Spectral Theory Notes

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1 Normed and Banach Spaces

Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. *A* seminorm *on X is a function $\|\cdot\| : X \to [0, \infty)$ satisfying:*

- $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$,
- $\|\alpha x\| = |\alpha| \cdot \|x\|$ *for all scalars* $\alpha \in \mathbb{F}, x \in X$.

If in addition $\|x\| = 0 \Rightarrow x = 0$, then $\|\cdot\|$ is a norm*, and X is a* normed space*.*

The norm induces a metric $d(x, y) := \|x - y\|$, which defines the *norm topology* on X.

Remark 1.2. *In $\mathbb{R}^n$, the open ball*

$$B_r(x) := \{y \in \mathbb{R}^n : \|x - y\| < r\}$$

is an open set in the standard topology. This illustrates how a norm induces a natural topology on normed spaces.

Definition 1.3. *A normed space X is a* Banach space *if it is complete with respect to the norm metric, i.e., every Cauchy sequence in X converges to a limit in X.*

Examples include:

- ℓ^p spaces for $1 \leq p \leq \infty$,
- $L^p(U)$,
- Sobolev spaces $H^1(U)$.

2 Inner Product and Hilbert Spaces

Definition 2.1. *An* inner product *on a complex vector space H is a map $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ such that for all $x, y, z \in H$ and $\alpha, \beta \in \mathbb{C}$,*

- $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$,
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$,
- $\langle x, x \rangle > 0$ *for all* $x \neq 0$.

The induced norm is $\|x\| := \sqrt{\langle x, x \rangle}$.

A vector space equipped with an inner product is a *pre-Hilbert space*. If it is complete with respect to the induced norm, it is called a *Hilbert space*.

Example 2.2. *The sequence space*

$$\ell^2(\mathbb{Z}) := \left\{ (a_n)_{n\in\mathbb{Z}} : \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty \right\}$$

is a Hilbert space with inner product

$$\langle a, b\rangle = \sum_{n\in\mathbb{Z}} a_n \overline{b_n}.$$

3 Bounded Operator

Definition 3.1 (Bounded Operator and Operator Norm)**.** *Let V be a normed vector space. A linear operator $A : V \to V$ is called* bounded *if*

$$\|A\| := \sup \{\|A\phi\| : \phi \in V,\ \|\phi\| = 1\} < \infty.$$

The set of bounded linear operators on V is denoted by $\mathcal{B}(V)$.

Remark 3.2. *$\mathcal{B}(V)$ is a normed vector space under the operator norm. In fact:*

- *If $A, B \in \mathcal{B}(V)$, and $\lambda_1, \lambda_2 \in \mathbb{C}$, then $\lambda_1 A + \lambda_2 B \in \mathcal{B}(V)$;*
- *$AB \in \mathcal{B}(V)$, and $\|AB\| \leq \|A\| \cdot \|B\|$.*

Definition 3.3 (Adjoint and Self-Adjoint Operators)**.** *Let H be a Hilbert space. The* adjoint *of a bounded linear operator $A \in \mathcal{B}(H)$ is the unique operator $A^* \in \mathcal{B}(H)$ such that*

$$\langle Ax, y\rangle = \langle x, A^* y\rangle \quad \text{for all } x, y \in H.$$

An operator is called self-adjoint *if $A = A^*$.*

Remark 3.4. *For $T, S \in \mathcal{B}(H)$, we have:*

$$\|T^*\| = \|T\|, \quad \|T^*T\| = \|T\|^2, \quad (aS + bT)^* = \bar{a}S^* + \bar{b}T^*, \quad (ST)^* = T^*S^*, \quad T^{**} = T.$$

Let $\mathscr{R}(A)$ and $\mathscr{N}(A)$ denote the range and null space of A, respectively. Then the following dualities hold:

$$\mathscr{R}(A)^\perp = \mathscr{N}(A^*), \quad \mathscr{N}(A)^\perp = \overline{\mathscr{R}(A^*)}.$$

Definition 3.5 (Closed Operator)**.** *Let $A : X \to Y$ be a linear operator between Banach spaces. We say that A is* closed *if whenever $u_n \to u$ in X and $Au_n \to v$ in Y, it follows that*

$$Au = v$$

.

