

# A MIRROR THEOREM FOR GROMOV-WITTEN THEORY WITHOUT CONVEXITY

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ABSTRACT. We prove a genus zero Givental-style mirror theorem for all complete intersections in proper toric Deligne-Mumford stacks, which provides an explicit slice called big  $I$ -function on Givental's Lagrangian cone for such targets. Here the convexity assumption is no longer need as in previous quantum Lefschetz theorem.

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## 1. INTRODUCTION

In the past few decades, following predictions from string theory [CDLOGP91], a series of results known as mirror theorems has been proven; an incomplete list is [Giv96, CCIT15, CG07, Zin08, Giv98, CCLT09, LLY99, GJR17]. These theorems reveal elegant

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patterns and deep structures encoded in the collection of Gromov-Witten invariants of a given symplectic manifold or orbifold  $X$ . However, the scope of these results, and much of Gromov-Witten theory in general, is limited to the world of toric geometry; in all cases above,  $X$  is a complete intersection in a toric variety or certain complete intersection<sup>1</sup> in a toric stack [CCIT19]. The essential reason for this is that one of most efficient way to compute Gromov-Witten invariants is to utilize the technique of the localization theorem [AB95, GP99], which requires the targets to be carried with a good torus action (e.g. toric variety/stack).

Smooth hypersurfaces (or complete intersections more general) in toric Deligne-Mumford stacks are the next class of spaces to consider, but much less is known in this situation. The main difficulty comes from that a hypersurface in a toric stack doesn't have any good torus action in general. Hence one can't directly apply localization theorem to compute the Gromov-Witten invariants of the toric hypersurface. Alternatively, the usual way to compute the Gromov-Witten invariants of a given hypersurface is to use *quantum Lefschetz principle* [KKP03], which relates the twisted Gromov-Witten invariants of an ambient space  $X$  to the Gromov-Witten invariants of its hypersurface  $Y$  which is the zero locus of a section of a given line bundle  $L$  on  $X$ . However, there is a technical assumption called *convexity* for the line bundle  $L$  to apply the *quantum Lefschetz principle*. The convexity says, for any stable map  $f : C \rightarrow X$ , one has

$$H^1(C, f^*L) = 0 ,$$

which holds, for example, when the ambient space  $X$  is a projective variety, the source curve  $C$  is of genus zero and  $L$  is a positive line bundle. Even worse, a counterexample was found in [CGI<sup>+</sup>12] that *quantum Lefschetz principle* can fail for positive hypersurfaces in orbifolds. So there are limited methods to compute the genus zero Gromov-Witten invariants of orbifold hypersurfaces where the convexity fails (see [Gué19] for a recent update for certain hypersurfaces in weighted projective spaces), and a mirror theorem<sup>2</sup> for these targets is lacking for a long time in the literature.

The aim of this paper is to prove a mirror theorem for all complete intersections in proper toric Deligne-Mumford stacks, where the convexity is not required as in previous such mirror theorem. One key ingredient involved in the proof is the use of two twisted graph spaces, which serves the bridge which is analogous to the quasimap wall-crossing conjecture for the big  $I$ -functions.

## 1.1. Main results and Ideas of proof.

1.1.1. *Big I-function.* Let  $X$  be a *proper toric Deligne-Mumford stack* constructed by a GIT data  $(W = \oplus_{\rho \in [n]} \mathbb{C}_\rho, G = (\mathbb{C}^*)^k, \theta)$ , and  $Y \subset X$  is a complete intersection with respect to a direct sum of line bundles  $\oplus_{b=1}^c L_{\tau_b}$  on  $X$  (See §3 for a more precise setting).

<sup>1</sup>See the discussion of convexity below.

<sup>2</sup>In Givental's formalism, we need to construct an explicit slice on the Lagrangian cone.

The big  $I$ -function (or  $I$ -function in short) of the toric stack complete intersections can be written down as follows, of which is obtained by quasimap theory [CFK16]:

$$(1.1) \quad \mathbb{I}(q, t, z) = \exp\left(\frac{1}{z} \sum_{i=1}^l t_i u_i(c_1(L_{\pi_j}) + \beta(L_{\pi_j})z)\right) \sum_{\beta \in \text{Eff}(W, G, \theta)} \frac{\prod_{\rho: \beta(L_\rho) < 0} \prod_{\beta(L_\rho) < i < 0} (D_\rho + (\beta(L_\rho) - i)z)}{\prod_{\rho: \beta(L_\rho) > 0} \prod_{0 \leq i < \beta(L_\rho)} (D_\rho + (\beta(L_\rho) - i)z)} \cdot \frac{\prod_{b: \beta(L_{\tau_b}) > 0} \prod_{i: 0 \leq i < \beta(L_{\tau_b})} (c_1(L_{\tau_b}) + (\beta(L_{\tau_b}) - i)z)}{\prod_{b: \beta(L_{\tau_b}) < 0} \prod_{i: \beta(L_{\tau_b}) < i < 0} (c_1(L_{\tau_b}) + (\beta(L_{\tau_b}) - i)z)} i_*(s_{E \geq 0, \text{loc}}^!([Z_\beta^{ss}/(G/\langle g_\beta^{-1} \rangle)])) .$$

See section §3 for the terminology appearing in  $\mathbb{I}(q, t, z)$ . We remark here  $i_*(s_{E \geq 0, \text{loc}}^!([Z_\beta^{ss}/(G/\langle g_\beta^{-1} \rangle)]))$  is an element in the twister sector  $H^*(\bar{I}_{g_\beta^{-1}} Y, \mathbb{Q})$  (see Corollary 3.3),  $t = \sum_{i=1}^l t_i u_i(c_1(L_{\pi_1}), \dots, c_1(L_{\pi_k}))$  is an element in  $H^*(Y, \mathbb{Q})[t_1, \dots, t_l]$ , where  $k = \text{rank}(G)$ ,  $\{\pi_j, 1 \leq j \leq k\}$  are standard representations of  $G = (\mathbb{C}^*)^k$ .  $l$  can be any nonnegative integers and  $u_i$  can be any polynomial on first Chern class of line bundles  $L_{\pi_1}, \dots, L_{\pi_k}$ .

Now we state our main theorem, which actually corresponds to the quasimap wall-crossing conjecture for big  $\mathbb{I}$ -function (c.f [CFK16]).

**Theorem 1.1** (Main Theorem).  *$-z\mathbb{I}(q, t, -z)$  is a slice on Givental's Lagrangian cone of  $Y$ . More explicitly, there exists a mirror transformation  $\mu(q, t, y)$  such that we have the following identity:*

$$(1.2) \quad \mathbb{I}(q, t, z) = J(q, \mu(q, t, y), z),$$

where  $J(q, \mu(q, t, y), z)$  is defined by the  $J$ -function<sup>3</sup>

$$J(q, \mathbf{t}, z) := \mathbb{I}_Y + \frac{\mathbf{t}(z)}{z} + \sum_{\beta \in \text{Eff}(W, G, \theta)} \sum_{m \geq 0} \frac{q^\beta}{m!} \phi^\alpha \langle \mathbf{t}(-\bar{\psi}_1), \dots, \mathbf{t}(-\bar{\psi}_m), \frac{\phi_\alpha}{z(z - \bar{\psi}_*)} \rangle_{0, [m] \cup \star, \beta} .$$

Here the input  $\mathbf{t}$  is an element in  $(q, t)H^*(\bar{I}_\mu Y, \mathbb{Q})[y][[t_1, \dots, t_l]][[\text{Eff}(AY, G, \theta)]]$ <sup>4</sup>, and  $\mathbf{t}(z)$  (resp.  $\mathbf{t}(-\bar{\psi}_i)$ ) means that we replace the variable  $y$  in  $\mathbf{t}$  by  $z$  (resp.  $-\bar{\psi}_i$ ).

Note here any degree  $\beta \in \text{Eff}(W, G, \theta)$  of  $X$  (c.f.2.4), we will denote the Gromov-Witten invariant

$$\phi^\alpha \langle \mathbf{t}(-\bar{\psi}_1), \dots, \mathbf{t}(-\bar{\psi}_m), \frac{\phi_\alpha}{z(z - \bar{\psi}_*)} \rangle_{0, [m] \cup \star, \beta}$$

<sup>3</sup>Here we treat  $J(q, \mathbf{t}, z)$  as a functional, which means, after fixing the input  $\mathbf{t}$ , we think  $J(q, \mathbf{t}, z)$  as a formal series on the Novikov variable  $q$  and the variable  $z$ , which doesn't involve the variable  $y$ .

<sup>4</sup>It means that  $\mathbf{t}$  admits an expression as  $\sum_{(\beta, I) \neq 0} q^d t^I f_{d, I}$ , where  $f_{d, I} \in H^*(\bar{I}_\mu Y, \mathbb{Q})[y]$ . This choice of input  $\mathbf{t}$  gives a much less general definition of Givental's  $J$ -function in the usual literature, but it suffices for the need in this paper.

to be

$$\sum_{\substack{d \in \text{Eff}(AY, G, \theta) \\ i_*(d) = \beta}} \phi^\alpha \langle \mathbf{t}(-\bar{\psi}_1), \dots, \mathbf{t}(-\bar{\psi}_m), \frac{\phi_\alpha}{z(z - \bar{\psi}_\star)} \rangle_{0, [m] \cup \star, d},$$

where  $\text{Eff}(AY, G, \theta)$  is semigroup of the degree of  $Y$ .

Now let's define precisely  $\mu(q, t, z)$  appearing in our main theorem, Write  $z\mathbb{I}(q, t, z)$  as a formal Laurent series in variable  $z$ :

$$\dots + \mathbb{I}_{-1}(q, t)z^2 + \mathbb{I}_0(q, t)z + \mathbb{I}_1(q, t) + \mathcal{O}(z^{-1}),$$

and define  $\mu(q, t, z)$  to be the truncation in nonnegative  $z$  powers:

$$\mu(q, t, z) := [z\mathbb{I}(q, t, z) - \mathbb{I}_Y z]_+ = \dots + \mathbb{I}_{-1}(q, t)z^2 + (\mathbb{I}_0(q, t) - \mathbb{I}_Y)z + \mathbb{I}_1(q, t).$$

By the definition of  $\mathbb{I}(q, t, z)$ ,  $z\mathbb{I}(q, t, z)$  admits an asymptotic expansion in  $q, t$ :

$$z\mathbb{I}(q, t, z) = z\mathbb{I}_Y + \mathcal{O}(q) + \mathcal{O}(t),$$

which implies that  $\mu(q, t, z) = \mathcal{O}(q) + \mathcal{O}(t)$ .

1.1.2. *Two special cases of the mirror theorem.* Using Corollary 3.3 in §3, we get two interesting special cases of the  $I$ -function. When  $Y$  is a hypersurface with respect to a line bundle  $L := L_\tau$  for some character  $\tau$ , the formula above becomes:

$$\begin{aligned} \mathbb{I}(q, t, z) = & \exp\left(\frac{1}{z} \sum_{i=1}^l t_i u_i(c_1(L_{\pi_j}) + \beta(L_{\pi_j})z)\right) \\ & \left( \sum_{\substack{\beta \in \text{Eff}(W, G, \theta) \\ \beta(L) \geq 0}} q^\beta \frac{\prod_{\rho: \beta(L_\rho) < 0} \prod_{\beta(L_\rho) \leq i < 0} (D_\rho + (\beta(L_\rho) - i)z)}{\prod_{\rho: \beta(L_\rho) > 0} \prod_{0 \leq i < \beta(L_\rho)} (D_\rho + (\beta(L_\rho) - i)z)} \right. \\ & \times \prod_{0 \leq i < \beta(L)} (c_1(L) + (\beta(L) - i)z) \mathbb{I}_{g_\beta^{-1}} \\ & + \sum_{\substack{\beta \in \text{Eff}(W, G, \theta) \\ \beta(L) \in \mathbb{Z}_{<0}}} q^\beta \frac{\prod_{\rho: \beta(L_\rho) < 0} \prod_{\beta(L_\rho) < i < 0} (D_\rho + (\beta(L_\rho) - i)z)}{\prod_{\rho: \beta(L_\rho) > 0} \prod_{0 \leq i < \beta(L_\rho)} (D_\rho + (\beta(L_\rho) - i)z)} \\ & \times \prod_{\beta(L) < i < 0} \frac{1}{(c_1(L) + (\beta(L) - i)z)} [Y_\beta^{ss} / (G / \langle g_\beta^{-1} \rangle)] \\ & + \sum_{\substack{\beta \in \text{Eff}(W, G, \theta) \\ \beta(L) \in \mathbb{Q}_{<0} \setminus \mathbb{Z}_{<0}}} q^\beta \frac{\prod_{\rho: \beta(L_\rho) < 0} \prod_{\beta(L_\rho) \leq i < 0} (D_\rho + (\beta(L_\rho) - i)z)}{\prod_{\rho: \beta(L_\rho) > 0} \prod_{0 \leq i < \beta(L_\rho)} (D_\rho + (\beta(L_\rho) - i)z)} \\ & \times \prod_{\beta(L) < i < 0} \frac{1}{(c_1(L) + (\beta(L) - i)z)} \mathbb{I}_{g_\beta^{-1}} \Big). \end{aligned} \tag{1.3}$$

When all the line bundles  $L_{\tau_b}$  are all semi-positive, we have the following corollary. The above formula specializes to:

$$(1.4) \quad \mathbb{I}(q, t, z) = \exp\left(\frac{1}{z} \sum_{i=1}^l t_i u_i (c_1(L_{\pi_j}) + \beta(L_{\pi_j})z)\right) \sum_{\beta \in \text{Eff}(W, G, \theta)} q^\beta \frac{\prod_{\rho: \beta(L_\rho) < 0} \prod_{\beta(L_\rho) \leq i < 0} (D_\rho + (\beta(L_\rho) - i)z)}{\prod_{\rho: \beta(L_\rho) > 0} \prod_{0 \leq i < \beta(L_\rho)} (D_\rho + (\beta(L_\rho) - i)z)} \times \prod_b \prod_{0 \leq i < \beta(L_{\tau_b})} (c_1(L_{\tau_b}) + (\beta(L_{\tau_b}) - i)z) \mathbb{1}_{g_\beta^{-1}}$$

The above formula matches the formula for positive hypersurfaces in toric stacks for which the convexity holds [CCIT14, §5] and the formula for a possibly ray divisor (given by a coordinate function corresponding to the ray) of a toric stack for which the convexity may fail [CCIT15, CCFK15]. See §7 for a non-positive example where the convexity fails.

1.1.3. *Sketch of the proof of the main theorem.* Now Before sketching the proof of the main theorem, let's analyze  $\mu(q, t, z)$  in more detail. Let  $\mathbb{I}(q, z) := \mathbb{I}(q, 0, z)$ , we can expand  $\mathbb{I}(q, z)$  as

$$\mathbb{I}(q, z) = \sum_{\beta \in \text{Eff}(W, G, \theta)} q^\beta \mathbb{I}_\beta(z) ,$$

where  $\mathbb{I}_\beta(z) \in H^*(\bar{I}_\mu Y, \mathbb{Q})[z, z^{-1}]$ . Then we can decompose  $\mathbb{I}(q, t, z)$  as a formal sum

$$\mathbb{I}(q, t, z) = \sum_{\beta \in \text{Eff}(W, G, \theta), p \in \mathbb{N}} q^\beta \frac{\mathbf{t}^p}{p! z^p} \mathbb{I}_\beta(z) .$$

where  $\mathbf{t} = \sum_{i=1}^l t_i u_i (c_1(L_{\pi_j}) + \beta(L_{\pi_j})z)$ . For nonzero pair  $(\beta, p)$ , set  $\mu_{\beta, p} := [\frac{t^p z \mathbb{I}_\beta(z)}{p! z^p}]_+$  as the truncation in nonnegative  $z$  powers. We note that  $\mu_{\beta, p}(z)$  is a polynomial in  $H^*(\bar{I}_\mu Y, \mathbb{Q})[t_0, \dots, t_l, z]$  of homogeneous degree  $p$  in variables  $t_1, \dots, t_l$ . For convenience we set  $\mu_0(z) = \mu_{(0,0)}(z) = 0$ . Then we have write  $\mu(q, t, z)$  as a sum

$$\mu(q, t, z) = \sum_{p \in \mathbb{Z}_{\geq 0}, \beta \in \text{Eff}(W, G, \theta), (\beta, p) \neq 0} q^\beta \mu_{\beta, p}(z) ,$$

Multiply by  $z$  on both sides of (1.2), we observe that, to prove the main theorem, it suffices to prove that, for arbitrary  $(\beta, p) \in \text{Eff}(W, G, \beta) \times \mathbb{N}$ , we have:

$$(1.5) \quad z \frac{\mathbf{t}^p}{p! z^p} \mathbb{I}_\beta(z) = \delta_{(\beta, p), 0} z + \mu_{(\beta, p)}(z) + \sum_{m=0}^{\infty} \sum_{\substack{\beta_0 + \beta_1 + \dots + \beta_m = \beta \\ p_1 + \dots + p_m = p}} \frac{1}{m!} \phi^\alpha \langle \mu_{(\beta_1, p_1)}(-\bar{\psi}_1), \dots, \mu_{(\beta_m, p_m)}(-\bar{\psi}_m), \frac{\phi_\alpha}{z - \bar{\psi}_\star} \rangle_{0, [m] \cup \star, \beta_0} .$$

To prove (1.2), it suffices to prove (1.5) for every  $(\beta, p) \in \text{Eff}(W, G, \theta) \times \mathbb{N}$ . Observe that when  $(\beta, p) = 0$ , (1.5) reduces to

$$z = z ,$$

which holds obviously. This will be the base case for our inductive proof of (1.5).

To prove (1.5), observe that the nonnegative parts in  $z$  on both sides of (1.5) are equal to  $\mu_{\beta,p}(z)$  by the very definition of  $\mu_{\beta,p}(z)$ . It follows that, in order to prove (1.5), it suffices to show the truncations in negative  $z$  powers of (1.5)

$$(1.6) \quad \left[ z \frac{t^p}{p! z^p} \mathbb{I}_\beta(z) \right]_- := \sum_{m=0}^{\infty} \sum_{\substack{\beta_0 + \beta_1 + \dots + \beta_m = \beta \\ p_1 + \dots + p_m = p}} \frac{1}{m!} \phi^\alpha \langle \mu_{(\beta_1, p_1)}(-\bar{\psi}_1), \dots, \mu_{(\beta_m, p_m)}(-\bar{\psi}_m), \frac{\phi_\alpha}{z - \bar{\psi}_\star} \rangle_{0, [m] \cup \star, \beta_0}^\infty$$

holds. Equivalently, it suffices to show, for any nonnegative integer  $c$ , one has

$$(1.7) \quad \left[ z \frac{t^p}{p! z^p} \mathbb{I}_\beta(z) \right]_{z^{-c-1}} := \sum_{m=0}^{\infty} \sum_{\substack{\beta_0 + \beta_1 + \dots + \beta_m = \beta \\ p_1 + \dots + p_m = p}} \frac{1}{m!} \phi^\alpha \langle \mu_{(\beta_1, p_1)}(-\bar{\psi}_1), \dots, \mu_{(\beta_m, p_m)}(-\bar{\psi}_m), \phi_\alpha \bar{\psi}_\star^c \rangle_{0, [m] \cup \star, \beta_0}.$$

The idea to prove (1.7) is to show that both sides of (1.7) satisfy the same recursive relations (see Theorem 6.2 and Theorem 6.4) by induction on  $\beta(L_\theta) + p$ . This is done by considering two master spaces (see §4.1 and §5.1), which are root stack modification of the twisted graph spaces found in [CJR17a, CJR17b]. Then we apply virtual localization to express two auxiliary cycles (see (6.1) and (6.9)) corresponding to two master spaces in graph sums and extract  $\lambda^{-1}$  coefficients ( $\lambda$  is an equivariant parameter). Finally, the polynomiality of the two auxiliary cycles implies that the coefficients for  $\lambda^{-1}$  term must vanish, thus they yields recursive relations (see also Theorem 6.2 and Theorem 6.4) to finish the proof of the quasimap wall-crossing.

**Remark 1.2.** Previously, the genus zero quasimap wall-crossing conjecture was proved for GIT targets with good torus action [CCFK15, CFK14], i.e. zero dimensional torus fixed orbits are finite and the one dimensional torus fixed orbits are isolated. Thus the targets (toric stack hypersurfaces) we treat here provides the examples with no good torus action.

The proof of the quasimap wall-crossing here is quite robust; the main geometrical construction including twisted graph space and root stack construction, and recursive relations can be directly generalized to all proper GIT targets considered in quasimap theory. Hence we expect the method developed here can be used to prove the genus zero quasimap wall-crossing conjecture for all proper GIT targets considered in quasimap theory.

The main geometrical input in the proof of wall-crossing here is inspired by the twisted graph space used in [CJR17b, CJR17a], where they use the genus zero quasimap wall-crossing as input to prove the high genus quasimap wall-crossing. So it may be surprising that certain modification of the twisted graph space can be used to prove the genus zero quasimap wall-crossing directly. However one novelty of Here the modification takes  $r$ -root of certain divisor on the twisted graph space, then the push-forward of Chern class of root bundle appears naturally when one applies the localization as in the work of double ramification cycle [JPPZ17, JPPZ18], where they

need the Pixton's polynomiality of the push-forward in  $r$  and extract  $r^0\lambda^{-1}$  ( $\lambda$  is an equivariant parameter) coefficient of the push-forward of Chern class as in the proof of double ramification cycle relations in loc.cit. However it suffices to extract the  $\lambda^{-1}$  coefficient in this paper, hence it does not need Pixton's polynomiality.

**Remark 1.3.** During the preparation of this work, the author learns that Yang Zhou has used a totally different method to prove the quasimap wall-crossing conjecture for all GIT quotients and all genera [Zho19]. The author also learns that Felix Janda and Nawaz Sultani have a different way of computing the (S-extended)  $I$ -functions for some hypersurfaces in weighted projective spaces and use them to calculate Gromov-Witten invariants.

**1.2. Outline.** The rest of this paper is organized as follows. In §2, we will recall the quasimap theory, the author wants to draw readers' attention to the language of  $\theta'$ -stable quasimaps (see Remark 2.3), where  $\theta'$  can be a *rational character*, because it is more suitable than the language of  $\epsilon$ -stable quasimaps for the later construction of the master space in §4. In §3, we collect some important facts about (rigidified) inertia stack of toric stack complete intersections, and compare them with rigidified inertia stack of toric stacks. Some special cycles in the inertia stack will be discussed as they will be appeared in our  $\mathbb{I}$ -function. In §4 and §5, we will construct two master spaces which carry  $\mathbb{C}^*$ -actions, a very explicit  $\mathbb{C}^*$ -localization computation which is based on localization computations [CJR17a, JPPZ17] will be presented, this part is technical, we encourage the reader to skip to go to §6 first and to refer back when needed. In §6, we will calculate two auxiliary cycles corresponding to the two master spaces via localization, they provide recursive relations to prove the genus zero quasimap wall-crossing conjecture for toric stack hypersurfaces.

**Notations:** In this paper, we will always assume that all algebraic stacks and algebraic schemes are locally of finite type over the base field  $\mathbb{C}$ . Given a GIT target  $(W, G, \theta)$ , we will use symbols  $\mathfrak{X}, \mathfrak{Y} \dots$  to mean the quotient stack  $[W/G]$ , symbols  $X, Y \dots$  to mean the corresponding GIT stack quotient  $[W^{ss}(\theta)/G]$ ,  $I_\mu X, I_\mu Y \dots$  to mean the corresponding (cyclotomic) inertia stacks, and  $\bar{I}_\mu X, \bar{I}_\mu Y \dots$  to mean the corresponding rigidified inertia stacks.

We will use the following construction a lot throughout this paper.

**Definition 1.4 (Borel Construction).** Let  $G$  be a linear algebraic group and  $W$  be a variety. Fix a right  $G$ -action on the variety  $W$ . For any character  $\rho$  of  $G$ , we will denote  $L_\rho$  to be the line bundle on the quotient stack  $[W/G]$  defined by

$$W \times_G \mathbb{C}_\rho := [(W \times \mathbb{C}_\rho)/G],$$

where  $\mathbb{C}_\rho$  is the 1-dimensional representation of  $G$  via  $\rho$  and the action is given by

$$(x, u) \cdot g = (x \cdot g, \rho(g)u) \in W \times \mathbb{C}_\rho$$

for all  $(x, u) \in W \times \mathbb{C}_\rho$  and  $g \in G$ . For any linear algebraic group  $T$ , if we have a left  $T$ -action on  $W$  which commutes with the right action of  $G$ , we will lift the line bundle  $L_\rho$  defined above to be a  $T$ -equivariant line bundle, which is induced from the (left)  $T$  action on  $W \times \mathbb{C}_\rho$  in the way that  $T$  acts on  $\mathbb{C}_\rho$  trivially. By abusing notations, we will use the same notation  $L_\rho$  to mean the corresponding invertible sheaf (or  $T$ -equivariant invertible sheaf) over  $[W/G]$  unless stated otherwise.

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## 2. BACKGROUND ON QUASIMAPS

We first recall the definition of a *quasimap* to a GIT target and then allow quasimaps to include  $p$ -fields, our main reference is [CFKM14, CCFK15, CFK16, KO18]. By a GIT target, we mean a triple  $(W, G, \theta)$ , where  $W$  is an irreducible affine variety with locally complete intersection (l.c.i) singularity,  $G$  is a reductive group equipped with a right  $G$ -action on  $W$  and  $\theta$  is an (integral) character of  $G$ . Denote by  $\mathfrak{X} := [W/G]$  the quotient stack. Denote by  $W^{ss}$  (or  $W^{ss}(\theta)$ ) the semistable locus in  $W$ , and by  $W^s$  (or  $W^s(\theta)$ ) the stable locus. Throughout out this paper, for a GIT target  $(W, G, \theta)$ , we will always assume that  $W^{ss}(\theta) = W^s(\theta)$  and the *GIT stack quotient*

$$X := [W^{ss}(\theta)/G]$$

is a smooth *Deligne-Mumford stack*, under which condition,  $X$  is always semi-projective, i.e. it’s proper over the affine GIT quotient  $\text{Spec}(\mathbb{C}[W]^G)$  by the proj-construction of GIT quotient [CCFK15, §2.2][MFK94]:

$$\underline{X} = \mathbf{Proj} \oplus_{n=0}^{\infty} \Gamma(W, W \times \mathbb{C}_{n\theta})^G.$$

Let  $\mathbf{e}$  be the least common multiple of the exponents  $|\text{Aut}(\bar{x})|$  of automorphism groups  $\text{Aut}(\bar{x})$  of all geometric points  $\bar{x} \rightarrow X$  of  $X$ . Then, for any character  $\rho$  of  $G$ , the line bundle  $L_{\rho}^{\otimes \mathbf{e}}$  is the pullback of a line bundle from the coarse moduli  $\underline{X}$  of  $X$ , here the line bundle  $L_{\rho}$  is defined by the Borel (mixed) construction 1.4.

**Definition 2.1.** *Given a scheme  $S$  over  $\text{Spec}(\mathbb{C})$ ,  $f = ((C, q_1, \dots, q_m), P, x)$  is called a quasimap over  $S$  (alternatively  $\theta$ -quasimap over  $S$ ) of class  $(g, m, \beta)$  if it consists of the following data:*

- (1)  $(C, q_1, \dots, q_m)$  is a flat family of genus  $g$  twist curves over  $S$  [AGV08, §4], and  $m$  gerbe marked sections  $q_1, \dots, q_m$  over  $S$ , here we don’t require the gerbe sections to be trivialized;
- (2)  $P$  is a principal  $G$ -bundle on  $C$ ;
- (3)  $x$  is a section of the affine  $W$ -bundle  $(P \times W)/G$  over  $C$  so that it determines a representable morphism  $[x] : C \rightarrow \mathfrak{X} = [W/G]$  as the composition

$$C \xrightarrow{x} (P \times W)/G \longrightarrow [W/G].$$

We say that the quasimap  $f$  is of degree  $\beta \in \text{Hom}_{\mathbb{Z}}(\text{Pic}(\mathfrak{X}), \mathbb{Q})$  if  $\beta(L) = \deg([x]^* L)$  for every line bundle  $L \in \text{Pic}(\mathfrak{X})$ ;

- (4) The base locus of  $[x]$  defined by  $[x]^{-1}(\mathfrak{X} \setminus X)$  is purely of relative dimension zero over  $S$ .



Sometimes we may also use the notation  $f : (C, \mathbf{q} = (q_i)) \rightarrow \mathfrak{X}$  or  $((C, q_1, \dots, q_m), [x])$  to mean a quasimap (or  $\theta$ -quasimap). A quasimap  $f$  is *prestable* (or  $\theta$ -*prestable*) if the base locus are away from nodes and markings.

**Remark 2.2.** We can extend the definition of  $\theta$ -prestable quasimap to allow any *rational character*  $\theta'$  such that  $\theta'$ -prestable quasimap is same as  $\alpha\theta'$ -prestable quasimap for any  $\alpha \in \mathbb{Q}_{>0}$ .

Consider a prestable quasimap  $f$ , since the base point is away from nodes and marking points, for each  $q \in C$ , as in [CFKM14, Definition 7.1.1], we define the length function  $l_\theta(q)$  as follows:

$$(2.1) \quad l_\theta(q) = \min \left\{ \frac{([x]^*s)_q}{n} \mid s \in \Gamma(W, W \times \mathbb{C}_{n\theta})^G, [x]^*s \neq 0, n \in \mathbb{Z}_{>0} \right\},$$

where  $([x]^*s)_q$  is the coefficient of the divisor  $([x]^*s)$  at  $q$ . Note here the length function  $l_\theta$  depends on the *integral* character  $\theta$ . We have the following important observation about the length function  $l_\theta$ : choose  $\alpha \in \mathbb{Q}_{>0}$  such that  $\theta' = \frac{1}{\alpha}\theta$  is another integral character. Then

$$l_\theta = \alpha l_{\theta'},$$

then the length function  $l_\theta$  can be defined for any *rational character*  $\theta'$ , i.e. choose  $\alpha \in \mathbb{Q}_{>0}$  and an integral character  $\theta$  such that  $\theta' = \alpha\theta$ , then we define

$$l_{\theta'} := \alpha l_\theta$$

as in [CFK16, Definition 2.4], note the definition of  $l_{\theta'}$  is independent of decomposition of  $\theta'$  as a product of positive rational number  $\alpha$  and an integral character  $\theta$  by the above observation.

Fix a positive rational number  $\epsilon \in \mathbb{Q}_{>0}$ . Given a prestable quasimap  $f$  over  $\text{Spec}(\mathbb{C})$ , we say  $f$  is a  $\epsilon$ -stable quasimap to  $X$  if  $f$  satisfies the following stability condition:

- (1) the  $\mathbb{Q}$ -line bundle  $(\phi_*([x]^*L_{\mathbf{e}\theta}))^{\frac{\epsilon}{\mathbf{e}}} \otimes \omega_{\underline{C}}^{\log}$  on the coarse moduli curve  $\underline{C}$  of  $C$  is ample. Here  $\phi : C \rightarrow \underline{C}$  is the coarse moduli map. Note the line bundle  $[x]^*L_{\mathbf{e}\theta}$  on  $C$  a pullback of a line bundle on the coarse curve  $\underline{C}$  by the choice of  $\mathbf{e}$  and the prestable condition. Here  $\omega_{\underline{C}}^{\log} = \omega_{\underline{C}}(\sum_{i=1}^m \underline{q}_i)$  is the log dualizing invertible sheaf of the coarse moduli  $\underline{C}$ ;
- (2)  $\epsilon l_\theta(q) \leq 1$  for any  $q \in C$ .

**Remark 2.3** ( $\theta'$ -quasimap). Using the above generalization of length function  $l_{\theta'}$  for a rational character  $\theta'$ , we can give the definition of  $\theta'$ -stable quasimap: given a  $\theta'$ -prestable quasimap  $f = ((C, q_1, \dots, q_m), [x])$ , we say  $f$  is a  $\theta'$ -stable quasimap to  $\mathfrak{X}$  if

- (1) the  $\mathbb{Q}$ -line bundle  $(\phi_*([x]^*L_{\mathbf{b}\theta'}))^{\frac{1}{\mathbf{b}\mathbf{e}}} \otimes \omega_{\underline{C}}^{\log}$  on the coarse moduli curve  $\underline{C}$  of  $C$  is ample. Here  $\phi : C \rightarrow \underline{C}$  is the coarse moduli map, and  $\mathbf{b}$  is a positive integer which makes  $\mathbf{b}\theta'$  an integral character. Note the ampleness is dependent of choice of the positive integer  $\mathbf{b}$ .
- (2)  $l_{\theta'}(q) \leq 1$  for any  $q \in C$ .

Given a GIT target  $(W, G, \theta)$ , following [CFK16, Propsition 2.7], an essentially equivalent definition about  $\epsilon$ -stable quasimaps to  $X$  is, but from a different point of view, the concept of a  $\epsilon\theta$ -stable quasimap to  $\mathfrak{X}$ . The concept of  $\theta'$ -stable quasimap will

play an important role in the construction of master space in section 4. For a *rational character*  $\theta'$  of  $G$ , we will use the notation  $Q_{g,m}^{\theta'}(\mathfrak{X}, \beta)$  to mean the moduli stack of  $\theta'$ -stable quasimaps to the quotient stack  $\mathfrak{X}$  of class  $(g, m, \beta)$ , which is same as the space  $Q_{0,m}^\epsilon([W^{ss}(\theta)/G], \beta)$  of  $\epsilon$ -stable quasimaps to the GIT stack quotient  $[W^{ss}(\theta)/G]$  of class  $(g, m, \beta)$ , where  $\theta' = \epsilon\theta$ ,  $\epsilon$  is a positive rational number and  $\theta$  is an integral character of  $G$ .

We call a prestable quasimap  $f$  over a scheme  $S$  is  $\epsilon$ -stable if for every  $\mathbb{C}$ -point  $s$  of  $S$ , the restriction of  $f$  over  $s$  is  $\epsilon$ -stable. We call  $f$  is 0+stable if  $f$  is  $\epsilon$ -stable for every positive rational number  $\epsilon \in \mathbb{Q}_{>0}$ .

