

# 量子场论期末论文答辩——态表示与场表示

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# 大纲

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## ④ 结论

# 态空间的表示

## 万有覆盖

为什么要考虑万有覆盖？

- 对于有限维情形，单连通李群的表示和它的李代数的表示是等价的 (Lie's second theorem)
- $G$  的 (不可约) 投影表示和  $\tilde{G}$  的 (不可约) 线性表示某种程度可以互相转化
  - 有限维情形，二者等价
  - 无限维情形， $\tilde{G}$  的 (不可约) 线性表示可以转化为  $G$  的 (不可约) 投影表示 (Schur 引理)
  - 反之并不永远成立。即，一般并没有： $G$  的 (不可约) 投影表示都来自于  $\tilde{G}$  的 (不可约) 线性表示。
  - 但 Bargmann 定理告诉我们， $SO(3)$ 、Poincare 群都是成立的！这涉及到李代数的上同调理论。
- 线性表示的理论比投影表示好很多：投影表示不能诱导、不能做直和 $\rightarrow$ 从而 Fock space 上不存在 Poincare 群的投影表示

我们需要的万有覆盖：

- The universal cover of  $SO^+(1, 3)$  is  $\lambda : SL(2, \mathbb{C}) \rightarrow SO^+(1, 3)$ . The kernel is  $\{\pm I\}$ .
- Let  $p : SL(2, \mathbb{C}) \rightarrow SO^+(1, 3)$  be the covering map. Then the preimage of  $SO(3)$  is  $SU(2)$ .
- The universal cover of  $\mathbb{R}^{1,3} \rtimes SO^+(1, 3)$  is  $\mathbb{R}^{1,3} \rtimes SL(2, \mathbb{C})$ . The action of  $SL(2, \mathbb{C})$  on  $\mathbb{R}^{1,3}$  is  $\Lambda, a \mapsto \lambda(\Lambda)a$ .

# 态空间的表示

万有覆盖

投影表示和万有覆盖的线性表示：

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\rho} & GL(V) \\ \pi \downarrow & & \downarrow r \\ G & \dashrightarrow & PGL(V) \end{array}$$

投影表示不能直和：

$$\rho(g) = \rho_1(g) \oplus \rho_2(g) \quad (1)$$

$$\begin{aligned} \rho(gh) &= \rho_1(gh) \oplus \rho_2(gh) = \omega \rho_1(g) \rho_1(h) \oplus \omega' \rho_2(g) \rho_2(h) \\ \rho(g)\rho(h) &= \rho_1(g) \oplus \rho_2(g) \cdot \rho_1(h) \oplus \rho_2(h) = \rho_1(g) \rho_1(h) \oplus \rho_2(g) \rho_2(h) \end{aligned} \quad (2)$$

直和两个分量的相位一般不相等，所以这两个式子差的并不是一个相位。

# 态空间的表示

## 诱导表示

### Definition 1

We say that  $(\rho, W)$  is induced from  $(\pi, V)$  if

$$W = \bigoplus_{i=1}^n g_i V$$

- Uniqueness:

$$\rho(g) \cdot \sum_{i=1}^n g_i v_i = \sum_{i=1}^n g_{j(i)} \pi(h_i) v_i$$

where for each  $g_i$  we find a unique pair of  $h_i \in H$  and  $j(i) \in \{1, \dots, n\}$  such that  
 $gg_i = g_{j(i)} h_i$ .

- Existence: Writing  $gg_i = g_{j(i)} h_i$  and  $g' g_{j(i)} = g_{k(i)} h'_i$ , so  $g' gg_i = g_{k(i)} h'_i h_i$ . So

$$\rho(g') \rho(g) g_i v = \rho(g') g_{j(i)} \pi(h_i) v = g_{k(i)} \pi(h'_i) \pi(h_i) v$$

and

$$\rho(g' g) g_i v = g_{k(i)} \pi(h'_i h_i) v$$

诱导表示仅对线性表示有效！投影表示的诱导，既不保证存在也不保证唯一！



# 态空间的表示

Wigner 分类定理

The Hilbert space  $\mathcal{H}$  of single particle states admits a projective, faithful, irreducible unitary representation of the Poincare group

$$U : \mathbb{R}^{1,3} \rtimes SO^+(1,3) \rightarrow \mathcal{P}\mathcal{U}(\mathcal{H})$$

•  $\tilde{U} : \mathbb{R}^{1,3} \rtimes SL(2, \mathbb{C}) \rightarrow \mathcal{U}(\mathcal{H})$  (3)

- Write  $\mathcal{H}_p = \text{span}\{\Psi_{p,\sigma} | \sigma \in B_p\}$  the eigenspace of  $P^\mu$  with eigenvalue  $p^\mu$ . Or equivalently, the eigenspace of  $\tilde{U}(I, a)$ 's with eigenvalues  $e^{-ia \cdot p}$ .