Equivalently, the graph of A,

$$\mathrm{Graph}(A) := \{(x, Ax) \in X \times Y : x \in \mathcal{D}(A)\},$$

is closed in $X \times Y$.

Theorem 3.6 (Closed Graph Theorem)**.** *Let $A : X \to Y$ be a closed linear operator between Banach spaces. Then A is bounded.*

Remark 3.7. *Let $A \in \mathcal{B}(X)$, and suppose $A - \lambda I$ is one-to-one and onto. Then $(A - \lambda I)^{-1} \in \mathcal{B}(X)$, i.e., the inverse is also bounded.*

Indeed, define $B := (A - \lambda I)^{-1}$. Since the set

$$\{(x, (A - \lambda I)x)\} \subset X \times X$$

is closed. Then its inverse graph

$$\mathrm{Graph}(B) = \{(y, By)\} = \{((A - \lambda I)x, x)\} \subset X \times X$$

is also closed. So the Closed Graph Theorem implies that $B \in \mathcal{B}(X)$.

4 Spectral Theory of Bounded Operators

Definition 4.1 (Resolvent, Spectrum, Spectral Radius). *Let $A \in \mathcal{B}(X)$ be a bounded linear operator. The* resolvent set *of A is*

$$\rho(A) := \{\lambda \in \mathbb{C} : (A - \lambda I) \textit{ is one-to-one and onto.}\}$$

The spectrum *of A is*

$$\sigma(A) := \mathbb{C} \setminus \rho(A).$$

And the spectral radius *is*

$$\mathrm{spr}(A) := \sup\{|z| : z \in \sigma(A)\}.$$

Definition 4.2 (Eigenvalue, Point Spectrum). *A complex number $\lambda \in \sigma(A)$ is called an* eigenvalue *of A if*

$$\ker(A - \lambda I) \neq \{0\}.$$

The set of all eigenvalues is called the point spectrum *of A, denoted $\sigma_p(A)$. If $w \neq 0$ satisfies $Aw = \lambda w$, then w is called an* eigenvector *associated with λ.*

Remark 4.3. *In finite-dimensional spaces, the spectrum $\sigma(A)$ coincides with the set of eigenvalues $\sigma_p(A)$. In infinite-dimensional spaces, however, this is no longer true: the spectrum may strictly contain non-eigenvalues.*

Example 4.4. *Consider the discrete Laplacian Δ on $\ell^2(\mathbb{Z})$, defined by*

$$[\Delta\psi](n) = \psi(n+1) + \psi(n-1), \quad \psi \in \ell^2(\mathbb{Z}).$$

We claim that Δ has no eigenvalues. Indeed, suppose $\Delta\psi = z\psi$ for some $\psi \neq 0$. Then:

$$z\psi(n) = \psi(n+1) + \psi(n-1),$$

whose characteristic equation is:

$$\chi^2 - z\chi + 1 = 0 \quad \Rightarrow \quad \chi = \frac{z \pm \sqrt{z^2 - 4}}{2}.$$

Let α, β be the two roots. Then a general solution is:

$$\psi(n) = C_0\alpha^n + C_1\beta^{-n}.$$

Since $|\alpha\beta| = 1$, either $|\alpha| \geq 1$ or $|\beta| \geq 1$, so the solution cannot lie in $\ell^2(\mathbb{Z})$ unless $\psi = 0$. Thus, $\ker(\Delta - zI) = \{0\}$, and Δ has no eigenvalues.

We will later prove that the spectrum of every bounded operator is nonempty. Therefore, in this example $\sigma(\Delta) \neq \sigma_p(\Delta)$.