**Definition 2.4.** A group homomorphism  $\beta \in \text{Hom}_{\mathbb{Z}}(\text{Pic } \mathfrak{X}, \mathbb{Q})$  is called  $L_\theta$ -effective if it is realized as a finite sum of classes of some quasimaps to  $X$ . Such elements form a semigroup with identity 0, denoted by  $\text{Eff}(W, G, \theta)$ .

We will need the following lemma proved in [CCFK15, Lemma 2.3].

**Lemma 2.5.** If  $((C, \mathbf{q}), [x])$  is a quasimap of degree  $\beta$ , then  $\beta(L_\theta) \geq 0$ . Moreover,  $\beta(L_\theta) = 0$  if and only if  $\beta = 0$ , if and only if the quasimap is constant (i.e.,  $[x]$  is a map into  $X$ , factored through an inclusion  $\mathbb{B}\Gamma \subset X$  of the classifying groupoid  $\mathbb{B}\Gamma$  of a finite group  $\Gamma$ ).

Now we will give the definition of *semi-positive line bundles* for quasimaps used in this paper.

**Definition 2.6.** Given a GIT data  $(W, G, \theta)$  which gives rise to a GIT stack quotient  $X$ , let  $L_\tau$  be the line bundle associated to a character  $\tau$  of  $G$ . Then we call  $L_\tau$  a *semi-positive line bundle* on  $\mathfrak{X}$  if

$$\beta(L_\tau) \geq 0$$

whenever  $\beta$  is the degree of a quasimap to  $X$ . Sometimes if the GIT model of  $X$  and the character  $\tau$  are clear in the context, we will also call the restricted line bundle  $L_\tau|_X$  a *semi-positive line bundle* on  $X$ . By abusing notations, we will denote the restricted line bundle on  $X$  to be  $L_\tau$  unless stated otherwise.

**Remark 2.7.** Note the definition of a semi-positive line bundle on  $X$  depends on the GIT model of  $X$  and the character  $\tau$ . By Lemma 2.5, the line bundle  $L_\theta$  (or positive tensor power of  $L_\theta$ ) is always a semi-positive line bundle. Note, using the standard GIT presentations of weighted projective spaces, the line bundles  $\mathcal{O}(n)$  ( $n > 0$ ) on weighted projective spaces are always semi-positive line bundles by the definition above.

In the following, we will give an explicit description of quasimaps to *toric Deligne-Mumford stacks*.

**Remark 2.8** (Quasimaps to toric stack). Recall the construction of a (semi-projective) toric Deligne-Mumford stack (or toric stack in short) by a GIT data  $(W, G, \theta)$ . Let  $G = (\mathbb{C}^*)^k$ , and  $W := \bigoplus_{i=1}^n \mathbb{C}_{\rho_i}$  be a direct sum of 1-dimensional representations of  $G$  given by the characters  $\rho_i \in \chi(G)$  for  $1 \leq i \leq n$ . We will denote  $[n]$  to be the collection of (not necessarily distinct) characters  $\rho_i$  of  $G$  for  $1 \leq i \leq n$ . The toric stack  $X$  is defined to be the GIT stack quotient

$$[W^{ss}(\theta)/G].$$

Since we always assume that  $W^{ss}(\theta) = W^s(\theta)$ , then  $X$  is a *semi-projective Deligne-Mumford stack*.

Then in the definition of quasimaps to the toric stack  $X$ , we can replace the principal  $G$ -bundle  $P$  by  $k$  line bundles  $(L_j : 1 \leq j \leq k)$  on  $C$ , and replace the section  $x$  in the definition of quasimap by  $n$  sections

$$\vec{x} = (x_i : 1 \leq i \leq n) \in \bigoplus_{\rho \in [n]} \Gamma(C, L_\rho) ,$$

where  $L_\rho$  is a line bundle on  $C$  defined by

$$L_\rho = \bigotimes_{j=1}^k L_j^{\otimes m_j} ,$$

where and the numbers  $(m_j : 1 \leq j \leq k)$  are determined by the unique relation

$$\rho = \sum_{j=1}^k m_j \pi_j$$

in the character group  $\chi(G)$  of  $G$ . Here  $(\pi_j : 1 \leq j \leq k)$  are the standard characters of  $G = (\mathbb{C}^*)^k$  by projecting to coordinates.

One novel application of  $\theta'$ -stable quasimap for a rational character  $\theta'$  is the use of the notion of  $(\theta, \varepsilon)$ -stable quasimap introduced in [CFK16].

**Definition 2.9.** *[( $\theta', \varepsilon$ )-stable quasimap] Given a tuple  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_p) \in (Q_{>0} \cap (0, 1]^p)$ , we will call *prestable quasimap*  $\mathbf{f} := (C, \mathbf{q}, f : C \rightarrow [W/G] \times [\mathbb{C}/\mathbb{C}^*]^p)$  a  $(\theta', \varepsilon)$ -stable quasimap to  $\mathfrak{X}$  of type  $(g, m, \beta)$  if  $\mathbf{f}$  defines  $\theta' \oplus \bigoplus_{i=1}^p \varepsilon_i \text{id}_{\mathbb{C}^*}$ -stable quasimap to  $[W/G] \times [\mathbb{C}/\mathbb{C}^*]^p$  of type  $(g, m, (\beta, 1, \dots, 1))$ . We will denote  $Q_{g, m|p}^{(\theta, \varepsilon)}(\mathfrak{X}, \beta)$  to be the moduli stack of  $(\theta, \varepsilon)$ -stable quasimaps to  $\mathfrak{X}$  of type  $(g, m, \beta)$ . We call  $\mathbf{f}$  is  $(\theta, (0+)^p)$ -stable if  $\mathbf{f}$  is  $(\theta, \varepsilon)$ -stable for all  $\varepsilon \in \mathbb{Q}_{>0}^p$ . And we will denote  $Q_{g, m|p}^{\theta, 0+}(\mathfrak{X}, \beta)$  to be the moduli stack of  $(\theta, (0+)^p)$ -stable quasimaps to  $\mathfrak{X}$  of type  $(g, m, \beta)$ .*

**Remark 2.10.** It's shown in [CFK16] that a  $(\theta, \varepsilon)$ -stable map to  $\mathfrak{X}$  is equivalent to a  $\varepsilon$ -weighted  $\theta$ -stable map to  $\mathfrak{X}$ , i.e. the source curve is allowed to be a Hassett-stable curve with additional  $p$   $\varepsilon$ -weighted markings. Thus the moduli stack  $Q_{g, m|p}^{\theta, \varepsilon}(\mathfrak{X}, \beta)$  is equipped with  $p$  additional universal evaluation maps to  $\mathfrak{X}$  (not only to  $X$ ). We will denote them by

$$\hat{e}v_j : Q_{g, m|p}^{(\theta, \varepsilon)}(\mathfrak{X}, \beta) \rightarrow \mathfrak{X}, \quad 1 \leq j \leq p .$$

**2.1. Quasimap invariants.** We define the quasimap invariants in this section following [CFK14, CFKM14, AGV08, CCFK15]. Consider an algebraic torus  $T$  action on  $W$ , which commutes with the given  $G$ -action on  $W$ , here  $T$  can be the identity group. We assume that the  $T$ -fixed loci  $\underline{X}_0^T$  of the affine quotient  $\underline{X}_0 = \text{Spec}(\mathbb{C}[W]^G)$  is 0-dimensional. Denote by  $K := \mathbb{Q}(\{\lambda_i\})$  the rational localized  $T$ -equivariant cohomology of  $\text{Spec } \mathbb{C}$ , with  $\{\lambda_1, \dots, \lambda_{\text{rank}(T)}\}$  corresponding to a basis for the characters of  $T$ . We denote

$$\Lambda_K := K[[\text{Eff}(W, G, \theta)]]$$

to be the corresponding Novikov ring. We write  $q^\beta$  for the element corresponding to  $\beta$  in  $\Lambda_K$  so that  $\Lambda_K$  is the  $q$ -adic completion.

For any two elements  $\alpha_1, \alpha_2$  in the  $T$ -equivariant *Chen-Ruan cohomology* of  $X$ ,

$$H_{\text{CR},T}^*(X, \mathbb{Q}) := H_T^*(\bar{I}_\mu X, \mathbb{Q}) ,$$

we define the Poincaré pairing in the *non-rigidified* cyclotomic inertia stack  $I_\mu X$  of  $X$ :

$$\langle \alpha_1, \alpha_2 \rangle_{\text{orb}} := \int_{\sum_{r \in \mathbb{N}_{\geq 1}} r^{-1} [\bar{I}_{\mu_r} X]} \alpha_1 \cdot \iota^* \alpha_2 ,$$

where  $\iota$  is the involution of  $\bar{I}_\mu X$  obtained from the inversion automorphisms. Thus, if  $\{\phi_\alpha\}$  is a basis of  $H_{\text{CR},T}^*(X, \mathbb{Q})$  with  $\{\phi^\alpha\}$  the dual basis with respect to the Poincaré pairing defined above, one has

$$\sum_{r=1}^{\infty} r [\Delta_{\bar{I}_{\mu_r} X}] = \sum_{\alpha} \phi_\alpha \otimes \phi^\alpha \text{ in } H^*(\bar{I}_\mu X \times \bar{I}_\mu X, \mathbb{Q}),$$

where the diagonal class  $[\Delta_{\bar{I}_{\mu_r} X}]$  is obtained via push-forward of the fundamental class by  $(\text{id}, \iota) : \bar{I}_{\mu_r} X \rightarrow \bar{I}_{\mu_r} X \times \bar{I}_{\mu_r} X$ .

Let  $\bar{\psi}_i$  be the first Chern class of the universal cotangent line whose fiber at  $((C, q_1, \dots, q_m), [x])$  is the cotangent space of the coarse moduli  $\underline{C}$  of  $C$  at  $i$ -th marking  $\underline{q}_i$ . For non-negative integers  $a_i$  and classes  $\alpha_i \in H_T^*(\bar{I}_\mu X, \mathbb{Q})$ ,  $\delta_j \in H^*(\mathfrak{X}, \mathbb{Q})$ , we write

$$\langle \alpha_1 \bar{\psi}^{a_1}, \dots, \alpha_m \bar{\psi}^{a_m}; \delta_1, \dots, \delta_l \rangle_{0,m,\beta}^{\theta', \epsilon} := \int_{[Q_{0,m}^{\theta', \epsilon}(X, \beta)]^{\text{vir}}} \prod_i ev_i^*(\alpha_i) \bar{\psi}_i^{a_i} \prod_j \hat{e} v_j^*(\delta_j) .$$

We will also need the quasimap Chen-Ruan classes

$$(2.2) \quad (\widetilde{ev}_j)_* = \iota_*(\mathbf{r}_j(ev_j)_*),$$

where  $\mathbf{r}_j$  is the order function of the band of the gerbe structure at the marking  $q_j$ . Define a class in  $H_*^T(\bar{I}_\mu X) \cong H_T^*(\bar{I}_\mu X)$  by

$$\begin{aligned} \langle \alpha_1, \dots, \alpha_m, - \rangle_{0,\beta}^\epsilon &:= (\widetilde{ev}_{m+1})_* \left( \left( \prod ev_i^* \alpha_i \right) \cap [Q_{0,m}^\epsilon(X, \beta)]^{\text{vir}} \right) \\ &= \sum_{\alpha} \phi^\alpha \langle \alpha_1, \dots, \alpha_m, \phi_\alpha \rangle_{0,m+1,\beta}^\epsilon . \end{aligned}$$

### 3. GEOMETRY OF COMPLETE INTERSECTIONS IN TORIC DELIGNE-MUMFORD STACKS

**From now on**, we will fix a GIT data  $(W, G, \theta)$ , which represents a proper toric Deligne-Mumford stack (or toric stack in short)  $X := [W^{ss}(\theta)/G]$  as in remark 2.8. We will also fix a vector bundle  $E$  over  $\mathfrak{X} := [W/G]$  which is a direct sum of line bundles  $\oplus_{b=1}^c L_{\tau_b}$  associated to characters  $(\tau_b)_{b=1}^c$  of  $G$ . Let  $s_b \in \Gamma(W, W \times \mathbb{C}_{\tau_b})^G$  be sections such that they cut off an irreducible complete intersection in  $W$  which is smooth in  $W^{ss}(\theta)$ . Denote by  $AY$  to be the zero loci of the section  $s := \oplus_b s_b$  in  $\Gamma(W, \oplus_{b=1}^c W \times \mathbb{C}_{\tau_b})$  and by  $AY^{ss}(\theta)$  (or  $AY^{ss}$ ) be semistable loci, then  $(AY, G, \theta)$  will also be a GIT target. Note  $AY^{ss}$  is equal to the intersection of  $W^{ss}$  and  $AY$ . Denote  $Y := [AY^{ss}(\theta)/G]$  to be the corresponding toric stack complete intersections inside  $X$  and denote  $\mathfrak{Y} := [AY/G]$  to be the corresponding quotient stack of  $Y$ .

It's well known the rigidified inertia stacks of  $Y$  and  $X$  are

$$\bar{I}_\mu Y = \sqcup_{g \in G} [AY^{ss}(\theta)^g / (G / \langle g \rangle)], \quad \bar{I}_\mu X = \sqcup_{g \in G} [W^{ss}(\theta)^g / (G / \langle g \rangle)] .$$

To describe the relationship of rigidified inertia stacks of  $X$  and  $Y$ , we will need the following lemma:

**Lemma 3.1.** *For any torsion element  $g \in G$ ,  $AY^{ss}(\theta)^g \subset W^{ss}(\theta)^g$  is a complete intersection with respect to the sections  $\{s_b|b : \tau_b(g) = 1\}$ .*

*Proof.* For any point  $p \in W^{ss}(\theta)^g$  such that  $s$  vanishes on  $p$ , we have the following short exact sequence of tangent spaces

$$0 \rightarrow T_p AY^{ss}(\theta) \rightarrow T_p W^{ss}(\theta) \rightarrow \bigoplus_{b=1}^c \mathbb{C}_{\tau_b} \rightarrow 0 ,$$

which is also exact as representations of the finite group generated by  $g$ . Taking the  $g$ -invariant subspace of the above exact sequence, we get

$$0 \rightarrow T_p AY^{ss}(\theta)^g \rightarrow T_p W^{ss}(\theta)^g \rightarrow \bigoplus_{b: \tau_b(g)=1} \mathbb{C}_{\tau_b} \rightarrow 0 ,$$

which imply the lemma.  $\square$

For any degree  $\beta \in \text{Eff}(W, G, \beta)$ , we will also need an element  $g_\beta \in G$ , and two special sub-varieties  $Y_\beta^{ss} \subset AY^{ss}$ ,  $Z_\beta^{ss} \subset W^{ss}$  in the statement of the mirror theorem:

$$g_\beta := (e^{2\pi\sqrt{-1}\beta(L_{\pi_1})}, \dots, e^{2\pi\sqrt{-1}\beta(L_{\pi_k})}) \in G = (\mathbb{C}^*)^k ,$$

$$Y_\beta^{ss} := (AY^{ss})^{g_\beta} \cap \{(x_i) \in W | x_i = 0 \ \forall i : \beta(L_{\rho_i}) \in \mathbb{Z}_{<0}\} ,$$

and

$$Z_\beta^{ss} := (W^{ss})^{g_\beta} \cap \{(x_i) \in W | x_i = 0 \ \forall i : \beta(L_{\rho_i}) \in \mathbb{Z}_{<0}\} .$$

In the end of this section, we will prove a lemma 3.2 relating the geometry of  $Y_\beta^{ss}$  and  $Z_\beta^{ss}$ .

The geometrical significance of introducing  $Y_\beta^{ss}$  and  $Z_\beta^{ss}$  is that the quotient stacks  $[Y_\beta^{ss}/G]$  and  $[Z_\beta^{ss}/G]$  describe an important class in the stacky loop spaces for  $X$  and  $Y$  which we now describe.

First of all, let's recall the definition of stacky loop space into the toric stack  $X$  [CCFK15]. Set  $U = \mathbb{C}^2 \setminus \{0\}$ , for any positive integer  $a$ , denote  $\mathbb{P}_{a,1}$  to be the quotient stack  $[U/\mathbb{C}^*]$  defined by the  $\mathbb{C}^*$ -action on  $U$  with weights  $[a, 1]$  so that  $0 := [0 : 1]$  is a non-stacky point and  $\infty := [1 : 0] \cong \mathbb{B}\mu_a$  is a stacky point. The stacky loop space into  $X$

$$Q_{\mathbb{P}_{a,1}}(X, \beta) \subset \text{Hom}_\beta^{\text{rep}}(\mathbb{P}_{a,1}, \mathfrak{X})$$

is defined to be the moduli stack of representable morphisms from  $\mathbb{P}_{a,1}$  to  $\mathfrak{X}$  of degree  $\beta$  such that the generic point of  $\mathbb{P}_{a,1}$  is mapped into  $X$ . By [CCFK15, Lemma 4.6], for such representable morphism to exist,  $a$  must be the order of the finite cyclic group generated by  $g_\beta$ . We note  $a$  is also the minimal positive integer making  $a\beta(L_\tau)$  an integer for all character  $\tau$  of  $G$ . We can define the stacky loop space into  $Y$  in a similar manner, denote

$$Q_{\mathbb{P}_{a,1}}(Y, \beta) \subset \text{Hom}_\beta^{\text{rep}}(\mathbb{P}_{a,1}, \mathfrak{Y})$$

by the moduli stack of representable morphisms from  $\mathbb{P}_{a,1}$  to  $\mathfrak{X}$  of degree  $\beta$  such that the generic point of  $\mathbb{P}_{a,1}$  is mapped into  $Y$ .

Let  $a$  be the integer associated to  $g_\beta$ . Let  $\mathbb{C}[z_1, z_2]$  be the polynomial ring on variables  $z_1$  and  $z_2$  with weights  $a$  and 1 respectively. Consider the finite dimensional vector space

$$W_\beta := \bigoplus_{\rho \in [n]} \mathbb{C}[z_1, z_2]_{a\beta(L_\rho)}$$

with the  $G$ -action given by the direct sum of the diagonal  $G$ -action on  $\mathbb{C}[z_1, z_2]_{a\beta(L_\rho)}$  (homogeneous degree  $a\beta(L_\rho)$  part of  $\mathbb{C}[z_1, z_2]$ ) by the weight  $\rho$ , so that  $\mathbb{C}[z_1, z_2]_{a\beta(L_\rho)} \cong \bigoplus \mathbb{C}_\rho$ . Using an element of  $W_\beta$ , we can naturally associate a morphism from  $\mathbb{P}_{a,1}$  to  $\mathfrak{X}$  of degree  $\beta$ . Then we have the equivalence of the following two stacks:

$$\mathrm{Hom}_\beta(\mathbb{P}_{a,1}, \mathfrak{X}) \cong \mathrm{Hom}_\beta^{\mathrm{rep}}(\mathbb{P}_{a,1}, \mathfrak{X}) \cong [W_\beta/G],$$

under which correspondence, we have

$$Q_{\mathbb{P}_{a,1}}(X, \beta) \cong [W_\beta^{ss}(\theta)/G].$$

Consider the  $\mathbb{C}^*$ -action on  $P_{a,1}$  defined by

$$t(\zeta_1, \zeta_2) = (t\zeta_1, \zeta_2),$$

for all  $(\zeta_1, \zeta_2) \in U$  and  $t \in \mathbb{C}^*$ . This induces a  $\mathbb{C}^*$ -action on  $Q_{\mathbb{P}_{a,1}}(X, \beta)$  as well as on  $Q_{\mathbb{P}_{a,1}}(Y, \beta)$ . Denote  $F_\beta(X)$  (resp.  $F_\beta(Y)$ ) to be the subspace of  $Q_{\mathbb{P}_{a,1}}(X, \beta)$  (resp.  $Q_{\mathbb{P}_{a,1}}(Y, \beta)$ ) in which the representable morphism  $f : \mathbb{P}_{a,1} \rightarrow \mathfrak{X}$  (resp.  $f : \mathbb{P}_{a,1} \rightarrow \mathfrak{Y}$ ) with all the degree concentrated on the point  $[0 : 1]$ . More explicitly,  $F_\beta(X)$  (resp.  $F_\beta(Y)$ ) is comprised of the morphisms in the form

$$f : \mathbb{P}_{a,1} \rightarrow \mathfrak{X} \quad (\text{resp. } \mathfrak{Y}), \quad (\zeta_1, \zeta_2) \mapsto (a_\rho \zeta_1^{\beta(L_\rho)})_{\rho \in [n]},$$

where  $(a_\rho \zeta_1^{\beta(L_\rho)} : \rho \in [n]) \in W_\beta^{ss}(\theta)$ . Note for such map to be well defined,  $a_\rho$  is equal to 0 when  $\beta(L_\rho) \notin \mathbb{Z}_{\geq 0}$ .

We can that  $F_\beta(X)$  is a component of the  $\mathbb{C}^*$ -fixed loci of  $Q_{\mathbb{P}_{a,1}}(X, \beta)$ , which we can describe more explicitly as follows. Define

$$Z_\beta := \bigoplus_{\rho \in [n], \beta(L_\rho) \in \mathbb{Z}_{\geq 0}} \mathbb{C} \cdot z_1^{\beta(L_\rho)} \subset W_\beta.$$

We have  $Z_\beta^{ss} \cong Z_\beta \cap W_\beta^{ss}(\theta)$ , and

$$F_\beta(X) \cong [Z_\beta^{ss}/G], \text{ and } F_\beta(Y) \cong [Y_\beta^{ss}/G].$$

Let  $g_\beta$  be as above, then we have another characterization of  $Z_\beta$  and  $Z_\beta^{ss}$ :

$$Z_\beta \cong W^{g_\beta} \cap \bigcap_{\substack{\rho \in [n] \\ \beta(L_\rho) \in \mathbb{Z}_{<0}}} D_\rho \quad \text{and} \quad Z_\beta^{ss} \cong W^{ss}(\theta)^{g_\beta} \cap \bigcap_{\substack{\rho \in [n] \\ \beta(L_\rho) \in \mathbb{Z}_{<0}}} D_\rho.$$

Here  $D_\rho$  is the divisor of  $W$  given by  $x_\rho = 0$ .

It's clear that  $Y_\beta^{ss}$  is the vanishing loci of the sections  $\{s_b | b : \beta(L_{\tau_b}) \in \mathbb{Z}\}$  on  $Z_\beta^{ss}$ , but this may not be a complete intersection. Indeed, one can show the following.

**Lemma 3.2.** *For any  $b$  such that  $\beta(L_{\tau_b}) \in \mathbb{Z}_{<0}$ , the section  $s_b$  vanishes on  $Z_\beta^{ss}$ . Thus  $Y_\beta^{ss}$  is merely the vanishing loci of sections  $\{s_b | b : \beta(L_{\tau_b}) \in \mathbb{Z}_{\geq 0}\}$  in  $Z_\beta^{ss}$ .*

*Proof.* For  $b$  with  $\beta(L_{\tau_b}) \in \mathbb{Z}_{\leq 0}$ , for any point  $\vec{x} = (a_\rho)_{\rho \in [n]} \in Z_\beta^{ss}$ , the corresponding morphism in  $F_\beta(X)$  is in the form

$$[\vec{x}] : \mathbb{P}_{a,1} \rightarrow \mathfrak{X} : [\zeta_1, \zeta_2] \rightarrow (a_\rho \zeta_1^{\beta(L_\rho)})_{\rho \in [n]}.$$

Then the pull back of section  $s_b$  to  $\mathbb{P}_{a,1}$  becomes  $s_b(\vec{x}) z_1^{\beta(L_{\tau_b})}$ . However as the pull-back line bundle  $[\vec{x}]^* L_{\tau_b}$  is of degree  $\beta(L_{\tau_b}) < 0$  on  $\mathbb{P}_{a,1}$ , hence there is no nonzero section in the line bundle  $[x]^* L$ , which implies that  $s_b(\vec{x}) = 0$ . Now the lemma follows.  $\square$

Denote  $E_{\geq 0} := \bigoplus_{b:\beta(L_{\tau_b}) \in \mathbb{Z}_{\geq 0}} L_{\tau_b}$  and  $s_{\geq 0} = \bigoplus_{b:\beta(L_{\tau_b}) \in \mathbb{Z}_{\geq 0}} s_b$ . Then we can define the Gysin morphism

$$s_{E_{\geq 0}, \text{loc}}^! : A_*([Z_{\beta}^{ss}/(G/\langle g_{\beta}^{-1} \rangle)]) \rightarrow A_*([Y_{\beta}^{ss}/(G/\langle g_{\beta}^{-1} \rangle)])$$

as the localized top Chern class [Ful84, §14.1] with respect to the vector bundle  $E_{\geq 0}$  over  $[Z_{\beta}^{ss}/(G/\langle g_{\beta}^{-1} \rangle)]$  and the section  $s_{\geq 0}$ . Let  $i : \bar{I}_{\mu} Y \rightarrow \bar{I}_{\mu} X$  be the natural inclusion. Now we discuss two implications of the above lemma:

**Corollary 3.3.** (1) *If the set  $\{b|\beta(L_{\tau_b}) \in \mathbb{Z}\}$  is exactly the set  $\{b|\beta(L_{\tau_b}) \in \mathbb{Z}_{\geq 0}\}$ , then we have*

$$i_*(s_{E_{\geq 0}, \text{loc}}^!([Z_{\beta}^{ss}/(G/\langle g_{\beta}^{-1} \rangle)])) = \left( \prod_{\rho:\beta(L_{\rho}) \in \mathbb{Z}_{\leq 0}} D_{\rho} \right) \cdot \mathbb{1}_{g_{\beta}^{-1}}$$

in  $A^*(\bar{I}_{g_{\beta}^{-1}} Y)$ , where  $\mathbb{1}_{g_{\beta}^{-1}}$  is the identity class of  $\bar{I}_{g_{\beta}^{-1}} Y$ ,  $D_{\rho}$  is the class of the hyperplane given by  $x_{\rho} = 0$ .

(2) *If the set  $\{b|\beta(L_{\tau_b}) \in \mathbb{Z}\}$  is empty, then we have  $Y_{\beta}^{ss} = Z_{\beta}^{ss}$ , and  $\bar{I}_{g_{\beta}^{-1}} Y = \bar{I}_{g_{\beta}^{-1}} X$ , and  $s_{E_{\geq 0}, \text{loc}}^!$  is the identity morphism. Thus*

$$i_*(s_{E_{\geq 0}, \text{loc}}^!([Z_{\beta}^{ss}/(G/\langle g_{\beta}^{-1} \rangle)])) = \left( \prod_{\rho:\beta(L_{\rho}) \in \mathbb{Z}_{\leq 0}} D_{\rho} \right) \cdot \mathbb{1}_{g_{\beta}^{-1}}$$

in  $A^*(\bar{I}_{g_{\beta}^{-1}} Y)$ .

## 4. MASTER SPACE I

**4.1. Construction of master space I.** In this section, we will construct a master space which is a root stack modification of the twisted graph space considered in [CJR17a]. Let  $(AY, G, \theta)$  be the GIT data which gives rise to a hypersurface in the toric stack  $X = [W^{ss}(\theta)/G]$  as in previous sections. *Since a positive scaling of the stability character  $\theta$  will not change the GIT quotient. Without loss of generality, let's assume that the line bundle  $L_{\theta}$  on  $Y = [AY^{ss}(\theta)/G]$  is the pullback of a positive line bundle on the coarse moduli space  $\underline{Y}$  of  $Y$ .* First we will consider the following quotient stack

$$\mathbb{P}\mathfrak{Y}_{r,p}^{\frac{1}{r},p} = [(AY \times \mathbb{C}^p \times \mathbb{C}^2)/(G \times (\mathbb{C}^*)^p \times \mathbb{C}^*)]$$

defined by the following (right) action

$$(\vec{x}, \vec{y}, z_1, z_2) \cdot (g, h, t) = (\vec{x} \cdot g, (h_j y_j)_{j=1}^p, \theta(g)^{-1} \left( \prod_{j=1}^p h_j^{-1} \right) t^r z_1, t z_2),$$

where  $(g, h = (h_j)_{j=1}^p, t) \in G \times (\mathbb{C}^*)^p \times \mathbb{C}^*$  ( $\vec{x}, \vec{y} = (y_j)_{j=1}^p, z_1, z_2) \in AY \times \mathbb{C}^p \times \mathbb{C}^2$ . For simplicity, we will write  $AY_p := AY \times \mathbb{C}^p$ , and  $G_p := G \times (\mathbb{C}^*)^p$  and  $\theta_p$  as the character of  $G_p$  defined by

$$\theta_p(g, h) = \theta(g) \prod_{j=1}^p h_j \text{ for all } (g, h) \in G_p.$$

Fix a positive rational number  $\epsilon \in \mathbb{Q}_{>0} \cap (0, 1]$  and a tuple of positive rational numbers  $\epsilon = (\epsilon, \dots, \epsilon) \in (\mathbb{Q}_{>0})^p$ , we consider the stability given by the rational character of  $G_p \times \mathbb{C}^*$  defined by

$$\tilde{\theta}(g, h, t) = \theta(g)^\epsilon \left( \prod_{i=1}^p h_i^\epsilon \right) t^{3r}$$

for  $(g, h, t) \in G_p \times \mathbb{C}^*$ . Then the GIT stack quotient  $[(AY_p \times \mathbb{C}^2)^{ss}(\tilde{\theta})/(G_p \times \mathbb{C}^*)]$  is the root stack of the  $\mathbb{P}^1$ -bundle  $\mathbb{P}_Y(\mathcal{O}(-D_\theta) \oplus \mathcal{O})$  over  $Y$  by taking  $r$ -th root of the infinity divisor  $D_\infty$  given by  $z_2 = 0$ . We will denote the GIT stack quotient  $[(AY_p \times \mathbb{C}^2)^{ss}(\tilde{\theta})/(G_p \times \mathbb{C}^*)]$  to be  $\mathbb{P}Y^{\frac{1}{r}}$ , which is equipped with the infinity section  $\mathcal{D}_\infty$  given by  $z_2 = 0$  and the zero section  $\mathcal{D}_0$  given by  $z_1 = 0$ . Note this GIT quotient is independent of the integer  $p$  as the semistable(=stable) loci  $(AY_p \times \mathbb{C}^2)^{ss}(\tilde{\theta}) = AY^{ss}(\theta) \times (\mathbb{C}^*)^p \times (\mathbb{C}^2 \setminus \{0\})$ . We will take  $p = 1$  as our standard GIT reference for  $\mathbb{P}Y^{\frac{1}{r}}$ , which will be canonically identified with other GIT quotients from  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}, p}$  by choosing the embedding  $AY \subset AY_p$  as  $AY \cong AY_p \cap \{y_i = 1 | i = 1, \dots, p\}$ .

The inertia stack  $I_\mu \mathbb{P}Y^{\frac{1}{r}}$  of  $\mathbb{P}Y^{\frac{1}{r}}$  admits a decomposition

$$I_\mu \mathbb{P}Y \bigsqcup \bigsqcup_{j=1}^{r-1} \sqrt[r]{L_\theta/Y}.$$

Let  $(\vec{x}, (g, t))$  be a point of  $I_\mu \mathbb{P}Y^{\frac{1}{r}}$ , if  $(\vec{x}, (g, t))$  appears in the first factor of the decomposition above, then the automorphism  $(g, t)$  lies in  $G \times \{1\}$ ; if  $(\vec{x}, (g, t))$  occurs in the second factor of the decomposition above, the automorphism  $(g, t)$  lies in  $G \times \{\mu_r^j : 1 \leq j \leq r-1\} \subset G \times \mu_r$ , and the point  $\vec{x}$  is in the infinity section  $\mathcal{D}_\infty$  defined by  $z_2 = 0$ . Here  $\mu_r = \exp(\frac{2\pi\sqrt{-1}}{r}) \in \mathbb{C}^*$  and  $\mu_r$  is the cyclic group generated by  $\mu_r$ .

For  $(g, t) \in G \times \mu_r$ , we will use the notation  $\bar{I}_{(g,t)} \mathbb{P}Y^{\frac{1}{r}}$  to mean the rigidified inertia stack component of  $I_\mu \mathbb{P}Y^{\frac{1}{r}}$  which has automorphism  $(g, t)$ .

Consider the moduli stack of  $\tilde{\theta}$ -stable quasimaps to  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}, p}$ :

$$Q_{0,m}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}, p}, (d, 1^p, \frac{\delta}{r})).$$

More concretely,

$$Q_{0,m}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}, p}, (d, 1^p, \frac{\delta}{r})) = \{(C; q_1, \dots, q_m; L_1, \dots, L_{k+p}, N; \vec{x}, \vec{y}, z_1, z_2)\},$$

where  $(C; q_1, \dots, q_m)$  is a  $m$ -pointed prestable balanced orbifold curve of genus 0 with possible nontrivial isotropy only at special points, i.e. marked gerbes or nodes, the line bundles  $(L_j : 1 \leq j \leq k+p)$  and  $N$  are orbifold line bundles on  $C$  with

$$(4.1) \quad \deg([\vec{x}]) = d \in \text{Hom}(\text{Pic}(\mathfrak{Y}), \mathbb{Q}), \quad \deg(N) = \frac{\delta}{r},$$

$$(4.2) \quad \deg(L_{k+j}) = 1, \quad 1 \leq j \leq p,$$

and

$$(\vec{x}, \vec{y}, \vec{z}) := (x_1, \dots, x_n, y_1, \dots, y_p, z_1, z_2) \in \Gamma \left( \bigoplus_{i=1}^n L_{\rho_i} \oplus \bigoplus_{j=1}^p L_{k+j} \oplus (L_{-\theta_p} \otimes N^{\otimes r}) \oplus N \right).$$



Here, for  $1 \leq i \leq n$ , the line bundle  $L_{\rho_i}$  is equal to

$$\otimes_{j=1}^k L_j^{m_{ij}},$$

where  $(m_{ij})$  ( $1 \leq i \leq n, 1 \leq j \leq k+p$ ) is given by the relation  $\rho_i = \sum_{j=1}^k m_{ij} \pi_j$ . The same construction applies to the line bundle  $L_{-\theta_p}$  on  $C$ . Note here  $\delta$  is an integer when  $Q_{0,m}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (d, 1^p, \frac{\delta}{r}))$  is nonempty as  $N^{\otimes r}$  is the pull-back of some line bundle on the coarse moduli curve  $\underline{C}$ .