•  $\tilde{U}(\Lambda, 0)\mathcal{H}_p = \mathcal{H}_{\lambda(\Lambda)p}, \quad \Lambda \in SL(2, \mathbb{C})$  (4)

$$U(I, a)U(\Lambda, 0)\Psi_{p,\sigma} = \square U(\Lambda, 0)U(I, \Lambda^{-1}a)\Psi_{p,\sigma} = \square e^{i\Lambda^{-1}a \cdot p}U(\Lambda, 0)\Psi_{p,\sigma} = \square e^{ia \cdot \Lambda p}U(\Lambda, 0)\Psi_{p,\sigma}$$

where  $\square$  is the phase coming from projectivity. So we can conclude nothing from this.

Indeed we have

$$\tilde{U}(I, a)\tilde{U}(\Lambda, 0)\Psi_{p,\sigma} = \tilde{U}(\Lambda, 0)\tilde{U}(I, \lambda(\Lambda)^{-1}a)\Psi_{p,\sigma} = e^{i\lambda(\Lambda)^{-1}a \cdot p}\tilde{U}(\Lambda, 0)\Psi_{p,\sigma} = e^{ia \cdot \lambda(\Lambda)p}\tilde{U}(\Lambda, 0)\Psi_{p,\sigma}$$

## Remark 1

Note that 'eigenspace of  $U(I, a)$  with eigenvalue  $e^{-ia \cdot p}$ ' is not well defined, as it is a projective rep. Again we see the power of Bargmann's theorem.

# 态空间的表示

Wigner 分类定理

## 定理 1

(Wigner's classification)

Every faithful(except  $(-I, 0)$ ), irreducible unitary representation  $\tilde{U}$  of the universal cover of Poincare group  $\mathbb{R}^{1,3} \rtimes SL(2, \mathbb{C})$ , when restricted to the universal cover of Lorentz group  $SL(2, \mathbb{C})$ , must be an induced representation of a small group  $\mathcal{W}(k)$  for a certain standard momentum  $k$ .

$$\mathcal{H} = \bigoplus_{p \in \text{the mass shell of } k} \tilde{U}(\tilde{L}(p), 0) \mathcal{H}_k \quad (5)$$

$$\begin{aligned} \tilde{U}(\Lambda, a) \Psi_{p,\sigma} &= e^{-ip \cdot a} N(p) \sum_{\sigma'} \tilde{D}_{\sigma' \sigma}(W(\Lambda, p)) \tilde{U}(L(\lambda(\Lambda)p)) \Psi_{k,\sigma'} \\ &= e^{-ip \cdot a} \left( \frac{N(p)}{N(\lambda(\Lambda)p)} \right) \sum_{\sigma'} \tilde{D}_{\sigma' \sigma}(W(\Lambda, p)) \Psi_{\lambda(\Lambda)p, \sigma'} \end{aligned} \quad (6)$$

# 态空间的表示 I

Fock 空间和产生湮灭算符

The whole Hilbert space is the Fock space defined below

$$F_\nu(\mathcal{H}) = \overline{\bigoplus_{n=0}^{\infty} S_\nu(\mathcal{H}^{\otimes n})} \quad (7)$$

这个空间上面装备了  $\mathbb{R}^{1,3} \rtimes SL(2, \mathbb{C})$  的表示，通过表示的对称方、交错方、直和构造。但它上面并没有 Poincare 群的投影表示！因为投影表示不能做直和，相位对不上。对于两个投影表示  $\rho_1 : G \rightarrow GL(V_1), \rho_2 : G \rightarrow GL(V_2)$ ，定义直和为

$$\rho(g) = \rho_1(g) \oplus \rho_2(g) \quad (8)$$

于是在线性表示的时候，有同态性

$$\rho(gh) = \rho_1(gh) \oplus \rho_2(gh) = \rho_1(g)\rho_1(h) \oplus \rho_2(g)\rho_2(h) = \rho(g)\rho(h) \quad (9)$$

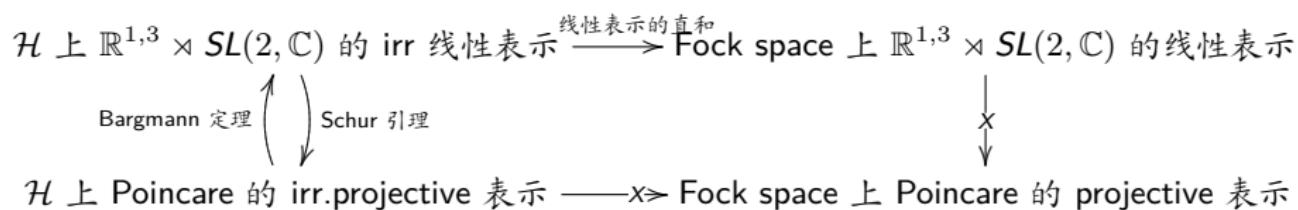
但在投影表示的情况，

$$\begin{aligned} \rho(gh) &= \rho_1(gh) \oplus \rho_2(gh) = \omega \rho_1(g)\rho_1(h) \oplus \omega' \rho_2(g)\rho_2(h) \\ \rho(g)\rho(h) &= \rho_1(g) \oplus \rho_2(g) \cdot \rho_1(h) \oplus \rho_2(h) = \rho_1(g)\rho_1(h) \oplus \rho_2(g)\rho_2(h) \end{aligned} \quad (10)$$

