$b = \max\{|\mathcal{M}(T)_{j,k}| : 1 \leq j, k \leq n\}.$$

Comparing $(*)$ and $(**)$, we have

$$\lambda c_j = \sum_{k=1}^{n} c_k \mathcal{M}(T)_{j,k}$$

for each $j \in \{1, \ldots, n\}$. Hence

$$\begin{aligned}|\lambda c_j| &= \left|\sum_{k=1}^{n} c_k \mathcal{M}(T)_{j,k}\right| \\ &\leq b \sum_{k=1}^{n} |c_k|\end{aligned}$$

for each $j \in \{1, \ldots, n\}$. The right side of the inequality above does not depend on j. Thus summing the inequality above from $j = 1$ to $j = n$ we get

$$|\lambda| \sum_{j=1}^{n} |c_j| \leq nb \sum_{k=1}^{n} |c_k|,$$

which we can rewrite as

$$|\lambda| \sum_{k=1}^{n} |c_k| \leq nb \sum_{k=1}^{n} |c_k|.$$

The inequality above implies that

$$|\lambda| \leq nb,$$

as desired.

17 Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{R}$. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of the complexification $T_{\mathbf{C}}$.

See Exercise 33 in Section 3B for the definition of $T_{\mathbf{C}}$.

SOLUTION First suppose λ is an eigenvalue of T. Then there exists $v \in V$ with $v \neq 0$ such that $Tv = \lambda v$. Thus $T_{\mathbf{C}} v = \lambda v$, which shows that λ is an eigenvalue of $T_{\mathbf{C}}$, completing one direction of the proof.

To prove the other direction, suppose now that λ is an eigenvalue of $T_{\mathbf{C}}$. Then there exist $u, v \in V$ with $u + iv \neq 0$ such that

$$T_{\mathbf{C}}(u + iv) = \lambda(u + iv).$$

The equation above implies that $Tu = \lambda u$ and $Tv = \lambda v$. Because $u \neq 0$ or $v \neq 0$, this implies that λ is an eigenvalue of T, completing the proof.

18 Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{C}$. Prove that λ is an eigenvalue of the complexification $T_{\mathbf{C}}$ if and only if $\bar{\lambda}$ is an eigenvalue of $T_{\mathbf{C}}$.

SOLUTION Suppose λ is an eigenvalue of the complexification $T_{\mathbf{C}}$. Thus there exist $u, v \in V$ such that $u + iv \neq 0$ and

$$T_{\mathbf{C}}(u + iv) = \lambda(u + iv).$$

Let $\lambda = \alpha + \beta i$, where $\alpha, \beta \in \mathbf{R}$.

The equation above implies that

$$Tu = \alpha u - \beta v \quad \text{and} \quad Tv = \alpha v + \beta u.$$

The equations above imply that

$$T_{\mathbf{C}}(u - iv) = (\alpha - \beta i)(u - iv) = \bar{\lambda}(u - iv).$$

Because $u - iv \neq 0$, the equation above implies that $\bar{\lambda}$ is an eigenvalue of $T_{\mathbf{C}}$.

We have shown that if λ is an eigenvalue of $T_{\mathbf{C}}$, then $\bar{\lambda}$ is also an eigenvalue of $T_{\mathbf{C}}$. Replacing λ with $\bar{\lambda}$ gives the implication in the other direction.

19 Show that the forward shift operator $T \in \mathcal{L}(\mathbf{F}^\infty)$ defined by

$$T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$$

has no eigenvalues.

SOLUTION The eigenvalue-eigenvector equation $Tz = \lambda z$ for this operator is

$$(0, z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \lambda z_3, \dots),$$

which is equivalent to

$$0 = \lambda z_1, \quad z_1 = \lambda z_2, \quad z_2 = \lambda z_3, \quad \dots.$$

The first equation above implies that $\lambda = 0$ or $z_1 = 0$. If $\lambda = 0$, then the rest of the equations imply that $0 = z_1 = z_2 = \dots$, which eliminates 0 as a possible eigenvalue. If $\lambda \neq 0$, then the equations above imply $z_1 = 0$, then $z_2 = 0$, then $z_3 = 0$, and so on, which eliminates all nonzero numbers λ as possible eigenvalues. Thus we conclude that T has no eigenvalues.