We require the the following conditions are satisfied for the above data:

- *Representability*: For every  $q \in C$  with isotropy group  $G_q$ , the homomorphism  $\mathbb{B}G_q \rightarrow \mathbb{B}(G_p \times \mathbb{C}^*)$  induced by the restriction of line bundles  $(L_j : 1 \leq j \leq k+p)$  and  $N$  to  $q$  is representable, Note the image of the homomorphism lies in the subgroup  $G \times \mathbb{C}^* \subset G_p \times \mathbb{C}^*$ .
- *Nondegeneracy*: The sections  $z_1$  and  $z_2$  never simultaneously vanish. Furthermore, for each point  $q$  of  $C$  at which  $z_2(q) \neq 0$ , the stability condition 2.3

$$l_{\tilde{\theta}}(q) \leq 1$$

for  $\tilde{\theta}$ -stable map to  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$  becomes the stability condition

$$(4.3) \quad l_{\epsilon\theta} \oplus \oplus_{j=1}^p \text{cid}_{\mathbb{C}^*}(q) \leq 1,$$

for the prestable quasimap  $[\vec{x}, \vec{y}] : C \rightarrow \mathfrak{Y} \times [\mathbb{C}/\mathbb{C}^*]^p$ . For each point  $q$  of  $C$  at which  $z_2(q) = 0$ , we have

$$(4.4) \quad \text{ord}_q(\vec{x}) = \text{ord}_q(\vec{y}) = 0.$$

We note that this can be phrased as the length condition (2.1) bounding the order of contact of  $(\vec{x}, \vec{y}, \vec{z})$  with the unstable loci of  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$  as in [CFK16, §2.1].

- *Stability*: The  $\mathbb{Q}$ -line bundle

$$(\phi_*(L_{\theta}))^{\otimes \epsilon} \otimes \bigotimes_{j=1}^p \phi_*(L_{k+j})^{\otimes \epsilon} \otimes \phi_*(N^{\otimes 3r}) \otimes \omega_{\underline{C}}^{\log}$$

on the coarse curve  $\underline{C}$  is ample. Here  $\phi : C \rightarrow \underline{C}$  is the coarse moduli map. Note here, the line bundles  $L_{\theta}$ ,  $(L_{k+j})_{j=1}^p$  and  $N^{\otimes 3r}$  are the pull back of line bundles on the coarse moduli of  $\underline{C}$ .

- *Vanishing*: The image of  $[\vec{x}] : C \rightarrow \mathfrak{X}$  lies in  $\mathfrak{Y}$ .

Let  $\vec{m} = (v_1, \dots, v_m) \in (G \times \mu_r)^m$ , we will denote  $Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (d, 1^p, \frac{\delta}{r}))$  to be:

$$Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (d, 1^p, \frac{\delta}{r})) \cap \text{ev}_1^{-1}(\bar{I}_{v_1} \mathbb{P}Y^{\frac{1}{r}}) \cap \dots \cap \text{ev}_m^{-1}(\bar{I}_{v_m} \mathbb{P}Y^{\frac{1}{r}}),$$

where

$$\text{ev}_i : Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r})) \rightarrow \bar{I}_{\mu} \mathbb{P}Y^{\frac{1}{r}}$$

are natural evaluation maps as before, by evaluating the sections  $(\vec{x}, \vec{z})$  at  $i$ th marking  $q_i$ . Using the vanishing loci of the section  $y_j$  of the degree one line bundle  $L_{k+j}$  for  $1 \leq j \leq p$ , which corresponds to a smooth non-orbifold point on  $C$ , one has another tuple of evaluation maps

$$(4.5) \quad \hat{\text{ev}}_j : Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r})) \rightarrow \mathfrak{Y},$$

for  $1 \leq j \leq p$ .

Because  $Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$  is a moduli space of stable quasimaps to a proper lci GIT quotient, it is a proper Deligne-Mumford stack equipped with a natural perfect obstruction theory relative to the stack  $\mathcal{D}_{0,m,\beta,1^p,\frac{\delta}{r}}$  of twisted curves equipped with a pair of line bundles by [CFKM14]. This obstruction theory has the form

$$(4.6) \quad R^\bullet \pi_*(u^* \mathbb{R}T_\rho).$$

Here, we denote the universal family over  $Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$  by

$$\begin{array}{ccc} \mathcal{V} & \xleftarrow[u]{\rho} & \mathcal{C} \\ & & \downarrow \pi \\ & & Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r})) \end{array}$$

where  $\mathcal{L}_{\rho_i}$  ( $1 \leq i \leq n$ ) and  $\mathcal{N}$  are the universal line bundles and

$$\mathcal{V} \subset \oplus_{i=1}^n \mathcal{L}_{\rho_i} \oplus (\oplus_{j=1}^p \mathcal{L}_{k+j}) \oplus (\mathcal{L}_{-\theta_p} \otimes \mathcal{N}^{\otimes r}) \oplus \mathcal{N}$$

is the subsheaf of sections taking values in the affine cone of  $Y$ . Somewhat more explicitly, (4.6) equals

$$(4.7) \quad \mathbb{E} \oplus R^\bullet \pi_*(\oplus_{j=1}^p \mathcal{L}_{k+j} \oplus (\mathcal{L}_{-\theta_p} \otimes \mathcal{N}^{\otimes r}) \oplus \mathcal{N}),$$

in which the sub-obstruction-theory  $\mathbb{E}$  comes from the deformations and obstructions of the sections  $\vec{x}$ , and  $\mathbb{E}$  fits into the following distinguished triangle:

$$\mathbb{E} \rightarrow R^\bullet \pi_*(\oplus_{i=1}^n \mathcal{L}_{\rho_i}) \rightarrow R^\bullet \pi_*(\oplus_{m=1}^b \mathcal{L}_{\tau_m}) \rightarrow .$$

**4.2.  $\mathbb{C}^*$ -action and fixed loci.** Consider the (left)  $\mathbb{C}^*$ -action on  $AY_p \times \mathbb{C}^2$  defined by:

$$\lambda(\vec{x}, \vec{y}, z_1, z_2) = (\vec{x}, \vec{y}, \lambda z_1, z_2) ,$$

this action descends to be an action on  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ . We will denote  $\lambda$  to be the equivariant class corresponding to the  $\mathbb{C}^*$ -action of weight 1. Let's first state a criteria for a morphism to  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$  to be  $\mathbb{C}^*$ -equivariant (see also [CLLL16, §2.2]), which will be important in the analysis of localization computations.

**Remark 4.1.** (*Equivariant morphism to  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$* ) Fix a stack  $S$  over  $\text{Spec}(\mathbb{C})$  with a left  $\mathbb{C}^*$ -action, then a  $\mathbb{C}^*$ -equivariant morphism from  $S$  to  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$  is equivalent to the following data: there exists  $k+p+1$   $\mathbb{C}^*$ -equivariant line bundles on  $S$

$$L_1, \dots, L_{k+p}, N$$

together with  $\mathbb{C}^*$ -invariant sections

$$(\vec{x}, \vec{y}, \vec{z}) := (x_1, \dots, x_n, y_{n+1}, \dots, y_{n+p}, z_1, z_2)$$

$$\in \Gamma \left( \oplus_{i=1}^n L_{\rho_i} \oplus (\oplus_{j=1}^p L_{k+j}) \oplus (L_{-\theta_p} \otimes N^{\otimes r} \otimes \mathbb{C}_\lambda) \oplus N \right)^{\mathbb{C}^*} .$$

Here  $L_{\rho_i}$  ( $1 \leq i \leq n$ ) and  $L_{-\theta_p}$  are constructed from  $(L_j)_{1 \leq j \leq k+p}$  as explained before,  $\mathbb{C}_\lambda$  is the trivial line bundle over  $S$  with  $\mathbb{C}^*$ -linearization of weight 1. These sections should also satisfy the vanishing condition imposed by the cone of  $Y$  as above.

Fix a nonzero degree  $\beta \in \text{Eff}(W, G, \theta)$  and a tuple of nonnegative integers  $(\delta_1, \dots, \delta_m) \in \mathbb{N}^m$ . Consider the tuple of multiplicities  $\vec{m} = (v_1, \dots, v_m) \in (G \times \mu_r)^m$ , where  $v_i = (g_i, \mu_r^{\delta_i})$ , we will denote  $Q_{0, \vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}, p}, (\beta, \frac{\delta}{r}))$  to be

$$\bigsqcup_{\substack{d \in \text{Eff}(AY, G, \theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta}} Q_{0, \vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}, p}, (d, 1^p, \frac{\delta}{r})),$$

where  $i_{\mathfrak{Y}} : \mathfrak{Y} \rightarrow \mathfrak{X}$  is the inclusion morphism. Thus  $Q_{0, \vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}, p}, (\beta, 1^p, \frac{\delta}{r}))$  inherits a  $\mathbb{C}^*$ -action as above.

Follow the presentation of [CJR17b, CJR17a], we can index the components of  $\mathbb{C}^*$ -fixed loci of  $Q_{0, \vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}, p}, (\beta, 1^p, \frac{\delta}{r}))$  by decorated graphs. A decorated graph  $\Gamma$  consists of vertices, edges, and  $m$  legs, and we decorate it as follows:

- Each vertex  $v$  is associated with an index  $j(v) \in \{0, \infty\}$ , a degree  $\beta(v) \in \text{Eff}(W, G, \theta)$  and a subset  $J_v \in \{1, \dots, p\}$ .
- Each edge  $e = \{h, h'\}$  is equipped with a degree  $\beta(e) \in \text{Eff}(W, G, \theta)$ , a subset  $J_e \subset \{1, \dots, p\}$  and  $\delta(e) \in \mathbb{N}$ .
- Each half-edge  $h$  and each leg  $l$  has an element (called multiplicity)  $m(h)$  or  $m(l)$  in  $G \times \mu_r$ .
- The legs are labeled with the numbers  $\{1, \dots, m\}$ .

By the “valence” of a vertex  $v$ , denoted  $\text{val}(v)$ , we mean the total number of incident half-edges, including legs.

The fixed locus in  $Q_{0, \vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}, p}, (\beta, 1^p, \frac{\delta}{r}))$  indexed by the decorated graph  $\Gamma$  parameterizes quasimaps of the following type:

- Each edge  $e$  corresponds to a genus-zero component  $C_e$  on which  $\deg(N|_{C_e}) = \frac{\delta(e)}{r}$  ( $\delta(e) > 0$ ),  $\deg(L_j|_{C_e}) = \beta(e)(L_{\pi_j})$  ( $1 \leq j \leq k$ ), and  $\deg(L_{k+j}|_{C_e}) = 1$  if and only if  $j \in J_e$ . We denote  $1^{J_e}$  to be the degree coming from the line bundles  $(L_{k+j} : 1 \leq j \leq p)$ . There are two distinguished points  $q_0$  and  $q_\infty$  on  $C_e$  such that  $q_\infty$  is the only point on  $C_e$  at which  $z_2$  vanishes, and  $q_0$  is the only point on  $C_e$  determined by the following conditions:
  - if  $C_e$  has base points,  $q_0$  is the only base point on  $C_e$ ;
  - if  $C_e$  does not have base points on it,  $q_0$  is the only point on  $C_e$  at which  $z_1$  vanishes.

We call them the ramification points<sup>5</sup>, and all of degree  $(\beta(e), 1^{J_e})$  is concentrated at the ramification point  $q_0$ . That is,

when  $x_i|_{C_e} \neq 0$ , we have  $\text{ord}_{q_0}(x_i) = \beta(e)(L_{\rho_i})$ , and  $\text{ord}_{q_0}(y_i) = 1$ , only when  $j \in J_e$ .

- Each vertex  $v$  for which  $j(v) = 0$  (with unstable exceptional cases noted below) corresponds to a maximal sub-curve  $C_v$  of  $C$  over which  $z_1 \equiv 0$ , and each vertex  $v$  for which  $j(v) = \infty$  (again with unstable exceptions) corresponds to

<sup>5</sup>The definition of the ramification point here is different from the definition in [CJR17a, Page 13], where they claim that  $z_1$  or  $z_2$  each vanish at exactly one point on  $C_e$ . We find that there is a missing case when  $q_0$  is a base point and  $\deg(L_1|_{C_e}) = \deg L_2|_{C_e} = \delta(e)$  in their setting, then  $z_1|_{C_e} \equiv 1$ , which does not vanish anywhere on  $C_e$ . But the author finds this missing case does not affect their main result in [CJR17a].

a maximal sub-curve over which  $z_2 \equiv 0$ . The label  $\beta(v)$  denotes the degree coming from the restriction map  $[\vec{x}]|_{C_v}$ , note here we count the degree  $\beta(v)$  in  $\text{Eff}(W, G, \theta)$ , but not in  $\text{Eff}(AY, G, \theta)$ . The subset  $J_v$  is equal to the set where  $\deg(L_{k+j}|_{C_v}) = 1$  for  $1 \leq j \leq p$ . We denote  $1^{J_e}$  to be the degree coming from the line bundles  $(L_{k+j}|_{C_v})_{j=1}^p$ .

- A vertex  $v$  is *unstable* if stable quasimap of the type described above do not exist (where, as always, we interpret legs as marked points and half-edges as half-nodes). In this case,  $v$  corresponds to a single point of the component  $C_e$  for each adjacent edge  $e$ , which may be a node at which  $C_e$  meets  $C_{e'}$ , a marked point of  $C_e$ , an unmarked point, or a basepoint on  $C_e$  of order  $\beta(v)$ , note the base point only appears as a vertex over 0 due to the nondegeneracy condition.
- The index  $m(l)$  on a leg  $l$  indicates the rigidified inertia stack component  $\bar{I}_{m(l)} \mathbb{P}Y^{\frac{1}{r}}$  of  $\mathbb{P}Y^{\frac{1}{r}}$  on which the marked point corresponding to the leg  $l$  is evaluated, this is determined by the multiplicity of  $L_1, \dots, L_k, N$  at the corresponding marked point.
- A half-edge  $h$  incident to a vertex  $v$  corresponds to a node at which components  $C_e$  and  $C_v$  meet, and  $m(h)$  indicates the rigidified inertia component  $\bar{I}_{m(h)} \mathbb{P}Y^{\frac{1}{r}}$  of  $\mathbb{P}Y^{\frac{1}{r}}$  on which the node on  $C_v$  corresponding to  $h$  is evaluated. If  $v$  is unstable, then  $h$  corresponds to a single point on a component  $C_e$ , then  $m(h)$  is the *inverse* in  $G \times \mu_r$  of the multiplicity of  $L_1, \dots, L_k, N$  at this point.

In particular, we note that the decorations at each stable vertex  $v$  yield a tuple

$$\vec{m}(v) \in (G \times \mu_r)^{\text{val}(v)}$$

recording the multiplicities of  $L_1, \dots, L_k, N$  at every special point of  $C_v$ . We have the following remarks:

**Remark 4.2.** The crucial observation, now, is the following. For a stable vertex  $v$  such that  $j(v) = 0$ , we have  $z_1|_{C_v} \equiv 0$ , so the stability condition (4.3) implies that  $l_{\epsilon\theta \oplus \bigoplus_{j=1}^p \text{eid}_{\mathbb{C}^*}}(q) \leq 1$  for each  $q \in C_v$ . That is, the restriction of  $(C; q_1, \dots, q_m; L_1, \dots, L_{k+p}; \vec{x}, \vec{y})$  to  $C_v$  gives rise to a  $\epsilon\theta \oplus \bigoplus_{j=1}^p \text{eid}_{\mathbb{C}^*}$ -stable quasimap to the quotient stack  $\mathfrak{Y}_p := [AY/G] \times [\mathbb{C}/\mathbb{C}^*]^p$  (c.f. 2.9) in

$$Q_{0, \vec{m}(v)|J_v|}^{(\epsilon\theta, \epsilon)}(\mathfrak{Y}_p, (\beta(v), 1^{J_v})) := \bigsqcup_{\substack{d \in \text{Eff}(AY, G, \theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta(v)}} Q_{0, \vec{m}(v)|J_v|}^{(\epsilon\theta, \epsilon)}(\mathfrak{Y}_p, (d, 1^{J_v})) .$$

In this case, let  $j \in J_v$ , the evaluation map considered in (4.5) coincides with  $\hat{e}v_j$  for  $Q_{0, \vec{m}(v)|J_v|}^{(\epsilon\theta, \epsilon)}(\mathfrak{Y}_p, (\beta(v), 1^{J_v}))$  in Remark 2.10. On the other hand, for a stable vertex  $v$  such that  $j(v) = \infty$ , we have  $z_2|_{C_v} \equiv 0$ , so the stability condition (4.4) implies that  $\text{ord}_q(\vec{x}) = \text{ord}_q(\vec{y}) = 0$  for each  $q \in C_v$ . Thus, the restriction of  $(C; q_1, \dots, q_m; L_1, \dots, L_k; \vec{x})$  to  $C_v$  gives rise to a usual twisted stable map in

$$\mathcal{K}_{0, \vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v)) := \bigsqcup_{\substack{d \in \text{Eff}(AY, G, \theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta(v)}} \mathcal{K}_{0, \vec{m}(v)}(\sqrt[r]{L_\theta/Y}, d) .$$

Here  $\sqrt[r]{L_\theta/Y}$  is the root gerbe of  $Y$  by taking  $r$ -th root of  $L_\theta$ .

**Remark 4.3.** For each edge  $e$ , the restriction of  $(\vec{x}, \vec{y})$  to  $C_e$  defines a constant map to  $Y$  (possibly with an additional basepoint at the ramification point  $q_0$ ). So if there is no basepoint on  $C_e$ ,  $(\vec{x}, \vec{y}, \vec{z})$  defines a representable map

$$C_e \rightarrow \mathbb{B}G_y \times \mathbb{P}_{r,1}$$

where  $y \in Y$  comes from  $\vec{x}$ ,  $G_y$  is the isotropy group of  $y \in Y$ . Then we have  $m(q_0) = (g^{-1}, 1)$  and  $m(q_\infty) = (g, \mu_r^{\delta(e)})$  for some  $g \in G_y$ . Note when  $r$  is a sufficiently large prime comparing to  $\delta(e)$ , assuming that the order of  $g$  is equal to  $a$ , we have  $C_e \cong \mathbb{P}_{ar,a}^1$  and the ramification point  $q_\infty$  must be a special point. Here  $\mathbb{P}_{ar,a}^1$  is the unique Deligne-Mumford stack with coarse moduli  $\mathbb{P}^1$ , isotropy group  $\mu_a$  at  $0 \in \mathbb{P}^1$ , isotropy group  $\mu_{ar}$  at  $\infty \in \mathbb{P}^1$ , and generic trivial stabilizer.

If  $q_0$  is a basepoint of degree  $(\beta, 1^{J_e})$ , the ramification point  $q_0$  can't be an orbifold point, thus  $m(q_0) = (1, 1) \in G \times \mu_r$ . In this case, by the representable condition, we have  $C_e \cong \mathbb{P}_{ar,1}$  and  $m(q_\infty) = (g_\beta, \mu_r^{\delta(e)})$  if  $r$  is a sufficiently large prime. Here  $a$  is minimal positive integer associated to  $g_\beta$  as in §3.

**Remark 4.4.** If there is a basepoint on the edge curve  $C_e$ , then the degree  $(\beta(e), 1^{J_e}, \frac{\delta(e)}{r})$  on  $C_e$  must satisfy the relation  $\delta(e) \geq \beta(e)(L_\theta) + |J_e|$ . Otherwise we have  $z_1|_{C_e} \equiv 0$ , given the fact  $z_2$  vanishes at  $q_\infty$ , this will violate the nondegeneracy condition for  $z_1$  and  $z_2$ .

**4.3. Localization analysis.** Fix  $\beta \in \text{Eff}(W, G, \theta)$ ,  $\delta \in \mathbb{Z}_{\geq 0}$ , we will consider the space  $Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ . The reason why we assume that the second degree is  $\frac{\delta}{r}$  is that  $Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$  corresponds to  $Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}, (\beta, \delta))$ , here  $\mathbb{P}\mathfrak{Y}$  is equal to  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$  for  $r = 1$  and  $p = 1$ . In the remaining section, we will always assume that  $r$  is a *sufficiently large prime*.

When we apply the virtual localization formula of Graber–Pandharipande [GP99] to

$$[Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))]^{\text{vir}},$$

we can write the above in terms of contributions from each decorated graph  $\Gamma$ :

$$(4.8) \quad [Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))]^{\text{vir}} = \sum_{\Gamma} \frac{1}{\mathbb{A}_{\Gamma}} \iota_{\Gamma*} \left( \frac{[F_{\Gamma}]^{\text{vir}}}{e^{\mathbb{C}^*}(N_{\Gamma}^{\text{vir}})} \right).$$

Here, for each graph  $\Gamma$ ,  $[F_{\Gamma}]^{\text{vir}}$  is obtained from the  $\mathbb{C}^*$ -fixed part of the restriction to the fixed loci of the obstruction theory on  $Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ , and  $N_{\Gamma}^{\text{vir}}$  is the equivariant Euler class of the  $\mathbb{C}^*$ -moving part of this restriction. Besides,  $\mathbb{A}_{\Gamma}$  is the automorphism factor for the graph  $\Gamma$ , which represents the degree of  $F_{\Gamma}$  into the corresponding open and closed  $\mathbb{C}^*$ -fixed substack in  $Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ .

We will do an explicit computation for the contributions of each graph  $\Gamma$  in this subsection. As for the contribution of a graph  $\Gamma$  to (4.8), one can first employ the normalization exact sequence to the relative obstruction theory (4.7), which decomposes the contribution from  $\Gamma$  to (4.8) into contributions from vertex, edge, and node factors. This includes all but the automorphisms and deformations within  $\mathcal{D}_{0,m,\beta,\frac{\delta}{r}}$ . The latter are distributed in the vertex, edge, and node factors as deformations of the vertex components and their line bundles, deformations of the edge components and their

line bundles, and deformations of smoothing the nodes, respectively. We also include automorphisms of the source curve in the edge contributions as part of gerbe structure of the edge moduli  $\mathcal{M}_e$ , then an additional factor from the gerbe structure of each edge moduli will be included in the automorphism factor  $\mathbb{A}_\Gamma$  (see (4.17) for the localization contribution of graph  $\Gamma$ ).

**4.3.1. Vertex contributions.** First of all, consider the stable vertex  $v$  over  $\infty$ , this vertex moduli  $\mathcal{M}_v$  corresponds to the moduli stack

$$\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v)) := \bigsqcup_{\substack{d \in \text{Eff}(AY, G, \theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta(v)}} \mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, d),$$

which parameterizes twisted stable maps to the root gerbe  $\sqrt[r]{L_\theta/Y}$  over  $Y$ .

Let

$$\pi : \mathcal{C}_\infty \rightarrow \mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))$$

be the universal curve over  $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))$ . In this case, on  $\mathcal{C}_\infty$ , we have  $\mathcal{L}_{-\theta} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda \cong \mathcal{O}_{\mathcal{C}_\infty}$  as  $z_1|_{\mathcal{C}_\infty} \equiv 1$ , hence we have  $\mathcal{N} \cong \mathcal{L}_\theta^{\frac{1}{r}} \otimes \mathbb{C}_{-\frac{\lambda}{r}}$ , here  $\mathcal{L}_\theta^{\frac{1}{r}}$  is the line bundle over  $\mathcal{C}_\infty$  that is the pull back of the universal root bundle over  $\sqrt[r]{L_\theta/Y}$  along the universal map  $f : \mathcal{C}_\infty \rightarrow \sqrt[r]{L_\theta/Y}$ . The movable part of the perfect obstruction theory comes from the deformation of  $z_2$ , thus the *inverse of Euler class* of the virtual normal bundle is equal to

$$e^{\mathbb{C}^*}((-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}}) \otimes \mathbb{C}_{-\frac{\lambda}{r}}).$$

When  $r$  is a sufficiently large prime, following [JPPZ18], the above Euler class has a representation

$$\sum_{d \geq 0} c_d(-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}}) \left( \frac{-\lambda}{r} \right)^{|E(v)|-1-d}.$$

Here the virtual bundle  $-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}}$  has virtual rank  $|E(v)| - 1$ , where  $|E(v)|$  is the number of edges incident to the vertex  $v$ . The fixed part of the perfect obstruction theory contributes to the virtual cycle

$$[\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))]^{\text{vir}}.$$

For the stable vertex  $v$  over 0, the vertex moduli  $\mathcal{M}_v$  corresponds to the moduli space

$$Q_{0,\vec{m}(v)}^{\epsilon\theta,\epsilon}(\mathfrak{Y}, (\beta(v), 1^{J_v})) := \bigsqcup_{\substack{d \in \text{Eff}(AY, G, \theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta(v)}} Q_{0,\vec{m}(v)}^{\epsilon\theta,\epsilon}(\mathfrak{Y}, (d, 1^{J_v})).$$

Let  $\pi : \mathcal{C}_0 \rightarrow Q_{0,\vec{m}(v)}^{\epsilon\theta,\epsilon}(\mathfrak{Y}, (\beta(v), 1^{J_v}))$  be the universal curve over  $Q_{0,\vec{m}(v)}^{\epsilon\theta,\epsilon}(\mathfrak{Y}, (\beta(v), 1^{J_v}))$ . In this case, the fixed part of the obstruction theory of the vertex moduli over 0 yields the virtual cycle

$$[Q_{0,\vec{m}(v)}^{\epsilon\theta,\epsilon}(\mathfrak{Y}, (\beta(v), 1^{J_v}))]^{\text{vir}}.$$

Note  $\mathcal{N}|_{\mathcal{C}_0} = \mathcal{O}_{\mathcal{C}_0}$  as  $z_2|_{\mathcal{C}_0} \equiv 1$ , therefore the virtual normal comes from the movable part of the infinitesimal deformations of the section  $z_1$ , which is a section of the line bundle  $\mathcal{L}_{-\theta_p}$  over  $\mathcal{C}_0$ , whose Euler class is equal to

$$e^{\mathbb{C}^*}((R^\bullet \pi_* \mathcal{L}_{-\theta_p}) \otimes \mathbb{C}_\lambda) .$$

**4.3.2. Edge contributions: basepoint case.** When there is a base point on the edge curve, it has degree  $(\beta(e), 1^{J_e}, \frac{\delta(e)}{r})$  with  $\beta(e) \neq 0$  and  $\delta(e) \geq \beta(e)(L_\theta) + |J_e|$  by Remark 4.4, we will write  $\beta(e)$  as  $\beta$  only in this subsection for simplicity unless stated otherwise. Then the multiplicity at  $q_\infty \in C_e$  is equal to  $(g, \mu_r^{\delta(e)}) \in G \times \mu_r$ , where  $g = g_\beta$  is defined in §3. Let  $a$  be the minimal positive integer associated to  $\beta$  as in §3, which is also the order of  $g_\beta$ . When  $r$  is sufficiently large, due to Remark 4.3,  $C_e$  must be isomorphic to  $\mathbb{P}_{ar,1}^1$  where the ramification point  $q_0$  for which  $z_1 = 0$  is an ordinary point, and the ramification point  $q_\infty$  for which  $z_2 = 0$  must be a special point, which is isomorphic to  $\mathbb{B}\mu_{ar}$ .

Recall that

$$[Y_\beta^{ss}/G] \cong [(Z_\beta^{ss} \cap AY)/G]$$

in §3. We now define the edge moduli  $\mathcal{M}_e$  to be

$${}^{a\delta(e)}\sqrt{L_{-\theta}/[Y_\beta^{ss}/G]} ,$$

which is the root gerbe over the stack  $[Y_\beta^{ss}/G] \subset [AY^{ss}(\theta)^g/G] \subset I_\mu Y$  by taking  $a\delta(e)$ th root of the line bundle  $L_{-\theta}$  on  $[Y_\beta^{ss}/G]$ .

The root gerbe  ${}^{a\delta(e)}\sqrt{L_{-\theta}/[Y_\beta^{ss}/G]}$  admits a representation as a quotient stack:

$$[(Y_\beta^{ss} \times \mathbb{C}^*)/(G \times \mathbb{C}_w^*)],$$

where the (right) action is defined by

$$(\vec{x}, v) \cdot (g, w) = (\vec{x} \cdot g, \theta(g)vw^{a\delta(e)}) ,$$

for all  $(g, w) \in G \times \mathbb{C}_w^*$  and  $(\vec{x}, v) \in A(Y)^g \times \mathbb{C}^*$ . Here  $\vec{x} \cdot g$  is given by the action as in the definition of  $[AY/G]$ . For every character  $\rho$  of  $G$ , we can define a new character of  $G \times \mathbb{C}_w^*$  by composing the projection map  $\text{pr}_G : G \times \mathbb{C}_w^* \rightarrow G$ . By an abuse of notation, we will continue to use the notation  $\rho$  to name the new character of  $G \times \mathbb{C}_w^*$ . Then the new character  $\rho$  will determines a line bundle  $L_\rho := [(Y_\beta^{ss} \times \mathbb{C}^* \times \mathbb{C}_\rho)/(G \times \mathbb{C}_w^*)]$  on  ${}^{a\delta(e)}\sqrt{L_{-\theta}/[Y_\beta^{ss}/G]}$ .

By virtue of its universal property of the root gerbe  ${}^{a\delta(e)}\sqrt{L_{-\theta}/[Y_\beta^{ss}/G]}$ , there is a universal line bundle  $\mathcal{R}$  that is the  $a\delta(e)$ th root of line bundle  $L_{-\theta}$  over the root gerbe. This root line bundle  $\mathcal{R}$  can also constructed by the Borel construction, i.e.  $\mathcal{R}$  is associated to the character  $p_2$ :

$$\text{pr}_{\mathbb{C}_w^*} : G \times \mathbb{C}_w^* \rightarrow \mathbb{C}_w^* \quad (g, w) \in G \times \mathbb{C}_w^* \mapsto w \in \mathbb{C}_w^* .$$

We have the relation

$$L_{-\theta} = \mathcal{R}^{a\delta(e)} .$$

Then the coordinate functions  $(\vec{x}, v) \in Y_\beta^{ss} \times \mathbb{C}^*$  descends to be tautological sections of vector bundle  $\bigoplus_{i=1}^n L_{\rho_i} \oplus (L_\theta \otimes \mathcal{R}^{a\delta(e)})$  on  ${}^{a\delta(e)}\sqrt{L_{-\theta}/[Y_\beta^{ss}/G]}$ .

We will construct a universal family of  $\mathbb{C}^*$ -fixed quasimaps to  $\mathbb{P}\mathfrak{Y}_{r,p}^{\frac{1}{r}}$  of degree  $(\beta, 1^{J_e}, \frac{\delta(e)}{r})$  over the edge moduli  $\mathcal{M}_e$ , which takes the form

$$\begin{array}{c} \mathcal{C}_e := \mathbb{P}_{ar,1}(\mathcal{R}^{\otimes a} \oplus \mathcal{O}_{\mathcal{M}_e}) \xrightarrow{ev} \mathbb{P}\mathfrak{Y}_{r,p}^{\frac{1}{r}} \\ \downarrow \pi \\ \mathcal{M}_e := \sqrt[{}^{a\delta(e)}]{L_{-\theta}/[Y_{\beta}^{ss}/G]} . \end{array}$$

The universal curve  $\mathcal{C}_e$  over the edge moduli  $\mathcal{M}_e$  is constructed as a quotient stack:

$$\mathcal{C}_e = [(Y_{\beta}^{ss} \times \mathbb{C}^* \times U)/(G \times \mathbb{C}_w^* \times \mathbb{C}_t^*)] ,$$

where the right action is defined by:

$$(\vec{x}, v, x, y) \cdot (g, w, t) = (\vec{x} \cdot g, \theta(g)vw^{a\delta(e)}, w^at^{ar}x, ty) ,$$

for all  $(g, w, t) \in G \times \mathbb{C}_w^* \times \mathbb{C}_t^*$  and  $(\vec{x}, v, (x, y)) = ((x_1, \dots, x_n), v, (x, y)) \in Y_{\beta}^{ss} \times \mathbb{C}^* \times U$ .

The universal map  $ev$  from  $\mathcal{C}_e$  to  $\mathbb{P}\mathfrak{Y}_{r,p}^{\frac{1}{r}}$  can be presented as follows:

$$\tilde{ev} : Y_{\beta}^{ss} \times \mathbb{C}^* \times U \rightarrow AY_p \times U ,$$

defined by:

(4.9)

$$(\vec{x}, v, (x, y)) \in Y_{\beta}^{ss} \times \mathbb{C}^* \times U \mapsto ((x_1x^{\beta(L_{\rho_1})}, \dots, x_nx^{\beta(L_{\rho_n})}), (x)_{j \in J_e}, v^{-1}x^{\delta(e)-\beta(L_{\theta})-|J_e|}, y^{a\delta(e)}) \in AY_p \times U .$$

Here  $(x)_{j \in J_e}$  are elements belonging to  $\mathbb{C}^p$  so that the  $j$ -th component is 1 if  $j \notin J_e$  and all the other components are  $x$ . Note that when  $\beta(L_{\rho_i}) \notin \mathbb{Z}_{\geq 0}$  for some  $i$ , we must have  $x_i = 0$  as  $\vec{x} \in Y_{\beta}^{ss}$ , so the  $\tilde{ev}$  is well defined. Then  $\tilde{ev}$  is equivariant with respect to the group homomorphism from  $G \times \mathbb{C}_w^* \times \mathbb{C}_t^*$  to  $G_p \times \mathbb{C}^*$  defined by:

(4.10)

$$(g, w, t) \in G \times \mathbb{C}_w^* \times \mathbb{C}_t^* \mapsto (g(t^{ar\beta(L_{\pi_1})}w^{a\beta(L_{\pi_1})}, \dots, t^{ar\beta(L_{\pi_k})}w^{a\beta(L_{\pi_k})}), (w^at^{ar})_{j \in J_e}, t^{a\delta(e)}) \in G_p \times \mathbb{C}^* .$$

Here  $(w^at^{ar})_{j \in J_e}$  is the element belonging to  $(\mathbb{C}^*)^p$  so that the  $j$ -th component is 1 if  $j \notin J_e$  and all the other components are  $w^at^{ar}$ . This gives the universal morphism  $f$  from  $\mathcal{C}_e$  to  $\mathbb{P}\mathfrak{Y}_{r,p}^{\frac{1}{r}}$  by descent.