直和两个分量的相位一般不相等，所以这两个式子差的并不是一个相位。

## 态空间的表示 II

Fock 空间和产生湮灭算符



产生湮灭算符的洛伦兹变换：

$$\tilde{U}_0(\Lambda, \alpha) a^\dagger(\mathbf{p}\sigma n) \tilde{U}_0^{-1}(\Lambda, \alpha) = e^{-i(\lambda(\Lambda)\mathbf{p}) \cdot \alpha} \sqrt{(\lambda(\Lambda)\mathbf{p})^0/p^0} \times \sum_{\bar{\sigma}} \tilde{D}_{\bar{\sigma}\sigma}^{(j)}(W(\Lambda, \mathbf{p})) a^\dagger(\mathbf{p}_\Lambda \bar{\sigma} n) \quad (11)$$

没有用投影表示书写的对应物。Weinberg 书上的写法不对。

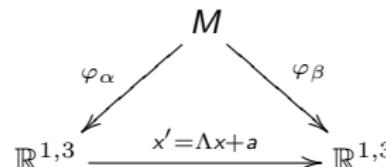
# 态空间的表示 I

一些讨论

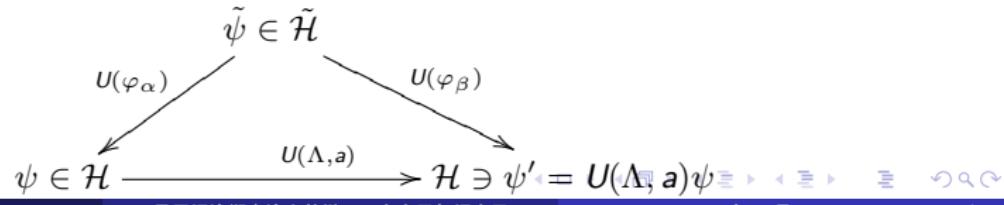
- 态空间上的庞加莱群的表示是什么物理意义？

设  $M$  是抽象时空流形 (manifold)。参考系是微分流形  $M$  上的坐标卡

$\varphi: M \rightarrow \mathbb{R}^{1,3}$ ，参考系之间的坐标变换是指  $\varphi_\beta \circ \varphi_\alpha^{-1}: \mathbb{R}^{1,3} \rightarrow \mathbb{R}^{1,3}$ 。见下图。



在 Schrodinger 绘景下，有一个抽象的 abstract 态空间  $\tilde{\mathcal{H}}$ ，而我们平时使用的是具体的 concrete 态空间  $\mathcal{H}$ 。确定时空流形  $M$  的一个参考系  $\varphi: M \rightarrow \mathbb{R}^{1,3}$ ，就有一个 trivialization  $U(\varphi): \tilde{\mathcal{H}} \rightarrow \mathcal{H}$ ，它的物理意义是，对于一个处于 abstract 态空间  $\tilde{\mathcal{H}}$  的 abstract 量子态，这个参考系的人看到的处于具体态空间  $\mathcal{H}$  的量子态是什么。而态表示  $U$  的意义就是，当两个参考系之间的坐标变换是  $\varphi_\beta \circ \varphi_\alpha^{-1}(x) = \Lambda x + a$  时，对于同一个固定的 abstract 量子态  $\tilde{\psi}$ ，这俩参考系分别看到的量子态  $\psi$  和  $\psi'$  之间的关系是  $\psi' = U(\Lambda, a)\psi$ 。如下图所示。



# 态空间的表示 II

一些讨论

以上抽象的理解的一个具体的例子就是量子力学的 Schrodinger 方程。设想有一个‘现在人’和一个‘未来人’。‘现在人’的参考系是  $\varphi_\alpha : M \rightarrow \mathbb{R}^{1,3}$ ，‘未来人’的参考系是  $\varphi_\beta : M \rightarrow \mathbb{R}^{1,3}$ 。对于同一个时空点，未来人会认为它的时间坐标比较小，现在人会认为比较大，(比如现在人认为是现在即  $t = 0$  的时空点，未来人认为是过去，也就是  $t < 0$ )。所以这个参考系变换是  $\varphi_\beta \circ \varphi_\alpha^{-1}(t, x) = (t - \Delta t, x)$ 。对，是减，不是加，这块没写错。所以俩人看到的同一个 abstract 量子态在 concrete 空间中的关系是  $\psi' = U(l, -\Delta t)\psi = e^{-iH\Delta t}\psi$ ，其中带撇的是未来人的。这便是熟知的 Schrodinger 方程！在之前量子力学的观点里，Schrodinger 方程的理解是，一个态随时间演化，而现在对它的理解是，未来人看到的 concrete 态和现在人看到的 concrete 态的区别。