20 Define the backward shift operator $S \in \mathcal{L}(\mathbf{F}^\infty)$ by

$$S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots).$$

(a) Show that every element of $\mathbf{F}$ is an eigenvalue of S.

(b) Find all eigenvectors of S.

SOLUTION Suppose λ is an eigenvalue of T. For this particular operator, the eigenvalue-eigenvector equation $Tz = \lambda z$ becomes the system of equations

$$\begin{aligned} z_2 &= \lambda z_1 \\ z_3 &= \lambda z_2 \\ z_4 &= \lambda z_3 \\ &\vdots \end{aligned}$$

From this we see that we can choose z_1 arbitrarily and then solve for the other coordinates:

$$\begin{aligned} z_2 &= \lambda z_1 \\ z_3 &= \lambda z_2 = \lambda^2 z_1 \\ z_4 &= \lambda z_3 = \lambda^3 z_1 \\ &\vdots \end{aligned}$$

Thus each $\lambda \in \mathbf{F}$ is an eigenvalue of T and the set of corresponding eigenvectors is

$$\{(w, \lambda w, \lambda^2 w, \lambda^3 w, \dots) : w \in \mathbf{F} \text{ and } w \neq 0\}.$$

21 Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) Prove that T and T^{-1} have the same eigenvectors.

SOLUTION

(a) First suppose λ is an eigenvalue of T. Thus there exists a nonzero vector $v \in V$ such that

$$Tv = \lambda v.$$

Applying T^{-1} to both sides of the equation above, we get $v = \lambda T^{-1}v$, which is equivalent to the equation $T^{-1}v = \frac{1}{\lambda}v$. Thus $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

To prove the implication in the other direction, replace T with T^{-1} and λ by $\frac{1}{\lambda}$ and then apply the result from the paragraph above.

(b) The proof of (a) shows that if v is an eigenvector of T, then v is also an eigenvector of T^{-1}. Replacing T by T^{-1}, we also see that if v is an eigenvector of T^{-1}, then v is also an eigenvector of T [which equals $(T^{-1})^{-1}$]. Thus T and T^{-1} have the same eigenvectors.

22 Suppose $T \in \mathcal{L}(V)$ and there exist nonzero vectors u and w in V such that

$$Tu = 3w \quad \text{and} \quad Tw = 3u.$$

Prove that 3 or -3 is an eigenvalue of T.

SOLUTION The equations above imply that

$$T(u + w) = 3(u + w) \quad \text{and} \quad T(u - w) = -3(u - w).$$

The vectors $u + w$ and $u - w$ cannot both be 0 (because otherwise we would have $u = w = 0$). Thus the equations above imply that 3 or -3 is an eigenvalue of T.

23 Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

SOLUTION Suppose $\lambda \in \mathbf{F}$ is an eigenvalue of ST. We want to prove that λ is an eigenvalue of TS. Because λ is an eigenvalue of ST, there exists a nonzero vector $v \in V$ such that

$$(ST)v = \lambda v.$$

Now

$$\begin{aligned}(TS)(Tv) &= T(STv)\\ &= T(\lambda v)\\ &= \lambda Tv.\end{aligned}$$

If $Tv \neq 0$, then the equation above shows that λ is an eigenvalue of TS, as desired.

If $Tv = 0$, then $\lambda = 0$ (because $S(Tv) = \lambda v$) and furthermore T is not invertible, which implies that TS is not invertible (by Exercise 11 in Section 3D), which implies that λ (which equals 0) is an eigenvalue of TS.

Regardless of whether or not $Tv = 0$, we have shown that λ is an eigenvalue of TS. Because λ was an arbitrary eigenvalue of ST, we have shown that every eigenvalue of ST is an eigenvalue of TS.

Reversing the roles of S and T, we conclude that every eigenvalue of TS is also an eigenvalue of ST. Thus ST and TS have the same eigenvalues.

24 Suppose A is an n-by-n matrix with entries in $\mathbf{F}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = Ax$, where elements of $\mathbf{F}^n$ are thought of as n-by-1 column vectors.

(a) Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigenvalue of T.

(b) Suppose the sum of the entries in each column of A equals 1. Prove that 1 is an eigenvalue of T.

