

## Exercises 1B Solutions: Definition of Vector Space

*Linear Algebra Done Right, 4th ed.*

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**Exercise 1.** Prove that  $-(-v) = v$  for every  $v \in V$ .

*Solution.* By definition,  $-(-v)$  is the additive inverse of  $(-v)$ , meaning it is the unique element satisfying  $(-v) + (-(-v)) = 0$ .

We claim that  $v$  satisfies this property. Indeed:

$$(-v) + v = v + (-v) = 0$$

where the first equality uses commutativity of addition.

Since  $v$  is an element such that  $(-v) + v = 0$ , and the additive inverse of  $(-v)$  is unique, we conclude that  $-(-v) = v$ . □ □

**Exercise 2.** Suppose  $a \in \mathbb{F}$ ,  $v \in V$ , and  $av = 0$ . Prove that  $a = 0$  or  $v = 0$ .

*Solution.* Suppose  $a \neq 0$ . We will show that  $v = 0$ .

Since  $a \neq 0$  and  $\mathbb{F}$  is a field,  $a$  has a multiplicative inverse  $a^{-1} \in \mathbb{F}$ .

From  $av = 0$ , multiply both sides by  $a^{-1}$ :

$$a^{-1}(av) = a^{-1} \cdot 0.$$

The left side simplifies using associativity of scalar multiplication:

$$a^{-1}(av) = (a^{-1}a)v = 1 \cdot v = v.$$

For the right side, we show  $a^{-1} \cdot 0_V = 0_V$ : Since  $0_V = 0_V + 0_V$ , we have

$$a^{-1} \cdot 0_V = a^{-1}(0_V + 0_V) = a^{-1} \cdot 0_V + a^{-1} \cdot 0_V.$$

Adding  $-(a^{-1} \cdot 0_V)$  to both sides gives  $0_V = a^{-1} \cdot 0_V$ .

Therefore  $v = 0$ .

We have shown: if  $a \neq 0$ , then  $v = 0$ . Equivalently,  $a = 0$  or  $v = 0$ . □ □

**Exercise 3.** Suppose  $v, w \in V$ . Explain why there exists a unique  $x \in V$  such that  $v + 3x = w$ .

**Solution:** We prove both existence and uniqueness.

**Existence:** Define  $x = \frac{1}{3}(w + (-v)) = \frac{1}{3}(w - v)$ .

We verify that this  $x$  satisfies  $v + 3x = w$ :

$$\begin{aligned}v + 3x &= v + 3 \cdot \frac{1}{3}(w - v) \\&= v + 1 \cdot (w - v) \\&= v + (w - v) \\&= v + w + (-v) \\&= w + v + (-v) \\&= w + 0 = w.\end{aligned}$$

**Uniqueness:** Suppose  $x_1$  and  $x_2$  both satisfy the equation:

$$v + 3x_1 = w \quad \text{and} \quad v + 3x_2 = w.$$

Then  $v + 3x_1 = v + 3x_2$ .

Adding  $(-v)$  to both sides:

$$3x_1 = 3x_2.$$

Multiplying both sides by  $\frac{1}{3}$ :

$$x_1 = x_2.$$

Therefore there exists a unique  $x \in V$  such that  $v + 3x = w$ :

$$\boxed{x = \frac{1}{3}(w - v)}$$

□

**Exercise 4.** The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

*Solution.* The empty set fails the **additive identity** axiom.

The additive identity axiom requires the existence of an element  $0 \in V$  such that  $v + 0 = v$  for all  $v \in V$ .

Since the empty set contains no elements, there is no element that could serve as the additive identity. Even though the condition “ $v + 0 = v$  for all  $v \in V$ ” is vacuously true (there are no  $v$  to check), the requirement that such an element *exists* fails.

All other axioms are vacuously satisfied because they are statements about elements of  $V$ , and the empty set has no elements to violate them.  $\square$   $\square$

**Why not additive inverse?** The additive inverse axiom says: “for every  $v \in V$ , there exists  $w \in V$  such that  $v + w = 0$ .” For the empty set, there are no elements  $v$  to consider, so this statement is vacuously true. The key difference is that additive identity requires something to *exist*, while additive inverse only makes claims about elements that are already there.

