19 Suppose $U \subseteq V$. Explain why

$$U^0 = \{\varphi \in V' : U \subseteq \operatorname{null} \varphi\}.$$

SOLUTION By definition,

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}.$$

However, $\varphi(u) = 0$ for all $u \in U$ if and only if $U \subseteq \operatorname{null} \varphi$. Thus

$$U^0 = \{\varphi \in V' : U \subseteq \operatorname{null} \varphi\}.$$

20 Suppose V is finite-dimensional and U is a subspace of V. Show that

$$U = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}.$$

SOLUTION If $u \in U$, then $\varphi(u) = 0$ for every $\varphi \in U^0$. Thus

$$U \subseteq \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}.$$

To prove the inclusion in the other direction, suppose $w \in V$ but $w \notin U$. Let $u_1, \dots, u_m$ be a basis of U. Because $w \notin U$, the list $u_1, \dots, u_m, w$ is linearly independent and hence can be extended to a basis of V. Thus by the linear map lemma (3.4), there exists $\psi \in V'$ such that

$$\psi(u_k) = 0 \text{ for } k = 1, \dots, m \quad \text{and} \quad \psi(w) = 1.$$

Thus $\psi \in U^0$ but $\psi(w) \neq 0$. Hence

$$w \notin \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}.$$

Thus

$$U \supseteq \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\},$$

which implies that

$$U = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\},$$

as desired.

21 Suppose V is finite-dimensional and U and W are subspaces of V.

(a) Prove that $W^0 \subseteq U^0$ if and only if $U \subseteq W$.

(b) Prove that $W^0 = U^0$ if and only if $U = W$.

SOLUTION

(a) First suppose $U \subseteq W$. Suppose $\varphi \in W^0$. If $u \in U$, then $u \in W$, and hence $\varphi(u) = 0$. Thus $\varphi \in U^0$. Hence $W^0 \subseteq U^0$.

To prove the implication in the other direction, now suppose $W^0 \subseteq U^0$. Suppose $u \in U$. Thus $\varphi(u) = 0$ for every $\varphi \in U^0$. Hence $\varphi(u) = 0$ for every $\varphi \in W^0$. Now Exercise 20 implies that $u \in W$. Thus $U \subseteq W$, as desired.

(b) If $U = W$, then clearly $W^0 = U^0$.

To prove the implication in the other direction, now suppose $W^0 = U^0$. Then $W^0 \subseteq U^0$ and $U^0 \subseteq W^0$. Now (a) implies that $U \subseteq W$ and $W \subseteq U$. Thus $U = W$, as desired.

22 Suppose V is finite-dimensional and U and W are subspaces of V.

(a) Show that $(U+W)^0 = U^0 \cap W^0$.

(b) Show that $(U \cap W)^0 = U^0 + W^0$.

SOLUTION

(a) First suppose $\varphi \in (U+W)^0$. Then $\varphi(u+w) = 0$ for all $u \in U$ and all $w \in W$. Taking $w = 0$ and then taking $u = 0$, in particular we see that $\varphi(u) = 0$ for all $u \in U$ and $\varphi(w) = 0$ for all $w \in W$. Thus $\varphi \in U^0$ and $\varphi \in W^0$. In other words, $\varphi \in U^0 \cap W^0$. Thus $(U+W)^0 \subseteq U^0 \cap W^0$.

To prove the inclusion in the other direction, suppose $\varphi \in U^0 \cap W^0$. If $u \in U$ and $w \in W$, then

$$\varphi(u+w) = \varphi(u) + \varphi(w) = 0 + 0 = 0.$$

Hence $\varphi \in (U+W)^0$. Thus $(U+W)^0 \supseteq U^0 \cap W^0$.

Thus $(U+W)^0 = U^0 \cap W^0$, as desired.