SOLUTION

(a) Let x be the n-by-1 matrix whose entries all equal 1. Then from the definition of matrix multiplication we see that the entry in row k, column 1 of Ax equals that sum of the entries in row k of A, which equals 1. Thus $Ax = x$, which implies that 1 is an eigenvalue of A.

(b) **Solution 1:** The columns of A equal the rows of the A^{t}, the transpose of A. From 3.132, we know that $\mathcal{M}(T') = A^{\mathrm{t}}$. Thus by (a), 1 is an eigenvalue of T'. Thus $T' - I$ is not injective, which implies that $T - I$ is not surjective (by 3.129), which implies that $T - I$ is not injective (by 5.7). Thus 1 is an eigenvalue of T.

Solution 2: For $x_1, \ldots, x_n \in \mathbf{F}$, we have

$$(x_1, \ldots, x_n) = \sum_{k=1}^{n} x_k e_k,$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbf{F}^n$. Thus

$$(*) \qquad (T - I)(x_1, \ldots, x_n) = \sum_{k=1}^{n} x_k (Te_k - e_k).$$

Because Te_k is the k^{th}-column of A, the coordinates of Te_k add up to 1. Thus the coordinates of $Te_k - e_k$ add up to 0. Hence the coordinates of the linear combination $\sum_{k=1}^{n} x_k (Te_k - e_k)$ add up to 0. Thus the equation $(*)$ above implies that $T - I$ is not surjective, which implies that $T - I$ is not injective (by 5.7). Thus 1 is an eigenvalue of T.

25 Suppose $T \in \mathcal{L}(V)$ and u, w are eigenvectors of T such that $u + w$ is also an eigenvector of T. Prove that u and w are eigenvectors of T corresponding to the same eigenvalue.

SOLUTION Suppose

$$Tu = \alpha u \quad \text{and} \quad Tv = \beta w \quad \text{and} \quad T(u + w) = \gamma(u + w).$$

Adding the first two equations above, we have $T(u + w) = \alpha u + \beta w$, which when combined with the third equation above gives

$$(\alpha - \gamma)u + (\beta - \gamma)w = 0.$$

If $\alpha \neq \beta$ then u, w is linearly independent (by 5.11) and the equation above then implies that $\alpha - \gamma = \beta - \gamma = 0$, which then implies that $\alpha = \beta$. Thus we see that $\alpha = \beta$.

26 Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

SOLUTION For each $v \in V \backslash \{0\}$, there exists $\lambda_v \in \mathbf{F}$ such that

$$Tv = \lambda_v v.$$

To show that T is a scalar multiple of the identity, we must show that λ_v is independent of v for each $v \in V \backslash \{0\}$. To do this, suppose $v, w \in V \backslash \{0\}$. We want to show that $\lambda_v = \lambda_w$. First consider the case where v, w is linearly dependent. Then there exists $b \in \mathbf{F}$ such that $w = bv$. We have

$$\begin{aligned} \lambda_w w &= Tw \\ &= T(bv) \\ &= bTv \\ &= b(\lambda_v v) \\ &= \lambda_v w, \end{aligned}$$

which shows that $\lambda_v = \lambda_w$, as desired.

Finally, consider the case where v, w is linearly independent. We have

$$\begin{aligned} \lambda_{v+w}(v + w) &= T(v + w) \\ &= Tv + Tw \\ &= \lambda_v v + \lambda_w w, \end{aligned}$$

which implies that

$$(\lambda_{v+w} - \lambda_v)v + (\lambda_{v+w} - \lambda_w)w = 0.$$

Because v, w is linearly independent, this implies that $\lambda_{v+w} = \lambda_v$ and $\lambda_{v+w} = \lambda_w$, so again we have $\lambda_v = \lambda_w$, as desired.

27 Suppose that V is finite-dimensional and $k \in \{1, \ldots, \dim V - 1\}$. Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V of dimension k is invariant under T. Prove that T is a scalar multiple of the identity operator.

SOLUTION Suppose T is not a scalar multiple of the identity operator. By Exercise 26, there exists $v_1 \in V\backslash\{0\}$ such that v_1 is not an eigenvector of T. Thus v_1, Tv_1 is linearly independent. Extend v_1, Tv_1 to a basis $v_1, Tv_1, v_2, \ldots, v_{\dim V - 1}$ of V. Let

$$U = \operatorname{span}(v_1, \ldots, v_k).$$

Then U is a subspace of V and $\dim U = k$. However, U is not invariant under T because $v_1 \in U$ but $Tv_1 \notin U$. This contradiction to our hypothesis about T shows that our assumption that T is not a scalar multiple of the identity was false.