$$t'(p) = t(p) - \Delta t, \quad \forall p \in M$$

- 为什么要求态空间上的表示是幺正 (unitary) 的？

因为我们希望对于两个 abstract 量子态  $\tilde{\psi}_1, \tilde{\psi}_2$ ，不同人看到的二者的 concrete 版本的内积模方  $|(\tilde{\psi}_1, \tilde{\psi}_2)|^2$  是不变的，因为内积的模方是有物理意义的量：把  $\tilde{\psi}_2$  扩充为一组互相正交的基底，然后用相应的物理量去测量  $\tilde{\psi}_1$ ，那么发现  $\tilde{\psi}_1$  坎缩到  $\tilde{\psi}_2$  的概率为  $|(\tilde{\psi}_1, \tilde{\psi}_2)|^2$ 。而这个实验结果应该不依赖于参考系的选取。由 Wigner 的另一个定理， $U$  要么是线性幺正的要么是反线性反幺正的，但是‘反幺正’连群都不构成：俩反幺正的复合起来就是幺正的了，并且‘反幺正’连线性变换都不是。所以很欣然地，我们要求  $U$  是幺正的。也就是说态空间上的表示我们要求它是幺正的。

# 场空间的表示

Lorentz 代数的有限维表示

$$\mathbf{A} = \frac{\mathbf{J} + i\mathbf{K}}{2}, \quad \mathbf{B} = \frac{\mathbf{J} - i\mathbf{K}}{2}. \quad (12)$$

$$[A_i, A_j] = i\varepsilon_{ijk}A_k, \quad [B_i, B_j] = i\varepsilon_{ijk}B_k, \quad [A_i, B_j] = 0, \quad (13)$$

$$so(1, 3) \hookrightarrow so(1, 3)_{\mathbb{C}} = su(2)_{\mathbb{C}} \oplus su(2)_{\mathbb{C}} = sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C}) \quad (14)$$

## 引理 2

Given two finite dimensional Lie algebras  $g_1, g_2$ , then any finite dimensional irreducible rep of  $g_1 \oplus g_2$  must come from the tensor product of two f.d irreducible reps of the two Lie algebras, i.e.

$$\pi(v_1, v_2) = \pi_1(v_1) \otimes I + I \otimes \pi_2(v_2) \quad (15)$$

We write  $\pi = \pi_1 \boxtimes \pi_2$ , differing from  $\otimes$  which we use for tensor product of two reps for the same Lie algebra.

Denote by  $A$  and  $B$  the spins of the two representations of  $su(2)$  or  $su(2)_{\mathbb{C}}$ .

$$\begin{aligned} \pi_{(A,B)}(J_i) &= J_i^{(A)} \otimes 1_{(2B+1)} + 1_{(2A+1)} \otimes J_i^{(B)} \\ \pi_{(A,B)}(K_i) &= -i(J_i^{(A)} \otimes 1_{(2B+1)} - 1_{(2A+1)} \otimes J_i^{(B)}), \end{aligned} \quad (16)$$

# 场空间的表示

Lorentz 代数的有限维表示

## 引理 3

The representation of  $SL(2, \mathbb{C})$  arising from the rep  $\bigoplus_{i=1}^n \pi_{A_i B_i}$  of its Lie algebra can be descended to an ordinary rep of  $SO^+(1, 3)$  iff  $A_i + B_i$  is an integer for all  $i$ , and can be descended to a projective rep of  $SO^+(1, 3)$  iff  $A_i + B_i$  is half an odd for all  $i$ .

## 例 4

Scalar representation has  $(A, B) = (0, 0)$ . So it is an ordinary rep of  $SO^+(1, 3)$ .

Vector representation has  $(A, B) = (\frac{1}{2}, \frac{1}{2})$ . So it is an ordinary rep of  $SO^+(1, 3)$ .

Dirac representation has  $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ . So it is a projective rep of  $SO^+(1, 3)$ .

$(\frac{1}{2}, 0) \oplus (0, 1)$  for example, is not even a projective rep of  $SO^+(1, 3)$ . It is just an ordinary rep of  $SL(2, \mathbb{C})$ .