There is a tautological line bundle  $\mathcal{O}_{C_e}(1)$  on  $C_e$  associated to the character  $\text{pr}_{\mathbb{C}_t^*}$  of  $G \times \mathbb{C}_w^* \times \mathbb{C}_t^*$  by the Borel construction. Here  $\text{pr}_{\mathbb{C}_t^*}$  is the projection map from  $G \times \mathbb{C}_w^* \times \mathbb{C}_t^*$  to  $\mathbb{C}_t^*$ .

We will define a (quasi<sup>6</sup> left)  $\mathbb{C}^*$ -action on  $\mathcal{C}_e$  such that the map  $ev$  constructed above is  $\mathbb{C}^*$ -equivariant. Define a (left)  $\mathbb{C}^*$ -action on  $\mathcal{C}_e$  which is induced from the  $\mathbb{C}^*$ -action on  $Y_{\beta}^{ss} \times \mathbb{C}^* \times U$ :

$$m : \mathbb{C}^* \times Y_{\beta}^{ss} \times \mathbb{C}^* \times U \rightarrow Y_{\beta}^{ss} \times \mathbb{C}^* \times U ,$$

$$t \cdot (x, v, (x, y)) = (x, v, (x, t^{\frac{-1}{ar\delta(e)}}y)) .$$

<sup>6</sup>This means we allow  $\mathbb{C}^*$ -action on  $C_e$  with fractional weight. See a similar discussion in [CLLL16, §2.2].



Note the morphism  $\pi$  is also  $\mathbb{C}^*$ -equivariant, where  $\mathcal{M}_e$  is equipped with trivial  $\mathbb{C}^*$ -action. By the universal property of the projectivized bundle  $\mathcal{C}_e$  over  $\mathcal{M}_e$ , the line bundle  $\mathcal{O}_{\mathcal{C}_e}(1)$  is equipped with a tautological section

$$(x, y) \in H^0((\mathcal{O}_{\mathcal{C}_e}(ar) \otimes \pi^* \mathcal{R}^{\otimes a}) \oplus (\mathcal{O}_{\mathcal{C}_e}(1) \otimes \mathbb{C}_{\frac{-1}{ar\delta(e)}})) ,$$

which is also a  $\mathbb{C}^*$ -invariant section. Here  $\mathcal{O}_{\mathcal{C}_e}(1)$  is the standard  $\mathbb{C}^*$ -equivariant line bundle on  $\mathcal{C}_e$  by the Borel construction.

Now we can check that  $ev$  is a  $\mathbb{C}^*$ -equivariant morphism from  $\mathcal{C}_e$  to  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}, p}$  with respect to the  $\mathbb{C}^*$ -actions for  $\mathcal{C}_e$  and  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}, p}$ . According to Remark 4.1,  $ev$  is equivalent to the following data:

- (1)  $k + p + 1$   $\mathbb{C}^*$ -equivariant line bundles on  $\mathcal{C}_e$ :

$$\mathcal{L}_j := \pi^* L_{\pi_j} \otimes \mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\pi_j})) \otimes \pi^* \mathcal{R}^{\otimes a\beta(L_{\pi_j})}, 1 \leq j \leq k ,$$

$$\mathcal{L}_{k+j} := \pi^* \mathcal{R}^{\otimes a} \otimes \mathcal{O}_{\mathcal{C}_e}(ar), j \in J_e, \mathcal{L}_{k+j} := \mathbb{C}, j \notin J_e$$

and

$$\mathcal{N} := \mathcal{O}_{\mathcal{C}_e}(a\delta(e)) \otimes \mathbb{C}_{\frac{-\lambda}{r}} ,$$

where the line bundles  $L_{\pi_j}$ ,  $\mathcal{R}$  are the standard  $\mathbb{C}^*$ -equivariant line bundle on  $\mathcal{M}_e$  by the Borel construction;

- (2) a universal section

(4.11)

$$\begin{aligned} (\vec{x}, \vec{y}, (\zeta_1, \zeta_2)) &:= ((x_1 x^{\beta(L_{\rho_1})}, \dots, x_n x^{\beta(L_{\rho_n})}), (x)_{J_e}, (v^{-1} x^{\delta(e) - \beta(L_{\theta}) - |J_e|}, y^{a\delta(e)})) \\ &\in H^0(\mathcal{C}_e, (\oplus_{i=1}^n \mathcal{L}_{\rho_i}) \oplus (\oplus_{j=1}^p \mathcal{L}_{k+j}) \oplus (\mathcal{L}_{-\theta_p} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_{\lambda}) \oplus \mathcal{N})^{\mathbb{C}^*} , \end{aligned}$$

where the line bundles  $\mathcal{L}_{-\theta_p}$  and  $\mathcal{L}_{\rho_i}$  are constructed from line bundles  $\mathcal{L}_j$  as before.

Equipped with these notations, now we compute the localization contribution from  $\mathcal{M}_e$ . Based on the perfect obstruction theory for quasimaps in  $Q_{0, \vec{m}}^{\theta}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}, p}, (\beta, 1^p, \frac{\delta}{r}))$ , the restriction of the perfect obstruction theory to  $\mathcal{M}_e$  decomposes into three parts: (1) the deformation theory of source curve  $\mathcal{C}_e$ ; (2) the deformation theory of the line bundles  $(\mathcal{L}_j)_{1 \leq j \leq k+p}$  and  $\mathcal{N}$ ; (3) the deformation theory for the section

$$(\vec{x}, \vec{y}, (\zeta_1, \zeta_2)) \in \Gamma \left( \oplus_{i=1}^n \mathcal{L}_{\rho_i} \oplus (\oplus_{j=1}^p \mathcal{L}_{k+j}) \oplus (\mathcal{L}_{-\theta_p} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_{\lambda}) \oplus \mathcal{N} \right) .$$

The virtual normal bundle comes from the movable part of the three parts, and the fixed part will contribute to the virtual cycle of  $\mathcal{M}_e$ . Recall that the complete intersection  $Y$  is cut off by the section  $s = \oplus_{b=1}^c s_b$  of the direct sums of line bundle  $\oplus_{b=1}^c L_{\tau_b}$  on the toric stack  $X$ , we will show that:

**Lemma 4.5.** *We have the following:*

$$[\mathcal{M}_e]^{\text{vir}} = i_{\mathcal{M}_e}^* (s_{E_{\geq 0, \text{loc}}}^!([Z_{\beta}^{ss}/G])) .$$

Here  $i_{\mathcal{M}_e} : \mathcal{M}_e \rightarrow [Y_{\beta}^{ss}/G]$  is the natural étale morphism by forgetting gerbe structure. Where  $E_{\geq 0} := \oplus_{b: \beta(L_{\tau_b}) \geq 0} L_{\tau_b}$ , and  $s_{E_{\geq 0, \text{loc}}}^! : A_*([Z_{\beta}^{ss}/G]) \rightarrow A_*([Y_{\beta}^{ss}/G])$  is the localized top Chern class [Ful84, §14.1] with respect to the vector bundle  $E_{\geq 0}$  over  $[Z_{\beta}^{ss}/G]$  and the section  $s_{\geq 0} := \oplus_{b: \beta(L_{\tau_b}) \in \mathbb{Z}_{\geq 0}} s_b$ .

**Remark 4.6.** We note that  $s_{\geq 0}$  and  $E_{\geq 0}$  can descend to a section and a vector bundle over  $[Y_{\beta}^{ss}/G]$ , thus  $s_{E_{\geq 0}, \text{loc}}^1([Z_{\beta}^{ss}/G])$  is also well defined.

First every fiber curve  $C_e$  in  $\mathcal{C}_e$  is isomorphic to  $\mathbb{P}_{ar,1}$ , which is rational. Then the infinitesimal deformations/obstructions of  $C_e$  and the line bundles  $L_j := \mathcal{L}_j|_{C_e}$ ,  $N := \mathcal{N}|_{C_e}$  are zero. Hence their contribution to the perfect obstruction theory solely comes from infinitesimal automorphisms. The infinitesimal automorphisms of  $C_e$  come from the space of vector field on  $C_e$  that vanishes on special points. Thus the  $\mathbb{C}^*$ -fixed part of the infinitesimal automorphisms of  $C_e$  comes from the 1-dimensional subspace of vector fields on  $C_e$  which vanish on the two ramification points, which, together with the infinitesimal automorphisms of line bundle  $N$ , will be canceled with the fixed part of infinitesimal deformation of sections  $(z_1, z_2) := (\zeta_1, \zeta_2)|_{C_e}$ . The movable part of infinitesimal automorphisms of  $C_e$  is nonzero only if at least one of ramification points on  $C_e$  is not a special point. By Remark 4.3, the ramification  $q_{\infty}$  must be a special point since it has nontrivial stacky structure when  $r$  is sufficiently large, and the ramification point  $q_0$  is not a special point, then the movable part of infinitesimal automorphisms of  $C_e$  contributes

$$\frac{\delta(e)}{\lambda - D_{\theta}}$$

to the virtual normal bundle.

Now let's turn to the localization contribution from sections. As for the deformations of  $z_2$ , we continue to use the tautological section  $(x, y)$  in (4.3.2). Sections of  $N$  is spanned by monomials  $(x^m y^n)|_{C_e}$  with  $arm + n = a\delta(e)$  and  $m, n \in \mathbb{Z}_{\geq 0}$ . Note  $x^m y^n$  may not be a global section of  $\mathcal{N}$  but always a global section of the line bundle  $R^{\otimes am} \otimes \mathcal{N} \otimes \mathbb{C}_{\frac{m}{\delta(e)}\lambda}$ . Then  $R^{\bullet} \pi_* \mathcal{N}$  will decompose as a direct sum of line bundles, each corresponds to the monomial  $x^m y^n$ , whose first chern class is

$$c_1(\mathcal{R}^{\otimes -am} \bigotimes \mathbb{C}_{\frac{m}{\delta(e)}\lambda}) = \frac{m}{\delta(e)}(D_{\theta} - \lambda).$$

So the total contribution is equal to

$$\prod_{m=0}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left( \frac{m}{\delta(e)}(D_{\theta} - \lambda) \right).$$

The term corresponding to  $m = 0$  in the above product is the  $\mathbb{C}^*$ -invariant part of  $R^{\bullet} \pi_* \mathcal{N}$ , it will contribute to the virtual cycle of  $\mathcal{M}_e$ . The rest contributes to the virtual normal bundle as

$$\prod_{m=1}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left( \frac{m}{\delta(e)}(D_{\theta} - \lambda) \right).$$

Note when  $r$  is sufficiently large, the above product becomes 1.

For the deformation of  $z_1$ , arguing in the same way as  $z_2$ , the Euler class of  $R^{\bullet} \pi_*(\mathcal{L}_{-\theta_p} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_{\lambda})$  is equal to

$$\prod_{m=0}^{\delta(e) - \beta(L_{\theta}) - |J_e|} \left( \frac{m}{\delta(e)}(-D_{\theta} + \lambda) \right).$$

The factor for  $m = 0$  appearing in the above product is the  $\mathbb{C}^*$ -fixed part of  $R^\bullet \pi_*(\mathcal{L}_{-\theta} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda)$ , it will contribute to the virtual cycle of  $\mathcal{M}_e$ . The rest contributes to the virtual normal bundle as

$$\prod_{m=1}^{\delta(e) - \beta(L_\theta) - |J_e|} \left( \frac{m}{\delta(e)} (-D_\theta + \lambda) \right).$$

Finally, let's turn to the localization contribution from the sections  $\vec{x}$  and  $\vec{y}$ . Before that, using the same argument, one can prove the following lemma:

**Lemma 4.7.** *When  $n \in \mathbb{Z}_{\geq 0}$ , we have*

$$e^{\mathbb{C}^*}(R\pi_*(\mathcal{O}_{\mathcal{C}_e}(n))) = \prod_{m=0}^{\lfloor \frac{n}{ar} \rfloor} \left( \frac{m}{\delta(e)} (D_\theta - \lambda) + \frac{n}{ar\delta(e)} \lambda \right).$$

When  $n \in \mathbb{Z}_{< 0}$ , we have

$$e^{\mathbb{C}^*}(R^\bullet \pi_*(\mathcal{O}_{\mathcal{C}_e}(n))) = \prod_{\frac{n}{ar} < m < 0} \frac{1}{\frac{m}{\delta(e)} (D_\theta - \lambda) + \frac{n}{ar\delta(e)} \lambda}.$$

Using the above lemma, we have the following description of  $e^{\mathbb{C}^*}(R^\bullet \pi_* \mathcal{L}_{\rho_i})$  for  $1 \leq i \leq n$ . Then for each  $\rho_i$ , we have:

(1) If  $\beta(L_{\rho_i}) \in \mathbb{Q}_{\geq 0}$ , one has

$$\begin{aligned} e^{\mathbb{C}^*}(R^\bullet \pi_*(\mathcal{L}_{\rho_i})) &= e^{\mathbb{C}^*}(R^\bullet \pi_*(\pi^*(L_{\rho_i}) \otimes \mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i})) \otimes \pi^*(\mathcal{R}^{\otimes a\beta(L_{\rho_i})}))) \\ &= e^{\mathbb{C}^*}(L_{\rho_i} \otimes \mathcal{R}^{\otimes a\beta(L_{\rho_i})} \otimes R^0 \pi_*(\mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i})))) \\ &= \prod_{m=0}^{\lfloor \beta(L_{\rho_i}) \rfloor} \left( D_{\rho_i} + \frac{\beta(L_{\rho_i})(-D_\theta)}{\delta(e)} + \frac{m}{\delta(e)} (D_\theta - \lambda) + \frac{\beta(L_{\rho_i})}{\delta(e)} \lambda \right) \\ &= \prod_{m=0}^{\lfloor \beta(L_{\rho_i}) \rfloor} \left( D_{\rho_i} + \frac{\beta(L_{\rho_i}) - m}{\delta(e)} (\lambda - D_\theta) \right). \end{aligned}$$

Hence we have

$$e^{\mathbb{C}^*}((R^\bullet \pi_* \mathcal{L}_{\rho_i})^{\text{mov}}) = \prod_{0 \leq m < \beta(L_{\rho_i})} \left( D_{\rho_i} + \frac{\beta(L_{\rho_i}) - m}{\delta(e)} (\lambda - D_\theta) \right).$$

Note the invariant part of  $R^\bullet \pi_* \mathcal{L}_{\rho_i}$  is nonzero only when  $\beta(L_{\rho_i}) \in \mathbb{Z}_{\geq 0}$ .

(2) If  $\beta(L_{\rho_i}) \in \mathbb{Q}_{< 0}$ , one has

$$\begin{aligned} e^{\mathbb{C}^*}(R^\bullet \pi_* \mathcal{L}_{\rho_i}) &= e^{\mathbb{C}^*}(R^\bullet \pi_*(\pi^* L_{\rho_i} \otimes \mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i})) \otimes \pi^* \mathcal{R}^{\otimes a\beta(L_{\rho_i})})) \\ &= \frac{1}{e^{\mathbb{C}^*}(L_{\rho_i} \otimes \mathcal{R}^{\otimes a\beta(L_{\rho_i})} \otimes R^1 \pi_*(\mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i}))))} \\ &= \prod_{\beta(L_{\rho_i}) < m < 0} \frac{1}{D_{\rho_i} + \frac{\beta(L_{\rho_i})(-D_\theta)}{\delta(e)} + \frac{m}{\delta(e)} (D_\theta - \lambda) + \frac{\beta(L_{\rho_i})}{\delta(e)} \lambda} \\ &= \prod_{\beta(L_{\rho_i}) < m < 0} \frac{1}{D_{\rho_i} + \frac{\beta(L_{\rho_i}) - m}{\delta(e)} (\lambda - D_\theta)}, \end{aligned}$$

which implies that

$$\begin{aligned} e^{\mathbb{C}^*}((R^\bullet \pi_* \mathcal{L}_{\rho_i})^{\text{mov}}) &= e^{\mathbb{C}^*}(R^\bullet \pi_* \mathcal{L}_{\rho_i}) \\ &= \prod_{\beta(L_{\rho_i}) < m < 0} \frac{1}{D_{\rho_i} + \frac{\beta(L_{\rho_i}) - m}{\delta(e)}(\lambda - D_\theta)}. \end{aligned}$$

The movable part of deformation of  $\vec{y}$  contributes to

$$e^{\mathbb{C}^*}(R^\bullet \pi_*(\oplus_{j=1}^p \mathcal{L}_{k+j})) = \left(\frac{\lambda - D_\theta}{\delta(e)}\right)^{|J_e|}.$$

to the virtual normal bundle and the fixed part of deformation of  $\vec{y}$  will be canceled with the automorphisms of line bundles  $(L_{k+j} : 1 \leq j \leq p)$ .

Recall that the complete intersection  $Y$  is cut off by the section  $s := \oplus_{b=1}^c s_b$  of direct sum of the line bundles  $E = \oplus_{b=1}^c L_{\tau_b}$  on  $X$  associated to the characters  $\tau_b$ . There is also an obstruction corresponding to the infinitesimal deformations of  $\vec{x}$  being moved away from  $[AY^{ss}(\theta)/G] \subset [W^{ss}(\theta)/G]$ , which contributes to the virtual normal bundle as the movable part of

$$\begin{aligned} e^{\mathbb{C}^*}(-(\oplus_b R^\bullet \pi_* \mathcal{L}_{\tau_b})) &= \frac{e^{\mathbb{C}^*}(R^1 \pi_* \oplus_{b:\beta(L_{\tau_b}) < 0} \mathcal{L}_{\tau_b})}{e^{\mathbb{C}^*}(R^0 \pi_* \oplus_{b:\beta(L_{\tau_b}) \geq 0} \mathcal{L}_{\tau_b})} \\ &= \frac{\prod_{b:\beta(L_{\tau_b}) < 0} \prod_{\beta(L_{\tau_b}) < m < 0} (c_1(L_{\tau_b}) + \frac{\beta(L_{\tau_b}) - m}{\delta(e)}(\lambda - D_\theta))}{\prod_{b:\beta(L_{\tau_b}) \geq 0} \prod_{0 \leq m \leq \beta(L_{\tau_b})} (c_1(L_{\tau_b}) + \frac{\beta(L_{\tau_b}) - m}{\delta(e)}(\lambda - D_\theta))}. \end{aligned}$$

Here  $m$  are all integers. One can see that the fixed part only comes from the summand corresponding to the terms  $b$  with  $\beta(L_{\tau_b}) \in \mathbb{Z}_{\geq 0}$ , for which there is one dimensional  $\mathbb{C}^*$ -fixed piece for each  $-R^\bullet \pi_* \mathcal{L}_{\tau_b}$ , which contributes to the virtual cycle of  $\mathcal{M}_e$ . Now combining the discussion above, the virtual cycle structure of  $\mathcal{M}_e$  solely comes automorphisms of line bundles  $(\mathcal{L}_j)_{j=1}^k$ , fixed part of deformations/obstructions of the sections  $\vec{x}$ . Then  $\mathbb{C}^*$ -fixed part of the obstruction theory  $\mathbb{E}^{\text{fix}}$  over  $\mathcal{M}_e$  fits into the following distinguished triangle

$$\mathbb{E}^{\text{fix}} \longrightarrow \mathbb{T}_{[Z_\beta^{ss}/G]}|_{\mathcal{M}_e} \xrightarrow{ds_{\geq 0}} E_{\geq 0} \longrightarrow$$

Here  $ds_{\geq 0}$  is the differential of the section  $s_{\geq 0} := \oplus_{b:\beta(L_{\tau_b}) \in \mathbb{Z}_{\geq 0}} s_b$  inside the vector bundle  $E_{\geq 0} =: \oplus_{b:\beta(L_{\tau_b}) \in \mathbb{Z}_{\geq 0}} L_{\tau_b}$  and  $\mathbb{T}_{[Z_\beta^{ss}/G]}|_{\mathcal{M}_e}$  is the pull back of the tangent complex  $\mathbb{T}_{[Z_\beta^{ss}/G]}$  along the morphism

$$\mathcal{M}_e \rightarrow {}^{a\delta(e)}\sqrt{L_{-\theta}/[Z_\beta^{ss}/G]} \rightarrow [Z_\beta^{ss}/G],$$

where the first arrow is the inclusion and the second arrow is the natural etale morphism by forgetting gerbe structure.

By Lemma 3.2,  $\mathcal{M}_e$  is the zero loci of the section  $s_{\geq 0}$  of the vector bundle  $E_{\geq 0} := \oplus_{b:\beta(L_{\tau_b}) \in \mathbb{Z}_{\geq 0}} L_{\tau_b}$  over  ${}^{a\delta(e)}\sqrt{L_{-\theta}/[Z_\beta^{ss}/G]}$ . Thus the virtual cycle  $[\mathcal{M}_e]^{vir}$  with respect to the perfect obstruction theory  $\mathbb{E}^{\text{fix}}$  can be obtained by the standard model of vector bundle  $E_{\geq 0}$  over  ${}^{a\delta(e)}\sqrt{L_{-\theta}/[Z_\beta^{ss}/G]}$  and the section  $s_{\geq 0}$  (c.f. [PT14, §Appendix]). Now

the Lemma 4.5 is immediate as the distinguished is pull back of a similar distinguished triangle on  $[Y^{ss}/G]$ .

We have the expression of virtual normal bundle from the movable part of curves, line bundles and sections as follows:

$$e^{\mathbb{C}^*}(N^{\text{vir}}) = \frac{\prod_{\rho: \beta(L_\rho) > 0} \prod_{0 \leq i < \beta(L_\rho)} (D_\rho + (\beta(L_\rho) - i) \frac{\lambda - D_\theta}{\delta(e)})}{\prod_{\rho: \beta(L_\rho) < 0} \prod_{\lfloor \beta(L_\rho) + 1 \rfloor \leq i < 0} (D_\rho + (\beta(L_\rho) - i) \frac{\lambda - D_\theta}{\delta(e)})} \cdot \frac{\delta(e)}{\lambda - D_\theta} \\ \cdot \frac{\prod_{b: \beta(L_{\tau_b}) < 0} \prod_{\beta(L_{\tau_b}) < m < 0} (c_1(L_{\tau_b}) + \frac{\beta(L_{\tau_b}) - m}{\delta(e)} (\lambda - D_\theta))}{\prod_{b: \beta(L_{\tau_b}) \geq 0} \prod_{0 \leq m < \beta(L_{\tau_b})} (c_1(L_{\tau_b}) + \frac{\beta(L_{\tau_b}) - m}{\delta(e)} (\lambda - D_\theta))} \cdot \prod_{m=1}^{\delta(e) - \beta(L_\theta) - |J_e|} \left( \frac{m}{\delta(e)} (-D_\theta + \lambda) \right).$$

We observe that, after taking the push-forward  $i: \mathcal{M}_e \rightarrow {}^{a\delta(e)}\sqrt{L_{-\theta}/I_{g_\beta}Y}$ , the localization contribution from edge moduli with a basepoint yields:

**Lemma 4.8.**

$$i_*(\text{Cont}_{\mathcal{M}_e}) = (\bar{i})^* \left( \iota_* \frac{(\frac{z}{z|J_e|} \mathbb{I}_\beta(q, z))|_{z = \frac{\lambda - D_\theta}{\delta(e)}}}{\prod_{m=1}^{\delta(e) - \beta(L_\theta) - |J_e|} (\frac{m}{\delta(e)} (-D_\theta + \lambda))} \right),$$

where  $\bar{i}_{I_{g_\beta}Y}: {}^{a\delta(e)}\sqrt{L_{-\theta}/I_{g_\beta}Y} \rightarrow \bar{I}_{g_\beta}Y := [Y_\beta^{ss}/(G/\langle g_\beta \rangle)]$  is the natural structure map by forgetting gerbe structure and taking rigidification,  $\iota$  is the involution of  $\bar{I}_\mu Y$  obtained from taking the inverse of the band, and  $\mathbb{I}_\beta$  is the coefficient of  $q^\beta$  of  $\mathbb{I}(q, 0, z)$  defined in the introduction.

**4.3.3. Edge contributions: without basepoint case.** The contribution from an edge without basepoint will not appear in the later analysis in §6. However we include the discussion for this case here for completeness. The reader is encouraged to skip this part in the first reading. In this case,  $J_e$  is empty. Assume that the multiplicity at  $q_\infty \in C_e$  is equal to  $(g, \mu_r^{\delta(e)}) \in G \times \mu_r$  and  $a_e$  (or  $a$  for simplicity) is the order of  $g$ . When  $r$  is sufficiently large, due to Remark 4.3,  $C_e$  must be isomorphic to  $\mathbb{P}_{ar,a}^1$  where the ramification point  $q_0$  for which  $z_1 = 0$  is isomorphic to  $\mathbb{B}\mu_a$ , and the ramification point  $q_\infty$  for which  $z_2 = 0$  must be a special point and is isomorphic to  $\mathbb{B}\mu_{ar}$ . The restriction of degree  $(\beta, \frac{\delta}{r})$  from  $C$  to  $C_e$  is equal to  $(0, \frac{\delta(e)}{r})$ , which is equivalent to:

$$\deg(L_j|_{C_e}) = 0 \quad \text{for } 1 \leq j \leq k, \quad \deg(N|_{C_e}) = \frac{\delta(e)}{r}.$$

Recall that the inertia stack component  $I_g Y$  of  $I_\mu Y$  is isomorphic to the quotient stack

$$[AY^{ss}(\theta)^g/G].$$

We construct the edge moduli  $\mathcal{M}_e$  as

$$\mathcal{M}_e := {}^{a\delta(e)}\sqrt{L_{-\theta}/I_g Y},$$

which is the root gerbe over the stack  $I_g Y$  by taking the  $a\delta(e)$ th root of the line bundle  $L_{-\theta}$ .

The root gerbe  ${}^{a\delta(e)}\sqrt{L_{-\theta}/I_g Y}$  admits a representation as a quotient stack:

$$(4.12) \quad [(AY^{ss}(\theta)^g \times \mathbb{C}^*)/(G \times \mathbb{C}_w^*)],$$

where the (right) action is defined by:

$$(\vec{x}, v) \cdot (g, w) = (\vec{x} \cdot g, \theta(g)vw^{a\delta(e)}) ,$$

for all  $(g, w) \in G \times \mathbb{C}_w^*$  and  $(\vec{x}, v) \in AY^{ss}(\theta)^g \times \mathbb{C}^*$ . Here  $\vec{x} \cdot g$  is given by the action as in the definition of  $[AY/G]$ , the torus  $\mathbb{C}_w^*$  is isomorphic to  $\mathbb{C}^*$  with variable  $w$ . For any character  $\rho$  of  $G$ , define a new character of  $G \times \mathbb{C}_w^*$  by composing the projection map  $\text{pr}_G : G \times \mathbb{C}_w^* \rightarrow G$ . By an abuse of notation, we will continue to use the notation  $\rho$  to mean the new character of  $G \times \mathbb{C}_w^*$ . Then  $\rho$  will determines a line bundle  $L_\rho := [(AY^{ss}(\theta)^g \times \mathbb{C}^* \times \mathbb{C}_\rho)/(G \times \mathbb{C}_w^*)]$  on  ${}^{a\delta(e)}\sqrt{L_{-\theta}/I_g Y}$  by the Borel construction.

By virtue of the universal property of root gerbe, on  $\mathcal{M}_e = {}^{a\delta(e)}\sqrt{L_{-\theta}/I_g Y}$ , there is a universal line bundle  $\mathcal{R}$  that is the  $a\delta(e)$ th root of the line bundle  $L_{-\theta}$ . The root bundle  $\mathcal{R}$  is associated to the character

$$\text{pr}_{\mathbb{C}^*} : G \times \mathbb{C}_w^* \rightarrow \mathbb{C}_w^*, \quad (g, w) \in G \times \mathbb{C}_w^* \mapsto w \in \mathbb{C}_w^*$$

by the Borel construction. We have the relation

$$L_{-\theta} = \mathcal{R}^{a\delta(e)} .$$

The coordinate functions  $\vec{x}$  and  $v$  of  $AY^{ss}(\theta)^g \times \mathbb{C}^*$  descends to be universal sections of line bundles  $\oplus_{\rho \in [n]} L_\rho$  and  $L_\theta \otimes \mathcal{R}^{\otimes a\delta(e)}$  over  $\mathcal{M}_e$ , respectively.

We will construct a universal family of  $\mathbb{C}^*$ -fixed quasimaps to  $\mathbb{P}\mathfrak{Y}_{r,p}^{\frac{1}{r}}$  of degree  $(0, 1^\emptyset, \frac{\delta(e)}{r})$  over  $\mathcal{M}_e$ :

$$\begin{array}{ccc} \mathcal{C}_e := \mathbb{P}_{ar,a}(\mathcal{R} \oplus \mathcal{O}_{\mathcal{M}_e}) & \xrightarrow{f} & \mathbb{P}\mathfrak{Y}_{r,p}^{\frac{1}{r}} \\ \pi \downarrow & & \\ \mathcal{M}_e := {}^{a\delta(e)}\sqrt{L_{-\theta}/I_g Y} & & \end{array}$$

Then the universal curve  $\mathcal{C}_e$  over  $\mathcal{M}_e$  can be represented as a quotient stack:

$$\mathcal{C}_e = [(AY^{ss}(\theta)^g \times \mathbb{C}^* \times U)/(G \times \mathbb{C}_w^* \times T)] ,$$

where  $T = \{(t_1, t_2) \in (\mathbb{C}^*)^2 \mid t_1^a = t_2^{ar}\}$ . The (right) action is defined by:

$$(\vec{x}, v, x, y) \cdot (g, w, (t_1, t_2)) = (\vec{x} \cdot g, \theta(g)vw^{a\delta(e)}, wt_1x, t_2y) ,$$

for all  $(g, w, (t_1, t_2)) \in G \times \mathbb{C}_w^* \times T$  and  $(\vec{x}, v, (x, y)) \in AY^{ss}(\theta)^g \times \mathbb{C}^* \times U$ . Then  $\mathcal{C}_e$  is a family of orbifold  $\mathbb{P}_{ar,a}$  parameterized by  $\mathcal{M}_e$ .

There are two standard characters  $\chi_1$  and  $\chi_2$  of  $T$ :

$$\chi_1 : (t_1, t_2) \in T \mapsto t_1 \in \mathbb{C}^*, \quad \chi_2 : (t_1, t_2) \in T \mapsto t_2 \in \mathbb{C}^* .$$

We can lift them to be new characters of  $G \times \mathbb{C}_w^* \times T$  by composing the projection map  $\text{pr}_T : G \times \mathbb{C}_w^* \times T \rightarrow T$ . By an abuse of notation, we continue to use  $\chi_1, \chi_2$  to denote the new characters. Then  $\chi_1, \chi_2$  defines two line bundles

$$M_1 := (AY^{ss}(\theta)^g \times \mathbb{C}^* \times U) \times_{G \times \mathbb{C}_w^* \times T} \mathbb{C}_{\chi_1}$$

and

$$M_2 := (AY^{ss}(\theta)^g \times \mathbb{C}^* \times U) \times_{G \times \mathbb{C}_w^* \times T} \mathbb{C}_{\chi_2}$$

on  $\mathcal{C}_e$  by the Borel construction, respectively. We have the relation  $M_1^{\otimes a} = M_2^{\otimes ar}$  on  $\mathcal{C}_e$ . The universal map  $f$  from  $\mathcal{C}_e$  to  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$  can be constructed as follows: let

$$\tilde{f} : AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \rightarrow AY \times U$$

be the morphism defined by:

$$(4.13) \quad \begin{aligned} (\vec{x}, v, x, y) &\in AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \mapsto \\ &((x_1, \dots, x_n), v^{-1}x^{a\delta(e)}, y^{a\delta(e)}) \in AY \times U \end{aligned}$$

Then  $\tilde{f}$  is equivariant with respect to the group homomorphism from  $G \times \mathbb{C}_w^* \times T$  to  $G \times \mathbb{C}^*$  defined by:

$$(4.14) \quad \begin{aligned} (g, w, (t_1, t_2)) &\in G \times \mathbb{C}_w^* \times T \mapsto \\ &(g((t_1^{-1}t_2^r)^{p_1}, \dots, (t_1^{-1}t_2^r)^{p_k}), t_2^{a\delta(e)}) \in G \times \mathbb{C}^*, \end{aligned}$$

where the tuple  $(p_1, \dots, p_k) \in \mathbb{N}^k$  satisfies that  $g = (\mu_a^{p_1}, \dots, \mu_a^{p_k}) \in G$ . Note  $\tilde{f}$  is well defined for  $\chi_1^{-1}\chi_2^r$  is a torsion character of  $T$  of order  $a$ . The above construction gives the universal morphism  $f$  from  $\mathcal{C}_e$  to  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$  by descent.

Now we define a (quasi left)  $\mathbb{C}^*$ -action on  $\mathcal{C}_e$  such that  $f$  is  $\mathbb{C}^*$ -equivariant. The  $\mathbb{C}^*$ -action on  $\mathcal{C}_e$  is induced by the  $\mathbb{C}^*$ -action on  $AY^{ss}(\theta)^g \times \mathbb{C}^* \times U$ :

$$\begin{aligned} m : \mathbb{C}^* \times AY^{ss}(\theta)^g \times \mathbb{C}^* \times U &\rightarrow AY^{ss}(\theta) \times \mathbb{C}^* \times U, \\ t \cdot (\vec{x}, v, (x, y)) &= (\vec{x}, v, (x, t^{\frac{-1}{ar\delta(e)}}y)). \end{aligned}$$

Note then  $\pi$  is  $\mathbb{C}^*$ -equivariant map, where  $\mathcal{M}_e$  is equipped with the trivial  $\mathbb{C}^*$ -action. By the universal property of the projectivized bundle  $\mathcal{C}_e$  over  $\mathcal{M}_e$ , one has a tautological section

$$(x, y) \in H^0(\mathcal{C}_e, (M_1 \otimes \pi^*\mathcal{R}) \oplus (M_2 \otimes \mathbb{C}_{\frac{-1}{ar\delta(e)}})),$$

which is also a  $\mathbb{C}^*$ -invariant section.

