Norm Equivalence of Rademacher Functions

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Problem 1. For $n = 0, 1, \dots$ and $x \in [0,1]$, we define

$$r_0(x) = 1,$$

and for $n = 1, 2, \dots,$

$$r_n(x) = \sum_{k=1}^{2^n} (-1)^{k-1} 1_{\left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(x),$$

where $1_I(x)$ denotes the indicator function of the interval I.

Let $\{r_n\}_{n=0}^{\infty}$ be defined as above.

i: Let $m \in \mathbb{N}$ and $j_1 \le j_2 \le \cdots \le j_{2m-1} \le j_{2m}$. Assume there exists at least one $k \in \{1, \dots, m\}$ such that $j_{2k-1} < j_{2k}$. Show that

$$\int_0^1 \prod_{\nu=1}^{2m} r_{j_\nu}(x)\, dx = 0.$$

ii: Show that for $0 < p < \infty$, there exists a constant $B_p > 0$ such that for all N and all $a_1, \dots, a_N \in \mathbb{R}$, we have

$$\left(\int_0^1 \left| \sum_{n=1}^{N} a_n r_n(x) \right|^p dx \right)^{1/p} \le B_p \left(\sum_{n=1}^{N} |a_n|^2 \right)^{1/2}.$$

Hint. The interesting case is $p > 2$. First prove the inequality when p is an even integer.

iii: Show that for $1 < p < \infty$, there exists a constant $A_p > 0$ such that for all N and all $a_1, \dots, a_N \in \mathbb{R}$, we have

$$\left(\sum_{n=1}^{N} |a_n|^2 \right)^{1/2} \le A_p \left(\int_0^1 \left| \sum_{n=1}^{N} a_n r_n(x) \right|^p dx \right)^{1/p}.$$

Hint. The left-hand side is equal to $\left\| \sum_{n=1}^{N} a_n r_n \right\|_2$. Apply Hölder's inequality when$p > 2$.

iv: Deduce that the inequality remains valid for all $p > 0$.

Solution to (i):

Suppose $j_{2k-1} < j_{2k}$ for some k. Consider the subsequence $j_{2k}, j_{2k+1}, \dots, j_{2m}$. The number of elements from $2k$ to $2m$ is odd.

Thus, among the functions $r_{j_{2k}}, \dots, r_{j_{2m}}$, there must exist at least one function that appears an odd number of times. In fact, because the sequence is ordered, we can find the largest index j_n within this subsequence which appears an odd number of times.

Now, observe the following key fact:

Key Fact: Suppose $q_1 \le q_2 \le \cdots \le q_k$ are non-negative integers, and assume that

$$q_k > q_1, \dots, q_{k-1}.$$

Then

$$\int_0^1 r_{q_1}(x) r_{q_2}(x) \cdots r_{q_k}(x)\, dx = 0.$$

Indeed, since r_{q_k} is constant on intervals of length 2^{-q_k} and alternates sign between adjacent intervals, while the product $r_{q_1} \cdots r_{q_{k-1}}$ is constant on much larger intervals, the presence of a strictly larger r_{q_k} forces the integral to vanish due to cancellation between adjacent intervals.

In our case, the largest index j_n among $j_{2k}, \dots, j_{2m}$ that appears an odd number of times is strictly greater than any other indices in the product. Thus, we can apply the Key Fact to conclude that the integral vanishes:

$$\boxed{\int_0^1 \prod_{\nu=1}^{2m} r_{j_\nu}(x)\, dx = 0.}$$

Remark:

If the condition were $j_{2k} < j_{2k+1}$ instead of $j_{2k-1} < j_{2k}$, the conclusion would not hold: all r_{j_n} could appear in pairs, leading to a nonzero integral.

Solution to (ii):

Step 1: The case $0 < p < 2$.

Apply Hölder's inequality with exponents $\dfrac{2}{p} > 1$ and $\dfrac{2}{2-p}$. We have

$$\left(\int_0^1 \left| \sum_{n=1}^N a_n r_n(x) \right|^p dx \right)^{1/p} \leq \left(\int_0^1 \left| \sum_{n=1}^N a_n r_n(x) \right|^2 dx \right)^{1/2} \left(\int_0^1 1 \, dx \right)^{\frac{1}{p} - \frac{1}{2}}.$$

$$\left(\int_0^1 \left| \sum_{n=1}^N a_n r_n(x) \right|^p dx \right)^{1/p} \leq \left(\sum_{n=1}^N |a_n|^2 \right)^{1/2}.$$

Thus, the desired inequality holds for $0 < p < 2$ with $B_p = 1$.