(b) First suppose $\varphi \in U^0 + W^0$. Thus $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in U^0$ and $\varphi_2 \in W^0$. If $v \in U \cap W$, then

$$\varphi(v) = \varphi_1(v) + \varphi_2(v) = 0 + 0 = 0.$$

Hence $\varphi \in (U \cap W)^0$. Thus

$$U^0 + W^0 \subseteq (U \cap W)^0.$$

To show that the inclusion above is an equality, we will show that both sides have the same dimension. We have

$$\begin{aligned}
\dim(U \cap W)^0 &= \dim V - \dim(U \cap W) \\
&= \dim V - (\dim U + \dim W - \dim(U+W)) \\
&= \dim U^0 + \dim W^0 - \dim(U+W)^0 \\
&= \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\
&= \dim(U^0 + W^0),
\end{aligned}$$

where the first equality comes from 3.125, the second equality comes from 2.43, the third equality comes from 3.125, the fourth equality comes from Exercise 22(a), and the fifth equality comes from 2.43.

The equality of dimensions along with the inclusion above show that

$$(U \cap W)^0 = U^0 + W^0,$$

as desired.

23 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_m \in V'$. Prove that the following three sets are equal to each other.

(a) $\operatorname{span}(\varphi_1, \dots, \varphi_m)$

(b) $\bigl((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\bigr)^0$

(c) $\{\varphi \in V' : (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) \subseteq \operatorname{null}\varphi\}$

SOLUTION Let A denote the set in (a), B denote the set in (b), and C denote the set in (c).

We will prove that $A = B$ by induction on m. For $m = 1$, we need to show that

$$\operatorname{span}(\varphi_1) = (\operatorname{null}\varphi_1)^0.$$

The inclusion $\operatorname{span}(\varphi_1) \subseteq (\operatorname{null}\varphi_1)^0$ follows from the definitions. To prove the inclusion in the other direction, suppose $\psi \in (\operatorname{null}\varphi_1)^0$. Thus $\operatorname{null}\varphi_1 \subseteq \operatorname{null}\psi$. Now Exercise 6 implies $\psi \in \operatorname{span}(\varphi_1)$, as desired.

Now suppose that $m > 1$ and that the desired result holds when m is replaced with $m - 1$. Then

$$\begin{aligned}\operatorname{span}(\varphi_1, \dots, \varphi_m) &= \operatorname{span}(\varphi_1, \dots, \varphi_{m-1}) + \operatorname{span}(\varphi_m)\\ &= \bigl((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_{m-1})\bigr)^0 + (\operatorname{null}\varphi_m)^0\\ &= \bigl((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\bigr)^0,\end{aligned}$$

where the second equality above comes from our induction hypothesis and the third equality above comes from Exercise 22(b). The last equality above completes the proof that $A = B$.

Now suppose $\varphi \in A$. Thus there exist $a_1, \dots, a_m$ such that

$$\varphi = a_1\varphi_1 + \cdots + a_m\varphi_m.$$

The equation above shows that if $v \in (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$, then $\varphi(v) = 0$ and hence $v \in \operatorname{null}\varphi$. Thus $\varphi \in C$, proving that $A \subseteq C$.

Now suppose $\varphi \in C$. Thus

$$(\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m) \subseteq \operatorname{null}\varphi. \tag{$*$}$$

To show that $\varphi \in B$, suppose $v \in (\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)$. Thus $(*)$ shows that $v \in \operatorname{null}\varphi$. Hence $\varphi(v) = 0$, which shows that $\varphi \in B$. Thus $C \subseteq B$, completing the proof.

24 Suppose V is finite-dimensional and $v_1, \ldots, v_m \in V$. Define a linear map $\Gamma\colon V' \to \mathbf{F}^m$ by $\Gamma(\varphi) = \bigl(\varphi(v_1), \ldots, \varphi(v_m)\bigr)$.

(a) Prove that $v_1, \ldots, v_m$ spans V if and only if Γ is injective.

(b) Prove that $v_1, \ldots, v_m$ is linearly independent if and only if Γ is surjective.

SOLUTION

(a) First suppose $v_1, \ldots, v_m$ spans V. If $\varphi \in V'$ and $\Gamma(\varphi) = 0$, then

$$\varphi(v_1) = \cdots = \varphi(v_m) = 0,$$

which implies that $\varphi(v) = 0$ for all $v \in V$, which implies that $\varphi = 0$. Thus Γ is injective.

To prove the implication in the other direction, we will prove its contrapositive. Thus suppose $v_1, \ldots, v_m$ does not span V. Thus there exists $\varphi \in V'$ such that φ equals 0 on $\operatorname{span}(v_1, \ldots, v_m)$ but $\varphi \neq 0$. Thus $\Gamma(\varphi) = 0$, which implies that Γ is not injective, as desired.

