MS&E311 | CME 307 - Project
Most diversified portfolio

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Abstract

Finding the optimal investment portfolio has been a research topic for decades. This paper seeks to provide a detailed resolution of the Maximum Diversity Portfolio (MDP) problem introduced by T. Froidure, Y. Choueifaty, J. Reynier [5], which translates into a portfolio diversification problem into a quadratic convex optimization one. We develop two approaches of the problem, depending on whether or not short-selling is allowed. The short-allowed strategy can be solved analytically while the long-only approach is obtained using an Alternating Direction Method of Multipliers. We study and report each portfolio performance on ten ETF across five asset classes, backtesting these over twelve years against the MSCI World Index.

1 Introduction & Context

Portfolio optimization has been an intensive area for research since Markovitz's Modern Potfolio Theory (1952). Even though Markovitz portfolio became a standard in the asset management industry, it proved to be less efficient than equally-weighted portfolios [2]. Hence new quantitative weight attribution methodologies have been developed in the last decade. The Most Diversified Portfolio aims to maximize a diversification ratio, defined as the ratio of the weighted average volatility and the volatility of the portfolio. In this paper, we do not delve into the mathematical properties of this portfolio and the intuition behind it. These can be found in [5]. Our focus is to solve the underlying optimization problem and compare the performances of the MDP against the MSCI World Index and a classical Inverse Volatility portfolio, on our own selection of ETFs. We obtain an analytic closed form for the MDP weights in case short-selling is allowed, and develop an ADMM solving for the optimization problem when long positions only are allowed.

Our code is made public on a GitHub repository [1].

2 Data

In this section, we present the data used to build an example portfolio and evaluate our model against a backtesting index.

Backtest Consistently with [5], our invetment universe for backtesting is the MSCI World, which is the reference market-cap weighted stock market index of 1,649 companies. This index covers various geographies (Western Europe, US, Singapore, Hong-Kong, Israel, Japan, Australia, New Zealand), different levels of market capitalizations across several sectors. The data was extracted from Bloomberg from 12/18/07 to 3/8/19.

[1] https://github.com/RaphAbb/Portfolio-Optimization

Preprint. Work in progress.

Portfolio components For diversification purposes, our portfolio was built using ETF (Exchanged-Traded fund) tracking four different types of assets, as reported in the table below. We added two FX rates to also include the currencies asset class.

Table 1: Components selected for the construction of the portfolio

Asset Class	Code	Name
Equity	SPY	SPRD S&P 500 ETF
	VTI	Vanguard Total Stock Market ETF
Bond	BND	Vanguard Total Bond Market ETF
	EMB	iShares J.P. Morgan USD Emerging Markets Bond ETF
Fixed Income	TLT	iShares 20+ Year Treasury Bond ETF
	MBB	iShares MBS Bond ETF
Commodities	IAU	iShares Gold Trust
	USO	United States Oil Fund
FX	USD\EUR	
	USD\JPY	

In the formation of this portfolio, we tried to pick, in each asset class, two standard ETF index with long-only strategies. To diminish the absolute correlation between our components, we also considered for each asset class two indices that differ either in the geography covered or in the features of the underlying asset.
We present below a short description of each component selected:

- SPY: reference ETF tracking the S&P 500, which covers the 500 biggest market capitalization in the US stock market (NYSE, Nasdaq, CBOE)
- VTI: ETF with exposure to the U.S. equity market, investing across all sectors in 1,523 different stocks
- BND: reference ETF tracking the US bond market (Treasury bills, corporate bonds, MBS, agency bonds) covering assets with all maturity lengths
- EMB: ETF with exposure to 441 all-term emerging market obligations issued in US dollars (Russia, Peru, Colombia, etc.)
- TLT: ETF providing exposure to 35 different long-term (20 years) US treasury
- MBB: Exposure to the mortgage backed security slice of the bond market, covering 2,340 assets
- IAU: ETF tracking the Gold commodity
- USO: ETF tracking the crude-oil commodity through futures

The rationale behind using ETFs instead of picking vanilla assets (Apple stock, Google corporate bond, etc.) is to hedge the unsystematic risk, specific to one security. Thus, we can get exposure to the systematic risk of a certain type of asset, with a specific criterion (geography, market capitalizations, sectors, maturities, etc), without being biased by the specific risk of a particular asset. As explained in [1], this is one of main goals achieved by passive investment.

3 Model

In this section, we delve into the mathematical structure of the most diversified portfolio and delve into the two approaches presented to build a portfolio. We define Σ the covariance matrix of the components of our portfolio, w the weights of the portfolio and σ the vector of volatility of the assets. The notations are consistent across the the whole part.

