Notes on Two Adjoint–Method Problems

1 Problem 1: Static Case

We consider

$$\begin{cases} u(x) + H(x, Du) = 0, & x \in \mathbb{T}^n, \\ u^\varepsilon(x) + H(x, Du^\varepsilon) = \varepsilon \Delta u^\varepsilon. \end{cases} \tag{1.1}$$

Differentiate twice with respect to x_i, $1 \le i \le n$, to obtain

$$\begin{aligned} &u^\varepsilon_{x_i x_i} + 2H_{x_i p_k}(x, Du^\varepsilon) Du^\varepsilon_{x_i x_k} + H_{x_i x_i}(x, Du^\varepsilon) \\ &\qquad + H_{p_k p_l}(x, Du^\varepsilon)\, u^\varepsilon_{x_i x_k} u^\varepsilon_{x_i x_l} + D_p H(x, Du^\varepsilon) \cdot Du^\varepsilon_{x_i x_i} = \varepsilon \Delta u^\varepsilon_{x_i x_i}. \end{aligned} \tag{1.2}$$

Since

$$H_{p_k p_l} u^\varepsilon_{x_i x_k} u^\varepsilon_{x_i x_l} \ge \theta |Du^\varepsilon_{x_i}|^2, \qquad 2H_{x_i p_k} u^\varepsilon_{x_i x_k} \le \tfrac{\theta}{2} |Du^\varepsilon_{x_i}|^2 + C, \tag{1.3}$$

and $H_{x_i x_i}$ is bounded, summing over i, we obtain

$$\Delta u^\varepsilon + D_p H(x, Du^\varepsilon) \cdot D(\Delta u^\varepsilon) - \varepsilon \Delta(\Delta u^\varepsilon) + \tfrac{\theta}{2} |D^2 u^\varepsilon|^2 \le C. \tag{1.4}$$

Let

$$z(x,t) = e^t \chi(t) \Delta u^\varepsilon, \qquad \chi(t) = t(T - t), \tag{1.5}$$

then

$$\partial_t z - \varepsilon \Delta z + D_p H(x, Du^\varepsilon) \cdot Dz + \tfrac{\theta}{2} e^t \chi(t) |D^2 u^\varepsilon|^2 \le C + \chi'(t) e^t \Delta u^\varepsilon(x). \tag{1.6}$$

Now consider

$$\begin{cases} -\partial_t \sigma^\varepsilon - \varepsilon \Delta \sigma^\varepsilon - \operatorname{div}(D_p H(x, Du^\varepsilon) \sigma^\varepsilon) = 0, \\ \sigma^\varepsilon_T = \delta_{x_0}. \end{cases} \tag{1.7}$$

Then

$$\frac{d}{dt}\int_{\mathbb{T}^n} z(x,t)\sigma^\varepsilon\,dx = \int_{\mathbb{T}^n} z_t\sigma^\varepsilon\,dx + \int_{\mathbb{T}^n} z\sigma^\varepsilon_t\,dx. \tag{1.8}$$

Since $\sigma^\varepsilon \geq 0$, by (1.6) and (1.7),

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^n} z\,\sigma^\varepsilon\,dx \leq{}& \overbrace{\int_{\mathbb{T}^n}\big(\varepsilon\Delta z - D_pH(x,Du^\varepsilon)\cdot Dz\big)\,\sigma^\varepsilon\,dx + \int_{\mathbb{T}^n} z\,\sigma^\varepsilon_t\,dx}^{=\,0}\\ &-\frac{\theta}{2}\int_{\mathbb{T}^n} e^t\chi(t)\,|D^2u^\varepsilon|^2\,\sigma^\varepsilon\,dx + \int_{\mathbb{T}^n}\chi'(t)e^t\Delta u^\varepsilon\,\sigma^\varepsilon\,dx + C.\end{aligned} \tag{1.9}$$

Applying [BKRS15, Cor. 7.2.3], when

$$\nu > \frac{n+2}{2}, \tag{1.10}$$

we have

$$\sigma^\varepsilon(x,t) \leq C\Big(1+\frac{1}{\varepsilon}\Big)^\nu t^{-\frac{n}{2}}\left(1+\frac{t^{2\nu}}{\varepsilon^\nu}\right). \tag{1.11}$$