Definition 4.5 (Operator Convergence)**.** *Let $\{A_n\} \subset \mathcal{B}(V)$, and $A \in \mathcal{B}(V)$. We say that:*

- *$A_n \to A$ in **norm** if $\|A_n - A\| \to 0$;*
- *$A_n \to A$ **strongly** if for all $\phi \in V$, we have $A_n\phi \to A\phi$ in norm. This is denoted by*

$$A = \text{s-}\lim_{n\to\infty} A_n.$$

Remark 4.6. *Norm convergence implies strong convergence, but the converse is false in general. This is one of the major differences between operator theory in finite- and infinite-dimensional settings.*

Example 4.7. *Let $V = \ell^2(\mathbb{Z})$, and define a sequence of operators $\{T_n\} \subset \mathcal{B}(V)$:*

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \cdots$$

Let $T = I$ be the identity operator, whose matrix is

$$T = \text{id} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Then:

- *For all n, we have $\|T_n\| = 1$;*
- *For $n \neq m$, we have $\|T_n - T_m\| = 1$, so $\{T_n\}$ is not a Cauchy sequence in norm;*
- *However, for any $q \in \ell^2(\mathbb{Z})$, we have $T_n q \to Tq$ strongly.*

Therefore, $T_n \to T$ strongly but not in norm. This shows that strong convergence does not imply norm convergence in infinite-dimensional spaces.

Proposition 4.8. *Let $\{A_n\} \subset \mathcal{B}(V)$ be a sequence of bounded operators such that*

$$\sum_{n=1}^{\infty} \|A_n\| < \infty.$$

Then the series $A := \sum_{n=1}^{\infty} A_n$ converges in norm to an element of $\mathcal{B}(V)$, and

$$\|A\| \leq \sum_{n=1}^{\infty} \|A_n\|.$$

Theorem 4.9. *Let $\mathcal{A}(V) \subset \mathcal{B}(V)$ denote the set of invertible operators. Suppose $A \in \mathcal{A}(V)$ and $B \in \mathcal{B}(V)$ satisfies*

$$\|B\| < \|A^{-1}\|^{-1}.$$

Then $A + B \in \mathcal{A}(V)$.

Sketch of Proof. Write $A + B = A(I + A^{-1}B)$. Define the Neumann series

$$R := \sum_{n=0}^{\infty} (-1)^n (A^{-1}B)^n.$$

This series converges in norm because $\|A^{-1}B\| < 1$. Then

$$(I + A^{-1}B)R = R(I + A^{-1}B) = I, \quad \text{so} \quad (A + B)^{-1} = RA^{-1}.$$

□

Remark 4.10. *This theorem implies that $\mathcal{A}(V) \subset \mathcal{B}(V)$ is an open subset.*

Define the *resolvent function* $R(A, z) := (A - zI)^{-1}$, mapping $\rho(A) \to \mathcal{B}(V)$.

Proposition 4.11 (Basic Resolvent Identities)**.** *Let $z, z' \in \rho(A)$, and $B \in \mathcal{B}(V)$. Then:*

(a) First resolvent identity:

$$R(A, z) - R(A, z') = (z' - z)R(A, z)R(A, z').$$

(b) Second resolvent identity:

$$R(A, z) - R(B, z) = R(A, z)(B - A)R(B, z).$$

Sketch of (a). Since $A - z$ is bijective, for any $y \in V$, we can find $x \in V$ such that

$$y = (A - z)x.$$

Then we observe:

$$y + (z - z')x = (A - z')x.$$
$$(A - z')^{-1}y = x - (z - z')(A - z')^{-1}x.$$

Thus,

$$R(A, z)y - R(A, z')y = x - \left[x - (z - z')(A - z')^{-1}x\right] = (z - z')(A - z')^{-1}x.$$

Substituting $x = R(A, z)y$, we conclude:

$$R(A, z) - R(A, z') = (z' - z)R(A, z)R(A, z').$$

Moreover, we can see $R(A, z)R(A, z') = R(A, z')R(A, z)$.

