Spectral Theory of Discrete One-Dimensional Ergodic Schrödinger Operators

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ABSTRACT. These notes represent a work-in-progress. They are incomplete and have not been proofread. Please do not pass them on.

Contents

CHAPTER 1

Baby Spectral Theory

1. The Hilbert Space $\ell^2(\mathbb{Z})$

In this section we introduce the Hilbert space $\ell^2(\mathbb{Z})$ and recall some key results from the general theory. We will not give any proofs.

DEFINITION 1.1. *The complex vector space of two-sided complex sequences with square-summable modulus is denoted by $\ell^2(\mathbb{Z})$, that is,*

$$\ell^2(\mathbb{Z}) = \left\{ \phi : \mathbb{Z} \to \mathbb{C} : \sum_{n\in\mathbb{Z}} |\phi(n)|^2 < \infty \right\}. \tag{1.1}$$

For $\phi, \psi \in \ell^2(\mathbb{Z})$, we define their inner product by

$$\langle \phi, \psi \rangle = \sum_{n\in\mathbb{Z}} \overline{\phi(n)} \psi(n). \tag{1.2}$$

The norm of an element $\phi \in \ell^2(\mathbb{Z})$ is given by

$$\|\phi\| = \langle \phi, \phi \rangle^{1/2}. \tag{1.3}$$

It is readily verified that (1.2) indeed defines an inner product, that is, for $\phi, \psi, \chi \in \ell^2(\mathbb{Z})$ and $a, b \in \mathbb{C}$, we have $\langle \phi, \phi \rangle \geq 0$; $\langle \phi, \phi \rangle = 0$ if and only if $\phi = 0$; $\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}$; and $\langle \phi, a\psi + b\chi \rangle = a\langle \phi, \psi \rangle + b\langle \phi, \chi \rangle$.

This implies that (1.3) indeed defines a norm, that is, for $\phi, \psi \in \ell^2(\mathbb{Z})$ and $a \in \mathbb{C}$, we have $\|\phi\| \geq 0$; $\|\phi\| = 0$ if and only if $\phi = 0$, $\|a\phi\| = |a|\|\phi\|$; and $\|\phi + \psi\| \leq \|\phi\| + \|\psi\|$.

This in turn induces a metric on $\ell^2(\mathbb{Z})$ via $\mathrm{dist}(\phi, \psi) = \|\phi - \psi\|$ and thus a topology. Convergence in $\ell^2(\mathbb{Z})$ will always be understood with respect to this topology. The space $\ell^2(\mathbb{Z})$ with this topology coming from an inner product is complete; hence it is a Hilbert space.

DEFINITION 1.2. *We say that $\phi, \psi \in \ell^2(\mathbb{Z})$ are orthogonal if $\langle \phi, \psi \rangle = 0$. A family $\{\phi_\gamma : \gamma \in \Gamma\} \subset \ell^2(\mathbb{Z})$ is called an orthonormal basis if $\langle \phi_{\gamma_1}, \phi_{\gamma_2} \rangle = \delta_{\gamma_1, \gamma_2}$ for $\gamma_1, \gamma_2 \in \Gamma$ and the set of finite linear combinations of elements of this family is dense in $\ell^2(\mathbb{Z})$. The standard orthonormal basis is $\{\delta_n : n \in \mathbb{Z}\}$, where $\delta_n \in \ell^2(\mathbb{Z})$ is defined by $\delta_n(m) = \delta_{n,m}$.*

It is not hard to see that any orthonormal basis of $\ell^2(\mathbb{Z})$ is countable and that the standard orthonormal basis is indeed an orthonormal basis.

2. Bounded Self-Adjoint Operators in $\ell^2(\mathbb{Z})$

DEFINITION 1.3. *A linear operator in $\ell^2(\mathbb{Z})$ is a pair $(D(A), A)$, where $D(A)$ is a subspace of $\ell^2(\mathbb{Z})$, called the domain of A, and A is a linear map from $D(A)$ to*

$\ell^2(\mathbb{Z})$, that is, we have $A(a\phi + b\psi) = aA(\phi) + bA(\psi)$ for $\phi, \psi \in \ell^2(\mathbb{Z})$ and $a, b \in \mathbb{C}$. A linear operator is called bounded if $D(A) = \ell^2(\mathbb{Z})$ and

$$\|A\| := \sup\{\|A\phi\| : \phi \in \ell^2(\mathbb{Z}),\ \|\phi\| = 1\} < \infty.$$

The space of all bounded operators in $\ell^2(\mathbb{Z})$ is denoted by $B(\ell^2(\mathbb{Z}))$. A bounded operator is called self-adjoint if $\langle A\phi, \psi\rangle = \langle \phi, A\psi\rangle$ for every $\phi, \psi \in \ell^2(\mathbb{Z})$.

We will often just write A for the linear operator $(D(A), A)$, especially if it is bounded. A bounded operator is continuous, that is, if $\phi_n \to \phi$, then $A\phi_n \to A\phi$.

Definition 1.4. *Given a bounded operator A, the resolvent set $\rho(A)$ consists of the complex numbers z for which $A - zI$ is one-to-one and onto. The spectrum of A is $\sigma(A) = \mathbb{C} \setminus \rho(A)$.*

Because A is bounded, the inverse $(A - zI)^{-1}$ is automatically bounded for every $z \in \rho(A)$ by the closed graph theorem. The resolvent of A is the following map

$$R(A, \cdot) : \rho(A) \to B(\ell^2(\mathbb{Z})),\ z \mapsto (A - zI)^{-1}.$$

Proposition 1.5 (Resolvent Identities). (a) *If $A \in B(\ell^2(\mathbb{Z}))$ and $z, z' \in \rho(A)$, then*

$$R(A, z) - R(A, z') = (z - z')R(A, z)R(A, z') = (z - z')R(A, z')R(A, z). \tag{1.4}$$

(b) *If $A, B \in B(\ell^2(\mathbb{Z}))$ and $z \in \rho(A) \cap \rho(B)$, then*

$$R(A, z) - R(B, z) = R(A, z)(B - A)R(B, z) = R(B, z)(B - A)R(A, z). \tag{1.5}$$

Proposition 1.6 (Properties of the Spectrum). *If $A \in B(\ell^2(\mathbb{Z}))$ is self-adjoint, then $\sigma(A)$ is a compact subset of $\mathbb{R}$. Moreover, we have*

$$\|A\| = \max\{|z| : z \in \sigma(A)\} = \sup\{|\langle \phi, A\phi\rangle| : \phi \in \ell^2(\mathbb{Z}),\ \|\phi\| = 1\}.$$

Next we discuss the spectral theorem. Suppose that $A \in B(\ell^2(\mathbb{Z}))$ is self-adjoint. We wish to apply a function f to A in order to form a new operator $f(A)$. It is obvious how to do this if f is a polynomial. Moreover, one can check that the resulting operator depends only on $f|_{\sigma(A)}$. Using the Weierstrass Theorem, it is then possible to define $f(A)$ for functions f that are continuous on $\sigma(A)$.

Now let $\phi \in \ell^2(\mathbb{Z})$. Then

$$C(\sigma(A)) \ni f \mapsto \langle \phi, f(A)\phi\rangle \in \mathbb{C}$$

is a positive linear functional and hence there is a unique measure μ_ϕ on the compact set $\sigma(A)$ with

$$\langle \phi, f(A)\phi\rangle = \int_{\sigma(A)} f(E)\, d\mu_\phi(E). \tag{1.6}$$

This in turn allows one to extend the map $f \mapsto f(A)$ and the formula (1.6) to all bounded measurable functions on $\sigma(A)$. The measure μ_ϕ is called the spectral measure associated with ϕ (and A).

This is summarized and put to further use in the following central theorem.

Theorem 1.7 (Spectral Theorem). *Suppose $A \in B(\ell^2(\mathbb{Z}))$ is self-adjoint. Then, for every $\phi \in \ell^2(\mathbb{Z})$, there is a unique measure μ_ϕ such that* (1.6) *holds for every bounded measurable function f on $\sigma(A)$.*

Moreover, there exist at most countably many $\phi_n \in \ell^2(\mathbb{Z})$ such that with $\mu_n = \mu_{\phi_n}$, there is a unitary map

$$U : \ell^2(\mathbb{Z}) \to \bigoplus_n L^2(\mathbb{R}, d\mu_n) \tag{1.7}$$

so that

$$(UAU^{-1}g)_n(E) = Eg_n(E).$$

Here we write $g(E) = \{g_n(E)\}_n$ for a function g in $\bigoplus_n L^2(\mathbb{R}, d\mu_n)$.

Let us sketch how these measures arise. We will revisit this when we discuss Schrödinger operators in the next section.

Definition 1.8. *An element ϕ of $\ell^2(\mathbb{Z})$ is called cyclic for the operator A if the finite linear combinations of the elements $\{A^n\phi\}_{n=0}^{\infty}$ are dense.*

Suppose now that $A \in B(\ell^2(\mathbb{Z}))$ is self-adjoint and ϕ is a cyclic vector for A. We wish to show that (1.7) can be defined with just the measure $\mu = \mu_\phi$. To define a map $U : \ell^2(\mathbb{Z}) \to L^2(\mathbb{R}, d\mu_\phi)$ with the desired property, one needs to invert the map $f \mapsto f(A)$ discussed above. Namely, if $f \in C(\sigma(A))$, set $U(f(A)\phi) = f$. This defines U on a dense set. One can then verify that U preserves norms and that $(UAU^{-1}f)(E) = Ef(E)$ for such f's, and then extend the definition and the relation to the whole space.

This gives the statement of the spectral theorem if there is a cyclic vector. Otherwise, one can decompose $\ell^2(\mathbb{Z})$ into a countable direct sum of subspaces, which A leaves invariant and which contain a cyclic vector. The spectral theorem then follows upon an application of the above in each of these subspaces.

3. Spectral Decompositions and Quantum Dynamics

Next we discuss spectral types. Recall that any measure μ on $\mathbb{R}$ has a unique Lebesgue decomposition

$$\mu = \mu_{\mathrm{pp}} + \mu_{\mathrm{sc}} + \mu_{\mathrm{ac}},$$

where μ_{pp} is a pure point measure, μ_{sc} is a singular continuous measure, and μ_{ac} is an absolutely continuous measure. This means that $\mu_{\mathrm{pp}}(\mathbb{R} \setminus C) = 0$ for some countable set C, $\mu_{\mathrm{sc}}(\{E\}) = 0$ for every $E \in \mathbb{R}$, $\mu_{\mathrm{sc}}(\mathbb{R} \setminus N) = 0$ for some set N of zero Lebesgue measure, and $\mu_{\mathrm{ac}}(N) = 0$ for every set N of zero Lebesgue measure. We also write $\mu_{\mathrm{c}} = \mu_{\mathrm{sc}} + \mu_{\mathrm{ac}}$ for the continuous part and $\mu_{\mathrm{s}} = \mu_{\mathrm{pp}} + \mu_{\mathrm{sc}}$ for the singular part of μ.

Definition 1.9. *Suppose $A \in B(\ell^2(\mathbb{Z}))$ is self-adjoint. Define the following spaces,*

$$\begin{aligned}
\ell^2(\mathbb{Z})_{\mathrm{pp}} &= \{\phi : \mu_\phi = \mu_{\phi,\mathrm{pp}}\},\\
\ell^2(\mathbb{Z})_{\mathrm{sc}} &= \{\phi : \mu_\phi = \mu_{\phi,\mathrm{sc}}\},\\
\ell^2(\mathbb{Z})_{\mathrm{ac}} &= \{\phi : \mu_\phi = \mu_{\phi,\mathrm{ac}}\},
\end{aligned}$$

and sets,

$$\begin{aligned}
\sigma_{\mathrm{pp}}(A) &= \sigma\left(A|_{\ell^2(\mathbb{Z})_{\mathrm{pp}}}\right),\\
\sigma_{\mathrm{sc}}(A) &= \sigma\left(A|_{\ell^2(\mathbb{Z})_{\mathrm{sc}}}\right),\\
\sigma_{\mathrm{ac}}(A) &= \sigma\left(A|_{\ell^2(\mathbb{Z})_{\mathrm{ac}}}\right).
\end{aligned}$$

These sets are called the point, singular continuous, and absolutely continuous spectrum of A, respectively.

We say that A has pure point spectrum if $\ell^2(\mathbb{Z})_{\mathrm{pp}} = \ell^2(\mathbb{Z})$, purely singular continuous spectrum if $\ell^2(\mathbb{Z})_{\mathrm{sc}} = \ell^2(\mathbb{Z})$, and purely absolutely continuous spectrum if $\ell^2(\mathbb{Z})_{\mathrm{ac}} = \ell^2(\mathbb{Z})$.

A non-zero vector $\phi \in \ell^2(\mathbb{Z})$ is called an eigenvector of an operator A if there is a so-called eigenvalue $E \in \mathbb{C}$ with the property $A\phi = E\phi$. If $A \in B(\ell^2(\mathbb{Z}))$ is self-adjoint, it is easy to see that every eigenvector belongs to $\ell^2(\mathbb{Z})_{\mathrm{pp}}$ and every eigenvalue belongs to $\sigma_{\mathrm{pp}}(A)$. In fact, we have $\sigma_{\mathrm{pp}}(A) = \overline{\{\text{eigenvalues of } A\}}$.

The Lebesgue decomposition of a spectral measure μ_ϕ is of importance in the study of the long-time asymptotics of solutions to the Schrödinger equation,

$$i\partial_t \phi(t) = A\phi(t), \ \phi(0) = \phi. \tag{1.8}$$

The following theorem establishes relations of this kind.

THEOREM 1.10 (RAGE Theorem). *Suppose $A \in B(\ell^2(\mathbb{Z}))$ is self-adjoint, $\phi \in \ell^2(\mathbb{Z})$, and $\phi(t)$ is a solution of the Schrödinger equation* (1.8).

(a) *We have $\mu_\phi = \mu_{\phi,\mathrm{pp}}$ if and only if for every $\varepsilon > 0$, there is $N \in \mathbb{Z}_+$ such that*

$$\sum_{|n|\geq N} |\langle \delta_n, \phi(t)\rangle|^2 < \varepsilon \quad \textit{for every } t \in \mathbb{R}.$$

(b) *We have $\mu_\phi = \mu_{\phi,\mathrm{c}}$ if and only if for every $N \in \mathbb{Z}_+$,*

$$\lim_{T\to\infty} \int_{-T}^{T} \sum_{|n|\leq N} |\langle \delta_n, \phi(t)\rangle|^2 \, dt = 0.$$

(c) *If $\mu_\phi = \mu_{\phi,\mathrm{ac}}$, then for every $N \in \mathbb{Z}_+$,*

$$\lim_{|t|\to\infty} \sum_{|n|\leq N} |\langle \delta_n, \phi(t)\rangle|^2 = 0.$$

In the mathematical formulation of quantum mechanics, A is given by a so-called Schrödinger operator and for a solution $\phi(\cdot)$ of the associated Schrödinger equation, $|\langle \delta_n, \phi(t)\rangle|^2$ corresponds to the probability of finding the particle at site n at time t. Thus, in this situation, we can interpret the results of the RAGE Theorem as follows. The spectral measure of the initial state is pure point if and only if with arbitrarily high probability, the particle can be found in some fixed compact region in space at any given time; the spectral measure of the initial state is purely continuous if and only if the particle leaves any fixed compact region in space in a time-averaged sense as time goes to infinity; and if the spectral measure of the initial state is purely absolutely continuous, then the particle leaves any fixed compact region in space as time goes to infinity.

While these connections serve as a motivation to study the standard Lebesgue decomposition of spectral measures, one sometimes wants to take this a step further. By decomposing a given spectral measure with respect to α-dimensional Hausdorff measure on $\mathbb{R}$, it is possible to prove quantitative estimates for the time evolution involving the dimension α.

Recall that the α-dimensional Hausdorff measure h^α is defined by

$$h^\alpha(S) = \lim_{\delta\to 0} \inf_{\delta\text{-covers}} \sum |I_m|^\alpha,$$

where $S \subseteq \mathbb{R}$ is a Borel set and a δ-cover is a countable collection of intervals I_m of length bounded by δ such that the union of these intervals contains the set in question. Note that h^1 coincides with Lebesgue measure and h^0 is the counting measure.

Given a finite Borel measure μ and $\alpha \in [0, 1]$, we define the *upper α-derivative of μ* by

$$D_\mu^\alpha(E) = \limsup_{\varepsilon \downarrow 0} \frac{\mu((E - \varepsilon, E + \varepsilon))}{(2\varepsilon)^\alpha}.$$

Denote $T_f = \{E : D_\mu^\alpha(E) < \infty\}$ and $T_\infty = \{E : D_\mu^\alpha(E) = \infty\}$. Then [**32**],

THEOREM 1.11 (Rogers-Taylor). *We have $h^\alpha(T_\infty) = 0$ and $\mu(S \cap T_f) = 0$ for any S with $h^\alpha(S) = 0$.*

This suggests the following decomposition of μ. Let $\mu_{\alpha c}(\cdot) = \mu(\cdot \cap T_f)$ and $\mu_{\alpha s}(\cdot) = \mu(\cdot \cap T_\infty)$. Then, $\mu = \mu_{\alpha c} + \mu_{\alpha s}$. We say that μ is α-continuous if $\mu_{\alpha s} = 0$ and α-singular if $\mu_{\alpha c} = 0$. We also say that μ is zero-dimensional if $\mu_{\alpha c} = 0$ for every $\alpha > 0$.

The following result was proved by Last [**28**]. Similar bounds were shown earlier under more restrictive assumptions by Guarneri and Combes.

THEOREM 1.12 (Guarneri-Combes-Last). *Suppose $A \in B(\ell^2(\mathbb{Z}))$ is self-adjoint, $\phi \in \ell^2(\mathbb{Z})$, and $\phi(t)$ is a solution of the Schrödinger equation* (1.8). *If $\mu_{\phi,\alpha c} \neq 0$, then for every $p \in \mathbb{Z}_+$, there is a constant $C = C(A, \phi, p)$ such that*

$$\frac{1}{2T} \int_{-T}^{T} \sum_{n \in \mathbb{Z}} |n|^p |\langle \delta_n, \phi(t) \rangle|^2 \, dt \geq C T^{\alpha p} \tag{1.9}$$

It should be noted that these are strictly one-sided bounds. That is, growth of the LHS in (1.9) does not imply any continuity properties for μ_ϕ. We will see explicit examples later on when we discuss the random dimer model.

4. Borel Transforms and Derivatives of Finite Measures

The RAGE Theorem and related results show that there is an intimate connection between the long-time asymptotics of the solution to the Schrödinger equation with some initial state ϕ and the continuity/singularity properties of the associated spectral measure $\mu = \mu_\phi$. These properties can be studied, in turn, by an investigation of the (upper of lower α-) derivative of the measure. Moreover, the spectral theorem also says that

$$\langle \phi, (A - z)^{-1} \phi \rangle = \int_{\mathbb{R}} \frac{d\mu(E)}{E - z} \tag{1.10}$$

when $z \notin \sigma(A)$. For example, the identity (1.10) holds when $z \in \mathbb{C} \setminus \mathbb{R}$.

In this section we discuss results that establish a relationship between the derivative of μ and the boundary behavior of the right-hand side of (1.10), that is, as $\Im z \downarrow 0$. As a consequence of this connection and the formula (1.10), we will then be able to derive statements about the long-time asymptotics of the solution to the Schrödinger equation with initial state ϕ from the boundary behavior of the left-hand side of (1.10).

Recall that the upper derivative $D_\mu^+(E) := D_\mu^1(E)$ of μ at $E \in \mathbb{R}$ is given by

$$D_\mu^+(E) = \limsup_{\varepsilon \downarrow 0} \frac{\mu((E - \varepsilon, E + \varepsilon))}{2\varepsilon}.$$

On the other hand, the lower derivative of μ at $E \in \mathbb{R}$ is given by

$$D_\mu^-(E) = \liminf_{\varepsilon \downarrow 0} \frac{\mu((E-\varepsilon, E+\varepsilon))}{2\varepsilon}.$$

When the upper and lower derivative of μ at $E \in \mathbb{R}$ coincide, we call their common value the derivative of μ at E and denote it by $D_\mu(E)$.

The following is a basic result in measure theory.

THEOREM 1.13. (a) *$D_\mu(E)$ exists for Lebesgue almost every $E \in \mathbb{R}$ and equals the Radon-Nikodym derivative of the absolutely continuous part of μ:*

$$\mu_{\mathrm{ac}}(S) = \int_S D_\mu(E)\, dE.$$

(b) *The set $\{E \in \mathbb{R} : D_\mu(E) \in (0, \infty)\}$ is a support for the absolutely continuous part of μ.*
(c) *The set $\{E \in \mathbb{R} : D_\mu(E) = \infty\}$ is a support for the singular part of μ.*

Given a finite and compactly supported measure μ on $\mathbb{R}$, we denote its Borel transform by

$$F_\mu(z) = \int \frac{d\mu(E)}{E-z}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

THEOREM 1.14. (a) *For every $E \in \mathbb{R}$, we have*

$$D_\mu^-(E) \le \liminf_{\varepsilon \downarrow 0} \frac{1}{\pi} \Im F_\mu(E + i\varepsilon) \le \limsup_{\varepsilon \downarrow 0} \frac{1}{\pi} \Im F_\mu(E + i\varepsilon) \le D_\mu^+(E).$$

Consequently, the limit

$$\Im F_\mu(E + i0) = \lim_{\varepsilon \downarrow 0} \Im F_\mu(E + i\varepsilon)$$

exists for almost every E with respect to both μ and Lebesgue measure, and we have

$$D_\mu(E) = \frac{1}{\pi} \Im F_\mu(E + i0)$$

whenever the right-hand side exists.
(b) *The absolutely continuous part of μ obeys*

$$d\mu_{\mathrm{ac}}(E) = \frac{1}{\pi} \Im F_\mu(E + i0)\, dE. \tag{1.11}$$

(c) *For the singular part of $d\mu$, we have that*

$$\mu_{\mathrm{s}} \text{ is supported on } \{E \in \mathbb{R} : \Im F_\mu(E + i0) = \infty\}. \tag{1.12}$$

(d) *If F_1, F_2 are Borel transforms and $F_1(E + i0) = F_2(E + i0)$ for $E \in A \subseteq \mathbb{R}$ with $\mathrm{Leb}(A) > 0$, then $F_1 = F_2$.*
(e) *If $\Re F_\mu(E + i0) = 0$ almost everywhere on some open interval $I \subseteq \mathbb{R}$, then F_μ has an analytic continuation through I and $F_\mu(E + i0) \neq 0$ for every $E \in I$.*

Remark. Note that by (c) and (e), if $\Re F_\mu(E + i0) = 0$ on I, then $\mu = \mu_{\mathrm{ac}}$ on I.

We close this section with a discussion of one-parameter families of Borel transforms that arise in the theory of rank one perturbations. Suppose that $A \in B(\ell^2(\mathbb{Z}))$ is self-adjoint and $\phi \in \ell^2(\mathbb{Z}) \setminus \{0\}$. With the associated spectral measure μ, we therefore have

$$F(z) := \langle \phi, (A - z)^{-1} \phi \rangle = \int \frac{d\mu(E)}{E - z}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

For $\lambda \in \mathbb{R}$, we consider the operator

$$A_\lambda = A + \lambda\langle \phi, \cdot \rangle \phi,$$

which is a rank one perturbation of A. Define F_λ and μ_λ by

$$F_\lambda(z) = \langle \phi, (A_\lambda - z)^{-1}\phi\rangle = \int \frac{d\mu_\lambda(E)}{E-z}, \quad z \in \mathbb{C}\setminus\mathbb{R}. \tag{1.13}$$

LEMMA 1.15. *We have*

$$F_\lambda(z) = \frac{F(z)}{1+\lambda F(z)}. \tag{1.14}$$

PROOF. Using the resolvent formula (1.5) in the second step, we see that

$$\begin{aligned}(A_\lambda - z)^{-1}\phi - (A-z)^{-1}\phi &= \left[(A_\lambda - z)^{-1} - (A-z)^{-1}\right]\phi \\ &= (A-z)^{-1}\,(A - A_\lambda)\,(A_\lambda - z)^{-1}\phi \\ &= (A-z)^{-1}\,(-\lambda\langle\phi,\cdot\rangle\phi)\,(A_\lambda - z)^{-1}\phi \\ &= (A-z)^{-1}\left(-\lambda\langle\phi,(A_\lambda - z)^{-1}\phi\rangle\phi\right) \\ &= -\lambda\langle\phi,(A_\lambda - z)^{-1}\phi\rangle(A-z)^{-1}\phi\end{aligned}$$

and hence, by taking the the inner product with ϕ on both sides,

$$F_\lambda(z) - F(z) = -\lambda F_\lambda(z)F(z).$$

Solving this for $F_\lambda(z)$, we obtain (1.14). ☐

THEOREM 1.16. *We have*

$$\int [d\mu_\lambda(E)]\, d\lambda = dE$$

in the sense that if $f \in L^1(\mathbb{R}, dE)$, then $f \in L^1(\mathbb{R}, d\mu_\lambda)$ for Lebesgue almost every λ, $\int f(E)\, d\mu_\lambda(E) \in L^1(\mathbb{R}, d\lambda)$, and

$$\int \left(\int f(E)\, d\mu_\lambda(E)\right) d\lambda = \int f(E)\, dE. \tag{1.15}$$

PROOF. Denote for $E, \lambda \in \mathbb{R}$ and $z \in \mathbb{C}\setminus\mathbb{R}$,

$$f_z(E) = \frac{1}{E-z} - \frac{1}{E+i}$$

and

$$h_z(\lambda) = \frac{1}{\lambda + F(z)^{-1}} - \frac{1}{\lambda + F(-i)^{-1}}.$$

By closing the contour in the upper half-plane, we see that

$$\int f_z(E)\, dE = \begin{cases} 2\pi i & \Im z > 0, \\ 0 & \Im z < 0. \end{cases}$$

By (1.13) and (1.14) we have

$$\begin{aligned}\int f_z(E)\, d\mu_\lambda(E) &= F_\lambda(z) - F_\lambda(-i) \\ &= \frac{1}{\lambda + F(z)^{-1}} - \frac{1}{\lambda + F(-i)^{-1}} \\ &= h_z(\lambda).\end{aligned}$$

Observe that if $\pm\Im z > 0$, then $\pm\Im F(z) > 0$ and hence $\pm\Im F(z)^{-1} < 0$. Thus, $h_z(\lambda)$ has either two poles in the lower half-plane (if $\Im z < 0$) or one in each half-plane (if $\Im z > 0$). Thus, the same contour integral calculation as above (with λ instead of E) shows

$$\int \left(\int f_z(E)\, d\mu_\lambda(E) \right) d\lambda = \int h_z(\lambda)\, d\lambda = \begin{cases} 2\pi i & \Im z > 0, \\ 0 & \Im z < 0, \end{cases}$$

which proves (1.15) for $f = f_z$. The general case then follows from the fact the the finite linear combinations of the f_z's are dense (which can be shown with the help of Stone-Weierstrass). $\square$

5. Schrödinger Operators in $\ell^2(\mathbb{Z})$

5.1. Definitions and Basic Formulae. Given a bounded map $V : \mathbb{Z} \to \mathbb{R}$, called the potential, we define the associated Schrödinger operator by

$$(H\phi)(n) = \phi(n+1) + \phi(n-1) + V(n)\phi(n). \tag{1.16}$$

It is easy to check that H is a bounded self-adjoint operator in $\ell^2(\mathbb{Z})$.

While a single δ_n is in general not a cyclic vector for H, each pair $\{\delta_n, \delta_{n+1}\}$ is cyclic in the sense that any compactly supported $\phi \in \ell^2(\mathbb{Z})$ (and such vectors form a dense subset of $\ell^2(\mathbb{Z})$) can be written as $\phi = p(H)\delta_n + q(H)\delta_{n+1}$ with suitable polynomials p, q. Consider, without loss of generality, the case $n = 0$. Notice first that

$$\delta_2 = -\delta_0 + H\delta_1 - V(1)\delta_1,$$

and then

$$\begin{aligned} \delta_3 &= -\delta_1 + H\delta_2 - V(2)\delta_2 \\ &= -\delta_1 + H(-\delta_0 + H\delta_1 - V(1)\delta_1) - V(2)(-\delta_0 + H\delta_1 - V(1)\delta_1) \\ &= -H\delta_0 + V(2)\delta_0 + H^2\delta_1 - (V(1) + V(2))H\delta_1 + (V(2)V(1) - 1)\delta_1. \end{aligned}$$

Carrying on, this shows inductively that $\delta_m = p(H)\delta_0 + q(H)\delta_1$ for $m > 1$ and, in a similar way, for $m < 0$. Taking finite linear combinations, the claim follows.

Using this observation, it can then be shown that the sum of the spectral measures associated with δ_0 and δ_1 can serve as a universal spectral measure for H. That is, taking $\mu = \mu_{\delta_0} + \mu_{\delta_1}$, any spectral measure μ_ϕ is absolutely continuous with respect to μ. When not indicated otherwise, the "spectral measure of H" will refer to this measure μ. The Borel transform of μ takes the form

$$M(z) := F_\mu(z) = \int \frac{d\mu(E)}{E - z} = \langle \delta_0, (H - zI)^{-1}\delta_0 \rangle + \langle \delta_1, (H - zI)^{-1}\delta_1 \rangle. \tag{1.17}$$

Our goal is to link the boundary behavior of M to the asymptotic behavior of solutions to the difference equation

$$u(n+1) + u(n-1) + V(n)u(n) = zu(n). \tag{1.18}$$

Note first that for fixed $z \in \mathbb{C}$, the solutions to (1.18) form a two-dimensional complex vector space. Moreover, there are unimodular matrices $M(n, z)$ such that $u : \mathbb{Z} \to \mathbb{C}$ solves (1.18) if and only if

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = M_z(n) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}. \tag{1.19}$$

The columns of $M_z(n)$ are given by $\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix}$ for the solution u of (1.18) that has $\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. Moreover, it is easy to check that for $n \geq 1$

$$M_z(n) = T_z(n) \times \cdots \times T_z(1), \tag{1.20}$$

where

$$T_z(m) = \begin{pmatrix} z - V(m) & -1 \\ 1 & 0 \end{pmatrix}.$$

For $n \leq -1$, as similar formula holds. As a consequence of (1.19), we see that for any pair of solutions $u, \tilde{u}$ to (1.18), their Wronskian

$$W(u, \tilde{u})(n) = \det \begin{pmatrix} u(n+1) & \tilde{u}(n+1) \\ u(n) & \tilde{u}(n) \end{pmatrix}$$

is independent of n and it is zero if and only if $u, \tilde{u}$ are linearly dependent.

If $\Im z \neq 0$, $z \notin \sigma(H)$. Thus, $R(H, z) = (H - zI)^{-1}$ exists and is a bounded operator. In particular, for every $\phi \in \ell^2(\mathbb{Z})$, $(H - zI)^{-1}\phi$ is an element of $\ell^2(\mathbb{Z})$ and hence goes to zero at infinity. Using constancy of the Wronskian, we find that any linearly independent solution must go to infinity at infinity. Moreover, if ϕ is compactly supported, then $u(n) = \langle \delta_n, (H - zI)^{-1}\phi \rangle$ solves (1.18) outside the support of ϕ. Taking this solution for $n > \mathrm{supp}\phi$ and using (1.18) to extend it to a solution on all of $\mathbb{Z}$, we obtain the (up to a multiple) unique solution u_z^+ to (1.18) that is square-summable at ∞. Similarly, we can construct the (up to a multiple) unique solution u_z^- to (1.18) that is square-summable at $-\infty$.

With these definitions, it is then easy to verify that for $n \leq m$,

$$G(n, m; z) := \langle \delta_n, (H - z)^{-1} \delta_m \rangle = \frac{u_z^-(n) u_z^+(m)}{u_z^-(0) u_z^+(1) - u_z^-(1) u_z^+(0)}.$$

Let

$$m^+(z) := -\frac{u_z^+(1)}{u_z^+(0)} \quad \text{and} \quad m^-(z) := -\frac{u_z^-(0)}{u_z^-(1)}. \tag{1.21}$$

Returning to (1.17), we find

$$M(z) = \frac{u_z^-(0) u_z^+(0) + u_z^-(1) u_z^+(1)}{u_z^-(0) u_z^+(1) - u_z^-(1) u_z^+(0)} = \frac{m^+(z) + m^-(z)}{1 - m^+(z) m^-(z)}.$$

Finally, we prove an important elementary lemma, which shows how the values a solution takes on a given interval may be expressed in terms of the values it takes on the (outer) boundary and the Green function of the operator localized to this interval. Concretely, if $[n_1, n_2] = \{n \in \mathbb{Z} : n_1 \leq n \leq n_2\}$ and

$$H_{[n_1, n_2]} = P_{[n_1, n_2]} H P^*_{[n_1, n_2]}$$

denotes the restriction of H to this interval, where

$$P_J : \ell^2(\mathbb{Z}) \to \ell^2(J)$$

is the canonical orthogonal projection, denote for $z \notin \sigma(H_{[n_1, n_2]})$ and $n, m \in [n_1, n_2]$,

$$G_{[n_1, n_2]}(n, m) = \langle \delta_n, \left(H_{[n_1, n_2]} - z\right)^{-1} \delta_m \rangle.$$

Then, the following formula holds.

LEMMA 1.17. *Suppose $n \in [n_1, n_2] \subset \mathbb{Z}$ and u is a solution of the difference equation* (1.18). *If $z \notin \sigma(H_{[n_1,n_2]})$ and $n, m \in [n_1, n_2]$, then*

$$u(n) = -G_{[n_1,n_2]}(n, n_1)u(n_1 - 1) - G_{[n_1,n_2]}(n, n_2)u(n_2 + 1). \tag{1.22}$$

PROOF. Since u is a solution of (1.18), we have $(H - z)u = 0$ (in difference equation sense) and hence

$$\begin{aligned} 0 &= P_{[n_1,n_2]}(H - z)u \\ &= P_{[n_1,n_2]}(H - z)P^*_{[n_1,n_2]}P_{[n_1,n_2]}u + P_{[n_1,n_2]}(H - z)P^*_{\mathbb{Z}\setminus[n_1,n_2]}P_{\mathbb{Z}\setminus[n_1,n_2]}u, \end{aligned}$$

which in turn implies

$$\left(H_{[n_1,n_2]} - z\right)(P_{[n_1,n_2]}u) = -\left(u(n_1 - 1)\delta_{n_1} + u(n_2 + 1)\delta_{n_2}\right).$$

Thus, with the inner product of $\ell^2([n_1, n_2])$, we find for $n \in [n_1, n_2]$,

$$\begin{aligned} u(n) &= \langle \delta_n, (P_{[n_1,n_2]}u)\rangle \\ &= \left\langle \delta_n, \left(H_{[n_1,n_2]} - z\right)^{-1}\left(H_{[n_1,n_2]} - z\right)(P_{[n_1,n_2]}u)\right\rangle \\ &= -\left\langle \delta_n, \left(H_{[n_1,n_2]} - z\right)^{-1}(u(n_1 - 1)\delta_{n_1} + u(n_2 + 1)\delta_{n_2})\right\rangle \\ &= -u(n_1 - 1)\left\langle \delta_n, \left(H_{[n_1,n_2]} - z\right)^{-1}\delta_{n_1}\right\rangle - u(n_2 + 1)\left\langle \delta_n, \left(H_{[n_1,n_2]} - z\right)^{-1}\delta_{n_2}\right\rangle \end{aligned}$$

which is (1.22). □

5.2. Spectrum and Generalized Eigenfunctions. Given H as in (1.16) we say that $E \in \mathbb{R}$ is a generalized eigenvalue if (1.18) has a non-trivial solution u_E, called the corresponding generalized eigenfunction, satisfying

$$|u_E(n)| \le C(1 + |n|)^\delta \tag{1.23}$$

for suitable finite constants C and δ, and every $n \in \mathbb{Z}$.

THEOREM 1.18. (a) *Every generalized eigenvalue of H belongs to $\sigma(H)$.*

(b) *Fix $\delta > \frac{1}{2}$. Then, for μ-almost every $E \in \mathbb{R}$, there exists a generalized eigenfunction satisfying* (1.23).

(c) *The spectrum of H is given by the closure of the set of generalized eigenvalues of H.*

PROOF. (a) Suppose E is a generalized eigenvalue of H with corresponding generalized eigenfunction u_E. The idea is that a cutoff at $\pm L$ will, after normalization, produce a Weyl sequence; essentially because the "integral" dominates the boundary terms for polynomially bounded sequences.

Explicitly, for $L \ge 1$, we let

$$u_E^{(L)} = \chi_{[-L,L]}u_E$$

and

$$\varphi_E^{(L)} = \frac{u_E^{(L)}}{\|u_E^{(L)}\|}.$$

Observe that

$$\begin{aligned} (H - E)u_E^{(L)}(L) &= u_E^{(L)}(L + 1) + u_E^{(L)}(L - 1) + (V(n) - z)u_E^{(L)}(L) \\ &= 0 + u_E(L - 1) + (V(n) - z)u_E(L) \\ &= -u(L + 1) \end{aligned}$$

since u_E is a solution. Similarly, we see that

$$\begin{aligned}(H-E)u_E^{(L)}(L+1) &= u_E(L)\\ (H-E)u_E^{(L)}(-L) &= -u_E(-L-1)\\ (H-E)u_E^{(L)}(-L-1) &= u_E(-L).\end{aligned}$$

Moreover, $(H-E)u_E^{(L)}(L+1)$ vanishes for all other values of n since both u_E and 0 solve the difference equation. We find

$$(H-E)u_E^{(L)}(n) = \delta_{L+1,n}u_E(L)+\delta_{-L-1,n}u_E(-L)-\delta_{L,n}u_E(L+1)-\delta_{-L,n}u_E(-L-1)$$

and therefore

$$\begin{aligned}\|(H-E)\varphi_E^{(L)}\|^2 &= \frac{\|(H-E)u_E^{(L)}\|^2}{\|u_E^{(L)}\|^2}\\ &= \frac{\|u_E^{(L+1)}\|^2 - \|u_E^{(L-1)}\|^2}{\|u_E^{(L)}\|^2}.\end{aligned}$$

We claim that there is a sequence $L_k \to \infty$ such that $\|(H-E)\varphi_E^{(L_k)}\| \to 0$ as $k \to \infty$. This produces a Weyl sequence and hence shows that $E \in \sigma(H)$.

Suppose there is no such sequence $\{L_k\}$. Then there exists $\delta > 0$ such that $\|(H-E)\varphi_E^{(L)}\|^2 > \delta$ for every $L \geq L_0$. This in turn implies for $L \geq L_0$,

$$\frac{\|u_E^{(L+1)}\|^2 - \|u_E^{(L-1)}\|^2}{\|u_E^{(L)}\|^2} > \delta,$$

that is,

$$\begin{aligned}\|u_E^{(L+1)}\|^2 &> \|u_E^{(L-1)}\|^2 + \delta\|u_E^{(L)}\|^2\\ &\geq (1+\delta)\|u_E^{(L-1)}\|^2.\end{aligned}$$

Consequently,

$$\|u_E^{(2L+L_0)}\|^2 \geq (1+\delta)^L\|u_E^{(L_0)}\|^2,$$

which contradicts the polynomial upper bound (1.23) for u_E.

(b) For a Borel set $B \subseteq \mathbb{R}$, denote

$$\mu_{n,m}(B) = \langle \delta_n, \chi_B(H)\delta_m\rangle$$

and

$$\rho(B) = \sum_{n\in\mathbb{Z}} \lambda_n \mu_{n,n}(B),$$

where

$$\lambda_n = c(1+|n|)^{-2\delta} \quad \text{with } c > 0 \text{ chosen so that } \sum_{n\in\mathbb{Z}} \lambda_n = 1.$$

Then, ρ is a Borel probability measure with $\rho(B) = 0$ if and only if $\mu(B) = 0$, that is, ρ and μ are mutually absolutely continuous. Since

$$|\mu_{n,m}(B)| \leq \mu_{n,n}(B)^{\frac{1}{2}}\mu_{m,m}(B)^{\frac{1}{2}}$$

by the Cauchy-Schwarz Inequality, it follows that $\mu_{n,m}$ is absolutely continuous with respect to ρ. By the Radon-Nikodym Theorem, there exists a measurable density $F_{n,m}$ with

$$\mu_{n,m}(B) = \int \chi_B(E) F_{n,m}(E)\, d\rho(E). \tag{1.24}$$

Since $\mu_{n,n} \geq 0$, we have $F_{n,n} \geq 0$ ρ-almost everywhere. From

$$\begin{aligned}\rho(B) &= \sum_{n\in\mathbb{Z}} \lambda_n \mu_{n,n}(B) \\ &= \sum_{n\in\mathbb{Z}} \lambda_n \int \chi_B(E) F_{n,n}(E)\, d\rho(E) \\ &= \int \chi_B(E) \left(\sum_{n\in\mathbb{Z}} \lambda_n F_{n,n}(E) \right) d\rho(E),\end{aligned}$$

we see that for ρ-almost every E, we have

$$\sum_{n\in\mathbb{Z}} \lambda_n F_{n,n}(E) = 1$$

and, consequently,

$$F_{n,n}(E) \leq \lambda_n^{-1}.$$

Thus,

$$\begin{aligned}\left| \int \chi_B(E) F_{n,m}(E)\, d\rho(E) \right| &= |\mu_{n,m}(B)| \\ &\leq \mu_{n,n}(B)^{\frac{1}{2}} \mu_{m,m}(B)^{\frac{1}{2}} \\ &= \left(\int \chi_B(E) F_{n,n}(E)\, d\rho(E) \right)^{\frac{1}{2}} \left(\int \chi_B(E) F_{m,m}(E)\, d\rho(E) \right)^{\frac{1}{2}} \\ &\leq \lambda_n^{-\frac{1}{2}} \lambda_m^{-\frac{1}{2}} \rho(B)\end{aligned}$$

and, consequently,

$$|F_{n,m}(E)| \leq \lambda_n^{-\frac{1}{2}} \lambda_m^{-\frac{1}{2}} \quad \text{for } \rho\text{-almost every } E. \tag{1.25}$$

On the other hand, equation (1.24) implies that for every bounded measurable function f, we have that

$$\langle \delta_n, f(H)\delta_m \rangle = \int f(E) F_{n,m}(E)\, d\rho(E).$$

In particular, if g is compactly supported and bounded, we may set $f(E) = Eg(E)$ and find

$$\begin{aligned}\int Eg(E) F_{n,m}(E)\, d\rho(E) &= \int g(E) \Big(E F_{n,m}(E) \Big)\, d\rho(E) \\ &= \langle \delta_n, Hg(H)\delta_m \rangle \\ &= \langle H\delta_n, g(H)\delta_m \rangle \\ &= \langle \delta_{n+1} + \delta_{n-1} + V(n)\delta_n, g(H)\delta_m \rangle \\ &= \langle \delta_{n+1}, g(H)\delta_m \rangle + \langle \delta_{n-1}, g(H)\delta_m \rangle + \langle V(n)\delta_n, g(H)\delta_m \rangle \\ &= \int g(E) \Big(H^{(n)} F_{n,m}(E) \Big)\, d\rho(E),\end{aligned}$$

where $H^{(n)}F_{n,m}(E)$ is the (formal) operator H applied to the two-sided sequence $n \mapsto F_{n,m}(E)$. It follows that for some arbitrary fixed $m \in \mathbb{Z}$,

$$u_E(n) = F_{n,m}(E) \text{ solves (1.18) for } \rho\text{-almost every } E \in \mathbb{R}.$$

From (1.25), we obtain the estimate

$$|u_E(n)| \lesssim \lambda_n^{-\frac{1}{2}} \lesssim (1+|n|)^{\delta},$$

as desired.

