A MIRROR THEOREM FOR GROMOV-WITTEN THEORY WITHOUT CONVEXITY

JUN WANG

ABSTRACT. We prove a Givental-style mirror theorem for all positive hypersurfaces in proper toric Deligne-Mumford stacks, which provides an explicit slice on Givental's Lagrangian cone for such targets. Here the convexity is not required here. Our proof relies on the quasimap theory and consists of two parts: (1) we use the technique of p-fields to compute small I-functions of the positive toric stack hypersurfaces; (2) we prove the genus zero quasimap wall-crossing conjecture for the small I-functions. We expect that the method developed here can be used to provide a mirror theorem for all hypersurfaces in proper toric Deligne-Mumford stacks.

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Date: Wednesday 6th November, 2019.

1. Introduction

In the past few decades, following predictions from string theory [CDLOGP91], a series of results known as mirror theorems has been proven; an incomplete list is [Giv96, CCIT15, CG07, Zin08, Giv98, CCLT09, LLY99, GJR17]. These theorems reveal elegant patterns and deep structures encoded in the collection of Gromov-Witten invariants of a given symplectic manifold or orbifold X. However, the scope of these results, and much of Gromov-Witten theory in general, is limited to the world of toric geometry; in all cases above, X is a complete intersection in a toric variety or certain complete intersection[1] in a toric stack [CCIT14]. The essential reason for this is that computing Gromov-Witten invariants of a toric variety (or stack) can be reduced, via technique of the localization theorem [AB95], to evaluating a certain sum over decorated graphs.

Smooth hypersurfaces in toric Deligne-Mumford stacks are the next class of spaces to consider, but less is known in this situation. The main difficulty comes from that the natural torus action on a toric stack usually does not preserve a given hypersurface in it. Hence we can't directly apply virtual localization to compute the Gromov-Witten invariants of the toric hypersurface. Alternatively, the usual way to compute the Gromov-Witten invariants of a given hypersurface is to use *quantum Lefschetz principle* [KKP03], which relates the twisted Gromov-Witten invariants of an ambient space X to the Gromov-Witten invariants of its hypersurface Y which is the zero locus of a section of a given line bundle L on X. However, there is a technical assumption called *convexity* for the line bundle L to apply the *quantum Lefschetz principle*. The convexity says, for any stable map $f : C \to X$, one has

$$H^1(C, f^*L) = 0 \ ,$$

which holds, for example, when the ambient space X is a projective variety, the source curve C is of genus zero and L is a positive line bundle. Even worse, a counterexample was found in [CGI+12] that *quantum Lefschetz principle* can fail for positive hypersurfaces in orbifolds. So there are limited methods to compute the genus zero Gromov-Witten invariants of orbifold hypersurfaces where the convexity fails (see [Gué19] for a recent update for certain hypersurfaces in weighted projective spaces), and a mirror theorem[2] for these targets is lacking for a long time in the literature.

The aim of this paper is to prove a mirror theorem for all positive[3] hypersurfaces in proper toric Deligne-Mumford stacks, where the convexity is not required. Our proof relies heavily on quasimap theory, and the proof consists of two parts: (1) we use the technique of p-fields to compute small I-functions of the positive toric stack hypersurfaces; (2) we prove the genus zero quasimap wall-crossing conjecture for the small I-functions.

1.1. Main results and Ideas of proof.

1.1.1. *Computation of small I-function.* Let X be a *semi-projective toric Deligne-Mumford stack* constructed by a GIT data (W, G, θ), where $W := \oplus_{\rho\in[n]}\mathbb{C}_\rho$, $G := (\mathbb{C}^*)^k$ and θ is an integral character of G as in Remark 2.8. Fix a character τ of G, which

[1] See the discussion of convexity below.

[2] In Givental's formalism, we need to construct an explicit slice on the Lagrangian cone.

[3] See Definition 2.6 for the definition of positive line bundles used in this paper.

determines a positive[1] line bundle $L := L_\tau$ on $\mathfrak{X} := [W/G]$ by Borel construction 1.3. View $W \times \mathbb{C}_\tau$ as a $G-$equivariant line bundle over W. Let $s \in \Gamma(W, W \times \mathbb{C}_\tau)^G$ be a $G-$invariant section, which cuts off a *nonsingular* hypersurface in the semistable loci $W^{ss}(\theta)$ and an irreducible hypersurface in W. Denote by AY the zero loci of s in W, then (AY, G, θ) will still be a GIT target, and the GIT stack quotient $Y := [AY^{ss}(\theta)/G]$ is the corresponding hypersurface of X with respect to the line bundle L and the section s.

The I-function $I(q, \boldsymbol{t}, z)$ of Y is defined by using the 0+-stable graph quasimap invariants of Y (see §3.1.1 for the precise definition). We will focus on the (restricted) small I-function of Y

$$I(q, z) :=_{\mathrm{Eff}(AY,G,\theta) \to \mathrm{Eff}(W,G,\theta)} I(q, 0, z) \ ,$$

where the long subscript here is to remind us that the corresponding homomorphism between Novikov rings should be applied to the RHS. We will compute $I(q, z)$ explicitly. The idea is based on the computation of small I-functions of toric stacks by using stacky loop space in [CCFK15], here we generalize all the settings there to include p-fields. To compute $I(q, z)$, first we will employ a quantum Lefschetz theorem for quasimaps (see Theorem 2.10) proved recently [KO18, CL18] to rewrite the small I-function of Y via the space of *graph quasimaps with fields* to X (see §2 for the definition, and also Lemma 3.7). Then a comparison of the space of *graph quasimaps with fields* to X and the *stacky loop space with fields* to X (see Lemma 3.6) further reduces the computation to a cosection localized analysis for the stacky loop space with fields as in §3.3.

The formula we obtained for the small I-function of Y is the following:

$$I(q, z) = \mathbb{1}_Y + \sum_{\beta \neq 0,\ \beta \in \mathrm{Eff(W,G,\theta)}} q^\beta \frac{\prod_{\rho:\beta(L_\rho)<0} \prod_{\beta(L_\rho) \leqslant i < 0} (D_\rho + (\beta(L_\rho) - i)z)}{\prod_{\rho:\beta(L_\rho)>0} \prod_{0 \leqslant i < \beta(L_\rho)} (D_\rho + (\beta(L_\rho) - i)z)}$$
$$\times \prod_{0 \leqslant i < \beta(L)} \big(c_1(L) + (\beta(L) - i)z\big) \mathbb{1}_{g_\beta^{-1}} \ .$$

See section 3.3 for the terminology appearing in $I(q, z)$. The above formula matches the formula for positive hypersurfaces in toric stacks where the convexity holds [CCIT14, §5] and the formula for a possibly positive ray divisor (given by a coordinate function corresponding to the ray) of a toric stack where the convexity may fail [CCIT15, CCFK15].

1.1.2. *Sketch of the proof of genus zero quasimap wall-crossing.* Now let's assume that X is a *proper toric Deligne-Mumford stack* constructed by a GIT data (W, G, θ) as above. Let $I(q, 0, z)$ be the small I-function of Y. Write $zI(q, 0, z)$ as a formal Laurent series in variable z:

$$\cdots + I_{-1}(q)z^2 + I_0(q)z + I_1(q) + \mathcal{O}(z^{-1}),$$

and define $\mu(q, z)$ to be the truncation in nonnegative z powers:

$$\mu(q, z) := [zI(q, 0, z) - z]_+ = \cdots + I_{-1}(q)z^2 + (I_0(q) - 1)z + I_1(q) \ .$$

By the definition of $I(q, 0, z)$, $zI(q, 0, z)$ admits an asymptotic expansion in q:

$$zI(q, 0, z) = z\mathbb{1}_Y + \mathcal{O}(q),$$

[1] See Definition 2.6 and Remark 2.7 about positive line bundles.

which implies that $\mu(q,z)=\mathcal{O}(q)$. Write $\mu(q,z)$ as

$$\mu(q,z)=\sum_{\substack{d\neq 0\\ d\in \mathrm{Eff}(AY,G,\theta)}}\mu_d(z)q^d$$

such that $\mu_d(z)$ is a polynomial in $H^*(\bar{I}_\mu Y,\mathbb{Q})[z]$. For convenience we set $\mu_0(z)=0$. Then the celebrated *genus zero quasimap wall-crossing conjecture* [CFK14, CCFK15] says:

Conjecture 1.1. *One have the following identity:*

$$I(q,0,z)=J(q,\mu(q,y),z), \tag{1.1}$$

where $J(q,\mu(q,y),z)$ is defined by the $J-$function $J(q,\mathbf{t},z)$[1]

$$\begin{aligned}J(q,\mathbf{t},z):=&\mathbb{1}_Y+\frac{\mathbf{t}(z)}{z}\\ &+\sum_{d\in\mathrm{Eff}(AY,G,\theta)}\sum_{m\geqslant 0}\frac{q^d}{m!}\phi^\alpha\langle\mathbf{t}(-\bar{\psi}_1),\cdots,\mathbf{t}(-\bar{\psi}_m),\frac{\phi_\alpha}{z(z-\bar{\psi}_\star)}\rangle^\infty_{0,[m]\cup\star,d}\ .\end{aligned}$$

Here the input $\mathbf{t}$ is an element in $qH^*(\bar{I}_\mu Y,\mathbb{Q})[y][[\mathbf{Eff}(AY,G,\theta)]]$[2]*, and $\mathbf{t}(z)$ (resp. $\mathbf{t}(-\bar{\psi}_i)$) means that we replace the variable y in $\mathbf{t}$ by z (resp. $-\bar{\psi}_i$). This implies that $-zI(q,0,-z)$ is a slice on Givental's Lagrangian cone of Y.*

We will prove a restricted version of the above conjecture after reindexing $\mathrm{Eff}(AY,G,\theta)$ as $\mathrm{Eff}(W,G,\theta)$. For any $\beta\in\mathrm{Eff}(W,G,\theta)$, we will denote

$$\phi^\alpha\langle\mathbf{t}(-\bar{\psi}_1),\cdots,\mathbf{t}(-\bar{\psi}_m),\frac{\phi_\alpha}{z(z-\bar{\psi}_\star)}\rangle^\infty_{0,[m]\cup\star,\beta}$$

to be

$$\sum_{\substack{d\in\mathrm{Eff}(AY,G,\theta)\\ (i_{\mathfrak{Y}})_*(d)=\beta}}\phi^\alpha\langle\mathbf{t}(-\bar{\psi}_1),\cdots,\mathbf{t}(-\bar{\psi}_m),\frac{\phi_\alpha}{z(z-\bar{\psi}_\star)}\rangle^\infty_{0,[m]\cup\star,d}\ ,$$

where $i_{\mathfrak{Y}}:\mathfrak{Y}:=[AY/G]\to\mathfrak{X}:=[W/G]$ is the natural inclusion map. We still use the notation $\mu(q,z)$ to denote

$$\sum_{\beta\in\mathrm{Eff}(W,G,\theta)}q^\beta\mu_\beta(z)\ ,$$

where

$$\mu_\beta(z)=\sum_{\substack{d\in\mathrm{Eff}(AY,G,\theta)\\ (i_{\mathfrak{Y}})_*(d)=\beta}}\mu_d(z)\ .$$

Note $\mu_\beta(z)$ is still a polynomial in $H^*(\bar{I}_\mu Y,\mathbb{Q})[z]$ by the formula $I(q,z)$ as above.

[1]Here we treat $J(q,\mathbf{t},z)$ as a functional, which means, after fixing the input $\mathbf{t}$, we think $J(q,\mathbf{t},z)$ as a formal series on the Novikov variable q and the variable z but not on the variable y.

[2]It means that $\mathbf{t}$ admits an expression as $\sum_{d\neq 0}q^d f_d$, where $f_d\in H^*(\bar{I}_\mu Y,\mathbb{Q})[y]$. This choice of input $\mathbf{t}$ gives a much less general definition of Givental's $J-$function in the usual literature, but it suffices for the need in this paper.

For arbitrary $\beta \in \mathrm{Eff}(W, G, \beta)$, multiplying both sides of (1.1) by z, and then considering the coefficients before q^β on both sides, we have:

(1.2)
$$\begin{aligned}\mathbf{Coeff}_{q^\beta}(zI(q,z)) &= \delta_{\beta,0}z + \mu_\beta(z)\\ &+\sum_{m=0}^{\infty}\sum_{\beta_0+\beta_1+\cdots+\beta_m=\beta}\frac{1}{m!}\phi^\alpha\langle\mu_{\beta_1}(-\bar{\psi}_1),\cdots,\mu_{\beta_m}(-\bar{\psi}_m),\frac{\phi_\alpha}{z-\bar{\psi}_\star}\rangle^{\infty}_{0,[m]\cup\star,\beta_0}\ .\end{aligned}$$

To prove (1.1), it suffices to prove (1.2) for every $\beta \in \mathrm{Eff}(\mathrm{W}, \mathrm{G}, \theta)$. Observe that when $\beta = 0$, (1.2) reduces to

$$z = z\ ,$$

which holds obviously. This will be the base case for our inductive proof of (1.2).

To prove (1.2), observe that the nonnegative parts in z on both sides of (1.2) are equal to $\mu_\beta(z)$ by the very definition of $\mu_\beta(z)$. It follows that, in order to prove (1.2), it suffices to show the truncations in negative z powers of (1.2)

$$\begin{aligned}&[\mathbf{Coeff}_{q^\beta}(zI(q,z))]_- =\\ &\sum_{m=0}^{\infty}\sum_{\beta_0+\beta_1+\cdots+\beta_m=\beta}\frac{1}{m!}\phi^\alpha\langle\mu_{\beta_1}(-\bar{\psi}_1),\cdots,\mu_{\beta_m}(-\bar{\psi}_m),\frac{\phi_\alpha}{z-\bar{\psi}_\star}\rangle^{\infty}_{0,[m]\cup\star,\beta_0}\end{aligned}\tag{1.3}$$

holds. Equivalently, it suffices to show, for any nonnegative integer c, one has

$$\begin{aligned}&[\mathbf{Coeff}_{q^\beta}(zI(q,z))]_{z^{-c-1}} =\\ &\sum_{m=0}^{\infty}\sum_{\beta_0+\beta_1+\cdots+\beta_m=\beta}\frac{1}{m!}\phi^\alpha\langle\mu_{\beta_1}(-\bar{\psi}_1),\cdots,\mu_{\beta_m}(-\bar{\psi}_m),\phi_\alpha\bar{\psi}_\star^c\rangle^{\infty}_{0,[m]\cup\star,\beta_0}\ .\end{aligned}\tag{1.4}$$

The idea to prove (1.4) is to show that both sides of (1.4) satisfy the same recursive relations (see Theorem 6.3 and Theorem 6.5) by induction on the degree β. This is done by considering two master spaces (see §4.1 and §5.1), which are root stack modification of the twisted graph spaces found in [CJR17a, CJR17b]. Then we apply virtual localization to calculate two auxiliary cycles (see (6.3) and (6.11)) corresponding to two master spaces and extract λ^{-1} coefficients (λ is an equivariant parameter). Finally, the polynomiality of the two auxiliary cycles provides recursive relations (see also Theorem 6.3 and Theorem 6.5) to finish the proof of the quasimap wall-crossing.

Remark 1.2. The proof of the wall-crossing here deals with the toric stack hypersurfaces directly and does not rely on twisted theory. As the targets (toric stack hypersurfaces) that we treat here do not have any good torus actions in general as opposed to previous proven examples (or convex hypersurfaces thereof) [CFK14, CCFK15], we expect the method developed here can be used to prove the genus zero quasimap wall-crossing conjecture for all GIT targets considered in quasimap theory.

The main geometrical input in the proof of wall-crossing here is inspired by the twisted graph space used in [CJR17b, CJR17a], where they use the genus zero quasimap wall-crossing as input to prove the high genus quasimap wall-crossing. So it may be surprising that certain modification of the twisted graph space can be used to prove the genus zero quasimap wall-crossing directly. Here the modification takes r-root of certain divisor on the twisted graph space, then the push-forward of Chern class of root bundle appears naturally when one applies the localization as in the work of double ramification cycle [JPPZ17, JPPZ18], where they need the Pixton's polynomiality of the push-forward in r and extract $r^0\lambda^{-1}$(λ is an equivariant parameter) coefficient of the push-forward of Chern class as in the proof of double ramification cycle relations in loc.cit. However it suffices to extract the λ^{-1} coefficient in this paper, hence it does not need Pixton's polynomiality.

Besides, Theorems 6.3 and 6.5 can be purely viewed as recursive type relations, which determine whether a formal series is a slice on the Lagrangian cone. This is analogous to Brown's recursive type's characterization of a slice on the Givental's Lagrangian cone [Bro13]. We will elaborate this viewpoint elsewhere to prove a mirror theorem for all hypersurfaces in toric stacks and other targets [Wan].

During the preparation of this work, the author learns that Yang Zhou has used a totally different method to prove the quasimap wall-crossing conjecture for all GIT quotients and all genera [Zho]. The author also learns that Felix Janda and Nawaz Sultani have a different way of computing the (S-extended) $I-$functions for some hypersurfaces in weighted projective spaces.

1.2. **Outline.** This rest of this paper is organized as follows. In §2, we will recall the quasimap theory and a generalization by adding p-fields, the author wants to draw readers' attention to the language of θ'-stable quasimaps (see Remark 2.3), where θ' can be a *rational character*, because it is more suitable than the language of ϵ-stable quasimaps for the later construction of the master space in §4. In §3, different types of graph spaces which are related to the I-function will be discussed and compared. Then we will finish the computation of the small I-functions of positive Toric stack hypersurfaces. In §4 and §5, we will construct two master spaces which carry $\mathbb{C}^*$-actions, a very explicit $\mathbb{C}^*$-localization computation which is based on localization computations [CJR17a, JPPZ17] will be presented, this part is technical, we encourage the reader to skip to go to §6 first and to refer back when needed. In §6, we will calculate two auxiliary cycles corresponding to the two master spaces via localization, they provide recursive relations to prove the genus zero quasimap wall-crossing conjecture for positive toric stack hypersurfaces.

Notations: In this paper, we will always assume that all algebraic stacks and algebraic schemes are locally of finite type over the base field $\mathbb{C}$. Given a GIT target (W, G, θ), we will use symbols $\mathfrak{X}, \mathfrak{Y}$... to mean the quotient stack $[W/G]$, symbols X, Y... to mean the corresponding GIT stack quotient $[W^{ss}(\theta)/G]$, $I_\mu X$, $I_\mu Y$... to mean the corresponding (cyclotomic) inertia stacks, and $\bar{I}_\mu X$, $\bar{I}_\mu Y$...to mean the corresponding rigidified inertia stacks.

We will use the following construction a lot throughout this paper.

Definition 1.3 (**Borel Construction**). *Let G be a linear algebraic group and W be a variety. Fix a right G-action on the variety W. For any character ρ of G, we will denote L_ρ to be the line bundle on the quotient stack $[W/G]$ defined by*

$$W \times_G \mathbb{C}_\rho := [(W \times \mathbb{C}_\rho)/G] \ ,$$

where $\mathbb{C}_\rho$ is the 1-dimensional representation of G via ρ and the action is given by

$$(x, u) \cdot g = (x \cdot g, \rho(g)u) \in W \times \mathbb{C}_\rho$$

for all $(x, u) \in W \times \mathbb{C}_\rho$ and $g \in G$. For any linear algebraic group T, if we have a left T-action on W which commutes with the right action of G, we will lift the line bundle L_ρ defined above to be a T-equivariant line bundle, which is induced from the (left) T action on $W \times \mathbb{C}_\rho$ in the way that T acts on $\mathbb{C}_\rho$ trivially. By abusing notations, we will use the same notation L_ρ to mean the corresponding invertible sheaf (or $T-$equivariant invertible sheaf) over $[W/G]$ unless stated otherwise.

1.3. **Acknowledgments.** First, I want to express my deep gratitude to my advisor Hsian-Hua Tseng for his support and guidance. I want to especially thank his excellent talk on relative Gromov-Witten invariants and orbifold Gromov-Witten invariants at OSU, which draws my attention to double ramification cycles, which eventually leads to the birth of this paper. I would like to thank the organizers of the 2017 FRG workshop "Crossing the Walls in Enumerative Geometry" at Columbia, which exposed me to the most recent process in the subject as well as many inspiring ideas. I also want to thank Yang Zhou and Cristina Manolache on discussions on quasimap theory and thank Honglu Fan and Fenglong You on discussions on Givental's formalism.

2. Quasimaps with fields

We first recall the definition of a *quasimap* to a GIT target and then allow quasimaps to include p-fields, our main reference is [CFKM14, CCFK15, CFK16, KO18]. By a GIT target, we mean a triple (W, G, θ), where W is an irreducible affine variety with locally complete intersection (l.c.i) singularity, G is a reductive group equipped with a right $G-$action on W and θ is an (integral) character of G. Denote by $\mathfrak{X} := [W/G]$ the quotient stack. Denote by W^{ss} (or $W^{ss}(\theta)$) the semistable locus in W, and by W^s (or $W^s(\theta)$) the stable locus. Throughout out this paper, for a GIT target (W, G, θ), we will always assume that $W^{ss}(\theta) = W^s(\theta)$ and the *GIT stack quotient*

$$X := [W^{ss}(\theta)/G]$$

is a smooth *Deligne-Mumford stack*, under which condition, X is always semi-projective, i.e. it's proper over the affine GIT quotient $\mathrm{Spec}(\mathbb{C}[W]^G)$ by the proj-construction of GIT quotient [CCFK15, §2.2][MFK94]:

$$\underline{X} = \mathbf{Proj} \oplus_{n=0}^{\infty} \Gamma(W, W \times \mathbb{C}_{n\theta})^G.$$

Let $\mathbf{e}$ be the least common multiple of the exponents $|\mathrm{Aut}(\bar{x})|$ of automorphism groups $\mathrm{Aut}(\bar{x})$ of all geometric points $\bar{x} \to X$ of X. Then, for any character ρ of G, the line bundle $L_\rho^{\otimes \mathbf{e}}$ is the pullback of a line bundle from the coarse moduli $\underline{X}$ of X, here the line bundle L_ρ is defined by the Borel (mixed) construction 1.3.

Definition 2.1. *Given a scheme S over Spec($\mathbb{C}$), $f = ((C, q_1, \cdots, q_m), P, x)$ is called a quasimap over S (alternatively θ-*quasimap* over S) of class (g, m, β) if it consists of the following data:*

(1) *$(C, q_1, \cdots, q_m)$ is a flat family of genus g twist curves over S [AGV08, §4], and m gerbe marked sections $q_1, \cdots, q_m$ over S, here we don't require the gerbe sections to be trivialized;*
(2) *P is a principal G-bundle on C;*

(3) *x is a section of the affine $W-$bundle $(P\times W)/G$ over C so that it determines a representable morphism $[x]: C\to\mathfrak{X}=[W/G]$ as the composition*

$$C \xrightarrow{x} (P\times W)/G \longrightarrow [W/G]\ .$$

We say that the quasimap f is of degree $\beta\in Hom_{\mathbb{Z}}(Pic(\mathfrak{X}),\mathbb{Q})$ if $\beta(L)=deg([x]^*L)$ for every line bundle $L\in Pic(\mathfrak{X})$;

(4) *The base locus of $[x]$ defined by $[x]^{-1}(\mathfrak{X}\backslash X)$ is purely of relative dimension zero over S.*

Sometimes we may also use the notation $f:(C,\boldsymbol{q}=(q_i))\to\mathfrak{X}$ or $((C,q_1,\cdots,q_m),[x])$ to mean a quasimap (or θ-quasimap). A quasimap f is *prestable* (or $\theta-$*prestable*) if the base locus are away from nodes and markings.

Remark 2.2. We can extend the definition of θ-prestable quasimap to allow any *rational character* θ' such that θ'-prestable quasimap is same as $\alpha\theta'$-prestable quasimap for any $\alpha\in\mathbb{Q}_{>0}$.

Consider a prestable quasimap f, since the base point is away from nodes and marking points, for each $q\in C$, as in [CFKM14, Definition 7.1.1], we define the length function $l_\theta(q)$ as follows:

$$l_\theta(q)=min\{\frac{([x]^*s)_q}{n}|\ s\in\Gamma(W,W\times\mathbb{C}_{n\theta})^G,\ [x]^*s\neq 0,\ n\in\mathbb{Z}_{>0}\}\ , \tag{2.1}$$

where $([x]^*s)_q$ is the coefficient of the divisor $([x]^*s)$ at q. Note here the length function l_θ depends on the *integral* character θ. We have the following important observation about the length function l_θ: choose $\alpha\in\mathbb{Q}_{>0}$ such that $\theta'=\frac{1}{\alpha}\theta$ is another integral character. Then

$$l_\theta=\alpha l_{\theta'}\ ,$$

then the length function l_θ can be defined for any *rational character* θ', i.e. choose $\alpha\in\mathbb{Q}_{>0}$ and an integral character θ such that $\theta'=\alpha\theta$, then we define

$$l_{\theta'}:=\alpha l_\theta$$

as in [CFK16, Definition 2.4], note the definition of $l_{\theta'}$ is independent of decomposition of θ' as a product of positive rational number α and an integral character θ by the above observation.

Fix a positive rational number $\epsilon\in\mathbb{Q}_{>0}$. Given a prestable quasimap f over $Spec(\mathbb{C})$, we say f is a ϵ-*stable* quasimap to X if f satisfies the following stability condition:

(1) the $\mathbb{Q}-$line bundle $(\phi_*([x]^*L_{\mathbf{e}\theta}))^{\frac{\epsilon}{\mathbf{e}}}\otimes\omega_{\underline{C}}^{log}$ on the coarse moduli curve $\underline{C}$ of C is ample. Here $\phi: C\to\underline{C}$ is the coarse moduli map. Note the line bundle $[x]^*L_{\mathbf{e}\theta}$ on C a pullback of a line bundle on the coarse curve $\underline{C}$ by the choice of $\mathbf{e}$ and the prestable condition. Here $\omega_{\underline{C}}^{log}=\omega_{\underline{C}}(\sum_{i=1}^m\underline{q_i})$ is the log dualizing invertible sheaf of the coarse moduli $\underline{C}$;

(2) $\epsilon l_\theta(q)\leqslant 1$ for any $q\in C$.

Remark 2.3 (θ'-*quasimap*)**.** Using the above generalization of length function $l_{\theta'}$ for a rational character θ', we can give the definition of θ'-*stable quasimap*: given a θ'-prestable quasimap $f=((C,q_1,\cdots,q_m),[x])$, we say f is a θ'-*stable* quasimap to $\mathfrak{X}$ if

(1) the $\mathbb{Q}-$line bundle $(\phi_*([x]^*L_{\mathbf{b}\epsilon\theta'}))^{\frac{1}{\mathbf{b}\epsilon}} \otimes \omega_{\underline{C}}^{log}$ on the coarse moduli curve $\underline{C}$ of C is ample. Here $\phi : C \to \underline{C}$ is the coarse moduli map, and $\mathbf{b}$ is a positive integer which makes $\mathbf{b}\theta'$ an integral character. Note the ampleness is dependent of choice of the positive integer $\mathbf{b}$.
(2) $l_{\theta'}(q) \leqslant 1$ for any $q \in C$.

Given a GIT target (W, G, θ), following [CFK16, Propsition 2.7], an essentially equivalent definition about ϵ-stable quasimaps to X is, but from a different point of view, the concept of a $\epsilon\theta-$*stable quasimap* to $\mathfrak{X}$. The concept of θ'-stable quasimap will play an important role in the construction of master space in section 4. For a *rational character* θ' *of* G, we will use the notation $Q^{\theta'}_{g,m}(\mathfrak{X}, \beta)$ to mean the moduli stack of θ'-stable quasimaps to the quotient stack $\mathfrak{X}$ of class (g, m, β), which is same as the space $Q^{\epsilon}_{0,m}([W^{ss}(\theta)/G], \beta)$ of ϵ-stable quasimaps to the GIT stack quotient $[W^{ss}(\theta)/G]$ of class (g, m, β), where $\theta' = \epsilon\theta$, ϵ is a positive rational number and θ is an integral character of G.

We call a prestable quasimap f over a scheme S is ϵ-*stable* if for every $\mathbb{C}$-point s of S, the restriction of f over s is ϵ-stable. We call f is $0+$*stable* if f is $\epsilon-$stable for every positive rational number $\epsilon \in \mathbb{Q}_{>0}$.

Definition 2.4. *A group homomorphism $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}\,\mathfrak{X}, \mathbb{Q})$ is called L_θ-*effective *if it is realized as a finite sum of classes of some quasimaps to X. Such elements form a semigroup with identity 0, denoted by $\mathrm{Eff}(W, G, \theta)$.*

We will need the following lemma proved in [CCFK15, Lemma 2.3].

Lemma 2.5. *If $((C, \boldsymbol{q}), [x])$ is a quasimap of degree β, then $\beta(L_\theta) \geqslant 0$. Moreover, $\beta(L_\theta) = 0$ if and only if $\beta = 0$, if and only if the quasimap is constant (i.e., $[x]$ is a map into X, factored through an inclusion $\mathbb{B}\Gamma \subset X$ of the classifying groupoid $\mathbb{B}\Gamma$ of a finite group Γ).*

Now we will give the definition of *positive line bundles* for quasimaps used in this paper.

Definition 2.6. *Given a GIT data (W, G, θ) which gives rise to a GIT stack quotient X, let $L := L_\tau$ be the line bundle associated to a character τ of G. Then we call L a positive line bundle on $\mathfrak{X}$ if*

$$\beta(L) \geqslant 0$$

whenever β is the degree of a quasimap to X. Sometimes if the GIT model of X and the character τ are clear in the context, we will also call the restricted line bundle $L|_X$ a positive line bundle on X. By abusing notations, we will denote the restricted line bundle on X to be L unless stated otherwise.

Remark 2.7. Note the definition of a positive line bundle on X depends on the GIT model of X and the character τ. By Lemma 2.5, the line bundle L_θ (or positive tensor power of L_θ) is always a positive line bundle. Note, using the standard GIT presentations of weighted projective spaces, the line bundles $\mathcal{O}(n)$ $(n > 0)$ on weighted projective spaces are always positive line bundles by the definition above.

In the following, we will give an explicit description of quasimaps to *toric Deligne-Mumford stacks.*

Remark 2.8 (Quasimaps to toric stack)**.** Recall the construction of a (semi-projective) toric Deligne-Mumford stack (or toric stack in short) by a GIT data (W, G, θ). Let $G = (\mathbb{C}^*)^k$, and $W := \oplus_{i=1}^n \mathbb{C}_{\rho_i}$ be a direct sum of 1-dimensional representations of G given by the characters $\rho_i \in \chi(G)$ for $1 \leqslant i \leqslant n$. We will denote $[n]$ to be the collection of (*not necessarily distinct*) characters ρ_i of G for $1 \leqslant i \leqslant n$. The toric stack X is defined to be the GIT stack quotient

$$[W^{ss}(\theta)/G].$$

Since we always assume that $W^{ss}(\theta) = W^s(\theta)$, then X is a *semi-projective Deligne-Mumford stack*.

Then in the definition of quasimaps to the toric stack X, we can replace the principal $G-$bundle P by k line bundles $(L_j : 1 \leqslant j \leqslant k)$ on C, and replace the section x in the definition of quasimap by n sections

$$\vec{x} = (x_i : 1 \leqslant i \leqslant n) \in \oplus_{\rho \in [n]} \Gamma(C, L_\rho) \ ,$$

where L_ρ is a line bundle on C defined by

$$L_\rho = \otimes_{j=1}^k L_j^{\otimes m_j} \ ,$$

where and the numbers $(m_j : 1 \leqslant j \leqslant k)$ are determined by the unique relation

$$\rho = \sum_{j=1}^k m_j \pi_j$$

in the character group $\chi(G)$ of G. Here $(\pi_j : \ 1 \leqslant j \leqslant k)$ are the standard characters of $G = (\mathbb{C}^*)^k$ by projecting to coordinates.

Now we extend the concept of quasimaps to include p-fields following [KO18].

Definition 2.9. *Consider a GIT data (W, G, θ). Let τ be a character of G, which determines a positive line bundle $L := L_\tau$ on X by the Borel construction 1.3. Let $s \in \Gamma(W, W \times \mathbb{C}_\tau)^G$ be a $G-$invariant section, which cuts off a* nonsingular *hypersurface in the semistable loci $W^{ss}(\theta)$ and an irreducible hypersurface in W. We denote AY to be the zero loci of s in W, then (AY, G, θ) will still be a GIT target, and the GIT stack quotient $Y := [AY^{ss}(\theta)/G]$ is the corresponding hypersurface of X with respect to the line bundle L and the section s. We will denote $\mathfrak{Y} := [AY/G]$ to be the quotient stack corresponding to Y.*

We call a pair (f, p) is a quasimap (resp. prestable quasimap, ϵ-stable quasimap to X) with field if $f = ((C, q_1, \cdots, q_m), [x])$ is a quasimap (resp. prestable quasimap, ϵ-stable quasimap to X), and p (called p-field) is a section of the line bundle $[x]^* L^\vee \otimes \omega_C$. We also call a quasimap with field (f, p) is a $\theta'-$stable quasimap with field to $\mathfrak{X}$ if the quasimap f is a $\theta'-$stable quasimap to $\mathfrak{X}$, note here θ' can be a rational number.

We will denote by $Q^\epsilon_{g,m}(X, \beta)^p$ the moduli stack of ϵ-stable quasimaps with fields to X of class (g, m, β) and by $Q^{\epsilon\theta}_{g,m}(\mathfrak{X}, \beta)^p$ the moduli stack of $\epsilon\theta$-stable quasimaps with fields to $\mathfrak{X}$. Similar to the case of stable maps with fields introduced in [CL11], there is a cosection defined for the perfect obstruction theory of $Q^\epsilon_{0,m}(X, \beta)^p$ (or $Q^{\epsilon\theta}_{g,m}(\mathfrak{X}, \beta)^p$) [KO18, CL18]. In [KO18, CL18], they prove a quantum Lefschetz theorem for $Q^\epsilon_{0,m}(X, \beta)^p$ (or $Q^{\epsilon\theta}_{g,m}(\mathfrak{X}, \beta)^p$).

Theorem 2.10. *For any degree $\beta \in Hom(\chi(G), \mathbb{Q})$, we have:*

$$[Q^{\epsilon}_{0,m}(Y,\beta)]^{\mathrm{vir}} = (-1)^{\chi(L)}[Q^{\epsilon}_{0,m}(X,\beta)^p]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*(Q^{\epsilon}_{0,m}(Y,\beta))_{\mathbb{Q}}$$

or

$$[Q^{\epsilon\theta}_{0,m}(\mathfrak{Y},\beta)]^{\mathrm{vir}} = (-1)^{\chi(L)}[Q^{\epsilon\theta}_{0,m}(\mathfrak{X},\beta)^p]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*(Q^{\epsilon}_{0,m}(Y,\beta))_{\mathbb{Q}} \ .$$

Remark 2.11. Consider a $\mathbb{C}^*$-action on W, which commutes with the action of G on W and preserves the zero locus $Z(s)$. Assume further that the cosection to define the cosection localized virtual cycle $[Q^{\epsilon}_{0,m}(X,\beta)^p]^{\mathrm{vir}}_{\mathrm{loc}}$ in [KO18, CL18] is also $\mathbb{C}^*$-equivariant. As a result, Theorem 2.10 also holds in the $\mathbb{C}^*-$equivariant setting. This will be used when the GIT target is a graph space in the next section.

2.1. **Quasimap invariants.** In this section, we recall the quasimap invariants for the orbifold case ($\mathfrak{X} = [W/G]$, $X = [W^{ss}(\theta)/G]$), closely following [CFK14, CFKM14, AGV08, CCFK15]. Fix an algebraic torus T action on W, which commutes with the given $G-$action. Here T can be the trivial group. Assume that the $T-$fixed loci $\underline{X}_0^T$ of the affine quotient $\underline{X}_0 = \mathrm{Spec}(\mathbb{C}[W]^G)$ is a finite set of points. Let $K := \mathbb{Q}(\{\lambda_i\})$ be the rational localized T-equivariant cohomology of $\mathrm{Spec}\,\mathbb{C}$, with $\{\lambda_1, \dots, \lambda_{\mathrm{rank}(T)}\}$ corresponding to a basis for the characters of T. The Novikov ring is defined to be

$$\Lambda_K := K[[\mathrm{Eff}(W,G,\theta)]].$$

We write q^β for the element corresponding to β in Λ_K so that Λ_K is the q-adic completion. We denote by $q\Lambda_K$ the maximal ideal generated by q^β, $\beta \neq 0$.

For any two elements α_1, α_2 in the $T-$equivariant *Chen-Ruan cohomology* of X,

$$H^*_{\mathrm{CR},T}(X,\mathbb{Q}) := H^*_T(\bar{I}_\mu X, \mathbb{Q}) \ ,$$

we define the Poincaré pairing in the *non-rigidified* cyclotomic inertia stack $I_\mu X$ of X" in the sense that

$$\langle \alpha_1, \alpha_2 \rangle_{\mathrm{orb}} := \int_{\sum_{r\in\mathbb{N}_{\geqslant 1}} r^{-1}[\bar{I}_{\mu_r}X]} \alpha_1 \cdot \iota^*\alpha_2 \ ,$$

with ι the involution of $\bar{I}_\mu X$ obtained from the inversion automorphisms. Let $\{\phi_\alpha\}$ be a basis of $H^*_{\mathrm{CR},T}(X,\mathbb{Q})$ and let $\{\phi^\alpha\}$ be the dual basis with respect to the Poincaré pairing defined above, then we have that

$$\sum_{r=1}^{\infty} r[\Delta_{\bar{I}_{\mu_r}X}] = \sum_\alpha \phi_\alpha \otimes \phi^\alpha \text{ in } H^*(\bar{I}_\mu X \times \bar{I}_\mu X, \mathbb{Q}),$$

where the diagonal class $[\Delta_{\bar{I}_{\mu_r}X}]$ is obtained via push-forward of the fundamental class by $(\mathrm{id},\iota) : \bar{I}_{\mu_r}X \to \bar{I}_{\mu_r}X \times \bar{I}_{\mu_r}X$.