A similar computation yields:

$$R(A, z) - R(B, z) = R(A, z)(B - A)R(B, z).$$

□

Proposition 4.12 (Spectral Radius Estimate)**.** *For any bounded operator $A \in \mathcal{B}(V)$, the spectral radius satisfies*

$$\operatorname{spr}(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\} \leq \|A\|.$$

Sketch of Proof. Consider

$$R(z) := \sum_{k=0}^{\infty} z^{-k-1} A^k,$$

which converges in operator norm for $|z| > \|A\|$. This shows that $R(z) = (A - zI)^{-1}$ exists for such z, so

$$\sup\{|\lambda| : \lambda \in \sigma(A)\} \leq \|A\|.$$

□

Example 4.13 (Spectral Radius of the Discrete Laplacian)**.** *Recall the discrete Laplacian Δ on $\ell^2(\mathbb{Z})$, defined by*

$$[\Delta\psi](n) = \psi(n+1) + \psi(n-1).$$

We have previously shown that Δ has no eigenvalues, i.e., $\sigma_p(\Delta) = \emptyset$. Moreover, it is known that the spectrum of Δ is

$$\sigma(\Delta) = [-2, 2].$$

To see that the spectral radius achieves the operator norm, consider the normalized vector

$$\psi^{(N)} := \frac{1}{\sqrt{2N+1}} \chi_{[-N,N]},$$

i.e., $\psi^{(N)}(n) = \frac{1}{\sqrt{2N+1}}$ for $|n| \leq N$, and zero otherwise. Then $\|\psi^{(N)}\| = 1$, and a direct computation shows

$$\|\Delta\psi^{(N)}\| \to 2 \quad \text{as } N \to \infty.$$

Hence, the operator norm $\|\Delta\| = 2$. Combining this with $\sigma(\Delta) = [-2, 2]$, we conclude

$$\operatorname{spr}(\Delta) = \|\Delta\| = 2.$$

Proposition 4.14. *Let $A \in \mathcal{B}(V)$. Then $\sigma(A) \neq \emptyset$.*

Proof. Suppose $\sigma(A) = \emptyset$, i.e., $\rho(A) = \mathbb{C}$. Then the resolvent

$$g(z) := (A - zI)^{-1}$$

is an entire $\mathcal{B}(V)$-valued function. Moreover, from the resolvent estimate, we know

$$\|g(z)\| = \|R(A, z)\| = O\left(\frac{1}{|z|}\right) \quad \text{as } |z| \to \infty.$$

By Liouville's theorem, this implies $g(z) \equiv 0$, which contradicts invertibility. □

Example 4.15. *Consider the right shift operator H on ℓ^1, defined by*

$$H(\xi_1, \xi_2, \xi_3, \dots) = (\xi_2, \xi_3, \xi_4, \dots).$$

Note that this operator is bounded but not self-adjoint, and ℓ^1 is not a Hilbert space, so this example lies outside the framework we discussed earlier.

We have:

- $\sigma(H) = \{z \in \mathbb{C} : |z| \leq 1\}$*;*
- *every $\lambda \in \mathbb{D}$ is an eigenvalue, with eigenvector $(1, \lambda, \lambda^2, \dots) \in \ell^1$.*

Its adjoint H^* acts on ℓ^∞ by

$$H^*(\xi_1, \xi_2, \xi_3, \dots) = (0, \xi_1, \xi_2, \dots).$$

Although $\sigma(H^*) = \sigma(H)$, the operator H^* has no eigenvalues.

Remark 4.16. *We conclude this section with three remarks:*

1. *Both **Theorem 4.9** and **Proposition 4.12** follow the same proof strategy: write the inverse operator as a Neumann series and then rigorously establish its convergence.*

2. *From **Theorem 4.9** we know that the resolvent set $\rho(A)$ is open, so the spectrum $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is closed. **Proposition 4.12** further shows that the spectrum is bounded. Therefore, $\sigma(A) \subset \mathbb{C}$ is compact.*

3. *One fundamental motivation for spectral theory is its application to solving PDEs of the form $Lu = f$. Understanding the existence and structure of the inverse operator L^{-1}, or more generally the resolvent $(L + \mu I)^{-1}$, is equivalent to analyzing the spectrum of L. For example, consider the second-order differential operator:*

   $$Lu = -u''(x),$$

   and the eigenvalue equation

   $$-u''(x) = \mu u(x), \quad \mu \in \mathbb{R}.$$

   We consider two boundary conditions:

   (a) *Dirichlet condition: $u(0) = u(1) = 0$.*

   (b) *Periodic condition: u is 1-periodic.*

   The general solution to $u'' + \mu u = 0$ depends on the sign of μ:

   $$u(x) = \begin{cases} C_1 e^{-\sqrt{-\mu}x} + C_2 e^{\sqrt{-\mu}x} & \mu < 0, \\ ax + b & \mu = 0, \\ C_1 \sin(\sqrt{\mu}x) + C_2 \cos(\sqrt{\mu}x) & \mu > 0. \end{cases}$$

   We now analyze which values of μ yield nontrivial solutions under the two boundary conditions:

   (i) *$\mu < 0$: No nontrivial solution satisfies either (a) or (b).*

   (ii) *$\mu = 0$: The general solution is linear. Under (a), the only solution is trivial. Under (b), constants are solutions, so $\mu = 0$ is an eigenvalue for (b) only.*

   (iii) *$\mu > 0$: Write $\mu = n^2\pi^2$. Then $u(x) = C\sin(n\pi x)$ satisfies (a), while $u(x) = C_1 \sin(n\pi x) + C_2 \cos(n\pi x)$ satisfies (b). Hence, for both (a) and (b), the eigenvalues are $\mu_n = n^2\pi^2$, and the eigenfunctions form a Fourier basis.*

5 Self-Adjoint Operators and Spectral Measures

For the sake of clarity, we omit certain technical proofs and focus on outlining the overall theoretical framework. The precise definition of self-adjoint operators and the proofs of the following results can be found in Section 1.4 of our textbook.

Proposition 5.1. *Let $A \in \mathcal{B}(H)$. Then:*

- $\|A^*\| = \|A\|$*;*
- $\|A^*A\| = \|A\|^2$*;*
- *For all $A, B \in \mathcal{B}(H)$, $a \in \mathbb{C}$, we have:*

$$(A+B)^* = A^* + B^*, \quad (aA)^* = \bar{a}A^*, \quad (AB)^* = B^*A^*.$$

Definition 5.2. *An operator $A \in \mathcal{B}(H)$ is called:*

- self-adjoint *if $A = A^*$;*
- unitary *if $AA^* = A^*A = I$;*
- normal *if $AA^* = A^*A$;*
- idempotent *if $A^2 = A$.*

Proposition 5.3. *If $A \in \mathcal{B}(H)$ is self-adjoint, then $\sigma(A) \subset \mathbb{R}$ is a compact set. Moreover, for any normal operator $A \in \mathcal{B}(H)$, we have:*

$$\|A\| = \operatorname{spr}(A).$$

Sketch of Proof. Let $z = x + iy$, with $y \neq 0$. Then for any $\phi \in H$, we compute:

$$\|(A-z)\phi\|^2 = \|(A-x)\phi - iy\phi\|^2 = \|(A-x)\phi\|^2 + y^2\|\phi\|^2 - \langle A\phi - x\phi, iy\phi\rangle - \langle iy\phi, A\phi - x\phi\rangle.$$

Since $\langle i\phi, \phi\rangle = -\langle \phi, i\phi\rangle$ and A is self-adjoint, we have:

$$\langle A\phi, i\phi\rangle = \langle \phi, iA\phi\rangle = -\langle i\phi, A\phi\rangle.$$

Hence the cross terms cancel, and we obtain:

$$\|(A-z)\phi\|^2 \geq y^2\|\phi\|^2,$$

which shows that $A - z$ is injective and bounded below.