Now we can check that  $f$  is a  $\mathbb{C}^*$ -equivariant morphism from  $\mathcal{C}_e$  to  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$  with respect to the  $\mathbb{C}^*$ -actions for  $\mathcal{C}_e$  and  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ . Using Remark 4.1,  $f$  is given by the following data:

- (1)  $k + p + 1$   $\mathbb{C}^*$ -equivariant line bundles  $\mathcal{C}_e$ :

$$\begin{aligned} \mathcal{L}_j &:= \pi^*L_{\pi_j} \otimes (M_1^\vee \otimes M_2^{\otimes r})^{p_j}, 1 \leq j \leq k, \\ \mathcal{L}_{k+j} &:= \mathbb{C}, 1 \leq j \leq p \end{aligned}$$

and

$$\mathcal{N} := M_2^{a\delta(e)} \otimes \mathbb{C}_{\frac{-\lambda}{r}},$$

where  $(L_{\pi_j})_{1 \leq j \leq k}$  are the standard  $\mathbb{C}^*$ -equivariant line bundles on  $\mathcal{M}_e$  by the Borel contribution,  $M_1, M_2$  are the standard  $\mathbb{C}^*$ -equivariant line bundles on  $\mathcal{C}_e$  by the Borel construction;

- (2) a universal section

$$(4.15) \quad \begin{aligned} (\vec{x}, \vec{y}, (\zeta_1, \zeta_2)) &:= ((x_1, \dots, x_n), 1^p, (v^{-1}x^{a\delta(e)}, y^{a\delta(e)})) \\ &\in H^0(\mathcal{C}_e, \bigoplus_{i=1}^n \mathcal{L}_{\rho_j} \oplus (\bigoplus_{j=1}^p \mathcal{L}_{k+j}) \oplus (\mathcal{L}_{-\theta_p} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda) \oplus \mathcal{N})^{\mathbb{C}^*}. \end{aligned}$$

Use the similar analysis as in the last section, we have that  $[\mathcal{M}_e]^{vir} = [\mathcal{M}_e]$  and the virtual normal bundle

$$e^{\mathbb{C}^*}(N^{\text{vir}}) = \prod_{m=1}^{\delta(e)} \left( \frac{m}{\delta(e)} (-D_\theta + \lambda) \right).$$

when  $r$  is a sufficiently large prime with the exception that the movable part of infinitesimal automorphisms of  $C_e$  contributes

$$\frac{\delta(e)}{\lambda - D_\theta}$$

to the virtual normal bundle when  $a = 1$ .

**4.3.4. Node contributions.** The deformations in  $Q_{0,\vec{m}}^{\bar{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, \frac{\delta}{r}))$  smoothing a node contribute to the Euler class of the virtual normal bundle as the first Chern class of the tensor product of the two cotangent line bundles at the branches of the node. For nodes at which a component  $C_e$  meets a component  $C_v$  over the vertex 0, this contribution is

$$\frac{\lambda - D_\theta}{a\delta(e)} - \frac{\bar{\psi}_v}{a};$$

for nodes at which a component  $C_e$  meets a component  $C_v$  over the vertex  $\infty$ , this contribution is

$$\frac{-\lambda + D_\theta}{ar\delta(e)} - \frac{\bar{\psi}_v}{ar};$$

for nodes at which two edge component  $C_e$  meets with a vertex  $v$  over 0, the node-smoothing contribution is

$$\frac{\lambda - D_\theta}{a\delta(e)} + \frac{\lambda - D_\theta}{a\delta(e')}.$$

The nodes at which two edge component  $C_e$  meets with a vertex  $v$  over  $\infty$  will not occur using a similar argument in [JPPZ17, Lemma 6] when  $r$  is sufficiently large.

As for the node contributions from the normalization exact sequence of relative obstruction theory (4.7), each node  $q$  (specified by a vertex  $v$ ) contributes the inverse of Euler class of

$$(4.16) \quad (R^0\pi_*(\mathcal{L}_\theta^\vee \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda)|_q)^{\text{mov}} \oplus (R^0\pi_*\mathcal{N}|_q)^{\text{mov}}$$

to the Euler class of the virtual normal bundle.

In the case where  $j(v) = 0$ ,  $z_2|_q = 1$  gives a trivialization of  $\mathcal{N}$  at  $q$ . Thus, the second factor in (4.16) is trivial, while the Euler class of the first factor equals

$$\frac{1}{\lambda - D_\theta}.$$

In the case where  $j(v) = \infty$ ,  $z_1|_q = 1$  gives a trivialization of the fiber  $(\mathcal{L}_\theta^\vee \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda)|_q$ . Hence we have  $\mathcal{N}|_q \cong \mathcal{L}_\theta^{\frac{1}{r}}|_q \otimes \mathbb{C}_{-\frac{\lambda}{r}}$ , this implies that it  $R^0\pi_*(\mathcal{N}|_q) = 0$  because of the nontrivial stacky structure when  $r$  is sufficiently large. Thus there is no localization contribution from the normalization sequence at the node over  $\infty$ .



**4.4. Total localization contributions.** For each decorated graph  $\Gamma$ , denote the moduli  $F_\Gamma$  to be the fiber product

$$\prod_{v:j(v)=0} \mathcal{M}_v \times_{\bar{I}_\mu Y} \prod_{e \in E} \mathcal{M}_e \times_{\bar{I}_\mu \sqrt[r]{L_\theta/Y}} \prod_{v:j(v)=\infty} \mathcal{M}_v$$

of the following diagram:

$$\begin{array}{ccc} F_\Gamma & \longrightarrow & \prod_{v:j(v)=0} \mathcal{M}_v \times \prod_{e \in E} \mathcal{M}_e \times \prod_{v:j(v)=\infty} \mathcal{M}_v \\ \downarrow & & \downarrow \text{ev}_{h_0}, \text{ev}_{p_0}, \text{ev}_{p_\infty}, \text{ev}_{h'_\infty} \\ \prod_E (\bar{I}_\mu Y \times \bar{I}_\mu \sqrt[r]{L_\theta/Y}) & \xrightarrow{(\Delta \times \Delta^{\frac{1}{r}})^{|E|}} & \prod_E (\bar{I}_\mu Y)^2 \times (\bar{I}_\mu \sqrt[r]{L_\theta/Y})^2, \end{array}$$

where  $\Delta = (id, \iota)$  (resp.  $\Delta^{\frac{1}{r}} = (id, \iota)$ ) is the diagonal map of  $\bar{I}_\mu Y$  (resp.  $\bar{I}_\mu \sqrt[r]{L_\theta/Y}$ ), where  $\iota$  is the involution on  $\bar{I}_\mu Y$  (resp.  $\bar{I}_\mu \sqrt[r]{L_\theta/Y}$ ) by taking the inverse of the band of gerbe structure. Here when  $v$  is a stable vertex, the vertex moduli  $\mathcal{M}_v$  is described in 4.3.1; when  $v$  is an unstable vertex over 0, we treat  $\mathcal{M}_v := \bar{I}_{m(h)} Y$  with the identical virtual cycle, where  $h$  is the half-edge incident to  $v$ ; when  $v$  is an unstable vertex over  $\infty$ , we treat  $\mathcal{M}_v := \bar{I}_{m(h)} \sqrt[r]{L_\theta/Y}$  with the identical virtual cycle, where  $h$  is the half-edge incident to  $v$ .

We define  $[F_\Gamma]^{\text{vir}}$  to be:

$$\prod_{v:j(v)=0} [\mathcal{M}_v]^{\text{vir}} \times_{\bar{I}_\mu Y} \prod_{e \in E} [\mathcal{M}_e]^{\text{vir}} \times_{\bar{I}_\mu \sqrt[r]{L_\theta/Y}} \prod_{v:j(v)=\infty} [\mathcal{M}_v]^{\text{vir}}.$$

where the fiber product over  $\bar{I}_\mu Y$  and  $\bar{I}_\mu \sqrt[r]{L_\theta/Y}$  imposes that the evaluation maps at the two branches of each node (Here we adopt the convention that a node can link a unstable vertex and an edge.) agree. Then the contribution of decorated graph  $\Gamma$  to the virtual localization is:

$$(4.17) \quad \text{Cont}_\Gamma = \frac{\prod_{e \in E} a_e}{\text{Aut}(\Gamma)} (\iota_\Gamma)^* \left( \frac{[F_\Gamma]^{\text{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\text{vir}})} \right).$$

Here  $\iota_F : F_\Gamma \rightarrow Q_{0, \vec{m}}^\theta(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}, p}, (\beta, \frac{\delta}{r}))$  is a finite etale map of degree  $\frac{\text{Aut}(\Gamma)}{\prod_{e \in E} a_e}$  into the corresponding  $\mathbb{C}^*$ -fixed loci in twisted graph space. The virtual normal bundle  $e^{\mathbb{C}^*}(N_\Gamma^{\text{vir}})$  is the product of virtual normal bundles from vertex contributions, edge contributions and node contributions.

**Remark 4.9.** Let  $u$  be a polynomial on  $c_1(L_{\pi_1}), \dots, c_1(L_{\pi_k})$ . In the contribution from the graph  $\Gamma$ , assume that  $j \in J_e$  for some edge  $e$ , then  $\hat{e}v_j|_{F_\Gamma}$  factors through the projection from  $F_\Gamma$  to  $\mathcal{M}_e$ . By abusing notations, we denote  $\hat{e}v_j : \mathcal{M}_e \rightarrow \mathfrak{Y}$ . Thus when we want to apply virtual localization to  $\prod_{j=1}^p \hat{e}v_j^*(g(c_1(L_{\pi_k})))$ , we replace  $\mathbb{I}_\beta(q, z)$  in the edge contribution in Lemma 4.8 by  $g(c_1(L_{\pi_k}) + \beta(L_{\pi_k})z)^{|J_e|} \mathbb{I}_\beta(q, z)$ . Indeed, use the setting in §4.3.2, let  $\underline{e}v := \text{pr}_{r,p} \circ ev : \mathcal{C}_e \rightarrow \mathfrak{Y}$ , where  $\text{pr}_{r,p} : \mathbb{P}\mathfrak{Y}^{\frac{1}{r}, p} \rightarrow \mathfrak{Y}$  is the natural projection map. Then we have

$$\underline{e}v^*(L_\tau) = \pi^*(L_\tau \otimes \mathcal{R}^{a\beta(L_\tau)}) \otimes \mathcal{O}_{\mathcal{C}_e}(ar\beta(L_\tau))$$

for any character  $\tau$  of  $G$ . Let  $D_0$  be the zero section of  $\mathcal{C}_e$  over  $\mathcal{M}_e$  given by  $x = 0$ . Then  $\hat{e}v_j = \underline{e}v|_{D_0}$ . Note that  $\mathcal{O}_{\mathcal{C}_e}(1)|_{D_0} = \mathbb{C}_{\frac{1}{ar\delta(e)}}$ . Using the fact  $\mathcal{R}^{a\delta(e)} = L_{-\theta}$ , we have  $c_1(\hat{e}v^*(L_\tau)) = c_1(L_\tau) + \frac{\beta(L_\tau)(\lambda - D_\theta)}{\delta(e)}$ .

## 5. MASTER SPACE II

**5.1. Construction of master space II.** Fix two different primes  $r, s \in \mathbb{N}$ , let  $\theta$  be as in previous section, let  $\mathbb{P}Y_{r,s}$  be the root stack of the  $\mathbb{P}^1$  bundle  $\mathbb{P}_Y(\mathcal{O}(-D_\theta) \oplus \mathcal{O})$  over  $Y$  by taking the  $s$ -th root of the zero section ( $z_1 = 0$ ) and  $r$ -th root of the infinity section ( $z_2 = 0$ ). Then the zero section  $\mathcal{D}_0 \subset \mathbb{P}Y_{r,s}$  is isomorphic to  $\sqrt[s]{L_{-\theta}/Y}$ , and the infinity section  $\mathcal{D}_\infty \subset \mathbb{P}Y_{r,s}$  is isomorphic to  $\sqrt[r]{L_\theta/Y}$ .

We give a more concrete presentation of  $\mathbb{P}Y_{r,s}$  as a quotient stack:

$$\mathbb{P}Y_{r,s} = [(\mathbb{C}^* \times AY^{ss}(\theta) \times U) / (G \times \mathbb{C}_\alpha^* \times \mathbb{C}_t^*)],$$

where the (right)  $G \times \mathbb{C}_\alpha^* \times \mathbb{C}_t^*$ -action on  $\mathbb{C}^* \times AY^{ss}(\theta) \times U$  is given by:

$$(u, \vec{x}, z_1, z_2) \cdot (g, \alpha, t) = (\alpha^{-s}\theta(g)^{-1}t^r u, \vec{x}g, \alpha z_1, tz_2),$$

for  $(g, \alpha, t) \in G \times \mathbb{C}_\alpha^* \times \mathbb{C}_t^*$ , and  $(u, \vec{x}, z_1, z_2) \in \mathbb{C}^* \times AY^{ss}(\theta) \times U$ . Here  $U = \mathbb{C}^2 \setminus \{0\}$ . This quotient stack presentation of  $\mathbb{P}Y_{r,s}$  comes from the root stack construction in [AGV08, Appendix B] after some simplification.

The inertia stack  $I_\mu \mathbb{P}Y_{r,s}$  of  $\mathbb{P}Y_{r,s}$  admits a decomposition

$$I_\mu \mathbb{P}Y \sqcup \bigsqcup_{i=1}^{s-1} \sqrt[s]{L_{-\theta}/Y} \sqcup \bigsqcup_{j=1}^{r-1} \sqrt[r]{L_\theta/Y}.$$

Let  $(\vec{x}, (g, \alpha, t))$  be a point of the inertia stack  $I_\mu \mathbb{P}Y_{r,s}$ , if the point  $(\vec{x}, (g, \alpha, t))$  appears in the first factor of the decomposition above, then the automorphism  $(g, \alpha, t)$  lies in  $G \times \{1\} \times \{1\}$ ; if the point  $(\vec{x}, (g, \alpha, t))$  occurs in the second factor of the decomposition above, then the automorphism  $(g, \alpha, t)$  lies in  $G \times \{\mu_s^i : 1 \leq i \leq s-1\} \times \{1\} \subset G \times \mathbb{C}_\alpha^* \times \mathbb{C}_t^*$ , and the point  $\vec{x}$  is in the zero section  $\mathcal{D}_0$  defined by  $z_1 = 0$ ; finally if the point  $(\vec{x}, (g, \alpha, t))$  shows in the third factor of the decomposition above, then the automorphism  $(g, \alpha, t)$  lies in  $G \times \{1\} \times \{\mu_r^j : 1 \leq j \leq r-1\} \subset G \times \mathbb{C}_\alpha^* \times \mathbb{C}_t^*$ , and  $\vec{x}$  is in the infinity section  $\mathcal{D}_\infty$  defined by  $z_2 = 0$ . Here  $\mu_r = \exp(\frac{2\pi\sqrt{-1}}{r}) \in \mathbb{C}^*$  and  $\mu_s = \exp(\frac{2\pi\sqrt{-1}}{s}) \in \mathbb{C}^*$ .

Fix  $(g, \alpha, t) \in G \times \mu_s \times \mu_r$ , we will use the notation  $\bar{I}_{(g, \alpha, t)} \mathbb{P}Y_{r,s}$  to mean the rigidified inertia stack component of  $\bar{I}_\mu \mathbb{P}Y_{r,s}$  which has automorphism  $(g, \alpha, t)$ . Note if  $\alpha$  and  $t$  are not equal to 1 simultaneously, then the corresponding rigidified inertia stack component is empty.

Let  $\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r}))$  be the moduli stack of  $m$ -pointed twisted stable maps to  $\mathbb{P}Y_{r,s}$  of degree  $(d, \frac{\delta}{r})$ . More concretely,

$$\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r})) = \{(C; q_1, \dots, q_m; L_1, \dots, L_k, N_1, N_2; u, \vec{x} := (x_1, \dots, x_m), z_1, z_2)\},$$

where  $(C; q_1, \dots, q_m)$  is a  $m$ -pointed prestable balanced twisted curve of genus 0 with nontrivial isotropy only at special points,  $(L_j : 1 \leq j \leq k)$  and  $N_1, N_2$  are orbifold line bundles on  $C$  with

$$\deg([\vec{x}]) = d \in \text{Hom}(\text{Pic}(\mathfrak{Y}), \mathbb{Q}), \quad \deg(N_2) = \frac{\delta}{r},$$

and

$$(u, (\vec{x}, \vec{z})) := (u, x_1, \dots, x_n, z_1, z_2) \in \Gamma \left( ((N_1^\vee)^{\otimes s} \otimes L_{-\theta} \otimes N_2^{\otimes r}) \oplus \bigoplus_{i=1}^n L_{\rho_i} \oplus N_1 \oplus N_2 \right).$$

Here, for  $1 \leq i \leq n$ , the line bundle  $L_{\rho_i}$  is equal to

$$\bigotimes_{j=1}^k L_j^{m_{ij}},$$

where  $(m_{ij})_{1 \leq i \leq n, 1 \leq j \leq k}$  is given by the relation  $\rho_i = \sum_{j=1}^k m_{ij} \pi_j$ . The same construction applies to the line bundle  $L_{-\theta}$  on  $C$ . Note here  $\delta$  is an integer when  $\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r}))$  is nonempty as  $N_2^{\otimes r}$  is the pullback of some line bundle on the coarse moduli curve  $\underline{C}$ .

We require this data to satisfy the following conditions:

- *Representability:* For every  $q \in C$  with isotropy group  $G_q$ , the homomorphism  $\mathbb{B}G_q \rightarrow \mathbb{B}(G \times \mathbb{C}_\alpha^* \times \mathbb{C}_t^*)$  given by the restriction of line bundles  $(L_j : 1 \leq j \leq k)$  and  $N_1, N_2$  on  $q$  is representable.
- *Nondegeneracy:* The sections  $z_1$  and  $z_2$  never simultaneously vanish, and we have

$$(5.1) \quad \text{ord}_q(\vec{x}) = 0.$$

for all  $q \in C$ . Furthermore, the section  $u$  never vanish, so we have  $(N_1^\vee)^{\otimes s} \otimes L_{-\theta} \otimes N_2^{\otimes r} \cong \mathcal{O}_C$ .

- *Stability:* the map  $[u, \vec{x}, \vec{z}] : (C, q_1, \dots, q_m) \rightarrow \mathbb{P}Y_{r,s}$  satisfies the usual stability condition defined by a twisted stable map;
- *Vanishing:* The image of  $[\vec{x}] : C \rightarrow \mathfrak{X}$  lies in  $\mathfrak{Y}$ .

Let  $\vec{m} = (v_1, \dots, v_m) \in (G \times \mu_s \times \mu_r)^m$ , we will denote  $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r}))$  to be:

$$\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r})) \cap \text{ev}_1^{-1}(\bar{I}_{v_1} \mathbb{P}Y_{r,s}) \cap \dots \cap \text{ev}_m^{-1}(\bar{I}_{v_m} \mathbb{P}Y_{r,s}),$$

where

$$\text{ev}_i : \mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r})) \rightarrow \bar{I}_{\mu} \mathbb{P}Y_{r,s},$$

are natural evaluation maps as before, by evaluating the sections  $(u, \vec{x}, \vec{z})$  at  $q_i$ . Moreover, since  $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r}))$  is a moduli space of stable quasimaps to a lci GIT quotient, the results of [CFKM14] imply that it is a proper Deligne–Mumford stack equipped with a natural perfect obstruction theory relative to the stack  $\mathcal{D}_{0,m,d,\frac{\delta}{r}}$  of twisted curves equipped with line bundles  $(L_j : 1 \leq j \leq k)$  and  $N_1, N_2$ . This obstruction theory is of the form

$$(5.2) \quad R^\bullet \pi_*(u^* \mathbb{R}T_\rho).$$

Here, we denote the universal family over  $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r}))$  by

$$\begin{array}{ccc} \mathcal{V} & \xleftarrow[u]{\rho} & \mathcal{C} \\ & & \downarrow \pi \\ & & \mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r})), \end{array}$$

where  $\mathcal{L}_{\rho_i}$  and  $\mathcal{N}_1, \mathcal{N}_2$  are the universal line bundles and

$$\mathcal{V} \subset ((\mathcal{N}_1^\vee)^{\otimes s} \otimes \mathcal{L}_{-\theta} \otimes \mathcal{N}_2^{\otimes r}) \oplus \bigoplus_{i=1}^n L_{\rho_i} \oplus \mathcal{N}_1 \oplus \mathcal{N}_2$$

is the subsheaf of sections taking values in the affine cone of  $Y$ . Somewhat more explicitly, (4.6) equals

$$(5.3) \quad R^\bullet \pi_*((\mathcal{N}_1^\vee)^{\otimes s} \otimes \mathcal{L}_{-\theta} \otimes \mathcal{N}_2^{\otimes r}) \oplus \mathbb{E} \oplus R^\bullet \pi_*(\mathcal{N}_1 \oplus \mathcal{N}_2),$$

in which the sub-obstruction-theory  $\mathbb{E}$  comes from the deformations and obstructions of the sections  $\vec{x}$ .

**5.2.  $\mathbb{C}^*$ -action and fixed loci.** Define a (left)  $\mathbb{C}^*$ -action on  $\mathbb{C}^* \times AY^{ss}(\theta) \times U$  given by

$$t \cdot (u, \vec{x}, (z_1, z_2)) = (tu, \vec{x}, (z_1, z_2)) .$$

This action descends to be a (left)  $\mathbb{C}^*$ -action on  $\mathbb{P}Y_{r,s}$ , which induces a  $\mathbb{C}^*$ -action on  $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r}))$ . The reason why we define this action is that this definition lifts the  $\mathbb{C}^*$ -action on  $\mathbb{P}Y$  defined in §4.1 along the canonical structure map  $\pi_{r,s} : \mathbb{P}Y_{r,s} \rightarrow \mathbb{P}Y$ . We will let  $\lambda$  to be equivariant parameter corresponding to the action of weight 1.

First we state a similar criteria for maps to  $\mathbb{P}Y_{r,s}$  to be  $\mathbb{C}^*$ -equivariant as in Remark 4.1.

**Remark 5.1.** Given a stack  $S$  over  $\text{Spec}(\mathbb{C})$  equipped with a (left)  $\mathbb{C}^*$ -action, then a  $\mathbb{C}^*$ -equivariant morphism from  $S$  to  $\mathbb{P}Y_{r,s}$  is equivalent to that the following: there exists  $k+2$   $\mathbb{C}^*$ -equivariant line bundles on  $S$

$$L_1, \dots, L_k, N_1, N_2$$

together with sections invariant under the  $\mathbb{C}^*$ -action;

$$(5.4) \quad (u, \vec{x}, \vec{z}) := (u, x_1, \dots, x_n, z_1, z_2) \\ \in \Gamma \left( ((N_1^\vee)^{\otimes s} \otimes L_{-\theta} \otimes N_2^{\otimes r} \otimes \mathbb{C}_\lambda) \oplus \bigoplus_{i=1}^n L_{\rho_i} \oplus N_1 \oplus N_2 \right)^{\mathbb{C}^*} .$$

Here the line bundles  $L_{\rho_i}$  and  $L_{-\theta}$  are defined similarly as before. The sections should satisfy the vanishing condition imposed by the affine cone of  $Y$  in the definition of stable maps to  $\mathbb{P}Y_{r,s}$ .

Fix a nonzero degree  $\beta \in \text{Eff}(W, G, \theta)$ , and two tuples of nonnegative integers

$$(\delta_1, \dots, \delta_m) \in \mathbb{N}^m$$

and

$$(\delta'_1, \dots, \delta'_m) \in \mathbb{N}^m .$$

Consider the tuple of multiplicities  $\vec{m} = (v_1, \dots, v_m) \in (G \times \boldsymbol{\mu}_r)^m$ , where  $v_i = (g_i, \mu_s^{\delta'_i}, \mu_r^{\delta_i})$ , we will denote  $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  to be

$$\bigsqcup_{\substack{d \in \text{Eff}(AY, G, \theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta}} \mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r})) ,$$

where  $i_{\mathfrak{Y}} : \mathfrak{Y} \rightarrow \mathfrak{X}$  is the inclusion morphism. Thus  $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  inherits a  $\mathbb{C}^*$ -action as above.

We will follow the presentation of [CJR17b, CJR17a] to describe the virtual localization for  $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  similar to  $Q_{0,\vec{m}}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, \frac{\delta}{r}))$ , but the edge contribution is easier to analyze as there is no basepoint occurring for twisted stable maps.

We index the components of  $\mathbb{C}^*$ -fixed loci of  $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  by decorated graphs. A decorated graph  $\Gamma$  consists of vertices, edges, and  $m$  legs with the following decorations it:

- Each vertex  $v$  is associated with an index  $j(v) \in \{0, \infty\}$ , and a degree  $\beta(v) \in \text{Eff}(W, G, \theta)$ .
- Each edge  $e = \{h, h'\}$  is equipped with a degree  $\delta(e) \in \mathbb{N}$ .
- Each half-edge  $h$  and each leg  $l$  has an element  $m(h)$  or  $m(l)$  in  $G \times \mu_s \times \mu_r$ .
- The legs are labeled with the numbers  $\{1, \dots, m\}$ .

By the “valence” of a vertex  $v$ , denoted  $\text{val}(v)$ , we mean the total number of incident half-edges, including legs.

The fixed locus in  $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  indexed by the decorated graph  $\Gamma$  parameterizes stable map of the following type:

- Each edge  $e$  corresponds to a genus-zero component  $C_e$  on which  $\deg(N_2) = \frac{\delta(e)}{r}$  for some integer  $\delta(e) \in \mathbb{Z}_{>0}$ , where there are two distinguished points  $q_0$  and  $q_\infty$  on  $C_e$  satisfying that  $z_2|_{q_\infty} = 0$  and  $z_1|_{q_0} = 0$ , respectively. We call them the “ramification points”. Note that we have  $\deg(L_j|_{C_e}) = 0$  for all  $1 \leq j \leq k$ .
- Each vertex  $v$  for which  $j(v) = 0$  (with unstable exceptional cases noted below) corresponds to a maximal sub-curve  $C_v$  of  $C$  over which  $z_1 \equiv 0$ , then the restriction of  $(C; q_1, \dots, q_m; L_1, \dots, L_k; \vec{x})$  to  $C_v$  defines a twisted stable map in

$$\mathcal{K}_{0,\text{val}(v)}(\sqrt[s]{L_{-\theta}/Y}, \beta(v)) := \bigsqcup_{\substack{d \in \text{Eff}(AY, G, \theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta(v)}} \mathcal{K}_{0,\text{val}(v)}(\sqrt[s]{L_{-\theta}/Y}, d) .$$

Each vertex  $v$  for which  $j(v) = \infty$  (again with unstable exceptions) corresponds to a maximal sub-curve for which  $z_2 \equiv 0$ , then the restriction of  $(C; q_1, \dots, q_m; L_1, \dots, L_k; \vec{x})$  to  $C_v$  defines a twisted stable map in

$$\mathcal{K}_{0,\text{val}(v)}(\sqrt[r]{L_{\theta}/Y}, \beta(v)) := \bigsqcup_{\substack{d \in \text{Eff}(AY, G, \theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta(v)}} \mathcal{K}_{0,\text{val}(v)}(\sqrt[r]{L_{\theta}/Y}, d) .$$

The label  $\beta(v)$  denotes the degree coming from the restriction  $[x]|_{C_v} : C_v \rightarrow \mathfrak{X}$ . Note here we count the degree  $\beta(v)$  in  $\text{Eff}(W, G, \theta)$ , but not in  $\text{Eff}(AY, G, \theta)$ .

- A vertex  $v$  is *unstable* if stable twisted maps of the type described above do not exist (where, as always, we interpret legs as marked points and half-edges as half-nodes). In this case,  $v$  corresponds to a single point of the component  $C_e$  for each adjacent edge  $e$ , which may be a node at which  $C_e$  meets  $C_{e'}$ , a marked point of  $C_e$ , or an unmarked point.
- The index  $m(l)$  on a leg  $l$  indicates the rigidified inertia stack component  $\bar{I}_{m(l)}\mathbb{P}Y_{r,s}$  of  $\mathbb{P}Y_{r,s}$  on which the marked point corresponding to the leg  $l$  is

evaluated, this is determined by the multiplicity of  $L_1, \dots, L_k, N_1, N_2$  at the corresponding marked points.

- A half-edge  $h$  incident to a vertex  $v$  corresponds to a node at which components  $C_e$  and  $C_v$  meet, and  $m(h)$  indicates the rigidified inertia component  $\bar{I}_{m(h)} \mathbb{P}Y_{r,s}$  of  $\mathbb{P}Y_{r,s}$  on which the node on  $C_v$  is evaluated, this is determined by the multiplicity of  $L_1, \dots, L_k, N_1, N_2$  at the corresponding node. If  $v$  is unstable and hence  $h$  corresponds to a single point on a component  $C_e$ , then  $m(h)$  is the *inverse* in  $G \times \mu_s \times \mu_r$  of the multiplicity of  $L_1, \dots, L_k, N_1, N_2$  at this point.

In particular, we note that the decorations at each stable vertex  $v$  yield a vector

$$\vec{m}(v) \in (G \times \mu_s \times \mu_r)^{\text{val}(v)}$$

recording the multiplicities of  $L_1, \dots, L_k, N_1, N_2$  at every special point of  $C_v$ .

**Remark 5.2.** For each edge  $e$ , the restriction of  $\vec{x}$  to  $C_e$  defines a constant map to  $Y$ . So the restriction of  $(u, \vec{x}, \vec{z})$  to  $C_e$  defines a representable map

$$f : C_e \rightarrow \mathbb{B}G_y \times \mathbb{P}_{r,s}$$

where  $y \in Y$  comes from  $\vec{x}$  and  $G_y$  is the isotropy group of  $y \in Y$ . Then we have  $m(q_0) = (g^{-1}, \mu_s^{\delta(e)}, 1)$  and  $m(q_\infty) = (g, 1, \mu_r^{\delta(e)})$  for some  $g \in G_y$ . Denote  $a$  to be the order of element  $g \in G$ . Note when  $r$  and  $s$  are sufficiently large primes comparing to  $\delta(e)$ , we must have  $C_e \cong \mathbb{P}_{ar,as}^1$  and  $q_0$  and  $q_\infty$  are special points as they are nontrivial stacky points. Here  $\mathbb{P}_{ar,as}^1$  is the unique Deligne-Mumford stack with coarse moduli  $\mathbb{P}^1$ , isotropy group  $\mu_{as}$  at  $0 \in \mathbb{P}^1$ , isotropy group  $\mu_{ar}$  at  $\infty \in \mathbb{P}^1$ , and generic trivial stabilizer.

**5.3. Localization analysis.** Fix  $\beta \in \text{Eff}(W, G, \theta)$ ,  $\delta \in \mathbb{Z}_{\geq 0}$  and  $\vec{m} = (v_1, \dots, v_m) \in (G \times \mu_s \times \mu_r)^m$ , we will consider the space  $\mathcal{K}_{0, \vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ . The reason why we assume that the second degree is  $\frac{\delta}{r}$  is that  $\mathcal{K}_{0, [m]}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  admits a natural morphism to  $\mathcal{K}_{0, [m]}(\mathbb{P}Y, (\beta, \delta))$  (c.f. [AJT15, TT16]). Here  $\mathbb{P}Y$  is equal to  $\mathbb{P}Y_{r,s}$  for  $r = s = 1$ . In this section, we will always assume that  $r$  and  $s$  are *sufficiently large primes*.

Now we analyze the  $\mathbb{C}^*$ -localization contribution for  $\mathcal{K}_{0, \vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  as in §4.3.

**5.3.1. Vertex contributions.** The analysis of localization contribution for the stable vertex  $v$  is similar to the analysis in §4.3.1.

For the stable vertex  $v$  over  $\infty$ , the vertex moduli  $\mathcal{M}_v$  corresponds to the moduli stack

$$\mathcal{K}_{0, \vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v)) := \bigsqcup_{\substack{d \in \text{Eff}(AY, G, \theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta(v)}} \mathcal{K}_{0, \vec{m}(v)}(\sqrt[r]{L_\theta/Y}, d),$$

which parameterizes twisted stable maps to the root gerbe  $\sqrt[r]{L_\theta/Y}$  over  $Y$ .

Let

$$\pi : \mathcal{C}_\infty \rightarrow \mathcal{K}_{0, \vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))$$

be the universal curve over  $\mathcal{K}_{0, \vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))$ . Follow the same discussion in §4.3.1, the *inverse of the Euler class* of the virtual normal bundle for the vertex moduli  $\mathcal{M}_v$  over  $\infty$  is equal to

$$e^{\mathbb{C}^*}((-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}}) \otimes \mathbb{C}_{-\frac{\lambda}{r}}).$$

When  $r$  is a sufficiently large prime, following [JPPZ18], the above Euler class has a representation

$$\sum_{d \geq 0} c_d(-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}}) \left( \frac{-\lambda}{r} \right)^{|E(v)|-1-d}.$$

Here the virtual bundle  $-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}}$  has virtual rank  $|E(v)| - 1$ , where  $|E(v)|$  is the number of edges incident to the vertex  $v$ . The fixed part of the obstruction theory contributes to the virtual cycle

$$[\mathcal{K}_{0, \vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))]^{\text{vir}}.$$

For the stable vertex  $v$  over 0, the vertex moduli  $\mathcal{M}_v$  corresponds to the moduli space

$$\mathcal{K}_{0, \vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v)) := \bigsqcup_{\substack{d \in \text{Eff}(AY, G, \theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta(v)}} \mathcal{K}_{0, \vec{m}(v)}(\sqrt[r]{L_\theta/Y}, d).$$

Let

$$\pi : \mathcal{C}_0 \rightarrow \mathcal{K}_{0, \vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))$$

be the universal curve over  $\mathcal{K}_{0, \vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))$ , and  $f : \mathcal{C}_0 \rightarrow \sqrt[r]{L_\theta/Y}$ . In this case, the fixed part of the perfect obstruction theory for the vertex moduli over 0 yields the virtual cycle

$$[\mathcal{K}_{0, \vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v))]^{\text{vir}}.$$

Note  $\mathcal{N}_2|_{\mathcal{C}_0} \cong \mathcal{O}_{\mathcal{C}_0}$  as  $z_2|_{\mathcal{C}_0} \equiv 1$ , the virtual normal bundle comes from the movable part of the infinitesimal deformations of  $z_1$ , which is a section of the line bundle  $\mathcal{L}_{-\theta}^{\frac{1}{s}}$  over  $\mathcal{C}_0$ , which is the pullback of the universal  $s$ -th root line bundle on  $\sqrt[r]{L_\theta/Y}$  via the universal map  $f$ . Then the *inverse of the Euler class* of the virtual normal bundle is equal to

$$e^{\mathbb{C}^*}((-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{s}}) \otimes \mathbb{C}_{\frac{\lambda}{s}}).$$

We will simplify the above presentation when  $\beta(v) \neq 0$ . First, we will state a simple vanishing lemma regarding a line bundle of negative degree on a genus zero twisted curve, of which the proof is proceeded by induction on the number of connected components.

