

# QUANTUM LEFSCHETZ THEOREM REVISITED

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ABSTRACT. Let  $X$  be any smooth Deligne-Mumford stack with projective coarse moduli, and  $Y$  be a smooth complete intersection in  $X$  associated with a direct sum of semi-positive line bundles. We will introduce a useful and broad class known as admissible series for discussing quantum Lefschetz theorem. For any admissible series on the Givental's Lagrangian cone of  $X$ , we will show that a hypergeometric modification of the series lies on the Lagrangian cone of  $Y$ . This confirms a prediction from Coates-Corti-Iritani-Tseng about the genus zero quantum Lefschetz theorem beyond convexity. In our quantum Lefschetz theorem, we use extended variables to formulate the hypergeometric modification, which may be of self-independent interest.

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## 1. INTRODUCTION

*Gromov-Witten (GW) invariants* count the (virtual) number of stable maps from curves to a target variety/stack  $X$  with prescribed conditions. *Quantum Lefschetz* is one of the central topics in Gromov-Witten theory and it compares the GW invariants of a complete intersection and its ambient space. On the other hand, in Givental's formalism[Giv04, CG07, Tse10], all information of genus-zero Gromov-Witten invariants is encoded in an *overruled Lagrangian cone*  $\mathcal{L}_X$  sitting inside an infinite dimensional symplectic vector space. Consequently, a natural approach to the quantum Lefschetz problem in genus zero is to find an explicit slice on Givental's Lagrangian cone of the complete intersection using a given slice on Givental's Lagrangian cone of the ambient space. This approach is often referred to as a *mirror theorem* in the literature. The earliest quantum Lefschetz theorem can be traced back to the verification of the famous mirror conjecture for quintic threefolds[CDLOGP91, Giv96, LLY99]. Since then, more cases related to quantum Lefschetz have been proved. In genus zero, Givental-style quantum Lefschetz theorems can be divided into two categories:

- (1) The complete intersection  $Y$  is associated with a direct sum of *convex* line bundles over the ambient space  $X$ , see e.g.,[Giv98, Lee01, CG07, CCIT19]. The reason for requiring

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the convexity condition is that we need to apply the so-called Quantum Lefschetz principle proved in [KKP03]. This principle expresses the Gromov-Witten invariants of the complete intersection as an Euler class of a certain bundle over the moduli stack of genus-zero stable maps to  $X$ . See also [CGI<sup>+</sup>12] for some examples where this principle can fail if we drop the convexity condition, where the examples are associated with semi-positive line bundles in the sense of this paper. We note that convexity is a very rare condition when  $X$  is a not a variety.

- (2) When convexity fails, proving a quantum Lefschetz theorem is much more difficult. There has been significant progress on this case recently by solving the genus zero quasimap wall-crossing conjecture [Wan19, Zho22]. However, it's important to note that we require the ambient space to be a GIT quotient, which is a prerequisite for applying quasimap theory [CFKM14, CCFK15].

The main objective of this paper is to prove a genus-zero<sup>1</sup> quantum Lefschetz theorem that goes beyond the scope of the two categories mentioned previously, where the ambient space can be a non-GIT target and the convexity condition may not hold. Actually our scope of quantum Lefschetz theorem is quite broad, there are not many restrictions on the ambient space and we only assume that the complete intersection is associated with semi-positive line bundles, which is much weaker than convexity (see [CGI<sup>+</sup>12]). The proof relies on (and generalizes) the *recursive relation* of a point on the Givental's Lagrangian cone discovered by the author in [Wan19], which was used to demonstrate the genus-zero quasimap wall-crossing conjecture for abelian GIT quotients. However, we need to carefully pick a space carried with a  $\mathbb{C}^*$ -action which could provide the recursive relation we want; in our case we will use a *root-stack modification of the space of deformation to the normal cone*. This is *different* from the spaces used in loc.cit and it has the advantage of generalizing the quantum Lefschetz theorem proved in loc.cit to non-GIT targets.

**1.1. Main theorem.** Let  $X$  be a smooth Deligne-Mumford stack over  $\mathbb{C}$  with projective coarse moduli. Let  $\bar{I}_\mu X$  be the rigidified (cyclotomic) inertia stack of  $X$ , where a  $\mathbb{C}$ -point of  $\bar{I}_\mu X$  can be written as a pair  $(x, g)$  where  $x$  is a  $\mathbb{C}$ -point of  $X$  and  $g$  is an element in the isotropy group  $\text{Aut}(x)$ . Denote by  $C$  the finite set of connected components of  $\bar{I}_\mu X$  with  $\bar{I}_c X$  being the component corresponding to the element  $c \in C$ . The involution on  $\bar{I}_\mu X$  by sending  $(x, g)$  to  $(x, g^{-1})$  also induces an involution on  $C$ , we write  $c^{-1}$  to be image of  $c$  under the involution.

The *overruled Lagrangian cone*  $\mathcal{L}_X$  introduced by A. Givental is comprised by the so-called (big)  $J$ -function:

$$(1.1) \quad J^X(q, \mathbf{t}(z), -z) := -z \mathbb{1}_X + \mathbf{t}(z) + \sum_{\beta \in \text{Eff}(X)} \sum_{m \geq 0} \frac{q^\beta}{m!} \phi^\alpha \langle \mathbf{t}(\bar{\psi}_1), \dots, \mathbf{t}(\bar{\psi}_m), \frac{\phi_\alpha}{-z - \bar{\psi}_\star} \rangle_{0, [m] \cup \star, \beta}^X,$$

where the input<sup>2</sup>  $\mathbf{t}(z) \in H_{CR}^*(X, \mathbb{C})[z]$  is a polynomial in  $z$  with coefficients in the Chen-Ruan cohomology  $H_{CR}^*(X, \mathbb{C}) = H^*(\bar{I}_\mu X, \mathbb{C})$ , the notation  $q^\beta$  stands for the Novikov variable corresponding to the degree  $\beta$  in the cone  $\text{Eff}(X)$  of effective curve classes of  $X$ . We note that such a choice of the input  $\mathbf{t}(z)$  usually lead to some (rather mild) convergence issue about  $J$ -function. One way to handle this is using formal geometry; let  $N$  be any positive integer,

<sup>1</sup>Recently, there have been significant advance in the high genus case, where the failure of convexity provides one main obstacle to the computation of high genus GW invariants, see e.g., [Zin08, GJR17, GJR18, CGL21, CJR22, LR22, CJR21] and their references therein.

<sup>2</sup>We will also use the notation  $\mu(z)$  in this paper.

we will choose our input  $\mathbf{t}$  in the  $\mathbb{Z}_2$ -graded space

$$(q, t_1, \dots, t_N) H^*(\bar{I}_\mu X, \mathbb{C})[z][[t_1, \dots, t_N]] [\text{Eff}(X)] ,$$

where each variable  $t_i$  is associated with a grading in  $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ . More precisely,  $\mathbf{t}(z)$  can be written in the form of

$$(1.2) \quad \sum_{\substack{\beta \in \text{Eff}(X) \\ \vec{k} = (k_1, \dots, k_N) \in \mathbb{Z}_{\geq 0}^N}} q^\beta t_1^{k_1} \dots t_N^{k_N} f_{\beta, \vec{k}}(z) ,$$

where  $f_{\beta, \vec{k}}(z) \in H^*(\bar{I}_\mu X, \mathbb{C})[z]$ ,  $f_{0, \vec{0}}(z) = 0$  and  $t_1^{k_1} \dots t_N^{k_N} f_{\beta, \vec{k}}(z)$  is of even grading. See §2 for more details.

Let  $Y \subset X$  be a smooth complete intersection associated with a direct sum of semi-positive line bundles  $\oplus_{j=1}^r L_j$ , i.e., the pairing  $\beta(L_j) := (c_1(L_j), \beta) \geq 0$  for any degree  $\beta \in \text{Eff}(X)$ . Then the age function  $\text{age}_g(L_j|_x)$  for each  $(x, g) \in \bar{I}_c X$  is constant on each connected component  $\bar{I}_c X$  of  $\bar{I}_\mu X$  and we will denote the constant value to be  $\text{age}_c(L_j)$ . To state the main theorem in this paper, we will introduce a useful category for discussing the quantum Lefschetz known as *admissible series*. This concept summarizes a common feature of all previous proved  $J$ -functions about quantum Lefschetz. Here we will describe the content about admissible series needed in the statement of the main theorem, see §2 for a more general definition of admissible series.

Endow each variable  $t_i$  and line bundle  $L_j$  with a weight  $w_{ij} \in \mathbb{Q}_{\geq 0}$ . We will call a tuple  $(\beta, \vec{k} = (k_1, \dots, k_N), c) \in \text{Eff}(X) \times \mathbb{Z}_{\geq 0}^N \times C$  an *admissible pair* if

$$\text{age}_c(L_j) \equiv \beta(L_j) + \sum_{i=1}^N w_{ij} k_i \pmod{\mathbb{Z}}$$

for all line bundles  $L_j$ . Denote by  $\text{Adm}$  the set of all admissible pairs. Write each component  $f_{\beta, \vec{k}}(\mathbf{t}, z)$  in 1.2 as a sum

$$f_{\beta, \vec{k}}(z) = \sum_{c \in C} f_{\beta, \vec{k}, c}(z)$$

where  $f_{\beta, \vec{k}, c}(z)$  belongs to the space  $H^*(\bar{I}_{c-1} X, \mathbb{C})[z]$ . Then we call the input  $\mathbf{t}(z)$  an *admissible (power) series* if  $\mathbf{t}(z)$  can be written as in 1.2 and  $f_{\beta, \vec{k}, c}(z) = 0$  whenever  $(\beta, \vec{k}, c)$  is *not* an admissible pair.

Let  $J^X(q, \mathbf{t}, -z)$  be a point on the Lagrangian cone  $\mathcal{L}_X$  of  $X$  with  $\mathbf{t}(z)$  being an admissible (power) series. Following the tradition in the literature, to prove a quantum Lefschetz theorem, it's more convenient to flip the sign of  $z$  in the function  $J(q, \mathbf{t}, -z)$ ; in our case,  $J^X(q, \mathbf{t}, z)$  will always have an asymptotic expansion in variables  $q^\beta t_1^{k_1} \dots t_N^{k_N}$  as:

$$z + \sum_{(\beta, \vec{k}, c) \in \text{Adm}} q^\beta t_1^{k_1} \dots t_N^{k_N} J_{\beta, \vec{k}, c}^X(\mathbf{t}, z) ,$$

where  $J_{\beta, \vec{k}, c}^X(\mathbf{t}, z) \in H^*(\bar{I}_{c-1} X, \mathbb{C})[z, z^{-1}]$ . Define  $J^{X, tw}(q, \mathbf{t}, z)$  to be the hypergeometric modification of  $J^X(q, \mathbf{t}, z)$ :

$$(1.3) \quad \begin{aligned} J^{X, tw}(q, \mathbf{t}, z) &:= z + \sum_{(\beta, \vec{k}, c) \in \text{Adm}} q^\beta t_1^{k_1} \dots t_N^{k_N} J_{\beta, \vec{k}, c}^X(\mathbf{t}, z) \\ &\cdot \prod_{j=1}^r \prod_{0 \leq m < \beta(L_j) + \sum_i w_{ij} k_i} (c_1(L_j) + (\beta(L_j) + \sum_i w_{ij} k_i - m)z) . \end{aligned}$$

Our main theorem in this paper will be the following:

**Theorem 1.1** (=Theorem 4.6). *Let  $i : \bar{I}_\mu Y \rightarrow \bar{I}_\mu X$  be the inclusion of rigidified inertia stacks, then the series  $i^* J^{X,tw}(q, \mathbf{t}, -z)$  lies on the Lagrangian cone  $\mathcal{L}_Y$  of  $Y$ .*

Here we are suppressing a change about the  $J$ -function defining the Lagrangian cone  $\mathcal{L}_Y$  of  $Y$  by using Novikov variables from  $\text{Eff}(X)$  rather than  $\text{Eff}(Y)$ , see §3.4 for more details.

Even when the convexity holds, our theorem still have some new results compared to previous quantum Lefschetz theorems. Our selection of the variables  $t_i$  with their respective weights  $w_{ij}$  plays the same role as the Novikov variables  $q^\beta$  with the corresponding numbers  $\beta(L_j)$ , which resembles much similarity with the usage of extended degrees for  $S$ -extended  $I$ -functions of toric stacks in [CCIT15, CCIT19]. Therefore we will call the pair  $(\beta, \vec{k})$  an *extended degree* and call  $t_i$  *extended variables*. However the appearance of extended variables beyond toric cases is new. We note that introducing extended variables in loc.cits is a vital aspect to establish a Laudau-Ginzburg mirror for the big (equivariant) quantum ring of toric stacks (see [CCI<sup>+</sup>20]). We expect our theorem will have the similar mirror result (at least for ambient cohomology) and we will return to this question in the future. It's worth noting that the method of introducing extended variables in loc.cits cannot generalized to general cases as it relies on the  $S$ -extended fan structure for toric stacks in loc.cits (see also [Jia08]). This reflects a novel aspect of our method.

**1.2. Relation to twisted Gromov-Witten invariants.** We will discuss a relationship of our quantum Lefschetz theorem to twisted Gromov-Witten invariants, in particular a conjecture of Coates-Corti-Iritani-Tseng.

Consider the  $\mathbb{C}^*$ -action on the vector bundle  $E := \oplus_{j=1}^r L_j$  via scaling the fibers. The  $\mathbb{C}^*$ -equivariant Euler class of  $E$  can be written in term of the Chern roots  $c_1(L_j)$  as

$$e^{\mathbb{C}^*}(E) := \prod_j (\kappa + c_1(L_j)) ,$$

where  $\kappa$  be the equivariant parameter corresponding to the standard representation of  $\mathbb{C}^*$ . Then one can define a *twisted Lagrangian cone*  $\mathcal{L}_X^{tw}$  using  $(e^{\mathbb{C}^*}, E)$ -*twisted Gromov-Witten invariants* (See [CG07] and [Tse10] for more details).