28 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has at most $1 + \dim \operatorname{range} T$ distinct eigenvalues.

SOLUTION Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T; let $v_1, \ldots, v_m$ be corresponding eigenvectors. If $\lambda_j \neq 0$, then

$$T(v_j/\lambda_j) = v_j.$$

Because at most one of $\lambda_1, \ldots, \lambda_m$ equals 0, this implies that at least $m - 1$ of the vectors $v_1, \ldots, v_m$ are in range T. These vectors are linearly independent (by 5.11), which implies that

$$m - 1 \leq \dim \operatorname{range} T.$$

Thus $m \leq 1 + \dim \operatorname{range} T$, as desired.

29 Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and -4, 5, and $\sqrt{7}$ are eigenvalues of T. Prove that there exists $x \in \mathbf{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

SOLUTION Because T has at most 3 distinct eigenvalues (by 5.12), the hypotheses imply that 9 is not an eigenvalue of T. Thus $T - 9I$ is surjective. In particular, there exists $x \in \mathbf{R}^3$ such that $(T - 9I)x = (-4, 5, \sqrt{7})$.

30 Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

SOLUTION Let $v \in V$ be an eigenvector of T corresponding to the eigenvalue λ. Then

$$\begin{aligned} 0 &= (T - 2I)(T - 3I)(T - 4I)v \\ &= (\lambda - 2)(\lambda - 3)(\lambda - 4)v. \end{aligned}$$

Because $v \neq 0$, the equation above implies that

$$(\lambda - 2)(\lambda - 3)(\lambda - 4) = 0.$$

Thus $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$, as desired.

31 Give an example of $T \in \mathcal{L}(\mathbf{R}^2)$ such that $T^4 = -I$.

SOLUTION The operator on $\mathbf{R}^2$ of counterclockwise rotation by 45° has the desired property. Specifically, define $T \in \mathcal{L}(\mathbf{R}^2)$ by

$$T(x, y) = \left(\tfrac{1}{\sqrt{2}}(x - y), \tfrac{1}{\sqrt{2}}(x + y)\right)$$

for each $(x, y) \in \mathbf{R}^2$. Then, as is easy to verify,

$$T^2(x, y) = (-y, x)$$

and thus

$$\begin{aligned} T^4(x, y) &= T^2(T^2(x, y)) \\ &= (-x, -y). \end{aligned}$$

Hence $T^4 = -I$, as desired.

32 Suppose $T \in \mathcal{L}(V)$ has no eigenvalues and $T^4 = I$. Prove that $T^2 = -I$.

SOLUTION Let $v \in V$. Then

$$0 = (T^4 - I)v = (T^2 - I)(T^2 + I)v = (T - I)(T + I)(T^2 + I)v.$$

Because neither 1 nor -1 is an eigenvalue of T, the operators $T - I$ and $T + I$ are both injective. Thus the equation above implies that

$$(T^2 + I)v = 0.$$

In other words, $T^2 v = -v$. Thus $T^2 = -I$, as desired.

33 Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

(a) Prove that T is injective if and only if T^m is injective.

(b) Prove that T is surjective if and only if T^m is surjective.

SOLUTION

(a) First suppose T is injective. We will prove that T^m is injective by induction on m. The desired result holds if $m = 1$. Thus suppose that $m > 1$ and the desired result holds for $m - 1$.

Suppose $v \in V$ and $T^m v = 0$. Thus

$$T(T^{m-1}v) = 0.$$

Because T is injective, the equation above implies that $T^{m-1}v = 0$. Our induction hypothesis now implies that $v = 0$, which proves that T^m is injective, as desired.

To prove the implication in the other direction, now suppose T^m is injective. Suppose $\in V$ and $Tv = 0$. Then $T^m v = 0$. Because T^m is injective, this implies that $v = 0$. Thus T is injective, as desired.

(b) First suppose T is surjective. We will prove that T^m is surjective by induction on m. The desired result holds if $m = 1$. Thus suppose that $m > 1$ and the desired result holds for $m - 1$.