(b) First suppose $v_1, \ldots, v_m$ is linearly independent. Then the list $v_1, \ldots, v_m$ can be extended to a basis of V. Hence the linear map lemma (3.4) implies that $\varphi(v_1), \ldots, \varphi(v_m)$ can take on any values we choose. Thus Γ is surjective.

Conversely, suppose Γ is surjective. Thus it is not possible to write any v_k as a linear combination of $v_1, \ldots, v_{k-1}$ because doing so would contradict the existence of $\varphi \in V'$ such that

$$\varphi(v_1) = \cdots = \varphi(v_{k-1}) = 0 \quad \text{but} \quad \varphi(v_k) = 1.$$

The linear dependence lemma (2.19) now implies $v_1, \ldots, v_m$ is linearly independent, as desired.

25 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_m \in V'$. Define a linear map $\Gamma\colon V \to \mathbf{F}^m$ by $\Gamma(v) = \bigl(\varphi_1(v), \dots, \varphi_m(v)\bigr)$.

(a) Prove that $\varphi_1, \dots, \varphi_m$ spans V' if and only if Γ is injective.

(b) Prove that $\varphi_1, \dots, \varphi_m$ is linearly independent if and only if Γ is surjective.

SOLUTION

(a) First suppose $\varphi_1, \dots, \varphi_m$ spans V'. If $v \in V$ and $\Gamma(v) = 0$, then

$$\varphi_1(v) = \cdots = \varphi_m(v) = 0,$$

which implies that $\varphi(v) = 0$ for all $\varphi \in V'$, which implies that $v = 0$. Thus Γ is injective.

Conversely, suppose that Γ is injective. Thus

$$(\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m) = \{0\},$$

which implies that

$$\bigl((\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)\bigr)^0 = V'.$$

Now Exercise 23 implies that $\varphi_1, \dots, \varphi_m$ spans V'.

(b) First we will prove that if $\varphi_1, \dots, \varphi_m$ is linearly independent, then Γ is surjective. This will be done by induction on m.

Consider first the case $m = 1$. The statement that the list φ_1 is linearly independent implies $\varphi_1 \neq 0$, which implies that $\operatorname{range} \Gamma = \mathbf{F}^1$, verifying the desired statement when $m = 1$.

Thus assume that $m > 1$ and that the desired statement holds for $m - 1$. Suppose $\varphi_1, \dots, \varphi_m$ is linearly independent. Hence $\varphi_1, \dots, \varphi_{m-1}$ is linearly independent. By our induction hypothesis,

$$\{(\varphi_1(v), \dots, \varphi_{m-1}(v)) : v \in V\} = \mathbf{F}^{m-1}. \tag{$*$}$$

Because $\varphi_1, \dots, \varphi_m$ is linearly independent, $\varphi_m \notin \operatorname{span}(\varphi_1, \dots, \varphi_{m-1})$. Thus Exercise 23 implies that

$$(\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}) \not\subseteq \operatorname{null} \varphi_m.$$

Hence there exists $v \in V$ such that $\varphi_1(v) = \cdots = \varphi_{m-1}(v) = 0$ but $\varphi_m(v) \neq 0$.

If $(c_1, \dots, c_m) \in \mathbf{F}^m$, then by $(*)$ there exists $u \in V$ such that $\varphi_k(u) = c_k$ for each $k = 1, \dots, m - 1$. Thus

$$\varphi_k\biggl(u + \frac{c_m - \varphi_m(u)}{\varphi_m(v)}\, v\biggr) = c_k$$

for every $k = 1, \dots, m$. Hence Γ is surjective, as desired.

To prove the implication in the other direction, we will prove its contrapositive. Thus suppose $\varphi_1, \dots, \varphi_m$ is linearly dependent. Hence there exists $n \in \{1, \dots, m\}$ such that φ_n is a linear combination of $\varphi_1, \dots, \varphi_{n-1}$. This means that each vector in $\mathbf{F}^m$ whose first $n - 1$ coordinates equal 0 and whose n^{th} coordinate equals 1 is not in the range of Γ. Hence Γ is not surjective, as desired.