To show surjectivity of $A - z$, observe that

$$\overline{\operatorname{Ran}(A-z)} = \ker((A-\bar{z})^*)^\perp = \ker(A-\bar{z})^\perp = \{0\}^\perp = H.$$

Thus, $\operatorname{Ran}(A-z)$ is dense. Since we already showed that $A - z$ is bounded below, it follows that its range is also closed. Hence,

$$\operatorname{Ran}(A-z) = H.$$

More explicitly, suppose $\psi_n = (A-z)\phi_n \to \psi$ in H. Then,

$$\|\phi_n - \phi_m\| \leq \frac{1}{|y|}\|\psi_n - \psi_m\|,$$

so $\{\phi_n\}$ is a Cauchy sequence, hence converges to some $\phi \in H$, and we have $\psi_n = (A-z)\phi_n \to (A-z)\phi$, thus $\psi = (A-z)\phi \in \mathrm{Ran}(A-z)$.

Therefore, $A-z$ is bijective, and $z \in \rho(A)$, i.e.,

$$\sigma(A) \subset \mathbb{R}.$$

Now when A is normal. First observe:

$$\|A^{2n}\| = \|A^*A\|^n = \|A\|^{2n}, \quad \text{so} \quad \|A^n\| = \|A\|^n.$$

By the general spectral radius formula,

$$\mathrm{spr}(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \lim_{n\to\infty} \|A\| = \|A\|.$$

□

Proposition 5.4 (Orthogonal Projection Operator)**.** *Suppose $V \subset H$ is a closed subspace of a Hilbert space, and let P denote the orthogonal projection onto V. Then:*

(a) $P \in \mathcal{B}(H)$, i.e., P is a bounded linear operator;

(b) $P^2 = P = P^*$, i.e., P is self-adjoint and idempotent;

(c) For any orthonormal basis $\{h_n\}$ of H, we have:

$$\dim V = \sum_n \langle h_n, Ph_n \rangle,$$

where the right-hand side is independent of the choice of orthonormal basis. We allow both sides to be infinite.

Remark 5.5. *If $A \in \mathbb{C}^{n\times n}$ is a self-adjoint matrix, then by the spectral theorem, we can write*

$$A = \sum_{\lambda\in\sigma(A)} \lambda P_\lambda,$$

where P_λ denotes the orthogonal projection onto the eigenspace $\ker(A-\lambda I)$. Since $P_\lambda P_\mu = 0$ for $\lambda \neq \mu$, and $P_\lambda^2 = P_\lambda$, for any polynomial ρ, we can see

$$\rho(A) = \sum_{\lambda\in\sigma(A)} \rho(\lambda) P_\lambda.$$

If f is any continuous function on $\mathbb{R}$, we can similarly define

$$f(A) := \sum_{\lambda\in\sigma(A)} f(\lambda) P_\lambda,$$

which shows that all information about A is captured by its spectrum $\sigma(A)$.

However, in the infinite-dimensional setting, the spectrum may not consist of eigenvalues at all, so the projections P_λ are not defined and the above formula does not make sense.

In general, if A is a self-adjoint operator on a Hilbert space and $f : \mathbb{R} \to \mathbb{C}$ is continuous, we use the fact that $\sigma(A) \subset \mathbb{R}$ is compact. By the Weierstrass approximation theorem, we can find a sequence of polynomials $\{p_n\}$ such that

$$p_n \to f \quad \textit{uniformly on } \sigma(A).$$

Then we define the continuous functional calculus by

$$f(A) := \lim_{n\to\infty} p_n(A).$$

The first thing we must verify is that the definition of $f(A)$ is independent of the choice of approximating polynomial sequence $\{p_n\}$.

Proposition 5.6. *Let $A \in \mathcal{B}(V)$ is normal, and let p be a polynomial. Then*

$$\sigma(p(A)) = p(\sigma(A)) := \{p(z) : z \in \sigma(A)\}.$$

Remark 5.7. *Combined with **Proposition 5.3**, if $A \in \mathcal{B}(H)$ is normal and p is a polynomial, then*

$$\|p(A)\| = \sup\{|p(z)| : z \in \sigma(A)\}.$$

Now suppose $\{p_n\}$, $\{q_n\}$ are two sequences of polynomials converging uniformly to a continuous function f on $\sigma(A)$. Then

$$\|p_n(A) - q_n(A)\| = \sup\{|p_n(z) - q_n(z)| : z \in \sigma(A)\} \to 0,$$

which shows that the operator

$$f(A) := \lim_{n\to\infty} p_n(A)$$

is well-defined and independent of the choice of approximating polynomials.

