Upper Division Tutoring Program Topic Review

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Section I: Definitions and Theorems Review

Inner Product Spaces

Throughout, we use $\mathbb{F}$ to denote a field, either $\mathbb{R}$ or $\mathbb{C}$, and V to denote a vector space over $\mathbb{F}$.

Definition 1 (inner product)**.** An *inner product* on V is a function $\langle -, - \rangle : V \times V \to \mathbb{F}$ that assigns each ordered pair (u, v) of vectors in V to a number $\langle u, v \rangle \in \mathbb{F}$ and has the following properties:

- **Positivity:** $\langle v, v \rangle \geq 0$ for all $v \in V$,
- **Definiteness:** $\langle v, v \rangle = 0$ if and only if $v = 0$,
- **Additivity in first slot:** $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$,
- **Homogeneity in first slot:** $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ for all $\lambda \in \mathbb{F}$ and all $u, v \in V$, and
- **Conjugate symmetry:** $\langle u, v \rangle = \overline{\langle v, u \rangle}$ for $u, v \in V$.

Remark. If V is a real vector space, the last condition states that $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$ since every real number equals its complex conjugate.

Definition 2 (inner product space)**.** An *inner product space* is a vector space V along with an inner product usually denoted by $\langle -, - \rangle$ on V.

From now on, we assume that V is an inner product space.

Definition 3 (norm)**.** For $v \in V$, the *norm* of v, denoted $||v||$, is defined by:

$$||v|| = \sqrt{\langle v, v \rangle}.$$

Definition 4 (dot product)**.** If $V = \mathbb{R}^n$, the *dot product* is an inner product defined by:

$$\langle u, v \rangle = u \cdot v = u_1 v_1 + u_2 v_2 + \dots u_n v_n$$

where $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$.

Definition 5 (orthogonal)**.** Two vectors $u, v \in V$ are called *orthogonal* if $\langle u, v \rangle = 0$.

Theorem 1 (Cauchy-Schwarz Inequality)**.** Suppose $u, v \in V$. Then

$$|\langle u, v \rangle| \leq ||u|| \; ||v||.$$

Equality is attained if and only if u and v are scalar multiples of each other.

Theorem 2 (Triangle Inequality)**.** Suppose $u, v \in V$. Then

$$||u + v|| \leq ||u|| + ||v||.$$

Equality is attained if and only if u and v are nonnegative scalar multiples of each other.

Theorem 3 (Parallelogram Equality)**.** Suppose $u, v \in V$. Then

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

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Definition 6 (orthonormal)**.** A list of vectors is called *orthonormal* if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list. I.E. we say $e_1, \ldots, e_n$ is orthonormal if

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Proposition 1. Every orthonormal list of vectors is linearly independent.

Definition 7 (orthonormal basis)**.** An *orthonormal basis* of V is an orthonormal list of vectors in V that is also a basis of V.

Remark. Orthonormal bases are very nice to work with since for example, if we $e_1, \ldots, e_n$ is an orthonormal basis of V and we write can decompose $v = a_1 e_1 + \ldots + a_n e_n$ for some $a_1, \ldots, a_n \in \mathbb{F}$, then

$$||V||^2 = |a_1|^2 + \ldots + |a_n|^2.$$

Furthermore, we can find these coefficients using

$$a_i = \langle v, e_i \rangle \text{ for each } 1 \leq i \leq n.$$

Theorem 4 (Gram-Schmidt Procedure)**.** Suppose $v_1, \ldots, v_n$ is a linearly independent list of vectors in V. Define $e_1 = v_1 / \|v_1\|$. For $k = 2, \ldots, n$, define f_k inductively by

$$f_k = v_k - \sum_{i=1}^{k-1} \langle v_k, e_i \rangle e_i.$$

Then let $e_k = f_k / \|f_k\|$. Then $e_1, \ldots, e_n$ is an orthonormal list of vectors in V such that

$$\text{span}(v_1, \ldots, v_k) = \text{span}(e_1, \ldots, e_k)$$

for all $k = 1, \ldots, n$.

This procedure is generally used to turn a basis for V into an orthonormal basis.

