A brief introduction to the mapping degree of manifolds

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A good geometrical or visual understanding
is the first step in almost any part of mathematics.

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Outline

Basic concepts

Smooth manifold

A smooth manifold can be thought of as a mathematical object similar to a "curved space". Although it may be complex as a whole, it can be approximated as a Euclidean space on a small enough scale, which can be represented by a set of coordinates on a Euclidean space, which allows us to perform derivative and integration operations.

Tangent space

For any point p in the manifold M, all the vectors tangent to the manifold at the p point form a set of linear spaces, which is the tangent space of the manifold at the p point, denoted as T_pM, and any vector in T_pM is the tangent vector of the manifold at the p point. Taking the spherical surface S^2 of three-dimensional space as an example, its tangent space at each point is the tangent plane of the sphere at that point.

Basic concepts

Vector field

A vector field is a smooth mapping $X : MtoT_pM$ of each point on the manifold to its tangent space, and the totality of all vector fields on M is denoted as $\mathfrak{X}(M)$. Intuitively, a vector field can be thought of as a set of "streamlines" on a manifold that describe the direction of a certain "flow".

图: Vector field on S^1

Mod 2 degree

Definition

Let $f: M^n \to N^n$ be a smooth map, where M is a compact edgeless manifold and N is a connected manifold. If $y \in N$ is a regular value of f, then the number of preimage points for y is $\# f^{-1}(y)$. Define $\# f^{-1}(y) mod 2$ as mod 2 degrees mapped to f.

Mod 2 degree

In order to illustrate that the definition of mod 2 degree is good, the following two points need to be verified:

1. $\#f^{-1}(y)$ is meaningful, i.e., $f^{-1}(y)$ is a finite set;
2. The definition of item modulus 2 degrees does not depend on the choice of the regular value y.

For the first point, we only need to note that $f^{-1}(y)$ is compact as a closed set in the compact manifold M, and because f is a locally smooth homomorphism from M to N near $x \in f^{-1}(y)$, $f^{-1}(y)$ is discrete, and it is known to be a finite set when combined with compactness. Further, we have the following lemma:

Mod 2 degree

Lemma 1

Let $A = \{y \in N : y$ is a regular value of$f\}$, then $\# f^{-1}(y)$ as a function on A is locally constant.

Proof

Let $y \in A$, $f^{-1}(y) = \{x_1, x_2, \ldots, x_k\}$. For each x_i take the unintersecting neighborhoods U_i, such that f differentially homogeneous U_i into $V_i$neighborhoods in N. Let

$$V = \bigcap_{i=1}^{k} V_i - f(M - \bigcup_{i=1}^{k} U_i).$$

The construct guarantees that V is a neighborhood that contains y in A, and that for any $y' \in V$, there is $f^{-1}(y') \subseteq \bigcup_{i=1}^{k} U_i$. According to the local differential homeomorphism of f, $\# f^{-1}(y) = \# f^{-1}(y') = k$.

Smoothly homotopic

Definition

Given the manifolds X, Y, if there is a smooth mapping $F : X \times [0,1] \to Y$ of the two mapping $f, g : X \to Y$ such that

$$F(x,0) = f(x), \quad F(x,1) = g(x), \quad \forall x \in X,$$

then f and g are smoothly homotopic, and denoted $f \sim g$.

Smoothly homotopic is the equivalence relationship between smooth mappings.

Smoothly homotopic

It is easy to verify that smoothly homotopic satisfies reflexivity and symmetry. For transitivity, let$f \sim g, g \sim h$, and in order to construct a new smooth map, we use the following technique to take the smooth function

$$\lambda(t) = \begin{cases} 0, & (t \leq 0) \\ e^{-\frac{1}{t}}, & (t > 0) \end{cases}$$

Let

$$\phi(t) = \frac{\lambda(t - \frac{1}{3})}{\lambda(t - \frac{1}{3}) + \lambda(\frac{2}{3} - t)}$$

then

$$\phi(t) = \begin{cases} 0, & (0 \leq t \leq \frac{1}{3}) \\ 1, & (\frac{2}{3} \leq t \leq 1) \end{cases}$$

Smoothly homotopic

If $f \sim g, g \sim h$， then there exists a smooth mapping:$F(x,t)$satisfies：

$$F(x,0) = f(x), \quad G(x,0)$$

令 $G(x,t) = F(x,\phi(t))$，则：

$$G(x,t) = \begin{cases} f(x), & (0 \leq t \leq \frac{1}{3}) \\ g(x), & (\frac{2}{3} \leq t \leq 1) \end{cases}$$

In the same way, there is a smooth mapping $H(x,t)$ satisfies :

$$H(x,t) = \begin{cases} g(x), & (0 \leq t \leq \frac{1}{3}) \\ h(x), & (\frac{2}{3} \leq t \leq 1) \end{cases}$$