Proposition 5.8. *Based on the previous discussion, for any self-adjoint operator $A \in \mathcal{B}(H)$, we can define a map*

$$f \in C(\sigma(A)) \mapsto f(A) \in \mathcal{B}(H)$$

via uniform approximation by polynomials. This assignment satisfies the following properties for all $f, g \in C(\sigma(A))$, $a \in \mathbb{C}$:

- $(f+g)(A) = f(A) + g(A)$,
- $(af)(A) = af(A)$,
- $(fg)(A) = f(A)g(A)$, *(which implies commutativity)*
- $\big(f(A)\big)^* = \overline{f}(A)$,
- $\mathbf{1}(A) = I$, *where* $\mathbf{1}(z) \equiv 1$.

*Moreover, by **Proposition 5.3** and the polynomial approximation procedure, we have the operator norm estimate:*

$$\|f(A)\| = \|f\|_{L^\infty(\sigma(A))}.$$

As usual, we begin with the finite-dimensional setting. Suppose $A \in \mathbb{C}^{n\times n}$ is a self-adjoint matrix. Then for any $v \in \mathbb{C}^n$, we have:

$$\langle v, f(A)v\rangle = \sum_{\lambda\in\sigma(A)} f(\lambda)\|P_\lambda v\|^2,$$

where P_λ denotes the orthogonal projection onto $\ker(A - \lambda I)$.

This expression can be interpreted as an integral of f against the Dirac measure supported on $\sigma(A)$, with weights $\|P_\lambda v\|^2$ assigned to each $\lambda \in \sigma(A)$. That is,

$$\langle v, f(A)v\rangle = \int_{\sigma(A)} f(\lambda)\, d\mu_v(\lambda),$$

where $\mu_v := \sum_{\lambda\in\sigma(A)} \|P_\lambda v\|^2 \delta_\lambda$.

Even though in infinite-dimensional Hilbert spaces we may not be able to guarantee the existence of eigenvalues, the expression $\langle v, f(A)v\rangle$ still defines a measure μ_v on $\sigma(A)$, known as the spectral measure associated to v.

Theorem 5.9 (Spectral Theorem). *Let $A \in \mathcal{B}(H)$ be self-adjoint. Then for all $\phi, \psi \in H$, there exists a complex Borel measure $\mu = \mu^A_{\phi,\psi}$ on $\sigma(A)$, such that*

$$\langle \phi, f(A)\psi\rangle = \int_{\sigma(A)} f(\lambda)\, d\mu(\lambda)$$

for all $f \in C(\sigma(A))$. In particular, when $\phi = \psi$, μ is a positive measure.

Proof. Consider the map

$$f \mapsto \langle \phi, f(A)\psi\rangle.$$

This is a linear bounded functional on $C(\sigma(A))$, since

$$|\langle \phi, f(A)\psi\rangle| \leq \|f(A)\|\|\phi\|\|\psi\| \leq \|f\|_{C(\sigma(A))}\|\phi\|\|\psi\|.$$

By the Riesz representation theorem, there exists a unique complex Borel measure μ such that

$$\langle \phi, f(A)\psi\rangle = \int_{\sigma(A)} f(\lambda)\, d\mu(\lambda).$$

Now suppose $\phi = \psi$ and $f \geq 0$ in $C(\sigma(A))$. Then we can find $g \in C(\sigma(A))$ such that $g^2 = f$. By the functional calculus (via polynomial approximation), we have

$$f(A) = g(A)^2, \quad \text{and} \quad g(A) = g(A)^*.$$

So

$$\langle \phi, f(A)\phi\rangle = \langle \phi, g(A)^2\phi\rangle = \|g(A)\phi\|^2 \geq 0.$$

This shows that $\mu_{\phi,\phi}$ is a nonnegative regular Borel measure. $\square$