The series  $J^X(q, \mathbf{t}, z)$  in our main theorems are related to points on the twisted Lagrangian cone  $\mathcal{L}_X^{tw}$  in the following way: we can associate  $J^X$  a series defined by

$$(1.4) \quad \begin{aligned} I^{X,tw} &:= z + \sum_{(\beta, \vec{k}, c) \in \text{Adm}} q^\beta t_1^{k_1} \cdots t_N^{k_N} J_{\beta, \vec{k}, c}^X(\mathbf{t}, z) \\ &\cdot \prod_{j=1}^r \prod_{0 \leq m < \beta(L_j) + \sum_i w_{ij} k_i} \kappa + (c_1(L_j) + (\beta(L_j) + \sum_i w_{ij} k_i - m)z) . \end{aligned}$$

Then we have

$$\lim_{\kappa \rightarrow 0} I^{X,tw} = J^{X,tw} .$$

Moreover, using the same idea in [CCIT19, Theorem 22] where we view our  $t_i$  here as their  $x_i$  in loc.cit, one can prove that  $I^{X,tw}$  lies on the twisted Lagrangian cone  $\mathcal{L}_X^{tw}$ . In general, Coates-Corti-Iritani-Tseng make the following conjecture (see [OP18, Conjecture 5.2]):

**Conjecture 1.2.** *Let  $X$  be a smooth proper Deligne-Mumford stack with projective coarse moduli and  $Y$  is a smooth complete intersection<sup>3</sup> of  $X$ . Let  $I^{tw}$  be a point on the twisted Lagrangian cone  $\mathcal{L}_X^{tw}$  of  $X$ , if the limit  $\lim_{\kappa \rightarrow 0} i^* I^{tw}$  exists, then  $\lim_{\kappa \rightarrow 0} i^* I^{tw}$  is a point on the Lagrangian cone  $\mathcal{L}_Y$  of  $Y$ .*

<sup>3</sup>Note that the conjecture doesn't require the line bundles defining the complete intersection to be semi-positive.

This conjecture is known to be false in general, where a counterexample is found in [SW22]. However we can still take this as a guiding principle towards genus zero quantum Lefschetz theorem in general case and we are able to verify this conjecture under mild hypotheses as in Theorem 4.6.

**1.3. Sketch of the main idea of the proof.** The proof will first reduce to the hypersurface case. Set  $\mu^{X,tw} := [J^{X,tw}(q, \mathbf{t}, z) - z]_+$  to be the truncation in nonnegative  $z$ -powers. Note that the  $J$ -function  $J^Y(q, \mathbf{t}, z)$  is of the form  $z + \mathbf{t} + \mathcal{O}(z^{-1})$ , we observe that, to prove the main theorem, we need to show that

$$J^Y(q, i^* \mu^{X,tw}, z) = i^* J^{X,tw}(q, \mathbf{t}, z),$$

for which we only need to consider their *negative  $z$ -powers*. We aim to demonstrate that both sides of the negative  $z$ -powers of the above equation satisfy the *same recursive relations* in each degree corresponding to  $q^\beta t_1^{k_1} \cdots t_N^{k_N}$  except a special consideration in low degrees (see Lemma 4.7). First we will consider two auxiliary spaces carried with  $\mathbb{C}^*$ -actions, which are root-stack modifications of a certain orbi- $\mathbb{P}^1$  bundle over  $Y$  (see §3.1) and the space of deformation to the normal cone (see §4.1). We then apply virtual localization to express two auxiliary cycles (see (3.7) and (4.3)) corresponding to the two spaces in graph sums and extract  $\lambda^{-1}$  coefficient and  $\lambda^{-2}$  coefficient respectively (where  $\lambda$  is an equivariant parameter). Finally, the polynomiality of the two auxiliary cycles implies that the coefficients must vanish, leading to the same type of recursive relations (see also Proposition 3.4 and Theorem 4.2).

**1.4. Outline.** The rest of this paper is organized as follows. In §2, we introduce the concept of admissible series in more details than the in the introduction and collect some background on Gromov-Witten theory. In §3, we will give a recursive relation of an admissible series on the Lagrangian cone. Based on this recursive relation, in §4, we will prove our main theorem. In the Appendix, we carry out a detailed computation about edge contributions needed in the localization formula in our proof.

**1.5. Notation and convention.** We work over the field  $\mathbb{C}$ . Let  $D$  be an effective divisor of a smooth stack  $X$ , we will denote the  $\mathcal{O}(D)$  or  $\mathcal{O}_X(D)$  to be the line bundle associated with  $D$ .

For any positive integer  $i$ , we will use the notation  $\mu_i$  to mean the finite cyclic subgroup of  $\mathbb{C}^*$  of order  $i$ , and use the notation  $[i]$  to mean the set  $\{1, 2, \dots, i\}$ . For any rational number  $a$ , we will use the notation  $e^a$  for to mean the exponential  $\exp(\frac{\sqrt{-1}a}{2\pi})$ .

Let  $\lambda$  be the equivariant class in  $H_{\mathbb{C}^*}^2(\{\text{Spec}(\mathbb{C})\})$  corresponding to the stand representation of  $\mathbb{C}^*$ . For any rational number  $q$ , we will write  $\mathbb{C}_{q\lambda}$  to be the trivial line bundle with a  $\mathbb{C}^*$ -action of weight  $q$ .

## 2. BACKGROUND ON GROMOV-WITTEN THEORY

**2.1. Admissible series.** Let  $X$  be a smooth DM stack over  $\mathbb{C}$  with projective coarse moduli and  $\bar{I}_\mu X := \sqcup_{a \geq 1} \bar{I}_{\mu_a} X$  be the associated rigidified (cyclotomic) inertia stack. For our purpose, a finer decomposition of the rigidified inertia stack  $\bar{I}_\mu X$  is needed. Given a set of line bundles  $L_1, \dots, L_r$  over  $X$ , we will choose a decomposition of the rigidified inertia stack into open-closed components

$$\bar{I}_\mu X := \bigsqcup_{c \in C} \bar{I}_c X$$

where  $C$  is a *finite* index set for the decomposition which satisfies the following assumption:

**Assumption 2.1.** For each  $c \in C$  and every  $\mathbb{C}$ -point  $(x, g) \in \bar{I}_c X$ , where  $x$  is a  $\mathbb{C}$ -point of  $X$  and  $g$  is an element in the isotropy group  $\text{Aut}(x)$ , we will assume that the age function<sup>4</sup>  $\text{age}_g L_j|_x$  and the order function  $\text{ord}(g)$  are all constant functions on  $\bar{I}_c X$ . Then we can define  $\text{age}_c L_j := \text{age}_g L_j|_x$  and  $a(c) := \text{ord}(g)$ , which are well-defined as they are independent of the choice of the point  $(x, g) \in \bar{I}_c X$ .

Moreover, we require that the index set  $C$  has an involution which is compatible with the involution  $\iota : \bar{I}_\mu X \rightarrow \bar{I}_\mu X$  sending  $(x, g)$  to  $(x, g^{-1})$  for any  $\mathbb{C}$ -point  $(x, g)$  of  $\bar{I}_\mu X$ ; namely, for each index  $c \in C$ , there exists a unique index in  $C$  and we will denote it to be  $c^{-1}$  such that  $\iota(\bar{I}_c X) = \bar{I}_{c^{-1}} X$ .

We will call an index set satisfying the above assumption a *good index set*.

**Remark 2.2.** We note that a good index set  $C$  always exists (but may not unique), e.g., we can decompose  $\bar{I}_\mu X$  into connected components, which corresponds to the biggest good index set. When  $X$  is a quotient stack  $[W/G]$  where  $W$  is a connected affine scheme and  $G$  is a finite group, then we can choose the conjugacy class  $\text{Conj}(G)$  of  $G$  to be the index set  $C$ .

When  $Y$  is a substack of  $X$ , we will usually use  $C$  to index  $\bar{I}_\mu Y$  as well, which is also a good index set with respect to the restriction of line bundles  $L_1, \dots, L_r$  to  $Y$ .

Let  $m$  be a nonnegative integer, for an (ordered) tuple  $\vec{m} = \{c_1, \dots, c_m\} \in C^m$ , we will denote

$$\mathcal{K}_{g, \vec{m}}(X, \beta) := \mathcal{K}_{g, m}(X, \beta) \cap \bigcap_{i=1}^m \text{ev}_i^{-1}(\bar{I}_{c_i} X) .$$

Here  $\mathcal{K}_{g, m}(X, \beta)$  is the moduli stack of genus  $g$  twisted stable maps to  $X$  with  $m$  (not necessarily trivialized) gerby-marked points of degree  $\beta$  as defined in [AV02] and  $\text{ev}_i : \mathcal{K}_{g, m}(X, \beta) \rightarrow \bar{I}_\mu X$  is the evaluation map at the  $i$ th marking. Then  $\mathcal{K}_{g, m}(X, \beta)$  can be written as a disjoint union

$$\mathcal{K}_{g, m}(X, \beta) := \bigsqcup_{\vec{m} \in C^m} \mathcal{K}_{g, \vec{m}}(X, \beta) .$$

Sometimes, we will use the notation  $[m] \cup \star$  (resp.  $\vec{m} \cup \star$ ) to mean  $[m+1]$  (resp.  $\overrightarrow{m+1}$ ) to distinguish the  $m+1$ -th marking, which we denote to be  $\star$ .

Now we introduce *admissible pairs*:

**Definition 2.3.** Given a set of line bundles  $L_1, \dots, L_r$  over  $X$ , a positive integer  $N$  and  $r$  weights  $\vec{w}_j = (w_{1j}, \dots, w_{Nj}) \in \mathbb{Q}_{\geq 0}^N$  corresponding to each  $L_j$ . Let  $C$  be a good index set. Let  $c \in C$ ,  $\vec{k} = (k_1, \dots, k_N) \in \mathbb{Z}_{\geq 0}^N$  and  $\beta \in \text{Eff}(X)$ , where  $\text{Eff}(X)$  is the cone of effective curve classes of  $X$ . we will call  $(\beta, \vec{k}, c)$  an *admissible pair* if and only if

$$\text{age}_c(L_j) \equiv \beta(L_j) + (\vec{w}_j, \vec{k}) \bmod \mathbb{Z}$$

for all line bundles  $L_j$ , where  $(\vec{w}_j, \vec{k}) = \sum_i w_{ij} k_i$ . We will denote  $\text{Adm}$  to be the set of all admissible pairs.

**Definition 2.4.** Let

$$f(z) = \sum_{(\beta, \vec{k}, c) \in \text{Eff}(X) \times \mathbb{Z}_{\geq 0}^N \times C} q^\beta t_1^{k_1} \dots t_N^{k_N} f_{\beta, \vec{k}, c}(z) ,$$

be a formal series in  $H^*(\bar{I}_\mu X, \mathbb{C})[z, z^{-1}][[t_1, \dots, t_N][[\text{Eff}(X)]]$  in which

$$f_{\beta, \vec{k}, c} \in H^*(\bar{I}_{c^{-1}} X, \mathbb{C})[z, z^{-1}] .$$

<sup>4</sup>Recall that if  $g$  acts on the fiber of  $L$  over  $x$  by multiplication by the number  $e^q$ , where  $q \in \mathbb{Q}$ , then we define the age  $\text{age}_g(L_j|_x)$  to be  $\langle q \rangle$ , which is the fractional part of  $q$ .

We further put a (not unique!)  $\mathbb{Z}_2(=\mathbb{Z}/2\mathbb{Z})$ -grading on each variable  $t_i$ , which we denote the grading to be  $\bar{t}_i$ . Note that we also require the super-commutativity:

$$t_i t_j = (-1)^{\bar{t}_i \bar{t}_j} t_j t_i .$$

We will put a  $\mathbb{Z}_2$ -grading on  $H^*(\bar{I}_\mu X, \mathbb{C})$ , which is induced from its ordinary cohomological grading, and put even grading on Novikov variables  $q^\beta$  and  $z$ .

We will call  $f(z)$  an admissible series if  $f(z)$  satisfies the following condition.

- (1)  $f_{\beta, \vec{k}, c}(z) = 0$  whenever  $(\beta, \vec{k}, c)$  is not an admissible pair;
- (2)  $f_{0, \vec{0}, c}(z) = 0$  for all  $c \in C$ .
- (3)  $f_{\beta, \vec{k}, c}(z)$  belongs to the space  $H^{\sum_i k_i \bar{t}_i}(\bar{I}_{c-1} X, \mathbb{C})[z]$ , i.e., when  $\sum_i k_i \bar{t}_i = 0 \in \mathbb{Z}_2$ ,  $H^{\sum_i k_i \bar{t}_i}(\bar{I}_{c-1} X, \mathbb{C})$  means the even degree part of the cohomology group of  $\bar{I}_{c-1} X$ ; when  $\sum_i k_i \bar{t}_i = 1 \in \mathbb{Z}_2$ ,  $H^{\sum_i k_i \bar{t}_i}(\bar{I}_{c-1} X, \mathbb{C})$  means the odd degree part of the cohomology group of  $\bar{I}_{c-1} X$ .

Furthermore, let  $g \in \mathbb{C}[z]$ , we say a formal series  $g + f(z)$  is an admissible series near  $g$  if  $f(z)$  is an admissible series as above.

For any pair  $(\beta, \vec{k})$  in  $\text{Eff}(X) \times \mathbb{Z}_{\geq 0}^N$ , we will write  $q^\beta \mathbf{t}^{\vec{k}} := q^\beta t_1^{k_1} \cdots t_N^{k_N}$  and call  $(\beta, \vec{k})$  an extended degree.

**Remark 2.5.** Examples of admissible series include  $J$ -functions[Giv04] and  $I$ -functions in quasimap theory[CCFK15, CFK14, Web18, Web21, Wan19]. Therefore admissible series consists of a large class of objects studied in Gromov-Witten theory.