26 Suppose V is finite-dimensional and Ω is a subspace of V'. Prove that

$$\Omega = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in \Omega\}^0.$$

SOLUTION Let $\varphi_1, \ldots, \varphi_m$ be a basis of Ω. Thus

$$\begin{aligned}
\Omega &= \operatorname{span}(\varphi_1, \ldots, \varphi_m) \\
&= \bigl((\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)\bigr)^0 \\
&= \{v \in V : \varphi_k(v) = 0 \text{ for } k = 1, \ldots, m\}^0 \\
&= \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in \Omega\}^0,
\end{aligned}$$

where the second line follows from Exercise 23.

27 Suppose $T \in \mathcal{L}\big(\mathcal{P}_5(\mathbf{R})\big)$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$, where φ is the linear functional on $\mathcal{P}_5(\mathbf{R})$ defined by $\varphi(p) = p(8)$. Prove that

$$\operatorname{range} T = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}.$$

SOLUTION We have $\dim \operatorname{null} T' = 1$, which by the fundamental theorem of linear maps (3.21) implies that

$$\dim \operatorname{range} T' = \dim\big(\mathcal{P}_5(\mathbf{R})\big)' - 1.$$

Using 3.130(a) and 3.111 and then applying the fundamental theorem of linear maps to φ, we can rewrite the equation above as

$$\begin{aligned} \dim \operatorname{range} T &= \dim \mathcal{P}_5(\mathbf{R}) - 1 \\ &= \dim \operatorname{null} \varphi. \end{aligned} \tag{$*$}$$

Because $\varphi \in \operatorname{null} T'$, we have $0 = T'(\varphi) = \varphi \circ T$. Thus

$$\operatorname{range} T \subseteq \operatorname{null} \varphi = \{p \in \mathcal{P}_5(\mathbf{R}) : p(8) = 0\}.$$

But these subspaces of $\mathcal{P}_5(\mathbf{R})$ have the same dimension by $(*)$, and hence they are equal, as desired.

28 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_m$ is a linearly independent list in V'. Prove that

$$\dim\bigl((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\bigr) = (\dim V) - m.$$

SOLUTION We have

$$\begin{aligned}
\dim\bigl((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\bigr) &= \dim V - \dim\bigl((\operatorname{null}\varphi_1) \cap \cdots \cap (\operatorname{null}\varphi_m)\bigr)^0 \\
&= \dim V - \dim\operatorname{span}(\varphi_1, \dots, \varphi_m) \\
&= \dim V - m,
\end{aligned}$$

where the first equality comes from 3.125, the second equality comes from Exercise 23, and the third equality holds because $\varphi_1, \dots, \varphi_m$ is linearly independent.

29 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$.

(a) Prove that if $\varphi \in W'$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$, then $\operatorname{range} T = \operatorname{null} \varphi$.

(b) Prove that if $\psi \in V'$ and $\operatorname{range} T' = \operatorname{span}(\psi)$, then $\operatorname{null} T = \operatorname{null} \psi$.

SOLUTION

(a) Suppose $\varphi \in W'$ and $\operatorname{null} T' = \operatorname{span}(\varphi)$.

First consider the case $\varphi = 0$. In this case, $\operatorname{null} T' = \{0\}$. Thus T' is injective. Thus T is surjective (by 3.129), which means that $\operatorname{range} T = W$. Therefore $\operatorname{range} T = \operatorname{null} \varphi$ because both sides of this equation equal W in this case where $\varphi = 0$.

Now consider the case $\varphi \neq 0$. Thus $\dim \operatorname{null} T' = 1$, which by the fundamental theorem of linear maps (3.21) implies that

$$\dim \operatorname{range} T' = \dim W' - 1.$$

Using 3.130(a) and 3.111 and then applying the fundamental theorem of linear maps to φ, we can rewrite the equation above as

$$\begin{aligned} \dim \operatorname{range} T &= \dim W - 1 \\ &= \dim \operatorname{null} \varphi. \end{aligned}$$

Because $\varphi \in \operatorname{null} T'$, we have $0 = T'(\varphi) = \varphi \circ T$. Thus $\operatorname{range} T \subseteq \operatorname{null} \varphi$. But these two subspaces of W have the same dimension by the displayed equation above, and hence they are equal, as desired.

(b) Suppose $\psi \in V'$ and $\operatorname{range} T' = \operatorname{span}(\psi)$.