(c) Denote the set of generalized eigenvalues of H by $\mathcal{E}$. Since the spectrum of H is closed, part (a) implies that

$$\overline{\mathcal{E}} \subseteq \sigma(H).$$

On the other hand, part (b) shows that

$$0 \leq \mu\left(\mathbb{R} \setminus \overline{\mathcal{E}}\right) \leq \mu\left(\mathbb{R} \setminus \mathcal{E}\right) = 0$$

and therefore, $\overline{\mathcal{E}}$ is closed set having full μ-measure. Since the spectrum of H is given by the intersection of all closed sets having full μ-measure, it follows that

$$\sigma(H) \subseteq \overline{\mathcal{E}},$$

concluding the proof. □

5.3. Subordinacy Theory. After writing $M(z)$ in terms of the two functions $m^+(z)$, $m^-(z)$, the next step is to relate the boundary behavior of $m^{\pm}(z)$ to the behavior of solutions to (1.18) at $\pm\infty$. This is accomplished by the Gilbert-Pearson Theory of Subordinacy, developed in [**11, 12**], and it is especially explicit in the Jitomirskaya-Last Inequality from [**17**]. Since the treatment of the left half-line is completely analogous, we focus on the behavior of the solutions on the right half-line and the function $m^+(z)$.

In the following, all solutions to (1.18) mentioned will be regarded as maps from $\mathbb{Z}_+$ to $\mathbb{C}$. In particular, whether or not they are ℓ^2 is decided at $+\infty$. On the other hand, once a solution is specified for $n \in \mathbb{Z}_+$, it can be extended uniquely to a solution on all of $\mathbb{Z}$. When we talk about $u(n)$ for some $n \notin \mathbb{Z}_+$, we refer to the value at n of the unique extension of u to $\mathbb{Z}$.

As pointed out above, for $\Im z > 0$, (1.18) has a (up to normalization) unique solution u_z^+ that is ℓ^2 (at $+\infty$). We normalize u_z^+ by letting $u_z^+(0) = 1$, and hence we have $u_z^+(1) = -m^+(z)$.

A solution u of (1.18) is called *subordinate* (at $+\infty$) if

$$\lim_{L\to\infty} \frac{\|u\|_L}{\|v\|_L} = 0$$

for any linearly independent solution v of (1.18), where $\|\cdot\|_L$ denotes the norm of the solution over a lattice interval of length L. That is, we define

$$\|u\|_L \equiv \left[\sum_{n=1}^{[L]} |u(n)|^2 + (L - [L])|u([L]+1)|^2\right]^{1/2},$$

where $[L]$ denotes the integer part of L.

By considering an ℓ^2 solution u in (1.18), multiplying both sides of the equation by $\overline{u(n)}$, taking imaginary parts, and summing both sides from 1 to ∞, we obtain the equality

$$\Im(u(0)\overline{u(1)}) = \Im z \sum_{n=1}^{\infty} |u(n)|^2,$$

which for $u = u_z^+$ implies

$$\frac{\Im m(z)}{\varepsilon} = \sum_{n=1}^{\infty} |u_z^+(n)|^2. \tag{1.26}$$

Write

$$\Re z = E,\ \Im z = \varepsilon.$$

The first step is to use variation of parameters to express u_z^+, which solves (1.18), in terms of u_1 and u_2, which solve

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n) \tag{1.27}$$

and obey the initial conditions

$$\begin{pmatrix} u_1(1) & u_2(1) \\ u_1(0) & u_2(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that

$$M_z(n) = \begin{pmatrix} u_1(n+1) & u_2(n+1) \\ u_1(n) & u_2(n) \end{pmatrix}. \tag{1.28}$$

Lemma 1.19. *For any $n > 0$, $u_z^+(n)$ obeys the equality*
(1.29)

$$u_z^+(n) = u_2(n) - m^+(z)u_1(n) - i\varepsilon u_2(n)\sum_{k=1}^{n} u_1(k)u_z^+(k) + i\varepsilon u_1(n)\sum_{k=1}^{n} u_2(k)u_z^+(k).$$

Proof. Let $\tilde{v}(n)$ be the right-hand side of (1.29) for $n > 0$ and let $\tilde{v}(0) = 1$. By taking into account the Wronskian conservation,

$$u_1(n+1)u_2(n) - u_2(n+1)u_1(n) = 1 \quad \text{for every } n,$$

it is easy to verify that $\{\tilde{v}(n)\}_{n=0}^{\infty}$ obeys

$$\tilde{v}(n+1) = -\tilde{v}(n-1) + (E - V(n))\tilde{v}(n) + i\varepsilon u_z^+(n) \tag{1.30}$$

for any $n > 0$. Since for $n > 0$,

$$u_z^+(n+1) = -u_z^+(n-1) + (E + i\varepsilon - V(n))u_z^+(n), \tag{1.31}$$

and since $\tilde{v}(0) = u_z^+(0)$, $\tilde{v}(1) = u_z^+(1)$, we see by induction that $\tilde{v}(n) = u_z^+(n)$ for any $n \geq 0$. $\square$

For $\varepsilon > 0$, define the length $L(\varepsilon) \in (0, \infty)$ by

$$\|u_1\|_{L(\varepsilon)}\|u_2\|_{L(\varepsilon)} = \frac{1}{2\varepsilon}. \tag{1.32}$$

At most one of either u_1 or u_2 may be in ℓ^2, and so the left-hand side of (1.32) is a monotonically increasing continuous function of L which vanishes for $L = 0$ and goes to infinity as L goes to infinity. Similarly, the right-hand side of (1.32) is a monotonically increasing continuous function of ϵ which goes to infinity as ϵ goes to 0. Thus the function $L(\epsilon)$ is well defined and it is a monotonically increasing continuous function which goes to infinity as ε goes to 0. The core result of Jitomirskaya and Last is the following inequality which relates the Weyl-Titchmarsh m-function (for z in the upper half plane) to the solutions u_1 and u_2.

THEOREM 1.20 (Jitomirskaya-Last 1999). *Let $E \in \mathbb{R}$ be given. Then the following inequality holds for $\varepsilon > 0$ small enough:*

$$\frac{5-\sqrt{24}}{|m^+(E+i\varepsilon)|} < \frac{\|u_1\|_{L(\varepsilon)}}{\|u_2\|_{L(\varepsilon)}} < \frac{5+\sqrt{24}}{|m^+(E+i\varepsilon)|}.$$

Remark. One can improve the constants $5 \pm \sqrt{24}$ somewhat (see [**20**]), but since we do not need this improvement, we follow the original proof from [**17**]. There is also closely related work by Remling [**31**].

PROOF. From (1.29) we see that for any $L > 1$,

$$\|u_z^+\|_L \geq \|u_2 - m^+(z)u_1\|_L - 2\varepsilon\|u_1\|_L\|u_2\|_L\|u_z^+\|_L.$$

By considering $L = L(\varepsilon)$, which implies $2\varepsilon\|u_1\|_L\|u_2\|_L = 1$, we obtain

$$2\|u_z^+\|_{L(\varepsilon)} \geq \|u_2 - m^+(z)u_1\|_{L(\varepsilon)}. \tag{1.33}$$

Squaring the two sides of (1.33) and noting that by (1.26)

$$\|u_z^+\|_{L(\varepsilon)}^2 < \|u_z^+\|_\infty^2 = \frac{\Im m^+(z)}{\varepsilon},$$

we obtain

$$\begin{aligned}\frac{4\Im m^+(z)}{\varepsilon} &> \|u_2 - m^+(z)u_1\|_{L(\varepsilon)}^2 \\ &\geq \|u_2\|_{L(\varepsilon)}^2 + |m^+(z)|^2\|u_1\|_{L(\varepsilon)}^2 - 2|m^+(z)|\|u_2\|_{L(\varepsilon)}\|u_1\|_{L(\varepsilon)}.\end{aligned} \tag{1.34}$$

By (1.32), we have $\|u_2\|_{L(\varepsilon)}\|u_1\|_{L(\varepsilon)} = 1/2\varepsilon$, and so multiplying the two sides of (1.34) by 2ε yields

$$8\Im m^+(z) > \frac{\|u_2\|_{L(\varepsilon)}}{\|u_1\|_{L(\varepsilon)}} + |m^+(z)|^2\frac{\|u_1\|_{L(\varepsilon)}}{\|u_2\|_{L(\varepsilon)}} - 2|m^+(z)|,$$

which implies

$$|m^+(z)|^2\frac{\|u_1\|_{L(\varepsilon)}}{\|u_2\|_{L(\varepsilon)}} - 10|m^+(z)| + \frac{\|u_2\|_{L(\varepsilon)}}{\|u_1\|_{L(\varepsilon)}} < 0. \tag{1.35}$$

Solving (1.35) as a quadratic inequality for the variable $|m^+(z)|$, one obtains

$$(5-\sqrt{24})\frac{\|u_2\|_{L(\varepsilon)}}{\|u_1\|_{L(\varepsilon)}} < |m^+(z)| < (5+\sqrt{24})\frac{\|u_2\|_{L(\varepsilon)}}{\|u_1\|_{L(\varepsilon)}},$$

from which the result follows. □

COROLLARY 1.21 (Gilbert-Pearson 1987). *For every $E \in \mathbb{R}$, $|m^+(E+i0)| = \infty$ if and only if the solution u_1 of (1.27) is subordinate. Moreover, $m^+(E+i0) = \cot\theta$ for some $\theta \in (0,\pi)$ if and only if the solution of (1.27) obeying $u(0) = -\sin\theta$ and $u(1) = \cos\theta$ is subordinate.*

PROOF. The first statement is an immediate consequence of the Jitomirskaya-Last Inequality from Theorem 1.20.

Consider the solutions $u_{1,\theta}$, $u_{2,\theta}$ of (1.18) that obey the initial conditions $u_{1,\theta}(0) = -u_{2,\theta}(1) = -\sin\theta$ and $u_{1,\theta}(1) = u_{2,\theta}(0) = \cos\theta$. Note that they are the u_1, u_2 solutions for the modified potential $V_\theta(n) = V(n) - \delta_{1,n}\tan\theta$ away from the perturbation: On the one hand, $u_{1,\theta}(0) = -\sin\theta$ and $u_{1,\theta}(1) = \cos\theta$ imply that

$$\begin{aligned} 0 &= u_{1,\theta}(2) + u_{1,\theta}(0) + (V(1) - z)u_{1,\theta}(1) \\ &= u_{1,\theta}(2) - \sin\theta + (V(1) - z)\cos\theta \\ &= u_{1,\theta}(2) + (V(1) - \tan\theta - z)\cos\theta \\ &= u_{1,\theta}(2) + (V_\theta(1) - z)u_{1,\theta}(1) \end{aligned}$$

and, on the other hand, by Wronskian conservation, $u_{2,\theta}$ is the solution obeying the normalized orthogonal initial condition.

We wish to apply the Jitomirskaya-Last Inequality to the operator with this perturbed potential. In view of (1.29), the appropriate m-function is defined by

$$u^+_{z,\theta} = u_{2,\theta} - m^+_\theta(z)u_{1,\theta},$$

where $u^+_{z,\theta}$ is the solution that is ℓ^2 at infinity and normalized by $u^+_{z,\theta}(0)\cos\theta + u^+_{z,\theta}(1)\sin\theta = 1$. We have the following relation between the m-functions,

$$m^+_\theta(z) = \frac{\sin\theta + m^+(z)\cos\theta}{\cos\theta - m^+(z)\sin\theta}, \tag{1.36}$$

since, by uniqueness up to multiples of the solution that is ℓ^2 at infinity,

$$\begin{aligned} m^+(z) &= -\frac{u^+_{z,\theta}(1)}{u^+_{z,\theta}(0)} \\ &= -\frac{u_{2,\theta}(1) - m^+_\theta(z)u_{1,\theta}(1)}{u_{2,\theta}(0) - m^+_\theta(z)u_{1,\theta}(0)} \\ &= -\frac{\sin\theta - m^+_\theta(z)\cos\theta}{\cos\theta + m^+_\theta(z)\sin\theta}, \end{aligned}$$

from which (1.36) follows by inversion.

Thus, the second statement follows from (1.36) and the Jitomirskaya-Last Inequality from Theorem 1.20, applied to the potential V_θ. □

COROLLARY 1.22 (Gilbert 1989). *The singular part, μ_{sing}, is supported on the set of real energies E for which* (1.27) *has a solution that is subordinate at both $\pm\infty$ and the absolutely continuous part, μ_{ac}, is supported on the set of real energies E for which no solution of* (1.27) *is subordinate at ∞ or no solution of* (1.27) *is subordinate at $-\infty$.*

PROOF. By (1.11) and (1.12), we need to investigate for which real energies E, we have $\Im M(E + i0) = \infty$ (resp, $0 < \Im M(E + i0) < \infty$). A few simple manipulations show that

$$M(z) = \frac{-1}{\dfrac{-1}{\dfrac{-1}{m^+(z)} + \dfrac{-1}{m^-(z)}} + \dfrac{-1}{m^+(z) + m^-(z)}}.$$

Now all statements of the theorem follow from Theorem 1.14 and the fact that M and $m^\pm$ are Herglotz functions (analytic functions from $\mathbb{C}_+$ to $\mathbb{C}_+$). Note that if F is a Herglotz function, then so is $-1/F(z)$ and

$$\Im\left(\frac{-1}{F(z)}\right) = \frac{\Im F(z)}{|F(z)|^2}.$$

Thus, $\Im(-1/F(E+i0)) = \infty$ if and only if $F(E+i0) = 0$. $\square$

CHAPTER 2

Ergodic Transformations

1. The Measurable Setting

1.1. Definitions. Suppose we are given a probability measure space $(\Omega, \mathcal{A}, \mu)$. We will sometimes leave the σ-algebra $\mathcal{A}$ implicit and just write (Ω, μ). Integration with respect to μ will be denoted by $\mathbb{E}(\cdot)$, that is, if $f \in L^1(\Omega, d\mu)$, then

$$\mathbb{E}(f) = \int f(\omega)\, d\mu(\omega).$$

Suppose further that $T : \Omega \to \Omega$ is an invertible measure-preserving transformation. That is, T is a one-to-one and onto map so that for every $A \in \mathcal{A}$, we have $TA, T^{-1}A \in \mathcal{A}$ and $\mu(A) = \mu(TA) = \mu(T^{-1}A)$. Such a map T is also called an automorphism of $(\Omega, \mathcal{A}, \mu)$. Conversely, given $(\Omega, \mathcal{A}, T)$, a probability measure μ with the invariance property above is called an invariant probability measure for T, or just T-invariant.

Examples. (a) Translation on a torus: Ω is the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $\mathcal{A}$ is the Borel σ-algebra, μ is normalized Lebesgue measure, denoted by Leb, T is given by a translation, that is,

$$T(\omega_1, \dots, \omega_d) = (\omega_1 + \alpha_1, \dots, \omega_d + \alpha_d),$$

where $\alpha_1, \dots, \alpha_d$ are real numbers, which can and will be chosen in the interval $[0, 1)$.

(b) Shift on a sequence space: Fix some compact interval $I \subset \mathbb{R}$ with the induced topology. Consider the infinite product $\Omega = I^{\mathbb{Z}}$ with the product topology and the Borel σ-algebra $\mathcal{A}$. The shift transformation $T : \Omega \to \Omega$ is given by

$$(T\omega)_n = \omega_{n+1}.$$

There are many T-invariant measures μ. An important class is obtained by taking $\mu = \nu^{\mathbb{Z}}$, where ν is a Borel probability measure on I.

We say that $(\Omega, \mathcal{A}, \mu, T)$ is ergodic if in addition every invariant function is constant. More precisely, this means that if f is a measurable function on Ω and $f(\omega) = f(T\omega) = f(T^{-1}\omega)$ for μ-almost every $\omega \in \Omega$, then there is a constant f_* and a set $\Omega_* \subseteq \Omega$ of full μ-measure so that $f(\omega) = f_*$ for every $\omega \in \Omega_*$.

1.2. The Birkhoff Pointwise Ergodic Theorem. In this subsection we study time-averages of an observable and relate them to a space average with respect to an ergodic measure. The celebrated Birkhoff Ergodic Theorem states that these averages are equal pointwise for almost every initial point. The proof given below is due to Katznelson and Weiss [**18**].

THEOREM 2.1 (Birkhoff). *Suppose* (Ω, μ, T) *is ergodic and* $f \in L^1(\Omega, d\mu)$. *Then, for* μ*-almost every* $\omega \in \Omega$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n\omega) = \mathbb{E}(f).$$

PROOF. It suffices to consider non-negative functions f. Let us define

$$\overline{f}(\omega) = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n\omega)$$

and

$$\underline{f}(\omega) = \liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n\omega).$$

Both $\overline{f}$ and $\underline{f}$ are T-invariant. Our goal is to show that

$$\mathbb{E}(\overline{f}) \leq \mathbb{E}(f) \leq \mathbb{E}(\underline{f}). \tag{2.1}$$

It follows from (2.1) that $\overline{f}(\omega) = \underline{f}(\omega) \equiv f^*(\omega)$ for μ-almost every $\omega \in \Omega$. Clearly, f^* is T-invariant and hence constant μ-almost everywhere. Thus, almost surely,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n\omega) = f^*(\omega) = \mathbb{E}(f^*) = \mathbb{E}(f).$$

The last step follows from (2.1), which implies $\mathbb{E}(\overline{f}) = \mathbb{E}(f) = \mathbb{E}(\underline{f}) = \mathbb{E}(f^*)$.

Thus we are left with proving (2.1). Fix some $M > 0$, $\varepsilon > 0$ and denote

$$\overline{f}_M(\omega) = \min\{\overline{f}(\omega), M\}.$$

Define $N(\omega)$ to be the least integer $N \geq 1$ for which

$$\overline{f}_M(\omega) \leq \frac{1}{N} \sum_{n=0}^{N-1} f(T^n\omega) + \varepsilon.$$

Since $\overline{f}$ is T-invariant, so is $\overline{f}_M$, and hence averaging gives

$$\sum_{n=0}^{N(\omega)-1} \overline{f}_M(T^n\omega) = N(\omega)\overline{f}_M(\omega) \leq \sum_{n=0}^{N(\omega)-1} f(T^n\omega) + N(\omega)\varepsilon. \tag{2.2}$$

Next choose N such that the set

$$A = \{\omega : N(\omega) > N\}$$

has μ-measure less than ε/M. Define

$$\tilde{f}(\omega) = \begin{cases} f(\omega) & \omega \notin A \\ \max\{f(\omega), M\} & \omega \in A \end{cases} \quad \text{and} \quad \tilde{N}(\omega) = \begin{cases} N(\omega) & \omega \notin A \\ 1 & \omega \in A \end{cases}$$

We have

$$\sum_{n=0}^{\tilde{N}(\omega)-1} \overline{f}_M(T^n\omega) \leq \sum_{n=0}^{\tilde{N}(\omega)-1} \tilde{f}(T^n\omega) + \tilde{N}(\omega)\varepsilon \tag{2.3}$$

since, when $\omega \notin A$, (2.3) is equivalent to (2.2), and when $\omega \in A$, it holds trivially. Observe that $\tilde{N}(\omega) \leq N$ and

$$\mathbb{E}(\tilde{f}) \leq \mathbb{E}(\chi_{\Omega\setminus A} f) + \mathbb{E}(\chi_A f) + \mathbb{E}(\chi_A \cdot M) \leq \mathbb{E}(f) + \varepsilon. \tag{2.4}$$

Choose L with $NM/L < \varepsilon$. Define $N_0(\omega) = 0$ and

$$N_k(\omega) = N_{k-1}(\omega) + \tilde{N}(T^{N_{k-1}(\omega)}\omega), \quad k \geq 1.$$

Finally, let $k(\omega)$ be the maximal k for which $N_k(\omega) \leq L-1$. Write

$$\begin{aligned}
\sum_{n=0}^{L-1} \overline{f}_M(T^n\omega) &= \sum_{k=1}^{k(\omega)} \sum_{n=N_{k-1}(\omega)}^{N_k(\omega)-1} \overline{f}_M(T^n\omega) + \sum_{n=N_{k(\omega)}(\omega)}^{L-1} \overline{f}_M(T^n\omega) \\
&= \sum_{k=1}^{k(\omega)} \sum_{n=0}^{N_k(\omega)-N_{k-1}(\omega)-1} \overline{f}_M(T^n(T^{N_{k-1}(\omega)}\omega)) + \sum_{n=N_{k(\omega)}(\omega)}^{L-1} \overline{f}_M(T^n\omega) \\
&= \sum_{k=1}^{k(\omega)} \sum_{n=0}^{\tilde{N}(T^{N_{k-1}(\omega)}\omega)-1} \overline{f}_M(T^n(T^{N_{k-1}(\omega)}\omega)) + \sum_{n=N_{k(\omega)}(\omega)}^{L-1} \overline{f}_M(T^n\omega).
\end{aligned}$$

Now apply (2.3) to each of the $k(\omega)$ terms in the first sum and estimate each of the $L - N_{k(\omega)}(\omega) \leq N-1$ terms in the second sum by M. This gives

$$\begin{aligned}
\sum_{n=0}^{L-1} \overline{f}_M(T^n\omega) &\leq \sum_{k=1}^{k(\omega)} \left[\sum_{n=0}^{\tilde{N}(T^{N_{k-1}(\omega)}\omega)-1} \tilde{f}(T^n(T^{N_{k-1}(\omega)}\omega)) + \tilde{N}(T^{N_{k-1}(\omega)}\omega)\varepsilon \right] + (N-1)M \\
&\leq \sum_{n=0}^{L-1} \tilde{f}(T^n\omega) + L\varepsilon + (N-1)M.
\end{aligned}$$

(In the last step, we used $f \geq 0$.) Integrating both sides, using that T is measure-preserving, dividing by L, and applying (2.4) gives

$$\mathbb{E}(\overline{f}_M) \leq \mathbb{E}(\tilde{f}) + \varepsilon + \frac{(N-1)M}{L} \leq \mathbb{E}(f) + 3\varepsilon.$$

Thus, letting $M \to \infty$ and $\varepsilon \to 0$, we obtain $\mathbb{E}(\overline{f}) \leq \mathbb{E}(f)$, which is the first inequality in (2.1). The second inequality can be shown in a similar way. □

1.3. Kingman's Subadditive Ergodic Theorem. Time-averages of an observable are additive in the sense that if a given finite piece of an orbit is broken up into two pieces, then the averages over the two shorter pieces add up to give the average over the long piece. This additivity property was crucial in the proof of the Birkhoff Ergodic Theorem. In this subsection we prove Kingman's Ergodic Theorem, which merely assumes a subadditivity condition. The proof given below follows Steele [**35**].

We will need the following basic lemma about subadditive sequences in the proof.

LEMMA 2.2. *Suppose $\{a_n\}_{n\geq 1}$ is a sequence of real numbers satisfying $a_n \gtrsim -n$ and the subadditivity condition $a_{n+m} \leq a_n + a_m$ for $n, m \geq 1$. Then*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n}.$$

PROOF. By assumption, $a_n \gtrsim n$ and hence $I = \inf_{n\geq 1} \frac{a_n}{n}$ is finite. We have to prove that $\frac{a_n}{n}$ converges and the limit is given by I.

Given $\varepsilon > 0$, choose $K \in \mathbb{Z}_+$ with

$$\left| \frac{a_K}{K} - I \right| < \frac{\varepsilon}{2}.$$

Next choose $M \in \mathbb{Z}_+$ with

$$\max_{1 \le r \le K-1} \frac{a_r}{KM} < \frac{\varepsilon}{2}.$$

Now let $N = KM$ and consider $n \ge N$. Write $n = sK + r$ with $s \ge M$ and $0 \le r \le K - 1$. Then, by subadditivity,

$$\begin{aligned}\frac{a_n}{n} &= \frac{a_{sK+r}}{sK+r} \\ &\le \frac{sa_K}{sK+r} + \frac{a_r}{sK+r} \\ &\le \frac{sa_K}{sK} + \frac{a_r}{KM} \\ &\le I + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.\end{aligned}$$

The result follows. □

THEOREM 2.3 (Kingman 1973). *Suppose* (Ω, μ, T) *is ergodic. If* $f_n : \Omega \to \mathbb{R}$ *are measurable, obey* $\|f_n\|_\infty \lesssim n$ *and the subadditivity condition*

$$f_{n+m}(\omega) \le f_n(\omega) + f_m(T^n\omega), \tag{2.5}$$

then

$$\lim_{n\to\infty} \frac{1}{n} f_n(\omega) = \inf_{n\ge 1} \frac{1}{n} \mathbb{E}(f_n)$$

for μ*-almost every* $\omega \in \Omega$.

PROOF. We may assume that $f_n(\omega) \le 0$ for all n and ω. For if not, we consider

$$f_n'(\omega) = f_n(\omega) - \sum_{m=0}^{n-1} f_1(T^m\omega).$$

Subadditivity, (2.5), implies that $f_n'(\omega) \le 0$ for all n and ω. Moreover, $\{f_n'\}$ satisfies (2.5) as well. Since we can apply Birkhoff's ergodic theorem to the second term of f_n', we see that almost everywhere convergence for $\{f_n\}$ is equivalent to almost everywhere convergence for $\{f_n'\}$.

Thus, we assume that f_n takes values in the interval $[-Mn, 0]$ for some $M > 0$. Let

$$f(\omega) = \liminf_{n\to\infty} \frac{1}{n} f_n(\omega).$$

Since

$$\frac{f_{n+1}(\omega)}{n} \le \frac{f_1(\omega)}{n} + \frac{f_n(T\omega)}{n},$$

we see that, taking the lim inf, $f(\omega) \le f(T\omega)$. This shows

$$\{\omega : f(\omega) > \gamma\} \subseteq T^{-1}\{\omega : f(\omega) > \gamma\}.$$

But since T preserves μ, these two sets coincide up to a set of measure zero. Consequently, $f(T\omega) = f(\omega)$ almost everywhere, and hence $f(\omega)$ is almost surely equal to a constant $f \le 0$ by ergodicity. It remains to show that

$$\limsup_{n\to\infty} \frac{1}{n} f_n(\omega) \le f \quad \mu - \text{almost surely} \tag{2.6}$$

and

$$f = \inf_{n\ge 1} \frac{1}{n} \mathbb{E}(f_n). \tag{2.7}$$

Given $\varepsilon > 0$ and $N \geq 1$, define (the "bad" set)

$$B_{N,\varepsilon} = \left\{ \omega : \frac{f_k(\omega)}{k} > f + \varepsilon \text{ for every } 1 \leq k \leq N \right\}.$$

We denote its complement by $G_{N,\varepsilon}$. Notice that for each $\varepsilon > 0$,

$$\lim_{N \to \infty} \mu\left(G_{N,\varepsilon}\right) = 1. \tag{2.8}$$

Our goal is to find a good estimate from above for $f_n(\omega)$. We will subdivide with the goal of having as many terms in $G_{n,\varepsilon}$ as possible and then apply subadditivity and non-positivity. To find a suitable subdivision, we proceed as follows. Consider the integers $1 \leq m \leq n-1$ and group them into intervals according to the following algorithm. Let m be the least integer in $[1, n)$ that is not in an interval already constructed. (That is, we start with 1.) Consider $T^m\omega$. If $T^m\omega \in G_{N,\varepsilon}$, then there is $k \leq N$ such that $f_k(T^m\omega) \leq k(f + \varepsilon)$. If $m + k \leq n$, we take $[m, m + k)$ as an element of our decomposition. If $m + k > n$, we just take the interval $[m, m + 1)$. If, on the other hand, $T^m\omega \in B_{N,\varepsilon}$, we also just take the interval $[m, m + 1)$.

This leads to the following estimate:

$$\begin{aligned} f_n(\omega) &\leq \sum_{j=1}^{u} f_{k_j}(T^{m_j}\omega) + \sum_{j=1}^{v} f_1(T^{m'_j}\omega) + \sum_{j=1}^{w} f_1(T^{n-j}\omega) \\ &\leq (f + \varepsilon) \sum_{j=1}^{u} k_j \\ &\leq f \sum_{j=1}^{u} k_j \; + n\varepsilon. \end{aligned}$$

Of course, the numbers u, v, w are n-dependent. From Birkhoff's ergodic theorem, we may infer for μ-almost every ω,

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{u} k_j \geq \lim_{n \to \infty} \frac{1}{n} \sum_{m=1}^{n} \chi_{G_{N,\varepsilon}}(T^m\omega) = \mu\left(G_{N,\varepsilon}\right).$$

Combining this with the above (and observing $f \leq 0$), we obtain

$$\limsup_{n \to \infty} \frac{1}{n} f_n(\omega) \leq \varepsilon + f\mu\left(G_{N,\varepsilon}\right) \quad \mu - \text{almost surely.}$$

Taking $N \to \infty$, (2.8) yields

$$\limsup_{n \to \infty} \frac{1}{n} f_n(\omega) \leq \varepsilon + f \quad \mu - \text{almost surely.}$$

Since this holds for every $\varepsilon > 0$, (2.6) follows.

Integrating (2.5), we obtain

$$\mathbb{E}(f_{n+m}) \leq \mathbb{E}(f_n) + \mathbb{E}(f_m),$$

that is, $\{\mathbb{E}(f_n)\}$ is a subadditive sequence. Lemma 2.2 then implies

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{E}(f_n) = \inf_{n \geq 1} \frac{1}{n} \mathbb{E}(f_n).$$

On the other hand, we know that $\lim \frac{1}{n} f_n(\omega) = f$ almost surely and hence, by dominated convergence, $\lim \frac{1}{n} \mathbb{E}(f_n) = f$. This shows (2.7). $\square$

2. The Topological Setting

2.1. Definitions. We will often be interested in spaces Ω that have additional structure. In this case, we are especially interested in those measures μ and transformations T that respect this structure.

For example, suppose Ω is a compact metric space and $T : \Omega \to \Omega$ is a homeomorphism. Let us consider the set $M_1(T)$ of T-invariant Borel probability measures. The following important results hold; see, for example, Walters [**36**].

- The set $M_1(T)$ is non-empty and convex.
- A measure $\mu \in M_1(T)$ is ergodic if and only if it is an extreme point of $M_1(T)$. We say that T is uniquely ergodic if $M_1(T)$ consists of a single measure, which then necessarily must be ergodic.
- T is uniquely ergodic if and only if for every $f \in C(\Omega)$, $n^{-1} \sum_{k=0}^{n-1} f(T^k\omega)$ converges uniformly on Ω to a constant as $n \to \infty$.

The homeomorphism T is minimal if for every $\omega \in \Omega$, the T-orbit of $\omega \in \Omega$, $\mathcal{O}(\omega) = \{T^n\omega : n \in \mathbb{Z}\}$, is dense in Ω.

Theorem 2.4. *The following are equivalent:*

(i) *(Ω, T) is uniquely ergodic.*

(ii) *For every $f \in C(\Omega)$,*

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n\omega)$$

converges uniformly to a constant.

(iii) *For every $f \in C(\Omega)$,*

$$\frac{1}{N} \sum_{n=0}^{N-1} f(T^n\omega)$$

converges everywhere pointwise to a constant.

(iv) *There exists a T-invariant Borel probability measure μ such that for every $f \in C(\Omega)$ and every $\omega \in \Omega$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n\omega) = \int f(\omega)\, d\mu(\omega).$$

Theorem 2.5 (Furman 1997). *Suppose (Ω, T) is uniquely ergodic. If $f_n : \Omega \to \mathbb{R}$ are continuous and obey the subadditivity condition $f_{n+m}(\omega) \le f_n(\omega) + f_m(T^n\omega)$, then*

$$\limsup_{n\to\infty} \frac{1}{n} f_n(\omega) \le \inf_{n\ge 1} \frac{1}{n} \mathbb{E}(f_n)$$

for every $\omega \in \Omega$ and uniformly on Ω.

Proof. This is a modification of the proof of Theorem 2.3. As above we may assume that $f_n(\omega) \le 0$ for all n and ω. For if not, we consider

$$f_n'(\omega) = f_n(\omega) - \sum_{m=0}^{n-1} f_1(T^m\omega).$$

Subadditivity implies that $f_n'(\omega) \le 0$ for all n and ω. Moreover, $\{f_n'\}$ is subaddditive as well. Since we can apply Theorem 2.4 to the second term of f_n', we see that the asserted uniform upper bound for $\{f_n\}$ is equivalent to the uniform upper bound for $\{f'_n\}$.

Given $\varepsilon > 0$, it is possible by (2.8) to choose N so that

$$\mu(G_{N,\varepsilon}) > 1 - \varepsilon.$$

By Urysohn's Lemma, there is a continuous function g with $0 \le g \le \chi_{G_{N,\varepsilon}}$ and

$$\int g(\omega)\, d\mu(\omega) > 1 - 2\varepsilon.$$

Choose $\tilde{N}$ so that

$$\frac{1}{n} \sum_{m=1}^{n} g(T^m \omega) > 1 - 3\varepsilon$$

for every $\omega \in \Omega$ and $n \ge \tilde{N}$.

With $f = \inf_{n \ge 1} \frac{1}{n}\mathbb{E}(f_n) \le 0$ and the induced decomposition introduced above it follows that we have for every $\omega \in \Omega$ and $n \ge \max\{N, \tilde{N}\}$,

$$\begin{aligned}
f_n(\omega) &\le \sum_{j=1}^{u} f_{k_j}(T^{m_j}\omega) + \sum_{j=1}^{v} f_1(T^{m'_j}\omega) + \sum_{j=1}^{w} f_1(T^{n-j}\omega) \\
&\le (f + \varepsilon) \sum_{j=1}^{u} k_j \\
&\le f \sum_{j=1}^{u} k_j \; + \, n\varepsilon \\
&\le f \sum_{m=1}^{n} \chi_{G_{N,\varepsilon}}(T^m \omega) + \, n\varepsilon \\
&\le f \sum_{m=1}^{n} g(T^m \omega) + \, n\varepsilon \\
&\le f n(1 - 3\varepsilon) + \, n\varepsilon
\end{aligned}$$

This is the desired uniform upper bound. □

3. Examples

THEOREM 2.6 (Translations on a torus). *Suppose $\Omega = \mathbb{T}^d$ and T is given by*

$$T(\omega_1, \dots, \omega_d) = (\omega_1 + \alpha_1, \dots, \omega_d + \alpha_d).$$

Then, Leb *is ergodic if and only if the numbers $1, \alpha_1, \dots, \alpha_d$ are rationally independent, that is, if $n_0 = n_1\alpha_1 + \cdots + n_d\alpha_d$ with integers $n_0, n_1, \dots, n_d$, then $n_0 = n_1 = \cdots = n_d = 0$.*

PROOF. Suppose first that the numbers $1, \alpha_1, \dots, \alpha_d$ are rationally independent. We want to show that every T-invariant measurable function is almost everywhere constant. It suffices to consider $f \in L^\infty(\mathbb{T}^j)$ (by cutting off and using invariance). Consider the Fourier series of such an f,

$$f(\omega) = \sum_{k \in \mathbb{Z}^d} c_k \exp\left(2\pi i \langle k, \omega \rangle\right),$$

where convergence is understood in L^2-sense. Then, the invariance condition $f = f \circ T$ implies

$$\begin{aligned}\sum_{k \in \mathbb{Z}^j} c_k \exp\left(2\pi i \langle k, \omega \rangle\right) &= \sum_{k \in \mathbb{Z}^j} c_k \exp\left(2\pi i \langle k, \omega + \alpha \rangle\right) \\ &= \sum_{k \in \mathbb{Z}^j} c_k \exp\left(2\pi i \langle k, \omega \rangle\right) \cdot \exp\left(2\pi i \langle k, \alpha \rangle\right)\end{aligned}$$

By the uniqueness of the Fourier coefficients and the assumption (iv) (which implies $\exp\left(2\pi i \langle k, \alpha \rangle\right) \neq 1$ for any non-zero $k \in \mathbb{Z}^d$), it follows that $c_k = 0$ for every non-zero $k \in \mathbb{Z}^d$. This shows that $f = c_0$ in L^2-sense and hence f is Lebesgue almost everywhere constant.

Assume now that the numbers $1, \alpha_1, \ldots, \alpha_d$ are not rationally independent. Then there exist integers $n_1, \ldots, n_d$, not all zero, such that $n_1\alpha_1 + \cdots + n_d\alpha_d \in \mathbb{Z}$. Consider the function

$$f(\omega) = \exp\left(2\pi i \sum_{k=1}^{d} n_k \omega_k\right).$$

Then, f is not constant on any set of full Lebesgue measure. On the other hand, f is invariant:

$$\begin{aligned}f(T\omega) &= \exp\left(2\pi i \sum_{k=1}^{d} n_k(\omega_k + \alpha_k)\right) \\ &= \exp\left(2\pi i \sum_{k=1}^{d} n_k \omega_k\right) \cdot \exp\left(2\pi i \sum_{k=1}^{d} n_k \alpha_k\right) \\ &= \exp\left(2\pi i \sum_{k=1}^{d} n_k \omega_k\right) \\ &= f(\omega).\end{aligned}$$

This shows that Leb is not ergodic. □

THEOREM 2.7 (Skew-shifts on a torus). *Suppose* $\Omega = \mathbb{T}^2$ *and* T *is given by*

$$T(\omega_1, \omega_2) = (\omega_1 + \alpha, \omega_1 + \omega_2),$$

where $\alpha \in (0, 1)$ *is irrational. Then,* Leb *is* T*-ergodic.*

PROOF. We want to show that every T-invariant measurable function is almost everywhere constant. Again, it suffices to consider $f \in L^\infty(\mathbb{T}^d)$ (by cutting off and using invariance). Consider the Fourier series of such an f,

$$f(\omega) = \sum_{k \in \mathbb{Z}^2} c_k \exp\left(2\pi i \langle k, \omega \rangle\right),$$

where convergence is understood in L^2-sense. Then, the invariance condition $f = f \circ T$ implies

$$\begin{aligned}\sum_{k \in \mathbb{Z}^2} c_k \exp\left(2\pi i \langle k, \omega \rangle\right) &= \sum_{k \in \mathbb{Z}^2} c_k \exp\left(2\pi i \langle k, T\omega \rangle\right) \\ &= \sum_{k \in \mathbb{Z}^2} c_k \exp\left(2\pi i \left(k_1(\omega_1 + \alpha) + k_2(\omega_1 + \omega_2)\right)\right)\end{aligned}$$

$$= \sum_{k \in \mathbb{Z}^2} c_k \exp\left(2\pi i \langle k, \omega \rangle\right) \exp\left(2\pi i \left(k_1 \alpha + k_2 \omega_1\right)\right)$$

By the uniqueness of the Fourier coefficients, it follows that $c_k = 0$ for every non-zero $k \in \mathbb{Z}^2$. This shows that $f = c_0$ in L^2-sense and hence f is Lebesgue almost everywhere constant. □

THEOREM 2.8 (Hyperbolic toral automorphism). *Suppose* $\Omega = \mathbb{T}^2$ *and* T *is given by*

$$T(\omega_1, \omega_2) = (2\omega_1 + \omega_2, \omega_1 + \omega_2).$$

Then, Leb *is* T*-ergodic.*

PROOF. The scheme of the proof is as above. Invariance of a bounded f implies

$$\begin{aligned} \sum_{k \in \mathbb{Z}^2} c_k \exp\left(2\pi i \langle k, \omega \rangle\right) &= \sum_{k \in \mathbb{Z}^2} c_k \exp\left(2\pi i \langle k, T\omega \rangle\right) \\ &= \sum_{k \in \mathbb{Z}^2} c_k \exp\left(2\pi i \left(k_1(2\omega_1 + \omega_2) + k_2(\omega_1 + \omega_2)\right)\right) \\ &= \sum_{k \in \mathbb{Z}^2} c_k \exp\left(2\pi i \left((2k_1 + k_2)\omega_1 + (k_1 + k_2)\omega_2\right)\right). \end{aligned}$$

By uniqueness of Fourier coefficients, we find that

$$c_{2k_1 + k_2, k_1 + k_2} = c_{k_1, k_2}$$

for every $k \in \mathbb{Z}^2$. Since

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

does not have any eigenvalues of modulus one, it therefore follows that if $c_k \neq 0$ for some non-zero $k \in \mathbb{Z}^2$, this value repeats for arbitrarily large index and hence the Fourier coefficients do not converge to zero, which is a contradiction. □

CHAPTER 3

General Results for Ergodic Schrödinger Operators

Suppose $(\Omega, \mathcal{A}, \mu, T)$ is ergodic and $f : \Omega \to \mathbb{R}$ is measurable and bounded. Define potentials,

$$V_\omega(n) = f(T^n\omega), \quad \omega \in \Omega,\ n \in \mathbb{Z} \tag{3.1}$$

and Schrödinger operators on $\mathcal{H} = \ell^2(\mathbb{Z})$,

$$[H_\omega\psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n). \tag{3.2}$$

Our goal is to study the spectral properties of the operators H_ω. We will mainly be interested in statements that hold μ-almost surely.

1. Non-Randomness of Spectra and Spectral Types

Our goal in this section is to prove that the spectrum of H_ω is independent of ω away from a set of zero μ-measure. The same statement holds for the absolutely continuous spectrum, the singular continuous spectrum, and the point spectrum.

Define the unitary operator U on $\mathcal{H}$ by $[U\psi](n) = \psi(n+1)$. Then we have

$$H_{T\omega} = UH_\omega U^*. \tag{3.3}$$

This relation extends readily to functions of the operators.

We say that a family $\{A_\omega\}_{\omega\in\Omega}$ is weakly measurable if $\omega \mapsto \langle\phi, A_\omega\psi\rangle$ is measurable for all $\phi, \psi \in \mathcal{H}$. Clearly, the family $\{H_\omega\}_{\omega\in\Omega}$ as defined above is weakly measurable. Denote by $\mathcal{P}_\omega(I)$ the spectral projection onto the interval $I \subseteq \mathbb{R}$ associated with H_ω, that is, $\mathcal{P}_\omega(I) = \chi_I(H_\omega)$ and particularly

$$\langle\delta_0, \mathcal{P}_\omega(I)\delta_0\rangle + \langle\delta_1, \mathcal{P}_\omega(I)\delta_1\rangle = \int_{\sigma(H_\omega)} \chi_I(E)\, d\mu_\omega(E).$$

Since all spectral measures of H_ω are absolutely continuous with respect to the measure μ_ω (the μ associated with H_ω) and $\sigma(H_\omega)$ is the topological support of μ_ω, it therefore follows that $E \in \sigma(H_\omega)$ if and only if $\mathcal{P}_\omega(E-\varepsilon, E+\varepsilon) \neq 0$ for every $\varepsilon > 0$. By restriction to the supports of the ac, sc, pp parts of μ_ω identified above, we also obtain the families $\mathcal{P}_\omega^{(\mathrm{ac})}(I)$, $\mathcal{P}_\omega^{(\mathrm{sc})}(I)$, $\mathcal{P}_\omega^{(\mathrm{pp})}(I)$. For example, $\mathcal{P}_\omega^{(\mathrm{ac})}(I) = \chi_{I\cap S_{\mathrm{ac}}}(H_\omega)$.