Define by $\bar{\psi}_i$ the first Chern class of the universal cotangent line whose fiber at $((C, q_1, ..., q_m), [x])$ is the cotangent space of the coarse moduli $\underline{C}$ of C at i-th marking $\underline{q}_i$. For non-negative integers a_i and classes $\alpha_i \in H^*_T(\bar{I}_\mu X, \mathbb{Q})$, $t = \sum_j t_j\gamma_j$ with formal variables t_j, we write

$$\langle \alpha_1\bar\psi^{a_1},...,\alpha_m\bar\psi^{a_l}\rangle^{\epsilon}_{0,m,\beta} := \int_{[Q^{\epsilon}_{0,m}(X,\beta)]^{\mathrm{vir}}} \prod_i ev_i^*(\alpha_i)\bar\psi_i^{a_i};$$

$$\langle\!\langle \alpha_1\bar\psi^{a_1},...,\alpha_m\bar\psi^{a_m}\rangle\!\rangle^{\epsilon}_{m,\beta} := \sum_{l\geqslant 0}\frac{1}{l!}\langle \alpha_1\bar\psi^{a_1},...,\alpha_m\bar\psi^{a_m},t,...,t\rangle^{\epsilon}_{0,l+m,\beta};$$

$$\langle\!\langle \alpha_1\bar\psi^{a_1},...,\alpha_m\bar\psi^{a_m}\rangle\!\rangle^{\epsilon}_{m} := \sum_{\beta,l}\frac{q^\beta}{l!}\langle \alpha_1\bar\psi^{a_1},...,\alpha_m\bar\psi^{a_m},t,...,t\rangle^{\epsilon}_{0,l+m,\beta}$$
$$\in \Lambda_K[[\{t_j\}_j]]\ .$$

We may also define quasimap Chen-Ruan classes. Write

$$(\widetilde{ev_j})_* = \iota_*(\boldsymbol{r}_j(ev_j)_*), \tag{2.2}$$

where $\boldsymbol{r}_j$ is the order function of the band of the gerbe structure at the marking q_j. Define a class in $H_*^T(\bar I_\mu X)\cong H^*_T(\bar I_\mu X)$ by

$$\langle \alpha_1,...,\alpha_m,-\rangle^{\epsilon}_{0,\beta} := (\widetilde{ev_{m+1}})_*\Big((\prod ev_i^*\alpha_i)\cap[Q^{\epsilon}_{0,m}(X,\beta)]^{\mathrm{vir}}\Big)$$
$$= \sum_i \phi^\alpha\langle \alpha_1,...,\alpha_m,\phi_\alpha\rangle^{\epsilon}_{0,m+1,\beta}\ .$$

these are well-defined without T-localization, even when the coarse moduli space $\underline{X}$ is not projective.

3. Computation of small I-function of positive toric stack hypersurfaces

3.1. **The auxiliary graph spaces.**

3.1.1. *J^ϵ-function and graph space.* For readers' convenience, we will recall the definition of J^ϵ-function here, see more details in [CCFK15, §3.2][CFK14, §4]. Given a GIT target (W,G,θ) with $X:=[W^{ss}(\theta)/G]$, fix $g,m\geqslant 0$ and $\epsilon\geqslant 0+$, let $QG^{\epsilon}_{0,m,\beta}(X)$ denote the moduli stack of *ϵ-stable graph quasimaps* to X. First, by a prestable graph quasimap we mean the data

$$((C,q_1,...,q_m),[x]:=([x]_1,[x]_2))$$

with $((C,q_1,...,q_m),[x]_1)$ a m-pointed, genus 0 prestable quasimap to X and a map $[x]_2 : C\to\mathbb{P}^1$ for which the coarse moduli map $\underline{[x]}_2 : \underline{C}\to\mathbb{P}^1$ is a degree 1 map, that is, there is a unique component of $\underline{C}$ isomorphic to $\mathbb{P}^1$ under $\underline{[x]_2}$. For $\epsilon\in\mathbb{Q}_{>0}$, the ϵ-stability for a prestable graph quasimap is defined by imposing the requirements that

$$\omega_{\underline{C}}(\sum \underline{q}_i)\otimes(\phi_*([x]_1^*L_{\mathbf{e}\theta}))^{\epsilon/\mathbf{e}}\otimes\underline{[x]_2}^*\mathcal{O}_{\mathbb{P}^1}(3)\ \text{ is ample} \tag{3.1}$$

and that

$$\epsilon l_\theta(q)\leq 1,\ \text{ for every } \mathbb{C}\text{-point } q\in C. \tag{3.2}$$

Again, $l_\theta(q)$ in (3.2) is the length of the quasimap at q defined in (2.1). When the requirement (3.1) is true for every small enough $\epsilon\in\mathbb{Q}_{>0}$, we say that the graph quasimap is $(0+)$-*stable* (the length inequality imposes no condition and is discarded for $\epsilon=0+$).

Fix a (left) $\mathbb{C}^*$-action on $\mathbb{P}^1$ defined by

$$t[\zeta_1, \zeta_2] = [t\zeta_1, \zeta_2] \ ,$$

for $t \in \mathbb{C}^*, [\zeta_1, \zeta_2] \in \mathbb{P}^1$. This action induces a $\mathbb{C}^*$-action on $QG^\epsilon_{0,m,\beta}(X)$. This gives rise to the canonical $\mathbb{C}^*-$equivariant perfect obstruction theory on $QG^\epsilon_{0,m,\beta}(X)$ via the action of $\mathbb{C}^*$ on the tangent complex $\mathbb{T}_{\mathfrak{X}} \boxtimes \mathbb{T}_{\mathbb{P}^1}$ of $\mathfrak{X} \times \mathbb{P}^1$.

We use the same notation for the evaluation maps on graph space

$$ev_i : QG^\epsilon_{0,m,\beta}(X) \to \bar{I}_\mu X \times \mathbb{P}^1, \quad i = 1, \cdots, m.$$

Let z denote the $\mathbb{C}^*$-equivariant Euler class of the line bundle associated to the character of $\mathbb{C}^*$ with weight 1.

In what follows, for an integer $m \geqslant 0$, $[m]$ will mean the index set $\{1, \cdots, m\}$. Let $F(X) := QG^\epsilon_{0,[m]\cup\star,\beta}(X)^{\mathbb{C}^*}$ denote the $\mathbb{C}^*$-fixed substack of $QG^\epsilon_{0,[m]\cup\star,\beta}(X)$. Then $F(X)$ is a union of open and closed substacks $F^{A_1,\beta_1}_{A_2,\beta_2}(X)$, where $A_1 \cup A_2 = [m] \cup \star, \beta_1 + \beta_2 = \beta$. The component $F^{A_1,\beta_1}_{A_2,\beta_2}(X)$ corresponds to the distribution of marked points A_1 and class β_1 over $0 := [0, 1]$ and marked points A_2 and class β_2 over $\infty := [1, 0]$. Let η_0, η_∞ be the $\mathbb{C}^*$-equivariant class of $\mathbb{P}^1$ defined by the property

$$\eta_0|_0 = z, \ \ \eta_\infty|_\infty = -z, \ \ \eta_0|_\infty = 0 = \eta_\infty|_0.$$

Definition 3.1. *Given* $\mathbf{t} \in H^*(\bar{I}_\mu X, \Lambda)$, *we define the* J^ϵ*-function by*

$$\begin{aligned} J^\epsilon(q, \boldsymbol{t}, z) := &\sum_{\beta \in \mathrm{Eff}(W,G,\theta), m \geqslant 0} \frac{q^\beta}{m!} (\widetilde{\mathrm{pr}_{\bar{I}_\mu X} \circ ev_\star})_* \\ &\Big((ev_\star)^*(\eta_\infty) \cap \big(\prod_{i=1}^m ev_i^*(\boldsymbol{t})\big) \cap \frac{[F^{m,\beta}_{\star,0}(X)]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^{m,\beta}_{\star,0}/QG^\epsilon_{0,[m]\cup\{\star\},\beta}(X)})}\Big). \end{aligned} \tag{3.3}$$

Here $\mathrm{pr}_{\bar{I}_\mu X}$ *means the projection map from* $\bar{I}_\mu X \times \mathbb{P}^1$ *to* $\bar{I}_\mu X$. *The notation* $\widetilde{\mathrm{pr}_{\bar{I}_\mu X} \circ ev_\star}$ *is defined as in* (2.2).

For simplicity, we will denote $I(q, \boldsymbol{t}, z) := J^{0+}(q, \boldsymbol{t}, z)$ to be the I-function and write $J(q, \boldsymbol{t}, z) := J^\infty(q, \boldsymbol{t}, z)$ to be the J-function.

3.1.2. *Graph space with fields.* Using the same setting as in Definition 2.9. If we want to compute the J^ϵ-function of Y using quasimaps with fields, it will be necessary to state a version of quantum Lefschetz theorem for $\epsilon-$stable graph quasimaps, for which we first introduce the notation of graph quasimaps with fields to X analogous to the quasimaps with fields.

We will denote $QG^\epsilon_{0,m,\beta}(X)^p$ to be the moduli stack of $\epsilon-$stable graph quasimaps with fields to X. By a prestable graph quasimap with field we mean the data

$$((C, q_1, ..., q_m), [x] := ([x]_1, [x]_2), p) \ ,$$

where $((C, q_1, ..., q_m), P, ([x]_1, [x]_2))$ is a prestable graph quasimap and p is a section of the invertible sheaf $[x]_1^* L^\vee \otimes \omega_C$. For $\epsilon \in \mathbb{Q}_{>0} \cup \{0+\}$, the ϵ-stability for a prestable graph quasimap with field is defined by requiring the underlying graph quasimap to be $\epsilon-$stable as above.

In [CFK16], they show that every that ϵ-stable quasimap to the GIT stack quotient $[W^{ss}(\theta)/G]$ can equivalently seen as an $\epsilon\theta$-stable quasimap to quotient stack $[W/G]$. The key observation here is that an ϵ-stable graph quasimap with field to $[W^{ss}(\theta)/G]$ can be view as an $(\epsilon\theta \times 3\mathrm{id}_{\mathbb{C}^*})$-stbale quasimap with field to the quotient stack $[(W \times \mathbb{C}^2)/(G \times \mathbb{C}^*)]$.[1] Here the quotient stack $[(W \times \mathbb{C}^2)/(G \times \mathbb{C}^*)]$ is defined by the action

$$(x, (z_1, z_2)) \cdot (g, t) = (xg, (tz_1, tz_2))$$

for $(g, t) \in G \times \mathbb{C}^*$ and $(x, (z_1, z_2)) \in W \times \mathbb{C}^2$. Thus graph quasimaps also belongs to the category of quasimaps with fields to a suitable GIT target introduced before.

Using the section s defining the hypersurface Y, one can construct a cosection of the perfect obstruction theory of the space $QG^{\epsilon}_{0,m,\beta}(X)^p$. Since the standard $\mathbb{C}^*-$action on the graph space $X \times \mathbb{P}^1$ preserves the subspace $Y \times \mathbb{P}^1 \subset X \times \mathbb{P}^1$ and only scale $\mathbb{P}^1$. Following Remark 2.11 and the above discussion, we have the following theorem.

Theorem 3.2. *Let notations be as above. For any curve class $\beta \in Hom(Pic(\mathfrak{X}), \mathbb{Q})$, We have*

$$[QG^{\epsilon}_{0,m,\beta}(X)^p]^{\mathrm{vir}}_{\mathrm{loc}} = (-1)^{\chi(L)}[QG^{\epsilon}_{0,m,\beta}(Y)]^{\mathrm{vir}} \in A^{\mathbb{C}^*}_*(QG^{\epsilon}_{0,m,\beta}(Y))_{\mathbb{Q}}$$

for every $\epsilon \geqslant 0+$.

3.1.3. *Stacky loop space with fields.* **From now on**, we will fix a GIT data (W, G, θ), which represents a proper toric Deligne-Mumford stack (or toric stack in short) $X := [W^{ss}(\theta)/G]$ as in remark 2.8. We will also fix a *positive line bundle* $L := L_\tau$ on $\mathfrak{X} := [W/G]$ associated to a character τ of G. Let $s \in \Gamma(W, W \times \mathbb{C}_\tau)^G$ be a section such that the zero locus of s in W is irreducible. Assume that the section s cuts off a nonsingular hypersurface in the semistable loci $W^{ss}(\theta)$. Denote by AY to be the zero loci of s in W and by $AY^{ss}(\theta)$ (or AY^{ss}) be semistable loci, then (AY, G, θ) will also be a GIT target. Denote $Y := [AY^{ss}(\theta)/G]$ to be the corresponding toric stack hypersurface inside X and denote $\mathfrak{Y} := [AY/G]$ to be the corresponding quotients stack of Y.

Inspired by the computation of the small I-functions of toric stacks by using *stacky loop space* in [CCFK15], to compute the small $I-$functions of toric stack hypersurfaces and Theorem 2.10, we generalize the stacky loop space construction to include p-fields.

First of all, let's recall the definition of stacky loop space. Set $U = \mathbb{C}^2\backslash\{0\}$, for any positive integer a, denote $\mathbb{P}_{a,1}$ to be the quotient stack $[U/\mathbb{C}^*]$ defined by the $\mathbb{C}^*$-action on U with weights $[a, 1]$ so that $0 := [0 : 1]$ is a non-stacky point and $\infty := [1 : 0] \cong \mathbb{B}\mu_a$ is a stacky point. The stacky loop space

$$Q_{\mathbb{P}_{a,1}}(X, \beta) \subset Hom^{rep}_\beta(\mathbb{P}_{a,1}, \mathfrak{X})$$

is defined to be the moduli stack of representable morphisms from $\mathbb{P}_{a,1}$ to $\mathfrak{X}$ of degree β such that the generic point of $\mathbb{P}_{a,1}$ is mapped into X. Note the stacky point ∞ on $\mathbb{P}_{a,1}$ can also map into the unstable loci $\mathfrak{X}\backslash X$ unlike in the quasimap case. We can also incorporate *p-fields* into $Q_{\mathbb{P}_{a,1}}(X, \beta)$. Denote $Q_{\mathbb{P}_{a,1}}(X, \beta)^p$ to be *stacky loop space with fields*, which parameterizes the data

$$(f : \mathbb{P}_{a,1} \to \mathfrak{X}, p \in \Gamma(\mathbb{P}_{a,1}, f^*L^\vee \otimes \omega_{\mathbb{P}_{a,1}})),$$

[1] This is actually shown when G acts on $W^{ss}(\theta)$ freely in [CFK16], but it can be adapted to orbifold quasimap theory.

where $f \in Q_{\mathbb{P}_{a,1}}(X, \beta)$.

Remark 3.3. For any $\beta \in \mathrm{Eff}(W, G, \theta)$, the important observation here is that, for every representable morphism $f : \mathbb{P}_{a,1} \to \mathfrak{X}$ in $Hom_\beta(\mathbb{P}_{a,1}, \mathfrak{X})$, the degree of the line bundle $\omega_{\mathbb{P}_{a,1}} \otimes f^*(L^\vee)$ is negative by the definition 2.6. This implies the vanishing of the p-field for the morphism f, hence we have

$$Hom_\beta(\mathbb{P}_{a,1}, \mathfrak{X})^p = Hom_\beta(\mathbb{P}_{a,1}, \mathfrak{X}), \quad Q_{\mathbb{P}_{a,1}}(X, \beta)^p = Q_{\mathbb{P}_{a,1}}(X, \beta).$$

We begin with the following Lemma (this is also showed in [CCFK15, Lemma 4.6], but we present a proof with more details here.), which helps us to understand the geometry of hom-stack $Hom_\beta(\mathbb{P}_{a,1}, \mathfrak{X})$.

Lemma 3.4. *Let f be a morphism of degree β from $\mathbb{P}_{a,1}$ to the stack $\mathfrak{X} = [W/G]$. Then there exists a unique group homomorphism $\widetilde{\beta} : \mathbb{C}^* \to G$ such that f is induced from an equivariant morpshim $\widetilde{f}$ from U to W with respect to $\widetilde{\beta}$. Here $\widetilde{\beta}$ is related to β in the following way: for any character $\rho \in \chi(G)$, the composition $\rho \circ \widetilde{\beta}$ determines a character of $\mathbb{C}^*$ of weight $a\beta(L_\rho)$, where L_ρ is a line bundle on $\mathfrak{X}$ associated to the character ρ. Furthermore, the morphism f is representable if and only if a is the minimal positive integer such that $a\beta(L_\rho) \in \mathbb{Z}$ for any $\rho \in \chi(G)$.*

Proof. First of all, by descent (see for example [KL11, Proposition 3.2],[Vis05, §3.8]), the morphism f is equivalent to the following data:

(1) a morphism $h : U \to [W/G]$;
(2) a 2-isomorphism η between the composition $h \circ \mathrm{pr}_U$ and the composition $h \circ \sigma$. Here $\mathrm{pr}_U : U \times \mathbb{C}^* \to U$ is the projection map and $\sigma : U \times \mathbb{C}^* \to U$ is defined by the (right) group action of $\mathbb{C}^*$ on U.

We need that the coboundary $\partial\eta$ defined by

$$(\delta^*\eta)(m^*\eta)^{-1}(\gamma^*\eta) \in Aut(h \circ pr_U \circ \gamma)$$

is trivial. Here γ, m and δ are morphisms from $U \times \mathbb{C}^* \times \mathbb{C}^*$ to $U \times \mathbb{C}^*$ defined by

$$\gamma(u, t_1, t_2) = (\sigma(u, t_1), t_2), \quad m(u, t_1, t_2) = (u, t_1 t_2), \quad \delta(u, t_1, t_2) = (u, t_1)$$

for $(u, t_1, t_2) \in U \times \mathbb{C}^* \times \mathbb{C}^*$. Observe that any principal G-bundle P over U is a trivial bundle, hence h is represented by a morphism $\widetilde{f} : U \to W$ such that the diagram

$$\begin{tikzcd} P = U \times G \arrow[r, "(u,g) \mapsto \widetilde{f}(u)g"] \arrow[d, "(u,g) \mapsto u"] & W \\ U & \end{tikzcd}$$

gives the morphism h. Let $g : U \times \mathbb{C}^* \to G$ be a morphism which represents the 2−isomorphism η, we have

$$\widetilde{f}(\sigma(u, t)) = \widetilde{f}(u) \cdot g(u, t) \tag{3.4}$$

for $(u, t) \in U \times \mathbb{C}^*$. Here the dot means the (right) group action of G on W. Note that invertible functions on U are given by constant functions on U, by a result of Rosenlicht on invertible functions over a product of varieties [ACG11, Page 380, Lemma 6.2], the morphism g factors over the projection $pr_{\mathbb{C}^*}$ from $U\times\mathbb{C}^*$ to the second factor $\mathbb{C}^*$, then we can write g as a composition

$$g = \widetilde{\beta}\circ \mathrm{pr}_{\mathbb{C}^*},$$

where $\widetilde{\beta}$ is a morphism from $\mathbb{C}^*$ to G. Then the condition for the coboundary $\partial\eta$ to be trivial can be rewritten in plain language:

$$\widetilde{\beta}(t_1)\widetilde{\beta}(t_1t_2)^{-1}\widetilde{\beta}(t_2) = \mathbb{1}_G$$

for all $t_1, t_2 \in \mathbb{C}^*$, which implies that $\widetilde{\beta}$ is a group homomorphism. Now we can rewrite (3.4) as

$$\widetilde{f}(\sigma(u,t)) = \widetilde{f}(u)\cdot\widetilde{\beta}(t)\ ,$$

which implies $\widetilde{f}$ is an equivariant morphism from U to W with respect to the group homomorphism $\widetilde{\beta}$.

Note that the fact that f is representable is equivalent to the fact that $\widetilde{\beta}(\mu_a)$ is an element of G of order a, where μ_a is the a-th root of unity. Write $\widetilde{\beta}(\mu_a)$ as

$$(\exp(\frac{2\pi\sqrt{-1}\cdot n_1}{a}),\cdots,\exp(\frac{2\pi\sqrt{-1}\cdot n_k}{a}))\in G=(\mathbb{C}^*)^k,$$

where $(n_j:\ 1\leqslant j\leqslant k)$[1] is defined by the rule

$$\widetilde{\beta}(t) = (t^{n_1},\cdots,t^{n_k})\in G$$

for $t\in\mathbb{C}^*$. Then $\widetilde{\beta}(\mu_a)$ has order a if and only if $\gcd(n_1,\cdots,n_k)$ and a are coprime, this implies that f is representable if and only if a is the minimal integer such that $a\beta(L_\rho)\in\mathbb{Z}$ for any $\rho\in\chi(G)$. $\square$

Using the above lemma, it's shown in [CCFK15, §5.2] that we have the following description of $Q_{\mathbb{P}_{a,1}}(X,\beta)$:

Proposition 3.5. *Given $\beta\in \mathrm{Eff}(W,G,\beta)$, let a be the minimal positive integer associated to β by Lemma 3.4. Consider the finite dimensional vector space*

$$W_\beta := \bigoplus_{\rho\in[n]}\mathbb{C}[z_1,z_2]_{a\beta(L_\rho)}$$

with the G-action given by the direct sum of the diagonal G-action on $\mathbb{C}[z_1,z_2]_{a\beta(L_\rho)}$ by the weight ρ, so that $\mathbb{C}[z_1,z_2]_{a\beta(L_\rho)}\cong\bigoplus\mathbb{C}_\rho$.

Then we have the equivalence of the following two stacks:

$$\mathrm{Hom}_\beta(\mathbb{P}_{a,1},\mathfrak{X})\cong \mathrm{Hom}^{\mathrm{rep}}_\beta(\mathbb{P}_{a,1},\mathfrak{X})\cong[W_\beta/G]\ ,$$

under which correspondence, we have

$$Q_{\mathbb{P}_{a,1}}(X,\beta)\cong[W^{ss}_\beta(\theta)/G]\ .$$

In what follows, we will define a cosection for the natural perfect obstruction theory of $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$ over the Picard stack $\mathcal{P}ic^G = Hom(\mathbb{P}_{a,1},\mathbb{B}G)$ via the polynomial $s\in\Gamma(W,W\times\mathbb{C}_\tau)^G$. Moreover, we will define a $\mathbb{C}^*$-action on $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$ (see (3.13)), and show that the cosection constructed is indeed $\mathbb{C}^*$-equivariant, thus it yields a $\mathbb{C}^*$-equivariant cosection localized virtual cycle.

[1] Actually $n_j = a\beta(L_{\pi_j})$ for all $1\leqslant j\leqslant k$.

Consider the following diagram:

$$\begin{tikzcd} \mathcal{C} := Q_{\mathbb{P}_{a,1}}(X,\beta)^p \times \mathbb{P}_{a,1} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathfrak{X} \\ Q_{\mathbb{P}_{a,1}}(X,\beta)^p, & \end{tikzcd}$$

where $\pi : \mathcal{C} \to Q_{\mathbb{P}_{a,1}}(X,\beta)^p$ is the universal curve over $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$, and f is the universal map from $\mathcal{C}$ to $\mathfrak{X}$. Let $\mathscr{L}_{\rho_i}$ be the tautological invertible sheaves coming from the pull back of the line bundle L_{ρ_i} on $\mathfrak{X}$ along the universal map f. Let $\mathscr{L}$ be the invertible sheaf on $\mathcal{C}$ coming from the pull back of the line bundle L on $\mathfrak{X}$ along f. Denote ω_π to be the relative dualizing sheaf of $\mathcal{C}$ over $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$. Let $\mathscr{P} = \mathscr{L}^\vee \otimes \omega_\pi$ be the auxiliary sheaf, and let

$$\mathfrak{u}_i = f^* x_i \in \Gamma(\mathcal{C}, \mathscr{L}_{\rho_i})\ 1 \leqslant i \leqslant n \quad \text{and} \quad \mathfrak{p} \in \Gamma(\mathcal{C}, \mathscr{P})$$

be the tautological coordinate functions and universal p-field, respectively. Then the section $((\mathfrak{u}_i), \mathfrak{p})$ defines a universal section of

$$\mathrm{Vb}((\oplus_{i=1}^n \mathscr{L}_{\rho_i}) \oplus \mathscr{P}) \to \mathcal{C}.$$

Note $\mathfrak{p}$ is identically zero by Remark 3.3.

We define a multi-linear bundle morpshim over $\mathcal{C}$:

$$W: \quad \mathrm{Vb}((\oplus_{i=1}^n \mathscr{L}_{\rho_i}) \oplus \mathscr{P}) \to \mathrm{Vb}(\omega_\pi), \quad W(x,p) = p \cdot s(x_1, \cdots, x_n) \tag{3.5}$$

where $(x,p) = ((x_i)_{i=1}^n, p) \in \mathrm{Vb}((\oplus_{i=1}^n \mathscr{L}_{\rho_i}) \oplus \mathscr{P})$. This morphism is based on dual pairing $(\oplus_{i=1}^n \mathscr{L}_{\rho_i}) \oplus \mathscr{P} \to \omega_\pi$.

The morphism W induces a homomorphism of relative tangent complexes:

$$dW : \mathbb{T}_{\mathrm{Vb}((\oplus_{i=1}^n \mathscr{L}_{\rho_i}) \oplus \mathscr{P})/\mathcal{C}} \to W^* \mathbb{T}_{\mathrm{Vb}(\omega_\pi)/\mathcal{C}}\ . \tag{3.6}$$

In explicit form, for any closed $\xi \in \mathcal{C}$ and $(x,p) \in \mathrm{Vb}((\oplus_{i=1}^n \mathscr{L}_{\rho_i}) \oplus \mathscr{P})|_\xi$, $dW|_{(x,p)}$ sends

$$(\mathring{x}, \mathring{p}) = ((\mathring{x}_i), \mathring{p}) \in \mathbb{T}_{\mathrm{Vb}((\oplus_{i=1}^n \mathscr{L}_{\rho_i}) \oplus \mathscr{P})/\mathcal{C}}|_{(x,p)} = ((\oplus_{i=1}^n \mathscr{L}_{\rho_i}) \oplus \mathscr{P}) \otimes_{\mathscr{O}_\mathcal{C}} \boldsymbol{k}(\xi)$$

to

$$dW|_{(x,p)}(\mathring{x}, \mathring{p}) = s(x) \cdot \mathring{p} + p \cdot \sum_{i=1}^n \frac{\partial s}{\partial x_i}(x) \cdot \mathring{x}_i\ . \tag{3.7}$$

On the other hand, by pulling back the morphism dW to $\mathcal{C}$ along the universal section $\eta = (\mathfrak{u}, \mathfrak{p}) = ((\mathfrak{u}_i), \mathfrak{p})$, one has

$$\eta^* dW : \eta^* \mathbb{T}_{\mathrm{Vb}((\oplus_{i=1}^n \mathscr{L}_{\rho_i}) \oplus \mathscr{P})/\mathcal{C}} \to \eta^* W^* \mathbb{T}_{\mathrm{Vb}(\omega_\pi)/\mathcal{C}}\ . \tag{3.8}$$

Because the left hand side is canonically isomorphic to $(\oplus_{i=1}^n \mathscr{L}_{\rho_i}) \oplus \mathscr{P}$, and the right hand side is canonically isomorphic to ω_π, applying $R^\bullet \pi_*$, we obtain

$$\sigma_1^\bullet : R^\bullet \pi_*((\oplus_{i=1}^n \mathscr{L}_{\rho_i}) \oplus \mathscr{P}) \to R^\bullet \pi_*(\omega_\pi)\ . \tag{3.9}$$

We define

$$\sigma_1 : R^1 \pi_*((\oplus_{i=1}^n \mathscr{L}_{\rho_i}) \oplus \mathscr{P}) \to R^1 \pi_*(\omega_\pi) \cong \mathscr{O}_{Q_{\mathbb{P}_{a,1}}(X,\beta)^p}\ . \tag{3.10}$$

More concretely, for any smooth cover T of $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$, let $u=(u_i)=(\mathfrak{u}_i|_{\mathcal{C}_T})$ and $p=\mathfrak{p}|_{\mathcal{C}_T}$ be the restriction of $\mathfrak{u}=(\mathfrak{u}_i)$ $(1\leqslant i\leqslant n)$ and $\mathfrak{p}$ to the curve $C_T=\mathcal{C}\times_{Q_{\mathbb{P}_{a,1}}(X,\beta)^p}T$. Then for any

$$\mathring{u}_i\in H^1(C_T,\mathscr{L}_{\rho_i}|_{C_T})\quad\text{and}\quad \mathring{p}\in H^1(C_T,\mathscr{P}|_{\mathcal{C}_T})\ ,$$

one has

$$\sigma_1|_T(\mathring{u},\mathring{p})=s(u)\cdot\mathring{p}+p\cdot\sum_{i=1}^n\frac{\partial s}{\partial x_i}(u)\cdot\mathring{u}_i\in H^1(C_T,\omega_\pi)=\mathbb{C}\ .$$

Observe that $p\equiv 0$ when L is a positive line bundle by Remark 3.3, the above simplifies to

$$\sigma_1|_T(\mathring{u},\mathring{p})=s(u)\cdot\mathring{p}\ . \tag{3.11}$$

Recall that the natural perfect relative obstruction theory (cf. [CL11, Proposition 2.5]) for $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$ over $\mathcal{P}ic^G$ is given by

$$\phi_{Q_{\mathbb{P}_{a,1}}(X,\beta)^p/\mathcal{P}ic^G}:\mathbb{T}_{Q_{\mathbb{P}_{a,1}}(X,\beta)^p/\mathcal{P}ic^G}\to\mathbb{E}_{Q_{\mathbb{P}_{a,1}}(X,\beta)^p/\mathcal{P}ic^G}:=R^\bullet\pi_*((\oplus_{i=1}^n\mathscr{L}_{\rho_i})\oplus\mathscr{P}),$$

hence (3.10) defines a cosection of the relative perfect obstruction theory of $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$ over $\mathcal{P}ic^G$.

This cosection (3.10) can be lifted to be a cosection of the absolute perfect obstruction theory

$$\phi_{Q_{\mathbb{P}_{a,1}}(X,\beta)^p}:\mathbb{T}_{Q_{\mathbb{P}_{a,1}}(X,\beta)^p}\to\mathbb{E}_{Q_{\mathbb{P}_{a,1}}(X,\beta)^p} \tag{3.12}$$

as argued in [CL11, Proposition 3.5]. However, as the domain curve in $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$ is rational, there is no infinitesimal obstructions or deformations to deform the line bundles, so we have

$$\mathcal{O}b_{Q_{\mathbb{P}_{a,1}}(X,\beta)^p}=\mathcal{O}b_{Q_{\mathbb{P}_{a,1}}(X,\beta)^p/\mathcal{P}ic^G}=R^1\pi_*((\oplus_{i=1}^n\mathscr{L}_{\rho_i})\oplus\mathscr{P}).$$

Hence we can continue to use (3.10) to construct the cosection σ for the absolute perfect obstruction theory (3.12).

Now we define a (left) $\mathbb{C}^*$-action on $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$. By Proposition 3.5 and Remark 3.3, we have

$$Q_{\mathbb{P}_{a,1}}(X,\beta)^p=Q_{\mathbb{P}_{a,1}}(X,\beta)=[W_\beta^{ss}(\theta)/G]$$

and

$$W_\beta:=\bigoplus_{\rho\in[n]}\mathbb{C}[z_1,z_2]_{a\beta(L_\rho)},$$

where an element of $\mathbb{C}[z_1,z_2]_n$ is a polynomial $f(z_1,z_2)$ in $\mathbb{C}[z_1,z_2]$ satisfying

$$f(t^az_1,tz_2)=t^nf(z_1,z_2)$$

for all $t\in\mathbb{C}^*$. Define a $\mathbb{C}^*$-action on W_β by

$$t\cdot\big(f_1(z_1,z_2),\cdots,f_n(z_1,z_2)\big)=(f_1(t^{-1}z_1,z_2),\cdots,f_n(t^{-1}z_1,z_2)) \tag{3.13}$$

for $t\in T$ and $f_i(z_1,z_2)\in\mathbb{C}[z_1,z_2]_{a\beta(L_{\rho_i})}$ with $1\leqslant i\leqslant n$. Note this $\mathbb{C}^*$-action preserves $W_\beta^{ss}(\theta)$, so it induces a $\mathbb{C}^*$-action on $Q_{\mathbb{P}_{a,1}}(X,\beta)$ as well as on $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$.

To show that the cosection defined in (3.10) is $\mathbb{C}^*-$equivariant, one needs to lift the $\mathbb{C}^*$-action on $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$ to be a $\mathbb{C}^*-$action on the universal curve $\mathcal{C}$ and the locally free sheaf $\oplus_{i=1}^n \mathscr{L}_{\rho_i} \oplus \mathscr{P}$. To begin with, we first define the T-action on the universal curve $\mathcal{C} = Q_{\mathbb{P}_{a,1}}(X,\beta)^p \times \mathbb{P}_{a,1}$: define a $\mathbb{C}^*$-action on $W_\beta \times U$ by the map

(3.14)
$$t \cdot (f_1(z_1,z_2), \cdots, f_n(z_1,z_2), (\zeta_1,\zeta_2)) = (f_1(t^{-1}z_1,z_2), \cdots, f_n(t^{-1}z_1,z_2), (t\zeta_1,\zeta_2)) \ ,$$

for $(\zeta_1,\zeta_2) \in U$ and $f_i(z_1,z_2) \in \mathbb{C}[z_1,z_2]_{a\beta(L_{\rho_i})}$ with $1 \leqslant i \leqslant n$. It descents to define a $\mathbb{C}^*$-action on the universal curve $\mathcal{C}$, which maps equivariantly to $\mathfrak{X}$, where $\mathfrak{X}$ is equipped with the trivial $\mathbb{C}^*$-action, indeed the universal map f is induced from the morphism from $W_\beta \times U$ to W defined by

$$(f_1(z_1,z_2), \cdots, f_n(z_1,z_2), (\zeta_1,\zeta_2)) \mapsto (f_1(\zeta_1,\zeta_2), \cdots, f_n(\zeta_1,\zeta_2)),$$

for all $(\zeta_1,\zeta_2) \in U$ and $f_i(z_1,z_2) \in \mathbb{C}[z_1,z_2]_{a\beta(L_{\rho_i})}$ with $1 \leqslant i \leqslant n$.

Denote by pr_G (resp. $\mathrm{pr}_{\mathbb{C}^*}$) the projection from $G \times \mathbb{C}^*$ to G (resp. $\mathbb{C}^*$). For any character ρ of G, using the GIT presentation of the universal curve

$$\mathcal{C} = [(W_\beta^{ss} \times U)/(G \times \mathbb{C}^*)] \ ,$$

we denote $\mathcal{L}_\rho$ to be the canonical $\mathbb{C}^*-$equivariant invertible sheaf on $\mathcal{C}$ associated to the character $\rho \circ \mathrm{pr_G}$ of $G \times \mathbb{C}^*$ by the Borel construction. For any character χ of $\mathbb{C}^*$, we denote $\mathcal{M}_\chi$ to be the canonical $\mathbb{C}^*$-equivariant invertible sheaf on $\mathcal{C}$ associated to the character $\chi \circ \mathrm{pr}_{\mathbb{C}^*}$ of $G \times \mathbb{C}^*$ by the Borel construction.

Then the tautological coordinates $\mathfrak{u} = (\mathfrak{u}_i)$ live in the $\mathbb{C}^*$-equivariant locally free sheaf $\oplus_{i=1}^n(\mathcal{L}_{\rho_i} \otimes \mathcal{M}_{a\beta(L_{\rho_i})})$, and the universal p-field $\mathfrak{p}$ lives in the $\mathbb{C}^*$-equivariant invertible sheaf $(\mathcal{L}_\tau \otimes \mathcal{M}_{a\beta(L)})^\vee \otimes \mathcal{M}_{-a-1} \otimes \mathbb{C}_{\frac{1}{a}}$, where $\mathbb{C}_{\frac{1}{a}}$ means the trivial invertible sheaf on $\mathcal{C}$ with $\mathbb{C}^*$-linearization of weight $\frac{1}{a}$. Besides, the relative dualizing sheaf ω_π is isomorphic to $\mathcal{M}_{-a-1} \otimes \mathbb{C}_{\frac{1}{a}}$ as $\mathbb{C}^*-$equivariant invertible sheaves. By an abuse of notation, we will continue to use the notation $\mathscr{L}_i$ and $\mathscr{P}$ to mean

$$\mathscr{L}_i = \mathcal{L}_{\rho_i} \otimes \mathcal{M}_{a\beta(L_{\rho_i})}, \quad \mathscr{P} = (\mathcal{L}_\tau \otimes \mathcal{M}_{a\beta(L)})^\vee \otimes \mathcal{M}_{-a-1} \otimes \mathbb{C}_{\frac{1}{a}}$$

as $\mathbb{C}^*$-equivariant line bundles over $\mathcal{C}$, respectively.

Equipped with these notations, one can easily check that the morphism W in (3.5) is equivariant, which implies that dW in (3.7) are $\mathbb{C}^*$-equivariant. Besides, view the universal section $\eta = (\mathfrak{u}, \mathfrak{p})$ as the $\mathbb{C}^*$-equivariant section of the $\mathbb{C}^*$-equivariant vector bundle $\mathrm{Vb}(\oplus_{i=1}^n \mathscr{L}_{\rho_i} \oplus \mathscr{P})$. Since we define the cosection (3.9) via first pulling back dW along the equivariant universal section η and then pushing forward along the $\mathbb{C}^*$-equivariant morphism π, these immediately implies the cosection (3.10) is also $\mathbb{C}^*$-equivariant.