# 量子场的构造 I

## 动机

The quantity that directly relates to experiments is the  $S$  operator, defined by

$$(\Phi_\beta, S\Phi_\alpha) \equiv (\Psi_\beta^-, \Psi_\alpha^+)$$

The core of QFT is to make it Lorentz invariant, i.e.,

$$\tilde{U}_0(\Lambda, a)^{-1} S \tilde{U}_0(\Lambda, a) = S$$

The famous Dyson series gives a formula for  $S$  operator

$$S = T \exp(-i \int_{-\infty}^{+\infty} dt V(t))$$

In order for this quantity to be Lorentz invariant, we hope to write

$$V(t) = \int d^3x \mathcal{H}(x, t)$$

and hope that



$$\tilde{U}_0(\Lambda, a)^{-1} \mathcal{H}(x) U_0(\Lambda, a) = \mathcal{H}(\Lambda x + a) \quad (17)$$

# 量子场的构造 II

## 动机

$$[\mathcal{H}(x), \mathcal{H}(x')] = 0, \quad \text{when } (x - x')^2 > 0 \quad (18)$$

Above purpose motivates the outline or philosophy for constructing quantum fields:

- (1). In order for the first to satisfy, we need quantum fields—creation and annihilation fields  $\psi^\mp(x)$ , which satisfy a so called 'Lorentz transformation of fields'. We use polynomials of these to construct  $\mathcal{H}(x)$ .
- (2). In order for the second to satisfy as well as the  $\mathcal{H}(x)$  being Hermitian, we need to combine these two fields into a single one  $\psi_l(x) = \kappa_l \psi_l^+(x) + \lambda_l \psi_l^-(x)$  and hope that

$$[\psi_l(x), \psi_{l'}(x')]_\mp = 0, \quad \text{when } (x - x')^2 > 0 \quad (19)$$

- (3). For the motivation of the existence of a conservative charge  $Q$ , we find

$$[Q, a(\mathbf{p}, \sigma, n)] = -q(n)a(\mathbf{p}, \sigma, n)$$

$$[Q, a^\dagger(\mathbf{p}, \sigma, n)] = +q(n)a^\dagger(\mathbf{p}, \sigma, n)$$

not satisfactory for our purpose, as the polynomial constructed from  $\psi$  do not have a simple commutation relation with  $Q$ . So we replace  $a^\dagger$  with  $a^{c\dagger}$ , which creates particles with charge opposite to that of  $a^\dagger$ , into the single quantum field.

Above is the soul or outline of this section.

# 量子场的构造

## 场的 Lorentz 变换

我们用待定系数定义湮灭场和产生场：

$$\begin{aligned}\psi_I^+(x) &= \sum_{\sigma n} \int d^3 p u_I(x; \mathbf{p}, \sigma, n) a(\mathbf{p}, \sigma, n), \\ \psi_I^-(x) &= \sum_{\sigma n} \int d^3 p v_I(x; \mathbf{p}, \sigma, n) a^\dagger(\mathbf{p}, \sigma, n)\end{aligned}\quad (20)$$

希望它满足： $\forall \Lambda \in SL(2, \mathbb{C})$ ,

$$\begin{aligned}\tilde{U}_0(\Lambda, a)^{-1} \psi_I^+(\lambda(\Lambda)x + a) \tilde{U}_0(\Lambda, a) &= D_{\bar{\ell}}(\Lambda) \psi_I^+(x) \\ \tilde{U}_0(\Lambda, a)^{-1} \psi_I^-(\lambda(\Lambda)x + a) \tilde{U}_0(\Lambda, a) &= D_{\bar{\ell}}(\Lambda) \psi_I^-(x)\end{aligned}\quad (21)$$

Weinberg 书上用的是以下错误写法： $\forall \Lambda \in SO^+(1, 3)$ ,

$$\begin{aligned}U_0(\Lambda, a)^{-1} \psi_I^+(\Lambda x + a) U_0(\Lambda, a) &= D_{\bar{\ell}}(\Lambda) \psi_I^+(x) \\ U_0(\Lambda, a)^{-1} \psi_I^-(\Lambda x + a) U_0(\Lambda, a) &= D_{\bar{\ell}}(\Lambda) \psi_I^-(x)\end{aligned}\quad (22)$$

说它错误是因为根本不存在 Poincare 群在 Fock space 上的投影表示。

而且按照以上公式的写法，马上会给人一个致命的误导：做两次变换， $U_0$  的两次合成之间是一个取逆的关系，所以说出现的相位抵消了。于是得知  $D$  其实是个 Lorentz 群的 ordinary 表示，而非投影表示。但是 Section 1.4 证明了 Dirac 表示是个投影表示而非线性表示，于是，我们喜欢的 Dirac 表示等等都不在考虑范围内！这显然是荒谬的。

# 量子场的构造

## 场的 Lorentz 变换

想要‘场的 Lorentz 变换’被满足的充要条件是：

$$\begin{aligned}\psi_I^+(x) &= \sum_{\sigma, n} (2\pi)^{-3/2} \int d^3 p u_I(\mathbf{p}, \sigma, n) e^{ip \cdot x} a(\mathbf{p}, \sigma, n) \\ \psi_I^-(x) &= \sum_{\sigma, n} (2\pi)^{-3/2} \int d^3 p v_I(\mathbf{p}, \sigma, n) e^{-ip \cdot x} a^\dagger(\mathbf{p}, \sigma, n)\end{aligned}\quad (23)$$