We mention without proof that weak measurability is inherited by the spectral projections (resp., the ac, sc, and pp parts).

Lemma 3.1. *Suppose that $\{P_\omega\}_{\omega\in\Omega}$ is a weakly measurable family of orthogonal projections satisfying*

$$P_{T\omega} = UP_\omega U^*. \tag{3.4}$$

Then, either $\dim \operatorname{Ran}(P_\omega) = 0$ *μ-almost surely, or* $\dim \operatorname{Ran}(P_\omega) = \infty$ *μ-almost surely.*

PROOF. Since P_ω is an orthogonal projection, its trace $\mathrm{tr} P_\omega \in [0, \infty]$ is uniquely defined and equal to $\dim \operatorname{Ran}(P_\omega)$. Moreover, it follows from (3.4) that

$$\mathrm{tr} P_{T\omega} = \sum_{n\in\mathbb{Z}} \langle \delta_n, P_{T\omega} \delta_n \rangle = \sum_{n\in\mathbb{Z}} \langle \delta_n, U P_\omega U^* \delta_n \rangle = \sum_{n\in\mathbb{Z}} \langle \delta_{n+1}, P_\omega \delta_{n+1} \rangle = \mathrm{tr} P_\omega.$$

Thus, $\mathrm{tr} P_\omega$ is μ-almost surely constant and hence $\mathrm{tr} P_\omega = \mathbb{E}(\mathrm{tr} P_\omega)$ for μ-almost every ω. By positivity and shift-invariance,

$$\mathbb{E}(\mathrm{tr} P_\omega) = \sum_{n\in\mathbb{Z}} \mathbb{E}(\langle \delta_n, P_\omega \delta_n \rangle) = \sum_{n\in\mathbb{Z}} \mathbb{E}(\langle \delta_0, P_{T^n\omega} \delta_0 \rangle) = \sum_{n\in\mathbb{Z}} \mathbb{E}(\langle \delta_0, P_\omega \delta_0 \rangle),$$

which is either 0 or ∞. □

THEOREM 3.2 (Pastur 1980). *Given an ergodic family $\{H_\omega\}_{\omega\in\Omega}$, there exists a set $\Sigma \subseteq \mathbb{R}$ such that for μ-almost every ω, $\sigma(H_\omega) = \Sigma$ and $\sigma_{\mathrm{disc}}(H_\omega) = \emptyset$. Moreover, for every E,*

$$\mu\left(\{\omega : E \in \sigma_{\mathrm{p}}(H_\omega)\}\right) = 0.$$

PROOF. For rational numbers $p < q$, we let $d(p, q) = 0$ if $\dim \operatorname{Ran}(\mathcal{P}_\omega((p, q))) = 0$ μ-almost surely and $d(p, q) = \infty$ if $\dim \operatorname{Ran}(\mathcal{P}_\omega((p, q))) = \infty$ μ-almost surely. By Lemma 3.1, $d(p, q)$ is well-defined.

Let

$$\Omega_{p,q} = \{\omega : \dim \operatorname{Ran}(\mathcal{P}_\omega((p, q))) = d(p, q)\}$$

and

$$\Omega_0 = \bigcap_{p,q\in\mathbb{Q},\, p<q} \Omega_{p,q}.$$

Since Ω_0 is given by a countable intersection of sets of full measure, we have $\mu(\Omega_0) = 1$.

Consider $\omega_1, \omega_2 \in \Omega_0$ and suppose $E \notin \sigma(H_{\omega_1})$. Then there are rational numbers E_1, E_2 such that $E \in (E_1, E_2) \subseteq \mathbb{R} \setminus \sigma(H_{\omega_1})$. Consequently,

$$0 = \dim \operatorname{Ran}(\mathcal{P}_{\omega_1}((E_1, E_2))) = \dim \operatorname{Ran}(\mathcal{P}_{\omega_2}((E_1, E_2))),$$

and therefore $E \notin \sigma(H_{\omega_2})$. It follows that $\sigma(H_{\omega_2}) \subseteq \sigma(H_{\omega_1})$ and hence equality by symmetry.

Now assume that there exists $E \in \sigma_{\mathrm{disc}}(H_\omega)$ for some $\omega \in \Omega_0$. Then there are rational numbers E_1, E_2 with $E \in (E_1, E_2)$ and

$$0 < \dim \operatorname{Ran}(\mathcal{P}_\omega((E_1, E_2))) < \infty.$$

This contradicts $\omega \in \Omega_0$.

Finally, again by Lemma 3.1, $\dim \operatorname{Ran}(\mathcal{P}_\omega(\{E\})) = 0$ μ-almost surely or $\dim \operatorname{Ran}(\mathcal{P}_\omega(\{E\})) = \infty$ μ-almost surely. Since the solution space is two-dimensional, it follows that $\dim \operatorname{Ran}(\mathcal{P}_\omega(\{E\})) \leq 2$ for every ω and hence $\dim \operatorname{Ran}(\mathcal{P}_\omega(\{E\})) = 0$ μ-almost surely. □

The same ideas, applied to $\mathcal{P}_\omega^\bullet(I) = \mathcal{P}_\omega(I)\mathcal{P}_\omega^\bullet$, $\bullet \in \{\text{ac, sc, pp}\}$, prove the following result:

THEOREM 3.3 (Kunz-Souillard 1980). *Given an ergodic family $\{H_\omega\}_{\omega\in\Omega}$, there exist sets $\Sigma_{\mathrm{ac}}, \Sigma_{\mathrm{ac}}, \Sigma_{\mathrm{pp}} \subseteq \mathbb{R}$ such that for μ-almost every ω, $\sigma_\bullet(H_\omega) = \Sigma_\bullet$, $\bullet \in \{ac,sc,pp\}$.*

2. The Integrated Density of States

For $\omega \in \Omega$ and $N \geq 1$, define the measures $dk_{\omega,N}$ and dk by

$$\int f(E)\, dk_{\omega,N}(E) = \frac{1}{N} \operatorname{tr}(f(H_\omega)\chi_{[0,N-1]}).$$

and

$$\int f(E)\, dk(E) = \mathbb{E}\left(\langle \delta_0, f(H_\omega)\delta_0 \rangle\right) \tag{3.5}$$

for bounded measurable f. The function k defined by

$$k(E) = \int \chi_{(-\infty,E]}(E')\, dk(E')$$

is called the *integrated density of states.*

THEOREM 3.4 (Avron-Simon 1983). *The almost sure spectrum is given by the points of increase of k, that is,* $\operatorname{supp}(dk) = \Sigma$.

PROOF. We begin with the inclusion $\operatorname{supp}(dk) \subseteq \Sigma$: If $E_0 \notin \Sigma$, there is a non-negative continuous function f with $f(E_0) = 1$ and $f \equiv 0$ on Σ (since Σ is closed). For μ-almost every ω, we have $\Sigma = \sigma(H_\omega)$ and hence $f(H_\omega) = 0$. It follows that

$$\int f(E)\, dk(E) = \mathbb{E}\left(\langle \delta_0, f(H_\omega)\delta_0 \rangle\right) = 0.$$

Thus, $E_0 \notin \operatorname{supp}(dk)$.

Let us now prove $\operatorname{supp}(dk) \supseteq \Sigma$: If $E_0 \notin \operatorname{supp}(dk)$, there is a non-negative continuous function f with $f(E_0) = 1$ and $\int f(E)\, dk(E) = 0$. But

$$\int f(E)\, dk(E) = \mathbb{E}\left(\langle \delta_0, f(H_\omega)\delta_0 \rangle\right) = \mathbb{E}\left(\langle \delta_0, f(H_{T^n\omega})\delta_0 \rangle\right) = \mathbb{E}\left(\langle \delta_n, f(H_\omega)\delta_n \rangle\right),$$

where the second step follows by T-invariance of $d\mu$. Thus, for μ-almost every ω and every $n \in \mathbb{Z}$, $\langle \delta_n, f(H_\omega)\delta_n \rangle = 0$ (since the integrand is non-negative). The operator inequality $f(H_\omega) \geq 0$ therefore shows that $f(H_\omega) = 0$ for μ-almost every ω. Consequently, $E_0 \notin \Sigma$. □

LEMMA 3.5. *For every bounded measurable f, there exists a set $\Omega_f \subseteq \Omega$ of full μ-measure such that for every $\omega \in \Omega_f$,*

$$\int f(E)\, dk_{\omega,N}(E) \to \int f(E)\, dk(E) \tag{3.6}$$

as $N \to \infty$.

PROOF. Fix a bounded measurable function f and define

$$\overline{f}(\omega) = \langle \delta_0, f(H_\omega)\delta_0 \rangle.$$

Clearly, $\overline{f} \in L^\infty(\Omega, d\mu) \subseteq L^1(\Omega, d\mu)$. Therefore,

$$\begin{aligned}\int f(E)\, dk_{\omega,N}(E) &= \frac{1}{N} \operatorname{tr}(f(H_\omega)\chi_{[0,N-1]}) \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \langle \delta_n, f(H_\omega)\delta_n \rangle\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{N} \sum_{n=0}^{N-1} \langle \delta_0, f(H_{T^n\omega})\delta_0 \rangle \\
&= \frac{1}{N} \sum_{n=0}^{N-1} \overline{f}(T^n\omega) \\
&\to \mathbb{E}(\overline{f}) \\
&= \int f(E)\, dk(E).
\end{aligned}
$$

In the second to last step, we applied Theorem 2.1. □

Theorem 3.6 (Delyon-Souillard 1984). *The integrated density of states is continuous.*

Proof. Assume that k is discontinuous at E_0. Thus, if f_n are continuous functions with $f_n(E_0) = 1$ and $f_n(E) \downarrow 0$ for $E \neq E_0$, then, by monotone convergence,

$$\int f_n(E)\, dk(E) \to \delta > 0. \tag{3.7}$$

However, $\langle \delta_0, f_n(H_\omega)\delta_0 \rangle \downarrow \langle \delta_0, \mathcal{P}_\omega(\{E_0\})\delta_0 \rangle$, and hence, again by monotone convergence,

$$\int f_n(E)\, dk(E) = \mathbb{E}\left(\langle \delta_0, f_n(H_\omega)\delta_0 \rangle\right) \to \mathbb{E}\left(\langle \delta_0, \mathcal{P}_\omega(\{E_0\})\delta_0 \rangle\right).$$

By Lemma 3.5, we have for ω's from a set of full μ-measure,

$$\mathbb{E}\left(\langle \delta_0, \mathcal{P}_\omega(\{E_0\})\delta_0 \rangle\right) = \lim_{N\to\infty} \frac{1}{N} \operatorname{tr}(\mathcal{P}_\omega(\{E_0\})\chi_{[0,N-1]}).$$

Since these traces are bounded by 2, the limit is zero and we obtain a contradiction to (3.7). □

This raises the natural question of what can be said about the modulus of continuity of k. Later we will show with the help of the Thouless formula, which establishes a connection between dk and the Lyapunov exponent, that k is actually log-Hölder continuous. In this generality, the result is essentially optimal. For specific examples, however, one may hope for even stronger continuity properties. The rule of thumb is that k becomes smoother with increasing randomness of the potentials.

A different approach to the integrated density of states is obtained if we restrict the operator to a finite box, rather than its spectral projection. Denote the restriction of H_ω to $[1, N]$ with Dirichlet boundary conditions by $H_\omega^{(N)}$. For $\omega \in \Omega$ and $N \geq 1$, define measures $d\tilde{k}_{\omega,N}$ by placing uniformly distributed point masses at the eigenvalues $E_\omega^{(N)}(1) < \cdots < E_\omega^{(N)}(N)$ of $H_\omega^{(N)}$, that is,

$$\int f(E)\, d\tilde{k}_{\omega,N}(E) = \frac{1}{N} \sum_{n=1}^{N} f(E_\omega^{(N)}(n)).$$

Lemma 3.7. *For μ-almost every $\omega \in \Omega$, the measures $d\tilde{k}_{\omega,N}$ converge weakly to dk as $N \to \infty$.*

PROOF. By boundedness of its support, the measure dk is determined uniquely by its moments. Thus, it suffices to prove, for μ-almost every $\omega \in \Omega$, that the moments $d\tilde{k}_{\omega,N}$ converge to the moments of dk as $N \to \infty$ (see, e.g., [**9**, Theorem 9.2]).

Now,

$$\begin{aligned}
\int E^p \, d\tilde{k}_{\omega,N}(E) &= \frac{1}{N} \sum_{n=1}^{N} (E_\omega^{(N)}(n))^p \\
&= \frac{1}{N} \operatorname{tr}((H_\omega^{(N)})^p) \\
&= \frac{1}{N} \sum_{n=1}^{N} \langle \delta_n, (H_\omega^{(N)})^p \, \delta_n \rangle \\
&= \frac{1}{N} \left[\sum_{n=1}^{N} \langle \delta_n, (H_\omega)^p \, \delta_n \rangle + O(1) \right] \\
&= \frac{1}{N} \left[\sum_{n=1}^{N} \langle \delta_0, (H_{T^n\omega})^p \, \delta_0 \rangle + O(1) \right]
\end{aligned}$$

By the Birkhoff ergodic theorem, it follows that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \langle \delta_0, (H_{T^n\omega})^p \, \delta_0 \rangle = \mathbb{E} \left(\langle \delta_0, (H_\omega)^p \delta_0 \rangle \right)$$

for μ-almost every $\omega \in \Omega$. Thus, for these ω's,

$$\lim_{N \to \infty} \int E^p \, d\tilde{k}_{\omega,N}(E) = \mathbb{E} \left(\delta_0, (H_\omega)^p \, \delta_0 \rangle \right) = \int E^p \, dk(E),$$

as desired. □

3. The Lyapunov Exponent

As we saw earlier, the spectral properties of a Schrödinger operator can be studied by looking at the asymptotic properties of the solutions to the associated family of difference equations (1.18). Moreover, we saw that such a difference equation can be written in vector-matrix form as in (1.19) and that the transfer matrices appearing there are given by a product of one-step transfer matrices as in (1.20).

Suppose now that we are given an ergodic family $\{H_\omega\}_{\omega \in \Omega}$ of Schrödinger operators in $\ell^2(\mathbb{Z})$. Naturally, we want to study the solutions of

$$u(n+1) + u(n-1) + V_\omega(n) u(n) = z u(n), \tag{3.8}$$

which, for $n \geq 1$, may be expressed via

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = M_z(n, \omega) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}, \tag{3.9}$$

where

$$M_z(n, \omega) = T_z(n, \omega) \times \cdots \times T_z(1, \omega), \quad T_z(m, \omega) = \begin{pmatrix} z - V_\omega(m) & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.10}$$

Similar formulas hold for $n \leq -1$.

Since V_ω is obtained by sampling along the orbit of ω under T, it is clear that the same is true for the matrices $T_z(m, \omega)$. As a consequence, the transfer matrices may be obtained by iteration of a suitable skew-product over T. Concretely, consider the map

$$(T, M_z) : \Omega \times \mathbb{C}^2 \to \Omega \times \mathbb{C}^2, \quad (\omega, v) \mapsto (T\omega, M_z(\omega)v),$$

where

$$M_z(\omega) = \begin{pmatrix} z - f(\omega) & -1 \\ 1 & 0 \end{pmatrix}.$$

Then it is easy to check that

$$(T, M_z)^n(\omega, v) = (T^n\omega, M_z(n, T^{-1}\omega)v)$$

and that the so-called cocycle condition holds,

$$M_z(m+n, \omega) = M_z(m, T^n\omega)M_z(n, \omega). \tag{3.11}$$

Since norms are submultiplicative, this shows that

$$f_n(\omega, z) = \log \|M_z(n, \omega)\|$$

satisfies the subadditivity condition

$$f_{n+m}(\omega, z) \le f_n(\omega, z) + f_m(T^n\omega, z)$$

that enables us to apply Kingman's Subadditive Ergodic Theorem. We find that there is a number $\gamma(z) \in [0, \infty)$, called the *Lyapunov exponent*, so that

$$\gamma(z) = \inf_{n \ge 1} \frac{1}{n} \mathbb{E}(\log \|M_z(n, \omega)\|) = \lim_{n \to \infty} \frac{1}{n} \log \|M_z(n, \omega)\| \text{ for } \mu\text{−almost every } \omega \in \Omega.$$

If $\gamma(z) > 0$, this shows that for μ-almost every ω, the norm of $M_E(n, \omega)$ grows exponentially in n. The next result, which is entirely deterministic, shows that the solutions must then have exponential behavior as well.

THEOREM 3.8 (Ruelle 1979). *Suppose* $A_n \in \mathrm{SL}(2, \mathbb{C})$ *obey*

$$\lim_{n \to \infty} \frac{1}{n} \|A_n\| = 0$$

and

$$\lim_{n \to \infty} \frac{1}{n} \log \|A_n \cdots A_1\| = \gamma > 0.$$

Then there exists a one-dimensional subspace $V \subset \mathbb{R}^2$ *such that*

$$\lim_{n \to \infty} \frac{1}{n} \log \|A_n \cdots A_1 v\| = -\gamma \quad \textit{for } v \in V \setminus \{0\}$$

and

$$\lim_{n \to \infty} \frac{1}{n} \log \|A_n \cdots A_1 v\| = \gamma \quad \textit{for } v \notin V.$$

PROOF. Write $T_n = A_n \cdots A_1$, $t_n = \|T_n\|$, $a_n = \|A_n\|$, and

$$u_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

Since $|T_n|$ is self-adjoint and unimodular, it has eigenvalues t_n and t_n^{-1}. Define θ_n by $|T_n| u_{\theta_n} = t_n^{-1} u_{\theta_n}$. Then, by self-adjointness again, $|T_n| u_{\theta_n + \frac{\pi}{2}} = t_n u_{\theta_n + \frac{\pi}{2}}$. It follows from the trigonometric formulae $\sin(x+y) = \sin x \cos y + \cos x \sin y$ and $\cos(x+y) = \cos x \cos y - \sin x \sin y$ that

$$u_\theta = \cos(\theta - \theta_n) u_{\theta_n} + \sin(\theta - \theta_n) u_{\theta_n + \frac{\pi}{2}}.$$

Thus, using $\|T_n u_\theta\| = \||T_n| u_\theta\|$,

$$\|T_n u_\theta\|^2 = t_n^2 \sin^2(\theta - \theta_n) + t_n^{-2} \cos^2(\theta - \theta_n). \tag{3.12}$$

By (3.12) (with n replaced by $n+1$),

$$\begin{aligned} t_{n+1}^2 \sin^2(\theta_n - \theta_{n+1}) &\le \|T_{n+1} u_{\theta_n}\|^2 \\ &\le a_{n+1}^2 \|T_n u_{\theta_n}\|^2 \\ &= a_{n+1}^2 t_n^{-2}. \end{aligned}$$

Since A_{n+1} is unimodular,

$$t_n = \|T_n\| \le \|T_{n+1}\| \, \|A_{n+1}^{-1}\| = \|T_{n+1}\| \, \|A_{n+1}\| = t_{n+1} a_{n+1},$$

and hence

$$t_n^2 \sin^2(\theta_n - \theta_{n+1}) \le a_{n+1}^4 t_n^{-2}.$$

Since $\sin^2(x) \gtrsim x^2$ for the values of x in question, we obtain

$$|\theta_n - \theta_{n+1}| \lesssim \frac{a_{n+1}^2}{t_n^2}. \tag{3.13}$$

It follows from our assumptions that $\sum |\theta_n - \theta_{n+1}| < \infty$ and hence $\theta_n \to \theta_\infty$, which obeys

$$|\theta_n - \theta_\infty| \lesssim \sum_{m=n}^{\infty} \frac{a_{m+1}^2}{t_m^2}. \tag{3.14}$$

Let $u_\infty = u_{\theta_\infty}$ and $v_\infty = u_{\theta_\infty + \frac{\pi}{2}}$. We claim that the assertion of the theorem holds with V given by the span of u_∞. Since $\|T_n v_\infty\| \le t_n$ and $\|T_n u_\infty\| \ge t_n^{-1}$, it suffices to show

$$\|T_n v_\infty\|^2 \ge \tfrac{1}{2} t_n^2 \quad \text{for } n \text{ large enough} \tag{3.15}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log \|T_n u_\infty\| \le -\gamma. \tag{3.16}$$

Since $\theta_n - \theta_\infty \to 0$, (3.15) follows from (3.12). For every $\varepsilon > 0$ and n large, we have by (3.12), (3.13), and (3.14),

$$\begin{aligned} \|T_n u_\infty\|^2 &= t_n^2 \sin^2(\theta_\infty - \theta_n) + t_n^{-2} \cos^2(\theta_\infty - \theta_n) \\ &\le t_n^2 |\theta_\infty - \theta_n|^2 + e^{-2(1-\varepsilon)\gamma n} \\ &\lesssim t_n^2 \left(\sum_{m=n}^{\infty} \frac{a_{m+1}^2}{t_m^2} \right)^2 + e^{-2(1-\varepsilon)\gamma n} \\ &\le e^{2(1+\varepsilon)\gamma n} \left(\sum_{m=n}^{\infty} \frac{e^{2\varepsilon\gamma m}}{e^{2(1-\varepsilon)\gamma m}} \right)^2 + e^{-2(1-\varepsilon)\gamma n} \\ &= e^{2(1+\varepsilon)\gamma n} \left(\sum_{m=n}^{\infty} e^{(4\varepsilon - 2)\gamma m} \right)^2 + e^{-2(1-\varepsilon)\gamma n} \end{aligned}$$

and hence $\limsup_{n\to\infty} \frac{1}{n} \log \|T_n u_\infty\| \le (-1+5\varepsilon)\gamma$. Since this holds for every $\varepsilon > 0$, we get (3.16). $\square$

Remark. Let us make some comments on "the other half-line." The reader may have noticed that the Lyapunov exponent was introduced as a quantity that measures growth at ∞. What about growth at $-\infty$? Clearly, we can mimic all the steps above and introduce a "left half-line Lyapunov exponent" $\gamma^-(z)$. This, however, will lead to the same quantity as the following argument shows. We have

$$\gamma^-(E) = \inf_{n \le -1} \frac{\mathbb{E}(\log \|M_z(n,\omega)\|)}{|n|}.$$

Here the matrix $M_z(n,\omega)$ obeys, for $n \le -1$,

$$\begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} = M_z(n,\omega) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix}$$

for all solutions to (3.8). It is easy to check that

$$M_z(n,\omega) = M_z(T^n\omega)^{-1} \cdots M_z(T^{-1}\omega)^{-1} = \left[M_z(T^{-1}\omega) \cdots M_z(T^n\omega)\right]^{-1}.$$

Thus,

$$\begin{aligned}
\mathbb{E}(\log \|M_z(n,\omega)\|) &= \mathbb{E}\left(\log \| \left[M_z(T^{-1}\omega) \cdots M_z(T^n\omega)\right]^{-1} \|\right) \\
&= \mathbb{E}\left(\log \| \left[M_z(T^{-n-1}\omega) \cdots M_z(\omega)\right]^{-1} \|\right) \\
&= \mathbb{E}\left(\log \|M_z(T^{-n-1}\omega) \cdots M_z(\omega)\|\right).
\end{aligned}$$

In the second step we used that T is measure-preserving and in the third step we used that $\|M^{-1}\| = \|M\|$ for every $M \in \mathrm{SL}(2,\mathbb{R})$. The latter fact follows from

$$M^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = (JMJ^{-1})^T,$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and the fact that J is unitary.

Therefore,

$$\gamma^-(z) = \inf_{n \le -1} \frac{\mathbb{E}(\log \|M_z(n,\omega)\|)}{|n|} = \inf_{n \ge 1} \frac{\mathbb{E}(\log \|M_z(n,\omega)\|)}{n} = \gamma(z).$$

4. Subharmonicity of the Lyapunov Exponent

An important property of the Lyapunov exponent is subharmonicity. A function $F : \mathbb{C} \to \mathbb{R} \cup \{-\infty\}$ is called *submean* if

$$F(z) \le \frac{1}{2\pi} \int_0^{2\pi} F(z + re^{i\theta})\, d\theta \tag{3.17}$$

for all $z \in \mathbb{C}$ and $r > 0$. F is called *upper-semicontinuous* if

$$z_n \to z \quad \Rightarrow \quad \limsup_{n\to\infty} F(z_n) \le F(z) \tag{3.18}$$

or, equivalently, if $\{z : F(z) < \delta\}$ is open for every $\delta \in \mathbb{R}$. F is called *subharmonic* if it is both submean and upper-semicontinuous.

Let us collect some properties of subharmonic functions that we will use. A useful reference is [**30**].

PROPOSITION 3.9. (a) *If the matrices $M(z)$ depend on the complex variable z analytically, then*

$$F(z) = \log \|M(z)\|_\infty$$

is subharmonic. Here, the matrix norm is given by $\|M\|_\infty = \max |M_{i,j}|$.
(b) *Let $(X, d\nu)$ be a measure space with $\nu(X) < \infty$. Let $G : \mathbb{C} \times X \to \mathbb{R} \cup \{-\infty\}$ be such that*

- *G is measurable on $\mathbb{C} \times X$,*
- *$z \mapsto G(z,x)$ is subharmonic for each $x \in X$,*
- *$z \mapsto \sup_{x\in X} G(z,x)$ is locally bounded above.*

Then the function

$$F(z) = \int G(z,x)\, d\nu(x)$$

is subharmonic.
(c) *If $F_n \geq 0$ are subharmonic and form a decreasing sequence, then*

$$F(z) = \inf_{n\geq 1} F_n(z)$$

defines a subharmonic function.
(d) *If F is subharmonic, then*

$$F(z_0) = \lim_{r\to 0} \frac{1}{\pi r^2} \int_{|z-z_0|\leq r} F(z)\, dz.$$

PROOF. (a) Each of $\log(|M_{i,j}(z)|)$ is subharmonic. (The function is even harmonic away from the zeros of $M_{i,j}(z)$ as the real part of an analytic function. To deal with zeros one can use Blaschke products; this is fairly standard and can be found in most Complex Analysis texts.) Thus the desired statement follows from

$$F(z) = \log(\max |M_{i,j}(z)|) = \max \log(|M_{i,j}(z)|).$$

(b) It suffices to prove that u is subharmonic on every bounded open subset U of $\mathbb{C}$. Fix such a set U. It then follows from the third assumption that $\sup_{x\in X} G(z,x)$ is bounded above on U and hence, subtracting a constant if necessary, we may assume that the upper bound is zero. This will allow the use of Fatou and Fubini below.

Suppose $z_n \to z \in U$. Then, using Fatou's Lemma in the second step and upper-semicontinuity of $G(\cdot,x)$ in the third step,

$$\begin{aligned}
\limsup_{n\to\infty} F(z_n) &= \limsup_{n\to\infty} \int_U G(z_n,x)\, d\nu(x) \\
&\leq \int_U \limsup_{n\to\infty} G(z_n,x)\, d\nu(x) \\
&\leq \int_U G(z,x)\, d\nu(x) \\
&= F(z).
\end{aligned}$$

This shows that F is upper-semicontinuous in U.

If $\overline{B(z_0,r)} \subset U$, then using the submean property of $G(\cdot,x)$ in the second step and Fubini in the third step,

$$F(z_0) = \int G(z_0,x)\, d\nu(x)$$

$$\begin{aligned}
&\leq \int \left(\frac{1}{2\pi} \int_0^{2\pi} G(z_0 + re^{i\theta}, x)\, d\theta \right) d\nu(x) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int G(z_0 + re^{i\theta}, x)\, d\nu(x)\, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} F(z_0 + re^{i\theta})\, d\theta
\end{aligned}$$

and hence F is submean in U.

(c) For every $\delta \in \mathbb{R}$, the set $\{z : F(z) < \delta\}$ is the union of the open sets $\{z : F_n(z) < \delta\}$ and hence open. Thus, F is upper-semicontinuous.

Moreover, for every $n \geq 1$,

$$F_n(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} F_n(z_0 + re^{i\theta})\, d\theta$$

by assumption and hence, letting $n \to \infty$ and using the monotone convergence theorem, we find that F is submean.

(d) Since F is submean, we have

$$F(z_0) \leq \liminf_{r \to 0} \frac{1}{\pi r^2} \int_{|z - z_0| \leq r} F(z)\, dz.$$

On the other hand, since F is upper-semicontinuous, it follows that

$$F(z_0) \geq \limsup_{r \to 0} \frac{1}{\pi r^2} \int_{|z - z_0| \leq r} F(z)\, dz.$$

Combining the two inequalities, we obtain the result. □

THEOREM 3.10 (Craig-Simon 1983). *The Lyapunov exponent is subharmonic.*

PROOF. This is an immediate consequence of the properties listed above. Part (a) shows that $z \mapsto \log \|M_z(n,\omega)\|_\infty$ is subharmonic for every n, ω. Thus, by part (b), $z \mapsto \mathbb{E}(\log \|M_z(n,\omega)\|_\infty)$ is subharmonic for every n. Moreover, by subadditivity,

$$F_n(z) = \frac{1}{2^n} \mathbb{E}(\log \|M_z(2^n,\omega)\|_\infty)$$

is decreasing and hence, by part (c),

$$\tilde{\gamma}(z) = \inf_{n \geq 1} \frac{1}{2^n} \mathbb{E}(\log \|M_z(2^n,\omega)\|_\infty)$$

is subharmonic. Since all matrix norms are equivalent, we see that $\gamma \equiv \tilde{\gamma}$ and hence the Lyapunov exponent is subharmonic. □

5. The Thouless Formula and Log-Hölder Continuity of the IDS

The next theorem establishes the *Thouless formula.* It was derived on a non-rigorous basis by Thouless and then rigorously proved by Avron and Simon. The proof given below is due to Craig and Simon. It exploits subharmonicity and is simpler than the Avron-Simon proof.

THEOREM 3.11 (Thouless formula). *For every $z \in \mathbb{C}$, we have*

$$\gamma(z) = \int \log |E - z|\, dk(E). \tag{3.19}$$

PROOF. The right-hand side of (3.19) defines a subharmonic function, $\tilde{\gamma}(z)$, by Proposition 3.9.(a) and (b). Thus, our goal is to show

$$\gamma(z) = \tilde{\gamma}(z) \tag{3.20}$$

for every $z \in \mathbb{C}$.

Let us first prove (3.20) for $z \in \mathbb{C} \setminus \mathbb{R}$. Recall from (1.28) that we can write

$$M_z(n, \omega) = \begin{pmatrix} P_{n,\omega}(z) & Q_{n-1,\omega}(z) \\ P_{n-1,\omega}(z) & Q_{n-2,\omega}(z) \end{pmatrix}$$

and that due to

$$M_z(n, \omega) = \begin{pmatrix} z - V_\omega(n) & -1 \\ 1 & 0 \end{pmatrix} M_E(n-1, \omega)$$

$P_{k,\omega}$ and $Q_{k,\omega}$ are monic polynomials of degree k. It follows from (3.9) that $P_{n,\omega}(z) = 0$ if and only if (3.8) has a solution u with $u(n+1) = u(0) = 0$. Similarly, $Q_{n,\omega}(z) = 0$ if and only if (3.8) has a solution u with $u(n+2) = u(1) = 0$. Thus,

$$P_{n,\omega}(z) = \prod_{j=1}^{n} (z - E_\omega^{(n)}(j)), \quad Q_{n,\omega}(z) = \prod_{j=1}^{n} (z - E_{T\omega}^{(n)}(j)),$$

where $E_\omega^{(n)}(1), \ldots, E_\omega^{(n)}(n)$ (resp., $E_{T\omega}^{(n)}(1), \ldots, E_{T\omega}^{(n)}(n)$) are the eigenvalues of $H_\omega^{(n)}$ (resp., $H_{T\omega}^{(n)}$); compare the previous section. Therefore,

$$\log |P_{n,\omega}(z)| = \int \log |E - z| \, d\tilde{k}_{\omega,n}(E)$$

and

$$\log |Q_{n,\omega}(z)| = \int \log |E - z| \, d\tilde{k}_{T\omega,n}(E)$$

Since $z \in \mathbb{C} \setminus \mathbb{R}$, the function $E \mapsto \log |E - z|$ is bounded and continuous. We find

$$\lim_{n \to \infty} \frac{1}{n} \log |P_{n,\omega}(z)| = \int \log |E - z| \, dk(E)$$

for almost every ω, and the same statement for P replaced by Q. Since all norms on 2×2 matrices are equivalent, this shows (3.20) for $z \in \mathbb{C} \setminus \mathbb{R}$.

Let us now consider $E \in \mathbb{R}$. Then

$$\frac{1}{\pi r^2} \int_{|z-E| \le r} \gamma(z) \, dz = \frac{1}{\pi r^2} \int_{|z-E| \le r} \tilde{\gamma}(z) \, dz$$

for every $r > 0$. Using Proposition 3.9.(d), we finally obtain

$$\gamma(E) = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{|z-E| \le r} \gamma(z) \, dz = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{|z-E| \le r} \tilde{\gamma}(z) \, dz = \tilde{\gamma}(E),$$

which is (3.20) for real E. ☐

The Thouless formula has the following application: Since $\gamma(E) \ge 0$, the integrated density of states, k, must have a certain explicit global modulus of continuity. The precise result is as follows.

THEOREM 3.12 (Craig-Simon 1983). *The integrated density of states is* log-*Hölder continuous, that is, there is some uniform constant C such that for real E_1, E_2 with $|E_1 - E_2| < 1/2$,*

$$|k(E_1) - k(E_2)| \le C \left(\log \left(|E_1 - E_2|^{-1} \right) \right)^{-1}.$$

Remark. In fact, one may choose

$$C = \log\left(|E_1| + |E_2| + \|f\|_\infty + 2\right).$$

This is uniformly bounded on the spectrum (recall that we assumed that the sampling function f is bounded).

Proof. Without loss of generality, assume $E_1 < E_2$. Then

$$\begin{aligned}
0 &\le \gamma(E_1) \\
&= \int \log|E - E_1|\, dk(E) \\
&= \int_{E\in(E_1,E_2)} \log|E - E_1|\, dk(E) + \int_{|E-E_1|\le 1, E\notin(E_1,E_2)} \log|E - E_1|\, dk(E) + \\
&\quad + \int_{|E-E_1|>1} \log|E - E_1|\, dk(E)
\end{aligned}$$

The second integral is negative and therefore

$$-\int_{E\in(E_1,E_2)} \log|E - E_1|\, dk(E) \le \int_{|E-E_1|>1} \log|E - E_1|\, dk(E).$$

Bounding both sides by taking out the integrand in L^∞, we obtain

$$-\log|E_2 - E_1| \int_{E\in(E_1,E_2)} dk(E) \le \log(|E_1| + \|f\|_\infty + 2) \int dk(E)$$

since $\mathrm{supp}(dk) = \Sigma \subseteq [-2 - \|f\|_\infty, 2 + \|f\|_\infty]$. □

6. Kotani Theory in the Measurable Setting

6.1. Lyapunov Exponent and Absolutely Continuous Spectrum. Given a set $A \subseteq \mathbb{R}$, the *essential closure* of A is defined as follows:

$$\overline{A}^{\mathrm{ess}} = \{E \in \mathbb{R} : |(E - \varepsilon, E + \varepsilon) \cap A| > 0 \text{ for every } \varepsilon > 0\}.$$

Here, $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}$. Note that, in particular, $\overline{A}^{\mathrm{ess}} = \emptyset$ if $|A| = 0$.

Let us denote

$$\mathcal{Z} = \{E \in \mathbb{R} : \gamma(E) = 0\}.$$

Theorem 3.13 (Ishii-Pastur-Kotani 1973-1980-1984).

$$\Sigma_{\mathrm{ac}} = \overline{\mathcal{Z}}^{\mathrm{ess}}.$$

The inclusion "$\subseteq$" was proved by Ishii and Pastur. The other inclusion was proved by Kotani and is a much deeper result. In fact, the Ishii-Pastur half of the result is really an immediate consequence of the general theory of one-dimensional Schrödinger operators.

Proof of the Ishii-Pastur half of Theorem 3.13. Note that when $\gamma(E) > 0$, Theorem 3.8 says that for almost every ω, there are solutions $u^+_{E,\omega}$ and $u^-_{E,\omega}$ of (3.8) such that $u^\pm_{E,\omega}$ is exponentially decaying, and hence subordinate, at $\pm\infty$. Applying Fubini's theorem, we see that for μ-almost every ω, the set of $E \in \mathbb{R} \setminus \mathcal{Z}$ for which the property just described fails, has zero Lebesgue measure. In other words, for these ω's, $\mathbb{R} \setminus \mathcal{Z} \subseteq S_\omega$ up to a set of zero Lebesgue measure. Since sets of zero Lebesgue measure have zero weight with respect to the absolutely continuous parts of any spectral measure, we obtain from Corollary 1.22 that for μ-almost every ω,

$$\mathcal{P}_\omega^{\mathrm{ac}}(\mathbb{R} \setminus \mathcal{Z}) = 0.$$

This shows that for μ-almost every ω, $\sigma_{\mathrm{ac}}(H_\omega) \subseteq \overline{\mathcal{Z}}^{\mathrm{ess}}$. $\square$

Let us now turn to the Kotani half of Theorem 3.13. Given $z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\}$ and $\omega \in \Omega$, recall that there are (up to a constant multiple) unique solutions $u^\pm_{z,\omega}$ of (3.8) such that $u^\pm_{z,\omega}$ is square-summable at $\pm\infty$. We must have $u^\pm_{z,\omega}(0) \neq 0$ and hence we can define

$$m_\pm(z, \omega) = -\frac{u^\pm_{z,\omega}(\pm 1)}{u^\pm_{z,\omega}(0)}. \tag{3.21}$$

Clearly,

$$m_\pm(z, T^n\omega) = -\frac{u^\pm_{z,\omega}(n \pm 1)}{u^\pm_{z,\omega}(n)}. \tag{3.22}$$

By Theorem 3.8, we have for μ-almost every ω,

$$\lim_{n\to\infty} \frac{1}{n} \log \left| \frac{u^\pm_{z,\omega}(n)}{u^\pm_{z,\omega}(0)} \right| = -\gamma(z).$$

By (3.22),

$$\log \left| \frac{u^\pm_{z,\omega}(n)}{u^\pm_{z,\omega}(0)} \right| = \sum_{m=0}^{n-1} \log |m_\pm(z, T^{\pm m}\omega)|$$

and hence Birkhoff's ergodic theorem implies

$$\mathbb{E}(\log |m_\pm(z, \omega)|) = -\gamma(z). \tag{3.23}$$

PROPOSITION 3.14. *We have that*

$$\mathbb{E}\left(\log\left(1 + \frac{\Im z}{\Im m_\pm(z,\omega)}\right)\right) = 2\gamma(z).$$

PROOF. By the difference equation (3.8) that $u^\pm_{z,\omega}$ obeys,

$$m_\pm(z, T^n\omega) = V_\omega(n) - z - [m_\pm(z, T^{n\mp 1}\omega)]^{-1}. \tag{3.24}$$

Taking imaginary parts,

$$\Im m_\pm(z,\omega) = -\Im z - \Im\left([m_\pm(z, T^{\mp 1}\omega)]^{-1}\right).$$

Dividing by $\Im m_+(z,\omega)$,

$$1 = -\frac{\Im z}{\Im m_\pm(z,\omega)} - \frac{\Im\left([m_\pm(z, T^{\mp 1}\omega)]^{-1}\right)}{\Im m_\pm(z,\omega)}.$$

Taking the logarithm,

$$\begin{aligned}
\log\left(1 + \frac{\Im z}{\Im m_\pm(z,\omega)}\right) &= \log\left(-\Im\left([m_\pm(z, T^{\mp 1}\omega)]^{-1}\right)\right) - \log\left(\Im m_\pm(z,\omega)\right) \\
&= \log\left(\frac{\Im m_\pm(z, T^{\mp 1}\omega)}{|m_\pm(z, T^{\mp 1}\omega)|^2}\right) - \log\left(\Im m_\pm(z,\omega)\right)
\end{aligned}$$

Taking expectations and using invariance,

$$\mathbb{E}\left(\log\left(1 + \frac{\Im z}{\Im m_\pm(z,\omega)}\right)\right) = \mathbb{E}\left(\log\left(\frac{\Im m_\pm(z, T^{\mp 1}\omega)}{|m_\pm(z, T^{\mp 1}\omega)|^2}\right) - \log\left(\Im m_\pm(z,\omega)\right)\right)$$

$$\begin{aligned} &= -2\,\mathbb{E}\left(\log|m_\pm(z,\omega)|\right) \\ &= 2\gamma(z), \end{aligned}$$

where we used (3.23) in the last step. □

Denote

$$b(z,\omega) = m_+(z,\omega) + m_-(z,\omega) + z - V_\omega(0).$$

PROPOSITION 3.15. *We have*

$$\mathbb{E}\left(\Im\left(\frac{1}{b(z,\omega)}\right)\right) = -\frac{\partial\gamma(z)}{\partial(\Im z)}.$$

PROOF. It follows from (3.24) that

$$\frac{u^-_{z,\omega}(1)}{u^-_{z,\omega}(0)} = m_-(z,\omega) + z - V_\omega(0). \tag{3.25}$$

We have for $n \le m$,

$$G_\omega(n,m;z) := \langle \delta_n, (H_\omega - z)^{-1}\delta_m\rangle = \frac{u_-(n,\omega)u_+(m,\omega)}{u_+(1,\omega)u_-(0,\omega) - u_-(1,\omega)u_+(0,\omega)}. \tag{3.26}$$

From (3.21), (3.25), (3.26), we get

$$-G_\omega(0,0;z)^{-1} = m_+(z,\omega) + m_-(z,\omega) + z - V_\omega(0) = b(z,\omega). \tag{3.27}$$

The definition of $G_\omega(n,m;z)$ gives

$$\mathbb{E}(G_\omega(0,0;z)) = \int \frac{1}{E'-z}\,dk(E'). \tag{3.28}$$

Thus,

$$\begin{aligned} \mathbb{E}\left(\Im\left(\frac{1}{b(z,\omega)}\right)\right) &= -\Im\mathbb{E}\left(G_\omega(0,0;z)\right) \\ &= -\Im\int \frac{1}{E'-z}\,dk(E') \\ &= -\frac{\partial}{\partial(\Im z)}\int \log|z-E'|\,dk(E') \\ &= -\frac{\partial\gamma(z)}{\partial(\Im z)}, \end{aligned}$$

where we used (3.27), (3.28), and the Thouless formula. □

Denote

$$n_\pm(z,\omega) = \Im m_\pm(z,\omega) + \tfrac{1}{2}\Im z.$$

PROPOSITION 3.16. *We have that*

$$\mathbb{E}\left(\frac{1}{n_\pm(z,\omega)}\right) \le \frac{2\gamma(z)}{\Im z} \tag{3.29}$$

and

$$\mathbb{E}\left(\frac{\left[\frac{1}{n_+}+\frac{1}{n_-}\right]\cdot\left[(n_+-n_-)^2 + (\Re b)^2\right]}{|b|^2}\right) \le 4\left[\frac{\gamma(z)}{\Im z} - \frac{\partial\gamma(z)}{\partial(\Im z)}\right]. \tag{3.30}$$

PROOF. For $x \geq 0$, consider the function

$$A(x) = \log(1+x) - \frac{x}{1+\frac{x}{2}}.$$

Clearly,

$$A(0) = 0 \quad \text{and} \quad A'(x) = \frac{1}{1+x} - \frac{1}{1+x+\frac{x^2}{4}} \geq 0.$$

Therefore,

$$\log(1+x) \geq \frac{x}{1+\frac{x}{2}} \qquad \text{for all } x \geq 0. \tag{3.31}$$

Thus,

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{n_\pm(z,\omega)}\right) &= \mathbb{E}\left(\frac{1}{\Im m_\pm(z,\omega) + \frac{1}{2}\Im z}\right) \\
&= \frac{1}{\Im z}\,\mathbb{E}\left(\frac{\frac{\Im z}{\Im m_\pm(z,\omega)}}{1+\frac{\frac{\Im z}{\Im m_\pm(z,\omega)}}{2}}\right) \\
&\leq \frac{1}{\Im z}\,\mathbb{E}\left(\log\left(1+\frac{\Im z}{\Im m_\pm(z,\omega)}\right)\right) \\
&= \frac{2\gamma(z)}{\Im z},
\end{aligned}$$

which is (3.29). We used (3.31) in the third step and Proposition 3.14 in the last step.