**Lemma 5.3.** *Let  $L$  be a line bundle of negative degree on a genus zero twisted curve  $C$ . Assume that the degree of the restriction of the line bundle  $L|_{C_i}$  to every irreducible component  $C_i$  is non-positive. Then we have  $H^0(C, L) = 0$ .*

**Remark 5.4.** For every fiber curve  $C_0$  of the universal curve  $\mathcal{C}_0$  over  $\mathcal{M}_v$ . The degree of the restricted line bundle  $\mathcal{L}_{-\theta}^{\frac{1}{s}}|_{C_0}$  to  $C_0$  is non-positive. Indeed,  $\mathcal{L}_{-\theta}^{\frac{1}{s}}$  is the pullback of the  $s$ -th root of the line bundle  $L_\theta$  on  $\sqrt[r]{L_\theta/Y}$ , where  $L_\theta$  is the pullback of an *anti-ample* line bundle on the coarse moduli of  $\sqrt[r]{L_\theta/Y}$ . Now assuming  $\beta(v) \neq 0$ , we have the degree of the restricted line bundle  $\mathcal{L}_{-\theta}^{\frac{1}{s}}|_{C_0}$  is negative by Lemma 2.5. By the above lemma, one has

$$R^0 \pi_* \mathcal{L}_{-\theta}^{\frac{1}{s}} = 0.$$

Then we have

$$-R^\bullet \pi_* \mathcal{L}_{-\theta}^{\frac{1}{s}} = R^1 \pi_* \mathcal{L}_{-\theta}^{\frac{1}{s}} ,$$

which implies that  $R^1 \pi_* \mathcal{L}_{-\theta}^{\frac{1}{s}}$  is a vector bundle. When  $s$  is sufficiently large, it has rank  $|E(v)| - 1$  where  $|E(v)|$  is the number of edges incident to the vertex  $v$ . Especially when  $|E(v)| = 1$ , it has rank 0, thus the Euler class becomes 1, this case will be important in the later simplification of the localization contribution in §6.2.

**5.3.2. Edge contributions.** Assume that the multiplicity at  $q_\infty \in C_e$  is equal to  $(g, \mu_s^{\delta(e)}, 1) \in G \times \mu_s \times \mu_r$  and  $a$  (or  $a_e$ ) is the order of  $g \in G$ . When  $r, s$  is sufficiently large primes, due to Remark 5.2,  $C_e$  must be isomorphic to  $\mathbb{P}_{ar,as}^1$  where the ramification point  $q_0$  for which  $z_1 = 0$  is isomorphic to  $\mathbb{B}\mu_{as}$ , and the ramification point  $q_\infty$  for which  $z_2 = 0$  is isomorphic to  $\mathbb{B}\mu_{ar}$ . The restriction of the degree  $(\beta, \frac{\delta}{r})$  from  $C$  to  $C_e$  is equal to  $(0, \frac{\delta(e)}{r})$ , which is equivalent to:

$$\deg(L_j|_{C_e}) = 0, \quad \text{for } 1 \leq j \leq k, \quad \deg(N_2|_{C_e}) = \frac{\delta(e)}{r} .$$

Recall that the inertia stack component  $I_g Y$  of  $I_\mu Y$  is isomorphic to

$$[AY^{ss}(\theta)^g/G] .$$

We define the edge moduli  $\mathcal{M}_e$  to be

$${}^{as\delta(e)}\sqrt{L_{-\theta}/I_g Y} = {}^{as\delta(e)}\sqrt{L_{-\theta}/[AY^{ss}(\theta)^g/G]} ,$$

which is the  $as\delta(e)$ th root gerbe over the inertia stack component  $I_g Y$  of  $I_\mu Y$  by taking the  $as\delta(e)$ th root of the line bundle  $L_{-\theta}$ .

The root gerbe  ${}^{as\delta(e)}\sqrt{L_{-\theta}/I_g Y}$  admits a representation as a quotient stack:

$$[AY^{ss}(\theta)^g \times \mathbb{C}^* / (G \times \mathbb{C}_w^*)],$$

where the (right) action is defined by:

$$(\vec{x}, v) \cdot (g, w) = (\vec{x}g, \theta(g)^{-1}vw^{-as\delta(e)}) ,$$

for all  $(g, w) \in G \times \mathbb{C}_w^*$  and  $(\vec{x}, v) \in AY^{ss}(\theta)^g \times \mathbb{C}^*$ . For every character  $\rho$  of  $G$ , we can define a new character of  $G \times \mathbb{C}_w^*$  by composing the projection map  $\text{pr}_G : G \times \mathbb{C}_w^* \rightarrow G$ , we will still use  $\rho$  to name the new character of  $G \times \mathbb{C}_w^*$  by an abuse of notation. Then  $\rho$  will determines a line bundle  $L_\rho := [(AY^{ss}(\theta)^g \times \mathbb{C}^* \times \mathbb{C}_\rho) / (G \times \mathbb{C}_w^*)]$  on  ${}^{as\delta(e)}\sqrt{L_{-\theta}/I_g Y}$  by the Borel construction.

By virtue of the universal property of root gerbe, on  $\mathcal{M}_e = {}^{as\delta(e)}\sqrt{L_{-\theta}/I_g Y}$ , there is a universal line bundle  $\mathcal{R}$  that is the  $as\delta(e)$ th root of the line bundle  $L_{-\theta}$ . The root bundle  $\mathcal{R}$  is determined by the character  $\text{pr}_{\mathbb{C}^*}$ :

$$\text{pr}_{\mathbb{C}^*} : G \times \mathbb{C}_w^* \rightarrow \mathbb{C}_w^* \quad (g, w) \in G \times \mathbb{C}_w^* \mapsto w \in \mathbb{C}_w^* .$$

We have the relation

$$L_{-\theta} = \mathcal{R}^{as\delta(e)} .$$

The coordinate functions  $\vec{x}$  and  $v$  of  $AY^{ss}(\theta)^g \times \mathbb{C}^*$  descends to be universal sections of line bundles  $\oplus_{\rho \in [n]} L_\rho$  and  $L_{-\theta} \otimes \mathcal{R}^{-\otimes as\delta(e)}$  over  $\mathcal{M}_e$ , respectively.



We will construct a universal family of  $\mathbb{C}^*$ -fixed twisted stable maps to  $\mathbb{P}Y_{r,s}$  of degree  $(0, \frac{\delta(e)}{r})$  over  $\mathcal{M}_e$ :

$$\begin{array}{ccc} \mathcal{C}_e & := \mathbb{P}_{ar,as}(\mathcal{R} \oplus \mathcal{O}_{\mathcal{M}_e}) & \xrightarrow{f} \mathbb{P}Y_{r,s} \\ & \downarrow \pi & \\ \mathcal{M}_e & := \sqrt[as\delta(e)]{L_{-\theta}/I_g Y} & \end{array}$$

Then the universal curve  $\mathcal{C}_e$  over  $\sqrt[as\delta(e)]{L_{-\theta}/I_g Y}$  can be represented as a quotient stack:

$$\mathcal{C}_e = [(AY^{ss}(\theta)^g \times \mathbb{C}^* \times U)/(G \times \mathbb{C}_w^* \times T)] ,$$

where  $T = \{(t_1, t_2) \in (\mathbb{C}^*)^2 \mid t_1^{as} = t_2^{ar}\}$ . The right action is defined by:

$$(\vec{x}, v, x, y) \cdot (g, w, (t_1, t_2)) = (g \cdot \vec{x}, \theta(g)^{-1} v w^{-as\delta(e)}, w t_1 x, t_2 y) ,$$

for all  $(g, w, (t_1, t_2)) \in G \times \mathbb{C}_w^* \times T$  and  $(\vec{x}, v, (x, y)) \in AY^{ss}(\theta)^g \times \mathbb{C}^* \times U$ . Then  $\mathcal{C}_e$  is a family of orbifold  $\mathbb{P}_{ar,as}$  parameterized by  $\mathcal{M}_e$ .

There are two standard characters of  $T$

$$\chi_1 : (t_1, t_2) \in T \mapsto t_1 \in \mathbb{C}^* \quad \chi_2 : (t_1, t_2) \in T \mapsto t_2 \in \mathbb{C}^* ,$$

we can lift them to be characters of  $G \times \mathbb{C}_w^* \times T$  by composing the projection map  $\text{pr}_T : G \times \mathbb{C}_w^* \times T \rightarrow T$ . By an abuse of notation, we continue to use  $\chi_1, \chi_2$  to denote the new characters. These two new characters defines two line bundles

$$M_1 := (AY^{ss}(\theta)^g \times \mathbb{C}^* \times U) \times_{G \times \mathbb{C}_w^* \times T} \mathbb{C}_{\chi_1}$$

and

$$M_2 := (AY^{ss}(\theta)^g \times \mathbb{C}^* \times U) \times_{G \times \mathbb{C}_w^* \times T} \mathbb{C}_{\chi_2}$$

on  $\mathcal{C}_e$  by the Borel construction, respectively. We have the relation  $M_1^{\otimes as} = M_2^{\otimes ar}$  on  $\mathcal{C}_e$ . The universal map  $f$  from  $\mathcal{C}_e$  to  $\mathbb{P}Y_{r,s}$  can be described as follows: Let

$$\tilde{f} : AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \rightarrow \mathbb{C}^* \times AY^{ss}(\theta) \times U$$

be the morphism defined by:

$$(5.5) \quad \begin{aligned} (\vec{x}, v, x, y) \in AY^{ss}(\theta)^g \times \mathbb{C}^* \times U &\mapsto \\ (v, (x_1, \dots, x_n), x^{a\delta(e)}, y^{a\delta(e)}) &\in \mathbb{C}^* \times AY^{ss}(\theta) \times U . \end{aligned}$$

Then  $\tilde{f}$  is equivariant with respect to the group homomorphism from  $G \times \mathbb{C}_w^* \times T$  to  $G \times \mathbb{C}_\alpha^* \times \mathbb{C}_t^*$  defined by:

$$(5.6) \quad \begin{aligned} (g, w, (t_1, t_2)) \in G \times \mathbb{C}_w^* \times T &\mapsto \\ (g((t_1^{-s} t_2^r)^{p_1}, \dots, (t_1^{-s} t_2^r)^{p_k}), (w t_1)^{a\delta(e)}, t_2^{a\delta(e)}) &\in G \times \mathbb{C}_\alpha^* \times \mathbb{C}_t^* , \end{aligned}$$

where the tuple  $(p_1, \dots, p_k) \in \mathbb{N}^k$  satisfies that  $g = (\mu_a^{p_1}, \dots, \mu_a^{p_k}) \in G$ . Note  $\tilde{f}$  is well-defined for  $\chi_1^{-s} \chi_2^r$  is a torsion character of  $T$  of order  $a$ . The above construction gives the universal morphism  $f$  from  $\mathcal{C}_e$  to  $\mathbb{P}Y_{r,s}$  by descent.

We will define a (quasi left)  $\mathbb{C}^*$ -action on  $\mathcal{C}_e$  such that the map  $f$  constructed above is  $\mathbb{C}^*$ -equivariant. Define a  $\mathbb{C}^*$ -action on  $\mathcal{C}_e$  induced by the  $\mathbb{C}^*$ -action on  $AY^{ss}(\theta)^g \times \mathbb{C}^* \times U$ :

$$m : \mathbb{C}^* \times AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \rightarrow AY^{ss}(\theta)^g \times \mathbb{C}^* \times U ,$$

$$t \cdot (\vec{x}, v, (x, y)) = (\vec{x}, v, (x, t^{\frac{-1}{ar\delta(e)}} y)) .$$

Note the morphism  $\pi$  is also  $\mathbb{C}^*$ -equivariant, where  $\mathcal{M}_e$  is equipped with trivial  $\mathbb{C}^*$ -action. By the universal property of the projectivized bundle  $\mathcal{C}_e$  over  $\mathcal{M}_e$ , one has a tautological section

$$(5.7) \quad (x, y) \in H^0((M_1 \otimes \pi^* \mathcal{R}) \oplus (M_2 \otimes \mathbb{C}_{\frac{-\lambda}{ar\delta(e)}})) ,$$

which is also a  $\mathbb{C}^*$ -invariant section.

Now we can check that  $f$  is a  $\mathbb{C}^*$ -equivariant morphism from  $\mathcal{C}_e$  to  $\mathbb{P}Y_{r,s}$  with respect to the  $\mathbb{C}^*$ -actions for  $\mathcal{C}_e$  and  $\mathbb{P}Y_{r,s}$ . According to Remark 5.1,  $f$  is equivalent to the following data:

- (1)  $k + 2$   $\mathbb{C}^*$ -equivariant line bundles on  $\mathcal{C}_e$ :

$$\mathcal{L}_j := \pi^* L_{\pi_j} \otimes (M_1^{-\otimes s} \otimes M_2^{\otimes r})^{p_j}, 1 \leq j \leq k$$

and

$$\mathcal{N}_1 := (M_1 \otimes \pi^* \mathcal{R})^{\otimes a\delta(e)} \quad \mathcal{N}_2 := M_2^{a\delta(e)} \otimes \mathbb{C}_{\frac{-\lambda}{r}} .$$

Where  $L_{\pi_j}$  are the standard  $\mathbb{C}^*$ -equivariant line bundles on  $\mathcal{M}_e$  by the Borel construction,  $M_1, M_2$  are the standard  $\mathbb{C}^*$ -equivariant line bundles on  $\mathcal{C}_e$  by the Borel construction.

- (2) a universal section

$$(5.8) \quad (u, \vec{x}, (\zeta_1, \zeta_2)) := (v, x_1, \dots, x_n, (x^{a\delta(e)}, y^{a\delta(e)})) \\ \in \Gamma((\mathcal{N}_1^\vee)^{\otimes s} \otimes \mathcal{L}_{-\theta} \otimes \mathcal{N}_2^{\otimes r} \otimes \mathbb{C}_\lambda) \oplus \bigoplus_{1 \leq i \leq n} \mathcal{L}_{\rho_i} \oplus \mathcal{N}_1 \oplus \mathcal{N}_2)^{\mathbb{C}^*} .$$

Here one only need to check  $v \in \Gamma((\mathcal{N}_1^\vee)^{\otimes s} \otimes \mathcal{L}_{-\theta} \otimes \mathcal{N}_2^{\otimes r} \otimes \mathbb{C}_\lambda)$ , which is easy to be verified.

Now we compute the localization contribution from  $\mathcal{M}_e$ . Based on the perfect obstruction theory for stable maps in  $\mathcal{K}_{0, \vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ , the restriction of the perfect obstruction theory to  $\mathcal{M}_e$  decomposes into three parts: (1) the deformation theory of source curve  $\mathcal{C}_e$ ; (2) the deformation theory of the lines bundles  $(\mathcal{L}_i)_{1 \leq j \leq k}$  and  $\mathcal{N}$ ; (3) the deformation theory for the section

$$(u, \vec{x}, (\zeta_1, \zeta_2)) \in \Gamma((\mathcal{N}_1^{-\otimes s} \otimes \mathcal{L}_{-\theta} \otimes \mathcal{N}_2^{\otimes r} \otimes \mathbb{C}_\lambda) \oplus \bigoplus_{1 \leq i \leq n} \mathcal{L}_{\rho_i} \oplus \mathcal{N}_1 \oplus \mathcal{N}_2) .$$

The  $\mathbb{C}^*$ -fixed part of three parts above will contribute to the virtual cycle of  $\mathcal{M}_e$ , we will show that  $[\mathcal{M}_e]^{\text{vir}} = [\mathcal{M}_e]$ . The virtual normal bundle comes from the  $\mathbb{C}^*$ -moving part of the above three parts.

First every fiber curve  $C_e$  in  $\mathcal{C}_e$  over a geometrical point in  $\mathcal{M}_e$  is isomorphic to  $\mathbb{P}_{ar, as}$ , which is rational. There are no infinitesimal deformations/obstructions for  $C_e$ , line bundles  $\mathcal{L}_j := \mathcal{L}_j|_{C_e}$ ,  $\mathcal{N}_1 := \mathcal{N}_1|_{C_e}$  and  $\mathcal{N}_2 := \mathcal{N}_2|_{C_e}$ . Hence their contribution to the perfect obstruction theory comes from infinitesimal automorphisms. The infinitesimal automorphisms of  $C_e$  come from the space of vector fields on  $C_e$  that vanish on special points. Thus the  $\mathbb{C}^*$ -fixed part of infinitesimal automorphisms of  $C_e$  comes from the 1-dimensional subspace of vector fields on  $C_e$  which vanish on the two ramification points. The movable part of infinitesimal automorphisms of  $C_e$  is nonzero only if one of ramification points on  $C_e$  is not a special point. by Remark 5.2, the ramifications

on  $C_e$  are both nontrivial stacky points when  $r$  and  $s$  are sufficiently large, hence they must be special points. So there is no movable part for infinitesimal automorphisms of  $C_e$ .

Now let's turn to the localizations from sections. First the infinitesimal deformations of sections  $(u, \vec{x})$  are fixed, which, together with fixed part of infinitesimal automorphisms of  $C_e$  and line bundles  $L_j$ ,  $N_1$ ,  $N_2$ , as well as fixed parts of infinitesimal deformations of sections  $(z_1, z_2) := (\zeta_1, \zeta_2)|_{C_e}$ , contribute to the virtual cycle  $[\mathcal{M}_e]^{\text{vir}}$ , which is equal to the fundamental class of  $\mathcal{M}_e$ . The localization contribution from the infinitesimal deformations of sections  $(z_1, z_2)$  to the virtual normal bundle is:

$$(R^\bullet \pi_*(\mathcal{N}_1 \oplus \mathcal{N}_2))^{\text{mov}}.$$

We first come to the deformations of  $z_2$ , we continue to use the tautological section  $(x, y)$  as in (5.7). For each fiber  $C_e$ , sections of  $N_2$  is spanned by monomials  $(x^{asm} y^n)|_{C_e}$  with  $arm + n = a\delta(e)$  and  $m, n \in \mathbb{Z}_{\geq 0}$ . Note  $x^{asm} y^n$  may not be a global section of  $\mathcal{N}_2$  but always a global section of  $\mathcal{R}^{\otimes asm} \otimes \mathcal{N}_2 \otimes \mathbb{C}_{\frac{m}{\delta(e)}} \lambda$ . Then  $R^\bullet \pi_* \mathcal{N}_2$  will decompose as a direct sum of line bundles, each corresponds to the monomial  $x^{asm} y^n$ , whose first chern class is

$$c_1(\mathcal{R}^{\otimes -asm} \otimes \mathbb{C}_{\frac{-m}{\delta(e)}} \lambda) = \frac{m}{\delta(e)}(D_\theta - \lambda).$$

So the total contribution is equal to

$$\prod_{m=0}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left( \frac{m}{\delta(e)}(D_\theta - \lambda) \right).$$

The factor for  $m = 0$  appearing in the above product is the  $\mathbb{C}^*$ -fixed part of  $R^\bullet \pi_* \mathcal{N}_2$ , it will contribute to the virtual cycle of  $\mathcal{M}_e$ . The rest contributes to the virtual normal bundle as

$$\prod_{m=1}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left( \frac{m}{\delta(e)}(D_\theta - \lambda) \right).$$

Note when  $r$  is sufficiently large, the above product becomes 1.

For the deformations of  $z_1$ , arguing in the same way as  $z_2$ , the Euler class of  $R^\bullet \pi_* \mathcal{N}_1$  is equal to

$$\prod_{n=0}^{\lfloor \frac{\delta(e)}{s} \rfloor} \left( \frac{n}{\delta(e)}(-D_\theta + \lambda) \right).$$

The factor for  $m = 0$  appearing in the above product is the  $\mathbb{C}^*$ -fixed part of  $R^\bullet \pi_* \mathcal{N}_1$ , it will contribute to the virtual cycle of  $\mathcal{M}_e$ . The Euler class of virtual normal bundle of  $\mathcal{M}_e$  comes from the movable part of deformations of section  $z_1$  is:

$$\prod_{n=1}^{\lfloor \frac{\delta(e)}{s} \rfloor} \left( \frac{n}{\delta(e)}(-D_\theta + \lambda) \right).$$

Note when  $s$  is sufficiently large, the above product becomes 1.

**5.3.3. Node contributions.** The deformations in  $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  smoothing a node contribute to the Euler class of the virtual normal bundle as the first Chern class of the tensor product of the two cotangent line bundles at the branches of the node. For nodes at which a component  $C_e$  meets a component  $C_v$  over the vertex 0, this contribution is

$$\frac{\lambda - D_\theta}{as\delta(e)} - \frac{\bar{\psi}_v}{as}.$$

For nodes at which a component  $C_e$  meets a component  $C_v$  at the vertex over  $\infty$ , this contribution is

$$\frac{-\lambda + D_\theta}{ar\delta(e)} - \frac{\bar{\psi}_v}{ar}.$$

The type of node at which two edge component  $C_e$  meets with a vertex  $v$  over 0 or  $\infty$  will not occur using a similar argument in [JPPZ17, Lemma 6].

As for the node contributions from the normalization exact sequence, each node  $q$  (specified by a vertex  $v$ ) contributes the Euler class of

$$(5.9) \quad (R^0\pi_*\mathcal{N}_1|_q)^{\text{mov}} \oplus (R^0\pi_*\mathcal{N}_2|_q)^{\text{mov}}$$

to the virtual normal bundle. In the case where  $j(v) = 0$ ,  $z_2|_q \equiv 1$  gives a trivialization of the fiber  $\mathcal{N}_2|_q$ , note that  $(\mathcal{N}_1^\vee)^{\otimes s} \otimes \mathcal{L}_{-\theta} \otimes \mathcal{N}_2^{\otimes r} \otimes \mathbb{C}_\lambda \cong \mathbb{C}$  we have  $\mathcal{N}_2|_q \cong \mathbb{C}$  and  $\mathcal{N}_1|_q \cong L_{-\theta}^{\frac{1}{s}} \otimes \mathbb{C}_{\frac{\lambda}{s}}$ , this implies that  $(R^0\pi_*\mathcal{N}_2|_q)^{\text{mov}} = 0$  and  $R^0\pi_*\mathcal{N}_1|_q = 0$ . The later vanishes because of the nontrivial stacky structure of the line bundle  $\mathcal{N}_1$  at  $q$  when  $s$  is sufficiently large. Hence there no localization contribution from the normalization at the node  $q$  over 0. Similarly, for each node  $q$  incident to a vertex  $v$  with  $j(v) = \infty$ , there is no localization contribution from the normalization at the node over  $\infty$ .

**5.4. Total localization contributions.** For each decorated graph  $\Gamma$ , denote  $F_\Gamma$  to be the fiber product

$$\prod_{v:j(v)=0} \mathcal{M}_v \times_{\bar{I}_\mu \sqrt[s]{L_{-\theta}/Y}} \prod_{e \in E} \mathcal{M}_e \times_{\bar{I}_\mu \sqrt[r]{L_\theta/Y}} \prod_{v:j(v)=\infty} \mathcal{M}_v$$

of the following diagram:

$$\begin{array}{ccc} F_\Gamma & \xrightarrow{\quad} & \prod_{v:j(v)=0} \mathcal{M}_v \times \prod_{e \in E} \mathcal{M}_e \times \prod_{v:j(v)=\infty} \mathcal{M}_v \\ \downarrow & & \downarrow \text{ev}_{h_e}, \text{ev}_{p_0^e}, \text{ev}_{p_\infty^e}, \text{ev}_{h_e'} \\ \prod_E \bar{I}_\mu \sqrt[s]{L_{-\theta}/Y} \times \bar{I}_\mu \sqrt[r]{L_\theta/Y} & \xrightarrow{(\Delta^{\frac{1}{s}} \times \Delta^{\frac{1}{r}})^E} & \prod_E \left( (\bar{I}_\mu \sqrt[s]{L_{-\theta}/Y})^2 \times (\bar{I}_\mu \sqrt[r]{L_\theta/Y})^2 \right), \end{array}$$

where  $\Delta^{\frac{1}{s}} = (id, \iota)$  (resp.  $\Delta^{\frac{1}{r}} = (id, \iota)$ ) is the diagonal map of  $\bar{I}_\mu \sqrt[s]{L_{-\theta}/Y}$  (resp.  $\bar{I}_\mu \sqrt[r]{L_\theta/Y}$ ) into  $\bar{I}_\mu \sqrt[s]{L_{-\theta}/Y} \times \bar{I}_\mu \sqrt[s]{L_{-\theta}/Y}$  (resp.  $\bar{I}_\mu \sqrt[r]{L_\theta/Y} \times \bar{I}_\mu \sqrt[r]{L_\theta/Y}$ ). Here when  $v$  is a stable vertex, the vertex moduli  $\mathcal{M}_v$  is described in 5.3.1; when  $v$  is an unstable vertex over 0, we treat  $\mathcal{M}_v := \bar{I}_{m(h)} \sqrt[s]{L_{-\theta}/Y}$  with the identical virtual cycle, where  $m(h)$  is the multiplicity of the half-edge incident to  $v$ ; when  $v$  is an unstable vertex over  $\infty$ , we treat  $\mathcal{M}_v := \bar{I}_{m(h)} \sqrt[r]{L_\theta/Y}$  with the identical virtual cycle, where  $m(h)$  is the multiplicity of the half-edge incident to  $v$ .

We define that  $[F_\Gamma]^{\text{vir}}$  to be:

$$\prod_{v:j(v)=0} [\mathcal{M}_v]^{\text{vir}} \times_{\bar{I}_\mu \sqrt[s]{L_{-\theta}/Y}} \prod_{e \in E} [\mathcal{M}_e]^{\text{vir}} \times_{\bar{I}_\mu \sqrt[r]{L_\theta/Y}} \prod_{v:j(v)=\infty} [\mathcal{M}_v]^{\text{vir}},$$

where the fiber product over  $\bar{I}_\mu \sqrt[s]{L_{-\theta}/Y}$  and  $\bar{I}_\mu \sqrt[r]{L_\theta/Y}$  imposes that the evaluation maps at the two branches of each node (here we adopt the convention that a node can link a unstable vertex and an edge.) agree. Then the contribution of decorated graph  $\Gamma$  to the virtual localization is is:

$$(5.10) \quad \text{Cont}_\Gamma = \frac{\prod_{e \in E} sa_e}{\text{Aut}(\Gamma)} (\iota_\Gamma)_* \left( \frac{[F_\Gamma]^{\text{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\text{vir}})} \right).$$

Here  $\iota_F : F_\Gamma \rightarrow \mathcal{K}_{0,\bar{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  is a finite etale map of degree  $\frac{\text{Aut}(\Gamma)}{\prod_{e \in E} sa_e}$  into the corresponding  $\mathbb{C}^*$ -fixed loci in  $\mathcal{K}_{0,\bar{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ . The virtual normal bundle  $e^{\mathbb{C}^*}(N_\Gamma^{\text{vir}})$  is the product of virtual normal bundles from vertex contributions, edge contributions and node contributions.

## 6. RECURSION RELATIONS FROM AUXILIARY CYCLES

For any  $\beta \in \text{Eff}(W, G, \theta)$ , for simplicity, we will denote

$$\mathcal{K}_{0,\bar{m}}(\bullet, \beta) := \bigsqcup_{\substack{d \in \text{Eff}(\bullet) \\ (i_\bullet)_*(d) = \beta}} \mathcal{K}_{0,\bar{m}}(\bullet, d),$$

where  $\bullet$  can be  $Y, \sqrt[r]{L_\theta/Y}$  and  $\sqrt[s]{L_{-\theta}/Y}$ , and  $i_\bullet$  is the natural structure map from  $\bullet$  to  $\mathfrak{X}$  which factors through the inclusion  $i_{\mathfrak{Y}} : \mathfrak{Y} \rightarrow \mathfrak{X}$ .

**6.1. Auxiliary cycle I.** Fix a nonzero  $\beta \in \text{Eff}(W, G, \theta)$ , and a positive rational number  $\epsilon$  and the tuple  $\epsilon = (\epsilon, \dots, \epsilon) \in (\mathbb{Q}_{>0})^p$  such that  $\epsilon\beta(L_\theta) + p\epsilon \leq 1$ . Set  $\delta = \beta(L_\theta) + p$ . For simplicity, we will denote

$$Q_{0,\star}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r})) := \bigsqcup_{\substack{d \in \text{Eff}(AY, G, \theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta}} Q_{0,1}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (d, 1^p, \frac{\delta}{r})) \cap ev_1^{-1}(\bar{I}_{(g_\beta, \frac{\delta}{r})} \mathbb{P}Y^{\frac{1}{r}}).$$

Recall that  $g_\beta \in G$  is defined in §3. We will always assume that  $r$  is a sufficiently large prime in this subsection.

For any nonnegative integer  $c$ , we will first consider the following auxiliary cycle:

$$(6.1) \quad \frac{1}{p!} (\widetilde{EV}_\star)_* \left( e^{\mathbb{C}^*} (R^1 \pi_* f^* L_\infty^\vee) \cap [Q_{0,\star}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))]^{\text{vir}} \cap \bar{\psi}_\star^c \cap \prod_{j=1}^p \hat{ev}_j^*(\hat{t}) \right).$$

Here an explanation of the notations is in order:

(1) the morphism

$$\pi : \mathcal{C} \rightarrow Q_{0,\star}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$$

is the universal curve and the morphism

$$f : \mathcal{C} \rightarrow \mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$$

is the universal map;

- (2) the  $\mathbb{C}^*$ -equivariant line bundle  $L_\infty$  corresponds to the invertible sheaf  $\mathcal{O}(\mathcal{D}_\infty)$  on  $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$  with the  $\mathbb{C}^*$ -linearization that  $\mathbb{C}^*$  acts on the fiber over  $\mathcal{D}_\infty$  (given by  $z_2 = 0$ ) with weight  $-\frac{\lambda}{r}$  and  $\mathbb{C}^*$  acts on the fiber over  $\mathcal{D}_0$  (given by  $z_1 = 0$ ) with weight 0. For every fiber curve  $C$  of the universal curve  $\mathcal{C}$  via  $\pi$ , the important observation here is that the restricted line bundle  $f^*L_\infty^\vee|_C$  has negative degree as  $\beta \neq 0$  and has non-positive degree on every irreducible components of  $C$ . Indeed, if the image of an irreducible component of  $C$  via  $f$  isn't contained in  $\mathcal{D}_\infty$ , the degree is obviously non-positive. If the image of an irreducible component of  $C$  under  $f$  is contained in  $\mathcal{D}_\infty$ , then using the fact that  $L_\infty^\vee$  is isomorphic to  $(L_\theta^{\frac{1}{r}})^\vee$  over

$$\mathcal{D}_\infty \cong \sqrt[r]{L_\theta/Y}$$

and a similar discussion as in remark 5.4, the degree is also non-positive. Thus by Lemma 5.3, we have  $R^0\pi_*f^*L_\infty^\vee = 0$ , which implies that  $R^1\pi_*f^*L_\infty^\vee$  is a vector bundle (of rank 0);

- (3) the morphism  $EV_\star$  is a composition of the following maps:

$$Q_{0,\star}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r})) \xrightarrow{ev_\star} \bar{I}_\mu \mathbb{P}Y^{\frac{1}{r}} \xrightarrow{\text{pr}_r} \bar{I}_\mu Y,$$

where  $\text{pr}_r : \bar{I}_\mu \mathbb{P}Y^{\frac{1}{r}} \rightarrow \bar{I}_\mu Y$  is the morphism induced from the natural structure map from  $\mathbb{P}Y^{\frac{1}{r}}$  to  $Y$  forgetting  $z_1, z_2$ .  $(\bar{EV}_\star)_*$  is defined by

$$\iota_*(r_\star(EV_\star)_*)$$

as in (2.2). Note here  $r_\star$  is the order of the band from the gerbe structure of  $\bar{I}_\mu Y$  but not  $\bar{I}_\mu \mathbb{P}Y^{\frac{1}{r}}$ .

- (4) recall that  $\hat{ev}_j$  is defined in (4.5). The cohomology class  $\hat{t} \in H^*(\mathfrak{Y}, \mathbb{Q})[t_0, \dots, t_l]$  is of the form  $\sum_{i=1}^l t_i u_i(c_1(L_{\pi_j}))$ . where  $t_i$  are formal variables and  $u_i$  are polynomials in the first chern class of the line bundles  $L_{\pi_j}$  associated to the standard characters  $\pi_j$  of  $G$ .

Apply virtual localization to  $Q_{0,\star}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ , we first prove the following vanishing result, where the idea is borrowed from [JPT].