Therefore, for all t, when $\sigma^\varepsilon(x,t)\geq 1$,

$$|\log\sigma^\varepsilon(x,t)| \leq C\big(1+|\log\varepsilon|+|\log t|\big). \tag{1.12}$$

Cosequently,

$$\begin{aligned}\int_{\mathbb{T}^n}\sigma^\varepsilon(x,t)\,|\log\sigma^\varepsilon(x,t)|\,dx ={}& \int_{\{x\in\mathbb{T}^n:\sigma^\varepsilon(x,t)\geq 1\}}\sigma^\varepsilon(x,t)\,|\log\sigma^\varepsilon(x,t)|\,dx\\ &+\int_{\{x\in\mathbb{T}^n:\sigma^\varepsilon(x,t)<1\}}\sigma^\varepsilon(x,t)\,|\log\sigma^\varepsilon(x,t)|\,dx\\ \leq{}& \int_{\mathbb{T}^n}\tfrac{1}{e}\,dx + C\big(1+|\log\varepsilon|+|\log t|\big)\int_{\mathbb{T}^n}\sigma^\varepsilon(x,t)\,dx\\ \leq{}& C\big(1+|\log\varepsilon|+|\log t|\big).\end{aligned} \tag{1.13}$$

Then

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{T}^n} \sigma^\varepsilon \log \sigma^\varepsilon \, dx &= \int_{\mathbb{T}^n} \sigma_t^\varepsilon \, dx + \int_{\mathbb{T}^n} \sigma_t^\varepsilon \log \sigma^\varepsilon \, dx \\
&= -\int_{\mathbb{T}^n} \varepsilon \Delta \sigma^\varepsilon \log \sigma^\varepsilon \, dx - \int_{\mathbb{T}^n} \operatorname{div}\big(D_pH(x, Du^\varepsilon)\sigma^\varepsilon\big) \log \sigma^\varepsilon \, dx \\
&= \varepsilon \int_{\mathbb{T}^n} \frac{|D\sigma^\varepsilon|^2}{\sigma^\varepsilon} \, dx + \int_{\mathbb{T}^n} D_pH(x, Du^\varepsilon) \cdot D\sigma^\varepsilon \, dx.
\end{aligned} \tag{1.14}$$

Thus

$$\begin{aligned}
\varepsilon \int_{\mathbb{T}^n} \frac{|D\sigma^\varepsilon|^2}{\sigma^\varepsilon} \, dx &= \frac{d}{dt}\int_{\mathbb{T}^n} \sigma^\varepsilon \log \sigma^\varepsilon \, dx - \int_{\mathbb{T}^n} D_pH(x, Du^\varepsilon) \cdot D\sigma^\varepsilon \, dx \\
&\le \frac{d}{dt}\int_{\mathbb{T}^n} \sigma^\varepsilon \log \sigma^\varepsilon \, dx + \frac{\varepsilon}{2}\int_{\mathbb{T}^n} \frac{|D\sigma^\varepsilon|^2}{\sigma^\varepsilon} \, dx + \frac{C}{\varepsilon}\int_{\mathbb{T}^n} \sigma^\varepsilon \, dx.
\end{aligned} \tag{1.15}$$

$$\frac{\varepsilon}{2}\int_{\mathbb{T}^n} \frac{|D\sigma^\varepsilon|^2}{\sigma^\varepsilon} \, dx \le \frac{d}{dt}\int_{\mathbb{T}^n} \sigma^\varepsilon \log \sigma^\varepsilon \, dx + \frac{C}{\varepsilon}. \tag{1.16}$$

By mean value theorem, $\exists\, t_1 \in [\varepsilon, 2\varepsilon],\ t_2 \in [T-2\varepsilon, T-\varepsilon]$, such that

$$\int_{\mathbb{T}^n} \frac{|D\sigma^\varepsilon(x,t_i)|^2}{\sigma^\varepsilon(x,t_i)} \, dx \le \frac{C}{\varepsilon^2}\,(1 + |\log \varepsilon|), \qquad i = 1, 2. \tag{1.17}$$