3.2. **A comparison of local geometry of three spaces.** In this section, fix β, let a be the minimal positive integer associated to β as in Lemma 3.4. We will compare the local geometry around the $\mathbb{C}^*$-fixed loci of two spaces $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$ and $QG^{0+}_{0,*,\beta}(X)^p$.

Denote F_β^p to be the subspace of $Q_{\mathbb{P}_{a,1}}(\mathfrak{X},\beta)^p$ in which the representable morphism $f: \mathbb{P}_{a,1} \to \mathfrak{X}$ has a base point at $[0:1]$ with length $\beta(L)$ and vanishing p field. More explicitly, F_β^p comprises the morphisms in the form

$$f: \mathbb{P}_{a,1} \to \mathfrak{X}, \quad (\zeta_1,\zeta_2) \mapsto (a_\rho \zeta_1^{\beta(L_\rho)})_{\rho \in [n]} \ ,$$

where $(a_\rho z_1^{\beta(L_\rho)} : \rho \in [n]) \in W_\beta^{ss}(\theta)$ and we treat $\zeta_1^{\beta(L_\rho)}$ as 0 when $\beta(L_\rho) \notin \mathbb{Z}_{\geqslant 0}$. In particular, by Remark 3.3, that F_β^p is a component of the fixed loci of $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$ under the $\mathbb{C}^*$-action defined in (3.13).

The $\mathbb{C}^*$-action on $X \times \mathbb{P}^1$ induces a natural $\mathbb{C}^*$-action on $QG^{0+}_{0,\star,\beta}(X)^p$ where the cosection is also $\mathbb{C}^*-$equivariant as the $\mathbb{C}^*$-action only scales the $\mathbb{P}^1$ factor. Among the $\mathbb{C}^*$-fixed loci of $QG^{0+}_{0,\star,\beta}(X)^p$, there is a special component $F^{\emptyset,\beta}_{\star,0}(X)^p$ consisting of the graph quasimaps with fields whose underlying graph quasimap belongs to $F^{\emptyset,\beta}_{\star,0}(X)$ as described in the definition of J^{0+} function of X. Note every source curve in $F^{\emptyset,\beta}_{\star,0}(X)^p$ is isomorphic to $\mathbb{P}_{a,1}$, then the p-field associated to the graph quasimap $(C,\star,([x]_1,[x]_2),p)$ in $F^{\emptyset,\beta}_{\star,0}(X)^p$ vanishes as the degree of the line bundle $[x]_1^*L^\vee \otimes \omega_{\mathbb{P}_{a,1}}$ is negative by the positivity of the line bundle L, then one has[1]

$$F^{\emptyset,\beta}_{\star,0}(X)^p = F^{\emptyset,\beta}_{\star,0}(X)\ .$$

Then we have the following comparison theorem of the local geometry around the fixed loci F_β^p and $F^{\emptyset,\beta}_{\star,0}(X)^p$ in $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$ and $QG^{0+}_{0,\star,\beta}(X)^p$, respectively.

Lemma 3.6. (1) *There is a $\mathbb{C}^*$-equivariant isomorphism between the an open neighborhood of F_β^p in $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$ and an open neighborhood of $F^{\emptyset,\beta}_{\star,0}(X)^p$ in the closed substack $(\mathrm{pr}_{\mathbb{P}^1}\circ ev_\star)^{-1}(\infty)$ of $QG^{0+}_{0,\star,\beta}(X)^p$, under which $F_\beta^p \cong F^{\emptyset,\beta}_{\star,0}(X)^p$. The isomorphism also preserves the $\mathbb{C}^*$-perfect obstruction theories with cosections.*

(2) *Under the natural isomorphism between F_β^p and $F^{\emptyset,\beta}_{\star,0}(X)^p$,*

$$\frac{[F_\beta^p]^{\mathrm{vir}}_{\mathrm{loc}}}{e^{\mathbb{C}^*}\big(N^{\mathrm{vir}}_{F_\beta^p/Q_{\mathbb{P}_{a,1}}(X,\beta)^p}\big)} = \frac{ev_\star^*(\eta_\infty)[F^{\emptyset,\beta}_{\star,0}(X)^p]^{\mathrm{vir}}_{\mathrm{loc}}}{e^{\mathbb{C}^*}\big(N^{\mathrm{vir}}_{F^{\emptyset,\beta}_{\star,0}(X)^p/QG^{0+}_{0,\star,\beta}(X)^p}\big)}\ .$$

Here again the localization residues are taken as sums over the connected components of F_β^p and $F^{\emptyset,\beta}_{\star,0}(X)^p$.

Proof. (1) By the vanishing property of p-fields for F_β^p (resp. $F^{\emptyset,\beta}_{\star,0}(X)^p$) of $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$ (resp. $QG^{0+}_{0,\star,\beta}(X)^p$), one can argue similarly as in [CCFK15, Lemma 4.8]: consider the open neighborhood in $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$ by requiring that the base points are away from the stacky point $[1,0] \in \mathbb{P}_{a,1}$ and the open neighborhood in $QG^{0+}_{0,\star,\beta}(X)^p$ in which the source curve of a graph quasimap is isomorphic to $\mathbb{P}_{a,1}$.

(2) We compare the $\mathbb{C}^*$ moving and fixed parts of both obstruction theories. First for $QG^{0+}_{0,\star,\beta}(X)^p$, we need to look at the fixed part of $R^\bullet\pi_*\big((\oplus_{\rho\in[n]}\mathcal{L}_\rho)\oplus(\mathcal{L}^\vee\otimes\omega_\pi)\big)\oplus(R^\bullet\pi_*[x]_2^*T\mathbb{P}^1)$ and the fixed part of the automorphisms/deformations of

[1]Actually, the $p-$fields vanish on the whole space $QG^{0+}_{0,\star,\beta}(X)^p$. Indeed, every source curve C in $QG^{0+}_{0,\star,\beta}(X)^p$ must be a chain of rational curves and the marking must lie on a rational tail of C, then the degree of the restriction of the line bundle $[x]_1^*L^\vee \otimes \omega_{\mathbb{P}_{a,1}}$ to every irreducible component of C is non-positive. So one has

$$QG^{0+}_{0,\star,\beta}(X)^p = QG^{0+}_{0,\star,\beta}(X)\ .$$

$(C, q_\star)$ and line bundles $\mathcal{L}_j$. Altogether its fixed parts with cosection coincides with the fixed part of $\mathbb{E}_{Q_{\mathbb{P}_{a,1}}(X,\beta)^p}$ with cosection, this implies that

$$[F^p_\beta]^{\mathrm{vir}}_{\mathrm{loc}} = [F^{\emptyset,\beta}_{\star,0}(X)^p]^{\mathrm{vir}}_{\mathrm{loc}} .$$

The Euler class of the moving part of them altogether becomes the product of the Euler class of the moving part of $\mathbb{E}_{Q_{\mathbb{P}_{a,1}}(X,\beta)^p}$ and $(-z)$.

□

Eventually, we will reduce the computation of I-function of Y to the evaluation of the following term

$$\frac{[F^p_\beta]^{\mathrm{vir}}_{\mathrm{loc}}}{e^{\mathbb{C}^*}\left(N^{\mathrm{vir}}_{F^p_\beta/Q_{\mathbb{P}_{a,1}}(X,\beta)^p}\right)} .$$

To achieve this, let's introduce some notations first: for any $\beta \in \mathrm{Eff}(W, G, \theta)$, define $F^{\emptyset,\beta}_{\star,0}(Y)$ to be

$$\coprod_{\substack{d\in \mathrm{Eff}(AY,G,\theta)\\ (i_{\mathfrak{Y}})_*(d)=\beta}} F^{\emptyset,d}_{\star,0}(Y)$$

and define $QG^{0+}_{0,\star,\beta}(Y)$ to be

$$\coprod_{\substack{d\in \mathrm{Eff}(AY,G,\theta)\\ (i_{\mathfrak{Y}})_*(d)=\beta}} QG^{0+}_{0,\star,d}(Y) .$$

Recall here $F^{\emptyset,d}_{\star,0}(Y)$ is defined for $QG^{0+}_{0,\star,d}(Y)$ in the section 3.1.1.

We will need the following lemma:

Lemma 3.7. *We have the equality*

$$\frac{ev_\star^*(\eta_\infty) \cap [F^{\emptyset,\beta}_{\star,0}(Y)]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^{\emptyset,\beta}_{\star,0}(Y)/QG^{0+}_{0,\star,\beta}(Y)})} = \frac{(-1)^{\chi(L)} ev_\star^*(\eta_\infty) \cap [F^{\emptyset,\beta}_{\star,0}(X)^p]^{\mathrm{vir}}_{\mathrm{loc}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^{\emptyset,\beta}_{\star,0}(X)^p/QG^{0+}_{0,\star,\beta}(X)^p})}$$

in $A_*(F^{\emptyset,\beta}_{\star,0}(Y)) \otimes_{\mathbb{Q}} \mathbb{Q}[z, z^{-1}]$.

Proof. Let $i_{QG} : QG^{0+}_{0,\star,\beta}(Y) \to QG^{0+}_{0,\star,\beta}(X)^p$ be the degeneracy loci, which is also $\mathbb{C}^*$-equivariant. By Theorem 2.10, we have:

$$[QG^{0+}_{0,\star,\beta}(Y)]^{\mathrm{vir}} = (-1)^{\chi(L)}[QG^{0+}_{0,\star,\beta}(X)^p]^{\mathrm{vir}}_{\mathrm{loc}}.$$

Let $[N_0 \to N_1]$ be a resolution of the virtual normal bundle N^{vir} of $F^{\emptyset,\beta}_{\star,0}(X)^p$ inside $QG^{0+}_{0,\star,\beta}(X)^p$, then the normal sheaf $\mathcal{N}_{F^{\emptyset,\beta}_{\star,0}/QG^{0+}_{0,\star,\beta}(X)^p}$ is contained in $h^1/h^0(N^{vir}[-1]) = ker\{N_0 \to N_1\}$, thus contained in N_0. Hence the normal cone $\mathfrak{C}_{F^{\emptyset,\beta}_{\star,0}(X)^p/QG^{0+}_{0,\star,\beta}(X)^p}$ is contained in N_0 as well. We can define the virtual pull-back

$$\iota^!_{F^{\emptyset,\beta}_{\star,0}(X)^p} : A^{\mathbb{C}^*}_*(QG^{0+}_{0,\star,\beta}(X)^p(\sigma)) \otimes_{\mathbb{Q}[z]} \mathbb{Q}[z, z^{-1}] \to A_*(F^{\emptyset,\beta}_{\star,0}(X)^p(\sigma)) \otimes_{\mathbb{Q}} \mathbb{Q}[z, z^{-1}]$$

for the inclusion $\iota : F^{\emptyset,\beta}_{\star,0}(X)^p \to QG^{0+}_{0,\star,\beta}(X)^p$ as in [CKL17, Man11]. Note here the degeneracy loci $QG^{0+}_{0,\star,\beta}(X)^p(\sigma)$ and $F^{\emptyset,\beta}_{\star,0}(X)^p(\sigma)$ are equal to $QG^{0+}_{0,\star,\beta}(Y)$ and $F^{\emptyset,\beta}_{\star,0}(Y)$, respectively.

First apply virtual localization for $QG^{0+}_{0,\star,\beta}(Y)$, we have:

$$\begin{aligned}
&\frac{\iota^!_{F^{\emptyset,\beta}_{\star,0}(X)^p}(i_{QG})_*\big(ev_\star^*(\eta_\infty)\cap [QG^{0+}_{0,\star,\beta}(Y)]^{\mathrm{vir}}\big)}{e^{\mathbb{C}^*}(N_0)}\\
&=\frac{\iota^!_{F^{\emptyset,\beta}_{\star,0}(X)^p}(i_{QG})_*\left(\sum_{\beta_1+\beta_2=\beta}\iota_{F^{\emptyset,\beta_1}_{\star,\beta_2}(Y)*}\big(\frac{ev_\star^*(\eta_\infty)\cap[F^{\emptyset,\beta_1}_{\star,\beta_2}(Y)]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^{\emptyset,\beta_1}_{\star,\beta_2}(Y)/QG^{0+}_{0,\star,\beta}(Y)})}\big)\right)}{e^{\mathbb{C}^*}(N_0)}\\
&=\sum_{\beta_1+\beta_2=\beta}\frac{\iota^!_{F^{\emptyset,\beta}_{\star,0}(X)^p}(i_{QG})_*\left(\iota_{F^{\emptyset,\beta_1}_{\star,\beta_2}(Y)*}\big(\frac{ev_\star^*(\eta_\infty)\cap[F^{\emptyset,\beta_1}_{\star,\beta_2}(Y)]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^{\emptyset,\beta_1}_{\star,\beta_2}(Y)/QG^{0+}_{0,\star,\beta}(Y)})}\big)\right)}{e^{\mathbb{C}^*}(N_0)}\\
&=\sum_{\beta_1+\beta_2=\beta}\frac{\iota^!_{F^{\emptyset,\beta}_{\star,0}(X)^p}\iota_{F^{\emptyset,\beta_1}_{\star,\beta_2}(X)^p*}\left(i|_{F^{\emptyset,\beta_1}_{\star,\beta_2}(Y)*}\big(\frac{ev_\star^*(\eta_\infty)\cap[F^{\emptyset,\beta_1}_{\star,\beta_2}(Y)]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^{\emptyset,\beta_1}_{\star,\beta_2}(Y)/QG^{0+}_{0,\star,\beta}(Y)})}\big)\right)}{e^{\mathbb{C}^*}(N_0)}\\
&=i|_{F^{\emptyset,\beta}_{\star,0}(Y)*}\left(\frac{ev_\star^*(\eta_\infty)\cap[F^{\emptyset,\beta}_{\star,0}(Y)]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^{\emptyset,\beta}_{\star,0}(Y)/QG^{0+}_{0,\star,\beta}(Y)})}\right)\\
&=\frac{ev_\star^*(\eta_\infty)\cap[F^{\emptyset,\beta}_{\star,0}(Y)]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N^{vir}_{F^{\emptyset,\beta}_{\star,0}(Y)/QG^{0+}_{0,\star,\beta}(Y)})}\ .
\end{aligned}$$

Here $i|_{F^{\emptyset,\beta_1}_{\star,\beta_2}(Y)} : F^{\emptyset,\beta_1}_{\star,\beta_2}(Y) \to F^{\emptyset,\beta_1}_{\star,\beta_2}(X)^p(\sigma)$ is the degeneracy loci, which is the identity map, this is used in the last equality above. $\iota_{F^{\emptyset,\beta}_{\star,0}(X)^p} : F^{\emptyset,\beta}_{\star,0}(X)^p(\sigma) \to QG^{0+}_{0,\star,\beta}(X)^p(\sigma)$ and $\iota_{F^{\emptyset,\beta_1}_{\star,\beta_2}(Y)} : F^{\emptyset,\beta_1}_{\star,\beta_2}(Y) \to QG^{0+}_{0,\star,\beta}(Y)$ are the natural inclusion morphisms.

In the above equation, the first equality holds due to the virtual localization theorem for $QG^{0+}_{0,\star,\beta}(Y)$ and the fact that only the fixed components where the marking $\star$ goes to $\infty \in \mathbb{P}^1$ contributes due to the capping with $ev_\star^*(\eta_\infty)$. In the third equality we use Theorem (cf. [CKL17, Lemma 3.8][1])

$$\iota^!_{F^{\emptyset,\beta}_{\star,0}(X)^p}\iota_{F^{\emptyset,\beta}_{\star,0}(X)^p}(\alpha) = \alpha \cap e^{\mathbb{C}^*}(N_0)$$

for all $\alpha \in A_*(F^{\emptyset,\beta}_{\star,0}(Y)) \otimes \mathbb{C}[z, z^{-1}]$, and the fact that $\iota^!_{F^{\emptyset,\beta}_{\star,0}(X)^p}\iota_{F^{\emptyset,\beta_1}_{\star,\beta_2}(X)^p} = 0$ when $\beta_2 \neq 0$.

[1]It's actually proved for the whole fixed substack $(QG^{0+}(X)^p_{0,\star,\beta})^{\mathbb{C}^*}$ rather than the open and closed component $F^{\emptyset,\beta}_{\star,0}(X)^p$, but the argument works for $F^{\emptyset,\beta}_{\star,0}(X)^p$ where the proof can be modified with the only change that $(QG^{0+}_{0,\star,\beta}(X)^p)^{\mathbb{C}^*}$ is replaced by $F^{\emptyset,\beta}_{\star,0}(X)^p$.

On the other hand, applying the cosection localized localization theorem [CKL17] for $QG^{0+}_{0,\star,\beta}(X)^p$, we have:

$$
\begin{aligned}
&\frac{\iota^!_{F^{\emptyset,\beta}_{\star,0}(X)^p}(i_{QG})_*\big(ev_\star^*(\eta_\infty)\cap[QG^{0+}_{0,\star,\beta}(Y)]^{\mathrm{vir}}\big)}{e^{\mathbb{C}^*}(N_0)}\\
&=(-1)^{\chi(L)}\frac{\iota^!_{F^{\emptyset,\beta}_{\star,0}(X)^p}\big(ev_\star^*(\eta_\infty)\cap[QG^{0+}_{0,\star,\beta}(X)^p]^{\mathrm{vir}}_{\mathrm{loc}}\big)}{e^{\mathbb{C}^*}(N_0)}\\
&=(-1)^{\chi(L)}\sum_{\beta_1+\beta_2=\beta}\frac{\iota^!_{F^{\emptyset,\beta}_{\star,0}(X)^p}\Bigg(ev_\star^*(\eta_\infty)\cap\iota_{F^{\emptyset,\beta_1}_{\star,\beta_2}(X)^p*}\Big(\frac{[F^{\emptyset,\beta_1}_{\star,\beta_2}(X)^p]^{\mathrm{vir}}_{\mathrm{loc}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^{\emptyset,\beta_1}_{\star,\beta_2}(X)^p/QG^{0+}_{0,\star,\beta}(X)^p})}\Big)\Bigg)}{e^{\mathbb{C}^*}(N_0)}\\
&=(-1)^{\chi(L)}\sum_{\beta_1+\beta_2=\beta}\frac{\iota^!_{F^{\emptyset,\beta}_{\star,0}(X)^p}\Bigg(\iota_{F^{\emptyset,\beta_1}_{\star,\beta_2}(X)^p*}\Big(ev_\star^*(\eta_\infty)\cap\frac{[F^{\emptyset,\beta_1}_{\star,\beta_2}(X)^p]^{\mathrm{vir}}_{\mathrm{loc}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^{\emptyset,\beta_1}_{\star,\beta_2}(X)^p/QG^{0+}_{0,\star,\beta}(X)^p})}\Big)\Bigg)}{e^{\mathbb{C}^*}(N_0)}\\
&=(-1)^{\chi(L)}\frac{ev_\star^*(\eta_\infty)\cap[F^{\emptyset,\beta}_{\star,0}(X)^p]^{\mathrm{vir}}_{\mathrm{loc}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^{\emptyset,\beta}_{\star,0}(X)^p/QG^{0+}_{0,\star,\beta}(X)^p})}\ .
\end{aligned}
$$

Combing the above two equations, the Lemma is proved. □

Recall that we define the (restricted) small $I-$function $I(q,z)$ to be

$$I(q,z):=_{\mathrm{Eff}(AY,G,\theta)\to\mathrm{Eff}(W,G,\theta)}I(q,0,z)$$

By the above discussion, the small I-function for Y can be rewritten as:

$$
\begin{aligned}
I(q,z)&=\sum_{\beta\in\mathrm{Eff}(W,G,\theta)}q^\beta(\widetilde{\mathrm{pr}_{\bar I_\mu X}\circ ev_\star})_*\left(\frac{ev_\star^*(\eta_\infty)\cap[F^{\emptyset,\beta}_{\star,0}(Y)]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^{\emptyset,\beta}_{\star,0}(Y)/QG^{0+}_{0,\star,\beta}(Y)})}\right)\\
&=\sum_{\beta\in\mathrm{Eff}(W,G,\theta)}q^\beta(\widetilde{\mathrm{pr}_{\bar I_\mu X}\circ ev_\star})_*\left(\frac{(-1)^{\chi(L)}(ev_\star)^*(\eta_\infty)\cap[F^{\emptyset,\beta}_{\star,0}(X)^p]^{\mathrm{vir}}_{\mathrm{loc}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^{\emptyset,\beta}_{\star,0}(X)^p/QG^{0+}_{0,\star,\beta}(X)^p})}\right)\\
&=\sum_{\beta\in\mathrm{Eff}(W,G,\beta)}q^\beta(\widetilde{ev_\infty})_*\left((-1)^{\chi(L)}\frac{[F^p_\beta]^{\mathrm{vir}}_{\mathrm{loc}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^p_\beta/Q_{P_{a,1}}(X,\beta)^p})}\right)\ ,
\end{aligned}
$$

where $ev_\infty: F^p_\beta\to\bar I_\mu Y$ is the evaluation map at the point $\infty\in\mathbb{P}_{a,1}$ and the notation $\widetilde{ev_\infty}_*$ is defined as in (2.2). In the above equation, the first equality in (3.2) is by Definition 3.1. The second equality in (3.2) is due to Lemma 3.7. The third equality in (3.2) is due to Lemma 3.6.

3.3. **Localization computation.** In this section, we will finish the computation of $I-$function by computing the following term

$$\frac{[F^p_\beta]^{\mathrm{vir}}_{\mathrm{loc}}}{e(N^{\mathrm{vir}}_{F^p_\beta/Q_{\mathbb{P}_{a,1}}(X,\beta)^p})}\ .$$

First we give a more concrete description of F^p_β. Following the notations in [CCFK15, §5.3], for any $\beta \in \mathrm{Eff}(W, G, \theta)$, let a be the minimal positive integer making $a\beta(L_\rho) \in \mathbb{Z}$ for all $\rho \in \chi(G)$. Recall that

$$W_\beta := \bigoplus_{\rho \in [n]} \mathbb{C}[z_1, z_2]_{a\beta(L_\rho)} \ .$$

Define

$$Z_\beta := \bigoplus_{\rho \in [n], \beta(L_\rho) \in \mathbb{Z}_{\geqslant 0}} \mathbb{C} \cdot z_1^{\beta(L_\rho)} \subset W_\beta .$$

and $Z^{ss}_\beta := Z_\beta \cap W^{ss}_\beta(\theta)$, then we have

$$F^p_\beta \cong F_\beta \cong [Z^{ss}_\beta / G] \ .$$

Let $g_\beta = (e^{2\pi\sqrt{-1}\beta(L_{\pi_1})}, \cdots, e^{2\pi\sqrt{-1}\beta(L_{\pi_k})}) \in G = (\mathbb{C}^*)^k$, then we have another characterization of Z_β and Z^{ss}_β:

$$Z_\beta \cong W^{g_\beta} \cap \bigcap_{\substack{\rho \in [n] \\ \beta(L_\rho) \in \mathbb{Z}_{<0}}} D_\rho \quad \text{and} \quad Z^{ss}_\beta \cong W^{ss}(\theta)^{g_\beta} \cap \bigcap_{\substack{\rho \in [n] \\ \beta(L_\rho) \in \mathbb{Z}_{<0}}} D_\rho \ .$$

Here D_ρ is the divisor of W given by $x_\rho = 0$.

Note, via the evaluation map ev_∞ at the point $\infty \in \mathbb{P}^1$ for F_β, one has the identification:

$$ev_\infty : F_\beta \cong I_{g_\beta} X := [W^{ss}(\theta)^{g_\beta} / G] \ ,$$

where $I_{g_\beta} X$ is the inertia component of $\bar{I}_\mu X$ given by the automorphism $g_\beta \in G$.

We will prove the following:

Lemma 3.8. (1) *If $\beta(L) \in \mathbb{Z}$, one has*

$$e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^p_\beta / Q_{\mathbb{P}_{a,1}}(X,\beta)^p}) = \frac{\prod_{\rho:\beta(L_\rho)>0} \prod_{0 \leqslant i < \beta(L_\rho)} (D_\rho + (\beta(L_\rho) - i)z)}{\prod_{\rho:\beta(L_\rho)<0} \prod_{\lfloor \beta(L_\rho)+1 \rfloor \leqslant i < 0} (D_\rho + (\beta(L_\rho) - i)z)} \times (-1)^{\chi(L)-1} \prod_{0 \leqslant i < \beta(L)} \frac{1}{(c_1(L) + (\beta(L) - i)z)} \ .$$

(2) *If $\beta(L) \notin \mathbb{Z}$, one has*

$$e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^p_\beta / Q_{\mathbb{P}_{a,1}}(X,\beta)^p}) = \frac{\prod_{\rho:\beta(L_\rho)>0} \prod_{0 \leqslant i < \beta(L_\rho)} (D_\rho + (\beta(L_\rho) - i)z)}{\prod_{\rho:\beta(L_\rho)<0} \prod_{\lfloor \beta(L_\rho)+1 \rfloor \leqslant i < 0} (D_\rho + (\beta(L_\rho) - i)z)} \times (-1)^{\chi(L)} \prod_{0 \leqslant i < \beta(L)} \frac{1}{(c_1(L) + (\beta(L) - i)z)} \ .$$

Here D_ρ is the hyperplane of W associated to ρ as well as the divisor in Y (or X) by descent and restriction.

Proof. Using the setting in 3.1.3, the virtual normal bundle of F^p_β in $Q_{\mathbb{P}_{a,1}}(X, \beta)^p$ comes form the movable parts of infinitesimal deformations of sections of the line bundles

$(\mathscr{L}_\rho : \rho \in [n])$ and p-field in $\mathscr{P}$. By virtual localization theorem. We have:

$$e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F^p_\beta/Q_{\mathbb{P}_{a,1}}(X,\beta)^p}) = \frac{e^{\mathbb{C}^*}((\bigoplus_{\rho\in[n]} R^0\pi_*\mathscr{L}_\rho)^{\mathrm{mov}})e^{\mathbb{C}^*}((R^0\pi_*\mathscr{P})^{\mathrm{mov}})}{e^{\mathbb{C}^*}((\bigoplus_{\rho\in[n]} R^1\pi_*\mathscr{L}_\rho)^{\mathrm{mov}})e^{\mathbb{C}^*}((R^1\pi_*\mathscr{P})^{\mathrm{mov}})} \ .$$

Observe that $R^i\pi_*\mathscr{L}_\rho = \mathcal{O}(D_\rho) \otimes H^i(\mathbb{P}_{a,1}, \mathcal{O}(a\beta(L_\rho))$ over F^p_β for $i = 0, 1$. For any character ρ of G, we have the following description of $H^\bullet(\mathbb{P}_{a,1}, \mathcal{O}(a\beta(L_\rho))$:

(1) If $\beta(L_\rho) \geqslant 0$, using Čech complex, we can see that $H^0(\mathbb{P}_{a,1}, \mathcal{O}(a\beta(L_\rho))$ is spanned by the basis:

$$z_2^{a\beta(L_\rho)}, z_1 z_2^{a(\beta(L_\rho)-1)}, \cdots, z_1^{\lfloor\beta(L_\rho)\rfloor} z_2^{a(\beta(L_\rho)-\lfloor\beta(L_\rho)\rfloor)} \ ,$$

whose equivariant weights are $\beta(L_\rho)z, \cdots, (\beta(L_\rho)-\lfloor\beta(L_\rho)\rfloor)z$, respectively, and $H^1(\mathbb{P}_{a,1}, \mathcal{O}(a\beta(L_\rho))$ vanishes. Here, for any rational number $q \in \mathbb{Q}$, $\lfloor q\rfloor$ is the maximal integer which is no larger than q;

(2) If $\beta(L_\rho) < 0$, one has $H^1(\mathbb{P}_{a,1}, \mathcal{O}(a\beta(L_\rho))$ is spanned by the following basis:

$$z_1^{\lfloor\beta(L_\rho)+1\rfloor} z_2^{a(\beta(L_\rho)-\lfloor\beta(L_\rho)+1\rfloor)}, \cdots, z_1^{-1} z_2^{a(\beta(L_\rho)+1)}$$

with equivariant weights $(\beta(L_\rho) - i)z$ with $\lfloor\beta(L_\rho) + 1\rfloor \leqslant i < 0$, respectively, and $H^0(\mathbb{P}_{a,1}, \mathcal{O}(a\beta(L_\rho))$ vanishes.

Then we have:

$$e^{\mathbb{C}^*}((\bigoplus_{i=1}^n R^\bullet\pi_*\mathscr{L}_\rho)^{\mathrm{mov}}) = \frac{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leqslant i<\beta(L_\rho)}(D_\rho + (\beta(L_\rho) - i)z)}{\prod_{\rho:\beta(L_\rho)<0}\prod_{\lfloor\beta(L_\rho)+1\rfloor\leqslant i<0}(D_\rho + (\beta(L_\rho) - i)z)} \ .$$

Similarly, we have $R^i\pi_*\mathscr{P} = L^\vee \otimes H^i(\mathbb{P}_{a,1}, \mathcal{O}(-a\beta(L)-a-1)) \otimes \mathbb{C}_{\frac{1}{a}}$ over F^p_β where $i = 0, 1$. Here the special twist $\mathbb{C}_{\frac{1}{a}}$ measures the difference between the dualizing sheaf $\omega_{\mathbb{P}_{a,1}}$ and the invertible sheaf $\mathcal{O}(-a-1)$ from Borel construction as $\mathbb{C}^*-$equivariant invertible sheaves.

As $\beta(L) \geqslant 0$ by the positivity of the line bundle L, $H^1(\mathbb{P}_{a,1}, L^\vee \otimes \omega_{\mathbb{P}_{a,1}})$ is spanned by the basis

$$z_1^{\lfloor-\beta(L)-\frac{1}{a}\rfloor} z_2^{a(-\beta(L)-1-\frac{1}{a}-\lfloor-\beta(L)-\frac{1}{a}\rfloor)}, \cdots, z_1^{-1} z_2^{-a\beta(L)-1} \ ,$$

whose equivariant weights are $(-\beta(L)-1-i)z$ with $\lfloor-\beta(L) - \frac{1}{a}\rfloor \leqslant i < 0$, respectively, and $H^0(\mathbb{P}_{a,1}, \mathcal{O}(-a\beta(L) - a - 1))$ is equal to zero. Then we have

$$e^{\mathbb{C}^*}(R^0\pi_*\mathscr{P}) = 1 \ .$$

and

$$e^{\mathbb{C}^*}(R^1\pi_*\mathscr{P}) = \prod_{\lfloor-\beta(L)-\frac{1}{a}\rfloor\leqslant i<0}\Bigg(-c_1(L) + (-\beta(L) - 1 - i)z\Bigg) \ .$$

So we immediately get the movable part:

(1) If $\beta(L) \in \mathbb{N}$, one has

$$\begin{aligned} e^{\mathbb{C}^*}((R^1\pi_*\mathscr{P})^{\mathrm{mov}}) &= \prod_{\lfloor -\beta(L)-\frac{1}{a}\rfloor < i < 0} \Big(-c_1(L) + (-\beta(L)-1-i)z \Big) \\ &= \prod_{-\beta(L) \leqslant i < 0} \Big(-c_1(L) + (-\beta(L)-1-i)z \Big) \\ &= (-1)^{\chi(L)-1} \prod_{0 \leqslant i < \beta(L)} \Big(c_1(L) + (\beta(L)-i)z \Big) . \end{aligned}$$

(2) If $\beta(L) \notin \mathbb{N}$, one has

$$\begin{aligned} e^{\mathbb{C}^*}((R^1\pi_*\mathscr{P})^{\mathrm{mov}}) &= \prod_{\lfloor -\beta(L)-\frac{1}{a}\rfloor \leqslant i < 0} \Big(-c_1(L) + (-\beta(L)-1-i)z \Big) \\ &= \prod_{\lfloor -\beta(L)-\frac{1}{a}\rfloor \leqslant i < 0} \Big(-c_1(L) + (-\beta(L)-1-i)z \Big) \\ &= (-1)^{\chi(L)} \prod_{0 \leqslant i < \beta(L)} \Big(c_1(L) + (\beta(L)-i)z \Big) . \end{aligned}$$

Note in the second case, we use the fact $a\beta(L) \in \mathbb{Z}$. Put all together, we get the expression for the Euler class of virtual normal bundle. $\square$

To complete the explicit computation for the *small I-function* of Y, one is remained to calculate $[F^p_\beta]^{\mathrm{vir}}_{\mathrm{loc}}$. Recall that the line bundle $L := L_\tau$ is associated to the character τ of G and s is a $G-$invariant section of the $G-$equivariant line bundle $W \times \mathbb{C}_\tau$ over W. We will begin with a lemma comparing inertia stack components of Y and X.

Lemma 3.9. (1) *For any $g \in G$ with $\tau(g) = 1 \in \mathbb{C}^*$, the section $s \in \Gamma(W, W \times \mathbb{C}_\tau)$ does not vanish identically on the space $W^{ss}(\theta)^g$, which implies that the inertia stack component I_gY of Y is a proper hypersurface inside the inertia stack component I_gX of X.*

(2) *For any torsion element $g \in G$ with $\tau(g) \neq 1$, the section s vanishes identically on the space $W^{ss}(\theta)^g$, which implies that the inertia stack component I_gY of Y is the same as the inertia stack component I_gX of X.*

Proof. (1) It suffices to assume that $W^{ss}(\theta)^g$ is proper subspace of $W^{ss}(\theta)$, note this implies W^g is a proper linear subspace of W as $W^{ss}(\theta)$ is open dense in W. Let $\vec{x} = (x_1, \cdots, x_n)$ be a coordinate of W given by the n characters $(\rho_i : 1 \leqslant i \leqslant n)$ defining the toric stack X. Assume that the subspace W^g is given by $\{\vec{x} : x_{l+1} = \cdots = x_n = 0\}$, where $l = dim(W^g)$. Assume that the section s defining Y vanishes identically on $W^{ss}(\theta)^g$, then s can be written as a polynomial of the form

$$s(\vec{x}) = x_{l+1}g_1(\vec{x}) + \cdots + x_n g_{n-l}(\vec{x}) .$$

Then, for any $p \in W^{ss}(\theta)^g$, one has

$$\frac{\partial s}{\partial x_i}(p) = 0$$

for $1 \leqslant i \leqslant l$. Since s cuts off a smooth hypersurface in $W^{ss}(\theta)$, there exists an integer i between 1 and $n-l$ so that

$$\frac{\partial s}{\partial x_{l+i}}(p) \neq 0 \ .$$

Without loss of generality, let's assume that $i = 1$. Write f uniquely in the form

$$s(\vec{x}) = x_{l+1} g_1(\vec{x}) + s'(\hat{x}_{l+1}) \ ,$$

where $\hat{x}_{l+1} = (x_1, \cdots, x_l, x_{l+2}, \cdots, x_n)$. Observe that

$$\frac{\partial s}{\partial x_{l+1}}(p) = g_1(p) \ ,$$

and

$$g_1(p) = g_1(p \cdot g) = \tau(g)\rho_{l+1}^{-1}(g) \cdot g_1(p) = \rho_{l+1}^{-1}(g) g_1(p) \ ,$$

where the second equality comes from the fact $s(p \cdot g) = \tau(g)s(p)$ as s is a $G-$invariant section of the $G-$equivariant line bundle $W \times \mathbb{C}_\tau$ over W. Note $\rho_{l+1}(g) \neq 1$ by the very definition of W^g, so we must have $g_1(p) = 0$, which is a contradiction. This shows that s does not vanish identically on $W^{ss}(\theta)^g$.

(2) For any point $p \in W^{ss}(\theta)^g$ such that s vanishes on p, we have the following short exact sequence of tangent spaces

$$0 \to T_p AY^{ss}(\theta) \to T_p W^{ss}(\theta) \to \mathbb{C}_\tau \to 0 \ ,$$

which is also exact as representations of the finite group generated by g. Taking the g-invariant subspace of the above exact sequence, we get

$$T_p AY^{ss}(\theta)^g = T_p W^{ss}(\theta)^g \ ,$$

which immediately implies $AY^{ss}(\theta)^g$ and $W^{ss}(\theta)^g$ have the same dimension. As a consequence, the section s vanishes identically on the space $W^{ss}(\theta)^g$.

□

Recall that $AY := Z(s) \subset W$, we will denote $Y_\beta^{ss} := Z_\beta^{ss} \cap AY$. By abusing notations, we use the same letter s and L to mean the corresponding section and the line bundle over $[Z_\beta^{ss}/G]$ by descent. Here $s^!$ is the localized top chern class (cf. [Ful84, Chapter 14]) of the line bundle L on $[Z_\beta^{ss}/G]$ with respect to s. Then we have the following result:

Lemma 3.10. *Using the natural identification $F_\beta^p \cong F_\beta \cong [Z_\beta^{ss}/G]$, and the dengenacy loci $F_\beta^p(\sigma) \cong [Y_\beta^{ss}/G]$. We have:*

(1) *when $\beta(L) \in \mathbb{Z}$, one has*

$$[F_\beta^p]^{\mathrm{vir}}_{\mathrm{loc}} = -s^!\big([Z_\beta^{ss}/G]\big)$$

in the chow group $A_*([Y_\beta^{ss}/G])$;

(2) *when $\beta(L) \notin \mathbb{Z}$, the section s vanishes identically on Z_β^{ss}, and one has*

$$[F_\beta^p]_{\mathrm{loc}}^{\mathrm{vir}} = [Y_\beta^{ss}/G]$$

in the chow group $A_*([Y_\beta^{ss}/G])$.