**2.2. Background on orbifold Gromov-Witten theory.** Now assume that  $X$  carries a algebraic torus  $T$  action (can be trivial), we can define the so-called *Chen-Ruan cohomology* of  $X$ , Given any two elements  $\alpha_1, \alpha_2$  in the  $T$ -equivariant

$$H_{\text{CR}, T}^*(X, \mathbb{C}) := H_T^*(\bar{I}_\mu X, \mathbb{C}) ,$$

We can define the Poincaré pairing in the *non-rigidified* inertia stack  $I_\mu X$  of  $X$ :

$$\langle \alpha_1, \alpha_2 \rangle_{\text{orb}} := \int_{\sum_{c \in C} a(c)^{-1} [\bar{I}_c X]} \alpha_1 \cdot \iota^* \alpha_2 .$$

Here  $\iota$  is the involution of  $\bar{I}_\mu X$  obtained from the inversion automorphisms of the band. Therefore, the diagonal class  $[\Delta_{\bar{I}_c X}]$  obtained via push-forward of the fundamental class by  $(\text{id}, \iota) : \bar{I}_c X \rightarrow \bar{I}_c X \times \bar{I}_{c-1} X$  can be written as

$$\sum_{c \in C} a(c) [\Delta_{\bar{I}_c X}] = \sum_{\alpha} \phi_\alpha \otimes \phi^\alpha \text{ in } H_T^*(\bar{I}_\mu X \times \bar{I}_\mu X, \mathbb{C}),$$

where  $\{\phi_\alpha\}$  is a basis of  $H_{\text{CR}, T}^*(X, \mathbb{C})$  with  $\{\phi^\alpha\}$  the dual basis with respect to the Poincaré pairing defined above. Set  $g_{\alpha\beta} = \langle \phi_\alpha, \phi_\beta \rangle_{\text{orb}}$  and  $g^{\alpha\beta} = \langle \phi^\alpha, \phi^\beta \rangle_{\text{orb}}$ .

Denote by  $\bar{\psi}_i$  the first Chern class of the universal cotangent line whose fiber at  $((C, q_1, \dots, q_m), [x])$  is the cotangent space of the coarse moduli  $\underline{C}$  of  $C$  at  $i$ -th marking  $\underline{q}_i$ . For non-negative integers  $a_i$  and classes  $\alpha_i \in H_T^*(\bar{I}_\mu X, \mathbb{C})$ , using the virtual cycle  $[\mathcal{K}_{0, \vec{m}}(X, \beta)]^{\text{vir}}$  defined in [LT98, BF97, AGV08], we write the Gromov-Witten invariant:

$$\langle \alpha_1 \bar{\psi}^{a_1}, \dots, \alpha_m \bar{\psi}^{a_m} \rangle_{g, \vec{m}, \beta}^X := \int_{[\mathcal{K}_{g, \vec{m}}(X, \beta)]^{\text{vir}}} \prod_i \text{ev}_i^*(\alpha_i) \bar{\psi}_i^{a_i} .$$

When the tuple  $(g, \vec{m}, \beta)$  gives rise to an empty stack  $\mathcal{K}_{g, \vec{m}}(X, \beta)$ , we define the above integral is zero.

We will also need the stablemap Chen-Ruan classes

$$(2.1) \quad (\widetilde{ev}_j)_* = \iota_*(\mathbf{r}_j(\text{ev}_j)_*),$$



where  $r_j$  is the order function of the band of the gerbe structure at the marking  $q_j$ . Define a class in  $H_*^T(\bar{I}_\mu X) \cong H_T^*(\bar{I}_\mu X)$  by

$$\begin{aligned} \langle \alpha_1, \dots, \alpha_m, - \rangle_{0,\beta}^X &:= (\widetilde{ev_{m+1}})_* \left( \left( \prod ev_i^* \alpha_i \right) \cap [Q_{0,m}^\epsilon(X, \beta)]^{\text{vir}} \right) \\ &= \sum_{\alpha} \phi^\alpha \langle \alpha_1, \dots, \alpha_m, \phi_\alpha \rangle_{0,m+1,\beta}^X. \end{aligned}$$

### 3. A RECURSIVE RELATION ABOUT THE LAGRANGIAN CONE

**3.1. A root-stack modification of the twisted graph space.** Let  $Y$  be any smooth Deligne-Mumford stack with projective coarse moduli. Let  $L$  be a semi-positive line bundle over  $Y$ , i.e.,  $\beta(L) \geq 0$  for any degree  $\beta \in \text{Eff}(Y)$ . Denote by  $\mathbb{P}Y_{r,s}$  the root stack of the projective bundle (also named twisted graph space<sup>5</sup>)  $\mathbb{P}_Y(L^\vee \oplus \mathbb{C})$  over  $Y$  by taking  $s$ -th root of the zero section  $D_0 := \mathbb{P}(0 \oplus \mathbb{C})$  and  $r$ -th root of the infinity section  $D_\infty := \mathbb{P}(L^\vee \oplus 0)$ . Let  $\mathcal{D}_0$  and  $\mathcal{D}_\infty$  be the corresponding root divisors of  $D_0$  and  $D_\infty$  respectively, then the zero section  $\mathcal{D}_0 \subset \mathbb{P}Y_{r,s}$  is isomorphic to the root stack  $\sqrt[s]{L^\vee/Y}$  with normal bundle isomorphic to the root bundle  $(L^\vee)^{\frac{1}{s}}$ , and the infinity section  $\mathcal{D}_\infty \subset \mathbb{P}Y_{r,s}$  is isomorphic to the root stack  $\sqrt[r]{L/Y}$  with normal bundle isomorphic to the root bundle  $L^{\frac{1}{r}}$ .

There are two morphisms

$$q_0, q_\infty : \mathbb{P}Y_{r,s} \rightarrow \mathbb{B}\mathbb{C}^*,$$

associated to the line bundles  $\mathcal{O}(\mathcal{D}_0)$  and  $\mathcal{O}(\mathcal{D}_\infty)$  respectively such that  $\mathcal{O}(\mathcal{D}_0) \cong q_0^*(\mathbb{L})$  and  $\mathcal{O}(\mathcal{D}_\infty) \cong q_\infty^*(\mathbb{L})$ , where  $\mathbb{L}$  is the universal line bundle over  $\mathbb{B}\mathbb{C}^*$ . This will induce morphisms on their rigidified inertia stack counterparts:

$$q_0, q_\infty : \bar{I}_\mu \mathbb{P}Y_{r,s} \rightarrow \bar{I}_\mu \mathbb{B}\mathbb{C}^*,$$

Note the rigidified inertia stack  $\bar{I}_\mu \mathbb{B}\mathbb{C}^*$  of  $\mathbb{B}\mathbb{C}^*$  can be written as the disjoint union

$$\bigsqcup_{c \in \mathbb{C}^*} \bar{I}_c \mathbb{B}\mathbb{C}^*.$$

Let  $c$  be a complex number, any  $\mathbb{C}$ -point of  $\bar{I}_c \mathbb{B}\mathbb{C}^*$  is isomorphic to the pair  $(1_{\mathbb{C}}, c)$ , where  $1_{\mathbb{C}}$  is the trivial principal  $\mathbb{C}$ -bundle over  $\mathbb{C}$  and  $c$  is an element in the automorphism group  $\text{Aut}_{\mathbb{C}}(1_{\mathbb{C}}) \cong \mathbb{C}^*$ . Let  $(y, g)$  be a  $\mathbb{C}$ -point of  $\bar{I}_\mu \mathbb{P}Y_{r,s}$ , where  $y \in \text{Ob}(\mathbb{P}Y_{r,s}(\mathbb{C}))$  and  $g \in \text{Aut}(y)$ . Assume that  $q_0((y, g)) = (1_{\mathbb{C}}, c)$ , then  $g$  acts on the fiber  $\mathcal{O}(\mathcal{D}_0)|_y$  via the multiplication by  $c$ . We have a similar relation for  $q_\infty$ .

Let  $\text{pr}_{r,s} : \mathbb{P}Y_{r,s} \rightarrow Y$  be the projection to the base, which is a composition of deroot-stackification and projection from  $\mathbb{P}_Y(L^\vee \oplus \mathbb{C})$  to  $Y$ .

Now choose a good index set  $C$  for  $\bar{I}_\mu Y$  satisfying the assumption 2.1. For each  $c \in C$ ,  $c_0, c_\infty \in \mathbb{C}^*$ , we will define the rigidified inertia component  $\bar{I}_{(c, c_0, c_\infty)} \mathbb{P}Y_{r,s}$  to be

$$\text{pr}_{r,s}^{-1}(\bar{I}_c Y) \cap q_0^{-1}(\bar{I}_{c_0} \mathbb{B}\mathbb{C}^*) \cap q_\infty^{-1}(\bar{I}_{c_\infty} \mathbb{B}\mathbb{C}^*).$$

We also denote  $(c, c_0, c_\infty)^{-1} = (c^{-1}, c_0^{-1}, c_\infty^{-1})$ .

**Definition 3.1.** For any degree  $\beta \in \text{Eff}(Y)$  and rational number  $\delta \in \mathbb{Q}_{\geq 0}$ , we say a stable map  $f : C \rightarrow \mathbb{P}Y_{r,s}$  is of degree  $(\beta, \frac{\delta}{r})$  if  $(\text{pr}_{r,s} \circ f)_*[C] = \beta$  and  $\deg(f^* \mathcal{O}(\mathcal{D}_\infty)) = \frac{\delta}{r}$ . We will denote  $\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  to be the corresponding moduli stack of stable maps to  $\mathbb{P}Y_{r,s}$  of degree  $(\beta, \frac{\delta}{r})$ . Note that the line bundle  $\mathcal{O}(\mathcal{D}_\infty)$  is semi-positive.

<sup>5</sup>The terminology of “twisted graph space” is taken from [CJR17a, CJR17b], see loc.cits for some other applications of the twisted graph space in Gromov-Witten theory.



**3.2. Localization analysis.** Consider the  $\mathbb{C}^*$ -action on the projection bundle  $\mathbb{P}_Y(L^\vee \oplus \mathbb{C})$  by scaling the fiber such that the normal bundle of  $D_0$  (resp.  $D_\infty$ ) in  $\mathbb{P}_Y(L^\vee \oplus \mathbb{C})$  is of  $\mathbb{C}^*$ -weight 1 (resp.  $-1$ ). This  $\mathbb{C}^*$ -action induces a  $\mathbb{C}^*$ -action on  $\mathbb{P}Y_{r,s}$  such that the normal bundle of  $\mathcal{D}_0$  (resp.  $\mathcal{D}_\infty$ ) in  $\mathbb{P}Y_{r,s}$  is of  $\mathbb{C}^*$ -weight  $\frac{1}{s}$  (resp.  $-\frac{1}{r}$ ). Then we have an induced  $\mathbb{C}^*$ -action on the moduli  $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  of twisted stable maps to  $\mathbb{P}Y_{r,s}$ . We will apply the virtual localization formula of Graber–Pandharipande [GP99] to  $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ , for which introduce the notation of decorated graph so that we index the components of  $\mathbb{C}^*$ -fixed loci of  $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  by decorated graphs (trees). First, a decorated graph  $\Gamma$  consists of vertexes, edges and legs with the following decorations:

- Each vertex  $v$  is associated with an index  $j(v) \in \{0, \infty\}$ , and a degree  $\beta(v) \in \text{Eff}(Y)$ .
- Each edge  $e$  consists of a pair of half-edges  $\{h_0, h_\infty\}$ , and  $e$  is equipped with a degree  $\delta(e) \in \mathbb{Q}_{>0}$ . Here we call  $h_0$  and  $h_\infty$  half edges, and  $h_0$  (resp.  $h_\infty$ ) is attached to a vertex labeled by 0 (resp.  $\infty$ ).
- Each half-edge  $h$  and each leg  $l$  has a multiplicity  $m(h)$  or  $m(l)$  in  $C \times \mathbb{C}^* \times \mathbb{C}^*$ .
- The legs are labeled with the numbers  $\{1, \dots, m\}$  and each leg is incident to a unique vertex.

By the “valence” of a vertex  $v$ , denoted  $\text{val}(v)$ , we mean the total number of incident half-edges and legs.

For each  $\mathbb{C}^*$ -fixed stable map  $f : (C; q_1, \dots, q_m) \rightarrow \mathbb{P}Y_{r,s}$  in  $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ , we can associate a decorated graph  $\Gamma$  in the following way.

- Each edge  $e$  corresponds to a genus-zero component  $C_e$  which maps constantly to the base  $Y$  and intersects  $\mathcal{D}_0$  and  $\mathcal{D}_\infty$  properly. The restriction  $f|_{C_e}$  satisfies that  $r \cdot \deg(f|_{C_e}^* \mathcal{O}(\mathcal{D}_\infty)) = s \cdot \deg(f|_{C_e}^* \mathcal{O}(\mathcal{D}_0)) = \delta(e)$ .
- Each vertex  $v$  for which  $j(v) = 0$  (with unstable exceptional cases noted below) corresponds to a maximal sub-curve  $C_v$  of  $C$  which maps totally into  $\mathcal{D}_0$ , then the restriction of  $f$  to  $C_v$  defines a twisted stable map in

$$\mathcal{K}_{0,\text{val}(v)}(\sqrt[s]{L^\vee/Y}, \beta(v)) .$$

Each vertex  $v$  for which  $j(v) = \infty$  (again with unstable exceptions) corresponds to a maximal sub-curve which maps totally into  $\mathcal{D}_\infty$ , then the restriction of  $f$  to  $C_v$  defines a twisted stable map in

$$\mathcal{K}_{0,\text{val}(v)}(\sqrt[r]{L/Y}, \beta(v)) .$$

The label  $\beta(v)$  denotes the degree coming from the restriction  $\text{pr}_{r,s} \circ f|_{C_v} : C_v \rightarrow Y$ .