Then

$$\int_{\mathbb{T}^n} |D\sigma^\varepsilon| \, dx \le \Big(\int_{\mathbb{T}^n} \frac{|D\sigma^\varepsilon|^2}{\sigma^\varepsilon} \, dx\Big)^{1/2}\Big(\int_{\mathbb{T}^n} \sigma^\varepsilon \, dx\Big)^{1/2} \le \frac{C}{\varepsilon}(1 + |\log \varepsilon|)^{1/2}. \tag{1.18}$$

Therefore,by (1.9)

$$\begin{aligned}
\frac{\theta}{2}\int_{t_1}^{t_2}\int_{\mathbb{T}^n} e^t \chi(t)\, |D^2u^\varepsilon|^2\, \sigma^\varepsilon \, dx\, dt \le \int_{\mathbb{T}^n} z(x,t_1)\, \sigma^\varepsilon(x,t_1) \, dx - \int_{\mathbb{T}^n} z(x,t_2)\, \sigma^\varepsilon(x,t_2) \, dx \\
+ \int_{t_1}^{t_2}\int_{\mathbb{T}^n} \chi'(t) e^t \Delta u^\varepsilon \, \sigma^\varepsilon \, dx\, dt + C.
\end{aligned} \tag{1.19}$$

Moreover, by (1.18)

$$\begin{aligned}
\Big|\int_{\mathbb{T}^n} z(x,t_i)\, \sigma^\varepsilon(x,t_i) \, dx\Big| &\le e^{t_i}\chi(t_i) \int_{\mathbb{T}^n} |Du^\varepsilon(t_i)|\, |D\sigma^\varepsilon(t_i)| \, dx \\
&\le \frac{C}{\varepsilon} t_i (T - t_i)(1 + |\log \varepsilon|)^{1/2}, \quad i = 1, 2.
\end{aligned} \tag{1.20}$$

Since $t_1 \in [\varepsilon, 2\varepsilon]$, $t_2 \in [T-2\varepsilon, T-\varepsilon]$,

$$\Big| \int_{\mathbb{T}^n} z(x,t_1)\, \sigma^\varepsilon(x,t_1)\, dx - \int_{\mathbb{T}^n} z(x,t_2)\, \sigma^\varepsilon(x,t_2)\, dx \Big| \le C\, (1+|\log\varepsilon|)^{1/2}. \quad (1.21)$$

And

$$\begin{aligned} \int_{t_1}^{t_2}\int_{\mathbb{T}^n} \chi'(t) e^t \Delta u^\varepsilon \sigma^\varepsilon\, dx\, dt &\le \frac{\theta}{4}\int_{t_1}^{t_2}\int_{\mathbb{T}^n} e^t \chi(t) |D^2 u^\varepsilon|^2 \sigma^\varepsilon\, dx\, dt \\ &\quad + \frac{C}{\theta}\int_{t_1}^{t_2} \frac{(\chi'(t))^2}{\chi(t)} e^t \int_{\mathbb{T}^n} \sigma^\varepsilon\, dx\, dt \\ &\le \frac{\theta}{4}\int_{t_1}^{t_2}\int_{\mathbb{T}^n} e^t \chi(t) |D^2 u^\varepsilon|^2 \sigma^\varepsilon\, dx\, dt + C\,(1+|\log\varepsilon|). \end{aligned} \quad (1.22)$$

Plugging all together, we get

$$\int_{t_1}^{t_2}\int_{\mathbb{T}^n} \chi(t) e^t |D^2 u^\varepsilon|^2\, \sigma^\varepsilon\, dx\, dt \le C\Big((1+|\log\varepsilon|)^{1/2} + (1+|\log\varepsilon|) \Big). \quad (1.23)$$

