NOTE ON SMALL QUANTUM PRODUCT

1. Introduction

Using Hodge theory, this note records a property that the ambient cohomology for a very ample hypersurface is closed under the small restricted quantum product(see Theorem 3.1 below for the precise statement). we also show that the restriction of the poicare pairing to the ambient cohomology part of a very ample hypersurface is also non-degenerate.

2. Setting

Assume that X is a smooth projective variety over $\mathbb{C}$, and $j : Y \to X$ be an inclusion of smooth hypersurface coming from a section s of a very ample line bundle L on X. In this note, we will also assume that X admits a Lefschetz pencil for Y, i.e. there exits another section t of L whose zero loci is smooth and meets transversally with Y. Consider the space $\mathcal{Y} \subset X \times \mathbb{P}^1$ cut off by the equation $at + bs$. where a, b are the homogeneous coordinates of $\mathbb{P}^1$. By the transversality condition, $\mathcal{Y}$ itself is smooth, and $\mathcal{Y}$ is generically smooth over $\mathbb{P}^1$. Note $\mathcal{Y}$ is also the blow up of X along the base locus $Z(s, t)$. They fit into the following diagram with all squares Cartesian:

$$\begin{tikzcd}
Y \arrow[r, "i"] \arrow[d] \arrow[rrrr, bend left, "j"] & \mathcal{Y}_U \arrow[r, "j_U"] \arrow[d, "\pi_U"] & \mathcal{Y} \arrow[r, hook] \arrow[d, "\pi"] & X \times \mathbb{P}^1 \arrow[r, "pr_X"] \arrow[dl] & X \\
\{0\} \arrow[r] & U \arrow[r] & \mathbb{P}^1 & &
\end{tikzcd}$$

where U is an open subset of $\mathbb{P}^1$ containing the point $[a : b] = [0 : 1]$ making π_U is a smooth morphism.

For any degree $\beta \in \mathrm{Nef}(X)$, we define the space

$$\mathcal{M}_{0,n}(Y, \beta)$$

to be

$$\sqcup_{d \in \mathrm{NEf}(Y), j_*(d) = \beta} \mathcal{M}_{0,n}(Y, d) \ .$$

Then We define the *restricted* small quantum product of Y to be: for any $\alpha_1, \alpha_2 \in H^*(Y, \mathbb{Q})$, we define the small quantum product $\alpha_1 \circ \alpha_2$ to be

$$\sum_{\beta \in Nef(X)} q^\beta ev_3(ev_1^*(\alpha_1) \cap ev_2^*(\alpha_2) \cap [\mathcal{M}_{0,3}(Y, \beta)]^{\mathrm{vir}}) \ .$$

A natural question about the restricted small quantum product defined above is whether it is closed for the ambient cohomology $j^*(H^*(X, \mathbb{Q}))$.

Date: Thursday 5^{th} March, 2020.

3. Main theorem

In this section, we will prove the closed-ness of the restricted small restricted quantum product for very ample hypersurfaces.

Theorem 3.1. *In the above setting, for any two classes in the ambient cohomology $\alpha_1, \alpha_2 \in j^*H^*(X, \mathbb{Q})$, we have that $\alpha_1 \circ \alpha_2$ also lies in the ambient cohomology.*

Proof. Let $\beta \in Nef(X)$, observe that we have the following diagram coming from the good pencil:

$$\begin{tikzcd}
\mathcal{M}_{0,3}(Y,\beta) \arrow[r] \arrow[d] & \mathcal{M}_{0,3}(\mathcal{Y}_U/U,\beta) \arrow[r, "ev_1 \times ev_2"] \arrow[d, "ev_3"] & X \times X \\
Y \arrow[r, "i"] \arrow[d] & \mathcal{Y}_U \arrow[d, "\pi_U"] & \\
\{0\} \arrow[r] & U &
\end{tikzcd}$$

where $ev_1 \times ev_2$ factors through the map from $\mathcal{Y}_U \times \mathcal{Y}_U$ to $X \times X$ as a composition of the inclusion from $\mathcal{Y}_U$ to $X \times \mathbb{P}^1$ and the projection from $X \times \mathbb{P}^1$ to X.

Let $\alpha_1, \alpha_2 \in H^*(X, Q)$. Using functoriality of virtual class, the class

$$ev_{3*}([\mathcal{M}_{0,3}(Y,\beta)]^{\mathrm{vir}} \cap ev_1^*(j^*\alpha_1) \cap ev_2^*(j^*\alpha_2)) \tag{3.1}$$

is equal to the class

$$i^* ev_{3*}([\mathcal{M}_{0,3}(\mathcal{Y}_U/U,\beta)]^{\mathrm{vir}} \cap (ev_1 \times ev_2)^*(\alpha_1 \times \alpha_2)) \ , \tag{3.2}$$

which implies[1] that 3.1 lies in the invariant part of $H^*(Y, \mathbb{Q})$ under the monodromy action of the fundamental group $\pi_1(U, 0)$, which is equal to

$$(j_U \circ i)^*(H^*(\mathcal{Y}, \mathbb{Q}))$$

by global invariant theorem (c.f. [Cat15, Theorem 1.2.2]). Finally using the fact that $\mathcal{Y}$ is the blow up of X along the base loci of a Lefschetz pencil, we can show that $(j_U \circ i)^*(H^*(\mathcal{Y}, \mathbb{Q}))$ is equal to $j^*(H^*(X, \mathbb{Q}))$, which finishes the proof. □

Using global invariant theorem and semisimplicity theorem for monodromy of a Lefschetz pencil, we show that the restriction of the poicare pairing to the ambient cohomology is also non-degenerate as follows.