Notice that $n_+(z,\omega)+n_-(z,\omega) = \Im b(z,\omega)$. Thus the integrand on the left-hand side of (3.30) is equal to

$$\begin{aligned}
\frac{\left[\frac{1}{n_+}+\frac{1}{n_-}\right]\cdot\left[(n_+ + n_-)^2 - 4n_+n_- + (\Re b)^2\right]}{|b|^2} &= \frac{\left[\frac{1}{n_+}+\frac{1}{n_-}\right]\cdot\left[|b|^2 - 4n_+n_-\right]}{|b|^2} \\
&= \frac{1}{n_+}+\frac{1}{n_-} - 4\,\frac{n_+ + n_-}{|b|^2} \\
&= \frac{1}{n_+}+\frac{1}{n_-} + 4\,\Im\left(\frac{1}{b}\right).
\end{aligned}$$

The bound (3.30) now follows from (3.29) and Proposition 3.15. $\square$

PROOF OF THE KOTANI HALF OF THEOREM 3.13. The Thouless formula says that

$$\gamma(z) = \int \log|E-z|\,dk(E) = \Re\int \log(E-z)\,dk(E)$$

and hence $-\gamma(z)$ is the real part of a function whose derivative is a Borel transform (namely, of the measure dk). By general properties of the Borel transform, it follows that the limit $\gamma'(E+i0)$ exists for Lebesgue almost every $E \in \mathbb{R}$ and, in particular, for almost every $E \in \mathcal{Z}$. For these E, we have that

$$\lim_{\varepsilon\downarrow 0}\frac{\gamma(E+i\varepsilon)}{\varepsilon} = \lim_{\varepsilon\downarrow 0}\frac{\gamma(E+i\varepsilon)-\gamma(E)}{\varepsilon - 0} = \lim_{\varepsilon\downarrow 0}\frac{\partial\gamma}{\partial(\Im z)}(E+i\varepsilon), \tag{3.32}$$

and in particular, the limit is finite. Thus, by (3.29),

$$\limsup_{\varepsilon\downarrow 0} \mathbb{E}\left(\frac{1}{\Im m_\pm(E+i\varepsilon,\omega)+\frac{1}{2}\varepsilon}\right) < \infty$$

for almost every $E \in \mathcal{Z}$. Since $m_\pm$ are Borel transforms as well (of spectral measures with respect to half-line restrictions of H_ω), we also have that, for every $\omega \in \Omega$, $m_\pm(E+i0,\omega)$ exists for Lebesgue almost every $E \in \mathbb{R}$, and hence, for almost every E, $m_\pm(E+i0,\omega)$ exists for almost every ω. Combining the last two observations with Fatou's lemma, we find that

$$\mathbb{E}\left(\frac{1}{\Im m_\pm(E+i0,\omega)}\right) < \infty \tag{3.33}$$

for almost every E in $\mathcal{Z}$. So, for almost every $\omega \in \Omega$ and $E \in \mathcal{Z}$, $\Im m_\pm(E+i0,\omega) > 0$.

On the other hand, $m_+(E+i\varepsilon,\omega)+m_-(E+i\varepsilon,\omega)+E+i\varepsilon-V_\omega(0)$ has a finite limit for almost every $\omega \in \Omega$ and $E \in \mathcal{Z}$.

Hence, (3.27) shows that $0 < \Im G_\omega(0,0;E+i0) < \infty$ for almost every $\omega \in \Omega$ and $E \in \mathcal{Z}$, which implies the result. □

6.2. The Absolutely Continuous Part of the IDS. Denote the spectral measure associated with H_ω and δ_0 by ν_ω, that is,

$$G_\omega(0,0;z) = \langle \delta_0, (H_\omega - z)^{-1}\delta_0\rangle = \int \frac{d\nu_\omega(E)}{E-z}$$

for $z \in \mathbb{C}_+$. The results above imply the following for μ-almost every ω:

$$\nu_\omega^{(\mathrm{ac})}(E) = 0 \text{ for Lebesgue almost every } E \in \mathbb{R}\setminus\mathcal{Z},$$
$$\nu_\omega^{(\mathrm{ac})}(E) > 0 \text{ for Lebesgue almost every } E \in \mathcal{Z}.$$

Here, $\nu_\omega^{(\mathrm{ac})}(E)$ denotes the density of the absolutely continuous part of ν_ω. Write $k^{(\mathrm{ac})}(E)$ for the density of the absolutely continuous part of the density of states measure.

There is a direct relation between these densities [**24**]:

THEOREM 3.17 (Kotani 1997). *For almost every $E \in \mathcal{Z}$,*

$$k^{(\mathrm{ac})}(E) = \mathbb{E}\left(\nu_\omega^{(\mathrm{ac})}(E)\right). \tag{3.34}$$

PROOF. The inequality "$\geq$" in (3.34) follows from (3.5) (i.e., the density of states measure is the average of the measures ν_ω) and the fact that the average of absolutely continuous measures is absolutely continuous.

To prove the opposite inequality, we first note that for almost every $E \in \mathcal{Z}$, (3.28), (3.32), and Cauchy-Riemann imply

$$k^{(\mathrm{ac})}(E) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\frac{\gamma(E+i\varepsilon)}{\varepsilon}. \tag{3.35}$$

Because of (3.32), (3.33) and Fatou's lemma, (3.30) implies that for almost every pair $(E,\omega) \in \mathcal{Z}\times\Omega$,

$$\Im m_+(E+i0,\omega) = \Im m_-(E+i0,\omega) \tag{3.36}$$

and

$$\Re m_+(E+i0,\omega) + \Re m_-(E+i0,\omega) + E - V_\omega(0) = 0. \tag{3.37}$$

Thus, for almost every $(E,\omega) \in \mathcal{Z} \times \Omega$,

$$
\begin{aligned}
\nu_\omega^{(\mathrm{ac})}(E) &= \frac{1}{\pi} \Im G_\omega(0,0;E+i0) \\
&= \frac{1}{\pi} \Im \frac{-1}{m_+(E+i0,\omega)+m_-(E+i0,\omega)+E-V_\omega(0)} \\
&= \frac{1}{\pi} \Im \frac{-1}{2i\Im m_+(E+i0,\omega)} \\
&= \frac{1}{2\pi} \frac{1}{\Im m_+(E+i0,\omega)}
\end{aligned} \tag{3.38}
$$

Let P_ε be the Poisson kernel for the upper half-plane, that is,

$$P_\varepsilon(E) = \frac{1}{\pi} \frac{\varepsilon}{E^2+\varepsilon^2}.$$

Write

$$C_\varepsilon(E) = \int_{\mathcal{Z}} P_\varepsilon(E-E')\, dE'$$

and

$$\tilde{P}_\varepsilon(E,E') = P_\varepsilon(E-E') C_\varepsilon(E)^{-1}.$$

Then, by (3.38) and Jensen's inequality, we obtain for almost every $(E,\omega) \in \mathcal{Z} \times \Omega$,

$$
\begin{aligned}
\int_{\mathbb{R}} \nu_\omega^{(\mathrm{ac})}(E')\, P_\varepsilon(E-E')\, dE' &\geq \int_{\mathcal{Z}} \nu_\omega^{(\mathrm{ac})}(E')\, P_\varepsilon(E-E')\, dE' \\
&= \int_{\mathcal{Z}} \left(\frac{1}{2\pi} \frac{1}{\Im m_+(E'+i0,\omega)} \right) P_\varepsilon(E-E')\, dE' \\
&= C_\varepsilon(E) \int_{\mathcal{Z}} \left(\frac{1}{2\pi} \frac{1}{\Im m_+(E'+i0,\omega)} \right) \tilde{P}_\varepsilon(E,E')\, dE' \\
&\geq C_\varepsilon(E) \left(\int_{\mathcal{Z}} \left(\frac{1}{2\pi} \frac{1}{\Im m_+(E'+i0,\omega)} \right)^{-1} \tilde{P}_\varepsilon(E,E')\, dE' \right)^{-1} \\
&\geq C_\varepsilon(E)^2 \left(\int_{\mathbb{R}} \left(\frac{1}{2\pi} \frac{1}{\Im m_+(E'+i0,\omega)} \right)^{-1} P_\varepsilon(E-E')\, dE' \right)^{-1} \\
&= \frac{C_\varepsilon(E)^2}{2\pi} \frac{1}{\Im m_+(E+i\varepsilon,\omega)}
\end{aligned}
$$

Thus, for almost every $E \in \mathcal{Z}$,

$$\int_{\mathbb{R}} \mathbb{E}\left(\nu_\omega^{(\mathrm{ac})}(E')\right) P_\varepsilon(E-E')\, dE' \geq \frac{C_\varepsilon(E)^2}{2\pi}\, \mathbb{E}\left(\frac{1}{\Im m_+(E+i\varepsilon,\omega)} \right),$$

and hence

$$\mathbb{E}\left(\nu_\omega^{(\mathrm{ac})}(E)\right) \geq \frac{1}{2\pi} \limsup_{\varepsilon \downarrow 0} \mathbb{E}\left(\frac{1}{\Im m_+(E+i\varepsilon,\omega)} \right) \tag{3.39}$$

since $C_\varepsilon(E) < 1$ and $C_\varepsilon(E) \to 1$ as $\varepsilon \downarrow 0$.

Using (3.35), Proposition 3.14, the inequality $\log(1+x) \leq x$ for $x \geq 0$, and then (3.39), we find that

$$k^{(\mathrm{ac})}(E) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \frac{\gamma(E+i\varepsilon)}{\varepsilon}$$

$$
\begin{aligned}
&= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \mathbb{E}\left(\log\left(1 + \frac{\varepsilon}{\Im m_\pm(E + i\varepsilon, \omega)}\right)\right) \\
&\le \frac{1}{2\pi} \limsup_{\varepsilon \downarrow 0} \mathbb{E}\left(\frac{1}{\Im m_\pm(E + i\varepsilon, \omega)}\right) \\
&\le \mathbb{E}\left(\nu_\omega^{(\mathrm{ac})}(E)\right),
\end{aligned}
$$

concluding the proof of "$\le$" in (3.34). □

Corollary 3.18. *The spectrum is almost surely purely absolutely continuous if and only if the integrated density of states is absolutely continuous and the Lyapunov exponent vanishes almost everywhere with respect to the density of states measure.*

Proof. Suppose that the spectrum is almost surely purely absolutely continuous. The integrated density of states is then absolutely continuous since it is the distribution function of an average of purely absolutely continuous measures. The Lyapunov exponent vanishes almost everywhere with respect to the density of states measure by Theorems 3.4 and 3.13.

Conversely, if the integrated density of states is absolutely continuous and the Lyapunov exponent vanishes almost everywhere with respect to the density of states measure, we have that

$$
\begin{aligned}
1 &= \int_\Sigma dk(E) \\
&= \int_{\mathcal{Z}} dk(E) \\
&= \int_{\mathcal{Z}} k^{(\mathrm{ac})}(E)\, dE \\
&= \int_{\mathcal{Z}} \mathbb{E}\left(\nu_\omega^{(\mathrm{ac})}(E)\right) dE \\
&= \mathbb{E}\left(\int_{\mathcal{Z}} \nu_\omega^{(\mathrm{ac})}(E)\, dE\right) \\
&\le \mathbb{E}\left(\int_\Sigma \nu_\omega^{(\mathrm{ac})}(E)\, dE\right) \\
&\le 1.
\end{aligned}
$$

Here we used Theorem 3.17 in the fourth step and the fact that

$$\int_\Sigma \nu_\omega^{(\mathrm{ac})}(E)\, dE \le 1$$

for almost every ω in the last step. We therefore find that these integrals must in fact be equal to one for almost every ω, which means that the absolutely continuous part of ν_ω has full weight and so the measure is purely absolutely continuous. □

6.3. Weakly Bounded Solutions.

Theorem 3.19 (Deift-Simon 1983). *For almost every pair $(E, \omega) \in \mathcal{Z} \times \Omega$, there are linearly independent solutions $\tilde{u}_\pm$ of $H_\omega u = Eu$ such that $\tilde{u}_+ = \overline{\tilde{u}_-}$ and*

$$0 < \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\tilde{u}_\pm(n)|^2 < \infty.$$

Proof. As we saw above, for almost every $E \in \mathcal{Z}$, we have (3.33) and, for almost every $\omega \in \Omega$, $m_\pm(E + i0, \omega)$ exist and are finite with $\Im m_\pm(E + i0, \omega) > 0$.

The solution $u_+(n, E+i\varepsilon, \omega)$ has a limit which defines $\tilde{u}_+$ up to a normalization. That is, consider the solution of $H_\omega u = Eu$ with

$$\tilde{u}_+(0, E, \omega) = \frac{1}{(\Im m_+(E + i0, \omega))^{1/2}}$$

and

$$\tilde{u}_+(1, E, \omega) = -\frac{m_+(E + i0, \omega)}{(\Im m_+(E + i0, \omega))^{1/2}}.$$

Similarly, we define $\tilde{u}_-$ through the initial conditions

$$\tilde{u}_-(0, E, \omega) = \frac{1}{(\Im m_-(E + i0, \omega))^{1/2}}$$

and

$$\tilde{u}_-(-1, E, \omega) = -\frac{m_-(E + i0, \omega)}{(\Im m_-(E + i0, \omega))^{1/2}}.$$

Then (cf. (3.25)),

$$\frac{\tilde{u}_-(1)}{\tilde{u}_-(0)} = m_-(E + i0, \omega) + E - V_\omega(0)$$

and hence $\tilde{u}_+ = \overline{\tilde{u}_-}$ follows from

$$m_-(E + i0, \omega) + E - V_\omega(0) = -\overline{m}_+(E + i0, \omega),$$

which in turn is a consequence of (3.36) and (3.37).

It follows from the Ricatti equation (3.24) that $|\tilde{u}_+(n, E, \omega)| = |\tilde{u}_+(0, E, T^n\omega)|$, and hence

$$|\tilde{u}_+(n, E, \omega)|^2 = \frac{1}{\Im m_+(E + i0, T^n\omega)}.$$

Thus, Birkhoff yields

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\tilde{u}_+(n, E, \omega)|^2 = \mathbb{E}\left(\frac{1}{\Im m_+(E + i0, \omega)}\right)$$

and the right-hand side belongs to $(0, \infty)$ by (3.33). □

6.4. The Support of the Singular Part. For every $\omega \in \Omega$, we know that the singular part of the universal spectral measure μ_ω associated with H_ω is supported by the set

$$\mathcal{S}_\omega = \left\{ E \in \mathbb{R} : \limsup_{\varepsilon \downarrow 0} \Big(\Im G_\omega(0, 0; E + i\varepsilon) + \Im G_\omega(1, 1; E + i\varepsilon) \Big) = \infty \right\}.$$

Recall Theorem, which states that for a given $E \in \mathbb{R}$, the set of ω's for which E is an eigenvalue of H_ω has zero measure. The following result from [**8**] establishes the analogue for the singular support.

Theorem 3.20 (Deift-Simon 1983). *The set $\mathcal{S}_\omega$ is T-invariant and for every $E \in \mathbb{R}$, we have*

$$\mu(\{\omega \in \Omega : E \in \mathcal{S}_\omega\}) = 0.$$

Corollary 3.21. *For $\mu \times \mu$-almost every $(\omega, \omega') \in \Omega \times \Omega$, the measures $\mu_\omega^{\mathrm{sing}}$ and $\mu_{\omega'}^{\mathrm{sing}}$ are mutually singular.*

Proof. By the previous theorem, we have

$$\int_{\mathbb{R}} \mu\left(\{\omega' \in \Omega : E \in \mathcal{S}_{\omega'}\}\right) d\mu_\omega(E) = 0$$

for every $\omega \in \Omega$. Indeed, the theorem says that the integrand vanishes identically. Integrating over ω, we find

$$\begin{aligned}
0 &= \int_\Omega \int_{\mathbb{R}} \mu\left(\{\omega' \in \Omega : E \in \mathcal{S}_{\omega'}\}\right) d\mu_\omega(E)\, d\mu(\omega) \\
&= \int_\Omega \int_{\mathbb{R}} \int_\Omega \chi_{\mathcal{S}_{\omega'}}(E)\, d\mu(\omega')\, d\mu_\omega(E)\, d\mu(\omega) \\
&= \int_\Omega \int_\Omega \int_{\mathbb{R}} \chi_{\mathcal{S}_{\omega'}}(E)\, d\mu_\omega(E)\, d\mu(\omega')\, d\mu(\omega) \\
&= \int_{\Omega\times\Omega} \mu_\omega\left(\mathcal{S}_{\omega'}\right) d(\mu\times\mu)(\omega',\omega).
\end{aligned}$$

Thus, for $\mu\times\mu$-almost every $(\omega',\omega) \in \Omega\times\Omega$, we have $\mu_\omega(\mathcal{S}_{\omega'}) = 0$ and hence, in particular, $\mu_\omega^{\mathrm{sing}}(\mathcal{S}_{\omega'}) = 0$. Since $\mu_{\omega'}^{\mathrm{sing}}$ is supported by the set $\mathcal{S}_{\omega'}$, the result follows. □

Before giving the proof of Theorem 3.20 we need relate the definition of the set $\mathcal{S}_\omega$ to the behavior of solutions.

Proof of Theorem 3.20. □

7. Kotani Theory in the Topological Setting

We now strengthen the previous result in the sense that we pass from a measurable setting to a topological setting. To fix a universal topology, we consider spaces of sequences, on which the topology will be given by pointwise convergence. Fix a compact interval R. Endow $R^{\mathbb{Z}}$ with product topology, which makes it a compact metric space. If $V \in R^{\mathbb{Z}}$, we define the functions $m_\pm$ by

$$m_\pm(z) = \mp \frac{u_\pm(1)}{u_\pm(0)},$$

where $u_\pm$ solves $(\Delta + V - z)u = 0$ and is ℓ^2 at $\pm\infty$. We will be interested in those V for which the functions m_+, m_- obey identities like (3.36) and (3.37) (note also (3.25)), that is,

$$m_-(E+i0) = -\overline{m_+(E+i0)} \tag{3.40}$$

for a rich set of energies. Thus, for a set $\mathcal{Z} \subseteq \mathbb{R}$, we let

$$\mathcal{D}(\mathcal{Z}) = \{V \in R^{\mathbb{Z}} : m_\pm \text{ associated with } V \text{ obey (3.40) for a.e. } E \in \mathcal{Z}\}.$$

On $R^{\mathbb{Z}}$, define the shift transformation $[S(V)](n) = V(n+1)$.

Lemma 3.22. *Suppose that $\mathcal{Z} \subset \mathbb{R}$ has positive Lebesgue measure. Then:*
(a) *$\mathcal{D}(\mathcal{Z})$ is S-invariant and closed in $R^{\mathbb{Z}}$.*
(b) *For $V \in \mathcal{D}(\mathcal{Z})$, denote the restrictions to $\mathbb{Z}_\pm$ by $V_\pm$. Then V_- determines V_+ uniquely among elements of $\mathcal{D}(\mathcal{Z})$ and vice versa.*
(c) *If there exist $V^{(m)}, V \in \mathcal{D}(\mathcal{Z})$ such that $V_-^{(m)} \to V_-$ pointwise, then $V_+^{(m)} \to V_+$ pointwise.*

PROOF. (a) If u_1, u_2 denote the solutions of the same equation that obey $u_1(0) = u_2(1) = 1$ and $u_1(1) = u_2(0) = 0$, then we can write (note that we may normalize $u_\pm$ by $u_\pm(0) = 1$)

$$u_\pm(n) = u_1(n) \mp m_\pm(z)u_2(n).$$

Let us denote the m-functions associated with $S(V)$ by $\tilde{m}_\pm$. Clearly,

$$\tilde{m}_\pm(z) = \mp\frac{u_1(2) \mp m_\pm(z)u_2(2)}{u_1(1) \mp m_\pm(z)u_2(1)}.$$

Since the $u_j(m)$ are polynomials in z with real coefficients, this shows that

$$\tilde{m}_-(E + i0) = -\overline{\tilde{m}_+(E + i0)}$$

for almost every $E \in \mathcal{Z}$ and hence $\mathcal{D}(\mathcal{Z})$ is S-invariant. It is closed since the mapping "half-line potential $\mapsto$ m-function" is one-to-one and continuous if we choose the topology of pointwise convergence for potentials and the topology of uniform convergence on compact subsets of $\mathbb{C}_+$ for m-functions; the latter fact can be seen by deriving a continued fraction expansion for the m-function with coefficients given by (the energy and) the potential values. (For a proof that the identity between boundary values is preserved after taking limits, see [**23**, Lemma 5].)
(b) This follows from (3.40), the assumption that $\mathcal{Z}$ has positive Lebesgue measure, and the fact that the boundary values of a half-line m-function on any set of positive Lebesgue measure uniquely determine the half-line potential the m-function in question comes from.
(c) By compactness, there is a subsequence of $\{V^{(m)}\}$ that converges pointwise, that is, there is $\tilde{V}$ such that $V^{(m_k)} \to \tilde{V}$ as $k \to \infty$. By part (a), $\tilde{V} \in \mathcal{D}(\mathcal{Z})$. By assumption, $V_- = \tilde{V}_-$. Thus, by part (b), $V_+ = \tilde{V}_+$, and hence $V = \tilde{V}$. Consequently, $V_+^{(m_k)} \to V_+$ pointwise. In fact, we claim that $V_+^{(m)} \to V_+$ pointwise. Otherwise, we could reverse the argument (i.e., go from right to left) and show that $V_-^{(\tilde{m}_k)} \not\to V_-$ for some other subsequence. $\square$

Given an ergodic dynamical system $(\Omega, d\mu, T)$ and a measurable bounded sampling function $f : \Omega \to \mathbb{R}$ defining potentials $V_\omega(n) = f(T^n\omega)$ as before, we associate the following dynamical system $(R^{\mathbb{Z}}, d\nu, S)$: R is a compact interval that contains the range of f, $d\nu$ is the Borel measure on $R^{\mathbb{Z}}$ induced by $d\mu$ via $\Phi(\omega) = V_\omega$ (i.e., $\nu(A) = \mu(\Phi^{-1}(A))$), and S is the shift transformation on $R^{\mathbb{Z}}$ introduced above. Recall that the *topological support* of $d\nu$, $\operatorname{supp} d\nu$, is given by the intersection of all compact sets $B \subseteq R^{\mathbb{Z}}$ with $\nu(B) = 1$. Clearly, $\operatorname{supp} d\nu$ is closed and S-invariant.

Our first application of Lemma 3.22 is the so-called support theorem; compare [**21**]. For an S-ergodic Borel measure ν on $R^{\mathbb{Z}}$, let $\Sigma_{\mathrm{ac}}(\nu) \subseteq \mathbb{R}$ denote the almost sure absolutely continuous spectrum, that is, $\sigma_{\mathrm{ac}}(\Delta+V) = \Sigma_{\mathrm{ac}}(\nu)$ for ν almost every V. If ν comes from (Ω, μ, T, f), then $\Sigma_{\mathrm{ac}}(\nu)$ coincides with the set Σ_{ac} introduced earlier. The support theorem says that $\Sigma_{\mathrm{ac}}(\nu)$ is monotonically decreasing in the support of ν.

THEOREM 3.23 (Kotani 1985). *For every* $V \in \operatorname{supp} \nu$, *we have* $\sigma_{\mathrm{ac}}(\Delta + V) \supseteq \Sigma_{\mathrm{ac}}(\nu)$, *and hence*

$$\Sigma_{\mathrm{ac}}(\nu) = \bigcap_{V \in \operatorname{supp} \nu} \sigma_{\mathrm{ac}}(\Delta + V).$$

In particular, $\operatorname{supp} \nu_1 \subseteq \operatorname{supp} \nu_2$ *implies that* $\Sigma_{\mathrm{ac}}(\nu_1) \supseteq \Sigma_{\mathrm{ac}}(\nu_2)$.

PROOF. The statement is obvious when $\mathcal{Z}$ has zero Lebesgue measure, so let us assume the its Lebesgue measure is positive. Then it follows from Lemma 3.22 that

$$\mathrm{supp}\ \nu \subseteq \mathcal{D}(\mathcal{Z}) = \{V \in R^{\mathbb{Z}} : m_\pm \text{ associated with } V \text{ obey (3.40) for a.e. } E \in \mathcal{Z}\}.$$

Bearing in mind the Riccati equation (3.25), a calculation like the one in (3.38) therefore shows that for every $V \in \mathrm{supp}\ \nu$, the Green function associated with the operator $\Delta + V$ obeys $\Im G(0, 0; E + i0) > 0$ for almost every $E \in \mathcal{Z}$. This implies $\overline{\mathcal{Z}}^{\mathrm{ess}} \subseteq \sigma_{\mathrm{ac}}(\Delta + V)$ and hence the result by Theorem 3.13. □

A different proof may be found in Last-Simon [**29**, Sect. 6]. Here is a typical application of the support theorem:

COROLLARY 3.24. *Let* Per_ν *be the set of* $V \in \mathrm{supp}\ \nu$ *that are periodic, that is,* $S^p V = V$ *for some* $p \in \mathbb{Z}_+$. *Then,*

$$\Sigma_{\mathrm{ac}}(\nu) \subseteq \bigcap_{V \in \mathrm{Per}_\nu} \sigma(\Delta + V).$$

If there are sufficiently many gaps in the spectra of these periodic operators, one can show in this way that $\Sigma_{\mathrm{ac}}(\nu)$ is empty.

THEOREM 3.25. *Assume that the set*

$$\mathcal{Z} = \{E \in \mathbb{R} : \gamma(E) = 0\}.$$

has positive Lebesgue measure. Then,
(a) *Each* $V \in \mathrm{supp}\, d\nu$ *is determined completely by* V_- *(resp.,* V_+*).*
(b) *If we let*

$$(\mathrm{supp}\, d\nu)_\pm = \{V_\pm : V \in \mathrm{supp}\, d\nu\},$$

then the mappings

$$(\mathrm{supp}\, d\nu)_\pm \ni V_\pm \mapsto V_\mp \in (\mathrm{supp}\, d\nu)_\mp$$

are continuous with respect to pointwise convergence.

PROOF. (a) By our earlier results, we know that $\mathcal{D}(\mathcal{Z})$ is compact and has full ν-measure. Thus, $\mathrm{supp}\, d\nu \subseteq \mathcal{D}(\mathcal{Z})$ and the assertion follows from Lemma 3.22.(b).
(b) This follows from Lemma 3.22.(c). □

We say that (Ω, μ, T, f) is *topologically deterministic* if there exist continuous mappings $E_\pm : (\mathrm{supp}\, \nu)_\pm \to (\mathrm{supp}\, \nu)_\mp$ that are formal inverses of one another and obey $V_-^{\#} \in \mathrm{supp}\, \nu$ for every $V_- \in (\mathrm{supp}\, \nu)_-$, where

$$V_-^{\#}(n) = \begin{cases} V_-(n) & n \le 0, \\ E_-(V_-)(n) & n \ge 1. \end{cases}$$

This also implies $V_+^{\#} \in \mathrm{supp}\, \nu$ for every $V_+ \in (\mathrm{supp}\, \nu)_+$, where

$$V_+^{\#}(n) = \begin{cases} V_+(n) & n \ge 1, \\ E_+(V_+)(n) & n \le 0. \end{cases}$$

Otherwise, (Ω, μ, T, f) is *topologically non-deterministic.*

COROLLARY 3.26. *If* (Ω, μ, T, f) *is topologically non-deterministic, we have that* $\Sigma_{\mathrm{ac}} = \emptyset$.

Our next theorem is a surprisingly general result stating that aperiodic ergodic potentials taking on only finitely many values cannot generate absolutely continuous spectrum.

THEOREM 3.27. *Assume that the sampling function f takes finitely many values and the potentials V_ω are not periodic (for a.e. ω). Then, $\mathcal{Z}$ has zero Lebesgue measure.*

PROOF. Assume to the contrary that $\mathcal{Z}$ has positive Lebesgue measure. We will show that $\operatorname{supp} d\nu$ is a finite set and hence the potentials must be periodic.

By Theorem 3.25.(b), it follows that for every $\varepsilon > 0$, there is a $\delta > 0$ and $M \geq 1$ such that $|V(n) - \tilde{V}(n)| < \delta$, $-M+1 \leq n \leq 0$, implies $V(1) = \tilde{V}(1)$ for $V, \tilde{V} \in \operatorname{supp} d\nu$. Since the range of all elements of $\operatorname{supp} d\nu$ is contained in the finite set $\operatorname{Ran} f$, it follows that, upon choosing ε and δ small enough,

$$V(n) = \tilde{V}(n), \quad -M+1 \leq n \leq 0 \quad \Rightarrow \quad V(1) = \tilde{V}(1), \quad V, \tilde{V} \in \operatorname{supp} d\nu.$$

Since $\operatorname{supp} d\nu$ is S-invariant, we can iterate and obtain that V_+ is determined by $\{V(n) : -M+1 \leq n \leq 0\}$. Therefore, the cardinality of $\operatorname{supp} d\nu$ does not exceed $|\operatorname{Ran} f|^M$ and, in particular, $\operatorname{supp} d\nu$ is a finite set. □

Next, we discuss an extension of this result from sampling functions taking finitely many values to discontinuous sampling functions. Suppose Ω is a compact metric space and $T : \Omega \to \Omega$ a homeomorphism. We say that $l \in \mathbb{R}$ is an *essential limit* of f at ω_0 if there exists a sequence $\{\Omega_k\}$ of sets each of positive measure such that for any sequence $\{\omega_k\}$ with $\omega_k \in \Omega_k$, both $\omega_k \to \omega_0$ and $f(\omega_k) \to l$. If f has more than one essential limit at ω_0, we say that f is *essentially discontinuous* at this point.

THEOREM 3.28. *Suppose there is an $\omega_0 \in \Omega$ such that f is essentially discontinuous at ω_0 but is continuous at all points $T^n\omega_0$, $n < 0$. Then, $\mathcal{Z}$ has zero Lebesgue measure.*

PROOF. We will show that there are potentials in $\operatorname{supp} d\nu$ which agree on $\mathbb{Z}_-$ but take different values at $n = 0$.

Let l be an essential limit of f at ω_0. As f is essentially discontinuous at ω_0, it suffices to construct a potential in $\operatorname{supp} d\nu$ that agrees with V_{ω_0} to the left of zero and takes the value l at $n = 0$. Let $\{\Omega_k\}$ be a sequence of sets which exhibits the fact that l is an essential limit of f. Since each has positive μ-measure, we can find points $\omega_k \in \Omega_k$ so that V_{ω_k} is in the support of $d\nu$; indeed, this is the case for almost every point in Ω_k. As $\omega_k \to \omega_0$ and f is continuous at each of the points $T^n\omega_0$, $n < 0$, it follows that $V_{\omega_k}(n) \to V_{\omega_0}(n)$ for each $n < 0$. Moreover, since $f(\omega_k)$ converges to l, we also have $V_{\omega_k}(0) \to l$. We can guarantee convergence of $V_{\omega_k}(n)$ for $n > 0$ by passing to a subsequence because $R^{\mathbb{Z}}$ is compact. Let us denote this limit potential by V. As each V_{ω_k} lies in $\operatorname{supp} d\nu$, so does V; moreover, $V(0) = l$ and $V(n) = V_{\omega_0}(n)$ for each $n < 0$. □

Application. If $\Omega = [0, 1)$, $\alpha \in (0, 1)$ irrational, $T : \Omega \to \Omega$, $T\omega = \omega + \alpha \mod 1$, then it is expected that, under suitable regularity assumptions on f, the operators with potentials $V_\omega(n) = \lambda f(T^n\omega)$ have purely absolutely continuous spectrum for sufficiently small coupling constant λ. This has been proven for analytic f's. The previous two results show that at least some regularity must be required from f. For example, if f has a finite number (≥ 1) of non-removable discontinuities, then

Theorem 3.28 shows that $\Sigma_{\rm ac}$ is empty for every non-zero coupling constant. In fact, it was shown in [**7**] that in the situation just described, $\sigma_{\rm ac}(H_\omega) = \emptyset$ for every (rather than almost every) $\omega \in \Omega$.

THEOREM 3.29. *Let Ω be a compact metric space, $T : \Omega \to \Omega$ a homeomorphism, and μ a continuous (i.e., non-atomic) T-ergodic Borel measure. Then, for a dense G_δ set of functions $f \in C(\Omega)$, $\mathcal{Z}$ has zero Lebesgue measure.*

SKETCH OF PROOF. The key observation is that, for every $r > 0$,

$$(L^1(\Omega) \cap B_r(L^\infty(\Omega)), \|\cdot\|_1) \to \mathbb{R}, \quad f \mapsto \mathrm{Leb}(\mathcal{Z}) \quad \text{is upper semi-continuous.} \tag{3.41}$$

For $\delta > 0$, we then define $M_\delta = \{f \in C(\Omega) : \mathrm{Leb}(\mathcal{Z}) < \delta\}$. By (3.41), M_δ is open. Moreover, M_δ is dense: Approximate a given $f \in C(\Omega)$ by a suitable step function s and apply Theorem 3.27 to s. Then approximate s by continuous functions and apply (3.41).

It follows that $\{f \in C(\Omega) : \mathrm{Leb}(\mathcal{Z}) = 0\} = \bigcap_{\delta > 0} M_\delta$ is a dense G_δ set. For details concerning (3.41), see [**1**]. $\square$

This shows that, regardless of the choice of the transformation T, non-smoothness can cause positive Lyapunov exponents. Notice that the proof does not extend to regularity C^ε for any $\varepsilon > 0$. It is an interesting open question whether the result itself extends.

To demonstrate that non-smoothness cannot be made up for by a sufficiently small coupling constant, we note that a variation of the argument above shows the following [**1**]:

THEOREM 3.30. *Let Ω be a compact metric space, $T : \Omega \to \Omega$ a homeomorphism, and μ a continuous (i.e., non-atomic) T-ergodic Borel measure. Then, for a dense G_δ set of functions $f \in C(\Omega)$, the set $\mathcal{Z}$ associated with the potentials $V_\omega(n) = \lambda f(T^n \omega)$ has zero Lebesgue measure for almost every $\lambda > 0$.*

CHAPTER 4

Proving Positive Lyapunov Exponents

1. Positive Lyapunov Exponents at Large Coupling

1.1. Herman's Subharmonicity Argument and Extensions. *Example.* A very important result that can be proven easily using a subharmonicity argument is the positivity of the Lyapunov exponent at all energies for the almost Mathieu operator at sufficiently large coupling. This result is due to Herman. The argument employs subharmonicity in the random parameter, rather than the energy, and the trick is to "complexify" this parameter. Recall that the potential of the almost Mathieu operator is given by $V_\omega(n) = 2\lambda\cos(2\pi(n\alpha+\omega))$. Setting $w = e^{2\pi i\omega}$, we see that

$$V_\omega(n) = \lambda\left(e^{2\pi i\alpha n}w + e^{-2\pi i\alpha n}w^{-1}\right).$$

Thus, the one-step transfer matrices have the form

$$M_z(T^n\omega) = \begin{pmatrix} z - \lambda\left(e^{2\pi i\alpha n}w + e^{-2\pi i\alpha n}w^{-1}\right) & -1 \\ 1 & 0 \end{pmatrix}$$

If we define

$$N_n(w) = w^n M_z(n,\omega) = (wM_z(T^{n-1}\omega))\cdots(wM_z(\omega)),$$

initially on $|w| = 1$, we see that N_n extends to an entire function and hence $w \mapsto \log\|N_n(w)\|$ is subharmonic. Thus,

$$\int_0^1 \log\|N_n(e^{2\pi i\omega})\|\,d\omega \ge \log\|N_n(0)\| = n\log|\lambda|.$$

Moreover, $\|N_n(e^{2\pi i\omega})\| = \|M_z(n,\omega)\|$. Thus,

$$\begin{aligned}\gamma(z) &= \inf_{n\ge 1}\frac{1}{n}\int_0^1 \log\|M_z(n,\omega)\|\,d\omega \\ &= \inf_{n\ge 1}\frac{1}{n}\int_0^1 \log\|N_n(e^{2\pi i\omega})\|\,d\omega \\ &\ge \log|\lambda|.\end{aligned}$$

It is not hard to extend the argument given in the previous section from $f(\omega) = 2\lambda\cos(2\pi\omega)$ to the case where cos is replaced by a trigonometric polynomial. For example, if

$$f(\omega) = \sum_{m=1}^{M} 2\lambda_m\cos(2\pi m(\omega+\theta_m)),$$

then the Lyapunov exponent associated with the corresponding family of one-frequency quasi-periodic Schrödinger operators satisfies

$$\gamma(z) \ge \log|\lambda_M| \quad \text{for every } z \in \mathbb{C}.$$

Notice that the estimate becomes useless when the leading coefficient is too small. In particular, one cannot get any meaningful estimates for infinite trigonometric series in this way directly. Nevertheless, the following result can be shown:

THEOREM 4.1 (Sorets-Spencer 1991). *Suppose g is real-analytic on $\mathbb{T}$. Then there exists λ_0 such that for $|\lambda| > \lambda_0$ and α irrational, the Lyapunov exponent $\gamma_{\lambda,\alpha}(\cdot)$ associated with the potentials $V_\omega(n) = \lambda g(\omega + n\alpha)$ satisfies the estimate*

$$\gamma_{\lambda,\alpha}(z) \geq \frac{1}{2} \log |\lambda|$$

for every $z \in \mathbb{C}$.

2. Fürstenberg's Theorem

In this section we want to show that extreme randomness forces positive Lyapunov exponents for Schrödinger cocycles. We will give a complete proof of this fact for the Anderson model, followed by a discussion of the general case of Schrödinger cocycles over (uniformly) hyperbolic base dynamics.

All results in this direction have their roots in a beautiful theorem of Fürstenberg, which, in the special case of $\mathrm{SL}(2,\mathbb{R})$ matrices, we will present first. We then show how Fürstenberg's Theorem gives positive Lyapunov exponents at all energies in the Anderson model. Generalizations of Fürstenberg's Theorem and their applications are given at the end of this section.

Let $d\nu$ be a probability measure on $\mathrm{SL}(2,\mathbb{R})$ which satisfies

$$\int \log \|M\| \, d\nu(M) < \infty. \tag{4.1}$$

Let us consider i.i.d. matrices $T_1, T_2, \ldots$, each distributed according to ν. Write $M_n = T_n \cdots T_1$. We are interested in the Lyapunov exponent $\gamma \geq 0$, given by

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \log \|M_n\|, \quad \nu^{\mathbb{Z}_+} - \text{a.s.}$$

We are interested in conditions that ensure $\gamma > 0$. To motivate the result below, let us give some examples with $\gamma = 0$:

- If $d\nu$ is supported in $\mathrm{SO}(2,\mathbb{R})$, then $\gamma = 0$.
- If
$$\nu\left\{\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}\right\} = \frac{1}{2} \quad \text{and} \quad \nu\left\{\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}\right\} = \frac{1}{2},$$
then $\gamma = 0$: We have that
$$M_n = \begin{pmatrix} m_n & 0 \\ 0 & m_n^{-1} \end{pmatrix},$$
where $\log m_n = a_1 + \cdots + a_n$ and $\{a_j\}$ are i.i.d. random variables taking values $\pm \log 2$, each with probability $1/2$. Thus, $\log \|M_n\| = |a_1 + \cdots + a_n|$ and the strong law of large numbers gives $\frac{1}{n} \log \|M_n\| \to 0$ almost surely.
- If $p \in (0,1)$ and
$$\nu\left\{\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}\right\} = p \quad \text{and} \quad \nu\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\} = 1 - p,$$
then $\gamma = 0$.

Furstenberg's Theorem shows that this list is essentially exhaustive in the sense that the two mechanisms above, no growth of norms or a finite (cardinality = 2) invariant set of directions, are the only ones that can preclude a positive Lyapunov exponent.

Call two non-zero vectors v_1, v_2 in $\mathbb{R}^2$ equivalent if $v_2 = \lambda v_1$ for some $\lambda \in \mathbb{R}$. The set of equivalence classes is denoted by $\mathbb{P}^1$. Since every $M \in \mathrm{SL}(2,\mathbb{R})$ is invertible, it induces a mapping from $\mathbb{P}^1$ to $\mathbb{P}^1$ in the obvious way.

Let $\mathcal{M}(\mathbb{P}^1)$ denote the set of probability measures m on $\mathbb{P}^1$. Given $M \in \mathrm{SL}(2,\mathbb{R})$ and $m \in \mathcal{M}(\mathbb{P}^1)$, we define $Mm \in \mathcal{M}(\mathbb{P}^1)$ by

$$\int f(v)\, d(Mm)(v) = \int f(Mv)\, dm(v).$$

Moreover, we define the convolution $\nu * m \in \mathcal{M}(\mathbb{P}^1)$ by

$$\int f(v)\, d(\nu * m)(v) = \iint f(Mv)\, d\nu(M)\, dm(v).$$

If $\nu * m = m$, then m is called *ν-invariant.* By a Krylov-Bogoliubov argument, ν-invariant measures always exist.