Step 2: The case where $p = 2M$ for some integer $M \geq 1$.

Using the conclusion of (i), we have

$$\int_0^1 \left| \sum_{n=1}^N a_n r_n(x) \right|^{2M} dx \leq (2M)! \left(\sum_{n=1}^N |a_n|^2 \right)^M,$$

Taking $1/2M$-th root, we get

$$\left(\int_0^1 \left| \sum_{n=1}^N a_n r_n(x) \right|^{2M} dx \right)^{1/2M} \leq B_p \left(\sum_{n=1}^N |a_n|^2 \right)^{1/2}.$$

Thus, the inequality holds for even integers $p = 2M$.

Step 3: The general case $p > 0$.

Suppose p is close to an even integer, say $p = 2M\varepsilon$ with $0 < \varepsilon < 1$.

Using Hölder's inequality again:

$$\left(\int_0^1 \left| \sum_{n=1}^N a_n r_n(x) \right|^{2M\varepsilon} dx \right) \leq \left(\int_0^1 \left| \sum_{n=1}^N a_n r_n(x) \right|^{2M} dx \right)^{\varepsilon} \left(\int_0^1 1 \, dx \right)^{1-\varepsilon}.$$

Thus,

$$\left(\int_0^1 \left| \sum_{n=1}^N a_n r_n(x) \right|^p dx \right)^{1/p} \leq \left((2M)! \left(\sum_{n=1}^N |a_n|^2 \right)^M \right)^{\varepsilon/p} = B_p \left(\sum_{n=1}^N |a_n|^2 \right)^{1/2}.$$

Hence, the desired inequality holds for all $p > 0$.

□

Solution to (iii), (iv):

Case 1: $p > 2$.

Applying Hölder's inequality,

$$\left(\int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right|^2 dx\right)^{1/2} \leq \left(\int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right|^p dx\right)^{1/p}.$$

Thus, the desired inequality holds with $A_p = 1$ when $p > 2$.

Case 2: $1 < p < 2$.

Let p' be the conjugate exponent, $\frac{1}{p} + \frac{1}{p'} = 1$, so that $p' > 2$.

Applying Hölder's inequality again:

$$\int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right|^2 dx = \int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right| \cdot \left|\sum_{n=1}^N a_n r_n(x)\right| dx$$

$$\leq \left(\int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right|^p dx\right)^{1/p} \left(\int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right|^{p'} dx\right)^{1/p'}.$$

Using part (ii), we know that

$$\left(\int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right|^{p'} dx\right)^{1/p'} \leq B_{p'} \left(\sum_{n=1}^N |a_n|^2\right)^{1/2}.$$

Thus,

$$\left\|\sum_{n=1}^N a_n r_n\right\|_{L^2} \leq A_p \left(\int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right|^p dx\right)^{1/p}.$$

Case 3: $0 < p < 1$.

First consider $\frac{1}{2} < p < 1$.

We rewrite:

$$\int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right|^2 dx = \int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right|^{\frac{3}{2}} \left|\sum_{n=1}^N a_n r_n(x)\right|^{\frac{1}{2}} dx.$$

Applying Hölder's inequality with exponents $2p$ and $2p/(2p-1)$, we get:

$$\leq \left(\int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right|^p dx\right)^{1/2p} \left(\int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right|^{\frac{3p}{2p-1}} dx\right)^{\frac{2p-1}{3p}\times\frac{3}{2}}.$$

Using part (ii) again, we can bound the second term, and obtain:

$$\left\|\sum_{n=1}^N a_n r_n\right\|_{L^2} \leq A_p \left(\int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right|^p dx\right)^{1/p}.$$

Applying the same strategy iteratively for all $\frac{1}{2^k} < p < \frac{1}{2^{k-1}}$, we conclude that:

$$\left\|\sum_{n=1}^N a_n r_n\right\|_{L^2} \leq A_p \left(\int_0^1 \left|\sum_{n=1}^N a_n r_n(x)\right|^p dx\right)^{1/p},$$

for all $p > 0$.

In fact, combining the results from (ii), (iii), and (iv), we see that for all $p > 0$,

$$\left\|\sum_{n=1}^N a_n r_n\right\|_{L^2} \asymp \left\|\sum_{n=1}^N a_n r_n\right\|_{L^p},$$

that is, the L^2 and L^p norms of finite Rademacher sums are equivalent.

$\square$