Proof. First, let's give a detailed description of the cosection over the smooth cover Z_β^{ss} of $Q_{\mathbb{P}_{a,1}}(X,\beta)^p$: following the notations in (3.11), the obstruction bundle over Z_β^{ss} is equal to

$$Z_\beta^{ss} \times \left(\oplus_{\rho\in[n]} H^1(\mathbb{P}_{a,1}, \mathcal{O}(a\beta(L_\rho))) \oplus H^1(\mathbb{P}_{a,1}, L^\vee \otimes \omega_{\mathbb{P}_{a,1}}) \right) .$$

Using the identification

$$L^\vee \otimes \omega_{\mathbb{P}_{a,1}} \cong \mathcal{O}(-a\beta(L) - a - 1) \otimes \mathbb{C}_{\frac{1}{a}}$$

as $\mathbb{C}^*$-equivariant invertible sheaves, the bundle $Z_\beta^{ss} \times H^1(\mathbb{P}_{a,1}, L^\vee \otimes \omega_{\mathbb{P}_{a,1}})$ is trivialized by the basis

$$z_1^{\lfloor -\beta(L) - \frac{1}{a} \rfloor} z_2^{a(-\beta(L)-1-\frac{1}{a}-\lfloor -\beta(L)-\frac{1}{a}\rfloor)}, \cdots, z_1^{-1} z_2^{-a\beta(L)-1} \tag{3.15}$$

with equivariant weights $(-\beta(L) + \lfloor \beta(L) \rfloor)z, \cdots, -\beta(L)z$, respectively. Fix a point $x = \oplus_{\rho\in[n]} a_\rho z_1^{\beta(L_\rho)} \in Z_\beta^{ss}$. The restriction of the cosection $\sigma|_x$ to x sends

$$\mathring{u}_i \in H^1(\mathbb{P}_{a,1}, L_{\rho_i}) \quad \text{and} \quad \mathring{p} \in H^1(\mathbb{P}_{a,1}, L^\vee \otimes \omega_{\mathbb{P}_{a,1}})$$

to

$$\sigma|_x(\mathring{u}, \mathring{p}) = s(x) \cdot \mathring{p} \in H^1(\mathbb{P}_{a,1}, \omega_{\mathbb{P}_{a,1}}) . \tag{3.16}$$

If we identify $\omega_{\mathbb{P}_{a,1}} \cong \mathcal{O}(-a-1) \otimes \mathbb{C}_{\frac{1}{a}}$ as $\mathbb{C}^*-$equivariant invertible sheaves, then $H^1(\mathbb{P}_{a,1}, \omega_{\mathbb{P}_{a,1}})$ is trivilized by $z_1^{-1} z_2^{-1}$, which is of $\mathbb{C}^*$-weight zero. Note the expression in (3.16) does not depend on the choice of $\mathring{u}$. Take $\mathring{p} = z_1^{-i-1} z_2^{-a\beta(L)-1+ai}$, where $0 \leqslant i < \beta(L)$, then $\sigma|_x(\mathring{u}, \mathring{p})$ is equal to

$$\mathbf{Coeff}_{z_1^{-1} z_2^{-1}} \ s((a_\rho)_{\rho\in[n]}) z_1^{\beta(L)-i-1} z_2^{-a\beta(L)-1+ai} ,$$

from which we see $\sigma|_x(\mathring{u}, \mathring{p})$ is not equal to zero only if $\beta(L) \in \mathbb{Z}$ and $\mathring{p} = z_1^{-\beta(L)-1} z_2^{-1}$.

By the above the discussion, when $\beta(L) \in \mathbb{Z}$, note the fixed part of the obstruction bundle only comes from the infinitesimal deformations of p-fields, and the $\mathbb{C}^*$-fix part of the obstruction bundle over F_β^p is equal to $L^\vee$ corresponding to the section $z_1^{\beta(L)-1} z_2^{-1} \in H^1(\mathbb{P}_{a,1}, \omega_{\mathbb{P}_{a,1}} \otimes L^\vee)$, and the cosection is given by the multiplication by the section s from the line bundle $L^\vee$ to the trivial line bundle $\mathcal{O}$ over F_β^p. Then one has

$$[F_\beta^p]_{\mathrm{loc}}^{\mathrm{vir}} = -s^!\left([Z_\beta^{ss}/G]\right) \in A_*([Y_\beta^{ss}/G]) .$$

When $\beta(L) \notin \mathbb{Z}$, first we claim that s vanishes on Z_β^{ss}. Indeed, the equivariant weights of the basis of $H^1(\mathbb{P}_{a,1}, L^\vee \otimes \omega_{\mathbb{P}_{a,1}})$ in (3.15) are nonzero while elements in $H^1(\mathbb{P}_{a,1}, \omega_{\mathbb{P}_{a,1}})$ are of weight zero, this makes the cosection σ is identically zero as σ is $\mathbb{C}^*$-equivariant. If s does not vanish identically on Z_β^{ss}, by serre duality, the restriction of s to some point $x \in Z_\beta^{ss}$ makes $\sigma|_x$ nonzero, which is a contradiction, which implies our claim. As there is no fixed part of the obstruction bundle over F_β^p, we have $[F_\beta^p]_{\mathrm{loc}}^{\mathrm{vir}} = [F_\beta^p] = [Y_\beta^{ss}/G]$. □

This implies the following:

Corollary 3.11. (1) *If $\beta(L)\in\mathbb{Z}$, we have*

$$(\widetilde{ev_\infty})_*([F_\beta^p]^{\mathrm{vir}}_{\mathrm{loc}}) = -\big(\prod_{\rho:\beta(L_\rho)\in Z_{<0}} D_\rho\big)\cdot \mathbb{1}_{g_\beta^{-1}}\ ,$$

(2) *If $\beta(L)\notin\mathbb{Z}$, we have*

$$(\widetilde{ev_\infty})_*([F_\beta^p]^{\mathrm{vir}}_{\mathrm{loc}}) = \big(\prod_{\rho:\beta(L_\rho)\in Z_{<0}} D_\rho\big)\cdot \mathbb{1}_{g_\beta^{-1}}\ .$$

Here $\mathbb{1}_{g_\beta^{-1}}$ is the fundamental class of $\bar{I}_{g_\beta^{-1}}Y = [AY^{ss}(\theta)^g/(G/\langle g_\beta^{-1}\rangle)]$.

Proof. This immediately follows from Lemma 3.9 and Lemma 3.10, where we take $g=g_\beta$ in 3.9 and use the relation $\exp(2\pi\sqrt{-1}\beta(L))=\tau(g_\beta)$. □

Combing the above all together, we have the following theorem:

Theorem 3.12. *The small I-function of Y is:*

$$\begin{aligned}
I(q,z) &= \sum_{\beta\in\mathrm{Eff(W,G,\theta)}} q^\beta(\widetilde{ev_\infty})_*\Big((-1)^{\chi(L)}\frac{[F_\beta^p]^{\mathrm{vir}}_{\mathrm{loc}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_{F_\beta^p/Q_{P_{\bullet,1}}(X,\beta)^p})}\Big)\\
&= \sum_{\substack{\beta\in\mathrm{Eff(W,G,\theta)}\\ \beta(L)\in\mathbb{N}}} q^\beta\frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\lfloor\beta(L_\rho)+1\rfloor\leqslant i<0}(D_\rho+(\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leqslant i<\beta(L_\rho)}(D_\rho+(\beta(L_\rho)-i)z)}\\
&\times\prod_{0\leqslant i<\beta(L)}\big(c_1(L)+(\beta(L)-i)z\big)(\widetilde{ev_\infty})_*\big(s^!([Z_\beta^{ss}/G])\big)\\
&+ \sum_{\substack{\beta\in\mathrm{Eff(W,G,\theta)}\\ \beta(L)\notin\mathbb{N}}} q^\beta\frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\lfloor\beta(L_\rho)+1\rfloor\leqslant i<0}(D_\rho+(\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leqslant i<\beta(L_\rho)}(D_\rho+(\beta(L_\rho)-i)z)}\\
&\times\prod_{0\leqslant i<\beta(L)}\big(c_1(L)+(\beta(L)-i)z\big)(\widetilde{ev_\infty})_*([AY_\beta^{ss}/G])\\
&= \sum_{\beta\in\mathrm{Eff(W,G,\theta)}} q^\beta\frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)\leqslant i<0}(D_\rho+(\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leqslant i<\beta(L_\rho)}(D_\rho+(\beta(L_\rho)-i)z)}\\
&\times\prod_{0\leqslant i<\beta(L)}\big(c_1(L)+(\beta(L)-i)z\big)\mathbb{1}_{g_\beta^{-1}}\ .
\end{aligned}$$

Here $\mathbb{1}_{g_\beta^{-1}}$ is the fundamental class of $\bar{I}_{g_\beta^{-1}}Y = [AY^{ss}(\theta)^g/(G/\langle g_\beta^{-1}\rangle)]$.

4. Master space I

4.1. **Construction of master space I.** In this section, we will construct a master space which is a root stack modification of the twisted graph space considered in [CJR17a]. Let (AY,G,θ) be the GIT data which gives rise to a hypersurface in the toric stack $X=[W^{ss}(\theta)/G]$ as in previous sections. Choose a positive integer $\mathbf{e}$ so that the line bundle $L_{\mathbf{e}\theta}$ on $Y = [AY^{ss}(\theta)/G]$ is the pullback of a positive line bundle on the coarse moduli space $\underline{Y}$ of Y. First we will consider the following quotient stack

$$\mathbb{P}\mathfrak{Y}^{\frac{1}{r}} = [(AY\times\mathbb{C}^2)/(G\times\mathbb{C}^*)]$$

defined by the following (right) action

$$(\vec{x}, z_1, z_2)\cdot(g,t) = (\vec{x}\cdot g, \theta'(g)^{-1}t^r z_1, tz_2)\ ,$$

where $(g,t)\in G\times\mathbb{C}^*$ $(\vec{x}, z_1, z_2)\in AY\times\mathbb{C}^2$, and $\theta' = \mathbf{e}\theta$.

Fix a positive rational number $\epsilon\in\mathbb{Q}_{>0}$, we consider the stability given by the rational character of $G\times\mathbb{C}^*$ defined by

$$\widetilde{\theta}(g,t) = \theta'(g)^{\epsilon}t^{3r}$$

for $(g,t)\in G\times\mathbb{C}^*$. Then the GIT stack quotient $[(AY\times\mathbb{C}^2)^{ss}(\widetilde{\theta})/(G\times\mathbb{C}^*)]$ is the root stack of the $\mathbb{P}^1-$bundle $\mathbb{P}_Y(\mathcal{O}(-D_{\theta'})\oplus\mathcal{O})$ over Y by taking r-th root of the infinity divisor D_∞ given by $z_2=0$. We will denote the GIT stack quotient $[(AY\times\mathbb{C}^2)^{ss}(\widetilde{\theta})/(G\times\mathbb{C}^*)]$ to be $\mathbb{P}Y^{\frac{1}{r}}$, which is equipped with the infinity section $\mathcal{D}_\infty$ given by $z_2=0$ and the zero section $\mathcal{D}_0$ given by $z_1=0$.

The inertia stack $I_\mu\mathbb{P}Y^{\frac{1}{r}}$ of $\mathbb{P}Y^{\frac{1}{r}}$ admits a decomposition

$$I_\mu\mathbb{P}Y\sqcup\bigsqcup_{j=1}^{r-1}\sqrt[r]{L_{\theta'}/Y}\ .$$

Let $(\vec{x},(g,t))$ be a point of $I_\mu\mathbb{P}Y^{\frac{1}{r}}$, if $(\vec{x},(g,t))$ appears in the first factor of the decomposition above, then the automorphism (g,t) lies in $G\times\{1\}$; if $(\vec{x},(g,t))$ occurs in the second factor of the decomposition above, the automorphism (g,t) lies in $G\times\{\mu_r^j : 1\leqslant j\leqslant r-1\}\subset G\times\boldsymbol{\mu}_r$, and the point $\vec{x}$ is in the infinity section $\mathcal{D}_\infty$ defined by $z_2=0$. Here $\mu_r=\exp(\frac{2\pi\sqrt{-1}}{r})\in\mathbb{C}^*$ and $\boldsymbol{\mu}_r$ is the cyclic group generated by μ_r.

For $(g,t)\in G\times\boldsymbol{\mu}_r$. we will use the notation $\bar{I}_{(g,t)}\mathbb{P}Y^{\frac{1}{r}}$ to mean the rigidified inertia stack component of $\bar{I}_\mu\mathbb{P}Y^{\frac{1}{r}}$ which has automorphism (g,t).

Consider the moduli stack of $\widetilde{\theta}-$stable quasimaps to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$:

$$Q^{\widetilde{\theta}}_{0,m}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(d,\frac{\delta}{r}))$$

.

More concretely,

$$Q^{\widetilde{\theta}}_{0,m}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(d,\frac{\delta}{r})) = \{(C;q_1,\ldots,q_m;L_1,\cdots,L_k,N;\vec{x}=(x_1,\ldots,x_m),z_1,z_2)\},$$

where $(C;q_1,\ldots,q_m)$ is a m-pointed prestable balanced orbifold curve of genus 0 with possible nontrivial isotropy only at special points, i.e. marked gerbes or nodes, the line bundles $(L_j : 1\leqslant j\leqslant k)$ and N are orbifold line bundles on C with

$$\deg([\vec{x}]) = d\in Hom(Pic(\mathfrak{Y}),\mathbb{Q}),\quad \deg(N)=\frac{\delta}{r}, \tag{4.1}$$

and

$$(\vec{x},\vec{z}) := (x_1,\ldots,x_n,z_1,z_2)\in\Gamma\left(\bigoplus_{i=1}^{n}L_{\rho_i}\oplus(L_{-\theta'}\otimes N^{\otimes r})\oplus N\right).$$

Here, for $1 \leqslant i \leqslant n$, the line bundle L_{ρ_i} is equal to

$$\otimes_{j=1}^{k} L_j^{m_{ij}} ,$$

where (m_{ij}) ($1 \leqslant i \leqslant n$,$1 \leqslant j \leqslant k$) is given by the relation $\rho_i = \sum_{j=1}^{k} m_{ij}\pi_j$. The same construction applies to the line bundle $L_{-\theta'}$ on C. Note here δ is an integer when $Q^{\widetilde{\theta}}_{0,m}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (d, \frac{\delta}{r}))$ is nonempty as $N^{\otimes r}$ is the pull-back of some line bundle on the coarse moduli curve $\underline{C}$.

We require this data to satisfy the following conditions:

- *Representability*: For every $q \in C$ with isotropy group G_q, the homomorphism $\mathbb{B}G_q \to \mathbb{B}(G \times \mathbb{C}^*)$ induced by the restriction of line bundles $(L_j : 1 \leqslant j \leqslant k)$ and N to q is representable.
- *Nondegeneracy*: The sections z_1 and z_2 never simultaneously vanish. Furthermore, for each point q of C at which $z_2(q) \neq 0$, the stability condition 2.3
$$l_{\widetilde{\theta}}(q) \leqslant 1$$
for $\widetilde{\theta}$-stable map to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ becomes the stability condition
$$l_{\epsilon\theta'}(q) \leqslant 1, \tag{4.2}$$
for the prestable quasimap $[\vec{x}] : C \to \mathfrak{Y}$. For each point q of C at which $z_2(q) = 0$, we have
$$\mathrm{ord}_q(\vec{x}) = 0. \tag{4.3}$$
We note that this can be phrased as the length condition (2.1) bounding the order of contact of $(\vec{x}, \vec{z})$ with the unstable loci of $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ as in [CFK16, §2.1].
- *Stability*: The $\mathbb{Q}-$line bundle
$$(\phi_*(L_{\theta'}))^{\otimes\epsilon} \otimes \phi_*(N^{\otimes 3r}) \otimes \omega_{\underline{C}}^{log}$$
on the coarse curve $\underline{C}$ is ample. Here $\phi : C \to \underline{C}$ is the coarse moduli map. Note here, by the definition of $\theta' = \mathbf{e}\theta$, the line bundle $L_{\theta'}$ is the pull back of a line bundle on the coarse moduli of $\underline{C}$.
- *Vanishing*: The image of $[\vec{x}] : C \to \mathfrak{X}$ lies in $\mathfrak{Y}$.

Let $\vec{m} = (v_1, \cdots, v_m) \in (G \times \boldsymbol{\mu}_r)^m$, we will denote $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (d, \frac{\delta}{r}))$ to be:

$$Q^{\widetilde{\theta}}_{0,m}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (d, \frac{\delta}{r})) \cap ev_1^{-1}(\bar{I}_{v_1}\mathbb{P}Y^{\frac{1}{r}}) \cap \cdots \cap ev_m^{-1}(\bar{I}_{v_m}\mathbb{P}Y^{\frac{1}{r}}) ,$$

where

$$ev_i : Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (\beta, \frac{\delta}{r})) \to \bar{I}_\mu\mathbb{P}Y^{\frac{1}{r}}$$

are natural evaluation maps as before, by evaluating the sections $(\vec{x}, \vec{z})$ at ith marking q_i. Moreover, since $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (d, \frac{\delta}{r}))$ is a moduli space of stable quasimaps to a lci GIT quotient, the results of [CFKM14] imply that it is a proper Deligne-Mumford stack equipped with a natural perfect obstruction theory relative to the stack $\mathcal{D}_{0,m,\beta,\frac{\delta}{r}}$ of twisted curves equipped with a pair of line bundles. This obstruction theory is of the form

$$R^\bullet\pi_*(u^*\mathbb{R}\mathbb{T}_\rho). \tag{4.4}$$

Here, we denote the universal family over $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(d,\frac{\delta}{r}))$ by

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\rho"'] & \mathcal{C} \arrow[l, bend right, "u"'] \arrow[d, "\pi"] \\ & Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(d,\frac{\delta}{r})) \ , \end{tikzcd}$$

where $\mathcal{L}_{\rho_i}$ $(1 \leqslant i \leqslant n)$ and $\mathcal{N}$ are the universal line bundles and

$$\mathcal{V} \subset \oplus_{i=1}^{n} \mathcal{L}_{\rho_i} \oplus (\mathcal{L}_{-\theta'} \otimes \mathcal{N}^{\otimes r}) \oplus \mathcal{N}$$

is the subsheaf of sections taking values in the affine cone of Y. Somewhat more explicitly, (4.4) equals

$$\mathbb{E} \oplus R^{\bullet}\pi_*\big((\mathcal{L}_{-\theta'} \otimes \mathcal{N}^{\otimes r}) \oplus \mathcal{N}\big), \tag{4.5}$$

in which the sub-obstruction-theory $\mathbb{E}$ comes from the deformations and obstructions of the sections $\vec{x}$.

4.2. $\mathbb{C}^*$**-action and fixed loci.** Consider the (left) $\mathbb{C}^*$-action on $W \times \mathbb{C}^2$ defined by:

$$\lambda(\vec{x}, z_1, z_2) = (\vec{x}, \lambda z_1, z_2) \ ,$$

this action descents to be an action on $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$. We will denote λ to be equivariant class corresponding to the $\mathbb{C}^*$-action of weight 1. Let's first state a criteria for a morphism to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ to be $\mathbb{C}^*$-equivariant (see also [CLLL16, §2.2]), which will be important in the analysis of localization computations.

Remark 4.1. (*Equivariant morphism to* $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$) Fix a stack S over $Spec(\mathbb{C})$ with a left $\mathbb{C}^*$-action, then a $\mathbb{C}^*$-equivariant morphism from S to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ is equivalent to the following data: there exists $k+1$ $\mathbb{C}^*$-equivariant line bundles on S

$$L_1, \cdots, L_k, N$$

together with $\mathbb{C}^*$-invariant sections

$$(\vec{x}, \vec{z}) := (x_1, \ldots, x_n, z_1, z_2) \in \Gamma\left(\bigoplus_{i=1}^{n} L_{\rho_i} \oplus (L_{-\theta'} \otimes N^{\otimes r} \otimes \mathbb{C}_\lambda) \oplus N\right)^{\mathbb{C}^*} .$$

Here L_{ρ_i} $(1 \leqslant i \leqslant n)$ and $L_{-\theta'}$ are constructed from $(L_j)_{1 \leqslant j \leqslant k}$ as explained before, $\mathbb{C}_\lambda$ is the trivial line bundle over S with $\mathbb{C}^*$−linearization of weight 1. These sections should also satisfy the vanishing condition imposed by the cone of Y as above.

Fix a nonzero degree $\beta \in \mathrm{Eff}(W, G, \theta)$ and a tuple of nonnegative integers $(\delta_1, \cdots, \delta_m) \in \mathbb{N}^m$. Consider the tuple of multiplicities $\vec{m} = (v_1, \cdots, v_m) \in (G \times \boldsymbol{\mu}_r)^m$, where $v_i = (g_i, \mu_r^{\delta_i})$, we will denote $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(\beta,\frac{\delta}{r}))$ to be

$$\bigsqcup_{\substack{d \in \mathrm{Eff}(AY,G,\theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta}} Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(d,\frac{\delta}{r})) \ ,$$

where $i_{\mathfrak{Y}} : \mathfrak{Y} \to \mathfrak{X}$ is the inclusion morphism. Thus $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (\beta, \frac{\delta}{r}))$ inherits a $\mathbb{C}^*$-action as above.

Follow the presentation of [CJR17b, CJR17a], the components of fixed loci of $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (\beta, \frac{\delta}{r}))$ under the $\mathbb{C}^*$-action can be indexed by decorated graphs, which we explain as follows. We denote such a graph by Γ which consists of vertices, edges, and m legs, and decorate it as follows:

- Each vertex $v \in V$ has an index $j(v) \in \{0, \infty\}$, and a degree $\beta(v) \in \mathrm{Eff}(W, G, \theta)$.
- Each edge e is equipped with degrees $\beta(e) \in \mathrm{Eff}(W, G, \theta)$ and $\delta(e) \in \mathbb{N}$.
- Each half-edge h (including the legs) has an element (called multiplicity) $m(h) \in G \times \boldsymbol{\mu}_r$.
- The legs are labeled with the numbers $\{1, \ldots, m\}$.

By the "valence" of a vertex v, denoted $\mathrm{val}(v)$, we mean the total number of incident half-edges, including legs.

The fixed locus in $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (\beta, \frac{\delta}{r}))$ indexed by the decorated graph Γ parameterizes quasimaps of the following type:

- Each edge e corresponds to a genus-zero component C_e on which $\deg(N|_{C_e}) = \frac{\delta(e)}{r}$ ($\delta(e) > 0$) and $deg(L_j|_{C_e}) = \beta(e)(L_{\pi_j})$, and there are two distinguished points q_0 and q_∞ on C_e such that q_∞ is the only point on C_e at which z_2 vanishes, and q_0 is the only point on C_e determined by the following conditions:
  - if C_e has base points, q_0 is the only base point on C_e;
  - if C_e does not have base points on it, q_0 is the only point on C_e at which z_1 vanishes.

  We call them the ramification points[1], and all of degree $\beta(e)$ is concentrated at the ramification point q_0. That is,
  $$\text{when } x_i|_{C_e} \neq 0, \quad \text{we have } \mathrm{ord}_{q_0}(x_i) = \beta(e)(L_{\rho_i}) \ .$$
  If both ramification points are special points (i.e. marked points or nodes), it follows that $\deg(L_j|_{C_e}) = 0$ for $1 \leqslant j \leqslant k$.
- Each vertex v for which $j(v) = 0$ (with unstable exceptional cases noted below) corresponds to a maximal sub-curve C_v of C over which $z_1 \equiv 0$, and each vertex v for which $j(v) = \infty$ (again with unstable exceptions) corresponds to a maximal sub-curve over which $z_2 \equiv 0$. The label $\beta(v)$ denotes the degree coming from the restriction map $[\vec{x}]|_{C_v}$, note here we count the degree $\beta(v)$ in $\mathrm{Eff}(W, G, \theta)$, but not in $\mathrm{Eff}(AY, G, \theta)$.
- A vertex v is *unstable* if stable quasimap of the type described above do not exist (where, as always, we interpret legs as marked points and half-edges as half-nodes). In this case, v corresponds to a single point of the component C_e for each adjacent edge e, which may be a node at which C_e meets $C_{e'}$, a marked point of C_e, a unmarked point, or a basepoint on C_e of order $\beta(v)$, note the base point only appears as a vertex over 0 due to the nondegeneracy condition.

---

[1] The definition of the ramification point here is different from the definition in [CJR17a, Page 13], where they claim that z_1 or z_2 each vanish at exactly one point on C_e. We find that there is a missing case when q_0 is a base point and $deg(L_1|_{C_e}) = degL_2|_{C_e} = \delta(e)$ in their setting, then $z_1|_{C_e} \equiv 1$, which does not vanish anywhere on C_e. But the author find this missing case does not affect their main result in [CJR17a].

- The index $m(l)$ on a leg l indicates the rigidified inertia stack component $\bar{I}_{m(l)}\mathbb{P}Y^{\frac{1}{r}}$ of $\mathbb{P}Y^{\frac{1}{r}}$ on which the marked point corresponding to the leg l is evaluated, this is determined by the multiplicity of $L_1, \cdots, L_k, N$ at the corresponding marked point.
- A half-edge h incident to a vertex v corresponds to a node at which components C_e and C_v meet, and $m(h)$ indicates the rigidified inertia component $\bar{I}_{m(h)}\mathbb{P}Y^{\frac{1}{r}}$ of $\mathbb{P}Y^{\frac{1}{r}}$ on which the node corresponding to h is evaluated. If v is unstable, then h corresponds to a single point on a component C_e, then $m(h)$ is the *inverse* in $G \times \boldsymbol{\mu}_r$ of the multiplicity of $L_1, \cdots, L_k, N$ at this point.

In particular, we note that the decorations at each stable vertex v yield a tuple

$$\vec{m}(v) \in (G \times \boldsymbol{\mu}_r)^{\mathrm{val}(v)}$$

recording the multiplicities of $L_1, \cdots, L_k, N$ at every special point of C_v. We have the following remarks:

Remark 4.2. The crucial observation, now, is the following. For a stable vertex v such that $j(v) = 0$, we have $z_1|_{C_v} \equiv 0$, so the stability condition (4.2) implies that $l_{\epsilon\theta'}(q)$ for each $q \in C_v$. That is, the restriction of $(C; q_1, \ldots, q_m; L_1, \cdots, L_k; \vec{x})$ to C_v gives rise to a $\epsilon\theta'$-stable quasimap to the quotient stack $\mathfrak{Y} = [AY/G]$ in

$$Q^{\epsilon\theta'}_{0,\vec{m}(v)}(\mathfrak{Y}, \beta(v)) := \coprod_{\substack{d \in \mathrm{Eff}(AY,G,\theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta(v)}} Q^{\epsilon\theta'}_{0,\vec{m}(v)}(\mathfrak{Y}, d) \ .$$

On the other hand, for a stable vertex v such that $j(v) = \infty$, we have $z_2|_{C_v} \equiv 0$, so the stability condition (4.3) implies that $\mathrm{ord}_q(\vec{x}) = 0$ for each $q \in C_v$. Thus, the restriction of $(C; q_1, \ldots, q_m; L_1, \cdots, L_k; \vec{x})$ to C_v gives rise to a usual twisted stable map in

$$\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_{\theta'}/Y}, \beta(v)) := \coprod_{\substack{d \in \mathrm{Eff}(AY,G,\theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta(v)}} \mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_{\theta'}/Y}, d) \ .$$

Here $\sqrt[r]{L_{\theta'}/Y}$ is the root gerbe of Y by taking r-th root of $L_{\theta'}$.

Remark 4.3. For each edge e, the restriction of $\vec{x}$ to C_e defines a constant map to Y (possibly with an additional basepoint at the ramification point q_0). So if there is no basepoint on C_e, $(\vec{x}, \vec{z})$ defines a representable map

$$C_e \to \mathbb{B}G_y \times \mathbb{P}_{r,1}$$

where $y \in Y$ comes from $\vec{x}$, G_y is the isotropy group of $y \in Y$. Then we have $m(q_0) = (g^{-1}, 1)$ and $m(q_\infty) = (g, \mu_r^{\delta(e)})$ for some $g \in G_y$. Note when r is a sufficiently large prime comparing to $\delta(e)$, assuming that the order of g is equal to a, we have $C_e \cong \mathbb{P}^1_{ar,a}$ and the ramification point q_∞ must be a special point. Here $\mathbb{P}^1_{ar,a}$ is the unique Deligne-Mumford stack with coarse moduli $\mathbb{P}^1$, isotropy group $\boldsymbol{\mu}_a$ at $0 \in \mathbb{P}^1$, isotropy group $\boldsymbol{\mu}_{ar}$ at $\infty \in \mathbb{P}^1$, and generic trivial stabilizer.

If q_0 is a basepoint of degree β, the ramification point q_0 can't be an orbifold point, thus $m(q_0) = (1, 1) \in G \times \boldsymbol{\mu}_r$. In this case, by the representable condition, we have $C_e \cong \mathbb{P}_{ar,1}$ and $m(q_\infty) = (g_\beta, \mu_r^{\delta(e)})$ if r is a sufficiently large prime. Here a is minimal positive integer associated to g_β as in Lemma 3.4.

Remark 4.4. If there is a basepoint on the edge curve C_e, then the degree $(\beta(e), \frac{\delta(e)}{r})$ on C_e must satisfy the relation $\delta(e) \geqslant \beta(e)(L_{\theta'})$. Otherwise we have $z_1|_{C_e} \equiv 0$, given the fact z_2 vanishes at q_∞, this will violate the nondegeneracy condition for z_1 and z_2.

4.3. **Localization analysis.** Fix $\beta \in \mathrm{Eff}(W, G, \theta), \delta \in \mathbb{Z}_{\geqslant 0}$, we will consider the space $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (\beta, \frac{\delta}{r}))$. The reason why we assume that the second degree is $\frac{\delta}{r}$ is that $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (\beta, \frac{\delta}{r}))$ corresponds to $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}, (\beta, \delta))$, here $\mathbb{P}\mathfrak{Y}$ is equal to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ for $r = 1$. In the remaining section, we will always assume that r is a *sufficiently large prime*.

Following the discussion in [CJR17a], the virtual localization formula of Graber–Pandharipande [GP99] expresses

$$[Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (\beta, \frac{\delta}{r}))]^{\mathrm{vir}}$$

in terms of contributions from each fixed-loci graph Γ:

$$[Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (\beta, \frac{\delta}{r}))]^{\mathrm{vir}} = \sum_{\Gamma} \frac{1}{\mathbb{A}_\Gamma} \iota_{\Gamma *} \left(\frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_\Gamma)} \right) . \tag{4.6}$$

Here, for each graph Γ, $[F_\Gamma]^{\mathrm{vir}}$ is obtained from the $\mathbb{C}^*$-fixed part of the restriction to the fixed loci of the obstruction theory on $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (\beta, \frac{\delta}{r}))$, and N^{vir}_Γ is the equivariant Euler class of the $\mathbb{C}^*$-moving part of this restriction. Besides, $\mathbb{A}_\Gamma$ is the automorphism factor for the graph Γ, which represents the degree of F_Γ into the corresponding open and closed $\mathbb{C}^*$-fixed substack in $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (\beta, \frac{\delta}{r}))$.

We will compute the contributions of each graph Γ explicitly in this subsection. To compute the contribution of a graph Γ to (4.6), one should first apply the normalization exact sequence to the relative obstruction theory (4.5), thus decomposing the contribution of Γ to (4.6) into vertex, edge, and node factors. This accounts for all but the automorphisms and deformations within $\mathcal{D}_{0,m,\beta,\frac{\delta}{r}}$. The latter are distributed in the vertex, edge, and node contributions as deformations of the vertex components and their line bundles, deformations of the edge components and their line bundles, and deformations smoothing the nodes, respectively, in what follows. We include the factors from automorphisms of the source curve also in the edge contributions as part of the gerbe structure of the edge moduli $\mathcal{M}_e$, then an additional factor from gerbe structure of each edge moduli will appear in the automorphism factor $\mathbb{A}_\Gamma$(see (4.15) for the localization contribution of graph Γ).

4.3.1. *Vertex contributions.* First of all, consider the stable vertex v over ∞, this vertex moduli $\mathcal{M}_v$ corresponds to the moduli stack

$$\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_{\theta'}/Y}, \beta(v)) := \bigsqcup_{\substack{d \in \mathrm{Eff}(AY,G,\theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta(v)}} \mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_{\theta'}/Y}, d) ,$$

which parameterizes twisted stable maps to the root gerbe $\sqrt[r]{L_{\theta'}/Y}$ over Y.

Let

$$\pi : \mathcal{C}_\infty \to \mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_{\theta'}/Y}, \beta(v))$$

be the universal curve over $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_{\theta'}/Y},\beta(v))$. In this case, on $\mathcal{C}_\infty$, we have $\mathcal{L}_{-\theta'}\otimes\mathcal{N}^{\otimes r}\otimes\mathbb{C}_\lambda\cong\mathcal{O}_{\mathcal{C}_\infty}$ as $z_1|_{\mathcal{C}_\infty}\equiv 1$, hence we have $\mathcal{N}\cong\mathcal{L}_{\theta'}^{\frac{1}{r}}\otimes\mathbb{C}_{-\frac{\lambda}{r}}$, here $\mathcal{L}_{\theta'}^{\frac{1}{r}}$ is the line bundle over $\mathcal{C}_\infty$ that is the pull back of the universal root bundle over $\sqrt[r]{L_{\theta'}/Y}$ along the universal map $f:\mathcal{C}_\infty\to\sqrt[r]{L_{\theta'}/Y}$. The movable part of the perfect obstruction theory comes from the deformation of z_2, thus the *inverse of Euler class* of the virtual normal bundle is equal to

$$e^{\mathbb{C}^*}((-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})\otimes\mathbb{C}_{-\frac{\lambda}{r}}).$$

When r is a sufficiently large prime, following [JPPZ18], the above Euler class has a representation

$$\sum_{d\geqslant 0}c_d(-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})(\frac{-\lambda}{r})^{|E(v)|-1-d}\ .$$

Here the virtual bundle $-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}}$ has virtual rank $|E(v)|-1$, where $|E(v)|$ is the number of edges incident to the vertex v. The fixed part of the perfecct obstruction theory contributes to the virtual cycle

$$[\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_{\theta'}/Y},\beta(v))]^{\mathrm{vir}}\ .$$

For the stable vertex v over 0, the vertex moduli $\mathcal{M}_v$ corresponds to the moduli space

$$Q^{\epsilon\theta'}_{0,\vec{m}(v)}(\mathfrak{Y},\beta(v)):=\bigsqcup_{\substack{d\in\mathrm{Eff}(AY,G,\theta)\\(i_{\mathfrak{Y}})_*(d)=\beta(v)}}Q^{\epsilon\theta'}_{0,\vec{m}(v)}(\mathfrak{Y},d)\ .$$

Let $\pi:\mathcal{C}_0\to Q^{\epsilon\theta'}_{0,\vec{m}(v)}(\mathfrak{Y},\beta(v))$ be the universal curve over $Q^{\epsilon\theta'}_{0,\vec{m}(v)}(\mathfrak{Y},\beta)$. In this case, the fixed part of the obstruction theory of the vertex moduli over 0 yields the virtual cycle

$$[Q^{\epsilon\theta'}_{0,\vec{m}(v)}(\mathfrak{Y},\beta(v))]^{\mathrm{vir}}\ .$$

Note $\mathcal{N}|_{\mathcal{C}_0}=\mathcal{O}_{\mathcal{C}_0}$ as $z_2|_{\mathcal{C}_0}\equiv 1$, therefore the virtual normal comes from the movable part of the infinitesimal deformations of the section z_1, which is a section of the line bundle $\mathcal{L}_{-\theta'}$ over $\mathcal{C}_0$, whose Euler class is equal to

$$e^{\mathbb{C}^*}((R^\bullet\pi_*\mathcal{L}_{-\theta'})\otimes\mathbb{C}_\lambda)\ .$$

4.3.2. *Edge contributions: without basepoint case.* Assume that the multiplicity at $q_\infty\in C_e$ is equal to $(g,\mu_r^{\delta(e)})\in G\times\boldsymbol{\mu}_r$ and a (or a_e) is the order of g. When r is sufficiently large, due to Remark 4.3, C_e must be isomorphic to $\mathbb{P}^1_{ar,a}$ where the ramification point q_0 for which $z_1=0$ is isomorphic to $\mathbb{B}\boldsymbol{\mu}_a$, and the ramification point q_∞ for which $z_2=0$ must be a special point and is isomorphic to $\mathbb{B}\boldsymbol{\mu}_{ar}$. The restriction of degree $(\beta,\frac{\delta}{r})$ from C to C_e is equal to $(0,\frac{\delta(e)}{r})$, which is equivalent to:

$$deg(L_j|_{C_e})=0\quad for\,1\leqslant j\leqslant k,\quad deg(N|_{C_e})=\frac{\delta(e)}{r}\ .$$

Recall that the inertia stack component I_gY of $I_\mu Y$ is isomorphic to the quotient stack

$$[AY^{ss}(\theta)^g/G]\ .$$

We construct the edge moduli $\mathcal{M}_e$ as

$$\mathcal{M}_e := \sqrt[a\delta(e)]{L_{-\theta'}/I_gY} \ ,$$

which is the root gerbe over the stack I_gY by taking the $a\delta(e)$th root of the line bundle $L_{-\theta'}$.

The root gerbe $\sqrt[a\delta(e)]{L_{-\theta'}/I_gY}$ admits a representation as a quotient stack:

$$[(AY^{ss}(\theta)^g \times \mathbb{C}^*)/(G\times \mathbb{C}^*_w)] \ , \tag{4.7}$$

where the (right) action is defined by:

$$(\vec{x}, v)\cdot(g,w) = (\vec{x}\cdot g, \theta'(g)vw^{a\delta(e)}) \ ,$$

for all $(g,w)\in G\times\mathbb{C}^*_w$ and $(\vec{x},v)\in AY^{ss}(\theta)^g\times\mathbb{C}^*$. Here $\vec{x}\cdot g$ is given by the action as in the definition of $[AY/G]$, the torus $\mathbb{C}^*_w$ is isomorphic to $\mathbb{C}^*$ with variable w. For any character ρ of G, define a new character of $G\times\mathbb{C}^*_w$ by composing the projection map $\mathrm{pr}_G : G\times\mathbb{C}^*_w \to G$. By an abuse of notation, we will continue to use the notation ρ to mean the new character of $G\times\mathbb{C}^*_w$. Then ρ will determines a line bundle $L_\rho := [(AY^{ss}(\theta)^g\times\mathbb{C}^*\times\mathbb{C}_\rho)/(G\times\mathbb{C}^*_w)]$ on $\sqrt[a\delta(e)]{L_{-\theta'}/I_gY}$ by the Borel construction.