3.1 The most Diversified Portfolio

The Most Diversified Portfolio implements a long-only strategy ($w \geq 0$) and can be found by solving the following problem:

$$\begin{array}{rl} \max & \frac{w^T\sigma}{\sqrt{w^T\Sigma w}} \\ \text{s.t.} & w^T e = 1 \\ & w \geq 0 \end{array}$$

where $e = (1, ..., 1)^T$.

This is equivalent to the minimization problem:

$$\begin{array}{rl} \min & \frac{\sqrt{w^T\Sigma w}}{w^T\sigma} \\ \text{s.t.} & w \geq 0 \end{array}$$

We can notice that our objective function is invariant by scalar multiplication on w, therefore if a portfolio w^* minimizes our objective function, then for any $\lambda \in \mathbb{R}$, λw^* is also a solution of the minimzation problem.

Therefore, giving a solution w^* of the problem, there exists λ such that $\lambda(w^*)^T\sigma = 1$, and we can thus equivalently study the problem:

$$\begin{array}{rl} \min & \sqrt{w^T\Sigma w} \\ \text{s.t.} & w^T\sigma = 1 \\ & w \geq 0 \end{array}$$

3.2 Analytic closed form when allowing short-selling

If we authorize short-selling, we can get ride of the constraint $w \geq 0$, and the problem becomes:

$$\begin{array}{rl} \min & \sqrt{w^T\Sigma w} \\ \text{s.t.} & w^T\sigma = 1 \end{array}$$

The Lagrangian of the problem is defined by:

$$\mathcal{L}(w, \lambda) = w^T\Sigma w - \lambda(w^T\sigma - 1)$$

The first-order optimality KKT conditions provide the two following equations:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial w} = 2\Sigma w - \lambda\sigma = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = -w^T\sigma - 1 = 0 \end{cases}$$

As Σ is positive-definite, it is thus non-singular. We have:

$$\begin{cases} w = \frac{1}{2}\lambda\Sigma^{-1}\sigma \\ w^T\sigma = \sigma^T w = 1 \end{cases}$$

Considering the dot product of the first equation with σ, we get:

$$\sigma^T w = 1 = \frac{1}{2}\lambda\sigma^T\Sigma^{-1}\sigma$$

from which we get $\lambda^* = \frac{2}{\sigma^T\Sigma^{-1}\sigma}$.

Re-injecting this formula in our first KKT equation, we obtain an analytical closed form for our vector of weights that minimizes the problem:

$$w^* = \frac{\Sigma^{-1}\sigma}{\sigma^T\Sigma^{-1}\sigma}$$

3.3 Alternating Direction Method of Multipliers for Long-Only portfolios

In this section, we delve into solving this problem using an ADMM (as per [3]) that splits the problem into several smaller sub problems that are solved iteratively.

We first get back to the original problem implementing long-only strategies (enforcing $w \geq 0$). Since we know that $f : x \to \frac{1}{2}\sqrt{x}$ is strictly increasing, the original most diversified long-only portfolio can be written as follows:

$$\begin{aligned} \max\ & \frac{1}{2} w^T \Sigma w \\ \text{s.t.}\ & w^T \sigma = 1 \\ & w \geq 0 \end{aligned} \tag{1}$$

Furthermore, we know that the ADMM is well suited for solving problems that can be divided in:

$$\begin{aligned} \min_{x,z} \quad & f(x) + g(z) \\ \text{s.t.} \quad & Ax + Bz = c \end{aligned}$$

To come to a form similar to above, we consider the Lagrangian of the original problem:

$$\mathcal{L}(w, \lambda, y) = \frac{1}{2} w^T \Sigma w + \lambda(w^T \sigma - 1) + y^T w$$

Then, for $\lambda \geq 0$ and $y \geq 0$, the Lagrangian relaxation problem becomes

$$\min_w \mathcal{L}(w, \lambda, y)$$

We know that, as the problem is convex, $\mathcal{L}$ is also convex. We know that there exists a couple of Lagrangian multipliers $(\lambda^*, y^*) \geq 0$ such that:

$$w^* \text{ solves(1)} \iff \begin{cases} w^* \in \arg\min \mathcal{L}(w, \lambda^*, y^*) \\ w^* \geq 0 \\ \lambda^*((w^*)^T \sigma - 1) = 0 \\ (y^*)^T w^* = 0 \end{cases}$$

As $\mathcal{L}(w, \lambda^*, y^*)$ is differentiable, the condition that $w^* \in \arg\min \mathcal{L}(w, \lambda^*, y^*)$ is equivalent to $\nabla \mathcal{L}(w^*, \lambda^*, y^*) = 0$. Furthermore, as shown in the section 3.2, for $\lambda^* = \frac{2}{\sigma^T \Sigma^{-1} \sigma}$ it holds that $(w^*)^T \sigma = 1$. Hence, the solution to the unconstrained relaxed convex problem with $\lambda = \lambda^*$ also satisfies the original problem (1).