Smoothly homotopic

At this point we make

$$Z(x,t)=\begin{cases}F(x,2t), & (0\le t\le \frac{1}{2})\\ G(x,2t-1), & (\frac{1}{2}<t\le 1)\end{cases}$$

then it is easy to know that $Z(x,t)$ is smooth and satisfies

$$Z(x,t)=\begin{cases}f(x), & (t=0)\\ h(x), & (t=1)\end{cases}$$

Homotopy lemma

Homotopy lemma

Let $f, g : M \to N$ be two smooth homotopy mappings, and $\dim(M) = \dim(N)$. If M is a compact edgeless manifold and $y \in N$ is a regular value of both f and g, then

$$\#f^{-1}(y) \equiv \#g^{-1}(y) \pmod 2.$$

Homotopy lemma

Let $F : M \times [0,1] \to N$ be smooth homotopy of f and g, and assume that y is also a regular value of F. then $F^{-1}(y)$ is a closed set in $M \times [0,1]$ that forms a one-dimensional compact manifold.

Since y is a regular value of f and g, $F^{-1}(y)$ is not tangent to the bottom $M \times 0$ or the top $M \times 1$. It follows that the boundary point of $F^{-1}(y)$ satisfies:

$$F^{-1}(y) \cap (M \times 0 \cup M \times 1) = f^{-1}(y) \times \{0\} \cup g^{-1}(y) \times \{1\}.$$

Homotopy lemma

According to the classification theorem of one-dimensional compact manifolds, the number of boundary points of any one-dimensional compact manifold is even. Therefore, the boundary points of $F^{-1}(y)$ satisfies:

$$\#f^{-1}(y) + \#g^{-1}(y) \text{ is even.}$$

i.e.,

$$\#f^{-1}(y) \equiv \#g^{-1}(y) \pmod 2.$$

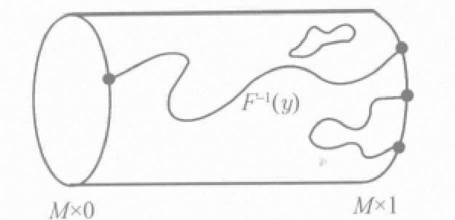

图: Boundary points are congruent with mod 2

Homotopy lemma

In the above proof, we are able to assume that y is also a regular value of F, relying on **Sard's theorem**. Sard's theorem ensures that the regular values of a smooth map are dense everywhere in the target manifold. Combined with **lemma 1**, we know that the number of preimage points of the regular value is a local constant in its neighborhood.
Therefore, a regular value y' can always be found near y that satisfies the conditions required for proof. From this, through the conclusion of $\#f^{-1}(y') \equiv \#g^{-1}(y') \pmod 2$, we finally derive the result we want.
With the above preparations, we can show that the definition of mod 2 degree does not depend on the choice of regular values:

Homotopy lemma

Theorem 1

If y, z are two regular values of f, then

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \pmod 2.$$

Proof

Since N is a connected manifold, according to the knowledge of manifold geometry, there is a single-parameter differential homomorphic population $h_t : N \to N$ satisfying $h_0 = \mathrm{id}$ and $h_1(y) = z$. At this point, $h_t \circ f$ depicts smooth homotopy between f and $h_1 \circ f$. From **homotopy lemma**, it can be seen that: $\#(h_1 \circ f)^{-1}(z) \equiv \#f^{-1}(z) \pmod 2$. And because $(h_1 \circ f)^{-1}(z) = f^{-1}(h_1^{-1}(z)) = f^{-1}(y)$, we have

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \pmod 2.$$

Homotopy invariant

Sard's theorem further guarantees that a common regular value y can always be found for smooth maps $f, g : M \to N$. Combined with **homotopy lemma**, we get that mod 2 degree is homotopy invariant.

Brouwer degree

Definition

Let M, N be two boundless, orientable n-dimensional manifolds, where M is compact and N is connected. For smooth maps $f\colon M \to N$, let $x \in M$ be the regular point of (f), then $df_x : T_xM \to T_{f(x)}N$ is a linear isomorphism between vectored spaces. Define it this way:

$$\text{sign}(df_x) = \begin{cases} 1 & \text{if } df_x \text{ remains oriented;} \\ -1 & \text{if } df_x \text{ reverses oriented.} \end{cases}$$

For any regular value $y \in N$ of f, defined

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \text{sign}(df_x).$$

Calling $\deg(f, y)$ is the Brouwer degree of f at the regular value (y).

Brouwer degree

Similar to mod 2 degrees, we want to demonstrate the following two properties:

Theorem 2

The integer $\deg(f, y)$ does not depend on the selection of the regular value y and is called the Brouwer degree of f and is denoted as $\deg f \deg f$.

Theorem 3

If f, g smooth homotopy, then $\deg f = \deg g$.

Brouwer ddegree

Lemma 2

If M is a tight and orientable manifold X, its orientation is induced by the boundary of X. Let $f: M \to N$ be smoothly expanded to $F: X \to N$, then for each regular value y, there is $\deg(f, y) = 0$.