且

$$\begin{aligned}\sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\bar{\sigma}} u_{\bar{l}}(\mathbf{p}_\Lambda, \bar{\sigma}, n) D_{\bar{\sigma}\sigma}^{(j_n)}(W(\Lambda, p)) &= \sum_l D_{\bar{l}l}(\Lambda) u_l(\mathbf{p}, \sigma, n) \\ \sqrt{\frac{(\Lambda p)^0}{p^0}} \sum_{\bar{\sigma}} v_{\bar{l}}(\mathbf{p}_\Lambda, \bar{\sigma}, n) D_{\bar{\sigma}\sigma}^{(j_n)}(W(\Lambda, p)) &= \sum_l D_{\bar{l}l}(\Lambda) v_l(\mathbf{p}, \sigma, n)\end{aligned}\quad (24)$$

翻译到表示论的语言：

### 定理 5

For 'Lorentz transformation of field' to satisfy, it is equivalent that the matrix  $\{u_l(\mathbf{p}, \sigma, n)\}_{l; \mathbf{p}, \sigma}$  is a homomorphism of representations of  $SL(2, \mathbb{C})$  from  $U_0|_{SL(2, \mathbb{C})}$  to  $D$ , and  $\{v_l(\mathbf{p}, \sigma, n)\}_{l; \mathbf{p}, \sigma}$  is a homomorphism of representations of  $SL(2, \mathbb{C})$  from the dual of  $U_0|_{SL(2, \mathbb{C})}$  to  $D$ .

# 量子场的构造

## 场的自旋与态的自旋

Frobenius reciprocity:

$$\begin{aligned} \text{Hom}_H(\pi, \text{Res}_H^G D) &\cong \text{Hom}_G(\text{Ind}_H^G \pi, D) \\ F &\mapsto (g_i v \mapsto D(g_i)F(v)) \\ G|_V &\leftarrow G \end{aligned} \tag{25}$$

### 定理 6

Equivalently, the matrix  $(u_l(0, \sigma, n))_{l; \sigma}$  is a homomorphism of representations of Lie algebra  $\text{su}(2)$  from  $dD^{(j_n)}$  to  $dD|_{\text{su}(2)}$ , and the matrix  $(v_l(0, \sigma, n))_{l; \sigma}$  is a homomorphism of representations of Lie algebra  $\text{su}(2)$  from the dual of  $dD^{(j_n)}$  to  $dD|_{\text{su}(2)}$ .

$$u_{\bar{l}}(\mathbf{q}, \sigma, n) = (m/q^0)^{1/2} \sum_l D_{\bar{l}l}(L(q)) u_l(0, \sigma, n) \tag{26}$$

$$v_{\bar{l}}(\mathbf{q}, \sigma, n) = (m/q^0)^{1/2} \sum_l D_{\bar{l}l}(L(q)) v_l(0, \sigma, n)$$

$$\sum_{\bar{\sigma}} u_{\bar{l}}(0, \bar{\sigma}, n) \mathbf{J}_{\bar{\sigma}\sigma}^{(j_n)} = \sum_l \mathcal{J}_{\bar{l}l} u_l(0, \sigma, n) \tag{27}$$

$$\sum_{\bar{\sigma}} v_{\bar{l}}(0, \bar{\sigma}, n) \mathbf{J}_{\bar{\sigma}\sigma}^{(j_n)*} = - \sum_l \mathcal{J}_{\bar{l}l} v_l(0, \sigma, n)$$

# 量子场的构造

## 场的自旋与态的自旋



$$Res_{span \mathbf{J}}^{so(1,3)\mathbb{C}}(\bigoplus_{i=1}^n \pi_{A_i} \boxtimes \pi_{B_i}) = \bigoplus_{i=1}^n \pi_{A_i} \otimes \pi_{B_i}$$

as representations of  $su(2)$ .

- $\pi_J$  is self dual:

$$-\mathbf{J}_{\bar{\sigma}\sigma}^{(j)*} = (-1)^{\sigma - \bar{\sigma}} \mathbf{J}_{-\bar{\sigma},\sigma}^{(j)} \quad (28)$$

$$v_l(0, \sigma) = (-1)^{j+\sigma} u_l(0, -\sigma) \quad (29)$$

### 定理 7

In order for the quantum field with spin  $\bigoplus_{i=1}^n (A_i, B_i)$  describing particles with spin  $j$  satisfying Axiom 3 not always zero, it is equivalent to the condition

$$j \in \{|A_i - B_i|, \dots, A_i + B_i - 1, A_i + B_i \mid i = 1, \dots, n\} \quad (30)$$

### 例 8

Scalar field is defined as the field where the field representation  $D$  is the trivial rep of the Lorentz group. So scalar field describes particles that have spin 0.