Suppose $v \in V$. Because T is surjective, there exists $u \in V$ such that $v = Tu$. By our induction hypotheses, T^m is surjective. Hence there exists $w \in V$ such that $u = T^{m-1}w$. Thus

$$v = Tu = T(T^{m-1}w) = T^m w.$$

Hence T^m is surjective, as desired.

To prove the implication in the other direction, now suppose T^m is surjective. Suppose $v \in V$. Thus there exists $u \in V$ such that $v = T^m u$. Now

$$v = T^m u = T(T^{m-1}u).$$

Thus T is surjective, as desired.

34 Suppose V is finite-dimensional and $v_1, \dots, v_m \in V$. Prove that the list $v_1, \dots, v_m$ is linearly independent if and only if there exists $T \in \mathcal{L}(V)$ such that $v_1, \dots, v_m$ are eigenvectors of T corresponding to distinct eigenvalues.

SOLUTION First suppose $v_1, \dots, v_m$ is linearly independent. Extend the list $v_1, \dots, v_m$ to a basis $v_1, \dots, v_m, \dots, v_n$ of V. By the linear map lemma (3.4), there exists $T \in \mathcal{L}(V)$ such that

$$Tv_k = kv_k$$

for each $k = 1, \dots, n$. Thus $v_1, \dots, v_m$ are eigenvectors of T corresponding to distinct eigenvalues.

Conversely, suppose there exists $T \in \mathcal{L}(V)$ such that $v_1, \dots, v_m$ are eigenvectors of T corresponding to distinct eigenvalues. Then by 5.11, $v_1, \dots, v_m$ is linearly independent.

35 Suppose that $\lambda_1, \dots, \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on $\mathbf{R}$.

Hint: Let $V = \operatorname{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$*, and define an operator* $D \in \mathcal{L}(V)$ *by* $Df = f'$*. Find eigenvalues and eigenvectors of* D*.*

SOLUTION Let $V = \operatorname{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$, and define $D \in \mathcal{L}(V)$ by $Df = f'$. This linear map really does map V into V because

$$D(e^{\lambda_k x}) = \lambda_k e^{\lambda_k x}.$$

The equation above also shows that for each $k = 1, \dots, n$, the vector $e^{\lambda_k x}$ is an eigenvector of D with eigenvalue λ_k. Thus 5.11 implies that $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is linearly independent.

36 Suppose that $\lambda_1, \ldots, \lambda_n$ is a list of distinct positive numbers. Prove that the list $\cos(\lambda_1 x), \ldots, \cos(\lambda_n x)$ is linearly independent in the vector space of real-valued functions on $\mathbf{R}$.

SOLUTION Let $V = \operatorname{span}(\cos(\lambda_1 x), \ldots, \cos(\lambda_n x))$, and define $T \in \mathcal{L}(V)$ by $Tf = f''$. This linear map really does map V into V because

$$T(\cos(\lambda_k x)) = -\lambda_k^2 \cos(\lambda_k x).$$

The equation above also shows that for each $k = 1, \ldots, n$, the vector $\cos(\lambda_k x)$ is an eigenvector of T with eigenvalue $-\lambda_k^2$.

Thus 5.11 implies that $\cos(\lambda_1 x), \ldots, \cos(\lambda_n x)$ is linearly independent.

37 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by

$$\mathcal{A}(S) = TS$$

for each $S \in \mathcal{L}(V)$. Prove that the set of eigenvalues of T equals the set of eigenvalues of $\mathcal{A}$.

SOLUTION First suppose λ is an eigenvalue of T. Thus there exists $v_1 \in V$ with $v_1 \neq 0$ such that $(T - \lambda I)v_1 = 0$. Extend the linearly independent list v_1 to a basis $v_1, v_2, \ldots, v_n$ of V. Let $S \in \mathcal{L}(V)$ be such that

$$Sv_k = \begin{cases} v_1 & \text{if } k = 1, \\ 0 & \text{if } k = 2, \ldots, n. \end{cases}$$

Then S is not the zero operator. However, $(T - \lambda I)Sv_k = 0$ for each $k = 1, \ldots, n$, and thus $(T - \lambda I)S = 0$. Thus $(\mathcal{A} - \lambda I)(S) = 0$, which implies that λ is an eigenvalue of $\mathcal{A}$.