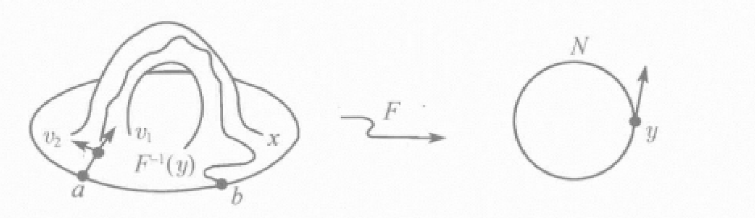

图: The orientation of$F^{-1}(y)$

Brouwer degree

Proof

Let y also be a regular value of F, then $F^{-1}(y)$ is a compact 1-dimensional manifold consisting of a finite union of circumferences and arcs, the edges of which must be at $M = \partial X$.
As shown in the figure, take the arc $A \subset F^{-1}(y)$ and its boundary is $\partial A = \{a, b\}$. If it can be proven

$$\text{sign}(df_a) + \text{sign}(df_b) = 0,$$

then summing all the arcs yields $\deg(f, y) = 0$.
Constrain the set of vector fields $(v_1, v_2, dots, v_n)$ on A so that at each point of A they form the positive basis of X, where v_1 is tangent to A and points inward at a and outward at b. Mapping dF_x maps $(v_2(x), dots, v_n(x))$ to a forward basis on N. Depending on the nature of boundary-induced orientation, there are:

$$\text{sign}(df_a) = -1, \quad \text{sign}(df_b) = 1.$$

Therefore, $\text{sign}(df_a) + \text{sign}(df_b) = 0$, then $\deg(f, y) = 0$.

Brouwer degree

Lemma 3

If f, g is homotopy, then for any regular value y, there is $\deg(f, y) = \deg(g, y)$.

Proof

Let's take the smooth homotopy $F(x, t)$ of f, g, and transform the problem into a discussion on the manifold $M \times [0, 1]$, where the orientation of M itself and the induced orientation obtained by $M \times 0, M \times 1$ must one remain the same, and the opposite is the opposite, according to **lemma 2**:

$$deg(F|_{M\times 0 \cup M\times 1}) = deg(f, y) - deg(g, y) = 0$$

With a discussion similar to mod 2 degrees, we can complete the proof of **theorem 2** and **theorem 3**. This suggests that the Brouwer degree is not only an irrelevant quantity chosen by the regular value, but also a topological invariant that remains constant in the sense of smooth homotopy.

Brouwer's fixed point theorem on D^n

Theorem 4

The circular surface is not the smooth and contractible core of the disk.

Proof

First, it is easy to observe that mod 2 degrees of the constant mapping $c : M \to M$ is 0, while mod 2 degrees of the constant mapping is 1. Therefore, the constant mapping of compact boundless manifolds cannot be smooth and homotopy with constant mapping.
When $M = S$, it means that there is no smooth mapping $f : D^{n+1} \to S^n$ to hold the points on the sphere in place. Assuming such a mapping, a smooth homotopy can be constructed:

$$F : S^n \times [0, 1] \to S^n, \quad F(x, t) = f(tx),$$

This leads to smooth homotopy between the constant and constant maps, which can lead to contradictions.

Brouwer's fixed point theorem on D^n

Theorem 5

The spherical surface S^n has a smooth vector field that is non-zero everywhere if and only if n is an odd number.

Proof

For an odd-dimensional sphere S^n, we can construct a tangent field that is non-zero everywhere:

$$X(x_1, x_2, \ldots, x_{n+1}) = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + \cdots + x_{n+1} \frac{\partial}{\partial x_n} - x_n \frac{\partial}{\partial x_{n+1}}.$$

This is a non-zero tangential field on the unit sphere.
When n is even, the Brouwer degree of the constant mapping is 1, and the Brouwer degree of the counter-diameter map is $(-1)^{n+1} = -1$, so they are not smooth homotopy. If there is a non-zero smooth vector field $v(x)$ on S^n, then consider the standard inner product on $\mathbb{R}^{n+1}$:

$$\forall x \in S^n, \quad x \cdot v(x) = 0.$$

Brouwer's fixed point theorem on D^n

Proof

Then, we can construct smooth homotopy

$$F(x,t) = x\cos\pi t + \frac{v(x)}{\|v(x)\|}\sin\pi t,$$

It connects the constant mapping and the diameter mapping, which leads to contradictions.

Brouwer's fixed point theorem on S^n

In fact, in order to guarantee that F is a well-defined smooth homotopy, we only need x and $v(x)$ to be non-collinear. Thus, by weakening the condition of $v(x)$, we can get Brouwer's fixed point theorem on S^n:

Theorem 6

If the even-dimensional sphere S^n to its own smooth mapping v does not map x to its radial point, then v must have an immobile point.

Proof

In this case, we still derive the contradiction by constructing smooth homotopy. If v does not have a fixed point, then x and $v(x)$ are not collinear everywhere. We construct smooth homotopy:

$$F(x,t) = x\cos\pi t + \frac{v(x) - \langle v(x), x\rangle x}{\|v(x) - \langle v(x), x\rangle x\|}\sin\pi t,$$

This again constructs smooth homotopy between the constant and counter-diameter mappings, leading to contradictions. The original proposition is proven.

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