Finally,

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^n} e^t \Delta u^\varepsilon\, \sigma^\varepsilon\, dx\, dt &\le \sqrt{n} \int_0^T \int_{\mathbb{T}^n} e^t |D^2 u^\varepsilon|\, \sigma^\varepsilon\, dx\, dt \\ &= \sqrt{n} \int_0^{t_1} \int_{\mathbb{T}^n} e^t |D^2 u^\varepsilon|\, \sigma^\varepsilon\, dx\, dt \\ &\quad + \sqrt{n} \int_{t_1}^{t_2} \int_{\mathbb{T}^n} e^t |D^2 u^\varepsilon|\, \sigma^\varepsilon\, dx\, dt \\ &\quad + \sqrt{n} \int_{t_2}^{T} \int_{\mathbb{T}^n} e^t |D^2 u^\varepsilon|\, \sigma^\varepsilon\, dx\, dt. \end{aligned} \quad (1.24)$$

Moreover, for the first interval we have

$$\begin{aligned} \int_0^{t_1} \int_{\mathbb{T}^n} e^t |D^2 u^\varepsilon|\, \sigma^\varepsilon\, dx\, dt &\le \left(\int_0^{t_1} \int_{\mathbb{T}^n} e^{2t} |D^2 u^\varepsilon|^2\, \sigma^\varepsilon\, dx\, dt \right)^{1/2} (t_1)^{1/2} \\ &\le \left(\tfrac{C}{\varepsilon}\right)^{1/2} \varepsilon^{1/2} = C. \end{aligned} \quad (1.25)$$

Similarly,

$$\int_{t_2}^{T} \int_{\mathbb{T}^n} e^t |D^2 u^\varepsilon|\, \sigma^\varepsilon\, dx\, dt \le C. \quad (1.26)$$

For the middle part, by Cauchy–Schwarz,

$$\left(\int_{t_1}^{t_2}\int_{\mathbb{T}^n} e^t|D^2u^\varepsilon|\,\sigma^\varepsilon\,dx\,dt\right)^2 \le \left(\int_{t_1}^{t_2}\int_{\mathbb{T}^n} t(T-t)e^t|D^2u^\varepsilon|^2\,\sigma^\varepsilon\,dx\,dt\right) \times \left(\int_{t_1}^{t_2} \frac{e^t}{t(T-t)}\,dt\right). \tag{1.27}$$

Since

$$\int_{t_1}^{t_2} \frac{e^t}{t(T-t)}\,dt \;\approx\; |\log\varepsilon|, \tag{1.28}$$

By (1.23), we obtain

$$\left(\int_{t_1}^{t_2}\int_{\mathbb{T}^n} e^t|D^2u^\varepsilon|\,\sigma^\varepsilon\,dx\,dt\right)^2 \le C(1+|\log\varepsilon|)^2. \tag{1.29}$$

Overall, combining (1.24)–(1.29), we deduce

$$\left|\int_0^T\int_{\mathbb{T}^n} e^t\Delta u^\varepsilon\,\sigma^\varepsilon\,dx\,dt\right| \le C(1+|\log\varepsilon|). \tag{1.30}$$

Remark 1. Looking back at (1.27), (1.22) and (1.20), we see that the estimate above is mainly influenced by the following three quantities:

$$\int_{t_1}^{t_2} \frac{1}{\chi(t)}\,dt, \qquad \int_{t_1}^{t_2} \frac{(\chi'(t))^2}{\chi(t)}\,dt, \qquad \chi(t_i)\,\frac{(1+|\log\varepsilon|)^{1/2}}{\varepsilon}. \tag{1.31}$$

Among them, the first two are the dominant ones; therefore, even if the bound for $\int_{\mathbb{T}^n}|D\sigma^\varepsilon(t_i)|\,dx$ (which underlies the third term) is relaxed, the conclusion still remains the same.

I also experimented with different choices of $\chi(t)$, for instance $\chi(t) = t^\alpha|\log t|^\beta$, but the final estimate did not improve.

Remark 2. In fact, by the method of Cirant–Goffi, for any $|t_1-t_2|<\varepsilon$ one can find $t_3\in(t_1,t_2)$ such that

$$\int_{\mathbb{T}^n} \frac{|D\sigma^\varepsilon(t_3)|^2}{\sigma^\varepsilon(t_3)}\,dx \le \frac{C}{\varepsilon}\,(1+|\log\varepsilon|). \tag{1.32}$$

Moreover, for times away from $t=T$, the fundamental solution σ^ε is smoother, so one should be able to obtain stronger estimates in that region.