To prove the implication in the other direction, now suppose that λ is an eigenvalue of $\mathcal{A}$. Thus there exists $S \in \mathcal{L}(V)$ such that $S \neq 0$ and

$$(\mathcal{A} - \lambda I)(S) = 0.$$

Thus $(T - \lambda I)S = 0$. In particular, if $v \in V$ is such that $Sv \neq 0$, then $(T - \lambda I)(Sv) = 0$, which implies that λ is an eigenvalue of T.

38 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T. The *quotient operator* $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(v + U) = Tv + U$$

for each $v \in V$.

(a) Show that the definition of T/U makes sense (which requires using the condition that U is invariant under T) and show that T/U is an operator on V/U.

(b) Show that each eigenvalue of T/U is an eigenvalue of T.

SOLUTION

(a) To show that the definition above of the quotient operator makes sense, we need to verify that if $v + U = w + U$, then $Tv + U = Tw + U$. Hence suppose $v + U = w + U$. Thus $v - w \in U$ (see 3.101). Because U is invariant under T, we also have $T(v - w) \in U$, which implies that $Tv - Tw \in U$, which implies that $Tv + U = Tw + U$. Thus the definition of T/U makes sense.

If $v, w \in U$, then

$$\begin{aligned}(T/U)\big((v + U) + (w + U)\big) &= (T/U)(v + w + U)\\ &= T(v + w) + U\\ &= Tv + Tw + U\\ &= (Tv + U) + (Tw + U)\\ &= (T/U)(v + U) + (T/U)(w + U).\end{aligned}$$

Similarly, if $c \in \mathbf{F}$ and $v \in V$, then

$$\begin{aligned}(T/U)\big(c(v + U)\big) &= (T/U)(cv + U)\\ &= T(cv) + U\\ &= cTv + U\\ &= c(Tv + U)\\ &= c(T/U)(v + U).\end{aligned}$$

Thus T/U is a linear map from V/U to V/U.

(b) Suppose $\lambda \in \mathbf{F}$ is an eigenvalue of T/U. Then there exists $w \in V$ such that $w \notin U$ (which is equivalent to the condition $w + U \neq 0 + U$) and

$$(T/U)(w + U) = \lambda(w + U).$$

The equation above could be rewritten as $Tw + U = \lambda w + U$. This implies that $(T - \lambda I)w \in U$.

Define $S\colon U + \operatorname{span}(w) \to V$ by

$$Sv = (T - \lambda I)v$$

for $v \in U + \operatorname{span}(w)$. Clearly S is a linear map.

If $u \in U$ and $c \in \mathbf{F}$, then $S(u + cw) = Tu - \lambda u + c(T - \lambda I)w \in U$, where the inclusion holds because U is invariant under T. Thus range $S \subseteq U$.

Because $w \notin U$, we know that $\dim(U + \operatorname{span}(w)) > \dim U$. The fundamental theorem of linear maps (3.21) applied to S now implies that $\dim \operatorname{null} S > 0$. Thus there exists $v \in U + \operatorname{span}(w)$ such that $v \neq 0$ and $Sv = 0$. Thus λ is an eigenvalue of T, as desired.

39 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has an eigenvalue if and only if there exists a subspace of V of dimension $\dim V - 1$ that is invariant under T.

SOLUTION First suppose T has an eigenvalue $\lambda \in \mathbf{F}$. Thus $\operatorname{range}(T - \lambda I) \neq V$. Hence there exists a subspace U of V such that $\dim U = \dim V - 1$ and $\operatorname{range}(T - \lambda I) \subseteq U$. Thus U is invariant under $T - \lambda I$ and hence under T, as desired.

To prove the implication in the other direction, now suppose there exists a subspace U of V that is invariant under T and has dimension $\dim V - 1$. Thus V/U has dimension one, which means that the operator T/U on V/U has an eigenvalue. By Exercise 38, that eigenvalue is also an eigenvalue of T.