First consider the case in which $\psi = 0$. In this case, $\operatorname{range} T' = \{0\}$. Thus $\dim \operatorname{range} T' = 0$, which implies that $\dim \operatorname{range} T = 0$ [by 3.130(a)]. Thus $T = 0$, which means that $\operatorname{null} T = V$. Therefore both sides of this equation equal V in this case where $\psi = 0$.

Now consider the case $\psi \neq 0$. Thus $\dim \operatorname{range} T' = 1$, which implies that $\dim \operatorname{range} T = 1$ [by 3.130(a)], which by the fundamental theorem of linear maps (3.21) implies that

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - 1 \\ &= \dim \operatorname{null} \psi. \end{aligned}$$

Because $\varphi \in \operatorname{range} T'$, we have $\varphi = T'(\psi)$ for some $\psi \in W'$. Thus $\varphi = \psi \circ T$, which implies that $\operatorname{null} T \subseteq \operatorname{null} \varphi$. But these two subspaces of V have the same dimension by the displayed equation above, and hence they are equal, as desired.

30 Suppose V is finite-dimensional and $\varphi_1, \dots, \varphi_n$ is a basis of V'. Show that there exists a basis of V whose dual basis is $\varphi_1, \dots, \varphi_n$.

SOLUTION From Exercise 28, we know that

$$\dim\bigl((\operatorname{null}\varphi_2) \cap \dots \cap (\operatorname{null}\varphi_n)\bigr) = 1$$

and

$$\dim\bigl((\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_n)\bigr) = 0.$$

Thus

$$(\operatorname{null}\varphi_2) \cap \dots \cap (\operatorname{null}\varphi_n) \not\subset (\operatorname{null}\varphi_1) \cap \dots \cap (\operatorname{null}\varphi_n).$$

In other words, there exists $x_1 \in V$ such that $x_1 \in (\operatorname{null}\varphi_2) \cap \dots \cap (\operatorname{null}\varphi_n)$ but $\varphi_1(x_1) \neq 0$. Multiplying x_1 by an appropriate scalar, we can assume that $\varphi_1(x_1) = 1$.

Similarly, for each $j = 2, \dots, n$, there exists $x_j \in V$ such that

$$\varphi_k(x_j) = 0 \text{ for } k \in \{1, \dots, n\} \setminus j \quad \text{and} \quad \varphi_j(x_j) = 1. \tag{$*$}$$

If $a_1, \dots, a_n \in \mathbf{F}$ and

$$0 = a_1 x_1 + \dots + a_n x_n,$$

then for each $j = 1, \dots, n$ we have

$$0 = \varphi_j(a_1 x_1 + \dots + a_n x_n) = a_j.$$

Hence $x_1, \dots, x_n$ is linearly independent and thus is a basis of V (by 2.38). Clearly $(*)$ shows that the dual basis of $x_1, \dots, x_n$ is $\varphi_1, \dots, \varphi_n$.

31 Suppose U is a subspace of V. Let $i\colon U \to V$ be the inclusion map defined by $i(u) = u$. Thus $i' \in \mathcal{L}(V', U')$.

(a) Show that $\operatorname{null} i' = U^0$.

(b) Prove that if V is finite-dimensional, then $\operatorname{range} i' = U'$.

(c) Prove that if V is finite-dimensional, then $\tilde{i}'$ is an isomorphism from V'/U^0 onto U'.

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space.

SOLUTION

(a) Suppose $\varphi \in V'$. Then

$$\begin{aligned} \varphi \in \operatorname{null} i' &\iff i'(\varphi) = 0 \\ &\iff \varphi \circ i = 0 \\ &\iff \varphi(u) = 0 \text{ for all } u \in U \\ &\iff \varphi \in U^0. \end{aligned}$$

Thus $\operatorname{null} i' = U^0$.

(b) Suppose V is finite-dimensional. Clearly i is injective. Thus by 3.131, T' is surjective. Hence $\operatorname{range} i' = U'$.

(c) Suppose V is finite-dimensional. By 3.107(b) and 3.107(b), $\tilde{i}'$ is an injective map from $V'/\operatorname{null} i'$ onto $\operatorname{range} i'$. Thus by (a) and (b) of this exercise, $\tilde{i}'$ is an isomorphism from V'/U^0 onto U'.