By virtue of the universal property of root gerbe, on $\mathcal{M}_e = \sqrt[a\delta(e)]{L_{-\theta'}/I_gY}$, there is a universal line bundle $\mathcal{R}$ that is the $a\delta(e)$th root of the line bundle $L_{-\theta'}$. The root bundle $\mathcal{R}$ is associated to the character

$$\mathrm{pr}_{\mathbb{C}^*} : G\times\mathbb{C}^*_w \to \mathbb{C}^*_w, \quad (g,w)\in G\times\mathbb{C}^*_w \mapsto w\in\mathbb{C}^*_w$$

by the Borel construction. We have the relation

$$L_{-\theta'} = \mathcal{R}^{a\delta(e)} \ .$$

The coordinate functions $\vec{x}$ and v of $AY^{ss}(\theta)^g\times\mathbb{C}^*$ descents to be universal sections of line bundles $\oplus_{\rho\in[n]}L_\rho$ and $L_{\theta'}\otimes\mathcal{R}^{\otimes a\delta(e)}$ over $\mathcal{M}_e$, respectively.

We will construct a universal family of $\mathbb{C}^*-$fixed quasimaps to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ of degree $(0,\frac{\delta(e)}{r})$ over $\mathcal{M}_e$:

$$\begin{array}{ccc} \mathcal{C}_e := \mathbb{P}_{ar,a}(\mathcal{R}\oplus\mathcal{O}_{\mathcal{M}_e}) & \xrightarrow{f} & \mathbb{P}\mathfrak{Y}^{\frac{1}{r}} \\ \Big\downarrow{\scriptstyle \pi} & & \\ \mathcal{M}_e := \sqrt[a\delta(e)]{L_{-\theta'}/I_gY}. & & \end{array}$$

Then the universal curve $\mathcal{C}_e$ over $\mathcal{M}_e$ can be represented as a quotient stack:

$$\mathcal{C}_e = [(AY^{ss}(\theta)^g\times\mathbb{C}^*\times U)/(G\times\mathbb{C}^*_w\times T)] \ ,$$

where $T=\{(t_1,t_2)\in(\mathbb{C}^*)^2| \quad t_1^a = t_2^{ar}\}$. The (right) action is defined by:

$$(\vec{x},v,x,y)\cdot(g,w,(t_1,t_2)) = (\vec{x}\cdot g, \theta'(g)vw^{a\delta(e)}, wt_1x, t_2y) \ ,$$

for all $(g,w,(t_1,t_2))\in G\times\mathbb{C}^*_w\times T$ and $(\vec{x},v,(x,y))\in AY^{ss}(\theta)^g\times\mathbb{C}^*\times U$. Then $\mathcal{C}_e$ is a family of orbifold $\mathbb{P}_{ar,a}$ parameterized by $\mathcal{M}_e$.

There are two standard characters χ_1 and χ_2 of T:

$$\chi_1 : (t_1,t_2)\in T\mapsto t_1\in\mathbb{C}^*, \quad \chi_2 : (t_1,t_2)\in T\mapsto t_2\in\mathbb{C}^* \ .$$

We can lift them to be new characters of $G\times\mathbb{C}^*_w\times T$ by composing the projection map $\mathrm{pr}_T : G\times\mathbb{C}^*_w\times T \to T$. By an abuse of notation, we continue to use χ_1, χ_2 to denote the new characters. Then χ_1, χ_2 defines two line bundles

$$M_1 := (AY^{ss}(\theta)^g \times\mathbb{C}^*\times U) \times_{G\times\mathbb{C}^*_w\times T} \mathbb{C}_{\chi_1}$$

and

$$M_2 := (AY^{ss}(\theta)^g \times\mathbb{C}^*\times U) \times_{G\times\mathbb{C}^*_w\times T} \mathbb{C}_{\chi_2}$$

on $\mathcal{C}_e$ by the Borel construction, respectively. We have the relation $M_1^{\otimes a} = M_2^{\otimes ar}$ on $\mathcal{C}_e$. The universal map f from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ can be constructed as follows: let

$$\tilde{f} : AY^{ss}(\theta)^g \times\mathbb{C}^*\times U \to AY\times U$$

be the morphism defined by:

$$\begin{aligned} (\vec{x}, v, x, y) &\in AY^{ss}(\theta)^g \times\mathbb{C}^*\times U \mapsto \\ &((x_1, \cdots, x_n), v^{-1}x^{a\delta(e)}, y^{a\delta(e)}) \in AY\times U \end{aligned} \tag{4.8}$$

Then $\tilde{f}$ is equivariant with respect to the group homomorphism from $G\times\mathbb{C}^*_w\times T$ to $G\times\mathbb{C}^*$ defined by:

$$\begin{aligned} (g, w, (t_1, t_2)) &\in G\times\mathbb{C}^*_w\times T \mapsto \\ &(g((t_1^{-1}t_2^r)^{p_1}, \cdots, (t_1^{-1}t_2^r)^{p_k}), t_2^{a\delta(e)}) \in G\times\mathbb{C}^* , \end{aligned} \tag{4.9}$$

where the tuple $(p_1, \cdots, p_k) \in \mathbb{N}^k$ satisfies that $g = (\mu_a^{p_1}, \cdots, \mu_a^{p_k}) \in G$. Note $\tilde{f}$ is well defined for $\chi_1^{-1}\chi_2^r$ is a torsion character of T of order a. The above construction gives the universal morphism f from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ by descent.

Now we define a (quasi[1] left) $\mathbb{C}^*-$action on $\mathcal{C}_e$ such that f is $\mathbb{C}^*-$equivariant. The $\mathbb{C}^*$-action on $\mathcal{C}_e$ is induced by the $\mathbb{C}^*-$action on $AY^{ss}(\theta)^g\times\mathbb{C}^*\times U$:

$$m : \mathbb{C}^* \times AY^{ss}(\theta)^g \times\mathbb{C}^*\times U \to AY^{ss}(\theta)\times\mathbb{C}^*\times U ,$$

$$t\cdot(\vec{x}, v, (x, y)) = (\vec{x}, v, (x, t^{\frac{-1}{ar\delta(e)}}y)) .$$

Note then π is $\mathbb{C}^*$-equivariant map, where $\mathcal{M}_e$ is equipped with the trivial $\mathbb{C}^*$-action. By the universal property of the projectivized bundle $\mathcal{C}_e$ over $\mathcal{M}_e$, one has a tautological section

$$(x, y) \in H^0(\mathcal{C}_e, (M_1\otimes\pi^*\mathcal{R}) \oplus (M_2\otimes\mathbb{C}_{\frac{-1}{ar\delta(e)}})) ,$$

which is also a $\mathbb{C}^*-$invariant section.

Now we can check that f is a $\mathbb{C}^*-$equivariant morphism from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ with respect to the $\mathbb{C}^*-$actions for $\mathcal{C}_e$ and $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$. According to Remark 4.1, f is equivalent to the following data:

(1) $k+1$ $\mathbb{C}^*-$equivariant line bundles $\mathcal{C}_e$:

$$\mathcal{L}_j := \pi^*L_{\pi_j} \otimes (M_1^\vee \otimes M_2^{\otimes r})^{p_j}, 1 \leqslant j \leqslant k ,$$

and

$$\mathcal{N} := M_2^{a\delta(e)} \otimes \mathbb{C}_{\frac{-\lambda}{r}} ,$$

[1] This means we allow $\mathbb{C}^*-$action on C_e with fractional weight. See a similar discussion in [CLLL16, §2.2].

where $(L_{\pi_j})_{1\leqslant j\leqslant k}$ are the standard $\mathbb{C}^*$-equivariant line bundles on $\mathcal{M}_e$ by the Borel contribution, M_1, M_2 are the standard $\mathbb{C}^*$-equivariant line bundles on C_e by the Borel construction;

(2) a universal section

$$\begin{aligned}(\vec{x},(\zeta_1,\zeta_2)) :=&((x_1,\cdots,x_n,(v^{-1}x^{a\delta(e)},y^{a\delta(e)}))\\ &\in H^0(\mathcal{C}_e,\bigoplus_{i=1}^n \mathcal{L}_{\rho_j}\oplus(\mathcal{L}_{-\theta'}\otimes\mathcal{N}^{\otimes r}\otimes\mathbb{C}_\lambda)\oplus\mathcal{N})^{\mathbb{C}^*}\ .\end{aligned} \tag{4.10}$$

Equipped with these notations, now we compute the localization contribution from $\mathcal{M}_e$. Based on the perfect obstruction theory for quasimaps in $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(\beta,\frac{\delta}{r}))$, the restriction of the prefect obstruction theory to $\mathcal{M}_e$ decomposes into three parts: (1) the deformation theory of source curve $\mathcal{C}_e$; (2) the deformation theory of the lines bundles $(\mathcal{L}_j)_{1\leqslant j\leqslant k}$ and $\mathcal{N}$; (3) the deformation theory of the section

$$(\vec{x},(\zeta_1,\zeta_2))\in\Gamma\left(\bigoplus_{i=1}^n \mathcal{L}_{\rho_i}\oplus(\mathcal{L}_{-\theta'}\otimes\mathcal{N}^{\otimes r}\otimes\mathbb{C}_\lambda)\oplus\mathcal{N}\right)\ .$$

The $\mathbb{C}^*-$fixed part of three parts above will contribute to the virtual cycle of $\mathcal{M}_e$, we will show that $[\mathcal{M}_e]^{\mathrm{vir}}=[\mathcal{M}_e]$. The virtual normal bundle comes from the $\mathbb{C}^*-$moving part of the three parts above.

First every fiber curve C_e of $\mathcal{C}_e$ over a geometrical point of $\mathcal{M}_e$ is isomorphic to $\mathbb{P}_{ar,a}$, which is rational. Then the infinitesimal deformations/obstructions of C_e and line bundles $L_j := \mathcal{L}_j|_{C_e}$, $N := \mathcal{N}|_{C_e}$ are zero. Thus their contribution to the perfect obstruction theory comes from infinitesimal automorphisms. The infinitesimal automorphisms of C_e come from the space of vector fields on C_e that vanishes on special points. Thus the $\mathbb{C}^*-$fixed part of the infinitesimal automorphism of C_e comes from the $1-$dimensional subspace of vector fields on C_e that vanish on the two ramification points, which, together with the infinitesimal automorphism of line bundle N, will be canceled with the fixed part of infinitesimal deformations of sections $(z_1,z_2)=(\zeta_1,\zeta_2)|_{C_e}$. The movable part of infinitesimal automorphisms of C_e is nonzero only if ramification point q_0 on C_e is not a special point by Remark 4.3. When the ramification point of C_e at 0 is not a special point, which happens when $a=1$, then it contributes

$$\frac{\delta(e)}{\lambda-D_{\theta'}}$$

to the virtual normal bundle.

Now let's turn to localization contributions from sections $(\vec{x},(\zeta_1,\zeta_2))$. First the deformations of sections $\vec{x}$ are fixed, which, together with the fixed parts of infinitesimal automorphisms of C_e and the line bundles L_j, N, as well as the fixed parts of infinitesimal deformations of sections (z_1,z_2), contributes to the virtual cycle $[\mathcal{M}_e]^{\mathrm{vir}}$ of the edge moduli $\mathcal{M}_e$, which is equal to the fundamental class of $\mathcal{M}_e$. The localization contribution from the infinitesimal deformations of sections (z_1,z_2) to the virtual normal bundle is the Euler class of

$$(R^\bullet\pi_*(\mathcal{L}_{-\theta'}\otimes\mathcal{N}^{\otimes r}\otimes\mathbb{C}_\lambda)\oplus R^\bullet\pi_*\mathcal{N})^{\mathrm{mov}}\ .$$

We first analyze the deformations of z_2, we continue to use the tautological section (x,y) as in (4.3.3). Sections of N are spanned by monomials $(x^{am}y^n)|_{C_e}$ with $arm+n = a\delta(e)$ and $m,n \in \mathbb{Z}_{\geqslant 0}$. Note $x^{am}y^n$ may not be a global section of $\mathcal{N}$ but always a global section of $R^{\otimes am} \otimes \mathcal{N} \otimes \mathbb{C}_{\frac{m}{\delta(e)}\lambda}$. Then $R^{\bullet}\pi_*\mathcal{N}$ will decompose as a direct sum of line bundles, each corresponds to the monomial $x^m y^n$, whose first chern class is

$$c_1(\mathcal{R}^{\otimes -am} \otimes \mathbb{C}_{\frac{-m}{\delta(e)}\lambda}) = \frac{m}{\delta(e)}(D_{\theta'} - \lambda) \ .$$

So the Euler class of $R^{\bullet}\pi_*\mathcal{N}$ is equal to

$$\prod_{m=0}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left(\frac{m}{\delta(e)}(D_{\theta'} - \lambda) \right) .$$

The factor for $m=0$ appearing in the above product is the $\mathbb{C}^*-$fixed part of $R^{\bullet}\pi_*\mathcal{N}$, it will contribute to the virtual cycle of $\mathcal{M}_e$. The rest contributes to the virtual normal bundle as

$$\prod_{m=1}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left(\frac{m}{\delta(e)}(D_{\theta'} - \lambda) \right) .$$

Note when r is sufficiently large, the above product becomes 1.

For the deformation of z_1, arguing in the same way as z_2, the Euler class of $R^{\bullet}\pi_*(\mathcal{L}_{-\theta'} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda)$ is equal to

$$\prod_{m=0}^{\delta(e)} \left(\frac{m}{\delta(e)}(-D_{\theta'} + \lambda) \right) .$$

The term corresponding $m=0$ in the above product is the $\mathbb{C}^*$-invariant part, which will contribute to the virtual cycle of $\mathcal{M}_e$. The rest contributes to the virtual normal bundle as

$$\prod_{m=1}^{\delta(e)} \left(\frac{m}{\delta(e)}(-D_{\theta'} + \lambda) \right) .$$

4.3.3. *Edge contributions: basepoint case.* When there is a base point on the edge curve, it has degree $(\beta(e), \frac{\delta(e)}{r})$ with $\beta(e) \neq 0$ and $\delta(e) \geqslant \beta(e)(L_{\theta'})$ by Remark 4.4, we will write $\beta = \beta(e)$ only in this subsection for simplicity unless stated otherwise. Then the multiplicity at $q_\infty \in C_e$ is equal to $(g, \mu_r^{\delta(e)}) \in G \times \boldsymbol{\mu}_r$, where $g = g_\beta$ is defined in §3.3. Let a (or a_e) be the minimal positive integer associated to β as in Lemma 3.4, which is also the order of g_β. When r is sufficiently large, due to Remark 4.3, C_e must be isomorphic to $\mathbb{P}^1_{ar,1}$ where the ramification point q_0 for which $z_1 = 0$ is an ordinary point, and the ramification point q_∞ for which $z_2 = 0$ must be a special point, which is isomorphic to $\mathbb{B}\boldsymbol{\mu}_{ar}$.

Recall that

$$[Y^{ss}_\beta / G] = [(Z^{ss}_\beta \cap AY)/G]$$

in §3.2. We define the edge moduli $\mathcal{M}_e$ to be

$$\sqrt[a\delta(e)]{L_{-\theta'}/[Y^{ss}_\beta/G]} \ ,$$

which is the root gerbe over the stack $[Y_\beta^{ss}/G] \subset [AY^{ss}(\theta)^g/G] \subset I_\mu Y$ by taking $a\delta(e)$th root of the line bundle $L_{-\theta'}$ on $[Y_\beta^{ss}/G]$.

The root gerbe $\sqrt[a\delta(e)]{L_{-\theta'}/[Y_\beta^{ss}/G]}$ admits a representation as a quotient stack:

$$[(Y_\beta^{ss} \times \mathbb{C}^*)/(G \times \mathbb{C}_w^*)],$$

where the (right) action is defined by

$$(\vec{x}, v) \cdot (g, w) = (\vec{x} \cdot g, \theta'(g) v w^{a\delta(e)}) \ ,$$

for all $(g, w) \in G \times \mathbb{C}_w^*$ and $(\vec{x}, v) \in A(Y)^g \times \mathbb{C}^*$. Here $\vec{x} \cdot g$ is given by the action as in the definition of $[AY/G]$. For every character ρ of G, we can define a new character of $G \times \mathbb{C}_w^*$ by composing the projection map $\mathrm{pr}_G : G \times \mathbb{C}_w^* \to G$. By an abuse of notation, we will continue to use the notation ρ to name the new character of $G \times \mathbb{C}_w^*$. Then the new character ρ will determines a line bundle $L_\rho := [(Y_\beta^{ss} \times \mathbb{C}^* \times \mathbb{C}_\rho)/(G \times \mathbb{C}_w^*)]$ on $\sqrt[a\delta(e)]{L_{-\theta'}/[Y_\beta^{ss}/G]}$.

By virtue of its universal property of the root gerbe $\sqrt[a\delta(e)]{L_{-\theta'}/[Y_\beta^{ss}/G]}$, there is a universal line bundle $\mathcal{R}$ that is the $a\delta(e)$th root of line bundle $L_{-\theta'}$ over the root gerbe. This root line bundle $\mathcal{R}$ can also constructed by the Borel construction, i.e. $\mathcal{R}$ is associated to the character p_2:

$$\mathrm{pr}_{\mathbb{C}_w^*} : G \times \mathbb{C}_w^* \to \mathbb{C}_w^* \quad (g, w) \in G \times \mathbb{C}_w^* \mapsto w \in \mathbb{C}_w^* \ .$$

We have the relation

$$L_{-\theta'} = \mathcal{R}^{a\delta(e)} \ .$$

Then the coordinate functions $(\vec{x}, v) \in Y_\beta^{ss} \times \mathbb{C}^*$ descents to be tautological sections of vector bundle $\bigoplus_{i=1}^n L_{\rho_i} \oplus (L_{\theta'} \otimes \mathcal{R}^{a\delta(e)})$ on $\sqrt[a\delta(e)]{L_{-\theta'}/[Y_\beta^{ss}/G]}$.

We will construct a universal family of $\mathbb{C}^*-$fixed quasimaps to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ of degree of $(\beta, \frac{\delta(e)}{r})$ over the edge moduli $\mathcal{M}_e$, which takes the form

$$\begin{tikzcd}
\mathcal{C}_e := \mathbb{P}_{ar,1}(\mathcal{R}^{\otimes a} \oplus \mathcal{O}_{\mathcal{M}_e}) \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}\mathfrak{Y}^{\frac{1}{r}} \\
\mathcal{M}_e := \sqrt[a\delta(e)]{L_{-\theta'}/[Y_\beta^{ss}/G]} \ . &
\end{tikzcd}$$

The universal curve $\mathcal{C}_e$ over the edge moduli $\mathcal{M}_e$ is constructed as a quotient stack:

$$\mathcal{C}_e = [(Y_\beta^{ss} \times \mathbb{C}^* \times U)/(G \times \mathbb{C}_w^* \times \mathbb{C}_t^*)] \ ,$$

where the right action is defined by:

$$(\vec{x}, v, x, y) \cdot (g, w, t) = (\vec{x} \cdot g, \theta'(g) v w^{a\delta(e)}, w^a t^{ar} x, ty) \ ,$$

for all $(g, w, t) \in G \times \mathbb{C}_w^* \times \mathbb{C}_t^*$ and $(\vec{x}, v, (x, y)) = ((x_1, \cdots, x_n), v, (x, y)) \in Y_\beta^{ss} \times \mathbb{C}^* \times U$.

The universal map f from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ can be presented as follows:

$$\tilde{f} : Y_\beta^{ss} \times \mathbb{C}^* \times U \to AY \times U \ ,$$

defined by:

$$(\vec{x}, v, (x,y)) \in Y_\beta^{ss} \times \mathbb{C}^* \times U \mapsto \big(\big(x_1 x^{\beta(L_{\rho_1})}, \cdots, x_n x^{\beta(L_{\rho_n})}\big), v^{-1} x^{\delta(e)-\beta(L_{\theta'})}, y^{a\delta(e)}\big) \in AY \times U \ . \tag{4.11}$$

Note that when $\beta(L_{\rho_i}) \notin \mathbb{Z}_{\geqslant 0}$, we must have $x_i = 0$ as $\vec{x} \in Y_\beta^{ss}$, so the $\tilde{f}$ is well defined. Then $\tilde{f}$ is equivariant with respect to the group homomorphism from $G \times \mathbb{C}_w^* \times \mathbb{C}_t^*$ to $G \times \mathbb{C}^*$ defined by:

$$(g, w, t) \in G \times \mathbb{C}_w^* \times \mathbb{C}_t^* \mapsto \big(g\big(t^{ar\beta(L_{\pi_1})} w^{a\beta(L_{\pi_1})}, \cdots, t^{ar\beta(L_{\pi_k})} w^{a\beta(L_{\pi_k})}\big), t^{a\delta(e)}\big) \in G \times \mathbb{C}^* \ . \tag{4.12}$$

This gives the universal morphism f from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ by descent.

There is a tautological line bundle $\mathcal{O}_{C_e}(1)$ on C_e associated to the character $\mathrm{pr}_{\mathbb{C}_t^*}$ of $G \times \mathbb{C}_w^* \times \mathbb{C}_t^*$ by the Borel construction. Here $\mathrm{pr}_{\mathbb{C}_t^*}$ is the projection map from $G \times \mathbb{C}_w^* \times \mathbb{C}_t^*$ to $\mathbb{C}_t^*$.

We will define a (quasi left) $\mathbb{C}^*-$action on $\mathcal{C}_e$ such that the map f constructed above is $\mathbb{C}^*-$equivariant. Define a (left) $\mathbb{C}^*-$action on $\mathcal{C}_e$ which is induced from the $\mathbb{C}^*-$action on $Y_\beta^{ss} \times \mathbb{C}^* \times U$:

$$m : \mathbb{C}^* \times Y_\beta^{ss} \times \mathbb{C}^* \times U \to Y_\beta^{ss} \times \mathbb{C}^* \times U \ ,$$

$$t \cdot (x, v, (x, y)) = (x, v, (x, t^{\frac{-1}{ar\delta(e)}} y)) \ .$$

Note the morphism π is also $\mathbb{C}^*$-equivariant, where $\mathcal{M}_e$ is equipped with trivial $\mathbb{C}^*$-action. By the universal property of the projectivized bundle $\mathcal{C}_e$ over $\mathcal{M}_e$, the line bundle $\mathcal{O}_{\mathcal{C}_e}(1)$ is equipped with tautological sections

$$(x, y) \in H^0\big(\big(\mathcal{O}_{\mathcal{C}_e}(ar) \otimes \pi^* \mathcal{R}^{\otimes a}\big) \oplus \big(\mathcal{O}_{\mathcal{C}_e}(1) \otimes \mathbb{C}_{\frac{-1}{ar\delta(e)}}\big)\big) \ ,$$

which is also a $\mathbb{C}^*-$invariant section. Here $\mathcal{O}_{\mathcal{C}_e}(1)$ are the standard $\mathbb{C}^*$-equivariant line bundle on $\mathcal{C}_e$ by the Borel construction.

Now we can check that f is a $\mathbb{C}^*-$equivariant morphism from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ with respect to the $\mathbb{C}^*-$actions for $\mathcal{C}_e$ and $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$. According to Remark 4.1, f is equivalent to the following data:

(1) $k+1$ $\mathbb{C}^*$-equivariant line bundles on $\mathcal{C}_e$:

$$\mathcal{L}_j := \pi^* L_{\pi_j} \otimes \mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\pi_j})) \otimes \pi^* \mathcal{R}^{\otimes a\beta(L_{\pi_j})}, 1 \leqslant j \leqslant k \ ,$$

and

$$\mathcal{N} := \mathcal{O}_{\mathcal{C}_e}(a\delta(e)) \otimes \mathbb{C}_{\frac{-\lambda}{r}} \ ,$$

where the line bundles L_{π_j}, $\mathcal{R}$ are the standard $\mathbb{C}^*$-equivariant line bundle on $\mathcal{M}_e$ by the Borel construction;

(2) a universal section

$$\big(\vec{x}, (\zeta_1, \zeta_2)\big) := \big(\big(x_1 x^{\beta(L_{\rho_1})}, \cdots, x_n x^{\beta(L_{\rho_n})}\big), (v^{-1} x^{\delta(e)-\beta(L_{\theta'})}, y^{a\delta(e)})\big) \in H^0\big(\mathcal{C}_e, (\oplus_{i=1}^n \mathcal{L}_{\rho_i}) \oplus (\mathcal{L}_{-\theta'} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda) \oplus \mathcal{N}\big)^{\mathbb{C}^*} \ , \tag{4.13}$$

where the line bundles $\mathcal{L}_{-\theta'}$ and $\mathcal{L}_{\rho_i}$ are constructed from line bundles $\mathcal{L}_j$ as before.

Equipped with these notations, now we compute the localization contribution from $\mathcal{M}_e$. Based on the perfect obstruction theory for quasimaps in $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(\beta,\frac{\delta}{r}))$, the restriction of the prefect obstruction theory to $\mathcal{M}_e$ decomposes into three parts: (1) the deformation theory of source curve $\mathcal{C}_e$; (2) the deformation theory of the lines bundles $(\mathcal{L}_j)_{1\leqslant j\leqslant k}$ and $\mathcal{N}$; (3) the deformation theory for the section

$$(\vec{x},(\zeta_1,\zeta_2))\in\Gamma\left(\bigoplus_{i=1}^{n}\mathcal{L}_{\rho_i}\oplus(\mathcal{L}_{-\theta'}\otimes\mathcal{N}^{\otimes r}\otimes\mathbb{C}_\lambda)\oplus\mathcal{N}\right)\ .$$

The virtual normal bundle comes from the movable part of the three parts, and the fixed part will contribute to the virtual cycle of $\mathcal{M}_e$. Recall that the hypersurface Y is cut off by a section of the line bundle L on the toric stack X, we will show that:

Lemma 4.5. *We have the following:*

(1) *when $\beta(L)\in\mathbb{Z}$, if s does not vanish identically on Z^{ss}_β, we have*

$$[\mathcal{M}_e]^{\mathrm{vir}}=[\mathcal{M}_e]\in A_*(\mathcal{M}_e)\ ;$$

otherwise, if s vanishes identically on Z^{ss}_β, then

$$[\mathcal{M}_e]^{\mathrm{vir}}=c_1(L)\cap[\mathcal{M}_e]\ .$$

In both situations, using the notations in Lemma 3.10, one has

$$[\mathcal{M}_e]^{\mathrm{vir}}=i^*_{\mathcal{M}_e}\big(s^!([Z^{ss}_\beta/G])\big)\ .$$

Here $i_{\mathcal{M}_e}:\mathcal{M}_e\to[Y^{ss}_\beta/G]$ is the natural étale morphism by forgetting gerbe structure;

(2) *when $\beta(L)\notin\mathbb{Z}$, we have*

$$[\mathcal{M}_e]^{\mathrm{vir}}=[\mathcal{M}_e]\in A_*(\mathcal{M}_e)\ .$$

Here $[\mathcal{M}_e]$ is the fundamental class of $\mathcal{M}_e$.

First every fiber curve C_e in $\mathcal{C}_e$ is isomorphic to $\mathbb{P}_{ar,1}$, which is rational. Then the infinitesimal deformations/obstructions of C_e and the line bundles $L_j:=\mathcal{L}_j|_{C_e}$, $N:=\mathcal{N}|_{C_e}$ are zero. Hence their contribution to the perfect obstruction theory solely comes from infinitesimal automorphisms. The infinitesimal automorphisms of C_e come from the space of vector field on C_e that vanishes on special points. Thus the $\mathbb{C}^*-$fixed part of the infinitesimal automorphisms of C_e comes from the $1-$dimensional subspace of vector fields on C_e which vanish on the two ramification points, which, together with the infinitesimal automorphisms of line bundle N, will be canceled with the fixed part of infinitesimal deformation of sections $(z_1,z_2):=(\zeta_1,\zeta_2)|_{C_e}$. The movable part of infinitesimal automorphisms of C_e is nonzero only if one of ramification points on C_e are not special points. by Remark 4.3, the ramification q_∞ must be a special point since it has nontrivial stacky structure when r is sufficiently large, and the ramification point q_0 is an ordinary point, then the movable part of infinitesimal automorphisms of C_e contributes

$$\frac{\delta(e)}{\lambda-D_{\theta'}}$$

to the virtual normal bundle.

Now let's turn to the localization contribution from sections. As for the deformations of z_2, we continue to use the tautological section (x, y) in (4.3.3). Sections of N is spanned by monomials $(x^m y^n)|_{C_e}$ with $arm + n = a\delta(e)$ and $m, n \in \mathbb{Z}_{\geqslant 0}$. Note $x^m y^n$ may not be a global section of $\mathcal{N}$ but always a global section of the line bundle $R^{\otimes am} \otimes \mathcal{N} \otimes \mathbb{C}_{\frac{m}{\delta(e)}}\lambda$. Then $R^\bullet \pi_* \mathcal{N}$ will decompose as a direct sum of line bundles, each corresponds to the monomial $x^m y^n$, whose first chern class is

$$c_1(\mathcal{R}^{\otimes -am} \bigotimes \mathbb{C}_{\frac{-m}{\delta(e)}}\lambda) = \frac{m}{\delta(e)}(D_{\theta'} - \lambda) \ .$$

So the total contribution is equal to

$$\prod_{m=0}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left(\frac{m}{\delta(e)}(D_{\theta'} - \lambda) \right) .$$

The term corresponding to $m = 0$ in the above product is the $\mathbb{C}^*-$invariant part of $R^\bullet \pi_* \mathcal{N}$, it will contribute to the virtual cycle of $\mathcal{M}_e$. The rest contributes to the virtual normal bundle as

$$\prod_{m=1}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left(\frac{m}{\delta(e)}(D_{\theta'} - \lambda) \right) .$$

Note when r is sufficiently large, the above product becomes 1.

For the deformation of z_1, arguing in the same way as z_2, the Euler class of $R^\bullet \pi_*(\mathcal{L}_{-\theta'} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda)$ is equal to

$$\prod_{m=0}^{\delta(e)-\beta(L_{\theta'})} \left(\frac{m}{\delta(e)}(-D_{\theta'} + \lambda) \right) .$$

The factor for $m = 0$ appearing in the above product is the $\mathbb{C}^*-$fixed part of $R^\bullet \pi_*(\mathcal{L}_{-\theta'} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda)$, it will contribute to the virtual cycle of $\mathcal{M}_e$. The rest contributes to the virtual normal bundle as

$$\prod_{m=1}^{\delta(e)-\beta(L_{\theta'})} \left(\frac{m}{\delta(e)}(-D_{\theta'} + \lambda) \right) .$$

Finally, let's turn to the localization contribution from the section $\vec{x}$. Before that, using the same argument, one can prove the following lemma:

Lemma 4.6. *When $n \in \mathbb{Z}_{\geqslant 0}$, we have*

$$e^{\mathbb{C}^*}\left(R\pi_*(\mathcal{O}_{\mathcal{C}_e}(n))\right) = \prod_{m=0}^{\lfloor \frac{n}{ar} \rfloor} \left(\frac{m}{\delta(e)}(D_{\theta'} - \lambda) + \frac{n}{ar\delta(e)}\lambda \right) .$$

When $n \in \mathbb{Z}_{<0}$, we have

$$e^{\mathbb{C}^*}\left(R^\bullet \pi_*(\mathcal{O}_{\mathcal{C}_e}(n))\right) = \prod_{\frac{n}{ar} < m < 0} \frac{1}{\frac{m}{\delta(e)}(D_{\theta'} - \lambda) + \frac{n}{ar\delta(e)}\lambda} .$$

Using the above lemma, we have the following description of $e^{\mathbb{C}^*}(R^\bullet \pi_* \mathcal{L}_{\rho_i})$ for $1 \leqslant i \leqslant n$. Then for each ρ_i, we have:

(1) If $\beta(L_{\rho_i}) \in \mathbb{Q}_{\geqslant 0}$, one has

$$\begin{aligned} e^{\mathbb{C}^*}\big(R^\bullet\pi_*(\mathcal{L}_{\rho_i})\big) &= e^{\mathbb{C}^*}\big(R^\bullet\pi_*(\pi^*(L_{\rho_i}) \otimes \mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i})) \otimes \pi^*(\mathcal{R}^{\otimes a\beta(L_{\rho_i})}))\big) \\ &= e^{\mathbb{C}^*}\big(L_{\rho_i} \otimes \mathcal{R}^{\otimes a\beta(L_{\rho_i})} \otimes R^0\pi_*(\mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i})))\big) \\ &= \prod_{m=0}^{\lfloor \beta(L_{\rho_i})\rfloor} \left(D_{\rho_i} + \frac{\beta(L_{\rho_i})(-D_{\theta'})}{\delta(e)} + \frac{m}{\delta(e)}(D_{\theta'} - \lambda) + \frac{\beta(L_{\rho_i})}{\delta(e)}\lambda\right) \\ &= \prod_{m=0}^{\lfloor \beta(L_{\rho_i})\rfloor} \left(D_{\rho_i} + \frac{\beta(L_{\rho_i}) - m}{\delta(e)}(\lambda - D_{\theta'})\right). \end{aligned}$$

Hence we have

$$e^{\mathbb{C}^*}((R^\bullet\pi_*\mathcal{L}_{\rho_i})^{\mathrm{mov}}) = \prod_{0 \leqslant m < \beta(L_{\rho_i})} \big(D_{\rho_i} + \frac{\beta(L_{\rho_i}) - m}{\delta(e)}(\lambda - D_{\theta'})\big)\ .$$

Note the invariant part of $R^\bullet\pi_*\mathcal{L}_{\rho_i}$ is nonzero only when $\beta(L_{\rho_i}) \in \mathbb{Z}_{\geqslant 0}$.

(2) If $\beta(L_{\rho_i}) \in \mathbb{Q}_{<0}$, one has

$$\begin{aligned} e^{\mathbb{C}^*}\big(R^\bullet\pi_*\mathcal{L}_{\rho_i}\big) &= e^{\mathbb{C}^*}\big(R^\bullet\pi_*\big(\pi^*L_{\rho_i} \otimes \mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i})) \otimes \pi^*\mathcal{R}^{\otimes a\beta(L_{\rho_i})}\big)\big) \\ &= \frac{1}{e^{\mathbb{C}^*}\big(L_{\rho_i} \otimes \mathcal{R}^{\otimes a\beta(L_{\rho_i})} \otimes R^1\pi_*(\mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i})))\big)} \\ &= \prod_{\beta(L_{\rho_i}) < m < 0} \frac{1}{D_{\rho_i} + \frac{\beta(L_{\rho_i})(-D_{\theta'})}{\delta(e)} + \frac{m}{\delta(e)}(D_{\theta'} - \lambda) + \frac{\beta(L_{\rho_i})}{\delta(e)}\lambda} \\ &= \prod_{\beta(L_{\rho_i}) < m < 0} \frac{1}{D_{\rho_i} + \frac{\beta(L_{\rho_i}) - m}{\delta(e)}(\lambda - D_{\theta'})}\ , \end{aligned}$$

which implies that

$$\begin{aligned} e^{\mathbb{C}^*}\big((R^\bullet\pi_*\mathcal{L}_{\rho_i})^{\mathrm{mov}}\big) &= e^{\mathbb{C}^*}\big(R^\bullet\pi_*\mathcal{L}_{\rho_i}\big) \\ &= \prod_{\beta(L_{\rho_i}) < m < 0} \frac{1}{D_{\rho_i} + \frac{\beta(L_{\rho_i}) - m}{\delta(e)}(\lambda - D_{\theta'})}\ . \end{aligned}$$

Recall that the hypersurface Y is cut off by a section of the line bundle L on X associated to the character τ. There is also an obstruction corresponding to the infinitesimal deformations of $\vec{x}$ being moved away from $[AY^{ss}(\theta)/G] \subset [W^{ss}(\theta)/G]$, which contributes the virtual localization as

$$\begin{aligned} e^{\mathbb{C}^*}\big(-R^\bullet\pi_*\mathcal{L}_\tau\big) &= e^{\mathbb{C}^*}\big(-R^0\pi_*\big(\pi^*L_\tau \otimes \mathcal{O}_{\mathcal{C}_e}(ar\beta(L_\tau)) \otimes \pi^*\mathcal{R}^{\otimes a\beta(L_\tau)}\big)\big) \\ &= \frac{1}{e^{\mathbb{C}^*}\big(L_\tau \otimes \mathcal{R}^{\otimes a\beta(L_\tau)} \otimes R^0\pi_*\mathcal{O}_{\mathcal{C}_e}(ar\beta(L_\tau))\big)} \\ &= \frac{1}{\prod_{m=0}^{\lfloor \beta(L_\tau)\rfloor}\big(c_1(L_\tau) + \frac{\beta(L_\tau)(-D_{\theta'})}{\delta(e)} + \frac{m}{\delta(e)}(D_{\theta'} - \lambda) + \frac{\beta(L_\tau)}{\delta(e)}\lambda\big)} \\ &= \frac{1}{\prod_{m=0}^{\lfloor \beta(L_\tau)\rfloor}\big(c_1(L_\tau) + \frac{\beta(L_\tau) - m}{\delta(e)}(\lambda - D_{\theta'})\big)}\ . \end{aligned}$$

When $\beta(L_\tau) \notin \mathbb{N}$ (same as $\tau(g) \neq 1$), there is no $\mathbb{C}^*-$invariant parts of $-R^\bullet\pi_*\mathcal{L}_\tau$. When $\beta(L_\tau) \in \mathbb{N}$ (same as $\tau(g) = 1$), there is one dimensional $\mathbb{C}^*-$fixed piece of $-R^\bullet\pi_*\mathcal{L}_\tau$, which contributes to the virtual cycle of $\mathcal{M}_e$. Now combing the discussion of invariant part from deformation of curves, line bundles and sections, Lemma 4.5 is immediate.