We now state the main result of this section:

THEOREM 4.2 (Furstenberg's Theorem). *Let ν be a probability measure on* $\mathrm{SL}(2,\mathbb{R})$ *which satisfies* (4.1). *Denote by G_ν the smallest closed subgroup of* $\mathrm{SL}(2,\mathbb{R})$ *which contains* $\operatorname{supp}\nu$.

Assume

(i) *G_ν is not compact.*

and one of the following conditions:

(ii) *There is no finite non-empty set $L \subseteq \mathbb{P}^1$ such that $M(L) = L$ for all $M \in G_\nu$.*

(ii') *There is no set $L \subseteq \mathbb{P}^1$ of cardinality* 1 *or* 2 *such that $M(L) = L$ for all $M \in G_\nu$.*

Then, $\gamma > 0$.

Remarks. (a) We will show that (i)+(ii) implies $\gamma > 0$.
(b) To so see that (ii) can be replaced by (ii'), we remark that (i)+(ii') implies (ii). To prove this, it is enough to show that if (i) holds and $v_1, \dots, v_k$ are distinct elements of $\mathbb{P}^1$ with

$$M(\{v_1, \dots, v_k\}) = \{v_1, \dots, v_k\} \quad \text{for every } M \in G_\nu,$$

then $k \le 2$. Each $M \in G_\nu$ induces a permutation $\pi(M)$ of $\{v_1, \dots, v_k\}$, and $\pi : G_\nu \to \mathcal{S}_k$ is a group homomorphism. The kernel H of π is a closed normal subgroup of G_ν, and G_ν/H is finite. By (i), H is not finite. If $k \ge 3$, consider representatives of v_1, v_2, v_3, which we denote by the same symbols. Write

$$v_3 = \alpha v_1 + \beta v_2, \quad \alpha, \beta \ne 0.$$

If $M \in H$, there are non-zero λ_i such that $Mv_i = \lambda_i v_i$, $i = 1, 2, 3$. This yields

$$\begin{aligned} \lambda_3 \alpha v_1 + \lambda_3 \beta v_2 &= \lambda_3 v_3 \\ &= M v_3 \\ &= \alpha M v_1 + \beta M v_2 \\ &= \alpha \lambda_1 v_1 + \beta \lambda_2 v_2. \end{aligned}$$

It follows that $\lambda_1 = \lambda_2 = \lambda_3$ and hence $M = \lambda_1 \mathrm{Id}$. Since $M \in \mathrm{SL}(2,\mathbb{R})$, this shows $H \subseteq \{\pm \mathrm{Id}\}$ and H is finite; contradiction.
(c) Note that the assumptions are monotonic in the support of the measure in the sense that if Theorem 4.2 applies to ν, then it applies to measures whose support contains the support of ν.
(d) Theorem 4.2 is a special case of a far more general result from [**10**].

LEMMA 4.3. *If ν satisfies the assumption* (ii) *of Theorem 4.2, then every ν-invariant measure is non-atomic.*

PROOF. Assume that m is ν-invariant with

$$w = \max\{m(\{v\}) : v \in \mathbb{P}^1\} > 0.$$

Let

$$L = \{v : m(\{v\}) = w\},$$

which is a finite non-empty subset of $\mathbb{P}^1$. It follows that, for $v_0 \in L$,

$$\begin{aligned} w &= m(\{v_0\}) \\ &= (\nu * m)(\{v_0\}) \\ &= \iint \chi_{\{v_0\}}(Mv)\, d\nu(M)\, dm(v) \\ &= \iint \chi_{\{M^{-1}v_0\}}(v)\, dm(v)\, d\nu(M) \\ &= \int m\left(\{M^{-1}v_0\}\right)\, d\nu(M), \end{aligned}$$

where we used ν-invariance in the second step.

On the other hand, $m\left(\{M^{-1}v_0\}\right) \leq w$ for all M by the definition of w. Since the integral is equal to w, we see that $m\left(\{M^{-1}v_0\}\right) = w$ for ν-almost every M. In other words, $\{M^{-1}v_0\} \in L$ for ν-almost every M. It follows that $M^{-1}(L) \subseteq L$ for ν-almost every M, and hence, by finiteness of L, for all M. This shows that $M(L) = L$ for all M, and so (ii) fails. □

From now on, we assume that ν satisfies (i) and (ii), and $m \in \mathcal{M}(\mathbb{P}^1)$ is ν-invariant (and hence non-atomic). Our first goal is to express the Lyapunov exponent in terms of these two measures.

LEMMA 4.4. *We have that*

$$\gamma = \iint \log \frac{\|Mv\|}{\|v\|}\, d\nu(M)\, dm(v).$$

PROOF. This follows quickly from Birkhoff and Osceledec-Ruelle: The shift σ on $\mathrm{SL}(2,\mathbb{R})^{\mathbb{Z}_+}$, the space of sequences $(T_1(\omega), T_2(\omega), \ldots)$, has the ergodic measure $\nu^{\mathbb{Z}_+}$. The skew-product

$$\tilde{\sigma} : \mathrm{SL}(2,\mathbb{R})^{\mathbb{Z}_+} \times \mathbb{P}^1 \to \mathrm{SL}(2,\mathbb{R})^{\mathbb{Z}_+} \times \mathbb{P}^1, \quad (\omega, v) \mapsto (\sigma\omega, T_1(\omega)v)$$

leaves invariant the measure $\nu^{\mathbb{Z}_+} \times m$. Consider the function

$$f(\omega, v) = \log \frac{\|T_1(\omega)v\|}{\|v\|}. \tag{4.2}$$

Then,

$$\frac{1}{n}\sum_{m=1}^{n} f(\tilde{\sigma}^m(\omega,v)) = \frac{1}{n}\log\frac{\|T_n(\omega)\cdots T_1(\omega)v\|}{\|v\|} = \frac{1}{n}\log\frac{\|M_n(\omega)v\|}{\|v\|},$$

which, on the one hand, converges to γ for almost every ω and almost every v by Osceledec-Ruelle, and, on the other hand, Birkhoff's Theorem gives that it converges to $\iint f\, d\nu^{\mathbb{Z}_+}\, dm$. Putting these two things together,

$$\gamma = \iint \log\frac{\|T_1(\omega)v\|}{\|v\|}\, d\nu^{\mathbb{Z}_+}(\omega)\, dm(v) = \iint \log\frac{\|Mv\|}{\|v\|}\, d\nu(M)\, dm(v), \tag{4.3}$$

as claimed. □

Let us prove two auxiliary lemmas, which will be useful later in the proof of Theorem 4.2.

Lemma 4.5. *If $m \in \mathcal{M}(\mathbb{P}^1)$ is non-atomic and $M_n \neq 0$ converge to $M \neq 0$, then $M_n m \to Mm$ weakly.*

Proof. Since $M \neq 0$, there is at most one direction v for which Mv is not defined (because the vectors in the direction of v are in $\operatorname{Ker} M$), similarly for all the M_n. Thus, for v outside a countable set $C \subseteq \mathbb{P}^1$, $M_n v$ and Mv are defined and we have $M_n v \to Mv$ as $n \to \infty$. If $f \in C(\mathbb{P}^1)$, we therefore get

$$\int f(v)\, d(M_n m)(v) = \int f(M_n v)\, dm(v) \to \int f(Mv)\, dm(v) = \int f(v)\, d(Mm)(v)$$

by dominated convergence and $m(C) = 0$. That is, $M_n m \to Mm$ weakly. □

Lemma 4.6. *If $m \in \mathcal{M}(\mathbb{P}^1)$ is non-atomic, then*

$$H = \{M \in \mathrm{SL}(2,\mathbb{R}) : Mm = m\}$$

is a compact subgroup of $\mathrm{SL}(2,\mathbb{R})$.

Proof. It is clear that H is a closed subgroup of $\mathrm{SL}(2,\mathbb{R})$ so we only need to prove boundedness. Assume that there are $M_n \in H$ such that $\|M_n\| \to \infty$. For a suitable subsequence, $\|M_{n_k}\|^{-1} M_{n_k}$ converges to a matrix $M_\infty \neq 0$, which obeys $M_\infty m = m$ by Lemma 4.5 since m is non-atomic. On the other hand,

$$\det M_\infty = \lim_{n\to\infty} \det \frac{M_{n_k}}{\|M_{n_k}\|} = \lim_{n\to\infty} \frac{1}{\|M_{n_k}\|^2} = 0,$$

so M_∞ is rank one and m must be a Dirac measure because of $M_\infty m = m$; which is a contradiction. □

We return to the proof of Theorem 4.2 and show that the measures $M_n(\omega)m \in \mathcal{M}(\mathbb{P}^1)$ converge weakly to a Dirac measure with a certain invariance property for almost every $\omega \in \mathrm{SL}(2,\mathbb{R})^{\mathbb{Z}_+}$.

Lemma 4.7. *For $\nu^{\mathbb{Z}_+}$-almost every ω, we have the following:*
(a) *There exists $m_\omega \in \mathcal{M}(\mathbb{P}^1)$ such that $M_n(\omega)m \to m_\omega$ weakly.*
(b) *For ν-almost every $M \in \mathrm{SL}(2,\mathbb{R})$, $M_n(\omega)Mm \to m_\omega$ weakly.*
(c) *There exists $v_\omega \in \mathbb{P}^1$ such that $m_\omega = \delta_{v_\omega}$.*

PROOF. (a) Fix $g \in C(\mathbb{P}^1)$ and define $G : \mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}$ by

$$G(M) = \int g(Mv)\, dm(v).$$

Let $\mathcal{F}_n$ be the σ-algebra of $\mathrm{SL}(2,\mathbb{R})^{\mathbb{Z}_+}$ formed by the cylinders of length n. Then $M_n(\cdot)$ is $\mathcal{F}_n$-measurable. We want to show that $G(M_n(\omega))$ converges for almost every ω. We will employ the Martingale Convergence Theorem; see [**9**, (2.10] for this result and [**9**, Sections 4.1 and 4.2] for background.

We have that

$$\begin{aligned}\mathbb{E}(G(M_{n+1})|\mathcal{F}_n) &= \int G(M_n M)\, d\nu(M) \\ &= \iint g(M_n M v)\, d\nu(M) dm(v) \\ &= \int g(M_n v)\, dm(v) \\ &= G(M_n),\end{aligned}$$

where we used ν-invariance of m in the last step. This shows that $\omega \mapsto G(M_n(\omega))$ is a martingale, and hence the limit

$$\Gamma_g(\omega) = \lim_{n\to\infty} G(M_n(\omega))$$

exists for almost every ω by the Martingale Convergence Theorem.

Now pick a countable dense subset $\{g_k\}$ of $C(\mathbb{P}^1)$ and take ω from the full measure subset where $\Gamma_{g_k}(\omega)$ exists for all k. Let m_ω be a weak accumulation point of the sequence $\{M_n(\omega)m\}$. Then

$$\int g_k\, dm_\omega = \lim_{j\to\infty} \int g_k\, d(M_{n_j}(\omega)m) = \lim_{j\to\infty} \int g_k \circ M_{n_j}(\omega) dm = \Gamma_{g_k}(\omega).$$

Since the limit is the same for every subsequence, we have in fact that $M_n(\omega)m \to m_\omega$ weakly, as desired.
(b) We have to show that for any $g \in C(\mathbb{P}^1)$,

$$\lim_{n\to\infty} \mathbb{E}(G(M_n M)) = \Gamma(g) = \lim_{n\to\infty} \mathbb{E}(G(M_n)), \quad \nu - \text{a.e. } M \in \mathrm{SL}(2,\mathbb{R}), \tag{4.4}$$

where $\mathbb{E}$ is integration over ω. We will show

$$\lim_{n\to\infty} \mathbb{E}([G(M_{n+1}) - G(M_n)]^2) = 0. \tag{4.5}$$

Since

$$\mathbb{E}([G(M_{n+1}) - G(M_n)]^2) = \mathbb{E}\left(\left[\iint (g(M_n M v) - g(M_n v)\, dm(v)\, d\nu(M)\right]^2\right),$$

we may deduce from (4.5), for almost every ω,

$$\lim_{n\to\infty} \int G(M_n(\omega)M) - G(M_n(\omega))\, d\nu(M) = \lim_{n\to\infty} \iint g(M_n(\omega)Mv) - g(M_n(\omega))\, dm(v)\, d\nu(M),$$

which implies (4.4).

Note that

$$\mathbb{E}\left([G(M_{n+1}) - G(M_n)]^2\right) = \mathbb{E}\left(G(M_{n+1})^2\right) + \mathbb{E}\left(G(M_n)^2\right) - 2\mathbb{E}\left(G(M_{n+1})G(M_n)\right)$$

and

$$\begin{aligned}\mathbb{E}\left(G(M_{n+1})G(M_n)\right) &= \mathbb{E}\left(\int g(M_{n+1}v)\,dm(v)\,\cdot\,\int g(M_n v)\,dm(v)\right)\\ &= \mathbb{E}\left(\iint g(M_n M v)\,dm(v)\,d\nu(M)\,\cdot\,\int g(M_n v)\,dm(v)\right)\\ &= \mathbb{E}\left(\left[\int g(M_n v)\,dm(v)\right]^2\right)\\ &= \mathbb{E}\left(G(M_n)^2\right).\end{aligned}$$

Thus,

$$\mathbb{E}\left([G(M_{n+1}) - G(M_n)]^2\right) = \mathbb{E}\left(G(M_{n+1})^2\right) - \mathbb{E}\left(G(M_n)^2\right)$$

and hence

$$\begin{aligned}\sum_{n=1}^{N}\mathbb{E}\left([G(M_{n+1}) - G(M_n)]^2\right) &= \sum_{n=1}^{N}\mathbb{E}\left(G(M_{n+1})^2\right) - \mathbb{E}\left(G(M_n)^2\right)\\ &= \mathbb{E}\left(G(M_{N+1})^2\right) - \mathbb{E}\left(G(M_1)^2\right)\\ &\le \|g\|_\infty.\end{aligned}$$

Thus, $\sum_{n=1}^{\infty}\mathbb{E}\left([G(M_{n+1}) - G(M_n)]^2\right) < \infty$ and (4.5) follows.
(c) Consider an ω for which (a) and (b) hold, that is,

$$M_n(\omega)m \to m_\omega \ \text{ and } \ M_n(\omega)Mm \to m_\omega, \quad \nu - \text{a.e. } M \in \mathrm{SL}(2,\mathbb{R}).$$

Let $M(\omega)$ be an accumulation point of the sequence $\{\|M_n(\omega)\|^{-1}M_n(\omega)\}$. Since ν is non-atomic, Lemma 4.5 implies

$$M(\omega)m = M(\omega)Mm = m_\omega, \quad \nu - \text{a.e. } M \in \mathrm{SL}(2,\mathbb{R}).$$

If $M(\omega)$ is invertible, it follows that

$$m = Mm, \quad \nu - \text{a.e. } M \in \mathrm{SL}(2,\mathbb{R}).$$

But

$$H = \{M \in \mathrm{SL}(2,\mathbb{R}) : m = Mm\}$$

is compact by Lemma 4.6, which contradicts assumption (i) from Theorem 4.2. It follows that $M(\omega)$ is not invertible, that is, its range is one-dimensional. But this implies the assertion since $M(\omega)m = m_\omega$. $\square$

The next step is to show that convergence to a Dirac measure implies norm growth.

LEMMA 4.8. *Let $m \in \mathcal{M}(\mathbb{P}^1)$ be non-atomic and let $\{M_n\} \in \mathrm{SL}(2,\mathbb{R})^{\mathbb{Z}_+}$ with $M_n m \to \delta_v$ weakly for some $v \in \mathbb{P}^1$. Then,*

$$\lim_{n\to\infty}\|M_n\| = \lim_{n\to\infty}\|M_n^*\| = \infty.$$

Moreover,

$$\frac{\|M_n^* w\|}{\|M_n^*\|} \to \left|\langle w, \tfrac{v}{\|v\|}\rangle\right|$$

for every $w \in \mathbb{R}^2$.

Proof. Assume first that there is a matrix M such that

$$\frac{M_n}{\|M_n\|} \to M.$$

By Lemma 4.5 we have that

$$\frac{M_n}{\|M_n\|} m \to Mm$$

weakly. Thus, by assumption, $Mm = \delta_v$. This shows that $\det M = 0$ since m is non-atomic (otherwise, $m = M^{-1}\delta_v = \delta_{M^{-1}v}$). Therefore,

$$0 = \det M = \lim_{n\to\infty} \det \frac{M_n}{\|M_n\|} = \lim_{n\to\infty} \frac{1}{\|M_n\|^2},$$

which yields the first assertion.

From $Mm = \delta_v$ we also infer that the range of M is the line in v-direction. Denote a unit vector in this direction by the same symbol, v. If $e_1 = (1,0)^T$ and $e_2 = (0,1)^T$, we can therefore write

$$Me_1 = \pm\|Me_1\|v, \quad Me_2 = \pm\|Me_2\|v.$$

Consider a vector $w \in \mathbb{R}^2$. We have that

$$\begin{aligned}\|M^*w\|^2 &= |\langle M^*w, e_1\rangle|^2 + |\langle M^*w, e_2\rangle|^2 \\ &= |\langle w, Me_1\rangle|^2 + |\langle w, Me_2\rangle|^2 \\ &= \left(\|Me_1\|^2 + \|Me_2\|^2\right) |\langle w, v\rangle|^2\end{aligned}$$

This shows $\|Me_1\|^2 + \|Me_2\|^2 = 1$ (set $w = v$ and use $\|M^*\| = 1$) and hence $\|M^*w\| = |\langle w, v\rangle|$ for all $w \in \mathbb{R}^2$, from which the second assertion follows.

If $M_n/\|M_n\|$ does not converge, we can perform the steps above for each convergent subsequence and get the same limits for all such subsequences. This gives the claims for the entire sequence. □

Our final goal is to prove that the norm growth must actually be exponentially fast. We first prove the following lemma, which will be helpful in this regard.

Lemma 4.9. *Let T be a measure preserving transformation on a probability space $(\Omega, d\mu)$. If $f \in L^1(\Omega, d\mu)$ is such that*

$$\lim_{n\to\infty} \sum_{m=0}^{n-1} f(T^m\omega) = \infty$$

for μ-almost every ω, then $\mathbb{E}(f) > 0$.

Proof. The Birkhoff Theorem shows that we may define a function $\tilde f$ for μ-almost every ω by

$$\tilde f(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} f(T^m\omega).$$

By assumption on f, $\tilde f \geq 0$. Assume that $\mathbb{E}(f) = 0$. Then, again by Birkhoff, $\tilde f(\omega) = 0$ for $\omega \in \Omega_0$, where Ω_0 is T-invariant and $\mu(\Omega_0) = 1$.

Write

$$s_n(\omega) = \sum_{m=0}^{n-1} f(T^m\omega).$$

For $\varepsilon > 0$, let

$$A_\varepsilon = \{\omega \in \Omega_0 : s_n(\omega) \geq \varepsilon \text{ for every } n \geq 1\}$$

and

$$B_\varepsilon = \bigcup_{k \geq 0} T^{-k}(A_\varepsilon).$$

For $\omega \in B_\varepsilon$, denote by $k(\omega) \geq 0$ the smallest integer with $T^{k(\omega)}\omega \in A_\varepsilon$. Then, for $n \geq k(\omega)$,

$$s_n(\omega) = s_{k(\omega)}(\omega) + s_{n-k(\omega)}(T^{k(\omega)}\omega) \geq s_{k(\omega)}(\omega) + \sum_{m=k(\omega)}^{n-1} \varepsilon \chi_{A_\varepsilon}(T^m \omega).$$

Dividing by n and taking n to infinity, we get

$$0 = \tilde{f}(\omega) \geq \varepsilon \tilde{\chi}_{A_\varepsilon}(\omega),$$

where $\tilde{\chi}_{A_\varepsilon}$ is defined through Birkhoff as before. Then,

$$\mu(A_\varepsilon) = \mathbb{E}(\tilde{\chi}_{A_\varepsilon}) = \mathbb{E}(\tilde{\chi}_{A_\varepsilon} \chi_{B_\varepsilon}) = 0.$$

Since T is measure-preserving, we get $\mu(B_\varepsilon) = 0$, which also implies

$$\mu\left(\bigcup_{\varepsilon > 0} B_\varepsilon\right) = 0.$$

This is a contradiction since $s_n(\omega) \to \infty$ implies $\omega \in \bigcup_{\varepsilon>0} B_\varepsilon$. $\square$

Proof of Theorem 4.2. Consider the function $f : \mathrm{SL}(2,\mathbb{R})^{\mathbb{Z}_+} \times \mathbb{P}^1 \to \mathrm{SL}(2,\mathbb{R})^{\mathbb{Z}_+} \times \mathbb{P}^1$, $f(\omega, v) = \log \frac{\|T_1(\omega)v\|}{\|v\|}$; compare (4.2). We saw above that $\gamma = \mathbb{E}(f)$; see (4.3). Therefore, our goal is to show that $\mathbb{E}(f) > 0$.

Note that ν^* satisfies the assumptions (i) and (ii) if ν does (since $Mv = w \Rightarrow M^*(w^\perp) = v^\perp$). Thus, by Lemmas 4.7 and 4.8, we have that

$$\sum_{m=0}^{n-1} f(\tilde{\sigma}^m(\omega, v)) = \log \frac{\|M_n^*(\omega)v\|}{\|v\|} \to \infty$$

for almost every ω and every $v \in \mathbb{P}^1 \setminus \{v_\omega^\perp\}$. In particular, this divergence holds $\nu^{\mathbb{Z}_+} \times m$-almost surely. Thus, it follows from Lemma 4.9 that $\mathbb{E}(f) > 0$, which concludes the proof. $\square$

Let us now apply Fürstenberg's Theorem to the Anderson model. Recall that it is given by $\Omega = R^{\mathbb{Z}}$, $R \subseteq \mathbb{R}$ a compact interval, T the standard shift, $\mu = \tilde{\nu}^{\mathbb{Z}}$, where $\tilde{\nu}$ is a probability measure on R so that $\operatorname{supp} d\tilde{\nu}$ has cardinality ≥ 2, and $f : \Omega \to \mathbb{R}$ is given by $f(\omega) = \omega(0)$. For every $E \in \mathbb{R}$, the measure $\tilde{\nu}$ induces a measure ν on $\mathrm{SL}(2,\mathbb{R})$ via

$$v \mapsto \begin{pmatrix} E - v & -1 \\ 1 & 0 \end{pmatrix}.$$

The definitions are such that the Lyapunov exponent associated with this ν at the beginning of this section is equal to $\gamma(E)$ defined earlier.

Theorem 4.10. *In the Anderson model, we have $\gamma(E) > 0$ for every $E \in \mathbb{R}$.*

PROOF. Let us check that Fürstenberg's Theorem applies. Fix any $E \in \mathbb{R}$. Since $\operatorname{supp} \tilde{\nu}$ has cardinality ≥ 2, $\operatorname{supp} \nu$ has cardinality ≥ 2, and hence G_ν contains at least two distinct elements of the form

$$M_x = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix},$$

for example, M_a and M_b with $a \neq b$. Note that

$$M^{(1)} = M_a M_b^{-1} = \begin{pmatrix} 1 & a-b \\ 0 & 1 \end{pmatrix} \in G_\nu.$$

Taking powers of the matrix $M^{(1)}$, we see that G_ν is not compact.

Consider the equivalence class of $e_1 = (1,0)^T$ in $\mathbb{P}^1$. Then $M^{(1)} e_1 = e_1$ and for every $v \in \mathbb{P}^1$, $(M^{(1)})^n v$ converges to e_1. Thus, if there is a finite invariant set of directions L, it must be equal to $\{e_1\}$. However,

$$M^{(2)} = M_a^{-1} M_b = \begin{pmatrix} 1 & 0 \\ a-b & 1 \end{pmatrix} \in G_\nu$$

and $M^{(2)} e_1 \neq e_1$; contradiction. Thus, the conditions (i) and (ii) of Theorem 4.2 hold and, consequently, $\gamma(E) > 0$. $\square$

CHAPTER 5

Random Potentials

1. Spectral Averaging and Localization

In this section we prove two localization results that are based on $|\mathcal{Z}| = 0$, which may be obtained through Kotani theory as we saw above, and spectral averaging. The first result concerns half-line restrictions of general ergodic Schrödinger operators for which one can prove $|\mathcal{Z}| = 0$. We have seen a number of examples already. The averaging in this case is with respect to the boundary condition at the origin. The second result concerns the Anderson model on the line. Such models are clearly non-deterministic, from which $|\mathcal{Z}| = 0$ follows by Corollary 3.26. The averaging is then with respect to a variation of the potential at one site. If the single-site distribution has an absolutely continuous component, this will imply localization. Recall that (spectral) localization means pure point spectrum with exponentially decaying eigenfunctions for almost every realization of the potentials, that is, for almost every ω from the underlying probability space. In all cases, where this two-step procedure applies, the averaging will force spectral measures to be concentrated on the set of energies, for which the Lyapunov exponent is positive and the Ruelle-Osceledec Theorem applies. That is, the exponential decay rate of the eigenfunctions is given by the (positive) Lyapunov exponent.

Let us consider the first scenario. We are given $(\Omega, d\mu, T, f)$ and we assume that the associated operators are such that

$$\mathrm{Leb}(\mathcal{Z}) = 0. \tag{5.1}$$

Consider the operators $H^+_{\omega,\theta}$ on $\ell^2(\mathbb{Z}_+)$ given by

$$[H^+_{\omega,\theta}\psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n)$$

and the boundary condition

$$\psi(0)\cos\theta + \psi(1)\sin\theta = 0.$$

Recall the a variation of the boundary condition can be phrased in terms of a modification of the potential at $n = 1$ and imposing a Dirichlet boundary condition. Thus, we can study the dependence of spectral measures on the boundary condition using the theory of rank one perturbation from Section 4.

Theorem 5.1. *Assume* $\mathrm{Leb}(\mathcal{Z}) = 0$. *Then, for almost every pair* (ω, θ), *the operator* $H^+_{\omega,\theta}$ *is spectrally localized.*

Remark. Given our earlier results, this theorem applies to a variety of cases, particularly those that are (measurably or topologically) non-deterministic. For example, we get localization for the half-line operators associated with an Anderson model or the doubling map.

PROOF. By assumption and the Osceledec-Ruelle Theorem we know that for almost every energy $E \in \mathbb{R}$ and for almost every $\omega \in \Omega$, we have $\gamma(E) > 0$ and there is a solution of the usual difference equation that decays like $e^{-\gamma(E)n}$ at infinity. Thus, by Fubini, we have that

$$\mathrm{Leb}(\mathbb{R} \setminus \mathcal{E}_\omega) = 0$$

for almost every ω, where

$$\mathcal{E}_\omega = \{E \in \mathbb{R} : \gamma(E) > 0 \text{ and there is a solution } u \text{ such that } |u(n)| \sim e^{-\gamma(E)n}\}.$$

For this full measure set of ω's, by spectral averaging, the spectral measure of the operator $H^+_{\omega,\theta}$ assigns zero weight to $\mathbb{R} \setminus \mathcal{E}_\omega$ for Lebesgue almost every θ. Thus, for almost every (ω, θ), the spectral measure of $H^+_{\omega,\theta}$ is supported by $\mathcal{E}_\omega$. By subordinacy theory, the spectrum must therefore be pure point with eigenfunctions decaying exponentially at the Lyapunov rate. □

Let us turn to the second scenario. We consider the Anderson model on the line and assume that the single-site distribution has an absolutely continuous component. That is, $\Omega = R^{\mathbb{Z}}$, $R \subseteq \mathbb{R}$ a compact interval, T the standard shift, $\mu = \nu^{\mathbb{Z}}$, where ν is a probability measure on R with $\nu_{\mathrm{ac}} \neq 0$, and $f : \Omega \to \mathbb{R}$ is given by $f(\omega) = \omega(0)$.

THEOREM 5.2. *For the Anderson model on the line with a single-site distribution that has an absolutely continuous component, we have that H_ω is spectrally localized for almost every $\omega \in \Omega$.*

PROOF. As above we have that, by assumption, the Osceledec-Ruelle Theorem, and Fubini,

$$\mathrm{Leb}(\mathbb{R} \setminus \mathcal{E}_\omega) = 0 \tag{5.2}$$

for almost every ω, where

$$\mathcal{E}_\omega = \{E \in \mathbb{R} : \gamma(E) > 0, \exists \text{ solutions } u_\pm \text{ with } |u_\pm(n)| \sim e^{-\gamma(E)|n|} \text{ as } n \to \pm\infty\}.$$

Note that the sets $\mathcal{E}_\omega$ are invariant with respect to a modification of V_ω on a finite set! We will perform such a modification, within the family $\{V_\omega\}$, on the set $\{0, 1\}$ because the pair $\{\delta_0, \delta_1\}$ is cyclic for each operator H_ω.

Denote the set of ω's for which (5.2) holds by Ω_0. We know that

$$\mu(\Omega_0) = 1. \tag{5.3}$$

For $\omega \in \Omega_0$, consider the operators

$$H_{\omega,\lambda_0,\lambda_1} = H_\omega + \lambda_0 \langle \delta_0, \cdot \rangle \delta_0 + \lambda_1 \langle \delta_1, \cdot \rangle \delta_1,$$

where $\lambda_0, \lambda_1 \in \mathbb{R}$. For every fixed λ_0, it follows from Theorem 1.16 and (5.2) that the spectral measure of the pair $(H_{\omega,\lambda_0,\lambda_1}, \delta_1)$ gives zero weight to the set $\mathbb{R} \setminus \mathcal{E}_\omega$ for Lebesgue almost every $\lambda_1 \in \mathbb{R}$. Similarly, for every fixed λ_1, the spectral measure of the pair $(H_{\omega,\lambda_0,\lambda_1}, \delta_0)$ gives zero weight to the set $\mathbb{R} \setminus \mathcal{E}_\omega$ for Lebesgue almost every $\lambda_0 \in \mathbb{R}$. As a consequence, we find that for Lebesgue almost every $(\lambda_0, \lambda_1) \in \mathbb{R}^2$, the universal spectral measure of $H_{\omega,\lambda_0,\lambda_1}$ (the sum of the spectral measures of δ_0 and δ_1) gives zero weight to the set $\mathbb{R} \setminus \mathcal{E}_\omega$. Write G_ω for this set of "good" pairs (λ_0, λ_1), so that

$$\mathrm{Leb}(\mathbb{R}^2 \setminus G_\omega) = 0. \tag{5.4}$$

Let

$$\Omega_1 = \{\omega + \lambda_0\delta_0 + \lambda_1\delta_1 : \omega \in \Omega_0,\ (\lambda_0, \lambda_1) \in G_\omega\}.$$

Since $\nu_{\mathrm{ac}} \neq 0$, it follows that from (5.3) and (5.4) that

$$\mu(\Omega_1) > 0.$$

Thus, by assumption on ν, with positive $\nu \times \nu$ probability, it follows from (5.2) that the whole-line spectral measure (corresponding to the sum of the δ_0 and δ_1 spectral measures) assigns no weight to $\mathbb{R} \setminus \mathcal{E}_\omega$ and hence, with positive μ probability, the operator H_ω is spectrally localized by subordinacy theory.

Since localization is a shift-invariant event, the operator H_ω must in fact be spectrally localized for μ-almost every ω. $\square$

CHAPTER 6

Periodic Potentials

1. Floquet Theory

In this section we discuss the special case of periodic potentials. Assume that $V : \mathbb{Z} \to \mathbb{R}$ obeys $V_p = V$. The associated hull Ω consists of the (finitely many) translates of V and the Haar measure is given by $\mu(\{V_j\}) = 1/p$ for every $j \in \{1, \ldots, p\}$. Since the Schrödinger operators whose potentials belong to the hull of V are unitarily equivalent, we will focus on the single operator $H = \Delta + V$ in what follows.

For this operator, the spectrum and the spectral type are easily determined with the help of the discriminant of the monodromy matrix. Let us define these quantities and show how the spectral analysis is carried out with their help.

The *monodromy matrix* is simply the transfer matrix associated with V over one period, $M_z(p)$, and the *discriminant* is its trace, $D(z) = \mathrm{Tr} M_z(p)$.

Theorem 6.1. (a) *The discriminant is a monic real polynomial of degree p.*

(b) *The Lyapunov exponent is given by*

$$\gamma(z) = \frac{1}{p} \log \left| \frac{D(z)}{2} + \sqrt{\frac{D(z)^2}{4} - 1} \right| ,$$

where the branch of the square-root is taken that maximizes the right-hand side.

(c) *If $D \in [-2, 2]$, then all solutions of $D(z) = D$ are real. If $D \in (-2, 2)$, then all roots of $D(z) = D$ are simple.*

(d) *The spectrum of H is given by*

$$\sigma(H) = \{z : D(z) \in [-2, 2]\} = \mathcal{Z}. \tag{6.1}$$

It consists of p compact intervals, $B_1, \ldots, B_p$, called bands, which are obtained by taking the closure of the p mutually disjoint open intervals whose union is $D^{-1}((-2, 2))$. Thus, there are $m \le p - 1$ bounded open intervals that separate bands, called open gaps, and $p - 1 - m$ points, where two bands overlap, called closed gaps.

(e) *(m-functions)*

(f) *The operator H has purely absolutely continuous spectrum.*

Proof. (a) This follows immediately from the definition.

(b) Since the monodromy matrix has determinant one, it has eigenvalues w, w^{-1} with $|w| \ge 1$. We have

$$D(z) = w + w^{-1}. \tag{6.2}$$

Suppose for the moment that $M_z(p)$ can be diagonalized:

$$U^{-1} M_z(p) U = \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix}.$$

Thus, we see that

$$U^{-1}M_z(np)U = \begin{pmatrix} w^n & 0 \\ 0 & w^{-n} \end{pmatrix}, \tag{6.3}$$

from which we find

$$\gamma(z) = \lim_{n\to\infty} \frac{1}{n} \log \|M_z(n)\| = \lim_{n\to\infty} \frac{1}{np} \log \|M_z(np)\| = \frac{1}{p} \log |w|.$$

When the monodromy matrix cannot be diagonalized, then $D(z) = \pm 2$ and there is a Jordan anomaly. In this case, we find $w = w^{-1} = \pm 1$ and

$$U^{-1}M_z(p)U = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}.$$

We obtain

$$U^{-1}M_z(np)U = \begin{pmatrix} (\pm 1)^n & (\pm 1)^{n-1}n \\ 0 & (\pm 1)^n \end{pmatrix},$$

and then

$$\gamma(z) = \lim_{n\to\infty} \frac{1}{n} \log \|M_z(n)\| = \lim_{n\to\infty} \frac{1}{np} \log \|M_z(np)\| = 0 = \frac{1}{p} \log |w|.$$

(c) Consider the map $w \mapsto w + w^{-1}$. It maps the unit circle $\partial\mathbb{D} = \{w \in \mathbb{C} : |w| = 1\}$ to $[-2, 2]$ and it maps both $\{w : 0 < |w| < 1\}$ and $\{w : |w| > 1\}$ to $\mathbb{C} \setminus [-2, 2]$. Explicitly, if $|w| = 1$, say $w = e^{i\varphi}$, then $w + w^{-1} = 2\cos(\varphi) \in [-2, 2]$. If w is real, then $|w + w^{-1}| \geq 2$ with equality if and only if $w = \pm 1$ (easy calculus exercise). Finally, if $|w| \neq 1$ and $w \notin \mathbb{R}$, it is easy to check that $w + w^{-1} \notin \mathbb{R}$ and in particular $\notin [-2, 2]$.

Now suppose $D(z) = D$. We have

$$D(\tilde{z}) = D + c_k(\tilde{z} - z)^k + O((\tilde{z} - z)^{k+1}) \tag{6.4}$$

for some $k \geq 1$ and $c_k \neq 0$. If $D \in [-2, 2]$, then w (in (6.2)) has modulus one, which implies $z \in \Sigma \subseteq \mathbb{R}$ since all transfer matrices are linearly bounded as we saw in part (b). Combining this observation with (6.4), we see that $k \leq 2$ since otherwise there would be a non-real $\tilde{z}$ near z with $D(\tilde{z}) \in [-2, 2]$; contradiction. If $D \in (-2, 2)$, we must even have $k = 1$. Otherwise we would be able to find a nearby non-real $\tilde{z}$ for which $D(\tilde{z}) \in (-2, 2)$, which is again a contradiction. Note also that if $D = \pm 2$, then $k = 2$ if and only if all real $\tilde{z}$ near z have $D(\tilde{z}) \in [-2, 2]$.

(d) As we saw above, when $D(z) \notin [-2, 2]$, then $|w| > 1$ and (6.3) shows that there are solutions u_+, u_- of $Hu = zu$ so that $u_\pm$ decays exponentially at $\pm\infty$ and increases exponentially at $\mp\infty$ and hence $\gamma(z) > 0$. Using these solutions, one can explicitly write down the kernel $\langle \delta_n, (H - z)^{-1}\delta_m \rangle$, which shows that $z \notin \sigma(H)$.

When $D(z) \in (-2, 2)$, then w and w^{-1} both have modulus one and are not equal. Thus, (6.3) shows that all solutions of $Hu = zu$ are bounded, which shows both $z \in \Sigma$ and $\gamma(z)$. The same argument applies when $D(z) = \pm 2$ and there is no Jordan anomaly.

Finally, when $D(z) = \pm 2$ and there is a Jordan anomaly, then $Hu = zu$ has a bounded solution and a linearly growing solution, which again shows $z \in \sigma(H)$ and $\gamma(z) = 0$.

By part (c), D runs from ± 2 to ∓ 2 in a strictly monotone way and hence, since it is a polynomials of degree p, $D^{-1}((-2, 2))$ consists of p disjoint open intervals.

(e) Recall from (1.21) that the m-functions $m^\pm$ are given by

$$m^+(z) = -\frac{u_z^+(1)}{u_z^+(0)} \quad \text{and} \quad m^-(z) = -\frac{u_z^-(0)}{u_z^-(1)},$$

where $z \in \mathbb{C}_+$ and $u_z^\pm$ is a solution of $Hu = zu$ which is square-summable at $\pm\infty$.

(f) Since all solutions are bounded for z in the interior of a band, subordinacy theory implies that the spectrum is purely absolutely continuous on the interior of any band. The endpoints of the finitely many bands form a finite set and hence cannot support any singular continuous spectrum. Since at these endpoints, there are no square-summable solutions by what we saw above, we conclude that the spectrum must be purely absolutely continuous. $\square$

For some purposes, the point of view discussed next will be useful. For $k \in \mathbb{R}$ and $l \in \mathbb{Z}$, define

$$A_l(k) = \begin{pmatrix} V(l+1) & 1 & & & e^{-ikp} \\ 1 & V(l+2) & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ e^{ikp} & & & 1 & V(l+p) \end{pmatrix}$$

and

$$A_l^- = \begin{pmatrix} V(l+2) & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & V(l+p) \end{pmatrix}$$

THEOREM 6.2. (a) *For every $k \in \mathbb{R}$, the p eigenvalues of $A_l(k)$ are independent of $l \in \mathbb{Z}$ and z is an eigenvalue of $A_l(k)$ if and only if $D(z) = 2\cos(kp)$. Consequently, we have*

$$\det(z - A_l(k)) = D(z) - 2\cos(kp) \tag{6.5}$$

and

$$\sigma(H) = \bigcup_{k \in \mathbb{R}} \sigma(A_l(k)) = \bigcup_{k \in [0, \frac{\pi}{p}]} \sigma(A_l(k)). \tag{6.6}$$

(b) *For every l, the $p-1$ eigenvalues of A_l^- are simple. For every l and k, the eigenvalues of $A_l(k)$ are at most doubly degenerate and the eigenvalues of A_l^- separate the eigenvalues of $A_l(k)$, that is,*

$$z_1(k) \le z_{l,1}^- \le z_2(k) \le z_{l,2}^- \le \cdots \le z_{l,p-1}^- \le z_p(k).$$

In particular, if a given eigenvalue of $A_l(k)$ is degenerate, then it is also an eigenvalue of A_l^-. Consequently, the eigenvalues of A_l^- lie between the open bands of $\sigma(H)$. That is, they either are band edges or belong to gaps of $\sigma(H)$.

PROOF. (a) By our results from the previous section, $z \in \sigma(A_l(k))$ and $D(z) = 2\cos(kp)$ are both equivalent to the existence of a solution to $Hu = zu$ obeying $u(n+p) = e^{ikp}u(n)$ for every $n \in \mathbb{Z}$. It follows that $\sigma(A_l(k))$ is independent of l. Moreover, we see that the two sides of (6.5) have the same roots and hence they are equal since they are both monic polynomials of degree p. The identity (6.6) is then just a reformulation of the identity $\sigma(H) = \{z : D(z) \in [-2, 2]\}$ established in (6.1).

(b) Assume that A_l^- has a degenerate eigenvalue. Choose two linearly independent eigenvectors. Note that they solve a "Dirichlet problem" and that a non-trivial linear combination starts off with a zero which, together with the Dirichlet condition, then generates the zero vector; contradiction. Thus, the eigenvalues of A_l^- are simple.

Next, define

$$C(z) = \frac{\det(z - A_l^-)}{\det(z - A_l(k))}. \tag{6.7}$$

Denote the eigenvalues of $A_l(k)$ (listed in increasing order and with repetitions accounting for multiplicity) by $z_1(k) \le \cdots \le z_p(k)$ and denote by $v_1(k), \dots, v_p(k)$ associated eigenvectors. Then, by Cramer's Rule and the Spectral Theorem,

$$\begin{aligned} C(z) &= \langle \delta_1, (z - A_l(k))^{-1} \delta_1 \rangle \\ &= \langle \delta_1, \sum_{j=1}^p (z - z_j(k))^{-1} \langle v_j, \delta_1 \rangle v_j \rangle \\ &= \sum_{j=1}^p \frac{|\langle \delta_1, v_j \rangle|^2}{z - z_j(k)}. \end{aligned} \tag{6.8}$$

This shows that all poles of $C(z)$ have order one. Given the simplicity of the eigenvalues of A_l^- and the formula (6.7), it therefore follows that the eigenvalues of $A_l(k)$ are at most doubly degenerate, and that if a given eigenvalue of $A_l(k)$ is degenerate, then it is also an eigenvalue of A_l^-.

It remains to show that for any consecutive pair of eigenvalues of $A_l(k)$, $z_j(k) \le z_{j+1}(k)$, we can find an eigenvalue $z_{l,j}^-$ of A_l^- with $z_j(k) \le z_{l,j}^- \le z_{j+1}(k)$. There are three cases:

Case 1: We have $z_j(k) = z_{j+1}(k)$. Then we saw above that their common value is an eigenvalue of A_l^-.

Case 2: We have $z_j(k) \ne z_{j+1}(k)$ and $\langle \delta_1, v_n \rangle = 0$ for $n \in \{j, j+1\}$. Then, the same eigenvalue $E_n(k)$ and "the same" eigenvector (modulo a truncation of the initial zero) v_n can serve for A_l^-.