We now formulate the Lagrangian relaxation problem to:

$$\begin{aligned} \min_{x,z}\ & \frac{1}{2} x^T \Sigma x + \lambda(z^T \sigma - 1) + y^T z \\ \text{s.t.}\ & x - z = 0 \end{aligned} \tag{2}$$

The augmented Lagrangian of (2) is defined as:

$$\mathcal{L}(x, z, \lambda, y) = \frac{1}{2} x^T \Sigma x + \lambda(z^T \sigma - 1) + y^T z + c(x - z) + \frac{\rho}{2} \|x - z\|_2^2$$

We then introduce the scaled dual variable $u = \frac{1}{\rho} y$, as proposed by [4]. This leads to:

$$\mathcal{L}(x, z, \lambda, y) = \frac{1}{2} x^T \Sigma x + \lambda(z^T \sigma - 1) + y^T z + \frac{\rho}{2} \|x - z + u\|_2^2 + \frac{\rho}{2} \|u\|_2^2$$

We then derive the following ADMM iterative scheme:

$$\begin{cases} x^{k+1} = \arg\min_x \frac{1}{2} x^T \Sigma x + \frac{\rho}{2} \|x - z^k + u^k\|_2^2 \\ z^{k+1} = \arg\min_z \lambda(z^T \sigma - 1) + y^T w + \frac{\rho}{2} \|x^{k+1} - z + u^k\|_2^2 \\ u^{k+1} = u^k + (x^{k+1} - z^{k+1}) \end{cases}$$

The x^k update will be carried out using Gradient Descent Method and the fact that:

$$\nabla f(x) = \Sigma x + \rho(x - z^k + u^k)$$

The z^k update will be carried out using Gradient Descent Method and the fact that:

$$\nabla g(z) = \lambda \sigma + y + \rho(x^{k+1} - z + u^k)$$

Implementation We implement the ADMM above in `model.py`. We run different models using $\lambda = 10^{-3}$ and $\lambda^* = \frac{2}{\sigma\Sigma^{-1}\sigma}$. To perform better convergence, we also apply a randomly permuted variant for the ADMM method, which consists in changing uniformly at random the order of the three updates above.

4 Results

We compare the Profit and Loss of three portfolios on our asset universe. The first one allocates weights proportionally to the inverse of the volatility of the assets (IV). The second one uses the analytic closed form of the Most Diversified Portfolio Methodology (MDP) variant in the case of short-allowed strategies. The last one uses an ADMM method to solve the MDP problem in its long-only original form. To build a benchmark for our two portfolios, we decided to use an inverse volatility weight distribution. Indeed, choosing a simple equally weighted allocation ($w_i = \frac{1}{10}$ for all i) does not make much sense for an asset universe with drastically different returns. We furthermore use the MSCI World Index as a reference to compare the returns and Sharpe obtained.

In our study, the weights have been rebalanced on a monthly basis, using 120 business days of data to compute volatilities and covariance matrix. The profits are not reinvested in the strategy.