**Lemma 6.1.** *The localization graph  $\Gamma$  that has more than one vertex labeled by  $\infty$  will contribute zero to (6.1).*

*Proof.* Assume by contradiction, by the connectedness of the graph, there is at least one vertex at 0 with valence at least two. Suppose  $f : C \rightarrow \mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$  is  $\mathbb{C}^*$ -fixed. Assume that  $C_0 \cap C_1 \cap C_2$  is part of curve  $C$ , where  $C_0$  contracted by  $f$  to  $\mathcal{D}_0$  (given by  $z_1 = 0$ ) and  $C_1, C_2$  are edges meeting with  $C_0$  at  $b_1$  and  $b_2$ . Then in the normalization sequence for  $R^\bullet\pi_*f^*(L_\infty^\vee)$ , it contains the part

$$\begin{aligned} & H^0(C_0, f^*(L_\infty^\vee)) \oplus H^0(C_1, f^*(L_\infty^\vee)) \oplus H^0(C_2, f^*(L_\infty^\vee)) \\ & \rightarrow H^0(b_1, f^*(L_\infty^\vee)) \oplus H^0(b_2, f^*(L_\infty^\vee)) \\ & \rightarrow H^1(C, f^*(L_\infty^\vee)). \end{aligned}$$

Hence there is one of the weight-0 pieces in  $H^0(b_1, f^*(L_\infty^\vee)) \oplus H^0(b_2, f^*(L_\infty^\vee))$  that is canceled with a weight-0 piece of  $H^0(C_0, f^*(L_\infty^\vee))$ , and the other is mapped injectively into  $H^1(C, f^*(L_\infty^\vee))$ . Thus  $H^1(C, f^*(L_\infty^\vee))$  contains a weight-0 piece with vanishing equivariant Euler class.  $\square$

Recall that we can write  $\mathbb{I}(q, t, z) = \sum_{\beta, p} q^\beta \mathbb{I}_{\beta, p}$ , where  $\mathbb{I}_{\beta, p} := \frac{t^p}{p! z^p} \mathbb{I}_\beta(z)$  is a Laurent polynomial in  $z$  with coefficient in degree  $p$  multi-polynomial in  $H^*(\bar{I}_\mu Y, \mathbb{Q})[t_0, \dots, t_l]$ . We will prove the following recursion relation by applying localization to (6.9).

**Theorem 6.2.** *For any nonnegative integer  $c$ ,  $[z\mathbb{I}_{\beta, p}]_{z^{-c-1}}$  satisfies the following relation:*

$$(6.2) \quad \begin{aligned} [z\mathbb{I}_{\beta, p}]_{z^{-c-1}} &= (e\widetilde{v}_\star)_*([\mathcal{K}_{0, \star}(Y, \beta)]^{\text{vir}} \cap ev_\star^*(\bar{\psi}_\star^c)) \\ &+ \left[ \sum_{m=1}^{\infty} \sum_{\substack{\beta_\star + \beta_1 + \dots + \beta_m = \beta \\ p_1 + \dots + p_m = p}} \frac{1}{m!} (e\widetilde{v}_\star)_* \left( \sum_{d=0}^{\infty} \epsilon_* (c_d(-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}})) \left( \frac{-\lambda}{r} \right)^{-1+m-d} \right. \right. \\ &\quad \left. \left. \cap [\mathcal{K}_{0, \vec{m}_r \cup \star}(\sqrt[r]{L_\theta/Y}, \beta_\star)]^{\text{vir}} \cap \prod_{i=1}^m \frac{ev_i^*(\frac{1}{\delta_i}(z\mathbb{I}_{\beta_i, p_i}(z))|_{z=\frac{\lambda-D_\theta}{\delta_i}})}{\frac{\lambda-ev_i^* D_\theta}{r\delta_i} + \frac{\bar{\psi}_i}{r}} \cap \bar{\psi}_\star^c \right) \right]_{\lambda^{-1}}. \end{aligned}$$

Here  $\delta_i = \beta_i(L_\theta) + p_i$  for  $1 \leq i \leq m$ ,  $\epsilon : \mathcal{K}_{0, \vec{m}_r \cup \star}(\sqrt[r]{L_\theta/Y}, \beta_\star) \rightarrow \mathcal{K}_{0, \vec{m} \cup \star}(Y, \beta_\star)$  is the natural structure morphism (c.f. [TT16]), where

$$\vec{m}_r \cup \star := ((g_{\beta_i}^{-1}, \mu_r^{-\beta_i(L_\theta)-p_i}) : 1 \leq i \leq m) \times \{(g_\beta, \mu_r^{\beta(L_\theta)+p})\} \in (G \times \mu_r)^{m+1},$$

and  $\vec{m} \cup \star := (g_{\beta_i}^{-1} : 1 \leq i \leq m) \times \{g_\beta\} \in G^{m+1}$ .

*Proof.* By Lemma 6.1, only decorated graph  $\Gamma$ , which has only one vertex  $v$  labeled by  $\infty$ , may have nonzero localization contribution to the (6.1). Note the only marking  $q_\star$  can only be incident to the vertex labeled by  $\infty$  due to the restriction on the multiplicity at  $q_\star$ . Furthermore, for such graph  $\Gamma$ , we claim there is no stable vertex labeled by 0. Indeed, for any vertex  $v$  over 0, its degree  $\beta(v)$  satisfies that  $\beta(v)(L_\theta) \leq \beta(L_\theta) \leq \frac{1}{\epsilon}$ , and it has valence 1 as no legs can attach to it and at most one edge is incident to it by Lemma 6.1, then the vertex  $v$  must be unstable. So the decorated graph  $\Gamma$  has only one vertex over  $\infty$  with possible several edges (can be empty) attached, and each vertex labeled by 0 corresponds to an edge in the graph  $\Gamma$ , which appears as an unmarked point (actually a base point as we will see). In the following, we analyze the localization contribution to (6.1) from the graph  $\Gamma$  described above. We have two cases which depends on whether the only vertex labeled by  $\infty$  on the graph  $\Gamma$  is stable or unstable.

- (1) If the only vertex  $v$  over  $\infty$  is unstable, then it's a vertex with valence 2, i.e., it's incident to a leg and an edge. In this case the degree  $(\beta, 1^p, \frac{\delta}{r})$  is concentrated on the single edge with the marked point  $q_\star$  appearing as the ramification point over  $\infty$  on the edge. Then it contributes

$$\frac{1}{\delta} (z\mathbb{I}_{\beta, p}(z))|_{z=\frac{\lambda-D_\theta}{\delta}} \cdot \left( \frac{\lambda-D_\theta}{\delta} \right)^c$$

to (6.1). Here we use the fact that the restriction of  $R^1\pi_*(f^*L_\infty^\vee)$  to  $F_\Gamma$  is a rank 0 vector bundle, so its equivariant Euler class is 1, and the restriction of  $\bar{\psi}_*$  to  $\mathcal{M}_e$  is equal to  $\frac{\lambda-D_\theta}{\delta}$ .

- (2) If the only vertex  $v$  over  $\infty$  is stable, then  $v$  is incident to only one leg and possible several edges (can be none). We assume that the vertex  $v$  has degree  $(\beta_*, \frac{\delta_*}{r})$  with  $\delta_* = \beta_*(L_\theta)$ . If there is no edges in the graph  $\Gamma$ , which happens if and only if  $\beta_* = \beta$  and  $p = 0$ , the corresponding graph has contribution

$$(6.3) \quad (e\widetilde{v}_*)_* \left( \sum_{d=0}^{\infty} \epsilon_*(c_d(-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}})) \left( \frac{-\lambda}{r} \right)^{-1-d} \cap [\mathcal{K}_{0,*}(\sqrt[r]{L_\theta/Y}, \beta_*)]^{\text{vir}} \cap \bar{\psi}_*^c \right).$$

to the (6.1). Otherwise we index all the edges attached to the vertex  $v$  from 1 to  $m$  such that the edge  $e_i$  corresponding to the index  $i$  has degree  $(\beta_i, 1^{J_{e_i}}, \frac{\delta_i}{r})$ . Since we assume that the total degree is  $(\beta, 1^p, \frac{\delta}{r}) = (\beta, 1^p, \frac{\beta(L_\theta)}{r})$ , and the degree on every edge satisfies the relation  $\delta_i \geq \beta_i(L_\theta) + p_i$  by Remark 4.4, where  $p_i = |J_{e_i}|$ . Therefore we must have  $\delta_i = \beta_i(L_\theta) + p_i$  for every edge  $e_i$ . It follows that all the edge has a base point and  $(\beta_i, p_i)$  is nonzero.

Equipped with these notations, by Remark 4.3, the vertex moduli  $\mathcal{M}_v$  over  $\infty$  is

$$\mathcal{K}_{0, \vec{m}_r \cup \star}(\sqrt[r]{L_\theta/Y}, \beta_*) = \mathcal{K}_{0, [m] \cup \star}(\sqrt[r]{L_\theta/Y}, \beta_*) \cap \bigcap_{i=1}^m ev_i^{-1}(\bar{I}_{(g_{\beta_i}^{-1}, \mu_r^{-\delta_i})} \sqrt[r]{L_\theta/Y}) \cap ev_\star^{-1}(\bar{I}_{(g_\beta, \mu_r^\delta)} \sqrt[r]{L_\theta/Y}).$$

Using the localization analysis in §4.3 and the fact that  $e^{\mathbb{C}^*}(R^1\pi_* f^* L_\infty^\vee) = 1$  as it's of rank zero on  $F_\Gamma$ . The localization contribution of such graph  $\Gamma$  to (6.1) is equal to

$$(6.4) \quad \frac{1}{\text{Aut}(\Gamma)} \tilde{e}v_* \left( \sum_{d=0}^{\infty} \epsilon_*(c_d(-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}})) \left( \frac{-\lambda}{r} \right)^{-1+m-d} \cap [\mathcal{K}_{0, \vec{m}_r \cup \star}(\sqrt[r]{L_\theta/Y}, \beta_*)]^{\text{vir}} \right. \\ \left. \cap \prod_{i=1}^m \frac{ev_i^* \left( \frac{1}{\delta_i} (z^{\frac{t p_i}{z^{p_i}}} \mathbb{I}_{\beta_i}(q, z)) \right) \Big|_{z=\frac{\lambda-D_\theta}{\delta_i}}}{-\frac{\lambda-ev_i^* D_\theta}{r \delta_i} - \frac{\bar{\psi}_i}{r}} \cap \bar{\psi}_*^c \right),$$

where  $t = \sum t_i u_i (c_1(L_{\tau_{ij}}) + \beta(L_{\tau_{ij}})z)$  and  $\epsilon : \mathcal{K}_{0, \vec{m}_r \cup \star}(\sqrt[r]{L_\theta/Y}, \beta_*) \rightarrow \mathcal{K}_{0, \vec{m} \cup \star}(Y, \beta_*)$  is the natural structure map. Here  $\vec{m} \cup \star = (g_{\beta_i}^{-1} : 1 \leq i \leq m) \times \{g_\beta\} \in G^{m+1}$ .

Fix  $\beta_*$  and  $m$ , the sum of (6.4) coming from all possible graph which has  $\infty$ -vertex  $v$  of degree  $\beta_*$  and  $m$  incident edges yields:

$$(6.5) \quad \sum_{\substack{\beta_* + \beta_1 + \dots + \beta_m = \beta \\ p_1 + \dots + p_m = p}} \frac{1}{m!} (\tilde{e}v_*)_* \left( \sum_{d=0}^{\infty} \epsilon_*(c_d(-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}})) \left( \frac{-\lambda}{r} \right)^{-1+m-d} \right. \\ \left. \cap [\mathcal{K}_{0, \vec{m}_r \cup \star}(\sqrt[r]{L_\theta/Y}, \beta_*)]^{\text{vir}} \cap \prod_{i=1}^m \frac{ev_i^* \left( \frac{1}{\delta_i} (z \mathbb{I}_{\beta_i, p_i}(z)) \right) \Big|_{z=\frac{\lambda-D_\theta}{\delta_i}}}{-\frac{\lambda-ev_i^* D_\theta}{r \delta_i} - \frac{\bar{\psi}_i}{r}} \cap \bar{\psi}_*^c \right).$$



In summary, the auxiliary cycle (6.1) is equal to:

$$\begin{aligned}
& \frac{1}{\delta} (z\mathbb{I}_{\beta,p}(z))|_{z=\frac{\lambda-D_\theta}{\delta}} \cdot \left(\frac{\lambda-D_\theta}{\delta}\right)^c \\
& + (e\widetilde{v}_\star)_* \left( \sum_{d=0}^{\infty} \epsilon_* (c_d(-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}}) \left(\frac{-\lambda}{r}\right)^{-1-d} \cap [\mathcal{K}_{0,\star}(\sqrt[r]{L_\theta/Y}, \beta_\star)]^{\text{vir}} \cap \bar{\psi}_\star^c \right) \\
(6.6) \quad & + \sum_{m=1}^{\infty} \sum_{\substack{\beta_\star + \beta_1 + \dots + \beta_m = \beta \\ p_1 + \dots + p_m = p}} \frac{1}{m!} (e\widetilde{v}_\star)_* \left( \sum_{d=0}^{\infty} \epsilon_* (c_d(-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}}) \left(\frac{-\lambda}{r}\right)^{-1+m-d} \right. \\
& \left. \cap [\mathcal{K}_{0,\widetilde{m}_r \cup \star}(\sqrt[r]{L_\theta/Y}, \beta_\star)]^{\text{vir}} \cap \prod_{i=1}^m \frac{ev_i^* \left( \frac{1}{\delta_i} (z\mathbb{I}_{\beta_i,p_i}(z)) \right)|_{z=\frac{\lambda-D_\theta}{\delta_i}}}{-\frac{\lambda-ev_i^* D_\theta}{r\delta_i} - \frac{\bar{\psi}_i}{r}} \cap \bar{\psi}_\star^c \right).
\end{aligned}$$

Observe that (6.1) does not have negative  $\lambda$  powers, then the  $\lambda^{-1}$  coefficient in the equation (6.6) is equal to zero. Note that the  $\lambda^{-1}$  coefficient in (6.6) is equal to

$$\begin{aligned}
& [z\mathbb{I}_{\beta,p}(z)]_{z^{-c-1}} - (e\widetilde{v}_\star)_* \left( r \epsilon_* ([\mathcal{K}_{0,\star}(\sqrt[r]{L_\theta/Y}, \beta_\star)]^{\text{vir}} \cap \bar{\psi}_\star^c) \right) \\
(6.7) \quad & - \left[ \sum_{m=1}^{\infty} \sum_{\substack{\beta_\star + \beta_1 + \dots + \beta_m = \beta \\ p_1 + \dots + p_m = p}} \frac{1}{m!} (e\widetilde{v}_\star)_* \left( \sum_{d=0}^{\infty} \epsilon_* (c_d(-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}}) \left(\frac{-\lambda}{r}\right)^{-1+m-d} \right. \right. \\
& \left. \left. \cap [\mathcal{K}_{0,\widetilde{m}_r \cup \star}(\sqrt[r]{L_\theta/Y}, \beta_\star)]^{\text{vir}} \cap \prod_{i=1}^m \frac{ev_i^* \left( \frac{1}{\delta_i} (z\mathbb{I}_{\beta_i,p_i}(z)) \right)|_{z=\frac{\lambda-D_\theta}{\delta_i}}}{\frac{\lambda-ev_i^* D_\theta}{r\delta_i} + \frac{\bar{\psi}_i}{r}} \cap \bar{\psi}_\star^c \right) \right]_{\lambda^{-1}}.
\end{aligned}$$

Now for the second term in (6.7), one uses the following identity:

$$(6.8) \quad \epsilon_* ([\mathcal{K}_{0,\star}(\sqrt[r]{L_\theta/Y}, \beta)]^{\text{vir}}) = \frac{1}{r} [\mathcal{K}_{0,\star}(Y, \beta)]^{\text{vir}},$$

which is proved in [TT16, Theorem 5.17]. Now (6.7) immediately implies the formula (6.2).  $\square$

**6.2. Auxiliary cycle II.** In this section, for any  $\beta \in \text{Eff}(W, G, \theta)$ , we will denote  $\mathcal{K}_{0,\widetilde{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  to be

$$\bigsqcup_{\substack{d \in \text{Eff}(AY, G, \theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta}} \mathcal{K}_{0,\widetilde{m}}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r})).$$

Fix  $\beta, \delta$  as in §6.1. Assume that  $r, s$  are sufficiently large primes. We will also compare (6.1) to the following auxiliary cycle:

$$\begin{aligned}
(6.9) \quad & \sum_{m=0}^{\infty} \sum_{\substack{\beta_\star + \beta_1 + \dots + \beta_m = \beta \\ p_1 + \dots + p_m = p}} \frac{1}{m!} (\widetilde{EV}_\star)_* \left( [\mathcal{K}_{0,\widetilde{m} \cup \star}(\mathbb{P}Y_{r,s}, (\beta_\star, \frac{\delta}{r}))]^{\text{vir}} \right. \\
& \left. \cap \prod_{i=1}^m ev_i^* (\text{pr}_{r,s}^* (\mu_{\beta_i,p_i}(-\bar{\psi}_i))) \cap \bar{\psi}_\star^c \cap e^{\mathbb{C}^*} (R^1 \pi_* f^* L_\infty^\vee) \right).
\end{aligned}$$

Here an explanation of the notations is in order:

- (1) the morphism

$$\pi : \mathcal{C} \rightarrow \mathcal{K}_{0, \vec{m} \cup \star}(\mathbb{P}Y_{r,s}, (\beta_\star, \frac{\delta}{r}))$$

is the universal curve and the morphism

$$f : \mathcal{C} \rightarrow \mathbb{P}Y_{r,s}$$

is the universal map;

- (2) for any nonnegative integer  $m$  and degrees  $\beta_\star, \beta_1, \dots, \beta_m$  in  $\text{Eff}(W, G, \theta)$ . Here  $\vec{m} = (m_i \in G \times \mu_s \times \mu_r : 1 \leq i \leq m)$ , in which  $m_i = (g_{\beta_i}^{-1}, \mu_s^{\beta_i(L_\theta)+p_i}, 1)$ , and  $m_\star = (g_\beta, 1, \mu_r^{\beta(L_\theta)+p}) \in G \times \mu_s \times \mu_r$ . So  $\mathcal{K}_{0, \vec{m} \cup \star}(\mathbb{P}Y_{r,s}, (\beta_\star, \frac{\delta}{r}))$  is defined to be:

$$\mathcal{K}_{0, m+1}(\mathbb{P}Y_{r,s}, (\beta_\star, \frac{\delta}{r})) \cap \bigcap_{i=1}^m ev_i^{-1}(\bar{I}_{m_i} \mathbb{P}Y_{r,s}) \cap ev_{m+1}^{-1}(\bar{I}_{m_\star} \mathbb{P}Y_{r,s});$$

- (3) the line bundle  $L_\infty$  corresponds to the invertible sheaf  $\mathcal{O}(\mathcal{D}_\infty)$  with the  $\mathbb{C}^*$ -linearization such that  $\mathbb{C}^*$  acts on the fiber over  $\mathcal{D}_\infty$  with weight  $-\frac{\lambda}{r}$  and acts on the fiber over  $\mathcal{D}_0$  with weight zero; Using the same reasoning in the case of auxiliary cycle I, we have  $R^0 \pi_* f^* L_\infty^\vee = 0$  and  $R^1 \pi_* f^* L_\infty^\vee$  is a vector bundle (of rank 0);
- (4) the morphism  $EV_\star$  is a composition of the following maps:

$$\mathcal{K}_{0, \vec{m} \cup \star}(\mathbb{P}Y_{r,s}, (\beta_\star, \frac{\delta}{r})) \xrightarrow{ev_\star} \bar{I}_\mu \mathbb{P}Y_{r,s} \xrightarrow{\text{pr}_{r,s}} \bar{I}_\mu Y,$$

where  $\text{pr}_{r,s} : \bar{I}_\mu \mathbb{P}Y_{r,s} \rightarrow \bar{I}_\mu Y$  is the morphism induced from the natural structure map from  $\mathbb{P}Y_{r,s}$  to  $Y$  forgetting  $u$  and  $z_1, z_2$ , and  $(\widehat{EV}_\star)_*$  is defined by

$$\iota_*(r_\star(EV_\star)_*)$$

as in 2.2. Note here  $r_\star$  is the order of the band from the gerbe structure of  $\bar{I}_\mu Y$  but not  $\bar{I}_\mu \mathbb{P}Y_{r,s}$ .

First we have a similar vanishing result as Lemma 6.1 by an analogous argument.

**Lemma 6.3.** *The localization graph  $\Gamma$  which has more than one vertex labeled by  $\infty$  contributes zero to (6.9).*

We will prove the following recursion relation by applying localization to (6.9).

**Theorem 6.4.** *Assume  $r, s$  are sufficiently large and prime. For any nonnegative integer  $c$ , the summation*

$$\sum_{m=0}^{\infty} \sum_{\substack{\beta_\star + \beta_1 + \dots + \beta_m = \beta \\ p_1 + \dots + p_m = p}} \frac{1}{m!} \phi^\alpha \langle \mu_{\beta_1, p_1}(-\bar{\psi}_1), \dots, \mu_{\beta_m, p_m}(-\bar{\psi}_m), \phi_\alpha \bar{\psi}_\star^c \rangle_{0, [m] \cup \star, \beta_\star}^\infty$$

satisfies the following relation:

$$\begin{aligned}
& \sum_{m=0}^{\infty} \sum_{\substack{\beta_{\star} + \beta_1 + \dots + \beta_m = \beta \\ p_1 + \dots + p_m = p}} \frac{1}{m!} \phi^{\alpha} \langle \mu_{\beta_1}(-\bar{\psi}_1), \dots, \mu_{\beta_m}(-\bar{\psi}_m), \phi_{\alpha} \bar{\psi}_{\star}^c \rangle_{0, [m] \cup \star, \beta_{\star}}^{\infty} \\
&= (e\tilde{v}_{\star})_* ([\mathcal{K}_{0, \star}(Y, \beta)]^{\text{vir}} \cap \bar{\psi}_{\star}^c) \\
(6.10) \quad &+ \left[ \sum_{m=1}^{\infty} \sum_{\substack{\beta_{\star} + \beta_1 + \dots + \beta_m = \beta \\ p_1 + \dots + p_m = p}} \frac{1}{m!} (e\tilde{v}_{\star})_* \left( \sum_{d=0}^{\infty} \epsilon_* (c_d(-R^{\bullet} \pi_{\star} \mathcal{L}_{\theta}^{\frac{1}{r}})) \left( \frac{-\lambda}{r} \right)^{-1+m-d} \right. \right. \\
&\quad \left. \left. \cap [\mathcal{K}_{0, \vec{m}_r \cup \star}(\sqrt{L_{\theta}/Y}, \beta_{\star})]^{\text{vir}} \cap \prod_{i=1}^m \frac{ev_i^* (\frac{1}{\delta_i} f_{\beta_i, p_i}(z))|_{z=\frac{\lambda-D_{\theta}}{\delta_i}}}{\frac{\lambda-ev_i^* D_{\theta}}{r\delta_i} + \frac{\bar{\psi}_i}{r}} \cap \bar{\psi}_{\star}^c \right) \right]_{\lambda^{-1}}.
\end{aligned}$$

Here  $f_{\beta_i, p_i}(z)$  is defined as follows:

$$\begin{aligned}
(6.11) \quad & f_{\beta_i, p_i}(z) := \mu_{\beta_i, p_i}(z) + \sum_{l=0}^{\infty} \sum_{\substack{\beta_0 + \beta_1 + \dots + \beta_l = \beta_i \\ p_1 + \dots + p_l = p_i}} \frac{1}{l!} \\
& (e\tilde{v}_0)_* \left( [\mathcal{K}_{0, [l] \cup \{0\}}(Y, \beta_0)]^{\text{vir}} \cap \bigcap_{j=1}^l ev_j^* (\mu_{\beta_j, p_j}(-\bar{\psi}_j)) \cap \frac{1}{z - \bar{\psi}_0} \right),
\end{aligned}$$

$\delta_i = \beta_i(L_{\theta}) + p_i$ ,  $\vec{m}_r \cup \star := ((g_{\beta_i}^{-1}, \mu_r^{-\beta_i(L_{\theta})-p_i}) : 1 \leq i \leq m) \cup \{g_{\beta}, \mu_r^{\beta(L_{\theta})+p}\}$ ,  $\vec{m}_r \cup \star := (g_{\beta_i}^{-1} : 1 \leq i \leq m) \cup \{g_{\beta}\}$  and  $\epsilon : \mathcal{K}_{0, \vec{m}_r \cup \star}(\sqrt{L_{\theta}/Y}, \beta_{\star}) \rightarrow \mathcal{K}_{0, \vec{m} \cup \star}(Y, \beta_{\star})$  is the natural structure morphism.

*Proof.* By Lemma 6.3, only decorated graph  $\Gamma$  that has only one vertex  $v$  labeled by  $\infty$  may have nonzero localization contribution to the (6.9). Let's denote by  $\beta_{\star}$  the degree of the unique vertex  $v$  labeled by  $\infty$  coming from  $\vec{x}$ . Note the marked point  $q_{\star}$  must lie on a vertex labeled by  $\infty$  due to the choice of multiplicity at the marking  $q_{\star}$ . Thus the vertex  $v$  can't be a node linking two edges. Note one can assume all the other legs appear in vertexes labeled by 0 due to the restriction of multiplicity on the other legs and the fact  $\mu_0 = 0$ . Hence there are only two types of graph  $\Gamma$  depending on whether  $v$  is a stable or unstable vertex.

- (1) If the only vertex  $v$  over  $\infty$  in  $\Gamma$  is unstable, in the case,  $v$  is of valence 2, i.e. it's incident to an edge and an leg corresponding to the marking  $q_{\star}$ . Then  $\Gamma$  has only one edge whose degree  $(0, \frac{\delta}{r})$ , and has only one vertex over 0, which is incident to the edge. The vertex over 0 can be stable or unstable. If the vertex over 0 is unstable, it must be a marked point with input  $\mu_{\beta, p}$ , then the graph  $\Gamma$  contributes

$$\frac{\mu_{\beta, p}(\frac{\lambda-D_{\theta}}{\delta})}{\delta} \cdot \left( \frac{\lambda-D_{\theta}}{\delta} \right)^c$$

to (6.9). If the vertex over 0 is stable, then this type of graphs contributes

$$\sum_{m=0}^{\infty} \sum_{\substack{\beta_0+\dots+\beta_m=\beta \\ p_1+\dots+p_m=p}} \frac{1}{m!} (e\widetilde{v}_0)_* \left( \sum_{d=0}^{\infty} \epsilon'_* (c_d(-R^\bullet \pi_* \mathcal{L}_{-\theta}^{\frac{1}{s}})) \left( \frac{\lambda}{s} \right)^{-d} \right. \\ \left. \cap [\mathcal{K}_{0, \vec{m}_s \cup \{0\}}(\sqrt{L_{-\theta}/Y}, \beta_0)]^{\text{vir}} \cap \bigcap_{i=1}^m ev_i^*(\mu_{\beta_i, p_i}(-\bar{\psi}_i)) \cap \frac{\frac{1}{\delta}(\frac{\lambda - ev_0^* D_\theta}{\delta})^c}{\frac{\lambda - ev_0^* D_\theta}{s\delta} - \frac{\bar{\psi}_0}{s}} \right)$$

to (6.9), where  $\vec{m}_s \cup \{0\} = ((g_{\beta_1}^{-1}, \mu_s^{\beta_1(L_\theta)+p_1}), \dots, (g_{\beta_m}^{-1}, \mu_s^{\beta_m(L_\theta)+p_m})) \cup \{(g_\beta, \mu_s^{\beta(L_\theta)+p})\}$ ,  $\vec{m} \cup \{0\} = (g_{\beta_1}^{-1}, \dots, g_{\beta_m}^{-1}) \cup \{g_\beta\}$  and  $\epsilon' : \mathcal{K}_{0, \vec{m}_s \cup \{0\}}(\sqrt{L_{-\theta}/Y}, \beta_0) \rightarrow \mathcal{K}_{0, \vec{m} \cup \{0\}}(Y, \beta_0)$  is the natural structure morphism (c.f. [TT16]). By Lemma 6.5 proved below, the above formula is equal to

$$\sum_{m=0}^{\infty} \sum_{\substack{\beta_0+\dots+\beta_m=\beta \\ p_1+\dots+p_m=p}} \frac{1}{m!} \phi^\alpha \langle \mu_{\beta_1, p_1}(-\bar{\psi}_1), \dots, \mu_{\beta_m, p_m}(-\bar{\psi}_m), \frac{\frac{1}{\delta}(\frac{\lambda - D_\theta}{\delta})^c \phi_\alpha}{\frac{\lambda - D_\theta}{\delta} - \bar{\psi}_0} \rangle_{0, [m] \cup \{0\}, \beta_0}^\infty.$$

In summary, the localization contribution from the decorated graphs of which the vertex over  $\infty$  is unstable contributes

(6.12)

$$\mu_{\beta, p} \left( \frac{\lambda - D_\theta}{\delta} \right) \cdot \left( \frac{\lambda - D_\theta}{\delta} \right)^c \\ + \sum_{m=0}^{\infty} \sum_{\substack{\beta_0+\beta_1+\dots+\beta_m=\beta \\ p_1+\dots+p_m=p}} \frac{1}{m!} \phi^\alpha \langle \mu_{\beta_1, p_1}(-\bar{\psi}_1), \dots, \mu_{\beta_m, p_m}(-\bar{\psi}_m), \frac{\frac{1}{\delta}(\frac{\lambda - D_\theta}{\delta})^c \phi_\alpha}{\frac{\lambda - D_\theta}{\delta} - \bar{\psi}_0} \rangle_{0, [m] \cup \{0\}, \beta_0}^\infty$$

to the (6.9). Here we use the fact  $R^1 \pi_*(f^* \mathcal{L}_\infty^\vee)$  is of rank 0 over  $F_\Gamma$ , so its Euler class is 1.

- (2) If the only vertex  $v$  over  $\infty$  in  $\Gamma$  is stable, then  $v$  is incident to only one leg (corresponding to the marking  $q_\star$ ) and possible several edges (can be none). Let's assume that  $v$  has degree  $(\beta_\star, \frac{\delta_\star}{r})$  with  $\delta_\star = \beta_\star(L_\theta)$ . If there is no edges in the graph  $\Gamma$ , which happens if and only if  $\beta_\star = \beta$  and  $p = 0$ , this has contribution:

$$(6.13) \quad (e\widetilde{v}_\star)_* \left( \sum_{d=0}^{\infty} \epsilon_* (c_d(-R^\bullet \pi_*(f^* \mathcal{L}_\theta^{\frac{1}{r}}))) \left( \frac{-\lambda}{r} \right)^{-1-d} \cap [\mathcal{K}_{0, \star}(\sqrt{L_\theta/Y}, \beta)]^{\text{vir}} \cap \bar{\psi}_\star^c \right)$$

to (6.9). Otherwise, there are  $m$  ( $m \geq 1$ ) edges attached to the vertex  $v$ , let's index them by  $[m] := \{1, \dots, m\}$ , with degree  $(0, \frac{\delta_i}{r})$  on the  $i$ th edge  $e_i$  for  $\delta_i \in \mathbb{Z}_{>0}$ . On each edge  $e_i$  there is exactly one vertex  $v_i$  over 0 incident to it, which can't be a unstable vertex of valence 1 (see Remark 5.2) or a node linking two edges by Lemma 6.3. So  $v_i$  is either a marking or a stable vertex with only one node incident to the edge  $e_i$  and possible  $l$  marked points ( $l$  can be zero) on it, let's label the legs incident to  $v_i$  by  $\{i1, \dots, il\} \subset [n]$  ( $n$  is the total number of legs on  $\Gamma$ ).

Assume that the vertex  $v_i$  has degree  $(\beta_{i0}, 0)$ . Since the insertion at the marking  $q_{ij}$  on the curve  $C_{v_i}$  corresponding to  $v_i$  is of the form  $\mu_{\beta_{ij}, p_{ij}}(-\bar{\psi}_{ij})$  in (6.9), let's say the leg for  $q_{ij}$  has *virtual degree*  $(\beta_{ij}, p_{ij})$  contribution to the

vertex  $v_i$ , denote  $\beta_i$  to be summation of  $\beta_{i0}$  and all the virtual degrees from the markings on  $C_{v_i}$ , similarly for  $p_i$ . We call  $(\beta_i, p_i)$  the *total degree* at the vertex  $v_i$ . From the (6.9), one has

$$\beta_\star + \beta_1 + \cdots + \beta_m = \beta, \quad p_1 + \cdots + p_m = p.$$

Note to ensure such a graph  $\Gamma$  exists, one must have

$$(6.14) \quad \beta_i(L_\theta) + p_i = \delta_i.$$

Indeed, by Riemann-Roch Theorem, one has

$$\deg(N_1|_{C_{v_i}}) = -\frac{\beta_{i0}(L_\theta)}{s} = (1 - \frac{\delta_i}{s}) + \sum_{j=1}^l \frac{\beta_{ij}(L_\theta) + p_{ij}}{s} \mod \mathbb{Z}.$$

Here the first term on the left hand is the age of  $N_1$  at the node of  $C_{v_i}$ , and the second term on the right is the sum of the ages of  $N_1$  at the marked points on  $C_{v_i}$ . As  $s$  is sufficiently large, one must have

$$\frac{\delta_i}{s} = \frac{\beta_{i0}(L_\theta)}{s} + \sum_{j=1}^l \frac{\beta_{ij}(L_\theta) + p_{ij}}{s},$$

which implies that  $\beta_i(L_\theta) + p_i = \delta_i$ .

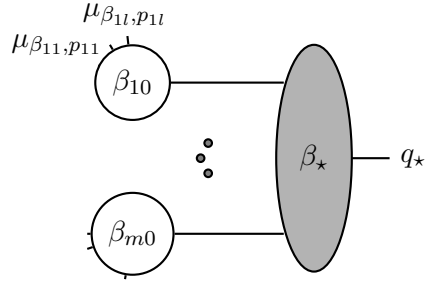


FIGURE 1. The ellipse dubbed gray on the right means the vertex labeled by  $\infty$  with a leg attached, and the two big circles on the left mean vertexes labeled by 0. The text inside the vertex means the decorated degree for this vertex. On the upper left vertex, texts near the legs mean the insertion terms. The three grey dots in the middle mean the other edges (together with its incident vertexes and legs on them) besides edges indexed by 1 and  $m$ .

We call a decorated graph  $\Gamma$  admissible if  $\Gamma$  has only one stable vertex  $v$  over  $\infty$  of degree  $\beta_\star$  and  $m$  ( $m \geq 1$ ) edges *labeled by*  $[m]$  incident to  $v$  such that the degree for each vertex over 0 satisfies (6.14). Note our definition of the admissible decorated graph has more decorations than the decorated graph introduced in Section 5 as we also label the edges. Then the automorphism group of an admissible decorated graph  $\Gamma$  is identity, which is usually smaller than the automorphism group of the corresponding decorated graph without labeling the edges. If we want to use admissible decorated graphs to compute

the localization contribution, we need to divide  $m!$  to offset the labeling as shown below.