Vector field has field rep  $(\frac{1}{2}, \frac{1}{2})$ , or  $D(\Lambda) = \Lambda$ . So vector field describes particles with spin 0 or 1.

Dirac field has field rep  $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ . So Dirac field describes particles with spin  $\frac{1}{2}$ .

# 量子场的构造

## 自旋统计关系

我们希望

$$[\psi_{ab}(x), \psi_{\tilde{a}\tilde{b}}^\dagger(y)]_\mp = 0, \quad (x-y)^2 > 0 \quad (31)$$

经计算

$$\begin{aligned} [\psi_{ab}(x), \tilde{\psi}_{\tilde{a}\tilde{b}}^\dagger(y)]_\mp &= (2\pi)^{-3/2} \int d^3 p (2p^0)^{-1} \pi_{ab, \tilde{a}\tilde{b}}(\mathbf{p}) \\ &\times [\kappa \tilde{\kappa}^* e^{ip \cdot (x-y)} \mp \lambda \tilde{\lambda}^* e^{-ip \cdot (x-y)}] \end{aligned} \quad (32)$$

其中定义自旋和

$$(2p^0)^{-1} \pi_{ab, \tilde{a}\tilde{b}}(\mathbf{p}) \equiv \sum_{\sigma} u_{ab}(\mathbf{p}, \sigma) \tilde{u}_{\tilde{a}\tilde{b}}^*(\mathbf{p}, \sigma) = \sum_{\sigma} v_{ab}(\mathbf{p}, \sigma) \tilde{v}_{\tilde{a}\tilde{b}}^*(\mathbf{p}, \sigma) \quad (33)$$

求解得

$$\begin{aligned} u_{ab}(\mathbf{p}, \sigma) &= \frac{1}{\sqrt{2p^0}} \sum_{a'b'} (\exp(-\hat{\mathbf{p}} \cdot \mathbf{J}^{(A)} \theta))_{aa'} (\exp(+\hat{\mathbf{p}} \cdot \mathbf{J}^{(B)} \theta))_{bb'} \\ &\times C_{AB}(j\sigma; a'b') \end{aligned} \quad (34)$$

$$v_{ab}(\mathbf{p}, \sigma) = (-1)^{j+\sigma} u_{ab}(\mathbf{p}, -\sigma)$$

# 量子场的构造

## 自旋统计关系

从而

$$\begin{aligned}\pi_{ab, \tilde{a}\tilde{b}}(\mathbf{p}) &= \sum_{a'b'} \sum_{\tilde{a}'\tilde{b}'} \sum_{\sigma} C_{AB}(j\sigma; a'b') C_{\tilde{A}\tilde{B}}(j\sigma; \tilde{a}'\tilde{b}') \\ &\quad \times (\exp(-\hat{\mathbf{p}} \cdot \mathbf{J}^{(A)}\theta))_{aa'} (\exp(\hat{\mathbf{p}} \cdot \mathbf{J}^{(B)}\theta))_{bb'} \\ &\quad \times (\exp(-\hat{\mathbf{p}} \cdot \mathbf{J}^{(A)}\theta))_{\tilde{a}\tilde{a}'} (\exp(\hat{\mathbf{p}} \cdot \mathbf{J}^{(B)}\theta))_{\tilde{b}\tilde{b}'}.\end{aligned}\tag{35}$$

我们可以证明

$$\pi_{ab, \tilde{a}\tilde{b}}(\mathbf{p}) = P_{ab, \tilde{a}\tilde{b}}(\mathbf{p}) + 2\sqrt{\mathbf{p}^2 + m^2} Q_{ab, a\tilde{a}\tilde{b}}(\mathbf{p}),\tag{36}$$

where  $P$  and  $Q$  are now polynomials in  $\mathbf{p}$  alone, with

$$\begin{aligned}P(-\mathbf{p}) &= (-)^{2A+2\tilde{B}} P(\mathbf{p}) \\ Q(-\mathbf{p}) &= -(-)^{2A+2\tilde{B}} Q(\mathbf{p})\end{aligned}\tag{37}$$

For  $x - y$  space-like, we can adopt a Lorentz frame in which  $x^0 = y^0$ , and thus

$$\begin{aligned}[\psi_{ab}(x), \psi_{\tilde{a}\tilde{b}}^\dagger(y)]_\mp &= [\kappa \tilde{\kappa}^* \mp (-)^{2A+2\tilde{B}} \lambda \tilde{\lambda}^*] P_{ab, \tilde{a}\tilde{b}}(-i\nabla) \Delta_+(\mathbf{x} - \mathbf{y}, 0) \\ &\quad + [\kappa \tilde{\kappa}^* \pm (-)^{2A+2\tilde{B}} \lambda \tilde{\lambda}^*] Q_{ab, a\tilde{a}\tilde{b}}(-i\nabla) \delta^3(\mathbf{x} - \mathbf{y})\end{aligned}\tag{38}$$

