From Jackson’ s Theorem to L^1 Convergence of Fourier Series

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1 Problem Statement

Let $f \in L^1(\mathbb{T})$, and assume that

$$\|f(\cdot + h) - f(\cdot)\|_1 = O(|h|^\alpha) \quad \text{for some } \alpha > 0.$$

Show that the Fourier partial sums $S_N f$ converge to f in the L^1 norm as $N \to \infty$.

2 Analogy with the Continuous Case

In class, we have seen that if $f \in C(\mathbb{T})$ satisfies a Hölder condition

$$\|f(\cdot + h) - f(\cdot)\|_\infty = O(|h|^\alpha),$$

then $S_N f \to f$ uniformly.

Theorem 2.1. Suppose that the modulus of continuity $\omega_f(\delta)$ satisfies

$$\omega_f(\delta) = o\left(\frac{1}{\log(1/\delta)}\right),$$

where

$$\omega_f(\delta) := \sup_{x \in \mathbb{T}} \sup_{|h| \le \delta} |f(x+h) - f(x)|.$$

Then $S_N f \to f$ uniformly on $\mathbb{T}$.

Theorem 2.2 (Jackson's Theorem)**.** Let $\mathcal{T}_N = \operatorname{span}\{e^{2\pi i k x} : |k| \le N\}$. Then for any $f \in C(\mathbb{T})$,

$$\operatorname{dist}_\infty(f, \mathcal{T}_N) := \inf_{p \in \mathcal{T}_N} \|f - p\|_\infty \le C w_f(1/N).$$

Proof of Theorem 2.1 using Jackson's Theorem:

Choose $p \in \mathcal{T}_N$ such that $\|f - p\|_\infty \le 2 \operatorname{dist}_\infty(f, \mathcal{T}_N)$. Then,

$$\|f - S_N f\|_\infty \le \|S_N f - S_N p\|_\infty + \|S_N p - p\|_\infty + \|p - f\|_\infty.$$

Since $p \in \mathcal{T}_N$, $S_N p = p$, so

$$\|f - S_N f\|_\infty \le \|S_N\|_1 \|f - p\|_\infty + \|f - p\|_\infty.$$

By Jackson's Theorem, $\|f-p\|_\infty \le c\,\omega_f(1/N)$, and $\|S_N\|_1 \le C_1 \log N$. Thus,

$$\|f - S_N f\|_\infty \le C_1 \log N\, \omega_f\left(\frac{1}{N}\right) + C_2\omega_f\left(\frac{1}{N}\right) \to 0$$

as $N \to \infty$, by the assumption on ω_f.

Then using **Theorem 2.1**, we can show $S_N f \to f$ uniformly on $\mathbb{T}$ when f satisfies Hölder condition.

The proof of Jackson's theorem relies on the existence of a strong kernel:

Lemma 2.3. Let $(k_n)_{n=1}^\infty$ be a sequence in $L^1(\mathbb{T})$ with $k_n \in \mathcal{T}_N$ satisfying:

1. $\displaystyle\int_{-1/2}^{1/2} k_n(t)dt = 1,$
2. There exists $\beta > 2$ such that $|k_n(t)| \le A\dfrac{n}{(1+n|t|)^\beta}$.

Then, for all $x \in \mathbb{T}$,

$$|f * k_n(x) - f(x)| \le c w_f(1/n).$$

We omit the proof of the existence of such a kernel here. However, in general, we do not need the condition that $k_n \in \mathcal{T}_N$. Nevertheless, since we use it to prove Jackson's theorem, we impose this requirement.

3 Application to L^1 Convergence

Imitating the continuous case, we would like to prove the following theorem.

First, we define the L^1-modulus of continuity:

$$\omega_f^{(1)}(\delta) = \sup_{|h|\le\delta} \|f(\cdot + h) - f(\cdot)\|_1.$$

Our assumption ensures:

$$\omega_f^{(1)}(\delta) \le C\delta^\alpha.$$

Theorem 3.1. Suppose that the modulus of continuity $\omega_f^{(1)}(\delta)$ satisfies

$$\omega_f^{(1)}(\delta) = o\left(\frac{1}{\log(1/\delta)}\right),$$

Then $S_N f$ converge to f in the L^1 norm.

Theorem 3.2. For any $f \in L^1(\mathbb{T})$,

$$\mathrm{dist}_1(f, \mathcal{T}_N) := \inf_{p\in\mathcal{T}_N} \|f-p\|_1 \le C w_f^{(1)}(1/N).$$

then

Using the kernel in **Lemma 2.3**, we obtain

$$\|f * k_n(x) - f(x)\|_1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} k_n(t)\big(f(x-t) - f(x)\big)dt \right| dx \le A\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{n}{(1+n|t|)^\beta}\omega_f^{(1)}(|t|)dt.$$

(i) When $|t| \le \dfrac{1}{2n}$, we estimate

$$\int_{-\frac{1}{2n}}^{\frac{1}{2n}} \frac{n}{(1+n|t|)^\beta}\omega_f^{(1)}(|t|)dt \le 2\omega_f^{(1)}\left(\frac{1}{2n}\right)\int_0^{\frac{1}{2n}} \frac{d(nt)}{(1+n|t|)^\beta} = 2\omega_f^{(1)}\left(\frac{1}{2n}\right)\int_0^{\frac{1}{2}} \frac{d\mu}{(1+\mu)^\beta} = C_1\omega_f^{(1)}\left(\frac{1}{2n}\right).$$

(ii) When $|t| > \dfrac{1}{2n}$, we use the inequality

$$\omega_f^{(1)}(x) \le \omega_f^{(1)}\big([xy^{-1}+1]y\big) \le \frac{[xy^{-1}+1]}{xy^{-1}}xy^{-1}\omega_f^{(1)}(y) \le 2xy^{-1}\omega_f^{(1)}(y), \quad x > y > 0.$$

Thus,

$$\omega_f^{(1)}(|t|) \le 4n|t|\omega_f^{(1)}\left(\frac{1}{2n}\right),$$

which leads to

$$\int_{\frac{1}{2n}\le|t|\le\frac{1}{2}} \frac{n}{(1+n|t|)^\beta}\omega_f^{(1)}(|t|)dt \le C\omega_f^{(1)}\left(\frac{1}{2n}\right)\int_{\frac{1}{2n}}^{\frac{1}{2}} \frac{n^2|t|dt}{(1+nt)^\beta} \le C\omega_f^{(1)}\left(\frac{1}{2n}\right)\int_{\frac{1}{2}}^{\frac{n}{2}} \frac{d\mu}{(1+\mu)^{\beta-1}} \le C_2\omega_f^{(1)}\left(\frac{1}{2n}\right).$$

Hence,

$$\|f * k_n(x) - f(x)\|_1 \le c\omega_f^{(1)}\left(\frac{1}{n}\right).$$

Similarly, we can prove **Lemma 2.3**.

Then, **Theorem 3.2** follows. As the strategy in Section 2, we conclude that $S_N f$ converges to f in the L^1 norm when f satisfies the Hölder condition in the L^1 norm.

More generally, the same strategy shows that analogous convergence results hold in $L^p(\mathbb{T})$ for every $1 \le p < \infty$, thus extending the scope of Jackson’s theorem from the continuous setting to the full scale of L^p spaces.