Problem 1. We aim to show that the inverse Fourier transform of $e^{-2\pi|\xi|}$ on $\mathbb{R}^n$ is

$$\phi(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}(1+|x|^2)^{-(n+1)/2}.$$

i: If $\beta \geq 0$, then

$$e^{-\beta} = \pi^{-1}\int_{-\infty}^{\infty}(1+t^2)^{-1}e^{-i\beta t}\,dt.$$

Proof: One way to derive this is by applying the residue theorem to the integral

$$\int_{-\infty}^{\infty}\frac{e^{-i\beta t}}{1+t^2}\,dt,$$

noting that the integrand has simple poles at $t = \pm i$. Alternatively, observe that the Fourier transform of $f(t) = \pi^{-1}(1+t^2)^{-1}$ satisfies

$$\widehat{f}(\xi) = e^{-2\pi|\xi|},$$

so that

$$e^{-\beta} = \widehat{f}\left(\frac{\beta}{2\pi}\right).$$

ii: If $\beta \geq 0$, then

$$e^{-\beta} = \int_0^{\infty}(\pi s)^{-1/2}e^{-s}e^{-\beta^2/4s}\,ds.$$

Proof: Using the identity

$$(1+t^2)^{-1} = \int_0^{\infty} e^{-(1+t^2)s}\,ds,$$

we substitute into the previous expression:

$$e^{-\beta} = \pi^{-1}\int_{-\infty}^{\infty}\left(\int_0^{\infty} e^{-(1+t^2)s}\,ds\right)e^{-i\beta t}\,dt = \int_0^{\infty}\pi^{-1}e^{-s}\left(\int_{-\infty}^{\infty} e^{-(t^2+i\beta t)/s}\,dt\right)ds.$$

Using Proposition 8.24 we obtain:

$$\int_{-\infty}^{\infty} e^{-(t^2+i\beta t)/s}\,dt = (\frac{\pi}{s})^{1/2}e^{-\beta^2/4s}.$$

Thus,

$$e^{-\beta} = \int_0^\infty (\pi s)^{-1/2} e^{-s} e^{-\beta^2/4s}\, ds.$$

iii: Let $\beta = 2\pi|\xi|$ with $\xi \in \mathbb{R}^n$.

Substituting into the formula from (ii), we find

$$e^{-2\pi|\xi|} = \int_0^\infty (\pi s)^{-1/2} e^{-s} e^{-\pi^2|\xi|^2/s}\, ds.$$

Recognizing that $e^{-\pi^2|\xi|^2/s}$ is the Fourier transform of a Gaussian, we use Proposition 8.24 again，then the inverse Fourier transform of $e^{-\pi^2|\xi|^2/s}$ is

$$(\pi/s)^{-n/2} e^{-|x|^2 s}.$$

Therefore,

$$\phi(x) = \int_0^\infty (\pi s)^{-1/2} e^{-s} (\pi/s)^{-n/2} e^{-|x|^2 s}\, ds = \pi^{-(n+1)/2} \int_0^\infty s^{(n-1)/2} e^{-(1+|x|^2)s}\, ds.$$

Recognizing this as a Gamma integral, we have

$$\phi(x) = \pi^{-(n+1)/2} (1+|x|^2)^{-(n+1)/2} \Gamma\left(\frac{n+1}{2}\right).$$

Thus, the final formula is

$$\boxed{\phi(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}} (1+|x|^2)^{-(n+1)/2}.}$$

Remark:

The main difficulty in this problem is that the function $e^{-2\pi|\xi|}$ involves the non-quadratic term $|\xi|$, which is hard to handle directly in Fourier analysis on $\mathbb{R}^n$.

The key idea is that in step (ii), we rewrite $e^{-2\pi|\xi|}$ using an integral representation where the exponent involves $|\xi|^2$ instead of $|\xi|$. This transformation allows us to exploit the well-known Fourier behavior of Gaussian functions, where $|\xi|^2$ naturally appears, and thus reduce the problem to a standard computation involving the Gamma function.