Theorem 3.2. *Let notations be as above, the restriction of the poicare pairing of Y to the ambient cohomology $j^*(H^*(X, \mathbb{Q}))$ is also non-degenerate.*

Proof. By Deligne's smooth decomposition theorem, first we have:

$$R^\bullet \pi_{U*}(IC_{\mathcal{Y}_U}) = \bigoplus_k R^k \pi_*(IC_{\mathcal{Y}_U})[-k] \tag{3.3}$$

[1]Strictly speaking, we interpret ev_{3*} in 3.2 as proper push-forward using Borel-Moore homology

On the other hand, we have that

$$
\begin{aligned}
R^\bullet\pi_{U*}(IC_{\mathcal{Y}_U}) &= R^\bullet\pi_{U*}\mathcal{D}_{\mathcal{Y}_U}\mathcal{D}_{\mathcal{Y}_U}(IC_{\mathcal{Y}_U}) \\
&= \mathcal{D}_U(\bigoplus_k R^k\pi_*(IC_{\mathcal{Y}_U})[-k]) \\
&= \bigoplus_j (R^j\pi_{U*}(IC_{\mathcal{Y}_U}))^\vee[j+2] \ ,
\end{aligned} \tag{3.4}
$$

where $\mathcal{D}_{\mathcal{Y}_U}$ or $\mathcal{D}_U$ are Verdier dual functors.

For a smooth variety W over $\mathbb{C}$, using the fact the intersection complex IC_W is just the shifted constant local system $\mathbb{Q}_W[dim_{\mathbb{C}}(W)]$, then we have, for any $k \in \mathbb{Z}$

$$
\begin{aligned}
j^*(H^k(X,\mathbb{Q})) &= H^k(Y,\mathbb{Q})^{\pi_1(U,0)} \\
&= H^0(U, R^{k-n}\pi_{U*}(IC_{\mathcal{Y}_U})) \\
&= H^0(U, (R^{n-k-2}\pi_{U*}(IC_{\mathcal{Y}_U}))^\vee) \\
&= (H^{2n-k-2}(Y,\mathbb{Q})^\vee)^{\pi_1(U,0)} \\
&= (H^{2n-k-2}(Y,\mathbb{Q})^{\pi_1(U,0)})^\vee \\
&= j^*(H^{2n-k-2}(X,\mathbb{Q}))^\vee
\end{aligned} \tag{3.5}
$$

Here the last second equality uses the semisimplicity of monodromy action of the Lefschetz pencil due to Deligne [BBDG18]. □

Remark 3.3. The theorem also extends to other nice orbifolds like when the ambient space is a proper smooth toric Deligne-Mumford stack (toric stack for short), i.e. $X := [W^{ss}(\theta)/G]$ is a proper toric DM stack constructed from a GIT data (W, G, θ), and Y is a very ample hypersurface equipped with a good Lefschetz pencil as above. Then the above diagrams have the inertia stack counterpart (the lemma 3.9 proved in [Wan19] may be useful). Besides, due to that the coarse moduli space of toric stacks (or hypersurfaces) are $V-$manifolds [PS08, Page 57, Example 2.42], thus rationally smooth [HTT08, Page 211, Definition 8.2.20], then the intersection complex coincide with constant local system shifted up to complex dimension. The above argument using global invariant theorem and semisimplicity theorem (which comes from a wonderful application of weight theory for perverse sheaf [BBDG18]) still works.

Acknowledgment: the author wants to thank that Professor Hiroshi Iritani asks him this question.

References

[BBDG18] Alexander Beilinson, Joseph Bernstein, Pierre Deligne, and Ofer Gabber, *Faisceaux pervers*, Société Mathématique de France., 2018. ↑3

[Cat15] Mark Andrea A. de Cataldo, *Perverse Sheaves and the Topology of Algebraic Varieties (2015 PCMI)*, CoRR (2015), available at 1506.03642. ↑2

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston, Inc., Boston, MA, 2008. Translated from the 1995 Japanese edition by Takeuchi. MR2357361 ↑3

[PS08] Chris A. M. Peters and Joseph H. M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 52, Springer-Verlag, Berlin, 2008. MR2393625 ↑3

[Wan19] Jun Wang, *A Mirror Theorem for Gromov-Witten Theory Without Convexity*, CoRR (2019), available at 1910.14440. ↑3

Department of Mathematics, The Ohio State Universty, Columbus, OH 43210, USA
E-mail address: wang.6410@osu.edu