- A vertex  $v$  is *unstable* if stable twisted maps of the type described above do not exist. In this case, we have  $\beta(v) = 0$  and  $v$  corresponds to a single point of the component  $C_e$  for each incident edge  $e$ , which may be a node at which  $C_e$  meets another edge curve  $C_{e'}$ , a marked point of  $C_e$ , or an unmarked point.
- Each leg  $l$  labeled by  $i$  corresponds to the  $i$ th marking  $q_i$ , and it's incident to the vertex  $v$  if  $q_i$  lies on  $C_v$ . The index  $m(l)$  on a leg  $l$  indicates the rigidified inertia stack component  $\bar{I}_{m(l)}\mathbb{P}Y_{r,s}$  of  $\mathbb{P}Y_{r,s}$  on which the gerby marked point corresponding to the leg  $l$  is evaluated.
- A half-edge  $h$  of an edge  $e$  corresponds a ramification point  $q \in C_e$ , i.e., there are two distinguished points (we call them ramification points)  $q_0$  and  $q_\infty$  on  $C_e$  satisfying that  $q_0$  maps to  $\mathcal{D}_0$  and  $q_\infty$  maps to  $\mathcal{D}_\infty$ , respectively. We associate half-edges  $h_0$  and  $h_\infty$  to  $q_0$  and  $q_\infty$  respectively. Then  $m(h)$  indicates the rigidified inertia component  $\bar{I}_{m(h)}\mathbb{P}Y_{r,s}$  of  $\mathbb{P}Y_{r,s}$  on which the ramification point  $q$  associated with  $h$  is evaluated. We will write the edge  $e$  as  $\{h_0, h_\infty\}$ .

Moreover, the decorated graph from a  $\mathbb{C}^*$ -fixed stable map should satisfy the following:

- (1) (Monodromy constraint for edge) For each edge  $e = \{h_0, h_\infty\}$ , we have that  $m(h_0) = (c^{-1}, e^{\frac{\delta(e)}{s}}, 1)$  and  $m(h_\infty) = (c, 1, e^{\frac{\delta(e)}{r}})$  for some  $c \in C$ . Moreover the choice of  $c$  should satisfy that

$$\delta(e) = \text{age}_c(L) \bmod \mathbb{Z} ,$$

which follows from the fact that the ages of line bundles  $f|_{C_e}^* \mathcal{O}(s\mathcal{D}_0)$  and  $f|_{C_e}^* (\mathcal{O}(r\mathcal{D}_\infty) \otimes \text{pr}_{r,s}^* L^\vee)$  at  $q_\infty$  should be equal, as the two line bundles on  $C_e$  are isomorphic.

- (2) (Monodromy constraint for vertex) For each vertex  $v$ , let  $I_v \subset [m]$  be the set of incident legs to  $v$ , and  $H_v$  be the set of incident half-edges to  $v$ . For each  $i \in I_v$ , write  $m(l_i) = (c_i, a_i, b_i)$ , and for each  $h \in H_v$ , write  $m(h) = (c_h, a_h, b_h)$ . If  $v$  is labeled by 0, we have that all  $b_i$  and  $b_h$  are equal to 1, and

$$(3.1) \quad e^{\frac{\beta(v)(L)}{s}} \times \prod_{i \in I_v} a_i \times \prod_{h \in H_v} a_h^{-1} = 1 .$$

If  $v$  is labeled by  $\infty$ , then we have all  $a_i$  and  $a_h$  are all equal to 1 and

$$(3.2) \quad e^{-\frac{\beta(v)(L)}{r}} \times \prod_{i \in I_v} b_i \times \prod_{h \in H_v} b_h^{-1} = 1 .$$

We will only consider decorated graph coming from a  $\mathbb{C}^*$ -fixed stable map. In particular, we note that the decorations at each stable vertex  $v$  yield a tuple

$$\overrightarrow{\text{val}}(v) \in (C \times \mathbb{C}^* \times \mathbb{C}^*)^{\text{val}(v)}$$

recording the multiplicities at every special point of  $C_v$ .<sup>6</sup> Then the restriction gives a stable map in

$$\mathcal{K}_{0, \overrightarrow{\text{val}}(v)}(\mathcal{D}_{j(v)}, \beta(v)) .$$

For each decorated graph  $\Gamma$ , we will associate each vertex  $v$  (resp. edge  $e$ ) a moduli space  $\mathcal{M}_v$  (resp.  $\mathcal{M}_e$ ) over which there is a family of  $\mathbb{C}^*$ -stable maps to  $\mathbb{P}Y_{r,s}$  with decorated degree (there are exceptions for unstable vertexes  $v$ , see section 3.2.1 for more details). Denote by  $F_\Gamma$  the space

$$\prod_{v: j(v)=0} \mathcal{M}_v \times_{\bar{I}_\mu \mathcal{D}_0} \prod_{e \in E} \mathcal{M}_e \times_{\bar{I}_\mu \mathcal{D}_\infty} \prod_{v: j(v)=\infty} \mathcal{M}_v ,$$

where the fiber product is taken by gluing the two branches at each node. By virtual localization formula [GP99], we can write

$$[\mathcal{K}_{0, \vec{m}}(\mathbb{P}Y_{r,s}, \beta)]^{\text{vir}} ,$$

in terms of contributions from each decorated graph  $\Gamma$ :

$$(3.3) \quad [\mathcal{K}_{0, \vec{m}}(\mathbb{P}Y_{r,s}, \beta)]^{\text{vir}} = \sum_{\Gamma} \frac{1}{\mathbb{A}_\Gamma} \iota_{\Gamma*} \left( \frac{[F_\Gamma]^{\text{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\text{vir}})} \right) .$$

Here, for each graph  $\Gamma$ ,  $[F_\Gamma]^{\text{vir}}$  is obtained via the  $\mathbb{C}^*$ -fixed part of the restriction to the fixed loci of the obstruction theory of  $\mathcal{K}_{0, \vec{m} \cup \star}(\mathbb{P}Y_{r,s}, \beta)$ , and  $N_\Gamma^{\text{vir}}$  is the equivariant Euler class of the  $\mathbb{C}^*$ -moving part of this restriction. Besides,  $\mathbb{A}_\Gamma$  is the automorphism factor for the graph  $\Gamma$ , which represents the degree of  $F_\Gamma$  into the corresponding open and closed  $\mathbb{C}^*$ -fixed substack  $i_\Gamma(F_\Gamma)$  in  $\mathcal{K}_{0, \vec{m} \cup \star}(\mathbb{P}Y_{r,s}, \beta)$ . In our case,  $\mathbb{A}_\Gamma$  is the product of the size of the automorphism group  $\text{Aut}(\Gamma)$  of  $\Gamma$  and the degrees from each edge moduli  $\mathcal{M}_e$  into the corresponding fixed loci.

Assume that  $r, s$  are sufficiently large primes. We will do an explicit computation for the contributions of each graph  $\Gamma$  in the following. As for the contribution of a graph  $\Gamma$  to (3.3), one

<sup>6</sup>For each node of  $C_v$ , let  $h$  be the incident half-edge, then we define the multiplicity at the branch of node at  $C_v$  to be  $m(h)^{-1}$ .

can first apply the normalization exact sequence to the obstruction theory, which decomposes the contribution from  $\Gamma$  to (3.3) into contributions from vertex, edge, and node factors.

**3.2.1. Vertex contribution.** Assume that  $v$  is a stable vertex. If the vertex is labeled by  $\infty$ , we define the vertex moduli  $\mathcal{M}_v$  to be  $\mathcal{K}_{0, \overrightarrow{val}(v)}(\mathcal{D}_\infty, \beta(v))$  and the fixed part of perfect obstruction theory gives rise to  $[\mathcal{K}_{0, \overrightarrow{val}(v)}(\mathcal{D}_\infty, \beta(v))]^{\text{vir}}$ . Let  $\pi : \mathcal{C} \rightarrow \mathcal{K}_{0, \overrightarrow{val}(v)}(\mathcal{D}_\infty, \beta(v))$  be the universal curve and  $f : \mathcal{C} \rightarrow \mathcal{D}_\infty$  be the universal map, then the movable part of the perfect obstruction theory yields the *inverse of the Euler class* of the virtual normal bundle which is equal to

$$e^{\mathbb{C}^*}((-R^\bullet \pi_* f^* L^{\frac{1}{r}}) \otimes \mathbb{C}_{-\frac{\lambda}{r}}).$$

When  $r$  is a sufficiently large prime and the multiplicity  $m(l)$  corresponding to each leg  $l$  incident to  $v$  is equal to  $(c_l, 1, \frac{\delta_l}{r})$  for some prefixed  $\delta_l \in \mathbb{Q}_{\geq 0}$  (note this implies  $\delta_l \ll r$ ) and  $c_l \in C$ , following a generalization of [JPPZ20] to the orbifold case, the above Euler class has a representation

$$\sum_{d \geq 0} c_d (-R^\bullet \pi_* f^* L^{\frac{1}{r}}) \left( \frac{-\lambda}{r} \right)^{|E(v)|-1-d}.$$

Here the virtual bundle  $-R^\bullet \pi_* f^* L^{\frac{1}{r}}$  has virtual rank  $|E(v)| - 1$ , where  $|E(v)|$  is the number of edges incident to the vertex  $v$ .

If the vertex  $v$  is labeled by 0, we define the vertex moduli  $\mathcal{M}_v$  to be  $\mathcal{K}_{0, \overrightarrow{val}(v)}(\mathcal{D}_0, \beta(v))$  and the fixed part of perfect obstruction theory gives rise to  $[\mathcal{K}_{0, \overrightarrow{val}(v)}(\mathcal{D}_0, \beta(v))]^{\text{vir}}$ . Let  $\pi : \mathcal{C} \rightarrow \mathcal{K}_{0, \overrightarrow{val}(v)}(\mathcal{D}_0, \beta(v))$  be the universal curve and  $f : \mathcal{C} \rightarrow \mathcal{D}_0$  be the universal map, then the movable part of the perfect obstruction theory yields the *inverse of the Euler class* of the virtual normal bundle which is equal to

$$e^{\mathbb{C}^*}((-R^\bullet \pi_* f^* (L^\vee)^{\frac{1}{s}}) \otimes \mathbb{C}_{\frac{\lambda}{s}}).$$

Now assume that there is only one edge incident to  $v$  and the multiplicity  $m(l)$  corresponding to each leg  $l$  incident to  $v$  is equal to  $(c_l, \frac{\delta_l}{s}, 1)$  for some prefixed  $\delta_l \in \mathbb{Q}_{\geq 0}$  (note this implies  $\delta_l \ll r$ ) and  $c_l \in G$ . By [Wan19, Lemma 5.2, Remark 5.3], the above Euler class is equal to 1 when  $\beta(v)(L) > 0$ , we note here the semi-positivity of  $L$  is essentially used in the proof of [Wan19, Lemma 5.2]. On the other hand, when  $\beta(v)(L) = 0$ , by the argument in the proof of [Wan19, Lemma 6.5], the above Euler class is still equal to 1 there is a special point (we will choose a node in our case) on  $C_v$  whose multiplicity  $m$  satisfies that  $\text{age}_m \mathcal{O}(\mathcal{D}_0) \neq 0$ .

When  $v$  is an unstable vertex over 0 (resp.  $\infty$ ), let  $h$  be the half-edge incident to  $v$ , the vertex moduli  $\mathcal{M}_v$  is defined to be  $\bar{I}_{m(h)-1} \mathcal{D}_0$  (resp.  $\bar{I}_{m(h)-1} \mathcal{D}_\infty$ ) with  $[\mathcal{M}_v]^{\text{vir}} = [\mathcal{M}_v]$  and zero virtual normal bundle.

**3.2.2. Edge contribution.** Let  $e = \{h_0, h_\infty\}$  be an edge in  $\Gamma$  with decorated degree  $\delta(e) \in \mathbb{Q}$  (we will write  $\delta(e)$  as  $\delta$  for simplicity if no confusion occurs) and decorated multiplicities  $m(h_0)$  and  $m(h_\infty)$ . Assume that  $m(h_\infty) = (c, 1, e^{\frac{\delta(e)}{r}})$ , denote  $a_e := a(c)$  be the number as Assumption 2.1. We define the edge moduli  $\mathcal{M}_e$  to be the root gerbe  ${}^{as\delta(e)}\sqrt{L^\vee/I_c Y}$  over  $I_c Y$ , then the virtual cycle  $[\mathcal{M}_e]^{\text{vir}}$  coming from the fix part of the obstruction theory is equal to the fundamental class  $[\mathcal{M}_e]$  of  $\mathcal{M}_e$  and *inverse of the Euler class* of virtual normal bundle is equal to 1 when  $r$  is a sufficiently large prime. We note that  $\mathcal{M}_e$  allows a finite étale map into the corresponding fixed-loci in  $\mathcal{K}_{0,2}(\mathbb{P}Y_{r,s}, (0, \frac{\delta(e)}{r}))$  of degree  $\frac{1}{as}$ . See appendix A.2 for more details.

**3.2.3. Node contributions.** The deformations in  $\mathcal{K}_{0, \overrightarrow{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  smoothing a node contribute to the Euler class of the virtual normal bundle as the first Chern class of the tensor product of

the two cotangent line bundles at the branches of the node. For nodes at which a component  $C_e$  meets a component  $C_v$  over the vertex labeled by  $X_0$ , this contribution is

$$\frac{\lambda - c_1(L)}{a_e s \delta(e)} - \frac{\bar{\psi}_v}{a_e s}.$$

For nodes at which a component  $C_e$  meets a component  $C_v$  at the vertex over  $\mathcal{D}_\infty$ , this contribution is

$$\frac{-\lambda + c_1(L)}{a_e r \delta(e)} - \frac{\bar{\psi}_v}{a_e r}.$$

We will not need node contributions from other types of nodes as the above types suffice the need in this paper.