Case 3: We have $z_j(k) \ne z_{j+1}(k)$, $\langle \delta_1, v_j \rangle \ne 0$, and $\langle \delta_1, v_{j+1} \rangle \ne 0$. Then (6.8) shows that $C(z)$ has simple poles at $z_j(k)$ and $z_{j+1}(k)$. There must be a zero between them, which by (6.7) is then an eigenvalue of A_l^-. □

For future reference, let us collect the statements on the behavior of solutions which we established along the way in the proofs above.

Theorem 6.3. *Consider the two-dimensional solution space of $Hu = zu$. There are four possible types of behavior:*

(i) *If z belongs to the resolvent set $\mathbb{C} \setminus \sigma(H)$, then there are linearly independent solutions u_+, u_- so that $u_\pm$ decays exponentially at $\pm\infty$ and grows exponentially at $\mp\infty$.*
(ii) *If z belongs to the interior of a band, then all solutions u are bounded.*
(iii) *If z is an endpoint of a band and borders an open gap, then there are linearly independent solutions u_b, u_l so that u_b is bounded and u_l is unbounded but linearly bounded.*

(iv) *If z is a closed gap, then all solutions u are bounded.*

PROOF. The statements (i) and (ii) were shown explicitly in the proof of Theorem 6.1 above. Consider an endpoint z of a band with $D(z) = 2$ (resp., $D(z) = -2$). The matrix $M_z(p)$ is diagonalizable if and only if the eigenvalue z of $A_l(0)$ (resp., $A_l(\frac{\pi}{p})$) is doubly degenerate, which in turn is equivalent to z being a closed gap by Theorem 6.2. This observation proves statements (iii) and (iv). □

2. Quantitative Estimates

In this section we aim for explicit estimates that hold in the general setting. Much of the material is taken from a series of papers of Last from the early 1990's [**25, 26, 27**]. As before we consider a real p-periodic potential V. The spectrum of $H = \Delta + V$ consists of p bands $B_1, \dots, B_p$, which are listed in increasing order and two consecutive bands can overlap in at most one point. Each of these bands is the image of a bijective map

$$\left[0, \frac{\pi}{p}\right] \ni k \mapsto z_j(k) \in B_j.$$

The p bands are separated by $p-1$ gaps $G_1, \dots, G_{p-1}$ which can be open or closed in the sense introduced above. That is, they are either given by a bounded open interval or by the empty set. Denote their unions by $B = B_1 \cup \dots \cup B_p$ $(= \sigma(H))$ and $G = G_1 \cup \dots \cup G_{p-1}$. Finally, we write $V_{\max} \max_n V(n)$, $V_{\min} = \min_n V(n)$, and $\mathrm{Var}_V = V_{\max} - V_{\min}$.

THEOREM 6.4. (a) *We have*

$$\mathrm{Leb}(G) \geq \mathrm{Var}_V \tag{6.9}$$

and

$$\mathrm{Leb}(\sigma(H)) \leq 4. \tag{6.10}$$

(b) *For every band B_j of $\sigma(H)$, we have*

$$\mathrm{Leb}(B_j) \leq \frac{4e}{D'\big(z_j^{(0)}\big)}, \tag{6.11}$$

where $z_j^{(0)}$ is the unique zero of the discriminant D in the band B_j.

PROOF. (a) By Theorem 6.2, we have for every pair $(l, m) \in \mathbb{Z}^2$

$$\begin{aligned}
\mathrm{Leb}(G) &= \sum_{j=1}^{p-1} \mathrm{Leb}(G_j) \\
&\geq \sum_{j=1}^{p-1} \left| z_{l,j}^- - z_{m,j}^- \right| \\
&\geq \Big| \sum_{j=1}^{p-1} z_{l,j}^- - \sum_{j=1}^{p-1} z_{m,j}^- \Big| \\
&= \big| \mathrm{Tr}(A_l^-) - \mathrm{Tr}(A_m^-) \big| \\
&= |V(m+1) - V(l+1)|.
\end{aligned}$$

Since this holds for every pair (l, m), the estimate (6.9) follows.

We have $\sigma(H) \subseteq [-2+V_{\min}, 2+V_{\max}]$ and hence

$$\mathrm{Leb}(\sigma(H)) \leq 4 + \mathrm{Var}_V - \mathrm{Leb}(G) \leq 4,$$

which is (6.10).

(b) Consider first the case where $2 \leq j \leq p-1$ and D is increasing on B_j. Define

$$L(z) = \frac{d \log D}{dz}(z) = \frac{D'(z)}{D(z)}.$$

Since

$$D(z) = \prod_{k=1}^{p} (z - z_k^{(0)}),$$

we have

$$L(z) = \sum_{k=1}^{p} \frac{1}{z - z_k^{(0)}}$$

and

$$L'(z) = -\sum_{k=1}^{p} \frac{1}{\left(z - z_k^{(0)}\right)^2}.$$

Dropping terms, we see that

$$L'(z) < -\frac{1}{\left(z - z_j^{(0)}\right)^2}.$$

Let us write $B_j = [a, b]$ and denote the first zero of D' to the right of B_j by c. Note that D has a local maximum at c and $L(c) = 0$. Thus, for every $z \in (z_j^{(0)}, c)$, we have

$$\begin{aligned} L(z) &= -\int_z^c L'(t)\,dt \\ &> \int_z^c \frac{dt}{\left(t - z_j^{(0)}\right)^2} \\ &= \frac{1}{z - z_j^{(0)}} - \frac{1}{c - z_j^{(0)}}. \end{aligned}$$

Now consider $z \in (z_j^{(0)}, b)$. Since $D(b) = 2$, we have

$$\begin{aligned} \log \frac{2}{D(z)} &= \log D(b) - \log D(z) \\ &= \int_z^b L(t)\,dt \\ &> \int_z^b \frac{1}{t - z_j^{(0)}} - \frac{1}{c - z_j^{(0)}}\,dt \\ &= \log\left(\frac{b - z_j^{(0)}}{z - z_j^{(0)}}\right) - \frac{b - z}{c - z_j^{(0)}} \\ &> \log\left(\frac{b - z_j^{(0)}}{z - z_j^{(0)}}\right) - 1. \end{aligned}$$

Thus,

$$\frac{2}{D(z)} > \frac{1}{e}\left(\frac{b - z_j^{(0)}}{z - z_j^{(0)}}\right),$$

or equivalently,

$$b - z_j^{(0)} < 2e\left(\frac{z - z_j^{(0)}}{D(z)}\right) = 2e\left(\frac{z - z_j^{(0)}}{D(z) - D\big(z_j^{(0)}\big)}\right).$$

Letting $z \to z_j^{(0)}$, we find that

$$b - z_j^{(0)} < \frac{2e}{D'\big(z_j^{(0)}\big)}.$$

Similar arguments work on the lower part $(a, z_j^{(0)})$ of the band B_j, yielding

$$z_j^{(0)} - a < \frac{2e}{D'\big(z_j^{(0)}\big)}.$$

Thus, we find the desired estimate

$$\operatorname{Leb}(B_j) = b - a = b - z_j^{(0)} + z_j^{(0)} - a < \frac{4e}{D'\big(z_j^{(0)}\big)}.$$

When $2 \le j \le p-1$ and D is decreasing on B_j, it is possible to prove this estimate in a similar way.

This leaves the case of the extremal bands, B_1 and B_p. Notice that the non-extremal part of an extremal band may be treated as before and that the extremal part of an extremal band is shorter than the non-extremal part since D' is strictly monotone on an extremal band! This gives the desired estimate for B_1 and B_p, concluding the proof. □

Here are estimates in the opposite direction:

THEOREM 6.5. (a) *We have*

$$\operatorname{Leb}(\sigma(H)) \ge \frac{4}{(4 + \operatorname{Var}_V)^{p-1}}. \tag{6.12}$$

(b) *In fact, for every band B_j of $\sigma(H)$, we have*

$$\operatorname{Leb}(B_j) > \frac{4}{p(4 + \operatorname{Var}_V)^{p-1}}. \tag{6.13}$$

PROOF. (a) Fix some $l \in \mathbb{Z}$. By the inequality of arithmetic and geometric means, we have

$$|\det(z - A_l^-)| = \prod_{j=1}^{p-1} |z - z_{l,j}^-| \le \left(\frac{1}{p-1}\sum_{j=1}^{p-1} |z - z_{l,j}^-|\right)^{p-1}. \tag{6.14}$$

Since for each j,

$$z_{l,j}^- \in [\min \sigma(H), \max \sigma(H)] \subseteq [-2 + V_{\min}, 2 + V_{\max}],$$

we have for every $z \in [\min \sigma(H), \max \sigma(H)]$,

$$\begin{aligned}\sum_{j=1}^{p-1} |z - z_{l,j}^-| &\leq \max \left\{ \sum_{j=1}^{p-1} |\min \sigma(H) - z_{l,j}^-|, \sum_{j=1}^{p-1} |\max \sigma(H) - z_{l,j}^-| \right\} \\ &= \max \left\{ \sum_{j=1}^{p-1} (z_{l,j}^- - \min \sigma(H)), \sum_{j=1}^{p-1} (\max \sigma(H) - z_{l,j}^-) \right\} \\ &= \max\{\mathrm{Tr}(A_l^-) - (p-1) \min \sigma(H), (p-1) \max \sigma(H) - \mathrm{Tr}(A_l^-)\} \\ &\leq (p-1)(4 + \mathrm{Var}_V). \end{aligned} \tag{6.15}$$

Combining (6.14) and (6.15), we obtain

$$|\det(z - A_l^-)| \leq (4 + \mathrm{Var}_V)^{p-1}. \tag{6.16}$$

For $\lambda \in \mathbb{R}$, let

$$B(\lambda) = \begin{pmatrix} V(l+1) - \lambda & 1 & & & e^{-\frac{i\pi}{2}} \\ 1 & V(l+2) & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ e^{\frac{i\pi}{2}} & & & 1 & V(l+p) \end{pmatrix}.$$

Then, by (6.5),

$$\begin{aligned}\det(z - B(\lambda)) &= \det(z - A_l(\tfrac{\pi}{2p})) - \lambda \det(z - A_l^-) \\ &= D(z) - \lambda \det(z - A_l^-)\end{aligned} \tag{6.17}$$

If we choose

$$\lambda = \frac{2}{(4 + \mathrm{Var}_V)^{p-1}},$$

we see from (6.16) that

$$|\lambda \det(z - A_l^-)| \leq 2$$

for every $z \in [\min \sigma(H), \max \sigma(H)]$. But D runs between -2 and 2 on each band of $\sigma(H)$, so that $\sigma(B(\lambda)) \subset \sigma(H)$ and $B(\lambda)$ has exactly one eigenvalue inside each band of $\sigma(H)$ (it has at least one in each band by the argument just given and hence exactly one since there is only a total of p eigenvalues). The same statement holds for $B(-\lambda)$ by the same argument and hence

$$\mathrm{Leb}(\sigma(H)) \geq |\mathrm{Tr}(B(\lambda)) - \mathrm{Tr}(B(-\lambda))| = 2\lambda = \frac{4}{(4 + \mathrm{Var}_V)^{p-1}},$$

which is (6.12).

(b) On B_j we have $D(z_j(k)) = 2\cos(kp)$ and hence we find by differentiation

$$\frac{dz_j}{dk}(k) = \frac{-2p \sin(kp)}{D'(z_j(k))}. \tag{6.18}$$

Since D' is a polynomial of degree $p-1$ which has leading order term pz^{p-1} and whose zeros separate the eigenvalues of $A_l(k)$, it follows by arguments similar to those which established (6.14) and (6.15) that

$$|D'(z)| \leq p\,(4 + \mathrm{Var}_V)^{p-1} \tag{6.19}$$

for every $z \in [\min \sigma(H), \max \sigma(H)]$. Explicitly, if we denote the zeros of D' by $\{z_j'\}_{j=1}^{p-1}$, then they all belong to $[\min \sigma(H), \max \sigma(H)] \subseteq [-2+V_{\min}, 2+V_{\max}]$ and therefore,

$$\begin{aligned} |D'(z)| &= p \prod_{j=1}^{p-1} |z - z_j'| \\ &\le p \left(\frac{1}{p-1} \sum_{j=1}^{p-1} |z - z_j'| \right)^{p-1} \\ &\le p \left(4 + \mathrm{Var}_V\right)^{p-1}. \end{aligned}$$

It follows from (6.18) and (6.19) that

$$\left| \frac{dz_j}{dk}(k) \right| \ge \frac{2\sin(kp)}{(4+\mathrm{Var}_V)^{p-1}}$$

and hence

$$\begin{aligned} \mathrm{Leb}(B_j) &= \max B_j - \min B_j \\ &= \int_{\min B_j}^{\max B_j} dz \\ &= \int_0^{\frac{\pi}{p}} \left| \frac{dz_j}{dk}(k) \right| dk \\ &\ge \int_0^{\frac{\pi}{p}} \frac{2\sin(kp)}{(4+\mathrm{Var}_V)^{p-1}} dk \\ &= \frac{4}{p(4+\mathrm{Var}_V)^{p-1}}, \end{aligned}$$

as claimed. □

LEMMA 6.6. *Suppose $j \in \{1, \dots, p\}$, $k \in (0, \frac{\pi}{p})$ and*

$$u_j^l(k) = (u_j(k, l+1), \dots, u_j(k, l+p))^T$$

is a normalized eigenvector corresponding to the eigenvalue $z_j(k)$ of $A_l(k)$. Then,

$$\frac{dz_j}{dk}(k) = 2p\Im\left(e^{-ikp} u_j(k, l+p)\overline{u_j(k, l+1)}\right).$$

PROOF. Since $k \in (0, \frac{\pi}{p})$, $z_j(k)$ is non-degenerate and hence depends on k analytically in this interval by general non-degenerate perturbation theory. Then,

$$\begin{aligned} \frac{dz_j}{dk}(k) &= \frac{d}{dk}\langle u_j^l(k), A_l(k) u_j^l(k)\rangle \\ &= \left\langle \frac{d}{dk} u_j^l(k), A_l(k) u_j^l(k) \right\rangle + \left\langle u_j^l(k), \frac{d}{dk} A_l(k) u_j^l(k) \right\rangle + \left\langle A_l(k) u_j^l(k), \frac{d}{dk} u_j^l(k) \right\rangle \\ &= z_j(k) \left\langle \frac{d}{dk} u_j^l(k), u_j^l(k) \right\rangle + \left\langle u_j^l(k), \frac{d}{dk} A_l(k) u_j^l(k) \right\rangle + z_j(k) \left\langle u_j^l(k), \frac{d}{dk} u_j^l(k) \right\rangle \\ &= \left\langle u_j^l(k), \frac{d}{dk} A_l(k) u_j^l(k) \right\rangle + z_j(k) \frac{d}{dk} \left\langle u_j^l(k), u_j^l(k) \right\rangle \\ &= \left\langle u_j^l(k), \frac{d}{dk} A_l(k) u_j^l(k) \right\rangle. \end{aligned}$$

(This is is known as the Feynman-Hellmann Theorem.)

We have

$$\frac{d}{dk}A_l(k) = \frac{d}{dk}\begin{pmatrix} V(l+1) & 1 & & & e^{-ikp} \\ 1 & V(l+2) & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ e^{ikp} & & & 1 & V(l+p) \end{pmatrix}$$

$$= ip \begin{pmatrix} 0 & 0 & & & -e^{-ikp} \\ 0 & 0 & 0 & & \\ & 0 & \ddots & \ddots & \\ & & \ddots & \ddots & 0 \\ e^{ikp} & & & 0 & 0 \end{pmatrix}.$$

Thus,

$$\begin{aligned} \frac{dz_j}{dk}(k) &= \left\langle u_j^l(k), \frac{d}{dk}A_l(k)u_j^l(k) \right\rangle \\ &= ip\left(-e^{-ikp}u_j(k,l+p)\overline{u_j(k,l+1)} + e^{ikp}u_j(k,l+1)\overline{u_j(k,l+p)}\right) \\ &= 2p\Im\left(e^{-ikp}u_j(k,l+p)\overline{u_j(k,l+1)}\right), \end{aligned}$$

as claimed. □

We can now prove an explicit upper bound for the norm of the monodromy matrix.

THEOREM 6.7. *Suppose $j \in \{1, \ldots, p\}$ and $k \in (0, \frac{\pi}{p})$. Then,*

$$\|M_{z_j(k)}(p)\| \le 2p \left| \frac{dz_j}{dk}(k) \right|^{-1}.$$

PROOF. As we saw earlier, the (real) matrix $M_{z_j(k)}(p)$ has the two distinct eigenvalues $e^{\pm ikp}$. The corresponding normalized eigenvectors $v^\pm$ are uniquely defined up to a phase and can in fact be chosen to be complex conjugates of each other. Thus, we may write

$$v^+ = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad v^- = \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}, \quad |v_1|^2 + |v_2|^2 = 1. \tag{6.20}$$

For $v \in \mathbb{C}^2$, choose $a, b \in \mathbb{C}$ with $v = av^+ + bv^-$. Then,

$$\begin{aligned} \frac{\|M_{z_j(k)}(p)v\|^2}{\|v\|^2} &= \frac{\|ae^{ikp}v^+ + be^{-ikp}v^-\|^2}{\|av^+ + bv^-\|^2} \\ &\le \frac{(|a|+|b|)^2}{|a|^2 - 2|a||b||\langle v^+, v^-\rangle| + |b|^2} \\ &= \frac{1}{\frac{|a|^2+|b|^2}{(|a|+|b|)^2} - \frac{2|a||b|}{(|a|+|b|)^2}|\langle v^+, v^-\rangle|} \\ &= \frac{2}{\frac{2|a|^2+2|b|^2}{(|a|+|b|)^2} - \frac{4|a||b|}{(|a|+|b|)^2}|\langle v^+, v^-\rangle|} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2}{1-|\langle v^+,v^-\rangle|} \\
&\leq \frac{2(1+|\langle v^+,v^-\rangle|)}{1-|\langle v^+,v^-\rangle|^2} \\
&\leq \frac{4}{1-|\langle v^+,v^-\rangle|^2},
\end{aligned}$$

where we used the obvious inequalities $2|a||b| \leq |a|^2 + |b|^2$ and $4|a||b| \leq (|a| + |b|)^2$ in the fifth step.

Denote the phase difference between v_1 and v_2 by γ, that is, $v_1 \bar{v}_2 = |v_1||v_2|e^{i\gamma}$. Then, recalling (6.20),

$$\begin{aligned}
1 - |\langle v^+, v^-\rangle|^2 &= 1 - |v_1^2 + v_2^2|^2 \\
&= (|v_1|^2 + |v_2|^2)^2 - |v_1^2 + v_2^2|^2 \\
&= 2|v_1|^2|v_2|^2 - 2\Re\left(v_1^2\overline{v_2^2}\right) \\
&= 2|v_1|^2|v_2|^2(1 - \cos 2\gamma) \\
&= 4|v_1|^2|v_2|^2 \sin^2\gamma \\
&= 4\left(\Im(v_1\bar{v}_2)\right)^2.
\end{aligned}$$

Combining the two observations, we find

$$\|M_{z_j(k)}(p)\| \leq \frac{1}{|\Im(v_1\bar{v}_2)|}. \tag{6.21}$$

We can extend $u(1) = v_1$, $u(2) = v_2$ to an eigenvector $(u(1), u(2), \ldots, u(p))$ of $A_0(k)$. Its norm N obeys

$$N^2 = \sum_{n=1}^{p} |u(n)|^2 \geq 1$$

because of (6.20). Since $\frac{1}{N}(u(1), u(2), \ldots, u(p))$ is a normalized eigenvector of $A_0(k)$, Lemma 6.6 gives

$$\left|\frac{dz_j}{dk}(k)\right| = \frac{2p}{N^2}\left|\Im\left(v_1\bar{v}_2\right)\right|.$$

Together with (6.21), this implies

$$\|M_{z_j(k)}(p)\| \leq \frac{2p}{N^2}\left|\frac{dz_j}{dk}(k)\right|^{-1} \leq 2p\left|\frac{dz_j}{dk}(k)\right|^{-1}$$

since $N \geq 1$, as desired. □

3. Periodic Approximations of General Ergodic Potentials

In this section we return to the general setting and consider an ergodic family of Schrödinger operators associated with (Ω, μ, T, f) as before. We will especially interested in the Lebesgue measure of the set $\mathcal{Z} = \{z : \gamma(z) = 0\}$. Recall that the almost sure absolutely spectrum, Σ_{ac}, is given by the essential closure of $\mathcal{Z}$.

Our first result is an extension of Theorem 6.4.

THEOREM 6.8 (Last 1992). $\mathrm{Leb}(\Sigma \setminus \mathcal{Z}) + \mathrm{Leb}(G) \geq \mathrm{Var}_V$.

COROLLARY 6.9 (Deift-Simon 1983). $\mathrm{Leb}(\mathcal{Z}) \leq 4$.

The next result establishes a lower bound for the Lebesgue measure of $\mathcal{Z}$. We will consider the following periodic approximations of the operators H_ω. Denote by $H_{\omega,p}$ the Schrödinger operator with p-periodic potential given by $V_{\omega,p}(n) = V_\omega(n)$ for $1 \le n \le p$ and by $D_{\omega,p}$ the associated discriminant.

THEOREM 6.10 (Last 1993). *For μ-almost every $\omega \in \Omega$, we have*

$$\mathrm{Leb}\left(\limsup_{p\to\infty} \sigma(H_{\omega,p}) \setminus \mathcal{Z}\right) = 0.$$

PROOF. For $\omega \in \Omega$, $p \ge 2$, $\varepsilon > 0$, define

$$S_{\omega,p}(\varepsilon) = \left\{ z : D_{\omega,p}(z) \in (-2,2), \ \left| \frac{dz_{\omega,p,j}}{dk}(k) \right| \ge \varepsilon \right\},$$

where $z_{\omega,p,j}$ denotes the j-th band function associated with the p-periodic potential $V_{\omega,p}$ and $j \in \{1, \dots, p\}$ and $k \in [0, \frac{\pi}{p}]$ are chosen so that the z in question is given by $z = z_{\omega,p,j}(k)$. Let

$$S_\omega = \limsup_{p\to\infty} S_{\omega,p}(p^{-2}).$$

We have

$$\mathrm{Leb}\left(\sigma(H_{\omega,p}) \setminus S_{\omega,p}(\varepsilon)\right) < \pi\varepsilon$$

since $\sigma(H_{\omega,p})$ consists of p bands and we have $0 \le k \le \frac{\pi}{p}$ on each of them. This implies that

$$\sum_{p\ge 2} \mathrm{Leb}\left(\sigma(H_{\omega,p}) \setminus S_{\omega,p}(p^{-2})\right) < \infty,$$

which in turn yields

$$\begin{aligned}
\mathrm{Leb}\left(\limsup_{p\to\infty} \sigma(H_{\omega,p}) \setminus S_{\omega,p}(p^{-2})\right) &= \mathrm{Leb}\left(\bigcap_{k\ge 1} \bigcup_{p\ge k} \sigma(H_{\omega,p}) \setminus S_{\omega,p}(p^{-2})\right) \\
&\le \inf_{k\ge 1} \sum_{p\ge k} \mathrm{Leb}\left(\sigma(H_{\omega,p}) \setminus S_{\omega,p}(p^{-2})\right) \\
&= 0.
\end{aligned}$$

This together with

$$S_\omega \subseteq \limsup_{p\to\infty} \sigma(H_{\omega,p})$$

and

$$\limsup_{p\to\infty} \sigma(H_{\omega,p}) \setminus S_\omega \subseteq \limsup_{p\to\infty} \left(\sigma(H_{\omega,p}) \setminus S_{\omega,p}(p^{-2})\right)$$

implies that S_ω and $\limsup_{p\to\infty} \sigma(H_{\omega,p})$ coincide up to sets of zero Lebesgue measure. Thus, it remains to show that

$$\lim_{p\to\infty} \frac{1}{p} \log \|M_z(p,\omega)\| = 0 \quad \text{for } \mu\text{-a.e. } \omega \in \Omega \text{ and Lebesgue a.e. } z \in S_\omega. \tag{6.22}$$

For every $\omega \in \Omega$, we have by the definition of S_ω and Theorem 6.7,

$$\|M_z(p_k,\omega)\| \le 2p_k^3$$

for $z \in S_\omega$ and a suitable sequence $p_k \to \infty$. Thus,

$$\liminf_{p\to\infty} \frac{1}{p} \log \|M_z(p,\omega)\| = 0 \quad \text{for every } \omega \in \Omega \text{ and every } z \in S_\omega.$$

On the other hand, for μ-almost every $\omega \in \Omega$ and Lebesgue almost every $z \in S_\omega$, the limit exists by results established earlier. The assertion (6.22) follows. $\square$

COROLLARY 6.11. *For μ-almost every $\omega \in \Omega$, we have*

$$\mathrm{Leb}(\mathcal{Z}) \geq \limsup_{p\to\infty} \mathrm{Leb}\left(\sigma(H_{\omega,p})\right).$$

PROOF. We have

$$\mathrm{Leb}\left(\bigcup_{p\geq 1} \sigma(H_{\omega,p})\right) < \infty$$

since the sets $\sigma(H_{\omega,p})$ are all contained in the interval $[-2-\|f\|_\infty, 2+\|f\|_\infty]$. Thus,

$$\bigcup_{p\geq k} \sigma(H_{\omega,p})$$

forms a decreasing sequence of sets of finite measure.

Therefore,

$$\begin{aligned}
\mathrm{Leb}(\mathcal{Z}) &\geq \mathrm{Leb}\left(\limsup_{p\to\infty} \sigma(H_{\omega,p})\right) \\
&= \mathrm{Leb}\left(\bigcap_{k\geq 1}\bigcup_{p\geq k} \sigma(H_{\omega,p})\right) \\
&= \lim_{k\to\infty} \mathrm{Leb}\left(\bigcup_{p\geq k} \sigma(H_{\omega,p})\right) \\
&\geq \limsup_{p\to\infty} \mathrm{Leb}\left(\sigma(H_{\omega,p})\right),
\end{aligned}$$

where we applied Theorem 6.10 in the first step and used the observation from above in the third step. $\square$

4. The Gordon Lemma and Applications

In this section we discuss a simple but far-reaching observation due to Gordon. Namely, if a potential is locally (close to) periodic for infinitely many periods, one can prove explicit estimates for the solutions to the associated difference equation which in turn may be used to exclude the existence of square-summable solutions. In other words, while Schrödinger operators with periodic potentials have purely absolutely continuous spectrum, a surprisingly weak local periodicity property of the potential already implies that the corresponding Schrödinger operator has purely continuous spectrum.

The heart of the argument is the Cayley-Hamilton Theorem, which for $\mathrm{SL}(2,\mathbb{C})$ matrices M takes the form

$$M^2 - \mathrm{Tr}M \cdot M + I = 0. \tag{6.23}$$

Recall that the transfer matrices belong to $\mathrm{SL}(2,\mathbb{C})$. We will apply (6.23) to these matrices when there are suitable local repetitions. Explicitly, the following version of Gordon's Lemma implements this.

LEMMA 6.12. *Suppose $V : \mathbb{Z} \to \mathbb{R}$ obeys $V(n+p) = V(n)$ for some $p \in \mathbb{Z}_+$ and $-p+1 \le n \le p$, $z \in \mathbb{C}$, and u solves*

$$u(n+1) + u(n-1) + V(n)u(n) = zu(n).$$

Then, we have

$$\max\left\{ \left\| \begin{pmatrix} u(-p+1) \\ u(-p) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u(p+1) \\ u(p) \end{pmatrix} \right\|, \left\| \begin{pmatrix} u(2p+1) \\ u(2p) \end{pmatrix} \right\| \right\} \ge \frac{1}{2} \left\| \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} \right\|. \tag{6.24}$$

PROOF. By assumption, we have

$$\begin{pmatrix} u(2p+1) \\ u(2p) \end{pmatrix} = M_z(2p) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} = M_z(p)^2 \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}$$

and similarly

$$\begin{pmatrix} u(p+1) \\ u(p) \end{pmatrix} = M_z(p)^2 \begin{pmatrix} u(-p+1) \\ u(-p) \end{pmatrix}.$$

Moreover, (6.23) implies

$$M_z(p)^2 - \mathrm{Tr} M_z(p) \cdot M_z(p) + I = 0.$$

Consequently, we have

$$\begin{pmatrix} u(2p+1) \\ u(2p) \end{pmatrix} - \mathrm{Tr} M_z(p) \begin{pmatrix} u(p+1) \\ u(p) \end{pmatrix} + \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{6.25}$$

and

$$\begin{pmatrix} u(p+1) \\ u(p) \end{pmatrix} - \mathrm{Tr} M_z(p) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} + \begin{pmatrix} u(-p+1) \\ u(-p) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.26}$$

The assertion (6.24) follows from (6.25) when $|\mathrm{Tr} M_z(p)| \le 1$ and it follows from (6.26) when $|\mathrm{Tr} M_z(p)| > 1$. $\square$

The estimate (6.24) can of course be used to exclude the existence of decaying solutions. Notice that the energy does not enter the argument. In particular, if the potential V has the required local periodicity for infinitely many values of p, we have the estimate (6.24) for infinitely many values of p. This in turn shows that no z can be an eigenvalue. It is clear that one can perturb about this situation a little bit and still deduce useful estimates. In light of this, the following definition is natural.

DEFINITION 6.13. *A bounded potential $V : \mathbb{Z} \to \mathbb{R}$ is called a Gordon potential if there are positive integers $q_k \to \infty$ such that*

$$\max_{1 \le n \le q_k} |V(n) - V(n \pm q_k)| \le k^{-q_k} \tag{6.27}$$

for every $k \ge 1$. Equivalently, there are positive integers $q_k \to \infty$ such that

$$\forall\, C > 0 : \lim_{k \to \infty} \max_{1 \le n \le q_k} |V(n) - V(n \pm q_k)| C^{q_k} = 0. \tag{6.28}$$

THEOREM 6.14. *Suppose V is a Gordon potential. Then, the operator $H = \Delta + V$ has purely continuous spectrum. More precisely, for every $z \in \mathbb{C}$ and every solution u of $Hu = zu$, we have*

$$\limsup_{|n| \to \infty} \left\| \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} \right\| \ge \frac{1}{2} \left\| \begin{pmatrix} u(1) \\ u(0) \end{pmatrix} \right\|. \tag{6.29}$$

PROOF. By assumption, there is a sequence $q_k \to \infty$ such that (6.28) holds. Given $z \in \mathbb{C}$, we consider a solution u of $Hu = zu$ and, for every k, a solution u_k of

$$u_k(n+1) + u_k(n-1) + V_k(n)u_k(n) = zu_k(n)$$

with $u_k(1) = u(1)$ and $u_k(0) = u(0)$, where V_k is the q_k-periodic potential that coincides with V on the interval $1 \le n \le q_k$.

It follows from Lemma 6.12 that u_k satisfies the estimate

$$\max\left\{\left\|\begin{pmatrix} u_k(-q_k+1) \\ u_k(-q_k)\end{pmatrix}\right\|, \left\|\begin{pmatrix} u_k(q_k+1) \\ u_k(q_k)\end{pmatrix}\right\|, \left\|\begin{pmatrix} u_k(2q_k+1) \\ u_k(2q_k)\end{pmatrix}\right\|\right\} \ge \frac{1}{2}\left\|\begin{pmatrix} u(1) \\ u(0)\end{pmatrix}\right\|.$$

Since V is very close to V_k on the relevant interval and u and u_k have the same initial conditions, we expect that they are close throughout the relevant interval and hence u obeys a similar estimate.

Let us make this observation explicit. Denote the transfer matrices associated with V_k by $M_{k,z}(n)$. We have

$$\begin{aligned}\max_{-q_k \le n \le 2q_k}\left\|\begin{pmatrix} u(n+1) \\ u(n)\end{pmatrix} - \begin{pmatrix} u_k(n+1) \\ u_k(n)\end{pmatrix}\right\| &\le \max_{-q_k \le n \le 2q_k}\|M_z(n) - M_{k,z}(n)\|\left\|\begin{pmatrix} u(1) \\ u(0)\end{pmatrix}\right\| \\ &\le C^{q_k}\max_{1 \le n \le q_k}|V(n) - V_k(n)|\left\|\begin{pmatrix} u(1) \\ u(0)\end{pmatrix}\right\|,\end{aligned}$$

which goes to zero by (6.28). □

An immediate application is the following. The special case $f(\omega) = 2\lambda\cos(2\pi\omega)$ was discussed by Avron and Simon in [**2**].

PROPOSITION 6.15. *Suppose $f : \mathbb{T} \to \mathbb{R}$ is Lipschitz continuous and T is rotation by the irrational number α, which is Liouville in the sense that there is a sequences of rational numbers p_k/q_k such that $q_k \to \infty$ and $|\alpha - \frac{p_k}{q_k}| < k^{-q_k}$. Then, V_ω is a Gordon potential for every $\omega \in \Omega$.*

Clearly, for any modulus of continuity, there is a corresponding Liouville condition such that an analogous statement holds. In particular, for sufficiently nice functions, the Gordon property holds for a residual set of α's.

In the following we describe work by Boshernitzan and Damanik on the metric and topological genericity of Gordon potentials for minimal homeomorphisms and continuous sampling functions; see [**3, 4**].

DEFINITION 6.16. *A sequence $\{\omega_k\}_{k\ge 0}$ in the compact metric space Ω has the repetition property if for every $\varepsilon > 0$ and $r > 0$, there exists $q \in \mathbb{Z}_+$ such that* $\mathrm{dist}(\omega_k, \omega_{k+q}) < \varepsilon$ *for $k = 0, 1, 2, \ldots, \lfloor rq \rfloor$.*

Of course, this definition makes sense in any metric space; but we remark that the compactness of Ω implies that the validity of the repetition property for any given sequence in Ω is independent of the choice of the metric.

DEFINITION 6.17. *We denote the set of points in the compact metric space Ω whose forward orbit with respect to the minimal homeomorphism T has the repetition property by $PRP(\Omega, T)$. That is,*

$$PRP(\Omega, T) = \{\omega \in \Omega : \{T^k\omega\}_{k\ge 0} \text{ has the repetition property}\}.$$

We say that (Ω, T) satisfies the topological repetition property (TRP) *if $PRP(\Omega, T) \ne \emptyset$.*

It is not hard to see that since (Ω, T) is minimal, $PRP(\Omega, T)$ being non-empty actually implies that it is residual. One can of course define $PRP(\Omega, T)$ also for a non-minimal topological dynamical system (Ω, T). In general, $PRP(\Omega, T)$ is always a G_δ set and so, in particular, a dense G_δ set in the closure of the orbit of any of its elements.

THEOREM 6.18. *Suppose* (Ω, T) *satisfies* (TRP). *Then there exists a residual subset* $\mathcal{F}$ *of* $C(\Omega)$ *such that for every* $f \in \mathcal{F}$*, there is a residual subset* $\Omega_f \subseteq \Omega$ *with the property that for every* $\omega \in \Omega_f$*,* V_ω *is a Gordon potential.*

PROOF. By assumption, there is a point $\omega \in \Omega$ whose forward orbit has the repetition property. For each $k \in \mathbb{Z}_+$, consider $\varepsilon = \frac{1}{k}$, $r = 3$, and the associated $q_k = q(\varepsilon, r)$. This ensures

$$q_k \to \infty.$$

Take an open ball B_k around ω with radius small enough so that

$$\overline{T^n(B_k)},\ 1 \le n \le 4q_k$$

are disjoint and, for every $1 \le j \le q_k$,

$$\bigcup_{l=0}^{3} T^{j+lq_k}(B_k)$$

is contained in some ball of radius 4ε. Define

$$\mathcal{C}_k = \left\{ f \in C(\Omega) : f \text{ is constant on each set } \bigcup_{l=0}^{3} T^{j+lq}(B_k),\ 1 \le j \le q_k \right\}$$

and let $\mathcal{F}_k$ be the open k^{-q_k} neighborhood of $\mathcal{C}_k$ in $C(\Omega)$. Notice that for each m,

$$\bigcup_{k \ge m} \mathcal{F}_k$$

is an open and dense subset of $C(\Omega)$. This follows since every $f \in C(\Omega)$ is uniformly continuous and the diameter of the set $\bigcup_{l=0}^{3} T^{j+lq_k}(B_k)$ goes to zero, uniformly in j, as $k \to \infty$. Thus,

$$\mathcal{F} = \bigcap_{m \ge 1} \bigcup_{k \ge m} \mathcal{F}_k$$

is a dense G_δ subset of $C(\Omega)$.

Consider some $f \in \mathcal{F}$. Then, $f \in \mathcal{F}_{k_l}$ for some sequence $k_l \to \infty$. Observe that for every $m \ge 1$,

$$\bigcup_{l \ge m} \bigcup_{j=1}^{q_{k_l}} T^{j+q_{k_l}}(B_{k_l})$$

is an open and dense subset of Ω since T is minimal and $q_{k_l} \to \infty$. Thus,

$$\Omega_f = \bigcap_{m \ge 1} \bigcup_{l \ge m} \bigcup_{j=1}^{q_{k_l}} T^{j+q_{k_l}}(B_{k_l})$$

is a dense G_δ subset of Ω.

It now readily follows that for every $f \in \mathcal{F}$ and $\omega \in \Omega_f$, V_ω is a Gordon potential. Explicitly, since $\omega \in \Omega_f$, ω belongs to $\bigcup_{j=1}^{q_{k_l}} T^{j+q_{k_l}}(B_{k_l})$ for infinitely many l. For each such l, we have by construction that

$$\max_{1 \le j \le q_{k_l}} |f(T^j\omega) - f(T^{j+q_{k_l}}\omega)| < 2k_l^{-q_{k_l}}$$

and

$$\max_{1 \le j \le q_{k_l}} |f(T^j\omega) - f(T^{j-q_{k_l}}\omega)| < 2k_l^{-q_{k_l}}$$

This shows that $V_\omega(n) = f(T^n\omega)$ is a Gordon potential. □

DEFINITION 6.19. *We say that* (Ω, T, μ) *satisfies the metric repetition property* (MRP) *if* $\mu(PRP(\Omega, T)) > 0$.

THEOREM 6.20. *Suppose* (Ω, T, μ) *satisfies* (MRP). *Then there exists a residual subset* $\mathcal{F}$ *of* $C(\Omega)$ *such that for every* $f \in \mathcal{F}$, *there is a subset* $\Omega_f \subseteq \Omega$ *of full* μ *measure with the property that for every* $\omega \in \Omega_f$, V_ω *is a Gordon potential.*

PROOF. Notice that, by ergodicity, the assumption $\mu(PRP(\Omega, T)) > 0$ implies $\mu(PRP(\Omega, T)) = 1$.

Let us inductively define a sequence of positive integers, $\{n_i\}_{i \ge 1}$, and a sequence of subsets of Ω, $\{\Omega_i\}_{i \ge 1}$.

Since $\mu(PRP(\Omega, T)) = 1$ we can choose $n_1 \in \mathbb{Z}_+$ large enough so that[1]

$$\Omega_1 = \Big\{\omega \in \Omega : \text{there exists } n \in [1, n_1) \text{ such that for } k_1, k_2 \in [1, n] \\ \text{with } k_1 \equiv k_2 \bmod n, \text{ we have } \operatorname{dist}(T^{k_1}\omega, T^{k_2}\omega) < 1\Big\}$$

obeys

$$\mu(\Omega_1) > 1 - 2^{-1}.$$

Once n_{i-1} and Ω_{i-1} have been determined, we can define n_i and Ω_i as follows. It is possible to find $n_i > \max\{n_{i-1}, 2^i\}$ such that

$$\Omega_i = \Big\{\omega \in \Omega : \text{there exists } n \in [n_{i-1}, n_i) \text{ such that for } k_1, k_2 \in [1, in] \\ \text{with } k_1 \equiv k_2 \bmod n, \text{ we have } \operatorname{dist}(T^{k_1}\omega, T^{k_2}\omega) < 2^{-i}\Big\}$$

obeys

$$\mu(\Omega_i) > 1 - 2^{-i}. \tag{6.30}$$

For $i \ge 5$, we set $m_i = (in_i)^2$ and choose (using the Rokhlin-Halmos Lemma [**6**, Theorem 1 on p. 242]) $O_i \subset \Omega$ in a way that T^jO_i, $1 \le j \le m_i$ are disjoint and

$$\mu\left(\bigcup_{j=1}^{m_i} T^jO_i\right) > 1 - 2^{-n_i-1}.$$

Next we further partition O_i into sets $S_{i,l}$, $1 \le l \le s_i$ such that for every $0 \le u \le m_i$,

$$\operatorname{diam}(T^uS_{i,l}) < \frac{1}{n_i}. \tag{6.31}$$

[1]The definition of Ω_1 contains redundancies whose purpose is to motivate the definition of the subsequent sets Ω_i.

Choose $K_{i,l} \subseteq S_{i,l}$ compact with

$$\mu(K_{i,l}) > \mu(S_{i,l}) \left(1 - 2^{-n_i - 1}\right).$$

Then, $T^u K_{i,l}$, $0 \le u \le m_i$, $1 \le l \le s_i$ are disjoint and their total measure is

$$\mu\left(\bigcup_{u,l} T^u K_{i,l}\right) \ge (1 - 2^{-n_i-1})^2 > 1 - 2^{-n_i}. \tag{6.32}$$

We will now collect a large subfamily of $\{T^u K_{i,l}\}$. For each l, we let u run from 0 upwards and ask for the corresponding $T^u K_{i,l}$ whether it has non-empty intersection with Ω_i. If it does, then there is a point ω and a corresponding $n \in (n_{i-1}, n_i)$. We add the current $T^u K_{i,l}$ to the subfamily we construct, along with the sets $T^{u+h} K_{i,l}$, $1 \le h \le in$. Then we continue with $T^{u+in+1} K_{i,l}$ and do the same. We stop when we are within n_i steps of the top of the tower. Let us denote the subfamily so constructed by $\mathcal{K}_i$. By (6.30), (6.32), and T-invariance of μ, we have that

$$\mu\left(\bigcup_{T^u K_{i,l} \in \mathcal{K}_i} T^u K_{i,l}\right) > 1 - 2^{-n_i} - 2^{-i} - \frac{n_i}{m_i}. \tag{6.33}$$

The next step is to group each of these "runs" into arithmetic progressions. Notice that locally we have n consecutive points that are $(i-1)$ times repeated up to some small error. Notice that this extends to the entire set if we make the allowed error a bit larger. Let us group these in sets into n arithmetic progressions of length i. The union of each of these arithmetic progressions of sets will constitute a new set $C_{i,m}$. By the definition of Ω_i and (6.31), we have

$$\operatorname{diam}(C_{i,m}) < \frac{2}{n_i} + 2^{-i}. \tag{6.34}$$

We will also consider the sets $\tilde{C}_{i,m}$ that are defined similarly, but with the first and the last set in the corresponding sequence of i sets deleted.