32 The *double dual space* of V, denoted by V'', is defined to be the dual space of V'. In other words, $V'' = (V')'$. Define $\Lambda\colon V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for each $v \in V$ and each $\varphi \in V'$.

(a) Show that Λ is a linear map from V to V''.

(b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.

(c) Show that if V is finite-dimensional, then Λ is an isomorphism from V onto V''.

Suppose V is finite-dimensional. Then V and V' are isomorphic, but finding an isomorphism from V onto V' generally requires choosing a basis of V. In contrast, the isomorphism Λ from V onto V'' does not require a choice of basis and thus is considered more natural.

SOLUTION

(a) A straightforward application of the definitions shows that Λ is a linear map from V to V''.

(b) Suppose $T \in \mathcal{L}(V)$. Suppose $v \in V$ and $\varphi \in V'$. Then

$$\begin{aligned}\bigl((T'' \circ \Lambda)(v)\bigr)(\varphi) &= \bigl(T''(\Lambda v)\bigr)(\varphi)\\ &= (\Lambda v \circ T')(\varphi)\\ &= (\Lambda v)\bigl(T'(\varphi)\bigr)\\ &= (\Lambda v)(\varphi \circ T)\\ &= (\varphi \circ T)(v)\\ &= \varphi(Tv).\end{aligned}$$

Also,

$$\begin{aligned}\bigl((\Lambda \circ T)(v)\bigr)(\varphi) &= \bigl(\Lambda(Tv)\bigr)(\varphi)\\ &= \varphi(Tv).\end{aligned}$$

Thus

$$\bigl((T'' \circ \Lambda)(v)\bigr)(\varphi) = \bigl((\Lambda \circ T)(v)\bigr)(\varphi),$$

for all $\varphi \in V'$, which implies that

$$(T'' \circ \Lambda)(v) = (\Lambda \circ T)(v),$$

for all $v \in V$, which implies that $T'' \circ \Lambda = \Lambda \circ T$, as desired.

(c) Suppose V is finite-dimensional. Suppose $v \in V$ and $\Lambda v = 0$. Thus $\varphi(v) = 0$ for every $\varphi \in V'$. Now Exercise 3 implies that $v = 0$. Thus Λ is injective.

Because $\dim V = \dim V' = \dim V''$ (by 3.111), we can conclude that Λ is an isomorphism of V onto V''.

33 Suppose U is a subspace of V. Let $\pi\colon V \to V/U$ be the usual quotient map. Thus $\pi' \in \mathcal{L}\big((V/U)', V'\big)$.

(a) Show that π' is injective.

(b) Show that $\operatorname{range} \pi' = U^0$.

(c) Conclude that π' is an isomorphism from $(V/U)'$ onto U^0.

The isomorphism in (c) is natural in that it does not depend on a choice of basis in either vector space. In fact, there is no assumption here that any of these vector spaces are finite-dimensional.

SOLUTION

(a) Suppose $\varphi \in (V/U)'$ and $\pi'(\varphi) = 0$. Then $\varphi \circ \pi = 0$, which means that $(\varphi \circ \pi)(v) = 0$ for every $v \in V$, which means that $\varphi(v + U) = 0$ for every $v \in V$. Thus $\varphi = 0$, which implies that π' is injective.

(b) Suppose $\varphi \in (V/U)'$. If $u \in U$, then

$$(\pi'(\varphi))(u) = (\varphi \circ \pi)(u) = \varphi(u + U) = \varphi(0) = 0$$

and thus $\pi'(\varphi) \in U^0$. Hence $\operatorname{range} \pi' \subseteq U^0$.

To show the inclusion in the other direction, suppose $\psi \in U^0$. Thus $\psi \in V'$ and $\psi(u) = 0$ for all $u \in U$. Define $\varphi \in (V/U)'$ by

$$\varphi(v + U) = \psi(v);$$

the condition that $\psi(u) = 0$ for all $u \in U$ shows that φ is well defined. The definitions now show that $\pi'(\varphi) = \psi$. Thus $\operatorname{range} \pi' \supseteq U^0$, completing the proof that $\operatorname{range} \pi' = U^0$.

(c) Now (a) and (b) immediately imply that π' is an isomorphism from $(V/U)'$ onto U^0.