**3.2.4. Total contribution.** For any decorated graph  $\Gamma$ , we define  $F_\Gamma$  to be the fiber product

$$\prod_{v:j(v)=0} \mathcal{M}_v \times_{\bar{I}_\mu \mathcal{D}_0} \prod_{e \in E} \mathcal{M}_e \times_{\bar{I}_\mu \mathcal{D}_\infty} \prod_{v:j(v)=\infty} \mathcal{M}_v$$

of the following diagram:

$$\begin{array}{ccc} F_\Gamma & \longrightarrow & \prod_{v:j(v)=0} \mathcal{M}_v \times \prod_{e \in E} \mathcal{M}_e \times \prod_{v:j(v)=\infty} \mathcal{M}_v \\ \downarrow & & \downarrow ev_{nodes} \\ \prod_E (\bar{I}_\mu \mathcal{D}_0) \times \bar{I}_\mu \mathcal{D}_\infty & \xrightarrow{(\Delta^0 \times \Delta^\infty)^{|E|}} & \prod_E (\bar{I}_\mu \mathcal{D}_0)^2 \times (\bar{I}_\mu \mathcal{D}_\infty)^2, \end{array}$$

where  $\Delta^0 = (id, \iota)$  (resp.  $\Delta^\infty = (id, \iota)$ ) is the diagonal map of  $\bar{I}_\mu \mathcal{D}_0$  (resp.  $\bar{I}_\mu \mathcal{D}_\infty$ ). Here the right-hand vertical map  $ev_{nodes}$  is the product of the evaluation maps at the two branches of each node. We note that there are two nodes corresponding to  $h_0$  and  $h_\infty$  for each edge  $e = \{h_0, h_\infty\}$ .

We define  $[F_\Gamma]^{\text{vir}}$  to be:

$$\prod_{v:j(v)=0} [\mathcal{M}_v]^{\text{vir}} \times_{\bar{I}_\mu \mathcal{D}_0} \prod_{e \in E} [\mathcal{M}_e]^{\text{vir}} \times_{\bar{I}_\mu \mathcal{D}_\infty} \prod_{v:j(v)=\infty} [\mathcal{M}_v]^{\text{vir}}.$$

Then the contribution of decorated graph  $\Gamma$  to the virtual localization is:

$$(3.4) \quad Cont_\Gamma = \frac{\prod_{e \in E} sa_e}{|\text{Aut}(\Gamma)|} (\iota_\Gamma)_* \left( \frac{[F_\Gamma]^{\text{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\text{vir}})} \right).$$

Here  $\iota_F : F_\Gamma \rightarrow \mathcal{K}_{0, \vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$  is a finite etale map of degree  $\frac{|\text{Aut}(\Gamma)|}{\prod_{e \in E} sa_e}$  into the corresponding  $\mathbb{C}^*$ -fixed loci. The virtual normal bundle  $e^{\mathbb{C}^*}(N_\Gamma^{\text{vir}})$  is the product of virtual normal bundles from vertex contributions, edge contributions and node contributions.

**3.3. Recursive relation.** Take  $L_1 = L$  and  $r = 1$  in 2.4 and write  $\vec{w} = \vec{w}_1^\rightarrow$  to be the weight. Let  $\mu(z) = \sum_{\beta, \vec{k}, c} q^\beta \mathbf{t}^{\vec{k}} \mu_{\beta, \vec{k}, c}(z)$  be an admissible (power) series as in 2.4. We further require that  $\mu_{\beta, \vec{k}, c} \in H^*(\bar{I}_{c^{-1}} X, \mathbb{C})[z]$  is a polynomial in  $z$ . For any admissible pair  $(\beta, \vec{k}, c)$  in  $\text{Eff}(Y) \times \mathbb{Z}_{\geq 0}^N \times C$ , define  $J_{\beta, \vec{k}, c}^Y(\mu, z)$  to be

$$(3.5) \quad \mu_{\beta, \vec{k}, c}(z) + \mathbf{Coeff}_{\mathbf{t}^{\vec{k}}} \left[ \sum_{m \geq 0} \sum_{\substack{(\beta_j, \vec{k}_j, c_j)_{j=1}^m \in \text{Adm}^m, \beta_* \in \text{Eff}(Y) \\ \beta_1 + \dots + \beta_m + \beta_* = \beta \\ \vec{k}_1 + \dots + \vec{k}_m = \vec{k}}} \mu_{\beta_1, \vec{k}_1, c_1}(-\bar{\psi}_1), \dots, \mu_{\beta_m, \vec{k}_m, c_m}(-\bar{\psi}_m), \frac{\phi_\alpha}{z - \bar{\psi}_*} \rangle_{0, \vec{m} \cup *, \beta_*}^Y \right],$$

where  $\vec{m} \cup \star = (c_1^{-1} \cdots, c_m^{-1}, c) \in C^{m+1}$ ,  $\vec{k}_j = (k_{j1}, \dots, k_{jN})$  and  $\vec{k}_1 + \cdots + \vec{k}_m = \vec{k}$  means the component-wise addition in  $\mathbb{Z}_{\geq 0}^N$ .

**Remark 3.2.** We can show that when  $(\beta, \vec{k}, c)$  is not an admissible pair and we define  $J_{\beta, \vec{k}, c}^Y(\mu, z)$  in the same way as above, we have  $J_{\beta, \vec{k}, c}^Y(\mu, z) = 0$ . Moreover  $J_{0, \vec{0}, c} = 0$ . This implies that  $J$ -function

$$J^Y(q, \mu, z) = z + \sum_{\beta, \vec{k}, c} q^\beta \mathbf{t}^{\vec{k}} J_{\beta, \vec{k}, c}^Y(\mu, z)$$

associated with the input  $\mu(z)$  above is an admissible series near  $z$ .

**Definition 3.3.** Let  $m, n$  be two nonnegative integers and  $(\beta, \vec{k}, c)$  be an admissible pair. We denote  $\Lambda_{\beta, \vec{k}, c, m, n}$  to be the set of tuples

$$(c, \beta_\star, ((\beta_1, \vec{k}_1, c_1), \dots, (\beta_{m+n}, \vec{k}_{m+n}, c_{m+n}))) \in C \times \text{Eff}(Y) \times \text{Adm}^{m+n},$$

where we require that  $\beta_\star + \sum_{i=1}^{m+n} \beta_i = \beta$ ,  $\sum_{i=1}^{m+n} \vec{k}_i = \vec{k}$ ,  $\beta_i(L) + (\vec{w}, \vec{k}_i) > 0$  for  $1 \leq i \leq m$  and  $\beta_i(L) + (\vec{w}, \vec{k}_i) = 0$  for  $m+1 \leq i \leq m+n$ . We call an element of  $\Lambda_{\beta, \vec{k}, c, m, n}$  stable if  $\beta_\star \neq 0$  or  $m+n \geq 2$  when  $\beta_\star = 0$ .

We note that  $\Lambda_{\beta, \vec{k}, c, m, n}$  is a finite set as  $\mathcal{K}_{0, m+n}(X, \beta)$  is finite type over  $\mathbb{C}$ , hence Noetherian.

For any admissible pair  $(\beta, \vec{k}, c)$  with  $\beta(L) + (\vec{w}, \vec{k}) > 0$ . We have the following recursive characterization about  $J_{\beta, \vec{k}, c}^Y(\mu, z)$ .

**Proposition 3.4.** For any admissible pair  $(\beta, \vec{k}, c)$  with  $\beta(L) + (\vec{w}, \vec{k}) > 0$ . Assume that  $r$  is a sufficiently large prime. Then for any nonnegative integer  $b$ , we have the following recursive relation:

$$\begin{aligned} & [J_{\beta, \vec{k}, c}^Y(\mu, z)]_{z^{-b-1}} \\ &= \left[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\substack{\Gamma \in \Lambda_{\beta, \vec{k}, c, m, n} \\ \Gamma \text{ is stable}}} \frac{1}{m!n!} (ev_\star)_* \left( \sum_{d=0}^{\infty} \epsilon_* (c_d (-R^\bullet \pi_* f^* L^{\frac{1}{r}})) \left( \frac{\lambda}{r} \right)^{-1+m-d} (-1)^d \right. \right. \\ (3.6) \quad & \left. \cap [\mathcal{K}_{0, \vec{m}+\vec{n} \cup \star}(\sqrt[r]{L/Y}, \beta_\star)]^{\text{vir}} \cap \prod_{i=1}^m \frac{ev_i^* (\frac{1}{\delta_i} \mathbf{t}^{\vec{k}_i} J_{\beta_i, \vec{k}_i, c_i}^Y(\mu, z)|_{z=\frac{\lambda-c_1(L)}{\delta_i}})}{\frac{\lambda-ev_i^* c_1(L)}{r\delta_i} + \frac{\bar{\psi}_i}{r}} \right. \\ & \left. \cap \prod_{i=m+1}^{m+n} ev_i^* (\mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}(-\bar{\psi}_i)) \cap \bar{\psi}_\star^b \right]_{\mathbf{t}^{\vec{k}} \lambda^{-1}}. \end{aligned}$$

Here  $\delta_i = \beta_i(L) + (\vec{w}, \vec{k}_i)$ , and  $\epsilon : \mathcal{K}_{0, \vec{m}+\vec{n} \cup \star}(\sqrt[r]{L/Y}, \beta_\star) \rightarrow \mathcal{K}_{0, \vec{m}+\vec{n} \cup \star}(Y, \beta_\star)$  is the natural structural morphism by forgetting the root structure of  $\sqrt[r]{L/Y}$  (c.f. [TT21]), where we choose the tuple  $\vec{m}+\vec{n} \cup \star$  for  $\mathcal{K}_{0, \vec{m}+\vec{n} \cup \star}(\sqrt[r]{L/Y}, \beta_\star)$  to be

$$((c_1^{-1}, 1, e^{\frac{-\delta_1}{r}}), \dots, (c_{m+n}^{-1}, 1, e^{\frac{-\delta_{m+n}}{r}}), (c, 1, e^{\frac{\delta}{r}})) \in (C \times \mathbb{C}^* \times \mathbb{C}^*)^{m+n+1},$$

and choose the tuple  $\vec{m}+\vec{n} \cup \star$  for  $\mathcal{K}_{0, \vec{m}+\vec{n} \cup \star}(Y, \beta_\star)$  to be

$$(c_1^{-1}, \dots, c_{m+n}^{-1}, c) \in C^{m+n+1}.$$

The proof of the above proposition is based on applying virtual localization to the following integral.

$$(3.7) \quad \sum_{m=0}^{\infty} \sum_{\substack{(\beta_j, \vec{k}_i, c_i)_{i=1}^m \in \text{Adm}^m, \beta_* \in \text{Eff}(Y) \\ \beta_* + \sum_{i=1}^m \beta_i = \beta \\ \vec{k}_1 + \dots + \vec{k}_m = \vec{k}}} \frac{1}{m!} (\widetilde{EV}_*)_* \left( \prod_{i=1}^m ev_i^* (\text{pr}_{r,s}^* (\mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i} (-\bar{\psi}_i))) \right) \\ \cap \bar{\psi}_*^b \cap [\mathcal{K}_{0, \vec{m} \cup \star}(\mathbb{P}Y_{r,s}, (\beta_*, \frac{\delta}{r}))]^{\text{vir}} .$$

Here an explanation of the notations is in order:

- (1) For degrees  $\beta_*, \beta_1, \dots, \beta_m$  in  $\text{Eff}(X)$  and tuples  $\vec{k}_1, \dots, \vec{k}_m$  and with  $\sum_{i=1}^m \beta_i + \beta_* = \beta$  and  $\vec{k}_1 + \dots + \vec{k}_m = \vec{k}$ , let  $(c_1, \dots, c_m) \in C^m$  such that  $(\beta_i, c_i)$  are admissible pairs. Write  $\delta_i = \beta_i(L) + (\vec{w}, \vec{k}_i)$  and  $\delta = \beta(L) + (\vec{w}, \vec{k})$ , we define  $\vec{m} \cup \star$  to be the  $m+1$ tuple

$$((c_1^{-1}, e^{\frac{\delta_i}{s}}, 1), \dots, (c_m^{-1}, e^{\frac{\delta_m}{s}}, 1), (c, 1, e^{\frac{\delta}{r}})) .$$

- (2) the morphism  $EV_*$  is a composition of the following maps:

$$\mathcal{K}_{0, \vec{m} \cup \star}(\mathbb{P}Y_{r,s}, (\beta_*, \frac{\delta}{r})) \xrightarrow{ev_*} \bar{I}_\mu \mathbb{P}Y_{r,s} \xrightarrow{\text{pr}_{r,s}} \bar{I}_\mu Y ,$$

and  $(\widetilde{EV}_*)_*$  is defined by

$$\iota_*(r_*(EV_*)_*).$$

Note here  $r_*$  is the order of the band from the gerbe structure of  $\bar{I}_\mu Y$  but not  $\bar{I}_\mu \mathbb{P}Y_{r,s}$ .

First we state a vanishing lemma regarding applying localization formula to 3.7.

**Lemma 3.5.** *Assume  $r, s$  is sufficiently large. If the localization graph  $\Gamma$  has more than one vertex labeled by  $\infty$ , then the corresponding fixed loci moduli  $F_\Gamma$  is empty, therefore it will contribute zero to (3.7).*

*Proof.* For any twisted stable map  $f : C \rightarrow \mathbb{P}Y_{r,s}$  in  $\mathcal{K}_{0, \vec{m} \cup \star}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ , denote  $N := f^*(\mathcal{O}(-\mathcal{D}_\infty))$ , first we show that  $H^1(C, N) = 0$ . Indeed, using orbifold Riemann-Roch, we have

$$\chi(N) = 1 + \deg(N) - \text{age}(N|_{q_*}) = 0 ,$$

as  $\deg(N) = -\frac{\delta}{r}$ , and  $\text{age}(N|_{q_*}) = 1 - \frac{\delta}{r}$ , then showing  $H^1(C, N) = 0$  is equivalent to show  $H^0(C, N) = 0$ . As the degree of  $N$  is negative on  $C$ , it remains to show that the degree of the restriction of the line bundle  $N$  to every irreducible component  $E$  of  $C$  is non-positive. Observe that the degree  $\deg(N|_E)$  is equal to intersection number  $-([E], [\mathcal{D}_\infty])$ . If the image of an irreducible component of  $C$  via  $f$  isn't contained in  $\mathcal{D}_\infty$ , the restricted degree of  $N$  to  $E$  is obviously non-positive. Otherwise, observe that  $N$  is isomorphic to  $(L^{\frac{1}{r}})^\vee$  over  $\mathcal{D}_\infty$ . As  $L$  is semi-positive, the claim follows. This finishes the proof of the part  $H^1(C, N) = 0$ .