We can now continue as before. Define

$$F_i = \{f \in C(\Omega) : f \text{ is constant on each set } C_{i,m}\}$$

and let $\mathcal{F}_i$ be the i^{-n_i} neighborhood of F_i in $C(\Omega)$. Notice that for each m,

$$\bigcup_{i \ge m} \mathcal{F}_i$$

is an open and dense subset of $C(\Omega)$. This follows from (6.33) and (6.34) since every $f \in C(\Omega)$ is uniformly continuous. Thus,

$$\mathcal{F} = \bigcap_{m \ge 1} \bigcup_{i \ge m} \mathcal{F}_i$$

is a dense G_δ subset of $C(\Omega)$.

If $f \in \mathcal{F}$, there is a sequence $i_k \to \infty$ such that $f \in \mathcal{F}_{i_k}$. For each k, f is within $i_k^{-n_{i_k}}$ of being constant on each set $C_{i_k,m}$. Recall that this set is the union of i sets in arithmetic progression relative to T.

If we instead consider $\tilde{C}_{i_k,m}$, we can go forward and backward one period and hence, by construction, this is exactly the Gordon condition at this level. Thus, it only remains to show that almost every $\omega \in \Omega$ belongs to infinitely many $\tilde{C}_{i_k,m}$.

This, however, follows from the measure estimates obtained above and the Borel-Cantelli Lemma. □

Let us now consider some examples and explore the validity of the various repetition properties for them.

THEOREM 6.21. *Every minimal shift $T\omega = \omega + \alpha$ on the torus $\mathbb{T}^d$ satisfies* (GRP)*, and hence also* (TRP) *and* (MRP)*.*

PROOF. By assumption, the orbit of $0 \in \mathbb{T}^d$ is dense. In particular, we can define $q_k \to \infty$ such that $T^{q_k}(0)$ is closer to 0 than any point $T^n(0)$, $1 \le n < q_k$. In particular, for every $\varepsilon > 0$ and every $r > 0$, there is $k(\varepsilon, r)$ such that for $k \ge k(\varepsilon, r)$, T^{q_k} is a shift on $\mathbb{T}^d$ with a shift vector of length bounded by ε. The repetition property now follows for the forward orbit of any choice of $\omega \in \mathbb{T}^d$. Thus, (GRP) is satisfied. □

Next we consider the standard skew-shift and generalizations thereof. First, we state and prove a result for the standard skew-shift. Recall that $\alpha \in \mathbb{T}$ is called badly approximable if there is a constant $c > 0$ such that

$$\langle \alpha q \rangle > \frac{c}{q}$$

for every $q \in \mathbb{Z} \setminus \{0\}$. Here, we write $\langle x \rangle = \mathrm{dist}_{\mathbb{T}}(x, 0)$ $(= \min\{|x - p| : p \in \mathbb{Z}\}$, where x denotes any representative in $\mathbb{R}$). The set of badly approximable α's has zero Lebesgue measure; see, for example, [**19**, Theorem 29 on p. 60]. In terms of the continued fraction expansion of α (cf. [**19**]), being badly approximable is equivalent to having bounded partial quotients.

THEOREM 6.22. *For a minimal skew-shift $T(\omega_1, \omega_2) = (\omega_1 + 2\alpha, \omega_1 + \omega_2)$ on the torus $\mathbb{T}^2$, the following are equivalent:*

(i) *α is not badly approximable.*
(ii) *(Ω, T) satisfies* (GRP).
(iii) *$(\Omega, T, \mathrm{Leb})$ satisfies* (MRP).
(iv) *(Ω, T) satisfies* (TRP).

PROOF. Iterating the skew-shift n times, we find

$$T^n(\omega_1, \omega_2) = (\omega_1 + 2n\alpha, \omega_2 + 2n\omega_1 + n(n-1)\alpha).$$

Thus,

$$T^{n+q}(\omega_1, \omega_2) - T^n(\omega_1, \omega_2) = (2q\alpha, 2q\omega_1 + q^2\alpha + 2nq\alpha - q\alpha). \tag{6.35}$$

<u>(i) ⇒ (ii)</u>: Assume that α is not badly approximable. This means that there is some sequence $q_k \to \infty$, such that

$$\lim_{k \to \infty} q_k \langle \alpha q_k \rangle = 0. \tag{6.36}$$

Let $(\omega_1, \omega_2) \in \mathbb{T}^2$, $\varepsilon > 0$, and $r > 0$ be given. We will construct a sequence $\tilde{q}_k \to \infty$ so that for $1 \le n \le r\tilde{q}_k$,

$$(2\tilde{q}_k\alpha, 2\tilde{q}_k\omega_1 + \tilde{q}_k^2\alpha + 2n\tilde{q}_k\alpha - \tilde{q}_k\alpha) \tag{6.37}$$

is of size $O(\varepsilon)$. Each $\tilde{q}_k$ will be of the form $m_k q_k$ for some $m_k \in \{1, 2, \dots, \lfloor \varepsilon^{-1} \rfloor + 1\}$.

It follows from (6.36) that in (6.37), every term except $2\tilde{q}_k\omega_1$ goes to zero as $k \to \infty$, regardless of the choice of m_k, and hence is less than ε for k large enough. To treat the remaining term, we can just choose m_k in the specified ε-dependent range so that $2\tilde{q}_k\omega_1 = m_k(2q_k\omega_1)$ is of size less than ε as well. Consequently, by (6.35), the orbit of (ω_1, ω_2) has the repetition property. Since (ω_1, ω_2) was arbitrary, it follows that (GRP) holds.

(ii) $\Rightarrow$ (iii): This is immediate.

(iii) $\Rightarrow$ (iv): This is immediate.

(iv) $\Rightarrow$ (i): Assuming (TRP), we see that there is a point ω such that $\{T^n\omega\}_{n\geq 0}$ has the repetition property. In particular, by (6.35), we see that for every $\varepsilon > 0$, there are $q_k \to \infty$ so that

$$\langle 2q_1\omega_1 + q_k^2\alpha + 2nq_k\alpha - q_k\alpha\rangle < \varepsilon \text{ for } 0 \leq n \leq q_k. \tag{6.38}$$

Evaluating this for $n = 0$, we find that $\langle 2q_1\omega_1 + q_k^2\alpha - q_k\alpha\rangle < \varepsilon$. Now vary n. Each time we increase n, we shift in the same direction by $\langle 2q_k\alpha\rangle$. If $\varepsilon > 0$ is sufficiently small, it follows from the estimate (6.38) that we cannot go around the circle completely and hence we have $\langle 2nq_k\alpha\rangle = n\langle 2q_k\alpha\rangle$ for every $0 \leq n \leq q_k$. We find that $\langle \alpha q_k\rangle \lesssim \frac{\varepsilon}{q_k}$, which shows that α is not badly approximable. □

The following theorem describes results that hold for generalizations of the standard skew-shift. Proofs may be found in [**4**].

THEOREM 6.23. *Suppose that* $\alpha \in \mathbb{T}$ *is irrational and the map* $T : \mathbb{T}^k \to \mathbb{T}^k$ *is given by* $T(\omega_1, \omega_2, \ldots, \omega_k) = (\omega_1 + \alpha, \omega_2 + \omega_1, \ldots, \omega_k + \omega_{k-1})$.

(a) *For every* $k \geq 2$, *the following are equivalent:*
- T *has* (TRP),
- $\liminf_{q\to\infty} q^{k-1}\langle q\alpha\rangle = 0$.

(b) *For* $k = 2$, *the following are equivalent:*
- T *has* (TRP),
- T *has* (MRP),
- T *has* (GRP),
- $\liminf_{q\to\infty} q\langle q\alpha\rangle = 0$.

(c) *If* $k \geq 3$, *then* T *does not have* (GRP).
(d) *If* $k \geq 3$, *then* T *does not have* (TRP) *for Lebesgue almost every* α.
(e) *If* $k = 3$, *then* T *has* (MRP) *for* α*'s from a residual subset of* $\mathbb{T}$.
(f) *If* $k \geq 4$, *then* T *does not have* (MRP).

CHAPTER 7

The Almost Mathieu Operator

In this chapter we will study the almost Mathieu operator

$$[H_\omega^{\lambda,\alpha}\psi](n) = \psi(n+1) + \psi(n-1) + 2\lambda\cos(2\pi(\omega + n\alpha))\psi(n).$$

Notice that the potential $2\lambda\cos(2\pi(\omega+n\alpha))$ is periodic if $\lambda = 0$ or α is rational. Since we have already studied the periodic case in detail, we will only consider the case where $\lambda \neq 0$ and α is irrational. Furthermore, by periodicity of the cosine, we consider α as an element of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Finally, it is also easy to see that $H_\omega^{\lambda,\alpha} = H_{\omega+\frac{1}{2}}^{-\lambda,\alpha}$. Thus, throughout this chapter, we will only consider the following range for our parameters:

$$\lambda > 0, \ \alpha \in \mathbb{T} \text{ irrational}, \ \omega \in \mathbb{T}.$$

Denote the almost sure spectrum of the operator family $\{H_\omega^{\lambda,\alpha}\}_{\omega\in\mathbb{T}}$ by $\Sigma^{\lambda,\alpha}$. Recall that $\sigma(H_\omega^{\lambda,\alpha}) = \Sigma^{\lambda,\alpha}$ for every $\omega \in \mathbb{T}$ by minimality of the irrational rotation and continuity of the cosine.

1. Aubry Duality

Consider the difference equation

$$u(n+1) + u(n-1) + 2\lambda\cos(2\pi(\omega + n\alpha))u(n) = Eu(n), \tag{7.1}$$

and also the dual difference equation

$$\tilde{u}(n+1) + \tilde{u}(n-1) + 2\lambda^{-1}\cos(2\pi(\tilde{\omega} + n\alpha))\tilde{u}(n) = (\lambda^{-1}E)\tilde{u}(n). \tag{7.2}$$

Lemma 7.1. (a) *Suppose $u \in \ell^1(\mathbb{Z})$ is a solution of* (7.1). *Consider its Fourier transform*

$$\hat{u}(\theta) = \sum_{m\in\mathbb{Z}} u(m)e^{2\pi i m\theta}.$$

Then, given any $\tilde{\omega} \in \mathbb{T}$, the sequence $\tilde{u}$ defined by

$$\tilde{u}(n) = \hat{u}(\tilde{\omega} + n\alpha)e^{2\pi i n\omega} \tag{7.3}$$

is a solution of (7.2).

(b) *Suppose $u \in \ell^2(\mathbb{Z})$ is a solution of* (7.1). *Then, for $\tilde{\omega}$ from a full-measure subset of $\mathbb{T}$, the sequence $\tilde{u}$ defined by* (7.3) *is a solution of* (7.2).

Proof. (a) If $u \in \ell^1(\mathbb{Z})$, $\hat{u} \in C(\mathbb{T})$ and we can evaluate it pointwise. We have

$$\begin{aligned}(\lambda^{-1}E)\tilde{u}(n) &= (\lambda^{-1}E)\hat{u}(\tilde{\omega} + n\alpha)e^{2\pi i n\omega} \\ &= (\lambda^{-1}E)\sum_{m\in\mathbb{Z}} u(m)e^{2\pi i m(\tilde{\omega}+n\alpha)}e^{2\pi i n\omega} \\ &= \lambda^{-1}\sum_{m\in\mathbb{Z}} (Eu(m))e^{2\pi i m(\tilde{\omega}+n\alpha)}e^{2\pi i n\omega}\end{aligned}$$

$$
\begin{aligned}
&= \lambda^{-1} \sum_{m \in \mathbb{Z}} \left[u(m+1) + u(m-1) + 2\lambda \cos(2\pi(\omega + m\alpha))u(m)\right] e^{2\pi i m(\tilde{\omega}+n\alpha)} e^{2\pi i n\omega} \\
&= \lambda^{-1} \sum_{m \in \mathbb{Z}} \left[u(m+1) + u(m-1) + \lambda \left(e^{2\pi i(\omega+m\alpha)} + e^{-2\pi i(\omega+m\alpha)}\right) u(m)\right] e^{2\pi i m\tilde{\omega}} e^{2\pi i n(\omega+m\alpha)} \\
&= \lambda^{-1} \sum_{m \in \mathbb{Z}} u(m+1) e^{2\pi i m\tilde{\omega}} e^{2\pi i n(\omega+m\alpha)} + \lambda^{-1} \sum_{m \in \mathbb{Z}} u(m-1) e^{2\pi i m\tilde{\omega}} e^{2\pi i n(\omega+m\alpha)} \\
&\quad + \sum_{m \in \mathbb{Z}} u(m) e^{2\pi i m\tilde{\omega}} e^{2\pi i(n+1)(\omega+m\alpha)} + \sum_{m \in \mathbb{Z}} u(m) e^{2\pi i m\tilde{\omega}} e^{2\pi i(n-1)(\omega+m\alpha)} \\
&= \lambda^{-1} \sum_{m \in \mathbb{Z}} u(m) e^{2\pi i(m-1)\tilde{\omega}} e^{2\pi i n(\omega+(m-1)\alpha)} + \lambda^{-1} \sum_{m \in \mathbb{Z}} u(m) e^{2\pi i(m+1)\tilde{\omega}} e^{2\pi i n(\omega+(m+1)\alpha)} \\
&\quad + \sum_{m \in \mathbb{Z}} u(m) e^{2\pi i m\tilde{\omega}} e^{2\pi i(n+1)(\omega+m\alpha)} + \sum_{m \in \mathbb{Z}} u(m) e^{2\pi i m\tilde{\omega}} e^{2\pi i(n-1)(\omega+m\alpha)} \\
&= \lambda^{-1} \left(e^{-2\pi i(\tilde{\omega}+n\alpha)} + e^{2\pi i(\tilde{\omega}+n\alpha)}\right) \sum_{m \in \mathbb{Z}} u(m) e^{2\pi i m(\tilde{\omega}+n\alpha)} e^{2\pi i n\omega} \\
&\quad + \sum_{m \in \mathbb{Z}} u(m) e^{2\pi i m(\tilde{\omega}+(n+1)\alpha)} e^{2\pi i(n+1)\omega} + \sum_{m \in \mathbb{Z}} u(m) e^{2\pi i m(\tilde{\omega}+(n-1)\alpha)} e^{2\pi i(n-1)\omega} \\
&= 2\lambda^{-1} \cos(2\pi(\tilde{\omega} + n\alpha))\tilde{u}(n) + \tilde{u}(n+1) + \tilde{u}(n-1),
\end{aligned}
$$

as claimed.

(b) If $u \in \ell^2(\mathbb{Z})$, $\hat{u}$ exists as an element of $L^2(\mathbb{T})$ and hence it is determined almost everywhere. Consider $\tilde{\omega}$ from the full measure set of elements for which all the quantities in the calculation above are determined. Then carry out the calculation to verify that $\tilde{u}$ is indeed a solution of (7.2) for the $\tilde{\omega}$ in question. □

Let us work out a version of duality that applies to the family $\{H_\omega^{\lambda,\alpha}\}_{\omega \in \mathbb{T}}$. The appropriate setting is as follows. Consider the Hilbert space $L^2(\mathbb{T} \times \mathbb{Z})$ and the operator

$$H^{\lambda,\alpha} : L^2(\mathbb{T} \times \mathbb{Z}) \to L^2(\mathbb{T} \times \mathbb{Z})$$

given by

$$[H^{\lambda,\alpha}\varphi](\omega, n) = \varphi(\omega, n+1) + \varphi(\omega, n-1) + 2\lambda \cos(2\pi(\omega + n\alpha))\varphi(\omega, n).$$

Introduce the duality transform

$$\mathcal{A} : L^2(\mathbb{T} \times \mathbb{Z}) \to L^2(\mathbb{T} \times \mathbb{Z}),$$

which is given by

$$[\mathcal{A}\varphi](\omega, n) = \sum_{m \in \mathbb{Z}} \int_{\mathbb{T}} e^{-2\pi i(\omega+n\alpha)m} e^{-2\pi i n\eta} \varphi(\eta, m)\, d\eta.$$

This definition assumes initially that φ is such that the sum in m converges, but note that in terms of the Fourier transform on $L^2(\mathbb{T} \times \mathbb{Z})$, we have

$$[\mathcal{A}\varphi](\omega, n) = \hat{\varphi}(n, \omega + n\alpha),$$

which may be used to extend the definition to all of $L^2(\mathbb{T} \times \mathbb{Z})$ and shows that $\mathcal{A}$ is unitary.

THEOREM 7.2. *Suppose $\lambda > 0$ and $\alpha \in \mathbb{T}$ is irrational.*

(a) *We have*

$$H^{\lambda,\alpha}\mathcal{A} = \lambda \mathcal{A} H^{\lambda^{-1},\alpha}.$$

(b) *For the integrated density of states, we have*

$$k^{\lambda,\alpha}(E) = \lambda k^{\lambda^{-1},\alpha}(\lambda^{-1}E) \quad \textit{for every } E \in \mathbb{R}.$$

The spectra of $H_\omega^{\lambda,\alpha}$ *and* $H_\omega^{\lambda^{-1},\alpha}$ *are therefore related as follow,*

$$\Sigma^{\lambda,\alpha} = \lambda \Sigma^{\lambda^{-1},\alpha}.$$

(c) *For the Lyapunov exponent, we have*

$$\gamma^{\lambda,\alpha}(E) = \gamma^{\lambda^{-1},\alpha}(\lambda^{-1}E) + \log \lambda \quad \textit{for every } E \in \mathbb{R}.$$

Proof. (a) Introduce the unitary translation and multiplication operators T and M, which act on $L^2(\mathbb{T} \times \mathbb{Z})$ as

$$T\varphi(\omega, n) = \varphi(\omega, n+1)$$

and

$$M\varphi(\omega, n) = e^{2\pi i(\omega + n\alpha)}\varphi(\omega, n).$$

We have

$$\begin{aligned}
[T\mathcal{A}\varphi](\omega, n) &= \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} e^{-2\pi i(\omega+(n+1)\alpha)m} e^{-2\pi i(n+1)\eta} \varphi(\eta, m)\, d\eta \\
&= \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} e^{-2\pi i(\omega+n\alpha)m} e^{-2\pi i n\eta} \left(e^{-2\pi i(\eta+m\alpha)} \varphi(\eta, m)\right)\, d\eta
\end{aligned}$$

and hence $[\mathcal{A}^{-1}T\mathcal{A}\varphi](\omega, n) = e^{-2\pi i(\omega+n\alpha)}\varphi(\omega, n)$. Similarly,

$$\begin{aligned}
[M\mathcal{A}\varphi](\omega, n) &= e^{2\pi i(\omega+n\alpha)} \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} e^{-2\pi i(\omega+n\alpha)m} e^{-2\pi i n\eta} \varphi(\eta, m)\, d\eta \\
&= \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} e^{-2\pi i(\omega+n\alpha)(m-1)} e^{-2\pi i n\eta} \varphi(\eta, m)\, d\eta \\
&= \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} e^{-2\pi i(\omega+n\alpha)m} e^{-2\pi i n\eta} \varphi(\eta, m+1)\, d\eta
\end{aligned}$$

and hence $[\mathcal{A}^{-1}M\mathcal{A}\varphi](\omega, n) = \varphi(\omega, n+1)$. Since $H^{\lambda,\alpha} = T + T^* + \lambda(M + M^*)$, we therefore find

$$\begin{aligned}
[\mathcal{A}^{-1}H^{\lambda,\alpha}\mathcal{A}\varphi](\omega, n) &= 2\cos(2\pi(\omega + n\alpha))\varphi(\omega, n) + \lambda(\varphi(\omega, n+1) + \varphi(\omega, n-1)) \\
&= \lambda\, H^{\lambda^{-1},\alpha}\, \varphi(\omega, n).
\end{aligned}$$

(b) For the function $\varphi_f \in L^2(\mathbb{T} \times \mathbb{Z})$ given by

$$\varphi_f(\omega, n) = \delta_{n,0},$$

we have

$$\begin{aligned}
[\mathcal{A}\varphi_f](\omega, n) &= \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} e^{-2\pi i(\omega+n\alpha)m} e^{-2\pi i n\eta} \varphi_f(\eta, m)\, d\eta \\
&= \int_{\mathbb{T}} e^{-2\pi i n\eta}\, d\eta \\
&= \delta_{n,0} \\
&= \varphi_f(\omega, n),
\end{aligned}$$

and hence $\mathcal{A}\varphi_f = \varphi_f$.

For $g \in C(\mathbb{R})$, we have

$$\begin{aligned}
\int g(E)\, dk^{\lambda,\alpha}(E) &= \int_{\mathbb{T}} \langle \delta_0, g(H_\omega^{\lambda,\alpha})\delta_0\rangle_{\ell^2(\mathbb{Z})}\, d\omega \\
&= \langle \varphi_f, g(H^{\lambda,\alpha})\varphi_f\rangle_{L^2(\mathbb{T}\times\mathbb{Z})} \\
&= \langle \mathcal{A}^*\varphi_f, \mathcal{A}^* g(H^{\lambda,\alpha})\varphi_f\rangle_{L^2(\mathbb{T}\times\mathbb{Z})} \\
&= \langle \mathcal{A}^*\varphi_f, \mathcal{A}^* g(H^{\lambda,\alpha})\mathcal{A}\mathcal{A}^*\varphi_f\rangle_{L^2(\mathbb{T}\times\mathbb{Z})} \\
&= \langle \varphi_f, \mathcal{A}^* g(H^{\lambda,\alpha})\mathcal{A}\varphi_f\rangle_{L^2(\mathbb{T}\times\mathbb{Z})} \\
&= \langle \varphi_f, g(\lambda H^{\lambda^{-1},\alpha})\varphi_f\rangle_{L^2(\mathbb{T}\times\mathbb{Z})} \\
&= \int_{\mathbb{T}} \langle \delta_0, g(\lambda H_\omega^{\lambda^{-1},\alpha})\delta_0\rangle_{\ell^2(\mathbb{Z})}\, d\omega \\
&= \int g(\lambda \tilde{E})\, dk^{\lambda^{-1},\alpha}(\tilde{E}) \\
&= \int g(E)\, d\left[\lambda k^{\lambda^{-1},\alpha}(\lambda^{-1}E)\right].
\end{aligned}$$

Here we used the unitarity of $\mathcal{A}^*$ in the third step, the invariance of φ_f in the fifth step and part (a) in the sixth step. This shows the identity for the integrated density of states, from which the identity for the spectrum follows since the latter is given by the points of non-constancy of the integrated density of states; compare Theorem 3.4.

(c) Using the Thouless formula and part (b), we find

$$\begin{aligned}
\gamma^{\lambda,\alpha}(E) &= \int \log|\tilde{E} - E|\, dk^{\lambda,\alpha}(\tilde{E}) \\
&= \int \log|\tilde{E} - E|\, d\left[\lambda k^{\lambda^{-1},\alpha}(\lambda^{-1}\tilde{E})\right] \\
&= \int \log|(\lambda E') - (\lambda\lambda^{-1}E)|\, dk^{\lambda^{-1},\alpha}(E') \\
&= \int \log|E' - \lambda^{-1}E|\, dk^{\lambda^{-1},\alpha}(E') + \log\lambda \\
&= \gamma^{\lambda^{-1},\alpha}(\lambda^{-1}E) + \log\lambda,
\end{aligned}$$

which is the desired identity for the Lyapunov exponent. □

Remark. Note that by non-negativity of the Lyapunov exponent, part (c) implies

$$\gamma^{\lambda,\alpha}(E) \ge \log\lambda$$

for all energies, and hence Aubry duality provides a different proof of this estimate.

Now that we have seen that the couplings λ and λ^{-1} are related via Aubry duality, we wish to explore if there is any relation between the spectral types of the two operator families. While the Aubry duality transform $\mathcal{A}$ is unitary, one should not be mislead to the belief that the almost sure spectral types of dual families are the same. In fact, almost the opposite is true! The following theorem from [**13**] describes a relation that demonstrates this fact.

THEOREM 7.3 (Gordon-Jitomirskaya-Last-Simon 1997). *Suppose $\alpha \in \mathbb{T}$ is irrational and $\lambda > 0$ is such that $H_\omega^{\lambda,\alpha}$ has pure point spectrum for almost every $\omega \in \mathbb{T}$. Then, $H_\omega^{\lambda^{-1},\alpha}$ has purely absolutely continuous spectrum for almost every $\omega \in \mathbb{T}$.*

A closely related result from the same paper is the following.

Theorem 7.4 (Gordon-Jitomirskaya-Last-Simon 1997). *Suppose $\alpha \in \mathbb{T}$ is irrational and $\lambda > 0$ is such that $H_\omega^{\lambda,\alpha}$ has some point spectrum for almost every $\omega \in \mathbb{T}$. Then, $H_\omega^{\lambda^{-1},\alpha}$ has some absolutely continuous spectrum for almost every $\omega \in \mathbb{T}$.*

Let λ and α be fixed. The technical key to the proof of these results is a measurable selection of eigenvalues and eigenfunctions. Suppose E is an eigenvalue of $H_\omega^{\lambda,\alpha}$ and $\psi \in \ell^2(\mathbb{Z})$ is an associated normalized eigenfunction. We say that $m \in \mathbb{Z}$ is the leftmost maximum for $|\psi|$ if $|\psi(n)| < |\psi(m)|$ for every $n < m$ and $|\psi(n)| \leq |\psi(m)|$ for every $n \geq m$. It is clear that the leftmost maximum exists and is unique. Recall the by constancy of the Wronskian, all eigenvalues are simple, that is, eigenspaces are always one-dimensional. We will uniquely fix the eigenfunction ψ by requiring $\psi(m) > 0$ and say that is is attached to m.

Lemma 7.5. *For every $m \in \mathbb{Z}$, the function N_m defined by*

$$N_m(\omega) = \#\{\textit{eigenvectors of } H_\omega^{\lambda,\alpha} \textit{ attached to } m\}$$

is measurable.

Denote

$$\mathbb{T}_{m,k} = \{\omega \in \mathbb{T} : N_m(\omega) \geq k\}.$$

Now start with $\mathbb{T}_{m,1}$. Let $\omega \in \mathbb{T}_{m,1}$ and define

$$\psi_1(n; \omega, m)$$

to be the eigenfunction of $H_\omega^{\lambda,\alpha}$ attached to m with maximal value of $\psi(m)$ and let

$$E_1(\omega, m)$$

be its eigenvalue. If there are multiple eigenfunctions with maximal $\psi(m)$ pick the one with the largest energy. This makes the choice unique, again by simplicity of eigenvalues. Notice also that by Bessel's inequality

$$\sum_{\text{eigenfunctions}} |\psi(m)|^2 = \sum_{\text{eigenfunctions}} |\langle \psi, \delta_m \rangle|^2 \leq \|\delta_m\|^2 = 1,$$

so that there are only finitely many eigenfunctions that have the same maximal value of $\psi(m)$.

Next consider $\mathbb{T}_{m,2}$. Define

$$\psi_2(n; \omega, m)$$

to be the eigenfunction attached to m with second-largest value of $\psi(m)$ and let

$$E_2(\omega, m)$$

be its eigenvalue. Break ties as above by considering the larger energy.

Continuing in this way, we select eigenfunctions $\psi_k(n; \omega, m)$ and eigenvalues $E_k(\omega, m)$ for $\omega \in \mathbb{T}_{m,k}$. Outside of $\mathbb{T}_{m,k}$, we set both functions equal to zero.

Lemma 7.6. *For fixed k, n, m, $\psi_k(n; \cdot, m)$ and $E_k(\cdot, m)$ are measurable functions on $\mathbb{T}$.*

For $l \in \mathbb{Z}$ and $\varphi \in L^2(\mathbb{T} \times \mathbb{Z})$, set

$$[S_l \varphi](\omega, n) = \varphi(\omega + \alpha l, n - l).$$

Clearly, $S : L^2(\mathbb{T} \times \mathbb{Z}) \to L^2(\mathbb{T} \times \mathbb{Z})$ is unitary.

LEMMA 7.7. *For every $\varphi \in L^2(\mathbb{T} \times \mathbb{Z})$, we have*

$$[\mathcal{A}S_l\varphi](\omega, n) = e^{-2\pi i\omega l}[\mathcal{A}\varphi](\omega, n).$$

PROOF. Consider a $\varphi \in L^2(\mathbb{T} \times \mathbb{Z})$ with $\varphi(\omega, n) = 0$ if $|n| \geq n_0$ for a suitable n_0. Then,

$$\begin{aligned}
[\mathcal{A}S_l\varphi](\omega, n) &= \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} e^{-2\pi i(\omega+n\alpha)m} e^{-2\pi i n\eta} (S_l\varphi(\eta, m))\, d\eta \\
&= \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} e^{-2\pi i(\omega+n\alpha)m} e^{-2\pi i n\eta} \varphi(\eta + \alpha l, m - l)\, d\eta \\
&= \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} e^{-2\pi i(\omega+n\alpha)(m+l)} e^{-2\pi i n(\eta-\alpha l)} \varphi(\eta, m)\, d\eta \\
&= e^{-2\pi i((\omega+n\alpha)l - n\alpha l)} \sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} e^{-2\pi i(\omega+n\alpha)m} e^{-2\pi i n\eta} \varphi(\eta, m)\, d\eta \\
&= e^{-2\pi i\omega l}[\mathcal{A}\varphi](\omega, n).
\end{aligned}$$

Since such φ's are dense, the lemma follows. □

LEMMA 7.8. *Suppose $\varphi \in L^2(\mathbb{T} \times \mathbb{Z})$ is such that*

$$\langle \varphi, S_l\varphi \rangle_{L^2(\mathbb{T}\times\mathbb{Z})} = 0 \quad \textit{for every } l \in \mathbb{Z} \setminus \{0\}.$$

Then,

$$g(\omega) = \sum_{n\in\mathbb{Z}} |[\mathcal{A}\varphi](\omega, n)|^2$$

is almost everywhere constant.

PROOF. Since $\varphi \in L^2(\mathbb{T} \times \mathbb{Z})$ and $\mathcal{A}$ is unitary, we have $g \in L^1(\mathbb{T})$. Let us show that all non-constant terms in the Fourier series of g vanish. This implies the assertion.

For $l \neq 0$, we have by assumption and the previous lemma,

$$\begin{aligned}
\int_{\mathbb{T}} e^{-2\pi i\omega l} g(\omega)\, d\omega &= \int_{\mathbb{T}} e^{-2\pi i\omega l} \sum_{n\in\mathbb{Z}} |[\mathcal{A}\varphi](\omega, n)|^2\, d\omega \\
&= \langle \mathcal{A}\varphi, e^{-2\pi i\omega l}\mathcal{A}\varphi \rangle_{L^2(\mathbb{T}\times\mathbb{Z})} \\
&= \langle \mathcal{A}\varphi, \mathcal{A}S_l\varphi \rangle_{L^2(\mathbb{T}\times\mathbb{Z})} \\
&= \langle \varphi, S_l\varphi \rangle_{L^2(\mathbb{T}\times\mathbb{Z})} \\
&= 0,
\end{aligned}$$

which yields the lemma as pointed out above. □

LEMMA 7.9. *Fix $k \in \mathbb{Z}_+$ and $m \in \mathbb{Z}$. Denote by*

$$\psi_k(n; \omega, m)$$

the measurable function constructed for $H_\omega^{\lambda,\alpha}$ above. Given $g \in L^2(\mathbb{T})$, set

$$\varphi(\omega, n) = g(\omega)\psi_k(n; \omega, m).$$

and

$$\psi_\omega(n) = [\mathcal{A}^*\varphi](\omega, n).$$

Then, the spectral measure $\mu_{H_\omega^{\lambda^{-1},\alpha},\psi_\omega}$ associated with the operator $H_\omega^{\lambda^{-1},\alpha}$ and the vector ψ_ω is almost surely ω-independent. As a consequence, the spectral measure $\mu_{H_\omega^{\lambda^{-1},\alpha},\psi_\omega}$ is purely absolutely continuous for almost every $\omega \in \mathbb{T}$.

PROOF. Note that for $l \in \mathbb{Z} \setminus \{0\}$,

$$\begin{aligned}
\langle \varphi, S_l \varphi \rangle_{L^2(\mathbb{T}\times\mathbb{Z})} &= \int_{\mathbb{T}} \left(\sum_{n\in\mathbb{Z}} \overline{\varphi(\omega,n)} \varphi(\omega+\alpha l, n-l) \right) d\omega \\
&= \int_{\mathbb{T}} \left(\sum_{n\in\mathbb{Z}} \overline{g(\omega)\psi_k(n;\omega,m)} g(\omega+\alpha l)\psi_k(n-l;\omega+\alpha l, m) \right) d\omega \\
&= \int_{\mathbb{T}} \overline{g(\omega)} g(\omega+\alpha l) \left(\sum_{n\in\mathbb{Z}} \overline{\psi_k(n;\omega,m)} \psi_k(n;\omega,m+l) \right) d\omega \\
&= 0.
\end{aligned}$$

Here we used that since $\psi_k(\cdot;\omega,m)$ and $\psi_k(\cdot;\omega,m+l)$ have their leftmost maximum at different points, their inner product vanishes by simplicity of eigenvalues and orthogonality of eigenfunctions corresponding to different eigenvalues.

Let $h \in C(\mathbb{R})$. By the definition of φ and the spectral theorem,

$$[h(H^{\lambda,\alpha})\varphi](\omega,n) = h(E_k(\omega,m))\varphi(\omega,n).$$

Thus, for $l \in \mathbb{Z} \setminus \{0\}$, we also have that $S_l\varphi$ is orthogonal to $h(H^{\lambda,\alpha})\varphi$ and hence, by arguments similar to those used in the proof of Lemma 7.8, we find that

$$\sum_{n\in\mathbb{Z}} \overline{[\mathcal{A}^*\varphi](\omega,n)} [\mathcal{A}^* h(H^{\lambda,\alpha})\varphi](\omega,n)$$

is almost everywhere constant.

Theorem 7.2, which established Aubry duality in $L^2(\mathbb{T}\times\mathbb{Z})$, implies that

$$\mathcal{A}^* h(H^{\lambda,\alpha}) = \lambda h(H^{\lambda^{-1},\alpha})\mathcal{A}^*.$$

Putting these two facts together we find that

$$\begin{aligned}
\int_{\mathbb{R}} h(E)\, d\mu_{H_\omega^{\lambda^{-1},\alpha},\psi_\omega} &= \langle \psi_\omega, h(H_\omega^{\lambda^{-1},\alpha})\psi_\omega \rangle_{\ell^2(\mathbb{Z})} \\
&= \sum_{n\in\mathbb{Z}} \overline{\psi_\omega(n)} [h(H_\omega^{\lambda^{-1},\alpha})\psi_\omega](n) \\
&= \sum_{n\in\mathbb{Z}} \overline{[\mathcal{A}^*\varphi](\omega,n)} [h(H^{\lambda^{-1},\alpha})\mathcal{A}^*\varphi](\omega,n) \\
&= \lambda^{-1} \sum_{n\in\mathbb{Z}} \overline{[\mathcal{A}^*\varphi](\omega,n)} [\mathcal{A}^* h(H^{\lambda,\alpha})\varphi](\omega,n)
\end{aligned}$$

is almost everywhere constant. Since this holds for arbitrary $h \in C(\mathbb{R})$ and $C(\Sigma^{\lambda^{-1},\alpha})$ is separable, it follows that the spectral measure $\mu_{H_\omega^{\lambda^{-1},\alpha},\psi_\omega}$ is almost surely ω-independent.

Recall that Corollary 3.21 implies that the singular parts of $\mu_{H_\omega^{\lambda^{-1},\alpha},\psi_\omega}$ and $\mu_{H_{\omega'}^{\lambda^{-1},\alpha},\psi_{\omega'}}$ are mutually singular for almost every $\omega, \omega' \in \mathbb{T}$. It therefore follows that $\mu_{H_\omega^{\lambda^{-1},\alpha},\psi_\omega}$ must be purely absolutely continuous for almost every $\omega \in \mathbb{T}$. $\square$

Let us now turn to the proof of the two duality statements for the spectral type.

Proof of Theorem 7.4. The eigenvectors

$$\{\psi_k(\cdot;\omega,m)\}_{k\in\mathbb{Z}_+,m\in\mathbb{Z}}$$

span the pure point subspace of $H_\omega^{\lambda,\alpha}$. Thus, if $H_\omega^{\lambda,\alpha}$ has some point spectrum for almost every $\omega\in\mathbb{T}$, then for some $m\in\mathbb{Z}$,

$$\psi_1(\cdot;\cdot,m)\neq 0 \quad \text{as an element of } L^2.$$

Now set $g(\omega)\equiv 1$ and apply Lemma 7.9. We obtain that the spectral measure $\mu_{H_\omega^{\lambda^{-1},\alpha},\psi_\omega}$ is purely absolutely continuous for almost every $\omega\in\mathbb{T}$. But since

$$\begin{aligned}\int_{\mathbb{T}}\int_{\mathbb{R}} d\mu_{H_\omega^{\lambda^{-1},\alpha},\psi_\omega}(E)\,d\omega &= \int_{\mathbb{T}}\langle\psi_\omega,\psi_\omega\rangle_{\ell^2(\mathbb{Z})}\,d\omega\\ &= \langle\mathcal{A}^*\varphi,\mathcal{A}^*\varphi\rangle_{L^2(\mathbb{T}\times\mathbb{Z})}\\ &= \langle\varphi,\varphi\rangle_{L^2(\mathbb{T}\times\mathbb{Z})}\\ &= \sum_{n\in\mathbb{Z}}\int_{\mathbb{T}}|\psi_1(n;\omega,m)|^2\,d\omega\\ &\neq 0,\end{aligned}$$

it follows that $\mu_{H_\omega^{\lambda^{-1},\alpha},\psi_\omega}\neq 0$ for a set of ω's of positive measure, and hence for a set of full measure due to almost everywhere constancy. This shows that $H_\omega^{\lambda^{-1},\alpha}$ has some absolutely continuous spectrum for almost every $\omega\in\mathbb{T}$. $\square$

Proof of Theorem 7.3. Fix an orthonormal basis $\{g_i\}$ of $L^2(\mathbb{T})$. By assumption, for almost every $\omega\in\mathbb{T}$,

$$\{\psi_k(\cdot;\omega,m)\}_{k,m}$$

is an orthonormal basis of $\ell^2(\mathbb{Z})$, where only those k,m are considered for which $\psi_k(\cdot;\omega,m)\neq 0$. Let

$$\varphi_{i,k,m}(\omega,n)=g_i(\omega)\psi_k(n;\omega,m).$$

Then, $\{\varphi_{i,k,m}\}_{i,k,m}$ is a complete orthogonal set in $L^2(\mathbb{T}\times\mathbb{Z})$. By unitarity of $\mathcal{A}^*$, $\{\mathcal{A}^*\varphi_{i,k,m}\}_{i,k,m}$ is a total orthogonal set in $L^2(\mathbb{T}\times\mathbb{Z})$ as well. But then, for almost every $\omega\in\mathbb{T}$, $\{[\mathcal{A}^*\varphi_{i,k,m}](\omega,\cdot)\}_{i,k,m}$ is a total orthogonal set in $\ell^2(\mathbb{Z})$. Since their spectral measures relative to $H_\omega^{\lambda^{-1},\alpha}$ are purely absolutely continuous for almost every $\omega\in\mathbb{T}$ as we have seen above, it follows that $H_\omega^{\lambda^{-1},\alpha}$ has purely absolutely continuous spectrum for almost every $\omega\in\mathbb{T}$. $\square$

2. The Case of Rational Frequency

3. The Lebesgue Measure of the Spectrum

3.1. Noncritical Coupling.

3.2. Critical Coupling: Unbounded Partial Quotients.

3.3. Critical Coupling: Recurrent Diophantine Condition.

4. The Spectral Type

4.1. Absence of Eigenvalues for Subcritical Coupling.

THEOREM 7.10 (Delyon 1987). *Suppose* $0 < \lambda < 1$. *Then, for every* α *and every* ω*, the almost Mathieu operator* $H_\omega^{\lambda,\alpha}$ *has purely continuous spectrum.*

PROOF. Assume to the contrary that there are $\lambda \in (0,1)$, $\alpha \in \mathbb{T}$ irrational, $\omega \in \mathbb{T}$, and $E \in \mathbb{R}$ such that the almost Mathieu difference equation (7.1) has a solution $u \in \ell^2(\mathbb{Z})$ with $\|u\|_{\ell^2(\mathbb{Z})} = 1$. For the Fourier transform $\hat{u}$ of u, we therefore have

$$\int_{\mathbb{T}} |\hat{u}(\theta)|^2 \, d\theta = 1. \tag{7.4}$$

Suppose $\tilde{\omega}$ belongs to the full-measure subset of $\mathbb{T}$, where the sequence $\tilde{u}$ defined by

$$\tilde{u}(n) = \hat{u}(\tilde{\omega} + n\alpha) e^{2\pi i n \omega}$$

is a solution of the dual almost Mathieu difference equation (7.2); see Lemma 7.1.(b).

From (7.4) we may infer that for $\varepsilon > 0$,

$$\begin{aligned}
\int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} |\tilde{u}(n)|^2 (1+|n|)^{-1-2\varepsilon} \, d\tilde{\omega} &= \sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{T}} |\hat{u}(\tilde{\omega}+n\alpha)|^2 \, d\tilde{\omega} \right) (1+|n|)^{-1-2\varepsilon} \\
&= \sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{T}} |\hat{u}(\theta)|^2 \, d\theta \right) (1+|n|)^{-1-2\varepsilon} \\
&= \sum_{n\in\mathbb{Z}} (1+|n|)^{-1-2\varepsilon} \\
&< \infty.
\end{aligned}$$

As a consequence, we obtain that for every $\varepsilon > 0$ and almost every $\tilde{\omega} \in \mathbb{T}$, there is a constant $C(\varepsilon, \tilde{\omega}) < \infty$ such that

$$|\tilde{u}(n)| \le C(\varepsilon, \tilde{\omega})(1+|n|)^{\frac{1}{2}+\varepsilon} \quad \text{for every } n \in \mathbb{Z}. \tag{7.5}$$

The solution $\tilde{u}$ cannot vanish identically for almost every such $\tilde{\omega}$ due to (7.4). Thus, there is a set $A \subset \mathbb{T}$ of positive measure so that for $\tilde{\omega} \in A$, $\tilde{u}$ exists, solves, is non-trivial and obeys the estimate (7.5).

On the other hand, the Lyapunov exponent associated with the operator family $\{H_{\tilde{\omega}}^{\lambda^{-1},\alpha}\}_{\tilde{\omega}\in\mathbb{T}}$ is strictly positive and hence, by Osceledec, for the energy in question, $\lambda^{-1}E$, and almost every $\tilde{\omega} \in \mathbb{T}$, all solutions of (7.2) behave exponentially. By construction, $\tilde{u}$ must therefore be exponentially decaying for almost every $\tilde{\omega}$ from the set A of positive measure and hence $\lambda^{-1}E$ is an eigenvalue of $H_{\tilde{\omega}}^{\lambda^{-1},\alpha}$ for almost every $\tilde{\omega} \in A$. This is a contradiction; compare Theorem 3.2. □

4.2. Localization for Supercritical Coupling. In this subsection we present a proof of Jitomirskaya's localization result. Recall that for $|\lambda| > 1$, the Lyapunov exponent is strictly bounded away from zero uniformly in z. Thus one expects exponential behavior of solutions. Combining this with the existence of generalized eigenfunctions, that is, solutions u_E obeying $|u_E(n)| \le C(1+|n|)$, for spectrally all energies E, one would therefore expect exponentially decaying eigenfunctions to exist for spectrally all energies. This picture cannot hold in complete generality, however, as we have seen that for Liouville α, there are no eigenvalues for any coupling and any phase.