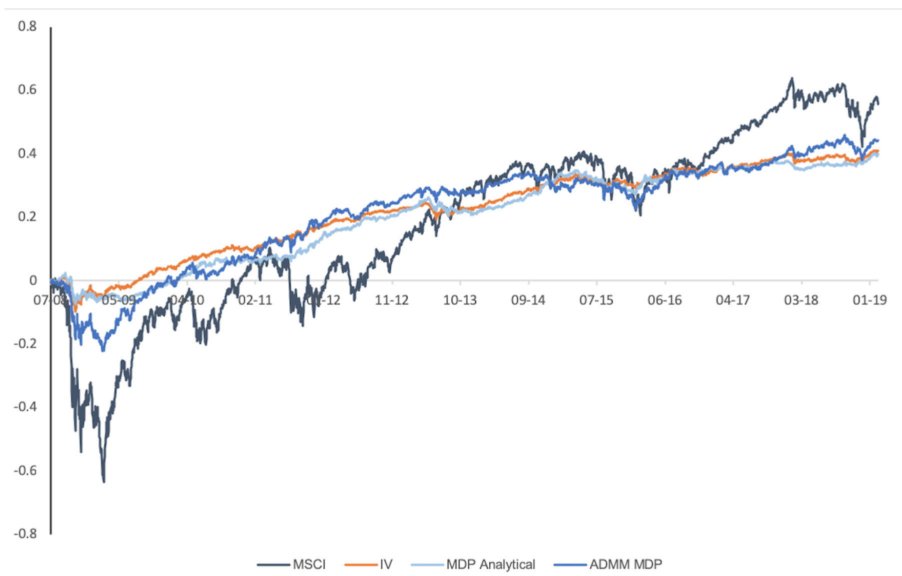

Figure 1: Last 12 years Profit Loss

Using market practices to compare the three portfolios, we compute their Sharpe Ratios on three different time windows (last year, last two years, last five years)[2]

Table 2: Results obtained using a short-selling-allowed closed form and long-only ADMM

Statistic	MSCI	Inverse Vol	MDP Analytical	ADMM MDP
Sharpe LY	0.724	**2.736**	2.070	1.753
Sharpe L2Y	0.556	1.939	**2.264**	0.803
Sharpe L5Y	0.189	0.583	**0.592**	0.428

To have a better idea of the stability of our performance, we can display the rolling Sharpe Ratio of our portfolios, computed on a one year time window.

[2]The L2Y and L5Y Sharpe ratio have not been annualized and are used as a comparison metric.

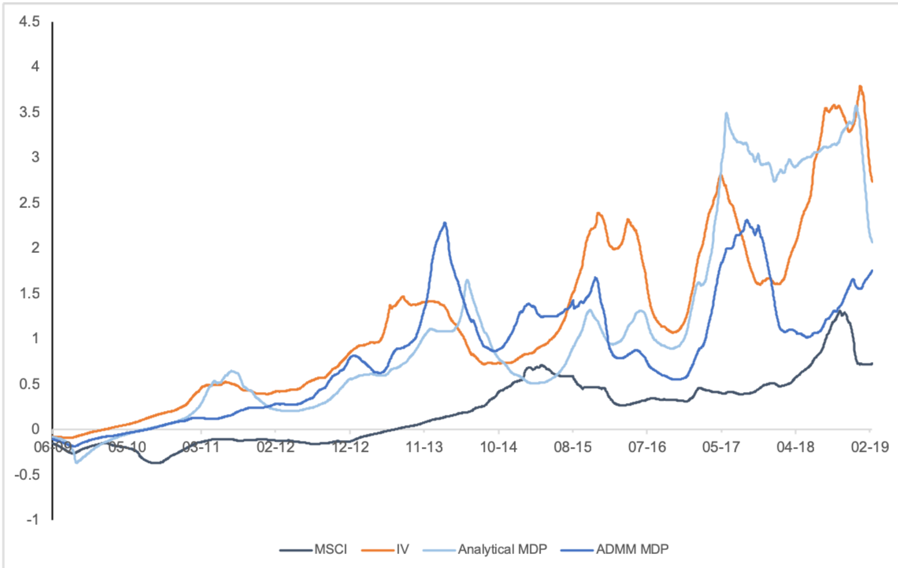

Figure 2: Last year Sharpe Ratio

We can notice that our Sharpes are pretty high, with an impressive spike for the MDP in February 2017.

5 Discussion

In this section, we provide several ideas that could be used to further analyze and improve MDP portfolio performances:

- Bootstrapping can be implemented to get better estimators of the asset's volatility and covariance matrix. For instance, as shown in [2], bootstrapping applied to Markowitz's portfolio lead to stronger returns.
- Our volatility estimation model come from classical vanilla sample estimators. More advanced ones would be an interesting study area in order to come up with a more intricate model (EWMA, GARCH).
- We use only ten ETFs as our asset universe: we are convinced that the MDP portfolio would provide even better results on a larger panel of assets.
- A deeper fine-tuning process of the model hyperparameters (rebalacing frequency, ADMM's hyperparameters, time-window on which volatilities and covariances are computed) could also provide better perfomances.
- On the evaluation side, it would be useful to consider a broader set of metrics (CAGR, Calmar Ratio, Sortino Ratio, etc.) in order to capture the mechanics and performance of the portfolios.

References

[1] Kenechukwu Anadu et al. "The Shift from Active to Passive Investing: Potential Risks to Financial Stability?" In: *Federal Reserve Bank of Boston* FEDS Working Paper No. 2018-060 (2018). FEDS: 2018.060. URL: http://dx.doi.org/10.17016/FEDS.2018.060.

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