Now assume by contradiction that the moduli of fixed-loci  $F_\Gamma$  is nonempty, by the connectness of the graph  $\Gamma$ , there is at least one vertex of the graph  $\Gamma$  labeled by 0 with at least two edges attached. Suppose  $f : C \rightarrow \mathbb{P}Y_{r,s}$  belongs to the moduli  $F_\Gamma$  of  $\mathbb{C}^*$ -fixed loci. Assume that  $C_0 \cap C_1 \cap C_2$  is part of curve  $C$ , where  $C_0$  is mapped by  $f$  to  $\mathcal{D}_0$  and  $C_1, C_2$  are edges meeting with  $C_0$  at  $b_1$  and  $b_2$ . Then in the normalization sequence for  $R^\bullet \pi_* N$ , it contains the part

$$\begin{aligned} & H^0(C_0, N) \oplus H^0(C_1, N) \oplus H^0(C_2, N) \\ & \rightarrow H^0(b_1, N) \oplus H^0(b_2, N) \\ & \rightarrow H^1(C, N) . \end{aligned}$$

Hence there is one of the weight-0 pieces in  $H^0(b_1, N) \oplus H^0(b_2, N)$  that is canceled with a weight-0 piece of  $H^0(C_0, N)$ , and the other is mapped injectively into  $H^1(C, N)$ , but this contradicts that  $H^1(C, N) = 0$ . So  $F_\Gamma$  is empty.  $\square$

Now we prove Proposition 3.4.

*Proof of Proposition 3.4.* The proof is almost identical to the proof in [Wan19, §6.2], we here sketch the main steps for the convenience of readers. First without loss of generality, we will

By Lemma 3.5, we only need to consider the decorated graph  $\Gamma$  that has only one vertex labeled by  $\infty$ . Note the marking  $q_\star$  corresponding to  $\star$  is incident to the vertex  $v_\star$  due to the choice of multiplicity at the marking  $q_\star$ , then the vertex  $v_\star$  can't be a node linking two edges. Such decorated graph is *star-shaped*, i.e., the vertex set  $V$  allows a decomposition

$$V = \{v_\star\} \sqcup V_0.$$

where the vertex  $v_\star$  is labeled by  $\infty$ , and all the vertexes in  $V_0$  are labeled by 0. Each vertex labeled by 0 is linked to  $v_\star$  by a unique edge. There are three types of star-shaped decorated graphs:

- (1) (Type I) The vertex  $v_\star$  is unstable and there is only one edge incident to  $v_\star$  in  $\Gamma$ , the unique vertex  $v$  labeled by 0 is unstable;
- (2) (Type II) The vertex  $v_\star$  is unstable and there is only one edge incident to  $v_\star$  in  $\Gamma$ , the unique vertex  $v$  labeled by 0 is stable;
- (3) (Type III) The vertex  $v_\star$  is stable.

Denote by  $\beta_\star$  the degree of the unique vertex  $v_\star$  labeled by  $\infty$ . Now let's compute the localization contribution from the above three types of graphs:

- (1) If the graph  $\Gamma$  is of type I, the vertex over 0 corresponds to a marked point with input  $\mu_{\beta, \vec{k}, c}$ , then the graph  $\Gamma$  contributes

$$\mathbf{t}^{\vec{k}} \frac{\mu_{\beta, \vec{k}, c}(\frac{\lambda - c_1(L)}{\delta})}{\delta} \cdot (\frac{\lambda - c_1(L)}{\delta})^b$$

to (4.3). Note the use the fact that the restriction of the psi-class  $\bar{\psi}_\star$  to  $\mathcal{M}_e$  is equal to  $\frac{\lambda - c_1(L)}{\delta}$  (see Remark A.4).

- (2) If the graph  $\Gamma$  is of type II,  $\Gamma$  has the same description with type I except that the vertex over 0 is stable. Then this type of graphs contributes

$$\sum_{\substack{(\beta_j, \vec{k}_j, c_j)_{j=1}^m \in \text{Adm}^m, \beta_\star \in \text{Eff}(Y) \\ \beta_1 + \dots + \beta_m + \beta_\star = \beta \\ k_1 + \dots + k_m = \vec{k}}} \frac{1}{m!} (e\widetilde{v}_\star)_* \left( \epsilon'_* ([\mathcal{K}_{0, \vec{m}_s \cup \star}(\sqrt[s]{L^\vee/Y}, \beta_\star)]^{\text{vir}}) \right. \\ \left. \cap \bigcap_{i=1}^m ev_i^* (\mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}(-\bar{\psi}_i)) \cap \frac{\frac{1}{\delta} (\frac{\lambda - ev_\star^* c_1(L)}{\delta})^b}{\frac{\lambda - ev_\star^* c_1(L)}{s\delta} - \frac{\bar{\psi}_\star}{s}} \right)$$

to (3.7), where  $\vec{m}_s \cup \star = ((c_1^{-1}, e^{\frac{\delta_1}{s}}, 1), \dots, (c_m^{-1}, e^{\frac{\delta_m}{s}}, 1), (c, e^{-\frac{\delta}{s}}, 1)) \in (C \times C^* \times C^*)^{m+1}$ ,  $\vec{m} \cup \star = (c_1^{-1}, \dots, c_m^{-1}, c) \in C^{m+1}$  and

$$\epsilon' : \mathcal{K}_{0, \vec{m}_s \cup \star}(\sqrt[s]{L^\vee/Y}, \beta_\star) \rightarrow \mathcal{K}_{0, \vec{m} \cup \star}(Y, \beta_\star)$$

is the natural structure morphism by forgetting root of  $\sqrt[s]{L^\vee/Y}$  (c.f. [TT21]). Note here we implicitly cancel out  $a_e$  factors from the node contribution and automorphic



factor everywhere and use the fact that the Euler class of virtual normal bundle for the vertex over 0 is equal to 1. It's proved that

$$\epsilon'_*([\mathcal{K}_{0,\vec{m}_s \cup \star}(\sqrt[s]{L^\vee/Y}, \beta_\star)]^{\text{vir}}) = \frac{1}{s}[\mathcal{K}_{0,\vec{m} \cup \star}(Y, \beta_\star)]^{\text{vir}},$$

in [TT21], then the above formula is equal to

$$\sum_{m \geq 0} \sum_{\substack{(\beta_j, \vec{k}_j, c_j)_{j=1}^m \in \text{Adm}^m, \beta_\star \in \text{Eff}(Y) \\ \beta_1 + \dots + \beta_m + \beta_\star = \beta \\ \vec{k}_1 + \dots + \vec{k}_m = \vec{k}}} \frac{1}{m!} \phi^\alpha \langle \mu_{\beta_1, \vec{k}_1, c_1}(-\bar{\psi}_1), \dots, \mu_{\beta_m, \vec{k}_m, c_m}(-\bar{\psi}_m), \frac{\frac{1}{\delta}(\frac{\lambda - c_1(L)}{\delta})^b \phi_\alpha}{\frac{\lambda - c_1(L)}{\delta} - \bar{\psi}_\star} \rangle_{0, \vec{m} \cup \star, \beta_\star}.$$

- (3) If the graph  $\Gamma$  is of type III, then  $v_\star$  is incident to the distinguished leg corresponding to the marking  $q_\star$  and  $m$  edges ( $m$  can be 0) and  $n$  legs. Each leg must be associated with the input  $\mathbf{t}^{\vec{k}} \mu_{\beta', \vec{k}', c'}(z)$  with  $\beta'(L) + (\vec{w}, \vec{k}') = 0$ , otherwise the support of the twisted sector  $\bar{I}_{c'-1}Y$  will avoid  $\mathcal{D}_\infty$ . Choose a labeling<sup>7</sup> of the  $m$  edges attached to the vertex  $v_\star$  by  $[m] := \{1, \dots, m\}$ . Let  $\frac{\delta_i}{r}$  be the decorated degree associated with the  $i$ th edge  $e_i$ . Let  $v_i$  be the vertex over 0 incident to  $e_i$ , then  $v_i$  can't be a unstable vertex of valence 1 as the corresponding ramification point  $q_0$  is a stacky point, or it can't be a node linking two edges by Lemma 3.5. Therefore  $v_i$  corresponds to either a marking point or a stable vertex. Assume that there are  $l$  legs ( $l$  can be zero) incident to  $v_i$ . let's label the legs incident to  $v_i$  by  $\{i1, \dots, il\} \subset [|\{\text{Legs of } \Gamma\}|]$ . Note that when  $v_i$  is unstable,  $l = 1$ .

Assume that the vertex  $v_i$  is decorated by the degree  $\beta_{i0}$ . Assume that the insertion at the marking  $q_{ij}$  on the curve<sup>8</sup>  $C_{v_i}$  corresponds to  $\mathbf{t}^{\vec{k}_{ij}} \mu_{\beta_{ij}, \vec{k}_{ij}, c_{ij}}(-\bar{\psi}_{ij})$  in (3.7), let's say the leg for  $q_{ij}$  has *virtual extended degree*  $(\beta_{ij}, \vec{k}_{ij})$  contribution to the vertex  $v_i$ , denote  $(\beta_i, \vec{k}_i)$  to be summation of  $(\beta_{i0}, 0)$  and all the virtual degrees from the legs incident to  $v_i$ . We call  $(\beta_i, \vec{k}_i)$  the *total extended degree* at the vertex  $v_i$ . From (3.7), one has

$$\beta_\star + \sum_{i=1}^m \beta_i = \beta, \quad \sum_{i=1}^m \vec{k}_i = \vec{k}.$$

Assume the multiplicity of the half edge incident to  $C_{v_i}$  is equal to  $(c_i^{-1}, e^{\frac{\delta_i}{s}}, 1)$ , here  $\delta_i$  is the decorated degree associated with the edge  $e_i$  and  $\delta_i = \text{age}_c(L) \bmod \mathbb{Z}$  by 3.2. Observe that to ensure such a graph  $\Gamma$  exists, one must have

$$(3.8) \quad \beta_i(L) + (\vec{w}, \vec{k}_i) = \delta_i.$$

Indeed, by orbifold Riemann-Roch Theorem, one has

$$\deg(f^* \mathcal{O}(\mathcal{D}_0)|_{C_{v_i}}) = -\frac{\beta_{i0}(L)}{s} = (1 - \frac{\delta_i}{s}) + \sum_{j=1}^l \frac{\beta_{ij}(L) + (\vec{w}, \vec{k}_{ij})}{s} \bmod \mathbb{Z}.$$

Here the first term on the right hand is the age of  $f^* \mathcal{O}(\mathcal{D}_0)$  at the node of  $C_{v_i}$ , and the second term on the right is the sum of the ages of  $f^* \mathcal{O}(\mathcal{D}_0)$  at the marked points on  $C_{v_i}$ . As  $s$  is sufficiently large, one must have

$$\frac{\delta_i}{s} = \frac{\beta_{i0}(L)}{s} + \sum_{j=1}^l \frac{\beta_{ij}(L) + (\vec{w}, \vec{k}_{ij})}{s},$$

<sup>7</sup>Such a labeling is not unique, but we will divide  $m!$  to offset the labeling in the end.

<sup>8</sup>When  $v$  is unstable, we just take  $v$  to be  $q_{i1}$ .

which implies that  $\beta_i(L) + (\vec{w}, \vec{k}_i) = \delta_i$ . Note that this also imply that  $(\beta_i, \vec{k}_i, c_i)$  is an admissible pair.

Now we can group the edge-labeled decorated graphs by the set  $\Lambda_{\beta, \vec{k}, c, m, n}$  defined in 3.3. For any element of  $\Lambda_{\beta, \vec{k}, c, m, n}$ , we can naturally associate a group of edge-labeled star-shaped decorated graphs such that the vertex incident to the edge labeled by  $i$  has total extended degree  $(\beta_i, \vec{k}_i)$  and the multiplicity at the half-edge  $h_i$  incident to  $v_i$  over 0 is  $m(h_i) := (c_i^{-1}, e^{\frac{\delta_i}{s}}, 1)$ . We may call an element  $\phi := (c, \beta_\star, ((\beta_1, \vec{k}_1, c_1), \dots, (\beta_{m+n}, \vec{k}_{m+n}, c_{m+n})))$  of  $\Lambda_{\beta, \vec{k}, c, m, n}$  a *meta graph*, and denote all the (star-shaped) edge-labeled decoration graph associated to  $\phi$  by  $\Gamma_\phi$ .