We have the expression of virtual normal bundle from the movable part of curves, line bundles and sections as follows:

$$e^{\mathbb{C}^*}(N^{\mathrm{vir}}) = \frac{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leqslant i<\beta(L_\rho)}(D_\rho + (\beta(L_\rho)-i)\frac{\lambda - D_{\theta'}}{\delta(e)})}{\prod_{\rho:\beta(L_\rho)<0}\prod_{\lfloor\beta(L_\rho)+1\rfloor\leqslant i<0}(D_\rho + (\beta(L_\rho)-i)\frac{\lambda - D_{\theta'}}{\delta(e)})}$$
$$\cdot \frac{1}{\prod_{0\leqslant i<\beta(L)}(c_1(L) + (\beta(L)-i)\frac{\lambda - D_{\theta'}}{\delta(e)})} \cdot \prod_{m=1}^{\delta(e)-\beta(L_{\theta'})}\Big(\frac{m}{\delta(e)}(-D_{\theta'}+\lambda)\Big).$$

Then the total contribution from edge moduli with a basepoint yields:

Lemma 4.7.

$$Cont_{\mathcal{M}_e} = (\bar{i}_{\mathcal{M}_e})^*\left(\iota_*\frac{(zI_\beta(q,z))|_{z=\frac{\lambda-D_{\theta'}}{\delta(e)}}}{\prod_{m=1}^{\delta(e)-\beta(L_{\theta'})}\big(\frac{m}{\delta(e)}(-D_{\theta'}+\lambda)\big)}\right),$$

where $\bar{i}_{\mathcal{M}_e} : \mathcal{M}_e \to [Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]$ is the natural structure map by forgetting gerbe structure and taking rigidification, ι is the involution of $\bar{I}_\mu Y$ obtained from taking the inverse of the band, and I_β is the coefficient of q^β in Theorem 3.12. Note that the definition of $\bar{i}_{\mathcal{M}_e}$ here is different from the definition of $i_{\mathcal{M}_e}$ in Lemma 4.5.

4.3.4. *Node contributions.* The deformations in $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}, (\beta, \frac{\delta}{r}))$ smoothing a node contribute to the Euler class of the virtual normal bundle as the first Chern class of the tensor product of the two cotangent line bundles at the branches of the node. For nodes at which a component C_e meets a component C_v over the vertex 0, this contribution is

$$\frac{\lambda - D_{\theta'}}{a\delta(e)} - \frac{\bar{\psi}_v}{a}\ ;$$

for nodes at which a component C_e meets a component C_v over the vertex ∞, this contribution is

$$\frac{-\lambda + D_{\theta'}}{ar\delta(e)} - \frac{\bar{\psi}_v}{ar}\ ;$$

for nodes at which two edge component C_e meets with a vertex v over 0, the node-smoothing contribution is

$$\frac{\lambda - D_{\theta'}}{a\delta(e)} + \frac{\lambda - D_{\theta'}}{a\delta(e')}\ .$$

The nodes at which two edge component C_e meets with a vertex v over ∞ will not occur using a similar argument in [JPPZ17, Lemma 6] when r is sufficiently large.

As for the node contributions from the normalization exact sequence, each node q (specified by a vertex v) contributes the Euler class of

$$(R^0\pi_*(\mathcal{L}^\vee_{\theta'} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda)|_q)^{\mathrm{mov}} \oplus (R^0\pi_*\mathcal{N}|_q)^{\mathrm{mov}} \tag{4.14}$$

to the virtual normal bundle.

In the case where $j(v)=0$, $z_2|_q=1$ gives a trivialization of $\mathcal{N}$ at q. Thus, the second factor in (4.14) is trivial, while the Euler class of the first factor equals

$$\frac{1}{\lambda - D_{\theta'}}\ .$$

In the case where $j(v)=\infty$, $z_1|_q=1$ gives a trivialization of the fiber $(\mathcal{L}_{\theta'}^\vee \otimes \mathcal{N}^{\otimes r}\otimes \mathbb{C}_\lambda)|_q$. Hence we have $\mathcal{N}|_q \cong \mathcal{L}_{\theta'}^{\frac{1}{r}}|_q \otimes \mathbb{C}_{-\frac{\lambda}{r}}$, this implies that it $R^0\pi_*(\mathcal{N}|_q)=0$ because of the nontrivial stacky structure when r is sufficiently large. Hence there is no localization contribution from the normalization sequence at the node over ∞.

4.4. **Total localization contributions.** For each decorated graph Γ, denote the moduli F_Γ to be the fiber product

$$\prod_{v:j(v)=0}\mathcal{M}_v \times_{\bar{I}_\mu Y} \prod_{e\in E}\mathcal{M}_e \times_{\bar{I}_\mu \sqrt[r]{L_{\theta'}/Y}} \prod_{v:j(v)=\infty}\mathcal{M}_v$$

of the following diagram:

$$\begin{tikzcd}
F_\Gamma \arrow[r] \arrow[d] & \prod_{v:j(v)=0}\mathcal{M}_v \times \prod_{e\in E}\mathcal{M}_e \times \prod_{v:j(v)=\infty}\mathcal{M}_v \arrow[d, "ev_{h_0},ev_{p_0},ev_{p_\infty},ev_{h'_\infty}"] \\
\prod_E(\bar{I}_\mu Y \times \bar{I}_\mu \sqrt[r]{L_{\theta'}/Y}) \arrow[r, "(\Delta\times\Delta^{\frac{1}{r}})^{|E|}"] & \prod_E(\bar{I}_\mu Y)^2 \times (\bar{I}_\mu \sqrt[r]{L_{\theta'}/Y})^2\ ,
\end{tikzcd}$$

where $\Delta=(id,\iota)$(resp. $\Delta^{\frac{1}{r}}=(id,\iota)$) is the diagonal map of $\bar{I}_\mu Y$ (resp. $\bar{I}_\mu \sqrt[r]{L_{\theta'}/Y}$), where ι is the involution on $\bar{I}_\mu Y$ (resp. $\bar{I}_\mu \sqrt[r]{L_{\theta'}/Y}$) by taking the inverse of the band of gerbe structure. Here when v is a stable vertex, the vertex moduli $\mathcal{M}_v$ is described in 4.3.1; when v is an unstable vertex over 0, we treat $\mathcal{M}_v := \bar{I}_{m(h)}Y$ with the identical virtual cycle, where h is the half-edge incident to v; when v is an unstable vertex over ∞, We treat $\mathcal{M}_v := \bar{I}_{m(h)}\sqrt[r]{L_{\theta'}/Y}$ with the identical virtual cycle, where h is the half-edge incident to v.

We define $[F_\Gamma]^{\mathrm{vir}}$ to be:

$$\prod_{v:j(v)=0}[\mathcal{M}_v]^{\mathrm{vir}} \times_{\bar{I}_\mu Y} \prod_{e\in E}[\mathcal{M}_e]^{\mathrm{vir}} \times_{\bar{I}_\mu \sqrt[r]{L_{\theta'}/Y}} \prod_{v:j(v)=\infty}[\mathcal{M}_v]^{\mathrm{vir}}\ .$$

where the fiber product over $\bar{I}_\mu \sqrt[s]{L_{-\theta'}/Y}$ and $\bar{I}_\mu \sqrt[r]{L_{\theta'}/Y}$ imposes that the evaluation maps at the two branches of each node (Here we adopt the convention that a node can link a unstable vertex and an edge.) agree. Then the contribution of decorated graph Γ to the virtual localization is:

$$Cont_\Gamma = \frac{\prod_{e\in E}a_e}{\mathrm{Aut}(\Gamma)}(\iota_\Gamma)_*\left(\frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})}\right)\ . \tag{4.15}$$

Here $\iota_F: F_\Gamma \to Q_{0,\vec{m}}^{\widetilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(\beta,\frac{\delta}{r}))$ is a finite etale map of degree $\frac{\mathrm{Aut}(\Gamma)}{\prod_{e\in E}a_e}$ into the corresponding $\mathbb{C}^*$-fixed loci in twisted graph space. The virtual normal bundle $e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})$ is the product of virtual normal bundles from vertex contributions, edge contributions and node contributions.

5. Master space II

5.1. **Construction of master space II.** Fix two different primes $r, s \in \mathbb{N}$, let θ' be as in previous section, let $\mathbb{P}Y_{r,s}$ be the root stack of the $\mathbb{P}^1$ bundle $\mathbb{P}_Y(\mathcal{O}(-D_{\theta'}) \oplus \mathcal{O})$ over Y by taking the s-th root of the zero section ($z_1 = 0$) and r-th root of the infinity section ($z_2 = 0$). Then the zero section $\mathcal{D}_0 \subset \mathbb{P}Y_{r,s}$ is isomorphic to $\sqrt[s]{L_{-\theta'}/Y}$, and the infinity section $\mathcal{D}_\infty \subset \mathbb{P}Y_{r,s}$ is isomorphic to $\sqrt[r]{L_{\theta'}/Y}$.

We give a more concrete presentation of $\mathbb{P}Y_{r,s}$ as a quotient stack:

$$\mathbb{P}Y_{r,s} = [(\mathbb{C}^* \times AY^{ss}(\theta) \times U)/(G \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t)] \ ,$$

where the (right) $G \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t$-action on $\mathbb{C}^* \times AY^{ss}(\theta) \times U$ is given by:

$$(u, \vec{x}, z_1, z_2) \cdot (g, \alpha, t) = (\alpha^{-s}\theta'(g)^{-1} t^r u, \vec{x}g, \alpha z_1, t z_2) \ ,$$

for $(g, \alpha, t) \in G \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t$, and $(u, \vec{x}, z_1, z_2) \in \mathbb{C}^* \times AY^{ss}(\theta) \times U$. Here $U = \mathbb{C}^2 \backslash \{0\}$. This quotient stack presentation of $\mathbb{P}Y_{r,s}$ comes from the root stack construction in [AGV08, Appendix B] after some simplification.

The inertia stack $I_\mu \mathbb{P}Y_{r,s}$ of $\mathbb{P}Y_{r,s}$ admits a decomposition

$$I_\mu \mathbb{P}Y \sqcup \bigsqcup_{i=1}^{s-1} \sqrt[s]{L_{-\theta'}/Y} \sqcup \bigsqcup_{j=1}^{r-1} \sqrt[r]{L_{\theta'}/Y} \ .$$

Let $(\vec{x}, (g, \alpha, t))$ be a point of the inertia stack $I_\mu \mathbb{P}Y_{r,s}$, if the point $(\vec{x}, (g, \alpha, t))$ appears in the first factor of the decomposition above, then the automorphism (g, α, t) lies in $G \times \{1\} \times \{1\}$; if the point $(\vec{x}, (g, \alpha, t))$ occurs in the second factor of the decomposition above, then the automorphism (g, α, t) lies in $G \times \{\mu_s^i : 1 \leqslant i \leqslant s-1\} \times \{1\} \subset G \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t$, and the point $\vec{x}$ is in the zero section $\mathcal{D}_0$ defined by $z_1 = 0$; finally if the point $(\vec{x}, (g, \alpha, t))$ shows in the third factor of the decomposition above, then the automorphism (g, α, t) lies in $G \times \{1\} \times \{\mu_r^j : 1 \leqslant i \leqslant r-1\} \subset G \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t$, and $\vec{x}$ is in the infinity section $\mathcal{D}_\infty$ defined by $z_2 = 0$. Here $\mu_r = \exp(\frac{2\pi\sqrt{-1}}{r}) \in \mathbb{C}^*$ and $\mu_s = \exp(\frac{2\pi\sqrt{-1}}{s}) \in \mathbb{C}^*$.

Fix $(g, \alpha, t) \in G \times \boldsymbol{\mu}_s \times \boldsymbol{\mu}_r$, we will use the notation $\bar{I}_{(g,\alpha,t)} \mathbb{P}Y_{r,s}$ to mean the rigidified inertia stack component of $\bar{I}_\mu \mathbb{P}Y_{r,s}$ which has automorphism (g, α, t). Note if α and t are not equal to 1 simultaneously, then the corresponding rigidified inertia stack component is empty.

Let $\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r}))$ be the moduli stack of m-pointed twisted stable maps to $\mathbb{P}Y_{r,s}$ of degree $(d, \frac{\delta}{r})$. More concretely, More concretely,

$$\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r})) = \{(C; q_1, \ldots, q_m; L_1, \cdots, L_k, N_1, N_2; u, \vec{x} := (x_1, \ldots, x_m), z_1, z_2)\},$$

where $(C; q_1, \ldots, q_m)$ is a m-pointed prestable balanced twisted curve of genus 0 with nontrivial isotropy only at special points, $(L_j : 1 \leqslant j \leqslant k)$ and N_1, N_2 are orbifold line bundles on C with

$$deg([\vec{x}]) = d \in Hom(Pic(\mathfrak{Y}), \mathbb{Q}), \quad \deg(N_2) = \frac{\delta}{r},$$

and

$$(u, (\vec{x}, \vec{z})) := (u, x_1, \ldots, x_n, z_1, z_2) \in \Gamma\left(\left((N_1^\vee)^{\otimes s} \otimes L_{-\theta'} \otimes N_2^{\otimes r}\right) \oplus \bigoplus_{i=1}^n L_{\rho_i} \oplus N_1 \oplus N_2\right).$$

Here, for $1 \leqslant i \leqslant n$, the line bundle L_{ρ_i} is equal to

$$\otimes_{j=1}^{k} L_j^{m_{ij}} ,$$

where $(m_{ij})_{1\leqslant i\leqslant n,\, 1\leqslant j\leqslant k}$ is given by the relation $\rho_i = \sum_{j=1}^{k} m_{ij}\pi_j$. The same construction applies to the line bundle $L_{-\theta'}$ on C. Note here δ is an integer when $\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))$ is nonempty as $N_2^{\otimes r}$ is the pullback of some line bundle on the coarse moduli curve $\underline{C}$.

We require this data to satisfy the following conditions:

- *Representability*: For every $q \in C$ with isotropy group G_q, the homomorphism $\mathbb{B}G_q \to \mathbb{B}(G\times\mathbb{C}^*_\alpha\times\mathbb{C}^*_t)$ given by the restriction of line bundles $(L_j : 1 \leqslant j \leqslant k)$ and N_1, N_2 on q is representable.
- *Nondegeneracy*: The sections z_1 and z_2 never simultaneously vanish, and we have

$$\mathrm{ord}_q(\vec{x}) = 0. \tag{5.1}$$

  for all $q \in C$. Furthermore, the section u never vanish, so we have $(N_1^\vee)^{\otimes s}\otimes L_{-\theta'}\otimes N_2^{\otimes r} \cong \mathcal{O}_C$.
- *Stability*: the map $[u,\vec{x},\vec{z}] : (C, q_1,\cdots,q_m) \to \mathbb{P}Y_{r,s}$ satisfies the usual stability condition defined by a twisted stable map;
- *Vanishing*: The image of $[\vec{x}] : C \to \mathfrak{X}$ lies in $\mathfrak{Y}$.

Let $\vec{m} = (v_1,\cdots,v_m) \in (G\times\boldsymbol{\mu}_s\times\boldsymbol{\mu}_r)^m$, we will denote $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))$ to be:

$$\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r})) \cap ev_1^{-1}(\bar{I}_{v_1}\mathbb{P}Y_{r,s}) \cap\cdots\cap ev_m^{-1}(\bar{I}_{v_m}\mathbb{P}Y_{r,s}) ,$$

where

$$ev_i : \mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r})) \to \bar{I}_\mu\mathbb{P}Y_{r,s},$$

are natural evaluation maps as before, by evaluating the sections $(u,\vec{x},\vec{z})$ at q_i. Moreover, since $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))$ is a moduli space of stable quasimaps to a lci GIT quotient, the results of [CFKM14] imply that it is a proper Deligne–Mumford stack equipped with a natural perfect obstruction theory relative to the stack $\mathcal{D}_{0,m,d,\frac{\delta}{r}}$ of twisted curves equipped with line bundles $(L_j : 1 \leqslant j \leqslant k)$ and N_1, N_2. This obstruction theory is of the form

$$R^\bullet\pi_*(u^*\mathbb{R}\mathbb{T}_\rho). \tag{5.2}$$

Here, we denote the universal family over $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))$ by

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\rho"'] & \mathcal{C} \arrow[l, bend right, "u"'] \arrow[d, "\pi"] \\ & \mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r})), \end{tikzcd}$$

where $\mathcal{L}_{\rho_i}$ and $\mathcal{N}_1,\mathcal{N}_2$ are the universal line bundles and

$$\mathcal{V} \subset \big((\mathcal{N}_1^\vee)^{\otimes s}\otimes\mathcal{L}_{-\theta'}\otimes\mathcal{N}_2^{\otimes r}\big) \oplus \bigoplus_{i=1}^{n} L_{\rho_i} \oplus \mathcal{N}_1 \oplus \mathcal{N}_2$$

is the subsheaf of sections taking values in the affine cone of Y. Somewhat more explicitly, (4.4) equals

$$R^\bullet\pi_*((\mathcal{N}_1^\vee)^{\otimes s}\otimes\mathcal{L}_{-\theta'}\otimes\mathcal{N}_2^{\otimes r})\oplus\mathbb{E}\oplus R^\bullet\pi_*(\mathcal{N}_1\oplus\mathcal{N}_2), \tag{5.3}$$

in which the sub-obstruction-theory $\mathbb{E}$ comes from the deformations and obstructions of the sections $\vec{x}$.

5.2. **$\mathbb{C}^*$-action and fixed loci.** Define a (left) $\mathbb{C}^*$-action on $\mathbb{C}^*\times AY^{ss}(\theta)\times U$ given by

$$t\cdot(u,\vec{x},(z_1,z_2))=(tu,\vec{x},(z_1,z_2))\ .$$

This action descends to be a (left) $\mathbb{C}^*$-action on $\mathbb{P}Y_{r,s}$, which induces a $\mathbb{C}^*$-action on $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))$. The reason why we define this action is that this definition lifts the $\mathbb{C}^*$-action on $\mathbb{P}Y$ defined in §4.1 along the canonical structure map $\pi_{r,s}:\mathbb{P}Y_{r,s}\to\mathbb{P}Y$. We will let λ to be equivariant parameter corresponding to the action of weight 1.

First we state a similar criteria for maps to $\mathbb{P}Y_{r,s}$ to be $\mathbb{C}^*$-equivariant as in Remark 4.1.

Remark 5.1. Given a stack S over $Spec(\mathbb{C})$ equipped with a (left) $\mathbb{C}^*$-action, then a $\mathbb{C}^*$-equivariant morphism from S to $\mathbb{P}Y_{r,s}$ is equivalent to that the following: there exists $k+2$ $\mathbb{C}^*$-equivariant line bundles on S

$$L_1,\cdots,L_k,N_1,N_2$$

together with sections invariant under the $\mathbb{C}^*$-action;

$$\begin{aligned}(u,\vec{x},\vec{z}):=&(u,x_1,\ldots,x_n,z_1,z_2)\\ \in&\,\Gamma\left(\left((N_1^\vee)^{\otimes s}\otimes L_{-\theta'}\otimes N_2^{\otimes r}\otimes\mathbb{C}_\lambda\right)\oplus\bigoplus_{i=1}^n L_{\rho_i}\oplus N_1\oplus N_2\right)^{\mathbb{C}^*}\ .\end{aligned}\tag{5.4}$$

Here the line bundles L_{ρ_i} and $L_{-\theta'}$ are defined similarly as before. The sections should satisfy the vanishing condition imposed by the affine cone of Y in the definition of stable maps to $\mathbb{P}_{r,s}$.

Fix a nonzero degree $\beta\in\mathrm{Eff}(W,G,\theta)$, and two tuples of nonnegative integers

$$(\delta_1,\cdots,\delta_m)\in\mathbb{N}^m$$

and

$$(\delta_1',\cdots,\delta_m')\in\mathbb{N}^m\ .$$

Consider the tuple of multiplicities $\vec{m}=(v_1,\cdots,v_m)\in(G\times\boldsymbol{\mu}_r)^m$, where $v_i=(g_i,\mu_s^{\delta_i'},\mu_r^{\delta_i})$, we will denote $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$ to be

$$\coprod_{\substack{d\in\mathrm{Eff}(AY,G,\theta)\\(i_{\mathfrak{Y}})_*(d)=\beta}}\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))\ ,$$

where $i_{\mathfrak{Y}}:\mathfrak{Y}\to\mathfrak{X}$ is the inclusion morphism. Thus $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$ inherits a $\mathbb{C}^*$-action as above.

We will follow the presentation of [CJR17b, CJR17a] to describe the virtual localization for $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$ similar to $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(\beta,\frac{\delta}{r}))$, but the edge contribution is easier to analyze as there is no basepoint occurring for twisted stable maps.

The components of fixed loci of $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$ under the $\mathbb{C}^*$-action can be indexed by decorated graph Γ, which we explain as follows. We denote such a graph by Γ which consists of vertices, edges, and m legs, and decorate it as follows:

- Each vertex $v \in V$ has an index $j(v) \in \{0,\infty\}$, and a degree $\beta(v) \in \mathrm{Eff}(W,G,\theta)$.
- Each edge e has a degree $\delta(e) \in \mathbb{N}$.
- Each half-edge h (including the legs) has an element $m(h) \in G \times \boldsymbol{\mu}_s \times \boldsymbol{\mu}_r$.
- The legs are labeled with the numbers $\{1,\dots,m\}$.

By the "valence" of a vertex v, denoted $\mathrm{val}(v)$, we mean the total number of incident half-edges, including legs.

The fixed locus in $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$ indexed by the decorated graph Γ parameterizes quasimaps of the following type:

- Each edge e corresponds to a genus-zero component C_e on which $\deg(N_2) = \frac{\delta(e)}{r}$ for some integer $\delta(e) \in \mathbb{Z}_{>0}$, where there are two distinguished points q_0 and q_∞ on C_e satisfying that $z_2|_{q_\infty} = 0$ and $z_1|_{q_0} = 0$, respectively. We call them the "ramification points". Note that we have $deg(L_j|_{C_e}) = 0$ for all $1 \leqslant j \leqslant k$.
- Each vertex v for which $j(v) = 0$ (with unstable exceptional cases noted below) corresponds to a maximal sub-curve C_v of C over which $z_1 \equiv 0$, then the restriction of $(C; q_1,\dots,q_m; L_1,\cdots,L_k;\vec{x})$ to C_v defines a twisted stable map in
$$\mathcal{K}_{0,val(v)}(\sqrt[s]{L_{-\theta'}/Y},\beta(v)) := \bigsqcup_{\substack{d\in \mathrm{Eff}(AY,G,\theta)\\ (i_{\mathfrak{Y}})_*(d)=\beta(v)}} \mathcal{K}_{0,val(v)}(\sqrt[s]{L_{-\theta'}/Y},d)\ .$$
Each vertex v for which $j(v) = \infty$ (again with unstable exceptions) corresponds to a maximal sub-curve for which $z_2 \equiv 0$, then the restriction of $(C; q_1,\dots,q_m; L_1,\cdots,L_k;\vec{x})$ to C_v defines a twisted stable map in
$$\mathcal{K}_{0,val(v)}(\sqrt[r]{L_{\theta'}/Y},\beta(v)) := \bigsqcup_{\substack{d\in \mathrm{Eff}(AY,G,\theta)\\ (i_{\mathfrak{Y}})_*(d)=\beta(v)}} \mathcal{K}_{0,val(v)}(\sqrt[r]{L_{\theta'}/Y},d)\ .$$
The label $\beta(v)$ denotes the degree coming from $[x]|_{C_v} : C_v \to \mathfrak{X}$. Note here we count the degree $\beta(v)$ in $\mathrm{Eff}(W,G,\theta)$, but not in $\mathrm{Eff}(AY,G,\theta)$.
- A vertex v is *unstable* if stable twisted maps of the type described above do not exist (where, as always, we interpret legs as marked points and half-edges as half-nodes). In this case, v corresponds to a single point of the component C_e for each adjacent edge e, which may be a node at which C_e meets $C_{e'}$, a marked point of C_e, or an unmarked point.
- The index $m(l)$ on a leg l indicates the rigidified inertia stack component $\bar{I}_{m(l)}\mathbb{P}Y_{r,s}$ of $\mathbb{P}Y_{r,s}$ on which the marked point corresponding to the leg l is evaluated, this is determined by the multiplicity of $L_1,\cdots,L_k,N_1,N_2$ at the corresponding marked points.
- A half-edge h incident to a vertex v corresponds to a node at which components C_e and C_v meet, and $m(h)$ indicates the rigidified inertia component $\bar{I}_{m(h)}\mathbb{P}Y_{r,s}$ of $\mathbb{P}Y_{r,s}$ on which the node on C_v is evaluated, this is determined by the multiplicity of $L_1,\cdots,L_k,N_1,N_2$ at the corresponding marked points. If v is unstable and hence h corresponds to a single point on a component C_e, then

$m(h)$ is the *inverse* in $G\times\boldsymbol{\mu_s}\times\boldsymbol{\mu}_r$ of the multiplicity of $L_1,\cdots,L_k,N_1,N_2$ at this point.

In particular, we note that the decorations at each stable vertex v yield a vector

$$\vec{m}(v)\in(G\times\boldsymbol{\mu}_s\times\boldsymbol{\mu}_r)^{\mathrm{val}(v)}$$

recording the multiplicities of $L_1,\cdots,L_k,N_1,N_2$ at every special point of C_v.

Remark 5.2. For each edge e, the restriction of $\vec{x}$ to C_e defines a constant map to Y. So the restriction of $(u,\vec{x},\vec{z})$ to C_e defines a representable map

$$f:C_e\to\mathbb{B}G_y\times\mathbb{P}_{r,s}$$

where $y\in Y$ comes from $\vec{x}$ and G_y is the isotropy group of $y\in Y$. Then we have $m(q_0)=(g^{-1},\mu_s^{\delta(e)},1)$ and $m(q_\infty)=(g,1,\mu_r^{\delta(e)})$ for some $g\in G_y$. Denote a to be the order of element $g\in G$. Note when r and s are sufficiently large primes comparing to $\delta(e)$, we must have $C_e\cong\mathbb{P}^1_{ar,as}$ and q_0 and q_∞ are special points as they are nontrivial stacky points. Here $\mathbb{P}^1_{ar,as}$ is the unique Deligne-Mumford stack with coarse moduli $\mathbb{P}^1$, isotropy group $\boldsymbol{\mu}_{as}$ at $0\in\mathbb{P}^1$, isotropy group $\boldsymbol{\mu}_{ar}$ at $\infty\in\mathbb{P}^1$, and generic trivial stabilizer.

5.3. **Localization analysis.** Fix $\beta\in\mathrm{Eff}(\mathrm{W},\mathrm{G},\theta),\delta\in\mathbb{Z}_{\geqslant 0}$ and $\vec{m}=(v_1,\cdots,v_m)\in(G\times\boldsymbol{\mu}_s\times\boldsymbol{\mu}_r)^m$, we will consider the space $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$. The reason why we assume that the second degree is $\frac{\delta}{r}$ is that $\mathcal{K}_{0,[m]}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$ admits a natural morphism to $\mathcal{K}_{0,[m]}(\mathbb{P}Y,(\beta,\delta))$(c.f.[AJT15, TT16]). Here $\mathbb{P}Y$ is equal to $\mathbb{P}Y_{r,s}$ for $r=s=1$. In this section, we will always assume that r and s are *sufficiently large primes*.

Following the discussion in [CJR17a], the virtual localization formula of Graber–Pandharipande [GP99] expresses

$$[\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))]^{\mathrm{vir}}$$

in terms of contributions from each fixed-loci graph Γ:

$$[\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))]^{\mathrm{vir}}=\sum_{\Gamma}\frac{1}{\mathbb{A}_\Gamma}\iota_{\Gamma*}\left(\frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})}\right). \tag{5.5}$$

Here, for each graph Γ, $[F_\Gamma]^{\mathrm{vir}}$ is obtained from the $\mathbb{C}^*$-fixed part of the restriction to the fixed loci of the obstruction theory on $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$, and N_Γ^{vir} as the equivariant Euler class of the $\mathbb{C}^*$-moving part of this restriction. Besides, $\mathbb{A}_\Gamma$ is the automorphism factor for the graph Γ, which represents the degree of F_Γ into the corresponding open and closed $\mathbb{C}^*$-fixed substack in $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$.

We will compute the contributions of each graph Γ explicitly in this subsection. To compute the contribution of a graph Γ to (5.5), one should first apply the normalization exact sequence to the relative obstruction theory (5.3), thus decomposing the contribution of Γ to (5.5) into vertex, edge, and node factors. This accounts for all but the automorphisms and deformations within $\mathcal{D}_{0,m,\beta,\frac{\delta}{r}}$. The latter are distributed in the vertex, edge, and node contributions as deformations of the vertex components and their line bundles, deformations of the edge components and their line bundles, and deformations smoothing the nodes, respectively, in what follows. We include the factors from automorphisms of the source curve also in the edge contributions as parts of the gerbe structures of the edge moduli $\mathcal{M}_e$, then an additional factor from gerbe structure of each edge moduli will appear in the automorphism factor $\mathbb{A}_\Gamma$(see (5.11) for the localization contribution of graph Γ).

5.3.1. *Vertex contributions.* The analysis of localization contribution for the stable vertex v is similar to the analysis in §4.3.1.

For the stable vertex v over ∞, the vertex moduli $\mathcal{M}_v$ corresponds to the moduli stack

$$\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_{\theta'}/Y}, \beta(v)) := \coprod_{\substack{d\in \mathrm{Eff}(AY,G,\theta)\\ (i_{\mathfrak{Y}})_*(d)=\beta(v)}} \mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_{\theta'}/Y}, d)\ ,$$

which parameterizes twisted stable maps to the root gerbe $\sqrt[r]{L_{\theta'}/Y}$ over Y.

Let

$$\pi : \mathcal{C}_\infty \to \mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_{\theta'}/Y}, \beta(v))$$

be the universal curve over $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_{\theta'}/Y}, \beta(v))$. Follow the same discussion in §4.3.1, the *inverse of the Euler class* of the virtual normal bundle for the vertex moduli $\mathcal{M}_v$ over ∞ is equal to

$$e^{\mathbb{C}^*}((-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})\otimes \mathbb{C}_{-\frac{\lambda}{r}})\ .$$

When r is a sufficiently large prime, following [JPPZ18], the above Euler class has a representation

$$\sum_{d\geqslant 0} c_d(-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})(\frac{-\lambda}{r})^{|E(v)|-1-d}\ .$$

Here the virtual bundle $-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}}$ has virtual rank $|E(v)|-1$, where $|E(v)|$ is the number of edges incident to the vertex v. The fixed part of the obstruction theory contributes to the virtual cycle

$$[\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_{\theta'}/Y}, \beta(v))]^{\mathrm{vir}}\ .$$

For the stable vertex v over 0, the vertex moduli $\mathcal{M}_v$ corresponds to the moduli space

$$\mathcal{K}_{0,\vec{m}(v)}(\sqrt[s]{L_{-\theta'}/Y}, \beta(v)) := \coprod_{\substack{d\in \mathrm{Eff}(AY,G,\theta)\\ (i_{\mathfrak{Y}})_*(d)=\beta(v)}} \mathcal{K}_{0,\vec{m}(v)}(\sqrt[s]{L_{-\theta'}/Y}, d)\ .$$

Let

$$\pi : \mathcal{C}_0 \to \mathcal{K}_{0,\vec{m}(v)}(\sqrt[s]{L_{-\theta'}/Y}, \beta(v))$$

be the universal curve over $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[s]{L_{-\theta'}/Y}, \beta(v))$, and $f:\mathcal{C}_0\to\sqrt[s]{L_{-\theta'}/Y}$. In this case, the fixed part of the perfect obstruction theory for the vertex moduli over 0 yields the virtual cycle

$$[\mathcal{K}_{0,\vec{m}(v)}(\sqrt[s]{L_{-\theta'}/Y}, \beta(v))]^{\mathrm{vir}}\ .$$

Note $\mathcal{N}_2|_{\mathcal{C}_0}\cong\mathcal{O}_{\mathcal{C}_0}$ as $z_2|_{\mathcal{C}_0}\equiv 1$, the virtual normal bundle comes from the movable part of the infinitesimal deformations of z_1, which is a section of the line bundle $\mathcal{L}_{-\theta'}^{\frac{1}{s}}$ over $\mathcal{C}_0$, which is the pullback of the universal $s-$th root line bundle on $\sqrt[s]{L_{-\theta'}/Y}$ via the universal map f. Then the *inverse of the Euler class* of the virtual normal bundle is equal to

$$e^{\mathbb{C}^*}((-R^\bullet \pi_* \mathcal{L}^{\frac{1}{s}}_{-\theta'}) \otimes \mathbb{C}_{\frac{\lambda}{s}}) \ .$$

We will simplify the above presentation when $\beta(v) \neq 0$. First, we will state a simple vanishing lemma regarding a line bundle of negative degree on a genus zero twisted curve.

Lemma 5.3. *Let L be a line bundle of negative degree on a genus zero twisted curve C. Assume that the degree of the restriction of the line bundle $L|_{C_i}$ to every irreducible component C_i is non-positive. Then we have $H^0(C, L) = 0$.*

Proof. We can prove Lemma by induction on the number of irreducible components of C. When the number is one, a nonzero section of L will give an effective divisor D so that the line bundle associated to the invertible sheaf $\mathcal{O}(D)$ is linearly equivalent to the line bundle L as C is irreducible, but this implies that degree of L is positive, a contradiction, hence finishes the case when the number is one. Assume that Lemma holds whenever the number of the irreducible components is less than n. Now assume that the curve C has n irreducible components. Choose a decomposition of C as a union of irreducible components

$$C_1 \cup \bigcup_{i=2}^{n} C_i$$

so that C_1 is a tail of C (i.e. C_1 has only one node), $\bigcup_{i=2}^{n} C_i$ (write $C_1' = \bigcup_{i=2}^{n} C_i$) is the union of the rest irreducible components and the degree of restricted line bundle $L|_{C_1'}$ is negative. Assume that C_1 and C_1' meet at a node q. Now consider the following normalization sequence

$$0 \to H^0(C, L) \to H^0(C_1, L|_{C_i}) \oplus H^0(C_1', L_{C_1'}) \to H^0(q, L|_q) \ .$$

When $deg(L|_{C_1}) < 0$, it follows by induction and the above sequence. So it remains to prove the case when $deg(L|_{C_1}) = 0$. Observe that the restriction map

$$H^0(C_1, L|_{C_1}) \to H^0(q, L|_q)$$

is injective when C_1 is of genus zero and $deg(L|_{C_1}) = 0$. Then the lemma follows by the above fact combing with the above normalization sequence and inductive assumption on C_1'. This finishes the proof. □

Remark 5.4. For every fiber curve C_0 of the universal curve $\mathcal{C}_0$ over $\mathcal{M}_v$. The restricted line bundle $\mathcal{L}^{\frac{1}{s}}_{-\theta'}|_{C_0}$ to C_0 has non-positive degree restricting to every irreducible components of C_0. Indeed $\mathcal{L}^{\frac{1}{s}}_{-\theta'}$ is the pullback of the s-th root of the line bundle $L_{-\theta'}$ on $\sqrt[s]{L_{-\theta'}/Y}$, where $L_{-\theta'}$ is the pullback of an *anti-ample* line bundle on the coarse moduli of $\sqrt[s]{L_{-\theta'}/Y}$. Now assuming $\beta(v) \neq 0$, we have the degree of the restricted line bundle $\mathcal{L}^{\frac{1}{s}}_{-\theta'}|_{C_0}$ is negative by Lemma 2.5. Using the above lemma, one has

$$R^0 \pi_* \mathcal{L}^{\frac{1}{s}}_{-\theta'} = 0 \ .$$

Then we have

$$-R^\bullet \pi_* \mathcal{L}^{\frac{1}{s}}_{-\theta'} = R^1 \pi_* \mathcal{L}^{\frac{1}{s}}_{-\theta'} \ ,$$

which implies that $R^1\pi_*\mathcal{L}_{-\theta'}^{\frac{1}{s}}$ is a vector bundle. When s is sufficiently large, it has rank $|E(v)|-1$ where $|E(v)|$ is the number of edges incident to the vertex v. Especially when $|E(v)|=1$, it has rank 0, thus the Euler class becomes 1, this case will be important in the later simplification of the localization contribution in §6.2.

5.3.2. *Edge contributions.* Assume that the multiplicity at $q_\infty \in C_e$ is equal to $(g, \mu_s^{\delta(e)}, 1) \in G\times\boldsymbol{\mu}_s\times\boldsymbol{\mu}_r$ and a (or a_e) is the order of $g\in G$. When r,s is sufficiently large primes, due to Remark 5.2, C_e must be isomorphic to $\mathbb{P}^1_{ar,as}$ where the ramification point q_0 for which $z_1=0$ is isomorphic to $\mathbb{B}\mu_{as}$, and the ramification point q_∞ for which $z_2=0$ is isomorphic to $\mathbb{B}\mu_{ar}$. The restriction of the degree $(\beta,\frac{\delta}{r})$ from C to C_e is equal to $(0,\frac{\delta(e)}{r})$, which is equivalent to:

$$deg(L_j|_{C_e})=0, \quad for \quad 1\leqslant j\leqslant k, \quad deg(N_2|_{C_e})=\frac{\delta(e)}{r} \ .$$

Recall that the inertia stack component I_gY of $I_\mu Y$ is isomorphic to

$$[AY^{ss}(\theta)^g/G] \ .$$

We define the edge moduli $\mathcal{M}_e$ to be

$$\sqrt[as\delta(e)]{L_{-\theta'}/I_gY} = \sqrt[as\delta(e)]{L_{-\theta'}/[AY^{ss}(\theta)^g/G]} \ ,$$

which is the $as\delta(e)$th root gerbe over the inertia stack component I_gY of $I_\mu Y$ by taking the $as\delta(e)$th root of the line bundle $L_{-\theta'}$.