40 Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Suppose $p \in \mathcal{P}(\mathbf{F})$ is a polynomial. Prove that

$$p(STS^{-1}) = Sp(T)S^{-1}.$$

SOLUTION First suppose m is a positive integer. Then

$$\begin{aligned}(STS^{-1})^m &= (STS^{-1})(STS^{-1})\cdots(STS^{-1})\\ &= ST(S^{-1}S)T(S^{-1}S)\cdots(S^{-1}S)TS^{-1}\\ &= ST^mS^{-1},\end{aligned}$$

which is our desired equation in the special case $p(z) = z^m$. Multiplying both sides of the equation above by a scalar and then summing a finite number of equations of the resulting form shows that $p(STS^{-1}) = Sp(T)S^{-1}$ for every polynomial $p \in \mathcal{P}(\mathbf{F})$.

41 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T. Prove that U is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbf{F})$.

SOLUTION Suppose $u \in U$. Because U in invariant under T, we know that $Tu \in U$.

Because U in invariant under T, we know that $T(Tu) \in U$. In other words, $T^2u \in U$.

Continuing in this fashion, we see that $T^n u \in U$ for every positive integer n.

Suppose $p \in \mathcal{P}(\mathbf{F})$ has degree n. Then

$$p(T)u \in \operatorname{span}(u, Tu, \ldots, T^n u).$$

Hence $p(T) \in U$. Thus U is invariant under $p(T)$.

42 Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \ldots, x_n) = (x_1, 2x_2, 3x_3, \ldots, nx_n)$.

(a) Find all eigenvalues and eigenvectors of T.

(b) Find all subspaces of $\mathbf{F}^n$ that are invariant under T.

SOLUTION

(a) For $k = 1, \ldots, n$, let e_k be the vector in $\mathbf{F}^n$ that equals 1 in the k^{th}-coordinate and 0 in the other coordinates. Then

$$Te_k = ke_k.$$

Thus the numbers $1, \ldots, n$ are all eigenvalues of T. By 5.12, T has no other eigenvalues.

It is easy to see that the set of eigenvectors corresponding to the eigenvalue k is the set of nonzero scalar multiples of e_k.

(b) If $k_1, \ldots, k_m$ are all in $\{1, \ldots, n\}$, then it is easy to see that

$$\operatorname{span}(e_{k_1}, \ldots, e_{k_m})$$

is a subspace invariant under T. Note that $\operatorname{span}(e_{k_1}, \ldots, e_{k_m})$ is the set of vectors in $\mathbf{F}^n$ whose k^{th}-coordinate equals 0 for all k in $\{1, \ldots, n\} \backslash \{k_1, \ldots, k_m\}$.

Every subspace invariant under T is of the form discussed in the paragraph above. To prove this, suppose U is a subspace of $\mathbf{F}^n$ that is invariant under T. Because U is invariant under T, we know that $p(T)(x) \in U$ for every $x \in U$ (by Exercise 41). For $p \in \mathcal{P}(\mathbf{F})$ and $(x_1, x_2, \ldots, x_n) \in \mathbf{F}^n$, we have

$$(p(T))(x_1, x_2, \ldots, x_n) = (p(1)x_1, p(2)x_2, \ldots, p(n)x_n).$$

Suppose $1 \leq k \leq n$ and the k^{th} coordinate of some vector $(x_1, \ldots, x_n) \in U$ is nonzero. Let $p \in \mathcal{P}(\mathbf{F})$ be such that $p(k) = 1/x_k$ and $p(j) = 0$ for all positive integers $j \leq n$ except k. Then

$$(p(T))(x_1, x_2, \ldots, x_n) = e_k.$$

Hence $e_k \in U$. Thus

$$U = \operatorname{span}(e_{k_1}, \ldots, e_{k_m}),$$

where $\{k_1, \ldots, k_m\}$ is the collection of positive integers $k \leq n$ such that the k^{th} coordinate of some vector in U is nonzero.

43 Suppose that V is finite-dimensional, $\dim V > 1$, and $T \in \mathcal{L}(V)$. Prove that $\{p(T) : p \in \mathcal{P}(\mathbf{F})\} \neq \mathcal{L}(V)$.

SOLUTION If $p, q \in \mathcal{P}(\mathbf{F})$, then

$$(p(T))(q(T)) = (q(T))(p(T))$$

by 5.17(b).

Exercise 16 in Section 3A thus shows that not every operator on V is of the form $p(T)$ for some $p \in \mathcal{P}(\mathbf{F})$.