**Exercise 5.** Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of  $V$ .

**Solution:** We must show that if  $V$  satisfies all vector space axioms except the additive inverse axiom, but does satisfy  $0v = 0$  for all  $v \in V$ , then the additive inverse axiom holds.

We need to show: for every  $v \in V$ , there exists  $w \in V$  such that  $v + w = 0$ .

Given  $v \in V$ , consider  $w = (-1)v$ .

We claim  $v + w = 0$ :

$$\begin{aligned} v + (-1)v &= 1 \cdot v + (-1)v && \text{(multiplicative identity axiom)} \\ &= (1 + (-1))v && \text{(distributivity over scalar addition)} \\ &= 0v && \text{(arithmetic in } \mathbb{F}) \\ &= 0 && \text{(our assumed condition)} \end{aligned}$$

Therefore  $(-1)v$  serves as the additive inverse of  $v$ .

This shows the additive inverse axiom is a consequence of  $0v = 0$  together with the other axioms. Hence the two formulations of vector space are equivalent.  $\square$

**Exercise 6.** Let  $\infty$  and  $-\infty$  denote two distinct objects, neither of which is in  $\mathbb{R}$ . Define an addition and scalar multiplication on  $\mathbb{R} \cup \{\infty, -\infty\}$  as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for  $t \in \mathbb{R}$  define

$$t \cdot \infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0; \end{cases} \quad t \cdot (-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0; \end{cases}$$

$$t + \infty = \infty + t = \infty, \quad t + (-\infty) = (-\infty) + t = -\infty,$$

$$\infty + \infty = \infty, \quad (-\infty) + (-\infty) = -\infty, \quad \infty + (-\infty) = 0.$$

Is  $\mathbb{R} \cup \{\infty, -\infty\}$  a vector space over  $\mathbb{R}$ ? Explain.

**Solution:** No,  $\mathbb{R} \cup \{\infty, -\infty\}$  is not a vector space over  $\mathbb{R}$ .

The **associativity of addition** fails.

Consider the following computation:

$$(1 + \infty) + (-\infty) = \infty + (-\infty) = 0.$$

But:

$$1 + (\infty + (-\infty)) = 1 + 0 = 1.$$

Since  $(1 + \infty) + (-\infty) = 0 \neq 1 = 1 + (\infty + (-\infty))$ , addition is not associative.

Therefore  $\mathbb{R} \cup \{\infty, -\infty\}$  is not a vector space. □

**Intuition:** The extended real line  $\mathbb{R} \cup \{\infty, -\infty\}$  is useful in analysis (limits, measure theory), but the rules  $\infty + (-\infty) = 0$  and  $t + \infty = \infty$  are inherently inconsistent with associativity. This is why  $\infty - \infty$  is considered an “indeterminate form” in calculus.

**Exercise 7.** Suppose  $S$  is a nonempty set and  $V$  is a vector space. Let  $V^S$  denote the set of functions from  $S$  to  $V$ . Define addition and scalar multiplication on  $V^S$  by

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) = \lambda f(x)$$

for all  $f, g \in V^S$ ,  $\lambda \in \mathbb{F}$ , and  $x \in S$ . Prove that  $V^S$  is a vector space.

**Solution:** We verify all eight vector space axioms for  $V^S$ . Each property is verified pointwise: two functions are equal if and only if they agree at every point  $x \in S$ .

*Addition axioms:*

1. **Commutativity:**  $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$ .
2. **Associativity:**  $((f + g) + h)(x) = (f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x)) = (f + (g + h))(x)$ .
3. **Additive identity:** Define  $\mathbf{0} \in V^S$  by  $\mathbf{0}(x) = 0_V$ . Then  $(f + \mathbf{0})(x) = f(x) + 0_V = f(x)$ .
4. **Additive inverse:** Define  $(-f)(x) = -f(x)$ . Then  $(f + (-f))(x) = f(x) + (-f(x)) = 0_V$ .