The root gerbe $\sqrt[as\delta(e)]{L_{-\theta'}/I_gY}$ admits a representation as a quotient stack:

$$[AY^{ss}(\theta)^g\times\mathbb{C}^*/(G\times\mathbb{C}^*_w)],$$

where the (right) action is defined by:

$$(\vec{x},v)\cdot(g,w)=(\vec{x}g,\theta'(g)^{-1}vw^{-as\delta(e)}) \ ,$$

for all $(g,w)\in G\times\mathbb{C}^*_w$ and $(\vec{x},v)\in AY^{ss}(\theta)^g\times\mathbb{C}^*$. For every character ρ of G, we can define a new character of $G\times\mathbb{C}^*_w$ by composing the projection map $\mathrm{pr}_G: G\times\mathbb{C}^*_w\to G$, we will still use ρ to name the new character of $G\times\mathbb{C}^*_w$ by an abuse of notation. Then ρ will determines a line bundle $L_\rho := [(AY^{ss}(\theta)^g\times\mathbb{C}^*\times\mathbb{C}_\rho)/(G\times\mathbb{C}^*_w)]$ on $\sqrt[as\delta(e)]{L_{-\theta'}/I_gY}$ by the Borel construction.

By virtue of the universal property of root gerbe, on $\mathcal{M}_e=\sqrt[as\delta(e)]{L_{-\theta'}/I_gY}$, there is a universal line bundle $\mathcal{R}$ that is the $as\delta(e)$th root of the line bundle $L_{-\theta'}$. The root bundle $\mathcal{R}$ is determined by the character $\mathrm{pr}_{\mathbb{C}^*}$:

$$\mathrm{pr}_{\mathbb{C}^*}: G\times\mathbb{C}^*_w\to\mathbb{C}^*_w \quad (g,w)\in G\times\mathbb{C}^*_w\mapsto w\in\mathbb{C}^*_w \ .$$

We have the relation

$$L_{-\theta'}=\mathcal{R}^{as\delta(e)} \ .$$

The coordinate functions $\vec{x}$ and v of $AY^{ss}(\theta)^g\times\mathbb{C}^*$ descents to be universal sections of line bundles $\oplus_{\rho\in[n]}L_\rho$ and $L_{-\theta'}\otimes\mathcal{R}^{-\otimes as\delta(e)}$ over $\mathcal{M}_e$, respectively.

We will construct a universal family of $\mathbb{C}^*-$fixed twisted stable maps to $\mathbb{P}Y_{r,s}$ of degree $(0, \frac{\delta(e)}{r})$ over $\mathcal{M}_e$:

$$
\begin{tikzcd}
\mathcal{C}_e := \mathbb{P}_{ar,as}(\mathcal{R} \oplus \mathcal{O}_{\mathcal{M}_e}) \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}Y_{r,s} \\
\mathcal{M}_e := \sqrt[as\delta(e)]{L_{-\theta'}/I_g Y}. &
\end{tikzcd}
$$

Then the universal curve $\mathcal{C}_e$ over $\sqrt[as\delta(e)]{L_{-\theta'}/I_g Y}$ can be represented as a quotient stack:

$$\mathcal{C}_e = [(AY^{ss}(\theta)^g \times \mathbb{C}^* \times U)/(G \times \mathbb{C}_w^* \times T)] \ ,$$

where $T = \{(t_1, t_2) \in (\mathbb{C}^*)^2 | \quad t_1^{as} = t_2^{ar}\}$. The right action is defined by:

$$(\vec{x}, v, x, y) \cdot (g, w, (t_1, t_2)) = (g \cdot \vec{x}, \theta'(g)^{-1} v w^{-as\delta(e)}, w t_1 x, t_2 y) \ ,$$

for all $(g, w, (t_1, t_2)) \in G \times \mathbb{C}_w^* \times T$ and $(\vec{x}, v, (x, y)) \in AY^{ss}(\theta)^g \times \mathbb{C}^* \times U$. Then $\mathcal{C}_e$ is a family of orbifold $\mathbb{P}_{ar,as}$ parameterized by $\mathcal{M}_e$.

There are two standard characters of T

$$\chi_1 : (t_1, t_2) \in T \mapsto t_1 \in \mathbb{C}^* \quad \chi_2 : (t_1, t_2) \in T \mapsto t_2 \in \mathbb{C}^* \ ,$$

we can lift them to be characters of $G \times \mathbb{C}_w^* \times T$ by composing the projection map $\mathrm{pr}_T : G \times \mathbb{C}_w^* \times T \to T$. By an abuse of notation, we continue to use χ_1, χ_2 to denote the new characters. These two new characters defines two line bundles

$$M_1 := (AY^{ss}(\theta)^g \times \mathbb{C}^* \times U) \times_{G \times \mathbb{C}_w^* \times T} \mathbb{C}_{\chi_1}$$

and

$$M_2 := (AY^{ss}(\theta)^g \times \mathbb{C}^* \times U) \times_{G \times \mathbb{C}_w^* \times T} \mathbb{C}_{\chi_2}$$

on $\mathcal{C}_e$ by the Borel construction, respectively. We have the relation $M_1^{\otimes as} = M_2^{\otimes ar}$ on $\mathcal{C}_e$. The universal map f from $\mathcal{C}_e$ to $\mathbb{P}Y_{r,s}$ can be described as follows: Let

$$\tilde{f} : AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \to \mathbb{C}^* \times AY^{ss}(\theta) \times U$$

be the morphism defined by:

$$
\begin{aligned}
&(\vec{x}, v, x, y) \in AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \mapsto \\
&\qquad (v, (x_1, \cdots, x_n), x^{a\delta(e)}, y^{a\delta(e)}) \in \mathbb{C}^* \times AY^{ss}(\theta) \times U \ .
\end{aligned}
\tag{5.6}
$$

Then $\tilde{f}$ is equivariant with respect to the group homomorphism from $G \times \mathbb{C}_w^* \times T$ to $G \times \mathbb{C}_\alpha^* \times \mathbb{C}_t^*$ defined by:

$$
\begin{aligned}
&(g, w, (t_1, t_2)) \in G \times \mathbb{C}_w^* \times T \mapsto \\
&\qquad \big(g\big((t_1^{-s} t_2^r)^{p_1}, \cdots, (t_1^{-s} t_2^r)^{p_k}\big), (w t_1)^{a\delta(e)}, t_2^{a\delta(e)}\big) \in G \times \mathbb{C}_\alpha^* \times \mathbb{C}_t^* \ ,
\end{aligned}
\tag{5.7}
$$

where the tuple $(p_1, \cdots, p_k) \in \mathbb{N}^k$ satisfies that $g = (\mu_a^{p_1}, \cdots, \mu_a^{p_k}) \in G$. Note $\tilde{f}$ is well-defined for $\chi_1^{-s} \chi_2^r$ is a torsion character of T of order a. The above construction gives the universal morphism f from $\mathcal{C}_e$ to $\mathbb{P}Y_{r,s}$ by descent.

We will define a (quasi left) $\mathbb{C}^*-$action on $\mathcal{C}_e$ such that the map f constructed above is $\mathbb{C}^*-$equivariant. Define a $\mathbb{C}^*$-action on $\mathcal{C}_e$ induced by the $\mathbb{C}^*-$action on $AY^{ss}(\theta)^g \times \mathbb{C}^* \times U$:

$$m : \mathbb{C}^* \times AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \to AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \ ,$$

$$t \cdot (\vec{x}, v, (x, y)) = (\vec{x}, v, (x, t^{\frac{-1}{ar\delta(e)}} y)) \ .$$

Note the morphism π is also $\mathbb{C}^*$-equivariant, where $\mathcal{M}_e$ is equipped with trivial $\mathbb{C}^*$-action. By the universal property of the projectivized bundle $\mathcal{C}_e$ over $\mathcal{M}_e$, one has a tautological section

$$(x, y) \in H^0\big((M_1 \otimes \pi^*\mathcal{R}) \oplus (M_2 \otimes \mathbb{C}_{\frac{-\lambda}{ar\delta(e)}})\big) \ , \tag{5.8}$$

which is also a $\mathbb{C}^*-$invariant section.

Now we can check that f is a $\mathbb{C}^*-$equivariant morphism from $\mathcal{C}_e$ to $\mathbb{P}Y_{r,s}$ with respect to the $\mathbb{C}^*-$actions for $\mathcal{C}_e$ and $\mathbb{P}Y_{r,s}$. According to Remark 5.1, f is equivalent to the following data:

(1) $k+2$ $\mathbb{C}^*$-equivariant line bundles on $\mathcal{C}_e$:

$$\mathcal{L}_j := \pi^* L_{\pi_j} \otimes (M_1^{-\otimes s} \otimes M_2^{\otimes r})^{p_j}, 1 \leqslant j \leqslant k$$

and

$$\mathcal{N}_1 := (M_1 \otimes \pi^*\mathcal{R})^{\otimes a\delta(e)} \quad \mathcal{N}_2 := M_2^{a\delta(e)} \otimes \mathbb{C}_{\frac{-\lambda}{r}} \ .$$

Where L_{π_j} are the standard $\mathbb{C}^*$-equivariant line bundles on $\mathcal{M}_e$ by the Borel construction, M_1, M_2 are the standard $\mathbb{C}^*$-equivariant line bundles on C_e by the Borel construction.

(2) a universal section

$$\begin{aligned} (u, \vec{x}, (\zeta_1, \zeta_2)) :=& (v, x_1, \cdots, x_n, (x^{a\delta(e)}, y^{a\delta(e)})) \\ & \in \Gamma\Big(\big((\mathcal{N}_1^\vee)^{\otimes s} \otimes \mathcal{L}_{-\theta'} \otimes \mathcal{N}_2^{\otimes r} \otimes \mathbb{C}_\lambda\big) \oplus \bigoplus_{1 \leqslant i \leqslant n} \mathcal{L}_{\rho_i} \oplus \mathcal{N}_1 \oplus \mathcal{N}_2\Big)^{\mathbb{C}^*} \ . \end{aligned} \tag{5.9}$$

Here one only need to check $v \in \Gamma((\mathcal{N}_1^\vee)^{\otimes s} \otimes \mathcal{L}_{-\theta'} \otimes \mathcal{N}_2^{\otimes r} \otimes \mathbb{C}_\lambda)$, which is easy to be verified.

Now we compute the localization contribution from $\mathcal{M}_e$. Based on the perfect obstruction theory for stable maps in $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$, the restriction of the perfect obstruction theory to $\mathcal{M}_e$ decomposes into three parts: (1) the deformation theory of source curve $\mathcal{C}_e$; (2) the deformation theory of the lines bundles $(\mathcal{L}_i)_{1 \leqslant j \leqslant k}$ and $\mathcal{N}$; (3) the deformation theory for the section

$$(u, \vec{x}, (\zeta_1, \zeta_2)) \in \Gamma\Big(\big(\mathcal{N}_1^{-\otimes s} \otimes \mathcal{L}_{-\theta'} \otimes \mathcal{N}_2^{\otimes r} \otimes \mathbb{C}_\lambda\big) \oplus \bigoplus_{1 \leqslant i \leqslant n} \mathcal{L}_{\rho_i} \oplus \mathcal{N}_1 \oplus \mathcal{N}_2\Big) \ .$$

The $\mathbb{C}^*-$fixed part of three parts above will contribute to the virtual cycle of $\mathcal{M}_e$, we will show that $[\mathcal{M}_e]^{\mathrm{vir}} = [\mathcal{M}_e]$. The virtual normal bundle comes from the $\mathbb{C}^*-$moving part of the above three parts.

First every fiber curve C_e in $\mathcal{C}_e$ over a geometrical point in $\mathcal{M}_e$ is isomorphic to $\mathbb{P}_{ar,as}$, which is rational. There are no infinitesimal deformations/obstructions for C_e, line bundles $L_j := \mathcal{L}_j|_{C_e}$, $N_1 := \mathcal{N}_1|_{C_e}$ and $N_2 := \mathcal{N}_2|_{C_e}$. Hence their contribution to the perfect obstruction theory comes from infinitesimal automorphisms. The infinitesimal automorphisms of C_e come from the space of vector fields on C_e that vanish on special points. Thus the $\mathbb{C}^*-$fixed part of infinitesimal automorphisms of C_e comes from the $1-$dimensional subspace of vector fields on C_e which vanish on the two ramification points. The movable part of infinitisimial automorphisms of C_e is nonzero only if one of ramification points on C_e is not a special point. by Remark 5.2, the ramifications on C_e are both nontrivial stacky points when r and s are sufficiently large, hence they must be special points. So there is no movable part for infinitesimal automorphisms of C_e.

Now let's turn to the localizations from sections. First the infinitesimal deformations of sections $(u, \vec{x})$ are fixed, which, together with fixed part of infinitesimal automorphisms of C_e and line bundles L_j, N_1, N_2, as well as fixed parts of infinitesimal deformations of sections $(z_1, z_2) := (\zeta_1, \zeta_2)|_{C_e}$, contribute to the virtual cycle $[\mathcal{M}_e]^{\mathrm{vir}}$, which is equal to the fundamental class of $\mathcal{M}_e$. The localization contribution from the infinitesimal deformations of sections (z_1, z_2) to the virtual normal bundle is:

$$(R^\bullet \pi_*(\mathcal{N}_1 \oplus \mathcal{N}_2))^{\mathrm{mov}} \ .$$

We first come to the deformations of z_2, we continue to use the tautological section (x, y) as in (5.8). For each fiber C_e, sections of N_2 is spanned by monomials $(x^{asm}y^n)|_{C_e}$ with $arm + n = a\delta(e)$ and $m, n \in \mathbb{Z}_{\geqslant 0}$. Note $x^{asm}y^n$ may not be a global section of $\mathcal{N}_2$ but always a global section of $\mathcal{R}^{\otimes asm} \otimes \mathcal{N}_2 \otimes \mathbb{C}_{\frac{m}{\delta(e)}\lambda}$. Then $R^\bullet\pi_*\mathcal{N}_2$ will decompose as a direct sum of line bundles, each corresponds to the monomial $x^{asm}y^n$, whose first chern class is

$$c_1(\mathcal{R}^{\otimes -asm} \bigotimes \mathbb{C}_{\frac{-m}{\delta(e)}\lambda}) = \frac{m}{\delta(e)}(D_{\theta'} - \lambda) \ .$$

So the total contribution is equal to

$$\prod_{m=0}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left(\frac{m}{\delta(e)}(D_{\theta'} - \lambda) \right) \ .$$

The factor for $m = 0$ appearing in the above product is the $\mathbb{C}^*-$fixed part of $R^\bullet\pi_*\mathcal{N}_2$, it will contribute to the virtual cycle of $\mathcal{M}_e$. The rest contributes to the virtual normal bundle as

$$\prod_{m=1}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left(\frac{m}{\delta(e)}(D_{\theta'} - \lambda) \right) \ .$$

Note when r is sufficiently large, the above product becomes 1.

For the deformations of z_1, arguing in the same way as z_2, the Euler class of $R^\bullet\pi_*\mathcal{N}_1$ is equal to

$$\prod_{n=0}^{\lfloor \frac{\delta(e)}{s} \rfloor} \left(\frac{n}{\delta(e)}(-D_{\theta'} + \lambda) \right) \ .$$

The factor for $m = 0$ appearing in the above product is the $\mathbb{C}^*-$fixed part of $R^\bullet\pi_*\mathcal{N}_1$, it will contribute to the virtual cycle of $\mathcal{M}_e$. The Euler class of virtual normal bundle of $\mathcal{M}_e$ comes from the movable part of deformations of section z_1 is:

$$\prod_{n=1}^{\lfloor \frac{\delta(e)}{s} \rfloor} \left(\frac{n}{\delta(e)}(-D_{\theta'} + \lambda) \right) \ .$$

Note when s is sufficiently large, the above product becomes 1.

5.3.3. *Node contributions.* The deformations in $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$ smoothing a node contribute to the Euler class of the virtual normal bundle as the first Chern class of the tensor product of the two cotangent line bundles at the branches of the node. For nodes at which a component C_e meets a component C_v over the vertex 0, this contribution is

$$\frac{\lambda - D_{\theta'}}{as\delta(e)} - \frac{\bar{\psi}_v}{as}.$$

For nodes at which a component C_e meets a component C_v at the vertex over ∞, this contribution is

$$\frac{-\lambda + D_{\theta'}}{ar\delta(e)} - \frac{\bar{\psi}_v}{ar}.$$

The type of node at which two edge component C_e meets with a vertex v over 0 or ∞ will not occur using a similar argument in [JPPZ17, Lemma 6].

As for the node contributions from the normalization exact sequence, each node q (specified by a vertex v) contributes the Euler class of

$$\left(R^0\pi_*\mathcal{N}_1|_q\right)^{\mathrm{mov}} \oplus \left(R^0\pi_*\mathcal{N}_2|_q\right)^{\mathrm{mov}} \tag{5.10}$$

to the virtual normal bundle. In the case where $j(v)=0$, $z_2|_q \equiv 1$ gives a trivialization of the fiber $\mathcal{N}_2|_q$, note that $(\mathcal{N}_1^\vee)^{\otimes s} \otimes \mathcal{L}_{-\theta'} \otimes \mathcal{N}_2^{\otimes r} \otimes \mathbb{C}_\lambda \cong \mathbb{C}$ we have $\mathcal{N}_2|_q \cong \mathbb{C}$ and $\mathcal{N}_1|_q \cong L_{-\theta'}^{\frac{1}{s}} \otimes \mathbb{C}_{\frac{\lambda}{s}}$, this implies that $(R^0\pi_*\mathcal{N}_2|_q)^{\mathrm{mov}} = 0$ and $R^0\pi_*\mathcal{N}_1|_q = 0$. The later vanishes because of the nontrivial stacky structure of the line bundle $\mathcal{N}_1$ at q when s is sufficiently large. Hence there no localization contribution from the normalization at the node q over 0. Similarly, for each node q incident to a vertex v with $j(v)=\infty$, there is no localization contribution from the normalization at the node over ∞.

5.4. **Total localization contributions.** For each decorated graph Γ, denote F_Γ to be the fiber product

$$\prod_{v:j(v)=0} \mathcal{M}_v \times_{\bar{I}_\mu \sqrt[s]{L_{-\theta'}/Y}} \prod_{e\in E} \mathcal{M}_e \times_{\bar{I}_\mu \sqrt[r]{L_{\theta'}/Y}} \prod_{v:j(v)=\infty} \mathcal{M}_v$$

of the following diagram:

$$\begin{array}{ccc}
F_\Gamma & \longrightarrow & \prod_{v:j(v)=0} \mathcal{M}_v \times \prod_{e\in E} \mathcal{M}_e \times \prod_{v:j(v)=\infty} \mathcal{M}_v \\
\downarrow & & \downarrow{\scriptstyle ev_{h_e}, ev_{p_0^e}, ev_{p_\infty^e}, ev_{h_e'}} \\
\prod_E \bar{I}_\mu \sqrt[s]{L_{-\theta'}/Y} \times \bar{I}_\mu \sqrt[r]{L_{\theta'}/Y} & \xrightarrow{(\Delta^{\frac{1}{s}} \times \Delta^{\frac{1}{r}})^E} & \prod_E \left((\bar{I}_\mu \sqrt[s]{L_{-\theta'}/Y})^2 \times (\bar{I}_\mu \sqrt[r]{L_{\theta'}/Y})^2 \right),
\end{array}$$

where $\Delta^{\frac{1}{s}} = (id,\iota)$(resp. $\Delta^{\frac{1}{r}} = (id,\iota)$) is the diagonal map of $\bar{I}_\mu \sqrt[s]{L_{-\theta'}/Y}$ (resp. $\bar{I}_\mu \sqrt[r]{L_{\theta'}/Y}$) into $\bar{I}_\mu \sqrt[s]{L_{-\theta'}/Y} \times \bar{I}_\mu \sqrt[s]{L_{-\theta'}/Y}$ (resp. $\bar{I}_\mu \sqrt[r]{L_{\theta'}/Y} \times \bar{I}_\mu \sqrt[r]{L_{\theta'}/Y}$). Here when v is a stable vertex, the vertex moduli $\mathcal{M}_v$ is described in 5.3.1; when v is an unstable vertex over 0, we treat $\mathcal{M}_v := \bar{I}_{m(h)} \sqrt[s]{L_{-\theta'}/Y}$ with the identical virtual cycle, where $m(h)$ is the multiplicity of the half-edge incident to v; when v is an unstable vertex over ∞, we treat $\mathcal{M}_v := \bar{I}_{m(h)} \sqrt[r]{L_{\theta'}/Y}$ with the identical virtual cycle, where $m(h)$ is the multiplicity of the half-edge incident to v.

We define that $[F_\Gamma]^{\mathrm{vir}}$ to be:

$$\prod_{v:j(v)=0}[\mathcal{M}_v]^{\mathrm{vir}}\times_{\bar{I}_\mu\sqrt[s]{L_{-\theta'}/Y}}\prod_{e\in E}[\mathcal{M}_e]^{\mathrm{vir}}\times_{\bar{I}_\mu\sqrt[r]{L_{\theta'}/Y}}\prod_{v:j(v)=\infty}[\mathcal{M}_v]^{\mathrm{vir}}\ ,$$

where the fiber product over $\bar{I}_\mu\sqrt[s]{L_{-\theta'}/Y}$ and $\bar{I}_\mu\sqrt[r]{L_{\theta'}/Y}$ imposes that the evaluation maps at the two branches of each node (here we adopt the convention that a node can link a unstable vertex and an edge.) agree. Then the contribution of decorated graph Γ to the virtual localization is is:

$$Cont_\Gamma=\frac{\prod_{e\in E}sa_e}{\mathrm{Aut}(\Gamma)}(\iota_\Gamma)_*\left(\frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})}\right)\ . \tag{5.11}$$

Here $\iota_F\,:\,F_\Gamma\to\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$ is a finite etale map of degree $\frac{Aut(\Gamma)}{\prod_{e\in E}sa_e}$ into the corresponding $\mathbb{C}^*$-fixed loci in $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$. The virtual normal bundle $e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})$ is the product of virtual normal bundles from vertex contributions, edge contributions and node contributions.

6. Recursion relations from auxiliary cycles

In this section, we will assume that Y is a hypersurface in a *proper* toric Deligne-Mumford stack X. For any $\beta\in\mathrm{Eff}(W,G,\theta)$, for simplicity, we will denote

$$\mathcal{K}_{0,\vec{m}}(\bullet,\beta):=\bigsqcup_{\substack{d\in\mathrm{Eff}(\bullet)\\(i_\bullet)_*(d)=\beta}}\mathcal{K}_{0,\vec{m}}(\bullet,d),$$

where $\bullet$ can be $Y,\sqrt[r]{L_{\theta'}/Y}$ and $\sqrt[s]{L_{-\theta'}/Y}$, and $i_\bullet$ is the natural structure map from $\bullet$ to $\mathfrak{X}$ which factors through the inclusion $i_{\mathfrak{Y}}:\mathfrak{Y}\to\mathfrak{X}$.

We will make the following remark regarding Gromov-Witten classes of $\mathcal{K}_{0,\vec{m}\cup\star}(\bullet,\beta)$ and $\mathcal{K}_{0,[m]\cup\star}(\bullet,\beta)$.

Remark 6.1. Let's first discuss the case when $\bullet=\sqrt[s]{L_{-\theta'}/Y}$.

Given $\vec{m}=(v_1,\cdots,v_m)\in(G\times\boldsymbol{\mu}_s)^m$ and $\star\in G\times\boldsymbol{\mu}_s$, then

$$\mathcal{K}_{0,\vec{m}\cup\star}(\sqrt[s]{L_{-\theta'}/Y},\beta)$$

is empty unless the following

$$\star\cdot\prod_{i=1}^m v_i=(g_\beta,\mu_s^{-\beta(L_{\theta'})})\in G\times\boldsymbol{\mu}_s\ . \tag{6.1}$$

holds. Recall here g_β is defined in 3.3. Indeed, write the root stack $\sqrt[s]{L_{-\theta'}/Y}$ as the quotient stack

$$[(AY^{ss}(\theta)\times\mathbb{C}^*)/(G\times\mathbb{C}^*)]$$

defined by the right action of $G\times\mathbb{C}^*$ on $AY\times\mathbb{C}^*$:

$$(\vec{x},u)\cdot(g,t)=(\vec{x}\cdot g,\theta'(g)^{-1}t^{-s}u)$$

for $(\vec{x},u)\in AY\times\mathbb{C}^*$ and $(g,t)\in G\times\mathbb{C}^*$. Then a genus zero twisted stable map $f:C\to\sqrt[s]{L_{-\theta'}/Y}$ determines a representable vector bundle E of rank $n+1$ on C, which is a direct sum of $n+1$ line bundles. Then the multiplicity at each marked point comes from the monodromy of E at this point. Apply Riemann-Roch Formula to the vector bundle E on C, (6.1) holds.

The above discussion tells that the multiplicities at m of the $m+1$ marked points determine the multiplicity of remaining point. Thus, given $\vec{m}=(v_1,\cdots,v_m)\in(G\times\boldsymbol{\mu}_s)^m$ and $\star\in G\times\boldsymbol{\mu}_s$ such that (6.1) holds, we have the identity of the following Gromov-Witten classes

$$\begin{aligned}&(\widetilde{ev}_0)_*([\mathcal{K}_{0,[m]\cup\{0\}}(\sqrt[s]{L_{-\theta'}/Y},\beta)]^{\mathrm{vir}}\cap\gamma\cap\prod_{i=1}^m ev_i^*(\alpha_i))\\ &=(\widetilde{ev}_0)_*([\mathcal{K}_{0,\vec{m}\cup\{0\}}(\sqrt[s]{L_{-\theta'}/Y},\beta)]^{\mathrm{vir}}\cap\gamma\cap\prod_{i=1}^m ev_i^*(\alpha_i))\\ &=(\widetilde{ev}_\star)_*([\mathcal{K}_{0,\vec{m}\cup\star}(\sqrt[s]{L_{-\theta'}/Y},\beta)]^{\mathrm{vir}}\cap\gamma\cap\prod_{i=1}^m ev_i^*(\alpha_i))\ ,\end{aligned}\tag{6.2}$$

where α_i is a class belonging to the twisted sector $\bar{I}_{v_i}\sqrt[s]{L_{-\theta'}/Y}$, γ is any rational chow cycle in $A^*(\mathcal{K}_{0,[m]\cup\{0\}}(\sqrt[s]{L_{-\theta'}/Y},\beta))_{\mathbb{Q}}$. Here

$$\mathcal{K}_{0,\vec{m}\cup\{0\}}(\sqrt[s]{L_{-\theta'}/Y},\beta)$$

is defined to be

$$\mathcal{K}_{0,\vec{m}\cup\{0\}}(\sqrt[s]{L_{-\theta'}/Y},\beta)=\mathcal{K}_{0,[m]\cup\{0\}}(\sqrt[s]{L_{-\theta'}/Y},\beta)\cap\bigcap_{i=1}^m ev_i^{-1}(\bar{I}_{v_i}\sqrt[s]{L_{-\theta'}/Y})\ .$$

When $\bullet=Y$, given $\vec{m}=(v_1,\cdots,v_m)\in G^m$ and $\star\in G$, (6.1) is changed to

$$\star\cdot\prod_{i=1}^m v_i=g_\beta\in G\ .$$

When $\bullet=\sqrt[r]{L_{\theta'}/Y}$, given $\vec{m}=(v_1,\cdots,v_m)\in(G\times\boldsymbol{\mu}_r)^m$ and $\star\in G\times\boldsymbol{\mu}_r$, (6.1) is changed to

$$\star\cdot\prod_{i=1}^m v_i=(g_\beta,\mu_r^{\beta(L_{\theta'})})\in G\times\boldsymbol{\mu}_r\ .$$

The proof of the above two equalities are similar to the case when $\bullet=\sqrt[r]{L_{\theta'}/Y}$. We also have equations of the same type as (6.2) with the only replacement of $\sqrt[s]{L_{-\theta'}/Y}$ by Y and $\sqrt[r]{L_{\theta'}/Y}$, respectively.

Convention : We will also use the notation $\mathcal{K}_{0,[m]\cup\star}(\bullet,\beta)$, for which we mute the multiplicities at all markings for the index set $[m]\cup\star$.

6.1. **Auxiliary cycle I.** Fix a nonzero $\beta\in\mathrm{Eff(W,G,\theta)}$, pick a positive rational number ϵ such that $\epsilon\beta(L_{\theta'})\leqslant 1$. Set $\delta=\beta(L_{\theta'})$. For simplicity, we will denote

$$Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(\beta,\frac{\delta}{r})):=\coprod_{\substack{d\in\mathrm{Eff}(AY,G,\theta)\\(i_{\mathfrak{Y}})_*(d)=\beta}}Q^{\widetilde{\theta}}_{0,1}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(\beta,\frac{\delta}{r}))\cap ev_1^{-1}(\bar{I}_{(g_\beta,\frac{\delta}{r})}\mathbb{P}Y^{\frac{1}{r}})\ .$$

Recall that $g_\beta\in G$ is defined in §3.3. We will always assume that r is a sufficiently large prime in this subsection.

For any nonnegative integer c, we will first consider the following auxiliary cycle:

$$(\widetilde{EV_\star})_*\left(e^{\mathbb{C}^*}\left(R^1\pi_*f^*L_\infty^\vee\right)\cap[Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(\beta,\frac{\delta}{r}))]^{\mathrm{vir}}\cap\bar{\psi}_\star^c\right). \tag{6.3}$$

Here an explanation of the notations is in order:

(1) the morphism

$$\pi:\mathcal{C}\to Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(\beta,\frac{\delta}{r}))$$

is the universal curve and the morphism

$$f:\mathcal{C}\to\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$$

is the universal map;

(2) the $\mathbb{C}^*$-equivariant line bundle L_∞ corresponds to the invertible sheaf $\mathcal{O}(\mathcal{D}_\infty)$ on $\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ with the $\mathbb{C}^*$-linearization that $\mathbb{C}^*$ acts on the fiber over $\mathcal{D}_\infty$ (given by $z_2=0$) with weight $-\frac{\lambda}{r}$ and $\mathbb{C}^*$ acts on the fiber over $\mathcal{D}_0$ (given by $z_1=0$) with weight 0. For every fiber curve C of the universal curve $\mathcal{C}$ via π, the important observation here is that the restricted line bundle $(f|_C)^*L_\infty^\vee$ has negative degree as $\beta\neq 0$ and has non-positive degree on every irreducible components of C. Indeed, if the image of an irreducible component of C via f isn't contained in $\mathcal{D}_\infty$, the degree is obviously non-positive. If the image of an irreducible component of C under f is contained in $\mathcal{D}_\infty$, then using the fact that $L_\infty^\vee$ is isomorphic to $(L_{\theta'}^{\frac{1}{r}})^\vee$ over

$$\mathcal{D}_\infty\cong\sqrt[r]{L_{\theta'}/Y}$$

and a similar discussion as in remark 5.4, the degree is also non-positive. Thus by Lemma 5.3, we have $R^0\pi_*f^*L_\infty^\vee=0$, which implies that $R^1\pi_*f^*L_\infty^\vee$ is a vector bundle (of rank 0);

(3) the morphism $EV_\star$ is a composition of the following maps:

$$Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(\beta,\tfrac{\delta}{r}))\xrightarrow{ev_\star}\bar{I}_\mu\mathbb{P}Y^{\frac{1}{r}}\xrightarrow{\mathrm{pr}_r}\bar{I}_\mu Y\ ,$$

where $\mathrm{pr}_r:\bar{I}_\mu\mathbb{P}Y^{\frac{1}{r}}\to\bar{I}_\mu Y$ is the morphism induced from the natural structure map from $\mathbb{P}Y^{\frac{1}{r}}$ to Y forgetting z_1,z_2. $(\widetilde{EV_\star})_*$ is defined by

$$\iota_*(r_\star(EV_\star)_*)$$

as in (2.2). Note here $r_\star$ is the order of the band from the gerbe structure of $\bar{I}_\mu Y$ but not $\bar{I}_\mu\mathbb{P}Y^{\frac{1}{r}}$.

Apply virtual localization to $Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r}},(\beta,\frac{\delta}{r}))$, we first prove the following vanishing result, where the idea is borrowed from [JPHH].

Lemma 6.2. *The localization graph Γ that has more than one vertex labeled by ∞ will contribute zero to* (6.3).

Proof. Assume by contradiction, by the connectedness of the graph, there is at least one vertex at 0 with valence at least two. Suppose $f:C\to\mathbb{P}\mathfrak{Y}^{\frac{1}{r}}$ is $\mathbb{C}^*$-fixed. Assume that $C_0\cap C_1\cap C_2$ is part of curve C, where C_0 contracted by f to $\mathcal{D}_0$ (given by $z_1=0$)

and C_1, C_2 are edges meeting with C_0 at b_1 and b_2. Then in the normalization sequence for $R^\bullet\pi_* f^*(L_\infty^\vee)$, it contains the part

$$\begin{aligned}&H^0\left(C_0, f^*\left(L_\infty^\vee\right)\right)\oplus H^0\left(C_1, f^*\left(L_\infty^\vee\right)\right)\oplus H^0\left(C_2, f^*\left(L_\infty^\vee\right)\right)\\ \to &H^0\left(b_1, f^*\left(L_\infty^\vee\right)\right)\oplus H^0\left(b_2, f^*\left(L_\infty^\vee\right)\right)\\ \to &H^1\left(C, f^*\left(L_\infty^\vee\right)\right).\end{aligned}$$

Hence there is one of the weight-0 pieces in $H^0\left(b_1, f^*\left(L_\infty^\vee\right)\right)\oplus H^0\left(b_2, f^*\left(L_\infty^\vee\right)\right)$ that is canceled with a weight-0 piece of $H^0\left(C_0, f^*\left(L_\infty^\vee\right)\right)$, and the other is mapped injectively into $H^1\left(C, f^*\left(L_\infty^\vee\right)\right)$. Thus $H^1\left(C, f^*\left(L_\infty^\vee\right)\right)$ contains a weight-0 piece with vanishing equivariant Euler class. $\square$

Recall that we can write $I(q,z)=\sum_\beta q^\beta I_\beta(z)$, where I_β will be a polynomial in z with coefficient in $H^*(\bar{I}_\mu Y,\mathbb{Q})$. We will prove the following recursion relation by applying localization to (6.11).

Theorem 6.3. *For any nonnegative integer c, $[zI_\beta(z)]_{z^{-c-1}}$ satisfies the following relation:*

(6.4)
$$\begin{aligned}[zI_\beta(z)]_{z^{-c-1}} = {}&(\widetilde{ev}_\star)_*([\mathcal{K}_{0,\star}(Y,\beta)]^{\mathrm{vir}}\cap ev_\star^*(\bar\psi_\star^c))\\ &+\Bigg[\sum_{m=1}^{\infty}\sum_{\beta_\star+\beta_1+\cdots+\beta_m=\beta}\frac{1}{m!}(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*(c_d(-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\\ &\cap[\mathcal{K}_{0,[m]\cup\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)]^{\mathrm{vir}})\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}(zI_{\beta_i}(z))|_{z=\frac{\lambda-D_{\theta'}}{\delta_i}})}{\frac{\lambda-ev_i^*D_{\theta'}}{r\delta_i}+\frac{\bar\psi_i}{r}}\cap\bar\psi_\star^c\Bigg)\Bigg]_{\lambda^{-1}}.\end{aligned}$$

Here $\delta_i=\beta_i(L_{\theta'})$ for $1\leqslant i\leqslant m$ and $\epsilon:\mathcal{K}_{0,[m]\cup\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)\to\mathcal{K}_{0,[m]\cup\star}(Y,\beta_\star)$ is the natural structure morphism (c.f.[TT16]).

Proof. By Lemma 6.2, only decorated graph Γ, which has only one vertex v labeled by ∞, may have nonzero localization contribution to the (6.3). Note the only marking $q_\star$ can only be incident to the vertex labeled by ∞ due to the restriction on the multiplicity at $q_\star$. Furthermore, for such graph Γ, we claim there is no stable vertex labeled by 0. Indeed, for any vertex v over 0, its degree $\beta(v)$ satisfies that $\beta(v)(L_{\theta'})\leqslant\beta(L_{\theta'})\leqslant\frac{1}{\epsilon}$, and it has valence 1 as no legs can attach to it and at most one edge is incident to it by Lemma 6.2, then the vertex v must be unstable. So the decorated graph Γ has only one vertex over ∞ with possible several edges (can be empty) attached, and each vertex labeled by 0 corresponds to an edge in the graph Γ, which appears as an unmarked point (actually a base point as we will see). In the following, we analyze the localization contribution to (6.3) from the graph Γ described above. We have two cases which depends on whether the only vertex labeled by ∞ on the graph Γ is stable or unstable.

(1) If the only vertex v over ∞ is unstable, then it's a vertex with valence 2, i.e, it's incident to a leg and an edge. In this case the degree $(\beta,\frac{\delta}{r})$ is concentrated on the single edge with the marked point $q_\star$ appearing as the ramification point

over ∞ on the edge. Then it contributes

$$\frac{1}{\delta}(zI_\beta(z))|_{z=\frac{\lambda-D_{\theta'}}{\delta}}\cdot(\frac{\lambda-D_{\theta'}}{\delta})^c$$

to (6.3). Here we use the fact that the restriction of $R^1\pi_*(f^*L_\infty^\vee)$ to F_Γ is a rank 0 vector bundle, so its equivariant Euler class is 1, and the restriction of $\bar{\psi}_\star$ to $\mathcal{M}_e$ is equal to $\frac{\lambda-D_{\theta'}}{\delta}$.