Remark 3. Alternatively, if instead of multiplying by e^t to convert the static problem into a time–dependent one, we mimic the role of $\chi(t)$ by introducing a spatial test function $\omega(x)$ that vanishes near x_0, then in the end we are led, as in Problem 2, to estimate

$$\int_{\mathbb{T}^n} |D(\Delta u^\varepsilon)|\, \sigma^\varepsilon \, dx.$$

Therefore, if Problem 2 can eventually be resolved, it should also provide a new approach to Problem 1.

2 Problem 2: Degenerate Viscous Case

We consider the problem

$$\begin{cases} u_t + H(x, Du) = a(x)\Delta u, \quad x \in \mathbb{T}^n,\ t > 0, \\ u(x,0) = g(x), \end{cases} \tag{2.1}$$

$$\begin{cases} u^\varepsilon_t + H(x, Du^\varepsilon) = (\varepsilon + a(x))\Delta u^\varepsilon, \\ u^\varepsilon(x,0) = g(x). \end{cases} \tag{2.2}$$

where $a(x) \in C^\infty(\mathbb{T}^n)$ and $a(x) \geq 0$.

Differentiating (2.2) with respect to ε yields

$$\partial_t u^\varepsilon_\varepsilon + D_p H(x, Du^\varepsilon) \cdot Du^\varepsilon_\varepsilon = (a(x) + \varepsilon)\Delta u^\varepsilon_\varepsilon + \Delta u^\varepsilon. \tag{2.3}$$

To analyze this, we introduce the adjoint equation

$$\begin{cases} -\partial_t \sigma^\varepsilon - \operatorname{div}\big(D_p H(x, Du^\varepsilon)\, \sigma^\varepsilon\big) - \Delta\big((a(x) + \varepsilon)\sigma^\varepsilon\big) = 0, \\ \sigma^\varepsilon|_{t=T} = \delta_{x_0}. \end{cases} \tag{2.4}$$

Equivalently, this can be written in divergence form as

$$\partial_t \sigma^\varepsilon + \operatorname{div}\big(\alpha(x)\sigma^\varepsilon + \beta(x) \cdot D\sigma^\varepsilon\big) = 0. \tag{2.5}$$

Hence the fundamental solution still satisfies

$$\begin{cases} \sigma^\varepsilon(x,t) \geq 0, \\ \int_{\mathbb{T}^n} \sigma^\varepsilon(x,t)\, dx = 1. \end{cases} \tag{2.6}$$

Proceeding as in (1.2)–(1.4), we obtain

$$\begin{aligned}\partial_t u^\varepsilon_{x_ix_i} &+ D_pH(x,Du^\varepsilon)\cdot Du^\varepsilon_{x_ix_i} + 2H_{x_ip_j}(x,Du^\varepsilon)\,u^\varepsilon_{x_ix_j} + H_{x_ix_i}(x,Du^\varepsilon)\\ &\qquad + H_{p_jp_k}(x,Du^\varepsilon)\,u^\varepsilon_{x_ix_j}u^\varepsilon_{x_ix_k}\\ &= (a(x)+\varepsilon)\Delta u^\varepsilon_{x_ix_i} + a_{x_ix_i}(x)\Delta u^\varepsilon + 2a_{x_i}(x)\Delta u^\varepsilon_{x_i}.\end{aligned} \tag{2.7}$$

Comparing with (1.2), we observe the presence of additional terms $2a_{x_i}\Delta u^\varepsilon_{x_i}$ and $a_{x_ix_i}\Delta u^\varepsilon$. Since $a\in C^\infty(\mathbb{T}^n)$, for the latter we have

$$a_{x_ix_i}\Delta u^\varepsilon(x) \le \delta|D^2u^\varepsilon|^2 + \frac{C}{\delta}. \tag{2.8}$$

Following the method used in (1.4), we obtain

$$\partial_t(\Delta u^\varepsilon) + D\phi\cdot D(\Delta u^\varepsilon) - (a(x)+\varepsilon)\Delta(\Delta u^\varepsilon) + \tfrac{\theta}{2}|D^2u^\varepsilon|^2 - 2Da\cdot D(\Delta u^\varepsilon) \le C. \tag{2.9}$$