DEFINITION 7.11. *An irrational number $\alpha \in \mathbb{T}$ is called Diophantine if there are constants $c = c(\alpha) > 0$ and $r = r(\alpha) > 1$ such that*

$$|\sin(2\pi n\alpha)| > \frac{c}{|n|^r} \quad \text{for every } n \in \mathbb{Z} \setminus \{0\}. \tag{7.6}$$

Given such an α, $\omega \in \mathbb{T}$ is called resonant if the relation

$$\left|\sin\left(2\pi\left(\omega + \frac{n}{2}\alpha\right)\right)\right| < \exp\left(-|n|^{\frac{1}{2r}}\right) \tag{7.7}$$

holds for infinitely many $n \in \mathbb{Z}$; otherwise ω is called non-resonant.

It is known that Lebesgue almost every α is Diophantine. Moreover, the set of resonant ω's is a dense G_δ set (as can be seen directly from the definition) of zero Lebesgue measure (by Borel-Cantelli).

THEOREM 7.12 (Jitomirskaya 1999). *Suppose $\lambda > 1$, $\alpha \in \mathbb{T}$ is Diophantine, and $\omega \in \mathbb{T}$ is non-resonant. Then, the almost Mathieu operator $H_\omega^{\lambda,\alpha}$ has pure point spectrum with exponentially decaying eigenfunctions.*

Thus, localization holds for the almost Mathieu operator at super-critical coupling for almost every frequency and almost every phase. By the Gordon and Jitomirskaya-Simon criteria, none of the almost everywhere statements can be strengthened to an everywhere statement.

In the following, the parameters are assumed to satisfy the assumptions of the theorem and they will be fixed throughout the proof. As a consequence, we will sometimes not make the dependence of various quantities on them explicit.

We first explain how the Diophantine condition on α enters the picture. Given a monotonically increasing increasing function ϕ, the points $\omega_1, \dots, \omega_n \in \mathbb{T}$ are called ϕ-uniformly distributed if for every $h \in C(\mathbb{T})$, we have

$$\left|\sum_{j=1}^{n} h(\omega_j) - n \int h(\omega)\, d\omega\right| \le \phi(n)\mathrm{Var}(h),$$

where $\mathrm{Var}(h)$ denotes the total variation of h.

PROPOSITION 7.13. (a) *Suppose $\alpha \in \mathbb{T}$ is Diophantine with constants $c = c(\alpha) > 0$ and $r = r(\alpha) > 1$. Then, for every $\omega \in \mathbb{T}$ and every $n \in \mathbb{Z}_+$, the points*

$$\omega, \omega + \alpha, \omega + 2\alpha, \dots, \omega + (n-1)\alpha \in \mathbb{T}$$

are ϕ-uniformly distributed with

$$\phi(n) = \tilde{c} n^{1-r^{-1}} \log n,$$

with a suitable constant $\tilde{c}$.

(b) *If ϕ and ψ are both $o(n)$, then for every $\varepsilon > 0$ and $n > n_0(\varepsilon, \phi, \psi)$, we have*

$$\begin{aligned} n(\log 2 - \varepsilon) &\le \sum_{1 \le i \le n, i \ne j_1, \dots, j_{\psi(n)}} \log|z - \cos(2\pi\omega_i)|^{-1} \\ &\le n(\log 2 + \varepsilon) - 2\phi(n) \log \min_{1 \le i \le n, i \ne j_1, \dots, j_{\psi(n)}} |z - \cos(2\pi\omega_i)| \end{aligned}$$

for $z \in [-1, 1]$, ϕ-uniformly distributed points $\omega_1, \dots, \omega_n$ and $j_1, \dots, j_{\psi(n)} \in \{1, 2, \dots, n\}$.

PROOF. The proof is outside the scope of this text; see [**16**, Lemmas 11 and 12 and Equation (5.8)] and references mentioned there. ☐

DEFINITION 7.14. *Let*

$$P_k(\omega, E) = \det\left[(H_\omega^{\lambda,\alpha} - E)\Big|_{[0,k-1]} \right]$$

and

$$\mathcal{K} = \left\{ k \in \mathbb{Z}_+ : \exists\, \omega \in \mathbb{T} \text{ with } |P_k(\omega, E)| \geq \frac{1}{\sqrt{2}} e^{k\gamma(E)} \right\}.$$

LEMMA 7.15. *There are coefficients b_j, $0 \leq j \leq k$, such that*

$$P_k(\omega, E) = \sum_{j=0}^{k} b_j \left(\cos\left(2\pi(\omega + \frac{k-1}{2}\alpha) \right) \right)^j.$$

PROOF. Since cos is an even function, it follows that the change of basis $\delta_j \mapsto \delta_{k-1-j}$ transforms

$$H^{\lambda,\alpha}_{\omega - \frac{k-1}{2}}\Big|_{[0,k-1]} \quad \text{into} \quad H^{\lambda,\alpha}_{-\omega - \frac{k-1}{2}}\Big|_{[0,k-1]}.$$

Thus,

$$P_k\left(\omega - \frac{k-1}{2}, E\right) = P_k\left(-\omega - \frac{k-1}{2}, E\right). \tag{7.8}$$

Thus, the Fourier expansion of $\omega \mapsto P_k(\omega - \frac{k-1}{2}, E)$ reads

$$P_k\left(\omega - \frac{k-1}{2}, E\right) = \sum_{j=0}^{k} \tilde{b}_j \cos\left(2\pi j \left(\omega + \frac{k-1}{2}\alpha \right) \right)$$

since all the sin terms are absent due to (7.8) and the degree obviously does not exceed k. From this representation the lemma follows since one can see inductively that the linear span of $\{1, \cos(2\pi x), \cos(2\pi 2x), \dots, \cos(2\pi k x)\}$ is equal to the linear span of $\{1, \cos(2\pi x), \cos^2(2\pi x) \dots, \cos^k(2\pi x)\}$. ☐

LEMMA 7.16. *For every $k \in \mathbb{Z}_+$, at least one of $k, k+1, k+2$ belongs to $\mathcal{K}$.*

PROOF. Recall that the transfer matrix $M_E(k, \omega)$ may be written as

$$M_E(k,\omega) = \begin{pmatrix} P_k(\omega, E) & -P_{k-1}(\omega+\alpha, E) \\ P_{k-1}(\omega, E) & -P_{k-2}(\omega+\alpha, E) \end{pmatrix}. \tag{7.9}$$

Therefore, the statement of lemma follows from Kingman's subadditive ergodic theorem; see Theorem 2.3. ☐

When the Lyapunov exponent is positive, on average the transfer matrices have exponentially large norm and hence some of the entries must be exponentially large. These entries in turn appear in a description of the Green's function of the operator restricted to a finite interval. Namely, by Cramer's Rule, we have for n_1, $n_2 = n_1 + k - 1$, and $n \in [n_1, n_2]$,

$$\left| G_{[n_1,n_2]}(n_1, n; E) \right| = \left| \frac{P_{n_2 - n}(\omega + (n+1)\alpha, E)}{P_k(\omega + n_1\alpha, E)} \right|, \tag{7.10}$$

$$\left| G_{[n_1,n_2]}(n, n_2; E) \right| = \left| \frac{P_{n - n_1}(\omega + n_1\alpha, E)}{P_k(\omega + n_1\alpha, E)} \right|. \tag{7.11}$$

We have the following uniform bound:

LEMMA 7.17. *For every $E \in \mathbb{R}$ and $\varepsilon > 0$, there exists $k(E,\varepsilon)$ such that*

$$|P_k(\omega, E)| < \exp\left((\gamma(E)+\varepsilon)k\right)$$

for every $k > k(E,\varepsilon)$ and every $\omega \in \mathbb{T}$.

PROOF. This is an immediate consequence of (7.9) and Theorem 2.5. □

DEFINITION 7.18. *Fix $E \in \mathbb{R}$ and $\gamma \in \mathbb{R}$. A point $n \in \mathbb{Z}$ will be called (γ,k)-regular if there exists an interval $[n_1,n_2]$, containing n such that*

(i) $n_2 = n_1 + k - 1,$

(ii) $n \in [n_1, n_2],$

(iii) $|n - n_i| > \dfrac{k}{5},$

(iv) $\left|G_{[n_1,n_2]}(n, n_i; E)\right| < \exp\left(-\gamma|n-n_i|\right).$

Otherwise, n is called (γ,k)-singular.

LEMMA 7.19. *Fix $E \in \mathbb{R}$. Suppose n is $(\gamma(E)-\varepsilon,k)$-singular for some $0 < \varepsilon < \frac{\gamma(E)}{3}$ and $k > 4k(E,\frac{\varepsilon}{4}) + 1$. Then, for every j with*

$$n - \tfrac{3}{4}k \le j \le n - \tfrac{3}{4}k + \tfrac{k+1}{2},$$

we have that

$$|P_k(\omega + j\alpha, E)| \le \exp\left(k\left(\gamma(E) - \frac{\varepsilon}{8}\right)\right).$$

PROOF. Since n is $(\gamma(E)-\varepsilon,k)$-singular, it follows that for every interval $[n_1,n_2]$ of length k containing n with $|n-n_i| > \frac{k}{5}$, we have that

$$\left|G_{[n_1,n_2]}(n, n_i; E)\right| \ge \exp\left(-\left(\gamma(E)-\varepsilon\right)|n-n_i|\right).$$

By (7.10), this means that

$$\left|\frac{P_{n_2-n}(\omega+(n+1)\alpha, E)}{P_k(\omega+n_1\alpha, E)}\right| \ge \exp\left(-\left(\gamma(E)-\varepsilon\right)|n-n_1|\right).$$

We can choose n_1 to be equal to the j in question and set $n_2 = j+k-1$. Then we find, using Lemma 7.17,

$$\begin{aligned}
|P_k(\omega+j\alpha,E)| &\le |P_{j+k-1-n}(\omega+(n+1)\alpha,E)| \exp\left(\left(\gamma(E)-\varepsilon\right)|n-j|\right)\\
&\le \exp\left((\gamma(E)+\tfrac{\varepsilon}{4})|j+k-1-n|\right)\exp\left(\left(\gamma(E)-\tfrac{\varepsilon}{4}\right)|n-j|\right)\\
&= \exp\left((k-1)\gamma(E) + \tfrac{\varepsilon}{4}(j+k-1-n)+(j-n)\right)\\
&\le \exp\left((k-1)\gamma(E) + \tfrac{\varepsilon}{4}(-\tfrac{3}{4}k + \tfrac{k+1}{2} + k - 1 - \tfrac{3}{4}k + \tfrac{k+1}{2})\right)\\
&\le \exp\left(k(\gamma(E) + \tfrac{\varepsilon}{4}(-\tfrac{3}{4} + 2 - \tfrac{3}{4}))\right)\\
&= \exp\left(k(\gamma(E) - \tfrac{\varepsilon}{8})\right)
\end{aligned}$$

and the lemma follows. □

LEMMA 7.20. *Suppose $n \in [n_1,n_2] \subset \mathbb{Z}$ and u is a solution of the difference equation $Hu = Eu$. Then,*

$$u(n) = -G_{[n_1,n_2]}(n,n_1;E)u(n_1-1) - G_{[n_1,n_2]}(n,n_2;E)u(n_2+1). \tag{7.12}$$

In particular, if u_E is a generalized eigenfunction, then every point $n \in \mathbb{Z}$ with $u_E(n) \neq 0$ is (γ,k)-singular for $k > k_1 = k_1(E,\gamma,\omega,n)$.

PROOF. The identity (7.12) was shown earlier; see (1.17). From this formula we see that when $u_E(n) \neq 0$ and u_E is polynomially bounded, one cannot have exponentially small Green function entries. Clearly, the largeness condition on k only depends on the specified quantities. $\square$

LEMMA 7.21. *For every $n \in \mathbb{Z}$, $\varepsilon > 0$, $\tau < 2$, there exists $k_2 = k_2(\omega, \alpha, n, \varepsilon, \tau, E)$ such that for every $k \in \mathcal{K}$ with $k > k_2$, we have that*

$$m, n \text{ are both } (\gamma(E) - \varepsilon, k)\text{-singular and } |m - n| > \frac{k+1}{2} \Rightarrow |m - n| > k^\tau.$$

PROOF. Assume that m_1 and m_2 are both $(\gamma(E) - \varepsilon, k)$-singular with

$$d = m_2 - m_1 > \frac{k+1}{2}. \tag{7.13}$$

We set

$$n_i = m_i - \left\lfloor \frac{3}{4} k \right\rfloor, \quad i = 1, 2. \tag{7.14}$$

By Lemma 7.15, there is a polynomial Q_k of degree k such that

$$P_k(\omega) = Q_k \left(\cos \left(2\pi(\omega + \frac{k-1}{2} \alpha \right) \right). \tag{7.15}$$

Here we suppress the dependence on E since the energy will be fixed throughout the proof.

Let

$$\omega_j = \begin{cases} \omega + \left(n_1 + \frac{k-1}{2} + j\right) \alpha, & j = 0, 1, \dots, \lfloor \frac{k+1}{2} \rfloor - 1, \\ \omega + \left(n_2 + \frac{k-1}{2} + j - \lfloor \frac{k+1}{2} \rfloor\right) \alpha, & j = \lfloor \frac{k+1}{2} \rfloor, \lfloor \frac{k+1}{2} \rfloor + 1, \dots, k. \end{cases}$$

Due to (7.13) and (7.14), the points $\omega_0, \omega_1, \dots, \omega_k$ are distinct. In fact, we will see below that even the points $\cos(2\pi\omega_0), \cos(2\pi\omega_1), \dots, \cos(2\pi\omega_k)$ are distinct.

Lagrange interpolation then shows

$$|Q_k(z)| = \left| \sum_{j=0}^{k} Q_k(\cos(2\pi\omega_j) \frac{\prod_{l \neq j}(z - \cos(2\pi\omega_l)}{\prod_{l \neq j}(\cos(2\pi\omega_j) - \cos(2\pi\omega_l))} \right|. \tag{7.16}$$

By Lemma 7.19 there exists k_3 such that for $k > k_3$

$$|Q_k(\cos(2\pi\omega_j)| < \exp\left(\frac{k}{8}(\gamma(E) - \varepsilon)\right), \quad j = 0, 1, \dots, k. \tag{7.17}$$

We claim that if $d < k^\tau$ for some $\tau < 2$, there exists k_4 (depending on τ and all the other parameters fixed up to this point) so that for $k > k_4$, we have

$$\frac{|\prod_{l \neq j}(z - \cos(2\pi\omega_l)|}{|\prod_{l \neq j}(\cos(2\pi\omega_j) - \cos(2\pi\omega_l))|} \leq \exp\left(\frac{k\varepsilon}{16}\right) \quad \text{for } z \in [-1, 1],\ 0 \leq j \leq k. \tag{7.18}$$

Assuming this claim for the moment, we can complete the proof of the repulsion of singular clusters. Given $\tau < 2$, consider $k \in \mathcal{K}$ with $k > \max\{k_3, k_4\}$ and $\tilde{\omega}$ with

$$|P_k(\tilde{\omega})| \geq \frac{1}{\sqrt{2}} e^{k\gamma(E)}.$$

But assuming $d < k^\tau$, we also have the following upper bound,

$$|P_k(\tilde{\omega})| \leq (k+1) \exp\left(\frac{k}{8}(\gamma(E) - \varepsilon)\right) \exp\left(\frac{k\varepsilon}{16}\right),$$

which follows from (7.15)–(7.18) for $z = \cos\left(2\pi(\tilde{\omega} + \frac{k-1}{2}\alpha\right)$. This contradiction shows that $d < k^\tau$ is impossible.

It remains to prove the claim (7.18). Using the trigonometric formula

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

we may write

$$\cos(2\pi\omega_j) - \cos(2\pi\omega_l) = -2\sin\left(2\pi\left(\omega + n_1\alpha + \frac{k-1+i_j+i_l}{2}\alpha\right)\right)\sin(2\pi(i_j - i_l)\alpha),$$

where

$$i_j = \frac{\omega_j - \omega}{\alpha} - \left(n_1 + \frac{k-1}{2}\right) \in \mathbb{Z}.$$

Since ω is non-resonant, we have

$$\left|\sin\left(2\pi\left(\omega + \frac{m}{2}\alpha\right)\right)\right| \geq \exp\left(-|m|^{\frac{1}{2r(\alpha)}}\right)$$

for m sufficiently large; compare (7.7). Combining this with the Diophantine condition (7.6), it follows that

$$|\cos(2\pi\omega_j) - \cos(2\pi\omega_l)| \geq \exp\left(-\left(d + \frac{k}{4} + 1\right)^{\frac{1}{2r(\alpha)}}\right)\frac{c(\alpha)}{(d+k+1)^{r(\alpha)}} \tag{7.19}$$

Together with Proposition 7.13.(b) (applied to $\frac{\varepsilon}{11}$ instead of ε), (7.19) implies

$$\begin{aligned}\left|\prod_{l\neq j}(z - \cos(2\pi\omega_l))\right| &= e^{\sum_{l\neq j}\log|\cos(2\pi\omega_j) - \cos(2\pi\omega_l)|}\\ &\leq e^{(k+1)\left(-\log 2 + \frac{\varepsilon}{33}\right)}\end{aligned}$$

and

$$\begin{aligned}\left|\prod_{l\neq j}(\cos(2\pi\omega_j) - \cos(2\pi\omega_l))\right| &= e^{\sum_{l\neq j}\log|\cos(2\pi\omega_j) - \cos(2\pi\omega_l)|}\\ &\geq e^{(k+1)\left(-\log 2 - \frac{\varepsilon}{33}\right) - 2\phi(k+1)\left(\left(d+\frac{k}{4}+1\right)^{\frac{1}{2r(\alpha)}} - \log\frac{c(\alpha)}{(d+k+1)^{r(\alpha)}}\right)},\end{aligned}$$

where ϕ is as in Proposition 7.13.(a).

For $\tau < 2$ and $d \leq k^\tau$, these estimates imply

$$\begin{aligned}\frac{|\prod_{l\neq j}(z - \cos(2\pi\omega_l)|}{|\prod_{l\neq j}(\cos(2\pi\omega_j) - \cos(2\pi\omega_l))|} &\leq \frac{e^{(k+1)\left(-\log 2 + \frac{\varepsilon}{33}\right)}}{e^{(k+1)\left(-\log 2 - \frac{\varepsilon}{33}\right) - 2\phi(k+1)\left(\left(d+\frac{k}{4}+1\right)^{\frac{1}{2r(\alpha)}} - \log\frac{c(\alpha)}{(d+k+1)^{r(\alpha)}}\right)}}\\ &= e^{(k+1)\frac{2\varepsilon}{33} + 2\phi(k+1)\left(\left(d+\frac{k}{4}+1\right)^{\frac{1}{2r(\alpha)}} - \log\frac{c(\alpha)}{(d+k+1)^{r(\alpha)}}\right)}\\ &= e^{(k+1)\frac{2\varepsilon}{33} + o(k)}\end{aligned}$$

since $\phi = o(1)$ and $r(\alpha) > 1$. This proves (7.18) and completes the proof of the lemma. $\square$

PROOF OF THEOREM 7.12. Let $E(\omega)$ be a generalized eigenvalue of $H_\omega^{\lambda,\alpha}$ and denote the corresponding generalized eigenfunction by u_E. Assume without loss of generality $u_E(0) \neq 0$ (otherwise replace zero by one).

By Lemmas 7.20 and 7.21, if

$$|n| > \max\{k_1(E, \gamma(E) - \varepsilon, \omega, 0), k_2(\omega, \alpha, 0, \varepsilon, 1.5, E)\} + 1,$$

the point n is $(\gamma(E) - \varepsilon, k)$-regular for some $k \in \{|n| - 1, |n|, |n| + 1\} \cap \mathcal{K} \neq \emptyset$, since 0 is $(\gamma(E) - \varepsilon, k)$-singular. Thus, there exists an interval $[n_1, n_2]$ of length k containing n such that

$$\frac{1}{5}(|n| - 1) \leq |n - n_i| \leq \frac{4}{5}(|n| + 1)$$

and

$$\left|G_{[n_1,n_2]}(n, n_i)\right| < e^{-(\gamma(E)-\varepsilon)|n-n_i|}.$$

From this and (7.12), we therefore see that

$$|u_E(n)| \leq 2C(u_E)(2|n| + 1)e^{-\left(\frac{\gamma(E)-\varepsilon}{5}\right)(|n|-1)}.$$

By the uniform lower bound $\gamma(E) \geq \log \lambda$, this implies exponential decay if ε is chosen small enough. □

5. The Cantor Structure of the Spectrum

5.1. Diophantine Frequencies: Localization and Duality. In this subsection, we present Puig's proof of the striking fact that localization for the operator family $\{H_\omega^{\lambda,\alpha}\}_{\omega\in\mathbb{T}}$, as established in Theorem 7.12, implies via Aubry duality and reducibility Cantor spectrum for the dual family $\{H_\omega^{\lambda^{-1},\alpha}\}_{\omega\in\mathbb{T}}$. Once one has shown Cantor spectrum for $0 < \lambda < 1$, Aubry duality applied again yields Cantor spectrum for $\lambda > 1$. Thus, Puig's proof derives the following statement from the localization result, Theorem 7.12.

THEOREM 7.22 (Puig 2004). *Suppose α is Diophantine and $\lambda \in (0, \infty) \setminus \{1\}$. Then, $\sigma(H_\omega^{\lambda,\alpha})$ is a Cantor set for every $\omega \in \mathbb{T}$.*

In the course of the proof we will consider solutions of the almost Mathieu difference equation (7.1) and its dual (7.2).

LEMMA 7.23. *Suppose u is a solution of (7.1). If $\omega = 0$ or $\omega = \frac{1}{2}$, then the reflected sequence $\tilde{u}$ given by $\tilde{u}(n) = u(-n)$ is a solution as well.*

PROOF. This is a simple consequence of the fact that cos is even. Explicitly, for $\omega = 0$, we have

$$\begin{aligned} E\tilde{u}(n) &= Eu(-n) \\ &= u(-n-1) + u(-n+1) + 2\lambda\cos(2\pi(-n\alpha))u(-n) \\ &= \tilde{u}(n+1) + \tilde{u}(n-1) + 2\lambda\cos(2\pi(n\alpha))\tilde{u}(n), \end{aligned}$$

whereas for $\omega = \frac{1}{2}$, we have

$$\begin{aligned} E\tilde{u}(n) &= Eu(-n) \\ &= u(-n-1) + u(-n+1) + 2\lambda\cos(2\pi(\tfrac{1}{2} - n\alpha))u(-n) \\ &= \tilde{u}(n+1) + \tilde{u}(n-1) + 2\lambda\cos(2\pi(-\tfrac{1}{2} + n\alpha))\tilde{u}(n) \\ &= \tilde{u}(n+1) + \tilde{u}(n-1) + 2\lambda\cos(2\pi(\tfrac{1}{2} + n\alpha))\tilde{u}(n). \end{aligned}$$

The assertion of the lemma follows. $\square$

The case $\omega = 0$ will be of particular interest, so let us list the equations

$$u(n+1) + u(n-1) + 2\lambda\cos(2\pi n\alpha)u(n) = Eu(n), \tag{7.20}$$

$$u(n+1) + u(n-1) + 2\lambda^{-1}\cos(2\pi n\alpha)u(n) = (\lambda^{-1}E)u(n) \tag{7.21}$$

separately for future reference. Let us discuss the content of Lemma 7.1 in a context relevant to the proof of Theorem 7.22 to show explicitly how exponentially decaying eigenfunctions and highly regular quasi-periodic solutions are related to each other via Aubry duality.

LEMMA 7.24. (a) *Suppose u is an exponentially decaying solution of* (7.20). *Consider its Fourier series*

$$\hat{u}(\omega) = \sum_{m\in\mathbb{Z}} u(m)e^{2\pi i m\omega}.$$

Then, $\hat{u}$ is real-analytic on $\mathbb{T}$, it extends analytically to a strip, and the sequence $\tilde{u}(n) = \hat{u}(n\alpha)$ is a solution of (7.21).

(b) *Conversely, suppose u is a solution of* (7.21) *of the form $u(n) = g(n\alpha)$ for some real-analytic function g on $\mathbb{T}$. Consider the Fourier series*

$$g(\omega) = \sum_{n\in\mathbb{Z}} \hat{g}(n)e^{2\pi i n\omega}.$$

Then, the sequence $\{\hat{g}(n)\}$ is an exponentially decaying solution of (7.20).

PROOF. (a) Since u is exponentially decaying, $\hat{u}$ extends to a function that is analytic in a neighborhood of the unit circle $\mathbb{T}$. The other statement follows from Lemma 7.1.

(b) Since u is a solution of (7.21) and we have $u(m) = g(m\alpha)$, we have

$$g((m+1)\alpha) + g((m-1)\alpha) + 2\lambda^{-1}\cos(2\pi m\alpha)g(m\alpha) = (\lambda^{-1}E)g(m\alpha).$$

Rewriting this in terms of the Fourier expansion, we find

$$\sum_{n\in\mathbb{Z}} \hat{g}(n)e^{2\pi i n(m+1)\alpha} + \sum_{n\in\mathbb{Z}} \hat{g}(n)e^{2\pi i n(m-1)\alpha} + \lambda^{-1}\left(e^{-2\pi i m\alpha} + e^{2\pi i m\alpha}\right)\sum_{n\in\mathbb{Z}} \hat{g}(n)e^{2\pi i n m\alpha}$$
$$= (\lambda^{-1}E)\sum_{n\in\mathbb{Z}} \hat{g}(n)e^{2\pi i n m\alpha}.$$

It follows that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} E\hat{g}(n)e^{2\pi i n m\alpha} &= \sum_{n\in\mathbb{Z}}\left(\lambda\hat{g}(n)e^{2\pi i n(m+1)\alpha} + \lambda\hat{g}(n)e^{2\pi i n(m-1)\alpha} + \left(e^{-2\pi i m\alpha} + e^{2\pi i m\alpha}\right)\hat{g}(n)e^{2\pi i n m\alpha}\right)\\ &= \sum_{n\in\mathbb{Z}} \left(2\lambda\cos(2\pi n\alpha)\hat{g}(n) + \hat{g}(n-1) + \hat{g}(n+1)\right)e^{2\pi i n m\alpha}\end{aligned}$$

Since g is real-analytic on $\mathbb{T}$ with analytic extension to a strip, it follows that the Fourier coefficients of g decay exponentially and satisfy the difference equation (7.20). $\square$

The next step is to use the information provided by the previous lemma to reduce to constant coefficients. We prove a general statement to this effect:

LEMMA 7.25. *Let $\alpha \in \mathbb{T}$ be Diophantine and suppose $A : \mathbb{T} \to \mathrm{SL}(2, \mathbb{R})$ is a real-analytic map, with analytic extension to the strip $|\Im\omega| < \delta$ for some $\delta > 0$. Assume that there is a non-vanishing real-analytic map $v : \mathbb{T} \to \mathbb{R}^2$ with analytic extension to the same strip $|\Im\omega| < \delta$ such that*

$$v(\omega + \alpha) = A(\omega)v(\omega) \quad \text{for every } \omega \in \mathbb{T}.$$

Then, there are a real number c and a real-analytic map $B : \mathbb{T} \to \mathrm{SL}(2, \mathbb{R})$ with analytic extension to the strip $|\Im\omega| < \delta$ such that with

$$C = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \tag{7.22}$$

we have

$$B(\omega + \alpha)^{-1}A(\omega)B(\omega) = C \quad \text{for every } \omega \in \mathbb{T}. \tag{7.23}$$

PROOF. Since v does not vanish, $d(\omega) = v_1(\omega)^2 + v_2(\omega)^2$ is strictly positive and hence we can define

$$B_1(\omega) = \begin{pmatrix} v_1(\omega) & -\frac{v_2(\omega)}{d(\omega)} \\ v_2(\omega) & \frac{v_1(\omega)}{d(\omega)} \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

for $\omega \in \mathbb{T}$. We have

$$A(\omega)B_1(\omega) = \begin{pmatrix} v_1(\omega + \alpha) & * \\ v_2(\omega + \alpha) & * \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) \tag{7.24}$$

and hence

$$A(\omega)B_1(\omega) = B_1(\omega + \alpha)\tilde{C}(\omega)$$

with

$$\tilde{C}(\omega) = \begin{pmatrix} 1 & \tilde{c}(\omega) \\ 0 & 1 \end{pmatrix},$$

where $\tilde{c} : \mathbb{T} \to \mathbb{R}$ is analytic. Indeed, by (7.24) the first column of $\tilde{C}(\omega)$ is determined and then its $(2, 2)$ entry must be one since $\tilde{C}(\omega) = B_1(\omega + \alpha)^{-1}A(\omega)B_1(\omega) \in \mathrm{SL}(2, \mathbb{R})$. Now let

$$c = \int_{\mathbb{T}} \tilde{c}(\omega)\, d\omega.$$

and define the matrix C as in (7.22).

We claim that we can find $b : \mathbb{T} \to \mathbb{R}$ analytic (with analytic extension to a strip) such that

$$b(\omega + \alpha) - b(\omega) = \tilde{c}(\omega) - c \quad \text{for every } \omega \in \mathbb{T}. \tag{7.25}$$

Indeed, expand both sides of the hypothetical identity (7.25) in Fourier series:

$$\sum_{k\in\mathbb{Z}} b_k e^{2\pi i(\omega+\alpha)k} - \sum_{k\in\mathbb{Z}} b_k e^{2\pi i\omega k} = \sum_{k\in\mathbb{Z}} \tilde{c}_k e^{2\pi i\omega k} - c.$$

Since we have $\tilde{c}_0 = c$, the $k = 0$ terms disappear on both sides and hence all we need to do is to require

$$b_k(e^{2\pi i\alpha k} - 1) = \tilde{c}_k \quad \text{for every } k \in \mathbb{Z} \setminus \{0\}.$$

In other words, if we set $b_0 = 0$ and

$$b_k = \frac{\tilde{c}_k}{e^{2\pi i\alpha k} - 1} \quad \text{for every } k \in \mathbb{Z} \setminus \{0\},$$

then

$$b(\omega) = \sum_{k \in \mathbb{Z}} b_k e^{2\pi i \omega k}$$

satisfies (7.25). Since $\tilde{c}(\cdot)$ has an analytic extension to a strip, the coefficients $\tilde{c}_k$ decay exponentially. On the other hand, the Diophantine condition which α satisfies ensures that the coefficients b_k decay exponentially as well and hence $b(\cdot)$ is real-analytic with an extension to the same open strip.

Setting

$$B_2(\omega) = \begin{pmatrix} 1 & b(\omega) \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}),$$

and using (7.25), we find

$$\begin{aligned} B_2(\omega+\alpha)^{-1}\tilde{C}(\omega)B_2(\omega) &= \begin{pmatrix} 1 & -b(\omega+\alpha) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \tilde{c}(\omega) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b(\omega) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -b(\omega+\alpha) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b(\omega)+\tilde{c}(\omega) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & b(\omega)+\tilde{c}(\omega)-b(\omega+\alpha) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \\ &= C \end{aligned}$$

for every $\omega \in \mathbb{T}$. Thus, setting $B(\omega) = B_1(\omega)B_2(\omega)$, we obtain (7.23). □

Proof of Theorem 7.22. Consider first a coupling constant $\lambda > 1$. We have seen above that Aubry duality maps the energy E to the dual energy $\lambda^{-1}E$. We will establish below that if E is an eigenvalue of $H_0^{\lambda,\alpha}$, then the dual energy $\lambda^{-1}E$ is an endpoint of a gap of the spectrum of $H_0^{\lambda^{-1},\alpha}$. Since we already know that for Diophantine α, $H_0^{\lambda,\alpha}$ has pure point spectrum, we can consider energies belonging to the countable dense set of eigenvalues. It then follows that the dual energies are all endpoints of gaps and hence the gaps are dense because the spectra are just related by uniform scaling.

Let us implement this strategy. Consider an eigenvalue E of $H_0^{\lambda,\alpha}$ and a corresponding exponentially decaying eigenfunction. Then, Lemma 7.24 yields the real-analytic function $\hat{u}$, which has an analytic extension to a strip, and a quasi-periodic solution of the dual difference equation at the dual energy. Using this as input to Lemma 7.25, we then obtain that

$$A(\omega) = \begin{pmatrix} \lambda^{-1}E - 2\lambda^{-1}\cos(2\pi\omega) & -1 \\ 1 & 0 \end{pmatrix}$$

may be analytically conjugated via $B(\cdot)$ to the constant

$$C = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$

Let us show that

$$c \neq 0. \tag{7.26}$$

Assume to the contrary $c = 0$. Then,

$$A(\omega) = B(\omega+\alpha)B(\omega)^{-1}$$

for every $\omega \in \mathbb{T}$ and therefore, all solutions of (7.21) are analytically quasi-periodic! Indeed,

$$\begin{aligned}\begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} &= A(\omega + (n-1)\alpha) \begin{pmatrix} u(n-1) \\ u(n-2) \end{pmatrix} \\ &= \cdots \\ &= A(\omega + (n-1)\alpha) \times \cdots \times A(\omega) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} \\ &= B(\omega + n\alpha) B(\omega)^{-1} \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix},\end{aligned}$$

that is,

$$u(n) = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, B(\omega + n\alpha) B(\omega)^{-1} \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix} \right\rangle,$$

and hence $u(n) = g(n\alpha)$ with a real-analytic function g on $\mathbb{T}$. Now consider two linearly independent solutions of (7.21) and associate with them via Lemma 7.24 the corresponding exponentially decaying solutions of the dual equation (7.20). They must be linearly independent too, which yields the desired contradiction since by constancy of the Wronskian there cannot be two linearly independent exponentially decaying solutions. We therefore have (7.26).

Let us now perturb the energy and consider

$$\tilde{A}(\omega) = \begin{pmatrix} (\lambda^{-1}E + \lambda^{-1}\delta) - 2\lambda^{-1}\cos(2\pi\omega) & -1 \\ 1 & 0 \end{pmatrix} = A(\omega) + \begin{pmatrix} \lambda^{-1}\delta & 0 \\ 0 & 0 \end{pmatrix}.$$

We will show that there is $\delta_0 > 0$ such that

$$(7.27) \qquad 0 < |\delta| < \delta_0 \text{ and } \delta c < 0 \quad \Rightarrow \quad \lambda^{-1}E + \lambda^{-1}\delta \notin \sigma(H_0^{\lambda^{-1},\alpha}).$$

Assuming (7.27), we can quickly finish the proof of the theorem. Since the E's in question are dense in $\sigma(H_0^{\lambda,\alpha})$, Aubry duality shows that the $\lambda^{-1}E$'s in question are dense in $\sigma(H_0^{\lambda^{-1},\alpha})$. By (7.27) all these energies are endpoints of gaps of $\sigma(H_0^{\lambda^{-1},\alpha})$. Thus, $\sigma(H_0^{\lambda^{-1},\alpha})$ does not contain an interval. Recall that by general principles, $\sigma(H_0^{\lambda^{-1},\alpha})$ is closed and does not contain isolated points. Consequently, $\sigma(H_0^{\lambda^{-1},\alpha})$ is a Cantor set. Then, by Aubry duality again, $\sigma(H_0^{\lambda,\alpha})$ is a Cantor set, too. Since $\sigma(H_\omega^{\lambda,\alpha})$ does not depend on ω, it follows that for every Diophantine α, every $\lambda \in (0,\infty) \setminus \{1\}$, and every $\omega \in \mathbb{T}$, $\sigma(H_\omega^{\lambda,\alpha})$ is a Cantor set.

To finish the proof we show (7.27). We have

$$B(\omega + \alpha)^{-1} A(\omega) B(\omega) = C,$$

and therefore

$$\begin{aligned}B(\omega+\alpha)^{-1}\tilde{A}(\omega)B(\omega) &= B(\omega+\alpha)^{-1}\left(A(\omega) + \begin{pmatrix} \lambda^{-1}\delta & 0 \\ 0 & 0 \end{pmatrix}\right) B(\omega) \\ &= C + \begin{pmatrix} b_{22}(\omega+\alpha) & -b_{12}(\omega+\alpha) \\ -b_{21}(\omega+\alpha) & b_{11}(\omega+\alpha) \end{pmatrix} \begin{pmatrix} \lambda^{-1}\delta & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_{11}(\omega) & b_{12}(\omega) \\ b_{21}(\omega) & b_{22}(\omega) \end{pmatrix} \\ &= C + \begin{pmatrix} b_{22}(\omega+\alpha) & -b_{12}(\omega+\alpha) \\ -b_{21}(\omega+\alpha) & b_{11}(\omega+\alpha) \end{pmatrix} \begin{pmatrix} \lambda^{-1}\delta b_{11}(\omega) & \lambda^{-1}\delta b_{12}(\omega) \\ 0 & 0 \end{pmatrix} \\ &= C + \begin{pmatrix} \lambda^{-1}\delta b_{11}(\omega) b_{22}(\omega+\alpha) & \lambda^{-1}\delta b_{12}(\omega) b_{22}(\omega+\alpha) \\ -\lambda^{-1}\delta b_{11}(\omega) b_{21}(\omega+\alpha) & -\lambda^{-1}\delta b_{12}(\omega) b_{21}(\omega+\alpha) \end{pmatrix}\end{aligned}$$

□

Bibliography

[1] A. Avila and D. Damanik, Generic singular spectrum for ergodic Schrödinger operators, *Duke Math. J.* **130** (2005), 393–400

[2] J. Avron and B. Simon, Singular continuous spectrum for a class of almost periodic Jacobi matrices, *Bull. Amer. Math. Soc.* **6** (1982), 81–85

[3] M. Boshernitzan, D. Damanik, Generic continuous spectrum for ergodic Schrödinger operators, preprint (arXiv:0708.1263)

[4] M. Boshernitzan, D. Damanik, The repetition property for sequences on tori generated by polynomials or skew-shifts, preprint (arXiv:0708.3234)

[5] J. Bourgain, *Green's Function Estimates for Lattice Schrödinger Operators and Applications*, Annals of Mathematics Studies, 158. Princeton University Press, Princeton (2005)

[6] I. Cornfeld, S. Fomin, and Ya. Sinai, *Ergodic Theory*, Grundlehren der Mathematischen Wissenschaften **245**, Springer-Verlag, New York, 1982

[7] D. Damanik and R. Killip, Ergodic potentials with a discontinuous sampling function are non-deterministic, *Math. Res. Lett.* **12** (2005), 187–192

[8] P. Deift and B. Simon, Almost periodic Schrödinger operators. III. The absolutely continuous spectrum in one dimension, *Commun. Math. Phys.* **90** (1983), 389–411

[9] R. Durrett, *Probability: Theory and Examples*, second edition, Duxbury Press, Belmont, CA, 1996

[10] H. Fürstenberg, Noncommuting random products, *Trans. Amer. Math. Soc.* **108** (1963), 377–428

[11] D. Gilbert, On subordinacy and analysis of the spectrum of Schrödinger operators with two singular endpoints, *Proc. Roy. Soc. Edinburgh A* **112** (1989), 213–229

[12] D. Gilbert and D. Pearson, On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators, *J. Math. Anal. Appl.* **128** (1987), 30–56

[13] A. Gordon, S. Jitomirskaya, Y. Last, and B. Simon, Duality and singular continuous spectrum in the almost Mathieu equation, *Acta Math.* **178** (1997), 169–183

[14] P. Halmos, *Measure Theory*, D. Van Nostrand Company, Inc., New York, NY, 1950

[15] G. Hardy and E. Wright, *An Introduction to the Theory of Numbers*, 5th edition, Oxford University Press, New York, 1979

[16] S. Jitomirskaya, Metal-insulator transition for the almost Mathieu operator, *Ann. of Math.* **150** (1999), 1159–1175

[17] S. Jitomirskaya and Y. Last, Power-law subordinacy and singular spectra. I. Half-line operators, *Acta Math.* **183** (1999), 171–189

[18] Y. Katznelson and B. Weiss, A simple proof of some ergodic theorems, *Israel J. Math.* **42** (1982), 291–296

[19] A. Khintchine, *Continued Fractions*, Dover, Mineola (1997)

[20] R. Killip, A. Kiselev, and Y. Last, Dynamical upper bounds on wavepacket spreading, *Amer. J. Math.* **125** (2003), 1165–1198

[21] S. Kotani, Support theorems for random Schrödinger operators, *Commun. Math. Phys.* **97** (1985), 443–452

[22] S. Kotani, One-dimensional random Schrödinger operators and Herglotz functions, *Probabilistic methods in mathematical physics* (Katata/Kyoto, 1985), 219–250, Academic Press, Boston, MA, 1987

[23] S. Kotani, Support theorems for random Schrödinger operators, *Commun. Math. Phys.* **97** (1985), 443–452

[24] S. Kotani, Generalized Floquet theory for stationary Schrödinger operators in one dimension, *Chaos Solitons Fractals* **8** (1997), 1817–1854

[25] Y. Last, On the measure of gaps and spectra for discrete 1D Schrödinger operators, *Commun. Math. Phys.* **149** (1992), 347–360

[26] Y. Last, A relation between a.c. spectrum of ergodic Jacobi matrices and the spectra of periodic approximants, *Commun. Math. Phys.* **151** (1993), 183–192

[27] Y. Last, Zero measure spectrum for the almost Mathieu operator, *Commun. Math. Phys.* **164** (1994), 421–432

[28] Y. Last, Quantum dynamics and decompositions of singular continuous spectra, *J. Funct. Anal.* **142** (1996), 406–445

[29] Y. Last and B. Simon, Eigenfunctions, transfer matrices, and absolutely continuous spectrum of one-dimensional Schrödinger operators, *Invent. Math.* **135** (1999), 329–367

[30] T. Ransford, *Potential Theory in the Complex Plane*, London Mathematical Society Student Texts **28**, Cambridge University Press, Cambridge, 1995

[31] C. Remling, Relationships between the m-function and subordinate solutions of second order differential operators, *J. Math. Anal. Appl.* **206** (1997), 352–363

[32] C. Rogers, *Hausdorff Measures*, Cambridge University Press, Cambridge, 1998

[33] B. Simon, Spectral analysis of rank one perturbations and applications, In *Mathematical Quantum Theory. II. Schrödinger operators* (*Vancouver, BC, 1993*), 109–149, CRM Proc. Lecture Notes **8**, Amer. Math. Soc., Providence, RI, 1995

[34] B. Simon, *Representations of Finite and Compact Groups*, Graduate Studies in Mathematics **10**, American Mathematical Society, Providence, RI, 1996

[35] J. Steele, Kingman's subadditive ergodic theorem, *Ann. Inst. H. Poincaré Probab. Statist.* **25** (1989), 93–98

[36] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, New York-Berlin, 1982