*Scalar multiplication axioms:*

5. **Multiplicative identity:**  $(1f)(x) = 1 \cdot f(x) = f(x)$ .
6. **Associativity:**  $((\alpha\beta)f)(x) = (\alpha\beta)f(x) = \alpha(\beta f(x)) = (\alpha(\beta f))(x)$ .
7. **Distributivity (vectors):**  $(\lambda(f+g))(x) = \lambda(f(x)+g(x)) = \lambda f(x) + \lambda g(x) = (\lambda f + \lambda g)(x)$ .
8. **Distributivity (scalars):**  $((\alpha + \beta)f)(x) = (\alpha + \beta)f(x) = \alpha f(x) + \beta f(x) = (\alpha f + \beta f)(x)$ .

All axioms are satisfied, so  $V^S$  is a vector space over  $\mathbb{F}$ . □

**Exercise 8.** Suppose  $V$  is a real vector space. The **complexification** of  $V$ , denoted  $V_{\mathbb{C}}$ , equals  $V \times V$ . An element of  $V_{\mathbb{C}}$  is an ordered pair  $(u, v)$ , where  $u, v \in V$ , but we write this as  $u + iv$ .

Define addition and complex scalar multiplication on  $V_{\mathbb{C}}$  by

$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$

$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$

for all  $u_1, v_1, u_2, v_2, u, v \in V$  and  $a, b \in \mathbb{R}$ .

Prove that with the definitions of addition and scalar multiplication as above,  $V_{\mathbb{C}}$  is a complex vector space.

**Solution:** We verify the vector space axioms for  $V_{\mathbb{C}}$  over  $\mathbb{C}$ . Each axiom reduces to properties of the real vector space  $V$ .

*Addition axioms* (straightforward from  $V$ ):

1. **Commutativity:**  $(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2) = (u_2 + u_1) + i(v_2 + v_1)$ .
2. **Associativity:** Follows from associativity in  $V$  applied to both components.
3. **Additive identity:**  $0_{V_{\mathbb{C}}} = 0 + i0$ . Then  $(u + iv) + (0 + i0) = u + iv$ .
4. **Additive inverse:**  $-(u + iv) = (-u) + i(-v)$ .

*Scalar multiplication axioms:*

5. **Multiplicative identity:**  $(1 + 0i)(u + iv) = (1 \cdot u - 0 \cdot v) + i(1 \cdot v + 0 \cdot u) = u + iv$ . ✓
6. **Associativity:** Let  $\alpha = a + bi$ ,  $\beta = c + di$ , so  $\alpha\beta = (ac - bd) + (ad + bc)i$ .

Computing  $(\alpha\beta)(u + iv)$ :

$$= ((ac - bd)u - (ad + bc)v) + i((ac - bd)v + (ad + bc)u).$$

Computing  $\alpha(\beta(u + iv))$  where  $\beta(u + iv) = (cu - dv) + i(cv + du)$ :

$$\begin{aligned} &= (a(cu - dv) - b(cv + du)) + i(a(cv + du) + b(cu - dv)) \\ &= (acu - adv - bcv - bdu) + i(acv + adu + bcu - bdv). \end{aligned}$$

Regrouping confirms equality. ✓

7. **Distributivity (vectors):** For  $\lambda = a + bi$ :

$$\begin{aligned} \lambda((u_1 + iv_1) + (u_2 + iv_2)) &= (a + bi)((u_1 + u_2) + i(v_1 + v_2)) \\ &= \lambda(u_1 + iv_1) + \lambda(u_2 + iv_2). \quad \checkmark \end{aligned}$$

8. **Distributivity (scalars):** For  $\alpha = a + bi$ ,  $\beta = c + di$ :

$$(\alpha + \beta)(u + iv) = \alpha(u + iv) + \beta(u + iv). \quad \checkmark$$

All axioms are satisfied, so  $V_{\mathbb{C}}$  is a complex vector space. □