(2) If the only vertex v over ∞ is stable, then v is incident to only one leg and possible several edges (can be none). We assume that the vertex v has degree $(\beta_*,\frac{\delta_*}{r})$ with $\delta_*=\beta_*(L_{\theta'})$. If there is no edges in the graph Γ, which happens if and only if $\beta_\star=\beta$, the corresponding graph has contribution

$$(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^\infty\epsilon_*(c_d(-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1-d}\cap[\mathcal{K}_{0,\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)]^{\mathrm{vir}})\cap\bar{\psi}_\star^c\Bigg). \tag{6.5}$$

to the (6.3). Otherwise we index all the edges attached to the vertex v from 1 to m such that the edge e_i corresponding to the index i has degree $(\beta_i,\frac{\delta_i}{r})$. Since we assume that the total degree is $(\beta,\frac{\delta}{r})=(\beta,\frac{\beta(L_{\theta'})}{r})$, and the degree on every edge satisfies the relation $\delta_i\geqslant\beta_i(L_{\theta'})$ by Remark 4.4, then we must have $\delta_i=\beta_i(L_{\theta'})$ for every edge e_i. It follows that all the edge has a base point.

Equipped with these notations, the vertex moduli $\mathcal{M}_v$ over ∞ is

$$\mathcal{K}_{0,\vec{m}\cup\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)=$$
$$\mathcal{K}_{0,[m]\cup\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)\cap\bigcap_{i=1}^m ev_i^{-1}(\bar{I}_{(g_{\beta_i}^{-1},\mu_r^{-\delta_i})}\sqrt[r]{L_{\theta'}/Y})\cap ev_\star^{-1}(\bar{I}_{(g_\beta,\mu_r^\delta)}\sqrt[r]{L_{\theta'}/Y}).$$

Using the localization analysis in §4.3 and the fact that $e^{\mathbb{C}^*}(R^1\pi_*f^*L_\infty^\vee)=1$ as it's of rank zero on F_Γ. The localization contribution of such graph Γ to (6.3) is equal to

$$\begin{aligned}&\frac{1}{Aut(\Gamma)}\tilde{ev}_*\Bigg(\sum_{d=0}^\infty\epsilon_*(c_d(-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\cap[\mathcal{K}_{0,\vec{m}\cup\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)]^{\mathrm{vir}})\\&\qquad\cap\prod_{i=1}^m\frac{ev_i^*(\frac{1}{\delta_i}(zI_{\beta_i}(q,z))|_{z=\frac{\lambda-D_{\theta'}}{\delta_i}})}{-\frac{\lambda-ev_i^*D_{\theta'}}{r\delta_i}-\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\Bigg),\end{aligned}\tag{6.6}$$

where $\epsilon:\mathcal{K}_{0,\vec{m}\cup\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)\to\mathcal{K}_{0,\vec{m}'\cup\star}(Y,\beta_\star)$ is the natural structure map. Here $\vec{m}'\cup\star=(g_{\beta_i}^{-1}:1\leqslant i\leqslant m)\times g_\beta\in G^{m+1}$.

Fix β_* and m, the sum of (6.6) coming from all possible graph which has $\infty-$vertex v of degree $\beta_\star$ and m incident edges yields:

(6.7)
$$\sum_{\substack{\beta_1,\cdots,\beta_m \\ \beta_\star+\beta_1+\cdots+\beta_m=\beta}} \frac{1}{m!}(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*\big(c_d(-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}$$
$$\cap[\mathcal{K}_{0,[m]\cup\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)]^{\mathrm{vir}}\big)\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}(zI_{\beta_i}(z))|_{z=\frac{\lambda-D_{\theta'}}{\delta_i}})}{-\frac{\lambda-ev_i^*D_{\theta'}}{r\delta_i}-\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\Bigg).$$

Note here we can drop the multiplicity for each marking for $\mathcal{K}_{0,\vec{m}\cup\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)$ in (6.7) by Remark 6.1.

In summary, the auxiliary cycle (6.3) is equal to:

(6.8)
$$\begin{aligned}
&\frac{1}{\delta}(zI_\beta(z))|_{z=\frac{\lambda-D_{\theta'}}{\delta}}\cdot(\frac{\lambda-D_{\theta'}}{\delta})^c\\
&+(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*(c_d(-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1-d})\cap[\mathcal{K}_{0,\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)]^{\mathrm{vir}}\cap\bar{\psi}_\star^c\Bigg)\\
&+\sum_{m=1}^{\infty}\sum_{\beta_\star+\beta_1+\cdots+\beta_m=\beta}\frac{1}{m!}(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*(c_d(-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\\
&\cap[\mathcal{K}_{0,[m]\cup\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)]^{\mathrm{vir}})\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}(zI_{\beta_i}(z))|_{z=\frac{\lambda-D_{\theta'}}{\delta_i}})}{-\frac{\lambda-ev_i^*D_{\theta'}}{r\delta_i}-\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\Bigg).
\end{aligned}$$

Observe that (6.3) does not have negative λ powers, the λ^{-1} coefficient in the equation (6.8) is equal to zero. Note that the λ^{-1} coefficient in (6.8) is equal to

(6.9)
$$\begin{aligned}
&[zI_\beta(z)]_{z^{-c-1}}-(\widetilde{ev}_\star)_*\Bigg(r\epsilon_*([\mathcal{K}_{0,\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)]^{\mathrm{vir}})\cap\bar{\psi}_\star^c\Bigg)\\
&-\Bigg[\sum_{m=1}^{\infty}\sum_{\beta_\star+\beta_1+\cdots+\beta_m}\frac{1}{m!}(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*(c_d(-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\\
&\cap[\mathcal{K}_{0,[m]\cup\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)]^{\mathrm{vir}})\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}(zI_{\beta_i}(z))|_{z=\frac{\lambda-D_{\theta'}}{\delta_i}})}{\frac{\lambda-ev_i^*D_{\theta'}}{r\delta_i}+\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\Bigg)\Bigg]_{\lambda^{-1}}.
\end{aligned}$$

Now for the second term in (6.9), one uses the following identity:

(6.10)
$$\epsilon_*([\mathcal{K}_{0,\star}(\sqrt[r]{L_{\theta'}/Y},\beta)]^{\mathrm{vir}})=\frac{1}{r}[\mathcal{K}_{0,\star}(Y,\beta)]^{\mathrm{vir}},$$

which is proved[1] in [TT16, Thereom 5.17]. Now (6.9) immediately implies the formula (6.4). □

[1]The proof in loc. cit. needs the assumption that Y is a proper DM stack, so that's why our wall-crossing proof is restricted to the class of proper toric DM stacks. But the small I-function only requires that Y is a semi-projective toric DM stack.

6.2. **Auxiliary cycle II.** In this section, for any $\beta \in \mathrm{Eff}(W, G, \theta)$, we will denote $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ to be

$$\coprod_{\substack{d \in \mathrm{Eff}(AY, G, \theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta}} \mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (d, \frac{\delta}{r})) \ .$$

Fix β, δ as in §6.1. Assume that r, s are sufficiently large primes. We will also compare (6.3) to the following auxiliary cycle:

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{\beta_\star + \beta_1 + \cdots + \beta_m = \beta} \frac{1}{m!} (\widetilde{EV_\star})_* \Big([\mathcal{K}_{0,\vec{m} \cup \star}(\mathbb{P}Y_{r,s}, (\beta_\star, \frac{\delta}{r}))]^{\mathrm{vir}} \\ \cap \prod_{i=1}^{m} ev_i^* \big(\mathrm{pr}_{r,s}^* (\mu_{\beta_i}(-\bar{\psi}_i)) \big) \cap \bar{\psi}_\star^c \cap e^{\mathbb{C}^*}(R^1 \pi_* f^* L_\infty^\vee) \Big). \end{aligned} \tag{6.11}$$

Here an explanation of the notations is in order:

(1) the morphism

$$\pi : \mathcal{C} \to \mathcal{K}_{0,\vec{m} \cup \star}(\mathbb{P}Y_{r,s}, (\beta_\star, \frac{\delta}{r}))$$

is the universal curve and the morphism

$$f : \mathcal{C} \to \mathbb{P}Y_{r,s}$$

is the universal map;

(2) for any nonnegative integer m and degrees $\beta_\star, \beta_1, \cdots, \beta_m$ in $\mathrm{Eff}(W, G, \theta)$. Here $\vec{m} = \{m_i \in G \times \boldsymbol{\mu}_s \times \boldsymbol{\mu}_r : 1 \leqslant i \leqslant m\}$, in which $m_i = (g_{\beta_i}^{-1}, \mu_s^{\beta_i(L_{\theta'})}, 1)$ for $1 \leqslant i \leqslant m$, and $m_\star = (g_\beta, 1, \mu_r^{\beta(L_{\theta'})}) \in G \times \boldsymbol{\mu}_s \times \boldsymbol{\mu}_r$. So $\mathcal{K}_{0,\vec{m} \cup \star}(\mathbb{P}Y_{r,s}, (\beta_\star, \frac{\delta}{r}))$ is defined to be:

$$\mathcal{K}_{0,m+1}(\mathbb{P}Y_{r,s}, (\beta_\star, \frac{\delta}{r})) \cap \bigcap_{i=1}^{m} ev_i^{-1}(\bar{I}_{m_i} \mathbb{P}Y_{r,s}) \cap ev_{m+1}^{-1}(\bar{I}_{m_\star} \mathbb{P}Y_{r,s});$$

(3) the line bundle L_∞ corresponds to the invertible sheaf $\mathcal{O}(\mathcal{D}_\infty)$ with the $\mathbb{C}^*$-linearization such that $\mathbb{C}^*$ acts on the fiber over $\mathcal{D}_\infty$ with weight $-\frac{\lambda}{r}$ and acts on the fiber over $\mathcal{D}_0$ with weight zero; Using the same reasoning in the case of auxiliary cycle I, we have $R^0 \pi_* f^* L_\infty^\vee = 0$ and $R^1 \pi_* f^* L_\infty^\vee$ is a vector bundle (of rank 0);

(4) the morphism $EV_\star$ is a composition of the following maps:

$$\mathcal{K}_{0,\vec{m} \cup \star}(\mathbb{P}Y_{r,s}, (\beta_\star, \tfrac{\delta}{r})) \xrightarrow{ev_\star} \bar{I}_\mu \mathbb{P}Y_{r,s} \xrightarrow{\mathrm{pr}_{r,s}} \bar{I}_\mu Y \ ,$$

where $\mathrm{pr}_{r,s} : \bar{I}_\mu \mathbb{P}Y_{r,s} \to \bar{I}_\mu Y$ is the morphism induced from the natural structure map from $\mathbb{P}Y_{r,s}$ to Y forgetting u and z_1, z_2, and $(\widetilde{EV_\star})_*$ is defined by

$$\iota_*(r_\star (EV_\star)_*)$$

as in 2.2. Note here $r_\star$ is the order of the band from the gerbe structure of $\bar{I}_\mu Y$ but not $\bar{I}_\mu \mathbb{P}Y_{r,s}$.

First we have a similar vanishing result as Lemma 6.2 by an analogous argument.

Lemma 6.4. *The localization graph* Γ *which has more than one vertex labeled by* ∞ *contributes zero to* (6.11)*.*

We will prove the following recursion relation by applying localization to (6.11).

Theorem 6.5. *Assume* r, s *are sufficiently large and prime. For any nonnegative integer* c*, the summation*

$$\sum_{m=0}^{\infty} \sum_{\beta_\star+\beta_1+\cdots+\beta_m=\beta} \frac{1}{m!}\phi^\alpha \langle \mu_{\beta_1}(-\bar\psi_1), \cdots, \mu_{\beta_m}(-\bar\psi_m), \phi_\alpha \bar\psi_\star^c \rangle^\infty_{0,[m]\cup\star,\beta_\star}$$

satisfies the following relation:

$$\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{\beta_\star+\beta_1+\cdots+\beta_m=\beta} \frac{1}{m!}\phi^\alpha \langle \mu_{\beta_1}(-\bar\psi_1), \cdots, \mu_{\beta_m}(-\bar\psi_m), \phi_\alpha \bar\psi_\star^c \rangle^\infty_{0,[m]\cup\star,\beta_\star} \\
&= (\widetilde{ev_\star})_*([\mathcal{K}_{0,\star}(Y,\beta)]^{\mathrm{vir}} \cap \bar\psi_\star^c) \\
&+ \Big[\sum_{m=1}^{\infty} \sum_{\beta_\star+\beta_1+\cdots+\beta_m} \frac{1}{m!} (e\tilde v_\star)_* \Big(\sum_{d=0}^{\infty} \epsilon_* \big(c_d(-R^\bullet \pi_* \mathcal{L}_{\theta'}^{\frac{1}{r}}) (\frac{-\lambda}{r})^{-1+m-d} \\
&\cap [\mathcal{K}_{0,[m]\cup\star}(\sqrt[r]{L_{\theta'}/Y}, \beta_\star)]^{\mathrm{vir}} \big) \cap \prod_{i=1}^{m} \frac{ev_i^*(\frac{1}{\delta_i} f_{\beta_i}(z)|_{z=\frac{\lambda - D_{\theta'}}{\delta_i}})}{\frac{\lambda - ev_i^* D_{\theta'}}{r\delta_i} + \frac{\bar\psi_i}{r}} \cap \bar\psi_\star^c \Big) \Big]_{\lambda^{-1}}.
\end{aligned} \tag{6.12}$$

Here $f_{\beta_i}(z)$ *is defined as follows:*

$$\begin{aligned}
f_{\beta_i}(z) &= \mu_{\beta_i}(z) + \sum_{l=0}^{\infty} \sum_{\beta_0+\beta_1+\cdots+\beta_l=\beta_i} \frac{1}{l!} \\
&(\widetilde{ev_0})_* \Big([\mathcal{K}_{0,[l]\cup\{0\}}(Y,\beta_0)]^{\mathrm{vir}} \cap \bigcap_{j=1}^{l} ev_j^*(\mu_{\beta_j}(-\bar\psi_j)) \cap \frac{1}{z-\bar\psi_0} \Big),
\end{aligned} \tag{6.13}$$

$\delta_i = \beta_i(L_{\theta'})$*, and* $\epsilon : \mathcal{K}_{0,[m]\cup\star}(\sqrt[r]{L_{\theta'}/Y}, \beta_\star) \to \mathcal{K}_{0,[m]\cup\star}(Y,\beta_\star)$ *is the natural structure morphism.*

Proof. By Lemma 6.4, only decorated graph Γ, which has only one vertex v labeled ∞, may have nonzero localization contribution to the (6.11). Let's denote by $\beta_\star$ the degree of the $\infty-$vertex v coming from $\vec{x}$. Note the marked point $q_\star$ must lie on a vertex labeled by ∞ due to the choice of multiplicity at the marking $q_\star$, then the marking $q_\star$ is incident to the only vertex v labeled by ∞. Thus the vertex v can't be a node linking two edges. Hence there are only two types of graph Γ depending on whether v is a stable or unstable vertex.

(1) If the only vertex v over ∞ in Γ is unstable, in the case, v is of valence 2, i.e. it's incident to an edge and the marking $q_\star$. Then Γ has only one edge whose degree $(0, \frac{\delta}{r})$, and has only one vertex over 0, which is incident to the edge. The vertex over 0 can be stable or unstable. If the vertex over 0 is unstable, it must be a marked point with input μ_β, then the graph Γ contributes

$$\frac{\mu_\beta(\frac{\lambda - D_{\theta'}}{\delta})}{\delta} \cdot (\frac{\lambda - D_{\theta'}}{\delta})^c$$

to (6.11). If the vertex over 0 is stable, then this type of graphs contributes

$$\sum_{m=0}^{\infty}\sum_{\beta_0+\cdots+\beta_m=\beta}\frac{1}{m!}(\widetilde{ev}_0)_*\Bigg(\sum_{d=0}^{\infty}\epsilon'_*(c_d(-R^\bullet\pi_*\mathcal{L}_{-\theta'}^{\frac{1}{s}})(\frac{\lambda}{s})^{-d}$$

$$\cap\,[\mathcal{K}_{0,\vec{m}\cup\{0\}}(\sqrt[s]{L_{-\theta'}/Y},\beta_0)]^{\mathrm{vir}})\bigcap_{i=1}^{m}ev_i^*(\mu_{\beta_i}(-\bar{\psi}_i))\cap\frac{\frac{1}{\delta}(\frac{\lambda-ev_0^*D_{\theta'}}{\delta})^c}{\frac{\lambda-ev_0^*D_{\theta'}}{s\delta}-\frac{\bar{\psi}_0}{s}}\Bigg)$$

to (6.11), where $\epsilon':\mathcal{K}_{0,[m]\cup\star}(\sqrt[s]{L_{-\theta'}/Y},\beta_0)\to\mathcal{K}_{0,[m]\cup\star}(Y,\beta_0)$ is the natural structure morphism studied in [TT16]. By Lemma 6.6 proved below, the above formula is equal to

$$\sum_{m=0}^{\infty}\sum_{\beta_0+\cdots+\beta_m=\beta}\frac{1}{m!}\phi^\alpha\langle\mu_{\beta_1}(-\bar{\psi}_1),\cdots,\mu_{\beta_m}(-\bar{\psi}_m),\frac{\frac{1}{\delta}(\frac{\lambda-D_{\theta'}}{\delta})^c\phi_\alpha}{\frac{\lambda-D_{\theta'}}{\delta}-\bar{\psi}_0}\rangle^{\infty}_{0,[m]\cup\{0\},\beta_0}\ .$$

In summary, the localization contribution from the decorated graphs of which the vertex over ∞ is unstable contributes

(6.14)
$$\mu_\beta(\frac{\lambda-D_{\theta'}}{\delta})\cdot(\frac{\lambda-D_{\theta'}}{\delta})^c$$
$$+\sum_{m=0}^{\infty}\sum_{\beta_0+\beta_1+\cdots+\beta_m=\beta}\frac{1}{m!}\phi^\alpha\langle\mu_{\beta_1}(-\bar{\psi}_1),\cdots,\mu_{\beta_m}(-\bar{\psi}_m),\frac{\frac{1}{\delta}(\frac{\lambda-D_{\theta'}}{\delta})^c\phi_\alpha}{\frac{\lambda-D_{\theta'}}{\delta}-\bar{\psi}_0}\rangle^{\infty}_{0,[m]\cup\{0\},\beta_0}$$

to the (6.11). Here we use the fact $R^1\pi_*(f^*\mathcal{L}_\infty^\vee)$ is of rank 0 over F_Γ, so its Euler class is 1.

(2) If the only vertex v over ∞ in Γ is stable, then v is incident to only one leg (corresponding to the marking $q_\star$) and possible several edges (can be none). Let's assume that v has degree $(\beta_\star,\frac{\delta_\star}{r})$ with $\delta_\star=\beta_\star(L_{\theta'})$. If there is no edges in the graph Γ, which happens if and only if $\beta_\star=\beta$, this has contribution:

$$(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*(c_d(-R^\bullet\pi_*(f^*\mathcal{L}_{\theta'}^{\frac{1}{r}}))(\frac{-\lambda}{r})^{-1-d}\cap[\mathcal{K}_{0,\star}(\sqrt[r]{L_{\theta'}/Y},\beta)]^{\mathrm{vir}})\cap\bar{\psi}_\star^c\Bigg)\tag{6.15}$$

to (6.11). Otherwise, there are m edges attached to the vertex v, let's index them by $[m]:=\{1,\cdots,m\}$, with degree $(0,\frac{\delta_i}{r})$ on the ith edge e_i for $\delta_i\in\mathbb{Z}_{>0}$. On each edge e_i there is exactly one vertex v_i over 0 incident to it, which can't be a unstable vertex of valence 1 (see Remark 5.2) or a node linking two edges by Lemma 6.4. So v_i is either a marking or a stable vertex with only one node incident to the edge e_i and possible l marked points (l can be zero) on it, let's label the legs incident to v_i by $\{i1,\cdots,il\}\subset[n]$ (n is the total number of legs on Γ).

Assume that the vertex v_i has degree $(\beta_{i0},0)$. Since the insertion at the marking q_{ij} on the curve C_{v_i} corresponding to v_i is of the form $\mu_{\beta_{ij}}(-\bar{\psi}_{ij})$ in (6.11), let's say the leg for q_{ij} has *virtual degree* β_{ij} contribution to the vertex v_i, denote β_i to be summation of β_{i0} and all the virtual degrees from the markings on C_{v_i}, we call β_i the *total degree* at the vertex v_i. From the (6.11), one has

$$\beta_\star+\beta_1+\cdots+\beta_m=\beta\ .$$

Note to ensure such a graph Γ exists, one must have

$$\beta_i(L_{\theta'}) = \delta_i \ . \tag{6.16}$$

Indeed, by Riemann-Roch Theorem, one has

$$deg(N_1|_{C_{v_i}}) = -\frac{\beta_{i0}(L_{\theta'})}{s} = (1 - \frac{\delta_i}{s}) + \sum_{j=1}^{l} \frac{\beta_{ij}(L_{\theta'})}{s} \quad mod \quad \mathbb{Z} \ .$$

Here the first term on the left hand is the age of N_1 at the node of C_{v_i}, and the second term on the right is the sum of the ages of N_1 at the marked points on C_{v_i}. As s is sufficiently large, one must have

$$\frac{\delta_i}{s} = \frac{\beta_{i0}(L_{\theta'})}{s} + \sum_{j=1}^{l} \frac{\beta_{ij}(L_{\theta'})}{s} \ ,$$

which implies that $\beta_i(L_{\theta'}) = \delta_i$.

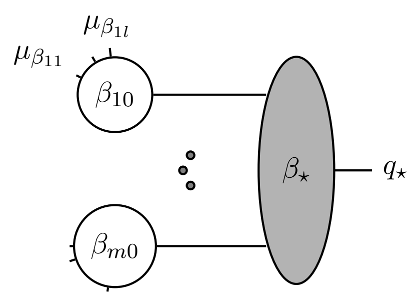

FIGURE 1. The ellipse dubbed gray on the right means the vertex labeled by ∞ with a leg attached, and the two big circles on the left mean vertexes labeled by 0. The text inside the vertex means the decorated degree for this vertex. On the upper left vertex, texts near the legs mean the insertion terms. The three grey dots in the middle mean the other edges (together with its incident vertexes and legs on them) besides edges indexed by 1 and m.

We call a decorated graph Γ admissible if Γ has only one stable vertex v over ∞ of degree $\beta_\star$ and m ($m \geqslant 1$) edges *labeled by* $[m]$ incident to v such that the degree for each vertex over 0 satisfies (6.16). Note our definition of the admissible decorated graph has more decorations than the decorated graph introduced in Section 5 as we also label the edges. Then the automorphism group of an admissible decorated graph Γ is identity, which is usually smaller than the automorphism group of the corresponding decorated graph without labeling the edges. If we want to use admissible decorated graphs to compute the localization contribution, we need to divide $m!$ to offset the labeling as shown below.

Now we can group the admissible decorated graphs by the triple

$$(m, \beta_\star, (\beta_1, \cdots, \beta_m)) \ .$$

Denote by $\Lambda_{(m,\beta_\star,(\beta_1,\cdots,\beta_m))}$ the collection of all the admissible decorated graphs such that the vertex incident to the edge labeled by i has total degree β_i.

Now using the locaization formula in §5.4 to compute the contribution from $\Lambda_{(m,\beta_\star,(\beta_1,\cdots,\beta_m))}$ to (6.11). For any admissible decorated graph Γ in $\Lambda_{(m,\beta_\star,(\beta_1,\cdots,\beta_m))}$, they all have the same localization contribution for vertex and nodes over ∞ and edges, as well as the same automorphism factor which comes from the gerbe structures of the edge moduli. Then the contribution from the vertex v_i together with node at v_i has localization contribution (after pushing forward to $\bar{I}_\mu Y$ along $\iota\circ(ev_{h_i})_*$, where h_i is the node on v_i incident to the edge e_i):

$$\mu_{\beta_i}(\frac{\lambda-D_{\theta'}}{\delta_i})+\sum_{l=0}^{\infty}\sum_{\beta_0+\beta_1+\cdots+\beta_l=\beta_i}\frac{1}{l!}(\widetilde{ev}_0)_*\Bigg(\sum_{d=0}^{\infty}\epsilon'_*(c_d(-R^\bullet\pi_*\mathcal{L}_{-\theta'}^{\frac{1}{s}})(\frac{\lambda}{s})^{-d}$$
$$\cap[\mathcal{K}_{0,[l]\cup\{0\}}(\sqrt[s]{L_{-\theta'}/Y},\beta_0)]^{\mathrm{vir}})\bigcap_{j=1}^{l}ev_j^*(\mu_{\beta_j}(-\bar{\psi}_j))\cap\frac{1}{\frac{\lambda-ev_0^*D_{\theta'}}{\delta_j s}-\frac{\bar{\psi}_0}{s}}\Bigg),$$

which, by Lemma 6.6 below, is equal to $f_{\beta_i}(z)|_{\frac{\lambda-D_{\theta'}}{\delta_i}}$ by our definition of $f_{\beta_i}(z)$. Put contributions from vertexes, edges together and nodes together, it yields

$$\frac{1}{m!}(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*(c_d(-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\cap[\mathcal{K}_{0,[m]\cup\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)]^{\mathrm{vir}})$$
$$\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}(f_{\beta_i}(z)\big|_{z=\frac{\lambda-D_{\theta'}}{\delta_i}})}{-\frac{\lambda-ev_i^*D_{\theta'}}{r\delta_i}-\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\Bigg).$$

Now go over all possible triples $(m,\beta_\star,(\beta_1,\cdots,\beta_m))$, it yields the summation:

$$\sum_{m=1}^{\infty}\sum_{\beta_\star+\beta_1+\cdots+\beta_m}\frac{1}{m!}(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*(c_d(-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d})\cap[\mathcal{K}_{0,\vec{m}\cup\star}(Y,\beta_\star)]^{\mathrm{vir}}$$
$$\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}(f_{\beta_i}(z)\big|_{z=\frac{\lambda-D_{\theta'}}{\delta_i}})}{-\frac{\lambda-ev_i^*D_{\theta'}}{r\delta_i}-\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\Bigg).$$

By the discussion above, we can write (6.11) as the following:

$$\begin{aligned}
&\frac{\mu_\beta(\frac{\lambda-D_{\theta'}}{\delta})}{\delta}\cdot(\frac{\lambda-D_{\theta'}}{\delta})^c+\sum_{m=0}^{\infty}\sum_{\beta_0+\beta_1+\cdots+\beta_m=\beta}\frac{1}{m!}(\widetilde{ev}_0)_*\\
&\left([\mathcal{K}_{0,[m]\cup\{0\}}(Y,\beta_0)]^{\mathrm{vir}}\cap\bigcap_{i=1}^{m}ev_i^*(\mu_{\beta_i}(-\bar{\psi}_i))\bigcap\frac{\frac{1}{\delta_0}(\frac{\lambda-ev_0^*D_{\theta'}}{\delta_0})^c}{\frac{\lambda-ev_0^*D_{\theta'}}{\delta_0}-\bar{\psi}_0}\right)\\
&+(\widetilde{ev}_\star)_*\left(\sum_{d=0}^{\infty}\epsilon_*(c_d(-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1-d}\cap[\mathcal{K}_{0,\star}(\sqrt[r]{L_{\theta'}/Y},\beta)]^{\mathrm{vir}})\cap\bar{\psi}_\star^c\right)\qquad(6.17)\\
&-\sum_{m=1}^{\infty}\sum_{\beta_\star+\beta_1+\cdots+\beta_m}\frac{1}{m!}(\widetilde{ev}_\star)_*\left(\sum_{d=0}^{\infty}\epsilon_*(c_d(-R^\bullet\pi_*\mathcal{L}_{\theta'}^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\right.\\
&\left.\cap[\mathcal{K}_{0,\vec{m}\cup\star}(\sqrt[r]{L_{\theta'}/Y},\beta_\star)]^{\mathrm{vir}})\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}(f_{\beta_i}(z)|_{z=\frac{\lambda-D_{\theta'}}{\delta_i}})}{\frac{\lambda-ev_i^*D_{\theta'}}{r\delta_i}+\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\right).
\end{aligned}$$

As (6.11) lies in $H^*(\bar{I}_\mu Y,\mathbb{Q})[\lambda]$, the coefficient of λ^{-1} term in (6.17) must vanish. Note that the coefficients before λ^{-1} in the first two terms in (6.17) yields (after replacing the index 0 by $\star$)

$$\sum_{m=0}^{\infty}\sum_{\beta_\star+\beta_1+\cdots+\beta_m=\beta}\frac{1}{m!}\phi^\alpha\langle\mu_{\beta_1}(-\bar{\psi}_1),\cdots,\mu_{\beta_m}(-\bar{\psi}_m),\phi_\alpha\bar{\psi}_\star^c\rangle_{0,[m]\cup\star,\beta_\star}^{\infty},$$

which is the left hand side of equality in (6.12). Then extract the coefficient of the λ^{-1} term in the third term in (6.17) and apply (6.10), this yields the first term on the right hand side of (6.12) up to a minus sign. Finally the λ^{-1} coefficient of the fourth term in (6.17) is equal to the second term on the right hand of equality in (6.12) up to a minus sign. This completes the proof of (6.12). □

Lemma 6.6. *Let notations be as above, when s is sufficiently large, one has*

$$\begin{aligned}
&(\widetilde{ev}_0)_*\left(\epsilon'_*(\sum_{d=0}^{\infty}c_d(-R^\bullet\pi_*\mathcal{L}_{-\theta'}^{\frac{1}{s}})(\frac{\lambda}{s})^{-d}\right.\\
&\left.\cap[\mathcal{K}_{0,[m]\cup\{0\}}(\sqrt[s]{L_{-\theta'}/Y},\beta_0)]^{\mathrm{vir}})\cap\gamma\cap\bigcap_{i=1}^{m}ev_i^*(\mu_{\beta_i}(-\bar{\psi}_i))\right)\qquad(6.18)\\
&=\frac{1}{s}(\widetilde{ev}_0)_*\big([\mathcal{K}_{0,[m]\cup\{0\}}(Y,\beta_0)]^{\mathrm{vir}}\cap\gamma\cap\bigcap_{i=1}^{m}ev_i^*(\mu_{\beta_i}(-\bar{\psi}_i))\big)\ ,
\end{aligned}$$

where $\epsilon':\mathcal{K}_{0,[m]\cup\{0\}}(\sqrt[s]{L_{-\theta'}/Y},\beta_0)\to\mathcal{K}_{0,[m]\cup\{0\}}(Y,\beta_0)$ is the natural structure map, γ is any chow cycle in $A^*(\mathcal{K}_{0,[m]\cup\{0\}}(\sqrt[s]{L_{-\theta'}/Y},\beta))_{\mathbb{Q}}$.

Proof. by Remark 6.1, it's equivalent to prove the following

(6.19)
$$(\widetilde{ev}_0)_*\Bigg(\sum_{d=0}^{\infty}\epsilon'_*(c_d(-R^\bullet\pi_*\mathcal{L}_{-\theta'}^{\frac{1}{s}})(\frac{\lambda}{s})^{-d}[\mathcal{K}_{0,\vec{m}\cup\{0\}}(\sqrt[s]{L_{-\theta'}/Y},\beta_0)]^{\mathrm{vir}})\cap\gamma$$
$$\cap\bigcap_{i=1}^m ev_i^*(\mu_{\beta_i}(-\bar{\psi}_i))\Bigg)=\frac{1}{s}(\widetilde{ev}_0)_*\Bigg([\mathcal{K}_{0,\vec{m}'\cup\{0\}}(Y,\beta_0)]^{\mathrm{vir}}\cap\gamma\cap\bigcap_{i=1}^m ev_i^*(\mu_{\beta_i}(-\bar{\psi}_i))\Bigg)\ ,$$

where $\vec{m}=(v_1,\cdots,v_m)\in(G\times\boldsymbol{\mu}_s)^m$ is defined by $v_i=(g_{\beta_i}^{-1},\mu_s^{\beta_i(L_{\theta'})})$ for $1\leqslant i\leqslant m$, and $\vec{m}'=(v_1',\cdots,v_m')\in G^m$ is defined by $v_i'=g_{\beta_i}^{-1}$. Then ϵ' sends $\mathcal{K}_{0,\vec{m}\cup\{0\}}(\sqrt[s]{L_{-\theta'}/Y},\beta_0)$ to $\mathcal{K}_{0,\vec{m}'\cup\{0\}}(Y,\beta_0)$. We will first show that $R^0\pi_*\mathcal{L}_{-\theta'}^{\frac{1}{s}}=0$ on $\mathcal{K}_{0,\vec{m}\cup\{0\}}(\sqrt[s]{L_{-\theta'}/Y},\beta_0)$, which implies that $R^1\pi_*\mathcal{L}_{-\theta'}^{\frac{1}{s}}=0$ as $R^\bullet\pi_*\mathcal{L}_{-\theta'}^{\frac{1}{s}}$ has virtual rank 0 when s is sufficiently large. By Remark 5.4, when $\beta_0\neq 0$, we have $R^0\pi_*\mathcal{L}_{-\theta'}^{\frac{1}{s}}=0$. So it remains to prove the case when $\beta_0=0$. Assume that $\beta_0=0$, since the vertex v labeled by ∞ is stable, there must be some marked points on C_v. Assume q_i is one of the marked points with insertion μ_{β_i}. Without loss of generality, we can assume $\beta_i\neq 0$ for all i as $\mu_0(z)=0$ by the very definition. Note we have
$$age_{q_i}((\mathcal{L}_{-\theta'}^{\frac{1}{s}})|_{C_v})=\frac{\beta_i(L_{\theta'})}{s}\neq 0\ ,$$
then the restricted line bundle $L_{-\theta'}^{\frac{1}{s}}:=(\mathcal{L}_{-\theta'}^{\frac{1}{s}})|_{C_v}$ can't have any nonzero section on C_v. Indeed the degree of the restriction of $L_{-\theta'}^{\frac{1}{s}}$ to every irreducible component is zero by Lemma 2.5 as the total degree β_0 is zero, then a nonzero section of $L_{-\theta'}^{\frac{1}{s}}$ will trivialize the line bundle $L_{-\theta'}^{\frac{1}{s}}$, this contradicts the fact that $L_{-\theta'}^{\frac{1}{s}}$ has nontrivial stacky structure at q_i.

Now as $-R^\bullet\pi_*\mathcal{L}_{-\theta'}^{\frac{1}{s}}=R^1\pi_*\mathcal{L}_{-\theta'}^{\frac{1}{s}}=0$, (6.19) follows immediately from the identity
$$\epsilon'_*([\mathcal{K}_{0,\vec{m}\cup\{0\}}(\sqrt[s]{L_{-\theta'}}/Y,\beta_0)]^{\mathrm{vir}})=\frac{1}{s}[\mathcal{K}_{0,\vec{m}'\cup\{0\}}(Y,\beta_0)]^{\mathrm{vir}}\ ,$$
which is proved in [TT16, Theorem 5.16]. □

6.3. **Proof of genus zero quasimap wall-crossing.** Using the notation in the introduction, we can now prove the quasimap wall-crossing conjecture:

Theorem 6.7. *After reindexing* $\mathrm{Eff}(AY,G,\theta)$ *by* $\mathrm{Eff}(W,G,\theta)$*, one has:*
$$I(q,z)=_{\mathrm{Eff}(AY,G,\theta)\to\mathrm{Eff}(W,G,\theta)}J(q,\mu(q,y),z),\tag{6.20}$$
where $J(q,\mu(q,y),z)$ *is defined by the* J*-function of* Y
$$J(q,\mathbf{t},z):=\mathbb{1}_Y$$
$$+\frac{\mathbf{t}(z)}{z}+\sum_{d\in\mathrm{Eff}(AY,G,\theta)}\sum_{m\geqslant 0}\frac{q^d}{m!}\phi^\alpha\langle\mathbf{t}(-\bar{\psi}_1),\cdots,\mathbf{t}(-\bar{\psi}_m),\frac{\phi_\alpha}{z(z-\bar{\psi}_\star)}\rangle_{0,[m]\cup\star,d}^\infty\tag{6.21}$$

Proof. According to the analysis in the introduction, it suffices to prove the following:

$$[zI_\beta(q,z)]_{z^{-c-1}} = \sum_{m=0}^{\infty} \sum_{\beta_0+\beta_1+\cdots+\beta_m=\beta} \frac{1}{m!}\phi^\alpha \langle \mu_{\beta_1}(-\bar\psi_1), \cdots, \mu_{\beta_m}(-\bar\psi_m), \phi_\alpha \bar\psi_\star^c\rangle^\infty_{0,[m]\cup\star,\beta_0}\ , \tag{6.22}$$

for any nonnegative integer c and degree β. Let's assume that (6.22) is proved for all degree $\beta' \in \mathrm{Eff}(W,G,\theta)$ with $\beta'(L_{\theta'}) < \beta(L_{\theta'})$. Then $f_{\beta_i}(z)$ in (6.12) is equal to $zI_{\beta_i}(z)$ by induction. Indeed, first notice that, in (6.12), $f_{\beta_i}(z)$ appears only for the graph Γ which has stable vertex over ∞ with degree $(\beta_\star, \frac{\delta_\star}{r})$. When $\beta_\star$ is nonzero, it immediately follows that $\beta_i(L_{\theta'}) < \beta(L_{\theta'})$; otherwise, when $\beta_\star = 0$, $f_{\beta_i}(z)$ only appears for the graph Γ where the only vertex v over ∞ is a stable vertex, then there are at least two edges in Γ, which implies that $\beta_i(L_{\theta'}) < \beta(L_{\theta'})$ as each vertex over 0 has nonzero total degree as it's equal to the degree $\delta(e)$ of the edge incident to the vertex by (6.16). Then (6.22) immediately follows from Theorem 6.3 and 6.5. ∎

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Department of Mathematics, The Ohio State Universty, Columbus, OH 43210, USA
E-mail address: wang.6410@osu.edu