By mimicking the argument in (1.19), we derive

$$\frac{\theta}{2}\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\chi(t)|D^2u^\varepsilon|^2\sigma^\varepsilon\,dxdt \le \int_{t_1}^{t_2}\int_{\mathbb{T}^n}\chi(t)\sigma^\varepsilon Da\cdot D(\Delta u^\varepsilon)\,dxdt + C(1+|\log\varepsilon|). \tag{2.10}$$

Moreover,

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\chi(t)\sigma^\varepsilon Da\cdot D(\Delta u^\varepsilon) = &-\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\chi(t)\sigma^\varepsilon\Delta a\,\Delta u^\varepsilon\,dxdt\\ &-\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\chi(t)D\sigma^\varepsilon\cdot Da\,\Delta u^\varepsilon\,dxdt.\end{aligned} \tag{2.11}$$

Hence,

$$\begin{aligned}\left|\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\chi(t)\sigma^\varepsilon Da\cdot D(\Delta u^\varepsilon)\,dxdt\right| \le &\frac{\theta}{4}\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\chi(t)|D^2u^\varepsilon|^2\sigma^\varepsilon\,dxdt\\ &+ C\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\chi(t)\frac{|D\sigma^\varepsilon|^2}{\sigma^\varepsilon}\,dxdt + C.\end{aligned} \tag{2.12}$$

In particular, using the boundedness of $|\Delta a|$ and $|Da|$, we deduce

$$\left|\int_{t_1}^{t_2}\int_{\mathbb{T}^n} \chi(t)\sigma^\varepsilon \Delta a\, \Delta u^\varepsilon \, dxdt\right| \leq \tfrac{\delta}{2}\int_{t_1}^{t_2}\int_{\mathbb{T}^n} \chi(t)|D^2u^\varepsilon|^2\sigma^\varepsilon \, dxdt + \frac{C_1}{\delta}\int_{t_1}^{t_2} \chi(t)\, dt, \tag{2.13}$$

and

$$\left|\int_{t_1}^{t_2}\int_{\mathbb{T}^n} \chi(t)D\sigma^\varepsilon\cdot Da\, \Delta u^\varepsilon \, dxdt\right| \leq \tfrac{\delta}{2}\int_{t_1}^{t_2}\int_{\mathbb{T}^n} \chi(t)|D^2u^\varepsilon|^2\sigma^\varepsilon \, dxdt + \frac{C_2}{\delta}\int_{t_1}^{t_2}\int_{\mathbb{T}^n} \chi(t)\frac{|D\sigma^\varepsilon|^2}{\sigma^\varepsilon}\, dxdt. \tag{2.14}$$

Since $\int_{t_1}^{t_2} \chi(t)\, dt$ is bounded, choosing $\delta = \frac{\theta}{4}$ completes the proof.

Remark 4. Because the conclusion of [BKRS15, Cor. 7.2.3] still applies, from (1.16) we know that

$$\int_{t_1}^{t_2}\int_{\mathbb{T}^n} \frac{|D\sigma^\varepsilon|^2}{\sigma^\varepsilon}\, dxdt \ \leq \ O\big(\tfrac{1}{\varepsilon^2}\big)\,.$$

Consequently, even if we approximate $\chi(t) \approx \varepsilon$, the estimate

$$\int_{t_1}^{t_2}\int_{\mathbb{T}^n} \chi(t)\frac{|D\sigma^\varepsilon|^2}{\sigma^\varepsilon}\, dxdt \ \leq \ O\big(\tfrac{1}{\varepsilon}\big)$$

still fails to meet the requirement needed for our argument. Therefore, just as discussed in Remark 2, an improvement of the original method is necessary.

References

[BKRS15] V. I. Bogachev, N. V. Krylov, M. Röckner, and S. V. Shaposhnikov, *Fokker–Planck–Kolmogorov equations*, Mathematical Surveys and Monographs, vol. 207, American Mathematical Society, Providence, RI, 2015.