Definition 8 (linear functional)**.** A *linear functional* on V is a linear map from V to $\mathbb{F}$. I.E. a linear functional is an element of $\mathcal{L}(V, \mathbb{F})$.

Theorem 5 (Riesz Representation Theorem)**.** Suppose V is finite-dimensional and φ is a linear functional on V. Then there is unique vector $u \in V$ such that:

$$\varphi(v) = \langle v, u \rangle$$

for every $v \in V$. In particular, this u is given by:

$$u = \overline{\varphi(e_1)} e_1 + \ldots + \overline{\varphi(e_n)} e_n$$

where $e_1, \ldots, e_m$ is an orthonormal basis of V.

Definition 9 (orthogonal complement)**.** If U is a subset of V, then the *orthogonal complement* of U, denoted $U^\perp$, is the set of all vectors in V that are orthogonal to every vector in U. I.E.

$$U^\perp = \{v \in V \mid \langle v, u \rangle = 0 \text{ for every } u \in U\}.$$

Proposition 2. If U is a finite-dimensional subspace of V, then

$$(U^\perp)^\perp = U \text{ and } V = U \oplus U^\perp.$$

Definition 10 (orthogonal projection)**.** Suppose U is a finite-dimensional subspace of V. The *orthogonal projection* of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as follows: For any $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^\perp$. Then $P_U v = u$.

Proposition 3. Suppose U is a finite-dimensional subspace of $V, v \in V$, and $u \in U$. Then

$$||v - P_U v|| \leq ||v - u||.$$

Furthermore, equality is attained if and only if $u = P_U v$.

Remark. The above proposition says that the orthogonal projection of v onto U is the closest vector in U to v.

Section II: Practice Exercises

1. Consider the space $V = \mathcal{P}_n(\mathbb{C})$. Show that

   $$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx$$

   defines an inner product on V.

2. Consider a fixed set of points $x_1, \ldots, x_n \in \mathbb{C}$. For two functions $f, g \in \mathcal{P}_n(\mathbb{C})$, is

   $$\langle f, g \rangle = \sum_{i=1}^{n} f(x_i)\overline{g(x_i)}$$

   an inner product? For which values of k is this an inner product on $\mathcal{P}_k(\mathbb{C})$?

3. (Axler 6.A.22) Show that if $u, v \in V$, then

   $$\|u + v\| \cdot \|u - v\| \leq \|u\|^2 + \|v\|^2.$$

4. (Axler 6.A.23) Suppose $v_1, \ldots, v_m \in V$ are such that $\|v_k\| \leq 1$ for each $k = 1, \ldots, m$. Show that there exists $a_1, \ldots, a_m \in \{1, -1\}$ such that

   $$\|a_1 v_1 + \ldots + a_m v_m\| \leq \sqrt{m}.$$

5. For $n = 1, 2, 3$, use the Gram-Schmidt algorithm to orthonormalize the basis $1, x, \ldots, x^n$ of $\mathcal{P}_n(\mathbb{C})$ equipped with the inner product of question 1.

6. (Axler 6.B.6a) Suppose $e_1, e_2, \ldots, e_n$ is an orthonormal basis of V. Prove that if $v_1, v_2, \ldots, v_n$ are vectors in V such that

   $$\|e_k - v_k\| < \frac{1}{\sqrt{n}}$$

   for each k, then $v_1, v_2, \ldots, v_n$ is a basis if V.

7. (Axler 6.B.18) Suppose $u_1, u_2, \ldots, u_m$ is a linearly independent list of vectors in V. Show that there exists $v \in V$ such that $\langle u_i, v \rangle = 1$ for all $i = 1, \ldots, m$.

8. Prove that every (pairwise) orthogonal list of non-zero vectors is linearly independent.

9. (Axler 6.C.10) Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V. Prove that U and $U^\perp$ are both invariant under T if and only if $P_U T = T P_U$.

10. (Axler 6.C.12) Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(0) = 0, p'(0) = 0$, and
$$\int_0^1 |2 + 3x - p(x)|^2 dx$$
is as small as possible.

11. (Axler 6.C.9) Suppose V is finite-dimensional. Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in $\text{null}(P)$ is orthogonal to every vector in $\text{range}(P)$. Prove that there exists a subspace U of V such that $P = P_U$.