Now we can group the admissible decorated graphs by the triple

$$(m, \beta_\star, ((\beta_1, p_1), \dots, (\beta_m, p_m))) .$$

Denote by  $\Lambda_{(m, \beta_\star, ((\beta_1, p_1), \dots, (\beta_m, p_m)))}$  the collection of all the admissible decorated graphs such that the vertex incident to the edge labeled by  $i$  has total degree  $(\beta_i, p_i)$ .

Now using the localization formula in §5.4 to compute the contribution from  $\Lambda_{(m, \beta_\star, ((\beta_1, p_1), \dots, (\beta_m, p_m)))}$  to (6.9). For any admissible decorated graph  $\Gamma$  in  $\Lambda_{(m, \beta_\star, ((\beta_1, p_1), \dots, (\beta_m, p_m)))}$ , they all have the same localization contribution for vertex and nodes over  $\infty$  and edges, as well as the same automorphism factor which comes from the gerbe structures of the edge moduli. Then the contribution from the vertex  $v_i$  together with node at  $v_i$  has localization contribution (after pushing forward to  $\bar{I}_\mu Y$  along  $\iota \circ (ev_{h_i})_*$ , where  $h_i$  is the node on  $v_i$  incident to the edge  $e_i$ ):

$$\begin{aligned} & \mu_{\beta_i, p_i} \left( \frac{\lambda - D_\theta}{\delta_i} \right) + \sum_{l=0}^{\infty} \sum_{\substack{\beta_0 + \beta_1 + \dots + \beta_l = \beta_i \\ p_1 + \dots + p_l = p_i}} \frac{1}{l!} (\widetilde{ev}_0)_* \left( \sum_{d=0}^{\infty} \epsilon'_*(c_d(-R^\bullet \pi_* \mathcal{L}_{-\theta}^{\frac{1}{s}}) \left( \frac{\lambda}{s} \right)^{-d} \right. \\ & \left. \cap [\mathcal{K}_{0, \bar{I} \cup \{0\}}(\sqrt{L_{-\theta}/Y}, \beta_0)]^{\text{vir}} \cap \bigcap_{j=1}^l ev_j^*(\mu_{(\beta_j, p_j)}(-\bar{\psi}_j)) \cap \frac{1}{\frac{\lambda - ev_0^* D_\theta}{\delta_j s} - \frac{\bar{\psi}_0}{s}} \right), \end{aligned}$$

which, by Lemma 6.5 below, is equal to  $f_{\beta_i, p_i}(z)|_{\frac{\lambda - D_\theta}{\delta_i}}$  by our definition of  $f_{\beta_i, p_i}(z)$ . Put contributions from vertexes, edges together and nodes together, it yields

$$\begin{aligned} & \frac{1}{m!} (\widetilde{ev}_\star)_* \left( \sum_{d=0}^{\infty} \epsilon_*(c_d(-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}}) \left( \frac{-\lambda}{r} \right)^{-1+m-d} \cap [\mathcal{K}_{0, \vec{m}_r \cup \star}(\sqrt{L_\theta/Y}, \beta_\star)]^{\text{vir}} \right. \\ & \left. \cap \prod_{i=1}^m \frac{ev_i^* \left( \frac{1}{\delta_i} (f_{\beta_i, p_i}(z)|_{z=\frac{\lambda - D_\theta}{\delta_i}}) \right)}{-\frac{\lambda - ev_i^* D_\theta}{r \delta_i} - \frac{\bar{\psi}_i}{r}} \cap \bar{\psi}_\star^c \right). \end{aligned}$$

Now go over all possible triples  $(m, \beta_\star, ((\beta_1, p_1), \dots, (\beta_m, p_m)))$ , it yields the summation:

$$\begin{aligned} & \sum_{m=1}^{\infty} \sum_{\substack{\beta_\star + \beta_1 + \dots + \beta_m = \beta \\ p_1 + \dots + p_m = p}} \frac{1}{m!} (\widetilde{ev}_\star)_* \left( \sum_{d=0}^{\infty} \epsilon_*(c_d(-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}}) \left( \frac{-\lambda}{r} \right)^{-1+m-d} \cap [\mathcal{K}_{0, \vec{m}_r \cup \star}(\sqrt{L_\theta/Y}, \beta_\star)]^{\text{vir}} \right. \\ & \left. \cap \prod_{i=1}^m \frac{ev_i^* \left( \frac{1}{\delta_i} (f_{\beta_i, p_i}(z)|_{z=\frac{\lambda - D_\theta}{\delta_i}}) \right)}{-\frac{\lambda - ev_i^* D_\theta}{r \delta_i} - \frac{\bar{\psi}_i}{r}} \cap \bar{\psi}_\star^c \right). \end{aligned}$$

By the discussion above, we can write (6.9) as the following:

$$\begin{aligned}
(6.15) \quad & \frac{\mu_\beta(\frac{\lambda-D_\theta}{\delta})}{\delta} \cdot (\frac{\lambda-D_\theta}{\delta})^c + \sum_{m=0}^{\infty} \sum_{\substack{\beta_0+\beta_1+\dots+\beta_m=\beta \\ p_1+\dots+p_m=p}} \frac{1}{m!} (\widetilde{ev}_0)_* \\
& \left( [\mathcal{K}_{0,[m]\cup\{0\}}(Y, \beta_0)]^{\text{vir}} \cap \bigcap_{i=1}^m ev_i^*(\mu_{\beta_i, p_i}(-\bar{\psi}_i)) \cap \frac{\frac{1}{\delta_0}(\frac{\lambda-ev_0^*D_\theta}{\delta_0})^c}{\frac{\lambda-ev_0^*D_\theta}{\delta_0} - \bar{\psi}_0} \right) \\
& + (\widetilde{ev}_\star)_* \left( \sum_{d=0}^{\infty} \epsilon_*(c_d(-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}}) (\frac{-\lambda}{r})^{-1-d} \cap [\mathcal{K}_{0,\star}(\sqrt{L_\theta/Y}, \beta)]^{\text{vir}} \cap \bar{\psi}_\star^c \right) \\
& - \sum_{m=1}^{\infty} \sum_{\substack{\beta_\star+\beta_1+\dots+\beta_m=\beta \\ p_1+\dots+p_m=p}} \frac{1}{m!} (\widetilde{ev}_\star)_* \left( \sum_{d=0}^{\infty} \epsilon_*(c_d(-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}}) (\frac{-\lambda}{r})^{-1+m-d} \right. \\
& \left. \cap [\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt{L_\theta/Y}, \beta_\star)]^{\text{vir}} \cap \prod_{i=1}^m \frac{ev_i^*(\frac{1}{\delta_i}(f_{\beta_i, p_i}(z)|_{z=\frac{\lambda-D_\theta}{\delta_i}}))}{\frac{\lambda-ev_i^*D_\theta}{r\delta_i} + \frac{\bar{\psi}_i}{r}} \cap \bar{\psi}_\star^c \right).
\end{aligned}$$

As (6.9) lies in  $H^*(\bar{I}_\mu Y, \mathbb{Q})[\lambda]$ , the coefficient of  $\lambda^{-1}$  term in (6.15) must vanish. Note that the coefficients before  $\lambda^{-1}$  in the first two terms in (6.15) yields (after replacing the index 0 by  $\star$ )

$$\sum_{m=0}^{\infty} \sum_{\substack{\beta_\star+\beta_1+\dots+\beta_m=\beta \\ p_1+\dots+p_m=p}} \frac{1}{m!} \phi^\alpha \langle \mu_{\beta_1, p_1}(-\bar{\psi}_1), \dots, \mu_{\beta_m, p_m}(-\bar{\psi}_m), \phi_\alpha \bar{\psi}_\star^c \rangle_{0, [m]\cup\star, \beta_\star}^\infty,$$

which is the left hand side of equality in (6.10). Then extract the coefficient of the  $\lambda^{-1}$  term in the third term in (6.15) and apply (6.8), this yields the first term on the right hand side of (6.10) up to a minus sign. Finally the  $\lambda^{-1}$  coefficient of the fourth term in (6.15) is equal to the second term on the right hand of equality in (6.10) up to a minus sign. This completes the proof of (6.10).  $\square$

**Lemma 6.5.** *when  $s$  is sufficiently large, one has*

$$\begin{aligned}
(6.16) \quad & (\widetilde{ev}_0)_* \left( \epsilon'_* \left( \sum_{d=0}^{\infty} c_d(-R^\bullet \pi_* \mathcal{L}_{-\theta}^{\frac{1}{s}}) (\frac{\lambda}{s})^{-d} \right. \right. \\
& \left. \left. \cap [\mathcal{K}_{0,\vec{m}_s\cup\{0\}}(\sqrt{L_{-\theta}/Y}, \beta_0)]^{\text{vir}} \cap \gamma \cap \bigcap_{i=1}^m ev_i^*(\mu_{\beta_i, p_i}(-\bar{\psi}_i)) \right) \right) \\
& = \frac{1}{s} (\widetilde{ev}_0)_* ([\mathcal{K}_{0,[m]\cup\{0\}}(Y, \beta_0)]^{\text{vir}} \cap \gamma \cap \bigcap_{i=1}^m ev_i^*(\mu_{\beta_i, p_i}(-\bar{\psi}_i))) ,
\end{aligned}$$

Here  $\epsilon' : \mathcal{K}_{0,\vec{m}_s\cup\{0\}}(\sqrt{L_{-\theta}/Y}, \beta_0) \rightarrow \mathcal{K}_{0,\vec{m}\cup\{0\}}(Y, \beta_0)$  is the natural structure map, where  $\vec{m}_s := ((g_{\beta_i}^{-1}, \mu_s^{\beta_i(L_\theta)+p_i}) : 1 \leq i \leq m) \cup \{(g_\beta, \mu_s^{-\beta(L_\theta)-\beta(\theta_0)-p})\}$ ,  $\vec{m} \cup \{0\} := (g_{\beta_i}^{-1} : 1 \leq i \leq m) \cup \{g_\beta\}$ ,  $\gamma$  is any chow cycle in  $A_*(\mathcal{K}_{0,\vec{m}\cup\{0\}}(\sqrt{L_{-\theta}/Y}, \beta))_{\mathbb{Q}}$ .

*Proof.* As  $\mu_{\beta_i, p_i}$  belongs to the twist sector  $H^*(\bar{I}_{g_{\beta_i}}^{-1}Y, \mathbb{Q})$ , by Riemann-Roch theorem, it's equivalent to prove the equality by replacing the right hand side of (6.16)

$$\frac{1}{s}(\widetilde{ev_0})_* \left( [\mathcal{K}_{0, \vec{m} \cup \{0\}}(Y, \beta_0)]^{\text{vir}} \cap \gamma \cap \bigcap_{i=1}^m ev_i^*(\mu_{\beta_i, p_i}(-\bar{\psi}_i)) \right).$$

We will first show that  $R^0\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}} = 0$  on  $\mathcal{K}_{0, \vec{m} \cup \{0\}}(\sqrt[s]{L_{-\theta}/Y}, \beta_0)$ , which implies that  $R^1\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}} = 0$  as  $R^\bullet\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}}$  has virtual rank 0 when  $s$  is sufficiently large. By Remark 5.4, when  $\beta_0 \neq 0$ , we have  $R^0\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}} = 0$ . So it remains to prove the case when  $\beta_0 = 0$ . Assume now that  $\beta_0 = 0$ , as the vertex  $v$  over 0 is stable, there must be some marked points on  $C_v$ . Assume  $q_i$  is one of the marked points with insertion  $\mu_{\beta_i, p_i}$ . Without loss of generality, we can assume  $(\beta_i, p_i) \neq 0$  for all  $i$  as  $\mu_0(z) = 0$  by the very definition. Note we have

$$age_{q_i}((\mathcal{L}_{-\theta}^{\frac{1}{s}})|_{C_v}) = \frac{\beta_i(L_\theta) + p_i}{s} \neq 0,$$

then the restricted line bundle  $L_{-\theta}^{\frac{1}{s}} := (\mathcal{L}_{-\theta}^{\frac{1}{s}})|_{C_v}$  can't have any nonzero section on  $C_v$ . Indeed the degree of the restriction of  $L_{-\theta}^{\frac{1}{s}}$  to every irreducible component is zero by Lemma 2.5 as the total degree  $\beta_0$  is zero, then a nonzero section of  $L_{-\theta}^{\frac{1}{s}}$  will trivialize the line bundle  $L_{-\theta}^{\frac{1}{s}}$ , this contradicts the fact that  $L_{-\theta}^{\frac{1}{s}}$  has nontrivial stacky structure at  $q_i$ .

Now as  $-R^\bullet\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}} = R^1\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}} = 0$ , (6.16) follows immediately from the identity

$$\epsilon'_*([\mathcal{K}_{0, \vec{m}_s \cup \{0\}}(\sqrt[s]{L_{-\theta}/Y}, \beta_0)]^{\text{vir}}) = \frac{1}{s}[\mathcal{K}_{0, \vec{m} \cup \{0\}}(Y, \beta_0)]^{\text{vir}},$$

which is proved in [TT16, Theorem 5.16].  $\square$

**6.3. Proof of Main Theorem.** Using the notation in the introduction, now we prove the main theorem:

**Theorem 6.6.** *After reindexing  $\text{Eff}(AY, G, \theta)$  by  $\text{Eff}(W, G, \theta)$ , one has:*

$$(6.17) \quad \mathbb{I}(q, t, z) =_{\text{Eff}(AY, G, \theta) \rightarrow \text{Eff}(W, G, \theta)} J(q, \mu(q, y), z),$$

where  $J(q, \mu(q, y), z)$  is defined by the  $J$ -function of  $Y$

$$(6.18) \quad J(q, \mathbf{t}, z) := \mathbb{1}_Y + \frac{\mathbf{t}(z)}{z} + \sum_{d \in \text{Eff}(AY, G, \theta)} \sum_{m \geq 0} \frac{q^\beta}{m!} \phi^\alpha \langle \mathbf{t}(-\bar{\psi}_1), \dots, \mathbf{t}(-\bar{\psi}_m), \frac{\phi_\alpha}{z(z - \bar{\psi}_\star)} \rangle_{0, [m] \cup \star, d}^\infty$$

*Proof.* According to the analysis in the introduction, it suffices to prove the following:

$$(6.19) \quad \left[ z \frac{\mathbf{t}^p}{p! z^p} \mathbb{I}_\beta(z) \right]_{z^{-c-1}} = \sum_{m=0}^{\infty} \sum_{\substack{\beta_0 + \beta_1 + \dots + \beta_m = \beta \\ p_1 + \dots + p_m = p}} \frac{1}{m!} \phi^\alpha \langle \mu_{\beta_1, p_1}(-\bar{\psi}_1), \dots, \mu_{\beta_m, p_m}(-\bar{\psi}_m), \phi_\alpha \bar{\psi}_\star^c \rangle_{0, [m] \cup \star, \beta_0}^\infty,$$



for any nonnegative integer  $c$  and degree  $(\beta, p)$ . Let's assume that (6.19) is proved for all degrees  $(\beta', p') \in \text{Eff}(W, G, \theta) \times \mathbb{N}$  with  $\beta'(L_\theta) + p' < \beta(L_\theta) + p$ . Then  $f_{\beta_i, p_i}(z)$  in (6.10) is equal to  $zI_{\beta_i, p_i}(z)$  by induction (note one can assume that  $(\beta_i, p_i) \neq 0$ ). Indeed, first notice that, in (6.10),  $f_{\beta_i, p_i}(z)$  appears only for the graph  $\Gamma$  which has stable vertex over  $\infty$  with degree  $(\beta_\star, \frac{\delta_\star}{r})$ . When  $\beta_\star$  is nonzero, it immediately follows that  $\beta_i(L_\theta) < \beta(L_\theta)$ , hence  $\beta_i(L_\theta) + p_i < \beta(L_\theta) + p$ ; otherwise, when  $\beta_\star = 0$ , as the unique vertex  $v$  over  $\infty$  is a stable vertex, then there are at least two edges in  $\Gamma$ , which implies that  $\beta_i(L_\theta) + p_i < \beta(L_\theta) + p$  as each vertex over 0 has nonzero total degree as it's equal to the degree  $\delta(e)$  of the edge incident to the vertex by (6.14). Then (6.19) immediately follows from Theorem 6.2 and 6.4.  $\square$

## 7. AN EXAMPLE

In this section, we will recover the quantum product computation by Corti for a cubic hypersurface  $Y$  which is cut off by the polynomial  $x_1^3 + x_2^3 + x_3^3 + x_4x_1$  in  $\mathbb{P}(1, 1, 1, 2)$ . The following is the table for (small) quantum product of  $Y$  obtained by Corti (see [MH14]):

	$\mathbb{1}$	$p$	$p^2$	$\mathbb{1}_{\frac{1}{2}}$
$\mathbb{1}$	$\mathbb{1}$	$p$	$p^2$	$\mathbb{1}_{\frac{1}{2}}$
$p$		$p^2 + 12r^2 + 3r\mathbb{1}_{\frac{1}{2}}$	$12r^2p$	$rp$
$p^2$			$108r^4 + 36r^3\mathbb{1}_{\frac{1}{2}}$	$12r^3$
$\mathbb{1}_{\frac{1}{2}}$				$\frac{1}{3}p^2 - 3r\mathbb{1}_{\frac{1}{2}}$

Here  $r = \frac{1}{2}q$ <sup>7</sup>,  $p$  is the hyperplane class of  $Y$  and  $\mathbb{1}_{\frac{1}{2}}$  is the fundamental class of the unique nontrivial twisted sector of  $H^*(\bar{I}_\mu Y, \mathbb{Q})$ . Due to the discussion in [MH14], the usual twisted  $I$ -function of  $\mathbb{P}(1, 1, 1, 2)$  only recovers the first two rows, and the rest two rows rely on Corti's key calculation

$$(7.1) \quad (\mathbb{1}_{\frac{1}{2}} \circ \mathbb{1}_{\frac{1}{2}}, \mathbb{1}_{\frac{1}{2}}) = \frac{-3}{2}r.$$

In the following, we will recover Corti's key calculation using the  $I$ -function by choosing a different GIT presentation of  $\mathbb{P}(1, 1, 1, 2)$ .

Choose the matrix

$$\rho = \begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} : \mathbb{Z}^2 \rightarrow \mathbb{Z}^5,$$

which gives the action of  $G := \mathbb{C}_t^* \times \mathbb{C}_z^*$  on  $W := \mathbb{C}^5$  so that the GIT (stack) quotient is still  $\mathbb{P}(1, 1, 1, 2)$  (with the choice of stability condition  $\theta = t^2z^3$ , this also corresponds to the S-extended data  $S = \{\frac{1}{2}\}$  in the sense of [CCIT19, CCIT15]). Consider the polynomial  $zx_1^3 + zx_2^3 + zx_3^3 + x_4x_1$ , then it cuts off a hypersurface in the new GIT stack quotient  $[W^{ss}(\theta)/G]$ , which is isomorphic to  $Y$ . Note this hypersurface doesn't come

<sup>7</sup>In [MH14], they use  $r = \frac{1}{2}q^{\frac{1}{2}}$ , their  $q^{\frac{1}{2}}$  corresponds our  $q$  here.

from a semi-positive line bundle on  $[W/G]$  in the sense of 2.6, in this case  $Y$  comes from the line bundle  $L_{t^3z}$  on  $[W/G]$ .

The monoid  $\text{Eff}(W, G, \theta)$  is generated by  $\beta_1, \beta_2 \in \text{Hom}(\chi(G), \mathbb{Q})$  such that

$$\begin{pmatrix} \beta_1(L_t) & \beta_1(L_z) \\ \beta_2(L_t) & \beta_2(L_z) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}.$$

Then we can think  $q := q^{\beta_1}$  generates the semigroup of degrees of *stable maps* to the hypersurface  $Y$  and  $x := q^{\beta_2}$  is a formal variable.

The  $I$ -function of  $Y$  using this new GIT presentation of  $\mathbb{P}(1, 1, 1, 2)$  is

$$\begin{aligned} I(q, x, z) = & \sum_{\substack{(l,k) \in \mathbb{N}^2 \\ \frac{3l-k}{2} \geq 0}} \frac{q^l x^k}{z^k k!} \frac{\prod_{i < 0} (p + (\frac{l-k}{2} - i)z)^3}{\prod_{i < \frac{l-k}{2}} (p + (\frac{l-k}{2} - i)z)^3} \\ & \frac{1}{\prod_{0 \leq i < l} (2p + (l-i)z)} \prod_{0 \leq i < \frac{3l-k}{2}} \left( 3p + (\frac{3l-k}{2} - i)z \right)^{\mathbb{1}_{\frac{k-l}{2}}} \\ & + \sum_{\substack{(l,k) \in \mathbb{N}^2 \\ \frac{3l-k}{2} \in \mathbb{Z}_{<0}}} \frac{q^l x^k}{z^k k!} \frac{\prod_{\frac{l-k}{2} < i < 0} (p + (\frac{l-k}{2} - i)z)^3}{\prod_{0 \leq i < l} (2p + (l-i)z)} \\ & \frac{1}{\prod_{\frac{3l-k}{2} < i < 0} \left( 3p + (\frac{3l-k}{2} - i)z \right)} \frac{1}{3} p^2 \\ & + \sum_{\substack{(l,k) \in \mathbb{N}^2 \\ \frac{3l-k}{2} \in Q_{<0} \setminus \mathbb{Z}_{<0}}} \frac{q^l x^k}{z^k k!} \frac{\prod_{\frac{l-k}{2} < i < 0} (p + (\frac{l-k}{2} - i)z)^3}{\prod_{0 \leq i < l} (2p + (l-i)z)} \\ & \frac{1}{\prod_{\frac{3l-k}{2} < i < 0} \left( 3p + (\frac{3l-k}{2} - i)z \right)} \frac{1}{3} p^2. \end{aligned} \tag{7.2}$$

where  $\mathbb{1}_{\frac{k-l}{2}} = \mathbb{1}_{\frac{1}{2}}$  if  $k-l$  is odd, otherwise  $\mathbb{1}_{\frac{k-l}{2}} = \mathbb{1}$ . We can show the following fact about  $I(q, x, z)$ :

$$I(q, x, z) = \mathbb{1} + \frac{x \mathbb{1}_{\frac{1}{2}} + qx \mathbb{1}}{z} + \mathcal{O}(x^3) + \mathcal{O}\left(\frac{1}{z^2}\right), \tag{7.3}$$

$$\frac{\partial I(q, x, z)}{\partial x} = \frac{\mathbb{1}_{\frac{1}{2}} + q \mathbb{1}}{z} + \frac{x(q^2 \mathbb{1} + \frac{1}{3}p^2 + \frac{q \mathbb{1}_{\frac{1}{2}}}{2})}{z^2} + \mathcal{O}(x^2) + \mathcal{O}\left(\frac{1}{z^3}\right), \tag{7.4}$$

and

$$\frac{\partial^2 I(q, x, z)}{\partial^2 x} = \frac{q^2 \mathbb{1} + \frac{1}{3}p^2 + \frac{q \mathbb{1}_{\frac{1}{2}}}{2}}{z^2} + \mathcal{O}(x) + \mathcal{O}\left(\frac{1}{z^3}\right). \tag{7.5}$$

Since  $-ze^{\frac{qx}{z}}I(q, x, -z)$  is a slice on the Givental's cone by string flow. Note we have the asymptotic expansion

$$(7.6) \quad ze^{\frac{-qx\mathbb{1}}{z}}I(q, x, z) = z\mathbb{1} + x\mathbb{1}_{\frac{1}{2}} + \mathcal{O}(x^3) + \mathcal{O}\left(\frac{1}{z}\right).$$

Then

$$(7.7) \quad ze^{\frac{-qx\mathbb{1}}{z}}I(q, x, z) = J^{Giv}(q, x\mathbb{1}_{\frac{1}{2}}, z) + \mathcal{O}(x^3),$$

where  $J^{Giv}(q, t, z)$  is Givental's  $J$ -function which has an asymptotic expansion

$$(7.8) \quad z\mathbb{1} + t + \mathcal{O}\left(\frac{1}{z}\right),$$

and  $t = \sum t^\alpha \phi_\alpha \in H^*(\bar{I}_\mu Y, \mathbb{Q})$ . We have the following standard fact about Givental's  $J$ -function (c.f. [Giv04]):

$$(7.9) \quad z \frac{\partial}{\partial t_\alpha} \frac{\partial}{\partial t_\beta} J^{Giv}(q, t, z) = \phi_\alpha \star_t \phi_\beta + \mathcal{O}(z^{-1}).$$

Now consider the function

$$(7.10) \quad z \frac{\partial^2}{\partial^2 x} \left( ze^{\frac{-qx\mathbb{1}}{z}} I(q, x, z) \right),$$

a direction computation using product rule yields:

$$(7.11) \quad q^2 e^{\frac{-qx\mathbb{1}}{z}} I(q, x, z) - 2zqe^{\frac{-qx\mathbb{1}}{z}} \frac{\partial}{\partial x} I(q, x, z) + z^2 e^{\frac{-qx\mathbb{1}}{z}} \frac{\partial^2}{\partial^2 x} I(q, x, z).$$

Apply (7.3), (7.4), (7.5) to the first term, second term and third term in (7.11), respectively, we have the following asymptotic expansion of (7.10):

$$(7.12) \quad q^2\mathbb{1} - 2q(\mathbb{1}_{\frac{1}{2}} + q\mathbb{1}) + (q^2\mathbb{1} + \frac{1}{3}p^2 + \frac{q\mathbb{1}_{\frac{1}{2}}}{2}) + \mathcal{O}(x) + \mathcal{O}(z^{-1}).$$

On the other hand, using equation (7.7), (7.9), one has another asymptotic expansion about (7.10):

$$(7.13) \quad \mathbb{1}_{\frac{1}{2}} \star_x \mathbb{1}_{\frac{1}{2}} + \mathcal{O}(x) + \mathcal{O}(z^{-1}).$$

Compare (7.12) and (7.13), after evaluating  $x = 0$  and ignoring all negative  $z$  powers, we have

$$\mathbb{1}_{\frac{1}{2}} \circ \mathbb{1}_{\frac{1}{2}} = \frac{1}{3}p^2 - \frac{3}{2}q\mathbb{1}_{\frac{1}{2}},$$

which recovers Corti's calculation<sup>8</sup> (7.1)!

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<sup>8</sup>Here the small quantum product  $\circ$  is defined by the specialization of the big quantum product  $\star_t$  to  $t = 0$ .

## REFERENCES

- [AGV08] Dan Abramovich, Tom Graber, and Angelo Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*, American Journal of Mathematics **130** (2008), no. 5, 1337–1398. [↑8](#), [11](#), [34](#)
- [AJT15] Elena Andreini, Yunfeng Jiang, and Hsian-Hua Tseng, *Gromov-Witten theory of root Gerbes I: Structure of genus 0 moduli spaces*, Journal of Differential Geometry **99** (2015Jan), no. 1, 1–45. [↑38](#)
- [AB95] M. F. Atiyah and R. Bott, *The Moment Map and Equivariant Cohomology*, Raoul Bott Collected Papers (1995), 241–268. [↑2](#)
- [CDLOGP91] Philip Candelas, Xenia C. De La Ossa, Paul S. Green, and Linda Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nuclear Physics B **359** (1991Jul), no. 1, 21–74. [↑1](#)
- [CLLL16] Huai-Liang Chang, Jun Li, Wei-Ping Li, and Chiu-Chu Melissa Liu, *An Effective Theory of Gw and Fjrw Invariants of Quintics Calabi-Yau Manifolds*, ArXiv e-prints (2016), available at [1603.06184](#). [↑18](#), [24](#)
- [CCFK15] Daewoong Cheong, Ionuț Ciocan-Fontanine, and Bumsig Kim, *Orbifold quasimap theory*, Mathematische Annalen **363** (2015Mar), no. 3-4, 777–816. [↑5](#), [6](#), [8](#), [10](#), [11](#), [13](#)
- [CFK14] Ionuț Ciocan-Fontanine and Bumsig Kim, *Wall-Crossing in Genus Zero Quasimap Theory and Mirror Maps*, Algebraic Geometry **1** (2014), no. 4, 400–448. [↑6](#), [11](#)
- [CFK16] Ionuț Ciocan-Fontanine and Bumsig Kim, *Big I-functions*, Development of Moduli Theory — Kyoto 2013, 2016, pp. 323–347. [↑3](#), [8](#), [9](#), [11](#), [17](#)
- [CFKM14] Ionuț Ciocan-Fontanine, Bumsig Kim, and Daves Maulik, *Stable quasimaps to GIT quotients*, Journal of Geometry and Physics **75** (2014Jan), 17–47. [↑8](#), [9](#), [11](#), [18](#), [35](#)
- [CJR17a] Emily Clader, Felix Janda, and Yongbin Ruan, *Higher-Genus Quasimap Wall-Crossing Via Localization*, ArXiv e-prints (2017), available at [1702.03427v1](#). [↑6](#), [7](#), [15](#), [19](#), [37](#)
- [CJR17b] ———, *Higher-Genus Wall-Crossing in the Gauged Linear Sigma Model*, CoRR (2017), available at [1706.05038](#). [↑6](#), [19](#), [37](#)
- [CG07] Thomas Coates and Alexander Givental, *Quantum Riemann-Roch, Lefschetz and Serre*, Annals of Mathematics **165** (2007), no. 1, 15–53. [↑1](#)
- [CCIT14] Tom Coates, Alessio Corti, Hiroshi Iritani, and Hsian-Hua Tseng, *Some Applications of the Mirror Theorem for Toric Stacks*, ArXiv e-prints (2014), available at [1401.2611v2](#). [↑5](#)
- [CCIT15] ———, *A mirror theorem for toric stacks*, Compositio Mathematica **151** (2015Jun), no. 10, 1878–1912. [↑1](#), [5](#), [57](#)
- [CCIT19] ———, *Some applications of the mirror theorem for toric stacks*, Advances in Theoretical and Mathematical Physics **23** (2019), no. 3, 767–802. [↑2](#), [57](#)
- [CCLT09] Tom Coates, Alessio Corti, Yuan-Pin Lee, and Hsian-Hua Tseng, *The quantum orbifold cohomology of weighted projective spaces*, Acta Mathematica **202** (2009), no. 2, 139–193. [↑1](#)
- [CGI<sup>+</sup>12] Tom Coates, Amin Gholampour, Hiroshi Iritani, Yunfeng Jiang, Paul Johnson, and Cristina Manolache, *The quantum Lefschetz hyperplane principle can fail for positive orbifold hypersurfaces*, Mathematical Research Letters **19** (2012), no. 5, 997–1005. [↑2](#)
- [Ful84] William Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 2, Springer-Verlag, Berlin, 1984. MR732620 [↑15](#), [25](#)
- [Giv96] Alexander Givental, *Equivariant Gromov-Witten invariants*, International Mathematics Research Notices **1996** (1996), no. 13, 613. [↑1](#)
- [Giv98] ———, *A Mirror Theorem for Toric Complete Intersections*, Topological Field Theory, Primitive Forms and Related Topics (1998), 141–175. [↑1](#)
- [Giv04] Alexander B. Givental, *Symplectic geometry of Frobenius structures*, Aspects of Mathematics (2004), 91–112. [↑59](#)
- [GP99] T. Graber and R. Pandharipande, *Localization of virtual classes*, Inventiones Mathematicae **135** (1999Jan), no. 2, 487–518. [↑2](#), [21](#)

- [GJR17] Shuai Guo, Felix Janda, and Yongbin Ruan, *A Mirror Theorem for Genus Two Gromov-Witten Invariants of Quintic Threefolds*, ArXiv e-prints (2017), available at [1709.07392](#).  
↑1
- [Gué19] Jérémy Guéré, *Hodge–Gromov–Witten theory* (2019), available at [1908.11409](#). ↑2
- [JPPZ17] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine, *Double ramification cycles on the moduli spaces of curves*, Publications mathématiques de l’IHÉS **125** (2017Jun), no. 1, 221–266. ↑6, 7, 32, 44
- [JPPZ18] ———, *Double Ramification Cycles With Target Varieties*, ArXiv e-prints (2018), available at [1812.10136](#). ↑6, 22, 39
- [JPT] P. Johnson, R. Pandharipande, and H.-H. Tseng, *Notes on local  $\mathbb{P}^1$ –orbifolds*. Available at <https://people.math.ethz.ch/~rahul/IPab.ps>. ↑46
- [KKP03] Bumsig Kim, Andrew Kresch, and Tony Pantev, *Functoriality in Intersection Theory and a Conjecture of Cox, Katz, and Lee*, Journal of Pure and Applied Algebra **179** (2003), no. 1-2, 127–136. ↑2
- [KO18] Bumsig Kim and Jeongseok Oh, *Localized Chern Characters for 2-periodic Complexes*, ArXiv e-prints (2018), available at [1804.03774v1](#). ↑8
- [LLY99] Bong H. Lian, Kefeng Liu, and S. T. Yau, *Mirror Principle I*, Surveys in Differential Geometry **5** (1999), no. 1, 405–454. ↑1
- [MH14] Guest Martin and Sakai Hironori, *Orbifold quantum D-modules associated to weighted projective spaces*, Commentarii Mathematici Helvetici **89** (2014), no. 2, 273–297 (English). ↑57
- [MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, Third, Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR1304906 ↑8
- [PT14] Rahul Pandharipande and Richard P Thomas, *13/2 ways of counting curves*, Moduli spaces **411** (2014), 282–333. ↑28
- [TT16] Xiang Tang and Hsian-Hua Tseng, *A Quantum Leray-Hirsch Theorem for Banded Gerbes*, ArXiv e-prints (2016), available at [1602.03564](#). ↑38, 47, 49, 52, 56
- [Zho19] Yang Zhou, *Quasimap Wall-Crossing for Git Quotients*, CoRR (2019), available at [1911.02745](#). ↑7
- [Zin08] Aleksey Zinger, *The Reduced Genus 1 Gromov-Witten Invariants of Calabi-Yau Hypersurfaces*, Journal of the American Mathematical Society **22** (2008), no. 3, 691–737. ↑1

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