# 量子场的构造

## 自旋统计关系

In order that this should vanish when  $x \neq y$ , we must have

$$\kappa \tilde{\kappa}^* = \pm (-1)^{2A+2\tilde{B}} \lambda \tilde{\lambda}^*. \quad (39)$$

Now let us consider the special case where  $\psi$  and  $\tilde{\psi}$  are the same, so in particular  $A = \tilde{A}$  and  $B = \tilde{B}$ .

$$|\kappa|^2 = \pm (-1)^{2A+2B} |\lambda|^2. \quad (40)$$

This is possible if and only if

$$\pm (-1)^{2A+2B} = +1 \quad (41)$$

and

$$|\kappa|^2 = |\lambda|^2 \quad (42)$$

### 定理 9

(Spin statistics)

In order for non-zero field  $\psi_{ab}(x)$  with field spin  $(A, B)$  to satisfy  $[\psi_{ab}(x), \psi_{\tilde{a}\tilde{b}}^\dagger(y)]_\mp = 0$  whenever  $(x - y)^2 > 0$ , it is equivalent to  $\pm(-1)^{2A+2B} = 1$ , and  $|\kappa| = |\lambda|$ . That's saying whether it is boson or fermion depends on whether  $2A + 2B$  is even or odd, or equivalently, whether  $2j$  is even or odd, as they have the same parity.

### 例 10

Scalar field has  $(A, B) = (0, 0)$  and  $j = 0$ . So scalar field describes bosons.

Vector field has  $(A, B) = (\frac{1}{2}, \frac{1}{2})$  and  $j = 0, 1$ . So vector field describes bosons.

Weyl field has  $(\frac{1}{2}, 0)$  or  $(0, \frac{1}{2})$  and  $j = \frac{1}{2}$ . So Weyl field describes fermions.

Whether Dirac field describe fermions needs CPT consideration.

- 态表示和场表示有何异同？

二者的异同主要总结为如下表格中。

群 表示空间 维数 是否 unitary 是否 projective 刻画参数	态表示 Poincare 量子态 Hilbert 空间 无穷维 必须是 可以是 (massive) 质量 $m > 0$ , 自旋 $j$	场表示 Lorentz 场空间 有限维 一定不是 可以是 自旋 $(A, B)$	联系 区别不大 俩东西 有本质区别 有本质区别 无区别 表示的嵌入
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- 如何理解态空间的 Poincare 生成元要求是厄米 (Hermitian) 的，而场空间的 Lorentz 生成元却一定不是厄米的？

因为表示空间是量子态空间！而因为量子态的内积（的模平方）是有物理意义的，所以我们要求 Poincare 群在态空间上的表示是保内积的也就是幺正的。

对于场空间。因为 Lorentz 群不是紧群！所以它没有有限维的忠实 (faithful) 的幺正 (unitary) 表示。所以导致它不厄米的罪魁祸首，是我们希望场空间是有限维的！但是场空间的表示不幺正是没问题的。因为场空间不是量子态的空间，不需要‘概率守恒’之类的东西，事实上因为表示肯定不幺正所以是不可能将场空间的向量诠释为量子态的。这也是为什么 Dirac 方程等等不理解为波函数的方程，而必须理解为场的方程！因为没有概率守恒。

## 结论 II

- 场的自旋与态的自旋之间是什么关系？

$$\bigoplus_{i=1}^n (A_i, B_i) \quad (43)$$

的场描述自旋

$$j \in \{|A_i - B_i|, \dots, A_i + B_i - 1, A_i + B_i \quad | \quad i = 1, \dots, n\} \quad (44)$$

的粒子。这是为了满足场的 Lorentz 变换性质。

- 自旋和它是玻色子还是费米子是什么关系？

场表示不可约时， $j$  是半奇数等价于它是费米子。这是为了满足  $\psi$  的类空对易、反对易关系。场表示可约时，需要 CPT。

- Weinberg 书上的处理有哪些 Bug？

- ▶ 首先是 Wigner 分类定理需要用一下 Bargmann 定理，转化为 universal cover 的线性表示去做，不然连第一步的本征空间都取不出来，以及 projective 表示不能诱导，所以之后的单粒子态公式推不出来（涉及到复杂的相位计算）。
- ▶ 因为投影表示不能做直和，所以根本不存在 Poincare 群在 Fock space 上的投影表示，只存在 universal cover 在 Fock space 上的线性表示。
- ▶ 从而产生湮灭算符的 Lorentz 变换、场的 Lorentz 变换都必须用 universal cover 的线性表示去写。并且场的表示  $D$  是 universal cover 的线性表示。Weinberg 书上把他们都当成 Poincare/Lorentz 的线性表示了，但连 Dirac 表示都不是。

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