Now we use the localization formula in §3.2.4 to compute the contribution from  $\Gamma_\phi$  to (3.7). Summing over the localization contribution of the vertex  $v_i$  together with branch of node  $h_i$  at  $v_i$  from all graphs in  $\Gamma_\phi$ , and pushing forward to  $\bar{I}_{c_i}^{-1}Y \cong \bar{I}_{m_{h_i}}\mathcal{D}_0$  along  $\iota \circ (ev_{h_i})_*$  (recall that  $\iota$  is the inversion map on Chen-Ruan cohomology  $H^*(\bar{I}_\mu Y)$ ), it yields

$$\begin{aligned} & \mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i} \left( \frac{\lambda - c_1(L)}{\delta_i} \right) + \sum_{m=0}^{\infty} \sum_{\substack{(\beta_j, \vec{k}_j, c_j)_{j=1}^m \in \text{Adm}^m, \beta_\star \in \text{Eff}(Y) \\ \beta_\star + \sum_{j=1}^m \beta_j = \beta_i \\ \vec{k}_1 + \dots + \vec{k}_m = \vec{k}}} \frac{1}{m!} \\ & (e\widetilde{v}_\star)_* \left( \epsilon'_* ([\mathcal{K}_{0, \vec{m}_s \cup \star}(\sqrt{L^V/Y}, \beta_\star)]^{\text{vir}}) \cap \bigcap_{j=1}^m ev_j^* (\mathbf{t}^{\vec{k}_j} \mu_{\beta_j, \vec{k}_j, c_j}(-\bar{\psi}_j)) \cap \frac{1}{\frac{\lambda - ev_\star^* c_1(L)}{\delta_j s} - \frac{\bar{\psi}_\star}{s}} \right), \end{aligned}$$

which is equal to  $\mathbf{t}^{\vec{k}_i} J_{\beta_i, \vec{k}_i, c_i}^Y(\mu, z)|_{\frac{\lambda - c_1(L)}{\delta_i}}$ . Note here we use the fact the Euler class of the virtual normal bundle for  $v_i$  is equal to 1 as the node has nontrivial isotropy action on the normal bundle  $N_{\mathcal{D}_0/\mathbb{P}Y_{r,s}}$ . Observe that all the edge-labeled decorated graphs in  $\Gamma_\phi$  have the same localization contributions from the unique vertex  $v_\star$  labeled by  $\infty$ , the edge  $e_i$  and the node over  $\infty$  incident to  $e_i$ . Moreover the localization formula for any graph in  $\Gamma_\phi$  depends multi-linearly on the localization contributions of vertexes over 0, then the localization from all meta graphs in  $\Lambda_{\beta, \vec{k}, c, m, n}$  to (3.7) yields the summation:

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\substack{\Gamma \in \Lambda_{\beta, \vec{k}, c, m, n} \\ \Gamma \text{ is stable}}} \frac{1}{m!n!} (e\widetilde{v}_\star)_* \left( \sum_{d=0}^{\infty} \epsilon_* (c_d(-R^\bullet \pi_* f^* L^{\frac{1}{r}}) \left( \frac{-\lambda}{r} \right)^{-1+m-d} \right. \\ (3.9) \quad & \left. \cap [\mathcal{K}_{0, \vec{m}+n \cup \star}(\sqrt{L/Y}, \beta_\star)]^{\text{vir}} \right) \cap \prod_{i=1}^m \frac{ev_i^* (\frac{1}{\delta_i} \mathbf{t}^{\vec{k}_i} J_{\beta_i, \vec{k}_i, c_i}(\mu, z)|_{z=\frac{\lambda - c_1(L)}{\delta_i}})}{\frac{\lambda - ev_i^* c_1(L)}{r\delta_i} - \frac{\bar{\psi}_i}{r}} \\ & \cap \prod_{i=m+1}^{m+n} ev_i^* (\mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}(-\bar{\psi}_i)) \cap \bar{\psi}_\star^b \Big). \end{aligned}$$

By the discussion above, we can write (3.7) in the following way:

$$\begin{aligned}
& \mathbf{t}^{\vec{k}} \frac{\mu_{\beta, \vec{k}, c}(\frac{\lambda - c_1(L)}{\delta})}{\delta} \cdot (\frac{\lambda - c_1(L)}{\delta})^b + \sum_{m=0}^{\infty} \sum_{\substack{(\beta_j, \vec{k}_j, c_j)_{j=1}^m \in \text{Adm}^m, \beta_* \in \text{Eff}(Y) \\ \beta_1 + \dots + \beta_m + \beta_* = \beta \\ \vec{k}_1 + \dots + \vec{k}_m = \vec{k}}} \frac{1}{m!} (e\widetilde{v}_*)_* \\
(3.10) \quad & \left( [\mathcal{K}_{0, \vec{m} \cup \star}(Y, \beta_*)]^{\text{vir}} \cap \bigcap_{i=1}^m ev_i^*(\mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}(-\bar{\psi}_i)) \cap \frac{\frac{1}{\delta}(\frac{\lambda - ev_*^* c_1(L)}{\delta})^b}{\frac{\lambda - ev_*^* c_1(L)}{\delta} - \bar{\psi}_*} \right) \\
& + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\substack{\Gamma \in \Lambda_{\beta, \vec{k}, c, m, n} \\ \Gamma \text{ is stable}}} \frac{1}{m!n!} (e\widetilde{v}_*)_* \left( \sum_{d=0}^{\infty} \epsilon_*(c_d(-R^\bullet \pi_* f^* L^{\frac{1}{r}})) \left( \frac{-\lambda}{r} \right)^{-1+m-d} \right. \\
& \cap [\mathcal{K}_{0, \vec{m} + n \cup \star}(\sqrt{L/Y}, \beta_*)]^{\text{vir}} \cap \prod_{i=1}^m \frac{ev_i^*(\frac{1}{\delta_i} \mathbf{t}^{\vec{k}_i} J_{\beta_i, \vec{k}_i, c_i}(\mu, z)|_{z=\frac{\lambda - c_1(L)}{\delta_i}})}{-\frac{\lambda - ev_i^* c_1(L)}{r\delta_i} - \frac{\bar{\psi}_i}{r}} \\
& \left. \cap \prod_{i=m+1}^{m+n} ev_i^*(\mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}(-\bar{\psi}_i)) \cap \bar{\psi}_*^b \right).
\end{aligned}$$

As (3.7) lies in  $\mathbf{t}^{\vec{k}} H^*(\bar{I}_\mu Y, \mathbb{Q})[\lambda]$ , the coefficient of  $\mathbf{t}^{\vec{k}} \lambda^{-1}$  term in (3.10) must vanish. Note that the coefficients before  $\lambda^{-1}$  in the first two terms in (3.10) yields

$$\sum_{m=0}^{\infty} \sum_{\substack{(\beta_j, \vec{k}_j, c_j)_{j=1}^m \in \text{Adm}^m, \beta_* \in \text{Eff}(Y) \\ \beta_1 + \dots + \beta_m + \beta_* = \beta \\ \vec{k}_1 + \dots + \vec{k}_m = \vec{k}}} \frac{1}{m!} \phi^\alpha \langle \mu_{\beta_1, \vec{k}_1, c_1}(-\bar{\psi}_1), \dots, \mu_{\beta_m, \vec{k}_m, c_m}(-\bar{\psi}_m), \phi_\alpha \bar{\psi}_*^b \rangle_{0, \vec{m} \cup \star, \beta_*},$$

which is the left hand side of equality in (4.4). Then we extract the coefficient of the  $\mathbf{t}^{\vec{k}} \lambda^{-1}$  term in the third term in (3.10), this yields the term on the right hand side of (4.4) up to a minus sign. This completes the proof of (4.4).  $\square$

**3.4. Specializing Novikov degrees.** When  $Y$  can be embedded into another smooth DM stack  $X$  with projective coarse moduli and  $L$  is a restriction of a line bundle of  $X$ , which we still denote to be  $l$  by an abuse of notation. For any degree  $\beta \in \text{Eff}(X)$ , we will denote  $\mathcal{K}_{0, \vec{m}}(Y, \beta)$  to be the disjoint union<sup>9</sup>

$$\bigsqcup_{\substack{d \in \text{Eff}(Y) \\ i_*(d) = \beta}} \mathcal{K}_{0, \vec{m}}(Y, d).$$

The same rule applies to the notation when we define Gromov-Witten invariants: for any degree  $\beta \in \text{Eff}(X)$ , we will denote the Gromov-witten invariants

$$\langle \dots \rangle_{0, \vec{m}, \beta}^Y := \sum_{d \in \text{Eff}(Y): i_*(d) = \beta} \langle \dots \rangle_{0, \vec{m}, d}^Y.$$

Now we replace the effective cone  $\text{Eff}(Y)$  by  $\text{Eff}(X)$  in the definition of  $\Lambda_{\beta, \vec{k}, c, m, n}$  and the recursion relation 3.6 and apply the rule of specialization of degrees, we can also apply the same strategy of proving of proposition 3.4 to show the following variation of 3.4:

<sup>9</sup>It's a finite disjoint union as  $\mathcal{K}_{0, m}(X, \beta)$  is finite type over  $\mathbb{C}$ , hence Noetherian.

**Proposition 3.6.** *Let*

$$\mu(z) = \sum_{\beta, \vec{k}, c} q^\beta \mathbf{t}^{\vec{k}} \mu_{\beta, \vec{k}, c}(z)$$

*be an admissible series in  $H^*(\bar{I}_\mu Y, \mathbb{C})[z][[t_1, \dots, t_N]]\llbracket \text{Eff}(X) \rrbracket$ . For any integer  $b \geq 0$  and admissible pair  $(\beta, \vec{k}, c)$  with  $\beta(L) + (\vec{w}, \vec{k}) > 0$ , we have the following recursive relation:*

$$\begin{aligned} & [J_{\beta, \vec{k}, c}^Y(\mu, z)]_{z^{-b-1}} \\ &= \left[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\substack{\Gamma \in \Lambda_{\beta, \vec{k}, c, m, n} \\ \Gamma \text{ is stable}}} \frac{1}{m!n!} (\widetilde{ev}_\star)_* \left( \sum_{d=0}^{\infty} \epsilon_* (c_d(-R^\bullet \pi_* f^* L^{\frac{1}{r}})) \left( \frac{\lambda}{r} \right)^{-1+m-d} (-1)^d \right. \right. \\ (3.11) \quad & \left. \cap [\mathcal{K}_{0, m+n} \rightarrow \mathcal{U}_\star(\sqrt[r]{L/Y}, \beta_\star)]^{\text{vir}} \cap \prod_{i=1}^m \frac{ev_i^* \left( \frac{1}{\delta_i} \mathbf{t}^{\vec{k}_i} J_{\beta_i, \vec{k}_i, c_i}^Y(\mu, z) \right) \Big|_{z=\frac{\lambda-c_1(L)}{\delta_i}}}{\frac{\lambda-ev_i^* c_1(L)}{r\delta_i} + \frac{\bar{\psi}_i}{r}} \right. \\ & \left. \cap \prod_{i=m+1}^{m+n} ev_i^* \left( \mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}(-\bar{\psi}_i) \right) \cap \bar{\psi}_\star^b \right]_{\mathbf{t}^{\vec{k}} \lambda^{-1}}. \end{aligned}$$

#### 4. PROOF OF THE MAIN THEOREM

In this section, we will first assume that  $X$  is a smooth Deligne-Mumford stack with projective moduli and  $Y \subset X$  is a smooth *hypersurface* such that the line bundle  $L := \mathcal{O}_X(Y)$  is *semi-positive*, i.e.,  $\beta(L) \geq 0$  for any degree  $\beta \in \text{Eff}(X)$ . Later in §4.4,  $Y$  will be assumed to be a complete intersection associated to a direct sum of semi-positive line bundles.

**4.1. A root stack modification of the space of the deformation to the normal cone.** Let  $P_X := \mathbb{P}_X(\mathbb{C} \oplus \mathbb{C})$  be the trivial  $\mathbb{P}^1$  bundle over  $X$ . It has two sections: one is the zero section  $X_0 := \mathbb{P}(0 \oplus \mathbb{C}) \subset P_X$ , and the other is the  $\infty$ -section  $X_\infty := \mathbb{P}(\mathbb{C} \oplus 0)$ . We will introduce the notation  $P_Y$  to mean the divisor  $\mathbb{P}_Y(\mathbb{C} \oplus \mathbb{C}) \subset P_X$ .

Let  $\mathfrak{Q}$  be the blow-up<sup>10</sup> of  $P_X$  along  $Y_0 := X_0 \cap P_Y$ . More explicitly,  $\mathfrak{Q}$  can be constructed as a hypersurface in the projective bundle  $\pi : \mathfrak{F} := \mathbb{P}_{P_X}(L^\vee \oplus \mathcal{O}_{P_X}(-1)) \rightarrow P_X$  associated to the section  $z_1 \cdot \pi^{-1}(s_{P_Y}) - z_2 \cdot \pi^{-1}(s_{X_0})$  of the line bundle  $\mathcal{O}_{\mathfrak{F}}(1)$ , where  $s_{P_Y}$  and  $s_{X_0}$  are the defining equations of  $P_Y$  and  $X_0$  in  $P_X$ , and  $z_1, z_2$  are the tautological sections of line bundles  $\pi^*(L^\vee) \otimes \mathcal{O}_{\mathfrak{F}}(1)$  and  $\pi^*\mathcal{O}_{P_X}(-1) \otimes \mathcal{O}_{\mathfrak{F}}(1)$  respectively. The space  $\mathfrak{Q}$  has four special smooth divisors: 1, the strict transformation of  $P_Y$ , which we will still denote it to be  $P_Y$ ; 2, the exceptional divisor, which we will denote to be  $E$  and it's isomorphic to the projective bundle  $\mathbb{P}_Y(L^\vee \oplus \mathbb{C})$ ; 3, the strict transformation of the 0-section  $X_0$ , we will still denote to be  $X_0$  by an abuse of notation, and it's isomorphic to  $X$  with normal bundle equal to  $L$ ; 4, the strict transformation of the  $\infty$ -section  $X_\infty$ , which we will still denote it to be  $X_\infty$ .

Let  $\mathbb{C}^*$  act on  $P_X$  by scaling the  $\mathbb{P}^1$ -fiber so that the weight of the  $\mathbb{C}^*$ -action on the normal bundle of  $X_\infty$  in  $P_X$  is  $-1$ . It induces a  $\mathbb{C}^*$ -action on  $\mathfrak{Q}$  with the  $\mathbb{C}^*$ -fixed loci  $X_0, X_\infty$  and  $D_0 := P_Y \cap E$ .

We will consider the root stack  $\mathfrak{R}$  of  $\mathfrak{Q}$  by taking  $s$ -th root of  $X_0$  and  $r$ -th root of  $P_Y$ . We will also assume that  $r, s$  are distinct primes. We will still use the notation  $X_0, X_\infty, E$  and  $P_Y$  to mean their corresponding divisors in  $\mathfrak{R}$  after taking roots. We note  $E$  is isomorphic to  $\mathbb{P}_{Y_{r,s}}$  so that  $\mathcal{D}_0 := X_0 \cap E$  is isomorphic to the root stack  $\sqrt[s]{L^\vee/Y}$  and  $\mathcal{D}_\infty := P_Y \cap E$  is isomorphic to the root stack  $\sqrt[r]{L/Y}$ .