Problem 2. **(i)**Define

$$U_{L,M}(\phi) := \int_{M^{-1}<|y|<M} \frac{1}{2y} \left(\int_{L^{-1}<|y+x|<L} \frac{\phi(x,y)}{y+x}\, dx + \int_{L^{-1}<|y-x|<L} \frac{\phi(x,y)}{y-x}\, dx \right) dy$$

and show that

$$\lim_{M\to\infty} \lim_{L\to\infty} U_{L,M}$$

exists in the sense of (tempered) distributions. Denote the limit by U. Let

$$\Omega = \{(x,y) \in \mathbb{R}^2 : |x| \neq |y|\}.$$

Verify that U coincides with the function $(y^2 - x^2)^{-1}$ on Ω.

Definition and Remark: Let $\tilde{\phi}(x,y) := -\phi(y,x)$ and define $\widetilde{U}(\phi) := U(\tilde{\phi})$.
Note that if we formally write

$$U(\phi) = \iint \frac{\phi(x,y)}{y^2 - x^2}\, dx\, dy,$$

then by a change of variables,

$$\widetilde{U}(\phi) = \iint \frac{\phi(x,y)}{y^2 - x^2}\, dy\, dx.$$

This calculation is rigorous if $\phi \in C_c^\infty(\Omega)$, and we then have $U = \widetilde{U}$ on Ω.

(ii) Compute the Fourier transforms of U and $\widetilde{U}$.

Hint: You may use that

$$\lim_{L\to\infty} \int_{L^{-1}<|x|<L} \frac{e^{ixs}}{x}\, dx = i\pi\, \mathrm{sign}(s),$$

which follows from Cauchy's theorem.

(iii) Show that $U \neq \widetilde{U}$ in the sense of tempered distributions on $\mathbb{R}^2$. Determine $U - \widetilde{U}$.

Remark 0.1. By direct computation, one finds that the Fourier transform of $U - \widetilde{U}$ is the constant function 1. Hence

$$U - \widetilde{U} = \delta_0,$$

the Dirac distribution at the origin.

This exercise thus provides a concrete example where Fubini's theorem fails in the distributional setting. It also illustrates the power of Fourier transform methods in identifying certain singular distributions.

Problem 3. Let $u \in \mathcal{D}'(\mathbb{R})$ be a distribution such that $u' = 0$. Then there exists a constant $C \in \mathbb{R}$ such that

$$\langle u, \phi \rangle = C \int_{\mathbb{R}} \phi(x)\, dx \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

Proof.

Since $u' = 0$, by the definition of distributional derivative, we have

$$\langle u, \phi' \rangle = -\langle u', \phi \rangle = 0 \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

Thus, u vanishes on all derivatives of test functions.

Now suppose $\psi \in \mathcal{D}(\mathbb{R})$ with $\int \psi = 0$. Define

$$\Phi(x) := \int_{-\infty}^{x} \psi(t)\, dt.$$

Since ψ is compactly supported and $\int \psi = 0$, it follows that $\Phi \in \mathcal{D}(\mathbb{R})$ and $\Phi' = \psi$. Hence u vanishes on all test functions with zero integral.

If $\langle u, \phi \rangle = 0$ for all ϕ, then $u = 0$. Otherwise, pick $\phi_0 \in \mathcal{D}(\mathbb{R})$ with $\langle u, \phi_0 \rangle \neq 0$. For any $\psi \in \mathcal{D}(\mathbb{R})$, define

$$\tilde{\psi} := \psi - \frac{\int \psi}{\int \phi_0} \phi_0,$$

so that $\int \tilde{\psi} = 0$, hence $\langle u, \tilde{\psi} \rangle = 0$, and thus

$$\langle u, \psi \rangle = \frac{\int \psi}{\int \phi_0} \langle u, \phi_0 \rangle.$$

Letting $C := \dfrac{\langle u, \phi_0 \rangle}{\int \phi_0}$, we conclude

$$\langle u, \psi \rangle = C \int \psi(x)\, dx,$$

as desired. $\square$

Remark.

In this proof, we used that any compactly supported smooth function on $\mathbb{R}$ with zero integral is the derivative of another test function.

I tried to generalize this argument to $\mathbb{R}^n$, but encountered difficulty: given $\psi \in \mathcal{D}(\mathbb{R}^n)$

with $\int \psi = 0$, is it always possible to find a compactly supported smooth function $\Phi \in \mathcal{D}(\mathbb{R}^n)$ such that the directional derivative $e_i \cdot \nabla\Phi = \psi$ for some direction e_i? If not always, under what conditions is this possible?