<sup>10</sup>This space is known as (a compactification of) the space of the deformation to the normal cone, c.f., [Ful84, ch5].

The  $\mathbb{C}^*$ -action on  $\mathfrak{Q}$  induces a  $\mathbb{C}^*$ -action on  $\mathfrak{R}$ . We see that the normal bundle of  $X_0$  in  $\mathfrak{R}$  is ( $\mathbb{C}^*$ -equivariantly) isomorphic to  $(L^\vee)^{\frac{1}{s}} \otimes \mathbb{C}_{\frac{\lambda}{s}}$ , the normal bundle of  $X_\infty$  in  $\mathfrak{R}$  is isomorphic to  $\mathbb{C}_{-\lambda}$ , and the normal bundle of  $\mathcal{D}_\infty$  in  $\mathfrak{R}$  is isomorphic to  $(L^{\frac{1}{r}} \otimes \mathbb{C}_{-\frac{\lambda}{r}}) \oplus \mathbb{C}_\lambda$ .

Let  $\text{pr}_{r,s} : \mathfrak{R} \rightarrow X$  be the morphism induced from composition of the following three maps: (1) a morphism  $\mathfrak{R} \rightarrow \mathfrak{Q}$  forgetting root structure; (2) the blow-down from  $\mathfrak{Q}$  to  $P_X$ ; (3) and the projection from  $P_X$  to the base  $X$ . Note  $\text{pr}_{r,s}$  induces a morphism from the rigidified inertia stack  $\bar{I}_\mu \mathfrak{R}$  to the rigidified inertia stack  $\bar{I}_\mu X$ .

There are two morphisms

$$q_0, q_\infty : \mathfrak{R} \rightarrow \mathbb{B}\mathbb{C}^*,$$

associated to the line bundle  $\mathcal{O}(X_0)$  and  $\mathcal{O}(P_Y)$  respectively such that  $\mathcal{O}(X_0) \cong q_0^*(\mathbb{L})$  and  $\mathcal{O}(P_Y) \cong q_\infty^*(\mathbb{L})$ , here  $\mathbb{L}$  is the universal line bundle over  $\mathbb{B}\mathbb{C}^*$ . This will induce morphisms on their rigidified inertia stack counterparts:

$$q_0, q_\infty : \bar{I}_\mu \mathfrak{R} \rightarrow \bar{I}_\mu \mathbb{B}\mathbb{C}^*,$$

Note the rigidified inertia stack  $\bar{I}_\mu \mathbb{B}\mathbb{C}^*$  of  $\mathbb{B}\mathbb{C}^*$  can be written as the disjoint union

$$\bigsqcup_{c \in \mathbb{C}^*} \bar{I}_c \mathbb{B}\mathbb{C}^*,$$

Let  $c$  be a complex number, any  $\mathbb{C}$ -point of  $\bar{I}_c \mathbb{B}\mathbb{C}^*$  is isomorphic to the pair  $(1_{\mathbb{C}}, c)$ , where  $1_{\mathbb{C}}$  is the trivial principal  $\mathbb{C}$ -bundle over  $\mathbb{C}$  and  $c$  is an element in the automorphism group  $\text{Aut}_{\mathbb{C}}(1_{\mathbb{C}}) \cong \mathbb{C}^*$ . Let  $(y, g)$  be a  $\mathbb{C}$ -point of  $\bar{I}_\mu \mathfrak{R}$ , where  $y \in \text{Ob}(\text{PY}_{r,s}(\mathfrak{R}))$  and  $g \in \text{Aut}(y)$ . Assume that  $q_0((y, g)) = (1_{\mathbb{C}}, c)$ , then  $g$  acts on the fiber  $\mathcal{O}(X_0)|_y$  via the multiplication by  $c$ . We have a similar relation for  $q_\infty$ .

Let  $C$  be a good index set for rigidified inertia stack  $\bar{I}_\mu X$  satisfying the assumption 2.1. Now we will use the index set  $C \times \mathbb{C}^* \times \mathbb{C}^*$  to index the components of the rigidified inertia stack  $\bar{I}_\mu \mathfrak{R}$ : for each  $c \in C$ ,  $c_0, c_\infty \in \mathbb{C}^*$ , we will define the rigidified inertia component  $\bar{I}_{(c, c_0, c_\infty)} \mathfrak{R}$  to be

$$\text{pr}_{r,s}^{-1}(\bar{I}_c X) \cap q_0^{-1}(\bar{I}_{c_0} \mathbb{B}\mathbb{C}^*) \cap q_\infty^{-1}(\bar{I}_{c_\infty} \mathbb{B}\mathbb{C}^*).$$

**Definition 4.1.** For any degree  $\beta \in \text{Eff}(X)$  and  $\delta, d \in \mathbb{Q}$ , we say a stable map  $f : C \rightarrow \mathfrak{R}$  is of degree  $(\beta, \frac{\delta}{r}, d)$  if  $(\text{pr}_{r,s} \circ f)_*[C] = \beta$ ,  $\deg(f^* \mathcal{O}(P_Y)) = \frac{\delta}{r}$  and  $\deg(f^* \mathcal{O}(X_\infty)) = d$ . We will denote  $\mathcal{K}_{0, \vec{m}}(\mathfrak{R}, (\beta, \frac{\delta}{r}, d))$  to be the corresponding moduli stack of twisted stable maps to  $\mathfrak{R}$  of degree  $(\beta, \frac{\delta}{r}, d)$ . We note  $\mathcal{K}_{0, \vec{m}}(\mathfrak{R}, (\beta, \frac{\delta}{r}, d))$  is proper as the degree data uniquely determines a homology class in  $H_2(\mathfrak{R}, \mathbb{Q})$ . Indeed, dual to  $H_2(\mathfrak{R}, \mathbb{Q})$ , we have  $H_2(\mathfrak{R}, \mathbb{Q})^\vee \cong H^2(\mathfrak{R}, \mathbb{Q}) \cong H^2(\mathfrak{Q}, \mathbb{Q})$ , and, by the blow-up construction of  $\mathfrak{Q}$ ,  $H^2(\mathfrak{Q}, \mathbb{Q})$  is isomorphic to the direct sum

$$H^2(X, \mathbb{Q}) \oplus \mathbb{Q}[X_\infty] \oplus \mathbb{Q}[P_Y],$$

where  $[P_Y]$ ,  $[X_\infty]$  (and  $[E]$ ) are the fundamental classes of  $P_Y$ ,  $X_\infty$  (and  $E$ ) respectively and the first summand is embedded into  $H^*(\mathfrak{Q}, \mathbb{Q})$  via the pullback  $\text{pr}_{r,s}^*$ . Note that we have  $r[P_Y] + [E] = \text{pr}_{r,s}^*([Y])$  in  $H^2(\mathfrak{R}, \mathbb{Q})$ . We also note that  $\mathcal{O}(X_\infty)$  is semi-positive, thus we can assume that  $d \geq 0$ .

**4.2. Localization analysis.** The  $\mathbb{C}^*$ -action on  $\mathfrak{R}$  induces a  $\mathbb{C}^*$ -action on the moduli of stable maps to  $\mathfrak{R}$ , we will use decorated graphs (trees) to index the components of  $\mathbb{C}^*$ -fixed loci of  $\mathcal{K}_{0, \vec{m}}(\mathfrak{R}, (\beta, \frac{\delta}{r}, d))$  similar to 3.2. The decorations on a graph  $\Gamma$  is given by the following data:

- Each vertex  $v$  is associated with an index  $j(v) \in \{X_0, \mathcal{D}_\infty, X_\infty\}$ , and a degree  $\beta(v) \in \text{Eff}(X)$ . In particular, we will also say the vertex  $v$  is labeled by 0 if  $j(v) = X_0$  and is labeled by  $\infty$  if  $j(v) = \mathcal{D}_\infty$ .

- Each edge  $e$  consists a pair of half-edges  $\{h_j, h_{j'}\}$ , and  $e$  is equipped with two numbers  $\delta(e), d(e) \in \mathbb{Q}$ . Here we call  $h_j$  and  $h_{j'}$  half edges and the subscript  $j$  or  $j'$  is taken in the set  $\{X_0, \mathcal{D}_\infty, X_\infty\}$ . A half-edge  $h_j$  labeled by  $j$  is incident to a vertex labeled by  $j$ .
- Each half-edge  $h$  and each leg  $l$  has an element  $m(h)$  or  $m(l)$  in  $C \times \mathbb{C}^* \times \mathbb{C}^*$ .
- The legs are labeled with the numbers  $\{1, \dots, m\}$  and each leg is incident to a unique vertex.

For  $j \in \{X_0, \mathcal{D}_\infty, X_\infty\}$ , we will use the symbol  $\mathfrak{R}_j$  to mean the space  $j$ . By the “valence” of a vertex  $v$ , denoted  $\text{val}(v)$ , we mean the total number of incident half-edges and legs. For each  $\mathbb{C}^*$ -fixed stable map  $f : (C; q_1, \dots, q_m) \rightarrow \mathfrak{R}$  in  $\mathcal{K}_{0, \vec{m}}(\mathfrak{R}, (\beta_\star, \frac{\delta}{r}, d))$ , we can associate a decorated graph  $\Gamma$  in the following way.

- Each edge  $e$  corresponds to a genus-zero component  $C_e$  of restricted stable-map degree  $(0, \frac{\delta(e)}{r}, d(e))$ . Thus  $f$  maps  $C_e$  constantly to the base. There are two distinguished points (we will also call ramification points)  $q_j$  and  $q_{j'}$  on  $C_e$  satisfying that  $q_j$  maps to  $\mathfrak{R}_j$  and  $q_{j'}$  maps to  $\mathfrak{R}_{j'}$ , respectively. There are two corresponding half-edges  $h_j$  and  $h_{j'}$  associated to  $q_j$  and  $q_{j'}$  respectively.
- Each vertex  $v$  for which  $j(v)$  (with unstable exceptional cases noted below) corresponds to a maximal sub-curve  $C_v$  of  $C$  which maps totally into  $\mathcal{X}_0$ , then the restriction of  $f$  to  $C_v$  defines a twisted stable map in

$$\mathcal{K}_{0, \text{val}(v)}(\mathfrak{R}_{j(v)}, \beta(v)) .$$

The label  $\beta(v)$  denotes the degree coming from the composition  $\text{pr}_{r,s} \circ f|_{C_v}$  of the restriction and projection to the base.

- A vertex  $v$  is *unstable* if stable twisted maps of the type described above do not exist. In this case,  $v$  corresponds to a single point of the component  $C_e$  for each incident edge  $e$ , which may be a node at which  $C_e$  meets another edge curve  $C_{e'}$ , a marked point of  $C_e$ , or an unmarked point.
- The index  $m(l)$  on a leg  $l$  indicates the rigidified inertia stack component  $\bar{I}_{m(l)}\mathfrak{R}$  of  $\mathfrak{R}$  on which the gerby marked point corresponding to the leg  $l$  is evaluated.
- A half-edge  $h$  of an edge  $e$  corresponds a ramification point  $q \in C_e$  such that  $f(q) \in \mathfrak{R}_j$  if  $h$  is labeled by  $j$ . Then  $m(h)$  indicates the rigidified inertia component  $\bar{I}_{m(h)}\mathfrak{R}$  of  $\mathfrak{R}$  on which the ramification point  $q$  associated with  $h$  is evaluated.

Moreover, the decorated graph from a fixed stable map should satisfy the following:

- (1) (Monodromy constraint for edge) For each edge  $e = \{h_0 := h_{x_0}, h_\infty := h_{\mathcal{D}_\infty}\}$  linking  $X_0$  and  $\mathcal{D}_\infty$ , we have that  $m(h_0) = (c^{-1}, e^{\frac{\delta(e)}{s}}, 1)$  and  $m(h_\infty) = (c, 1, e^{\frac{\delta(e)}{r}})$  for some  $c \in C$ . Moreover the choice of  $c$  should satisfy that

$$\delta(e) = \text{age}_c(L) \bmod \mathbb{Z} ,$$

which follows from the fact that we have an isomorphism of line bundles  $f|_{C_e}^* \mathcal{O}(sX_0) \cong f|_{C_e}^* (\mathcal{O}(rP_Y) \otimes \text{pr}_{r,s}^* L^\vee)$  on  $C_e$  and then the ages of both line bundles at  $q_\infty$  should be equal.

- (2) (Monodromy constraint for vertex) For each vertex  $v$ , let  $I_v \subset [m]$  be the set of incident legs, and  $H_v$  be the set of incident half-edges. For each  $i \in I_v$ , write  $m(l_i) = (c_i, a_i, b_i)$ , and for each  $h \in H_v$ , write  $m(h) = (c_h, a_h, b_h)$ . If  $v$  is labeled by 0, we have that  $b_i, b_h$  are equal to 1, and

$$e^{\frac{\beta(v)(L)}{s}} \times \prod_{i \in I_v} a_i \times \prod_{h \in H_v} a_h^{-1} = 1 .$$

If  $v$  is labeled by  $\infty$ , then we have  $a_i$  and  $a_h$  are all equal to 1 and

$$e^{-\frac{\beta(v)(L)}{r}} \times \prod_{i \in I_v} b_i \times \prod_{h \in E_v} b_h^{-1} = 1 .$$

In particular, we note that the decorations at each stable vertex  $v$  yield a tuple

$$\overrightarrow{val}(v) \in (C \times \mathbb{C}^* \times \mathbb{C}^*)^{\text{val}(v)}$$

recording the multiplicities at every special point of  $C_v$ . Then the restriction gives a stable map in

$$\mathcal{K}_{0, \overrightarrow{val}(v)}(\mathfrak{R}_{j(v)}, \beta(v)) .$$

Assume that  $r, s$  are sufficiently large primes. We will do a similar localization analysis as in §3.2.

**4.2.1. Vertex contribution.** Assume that  $v$  is a stable vertex, the localization contributions for  $X_0$  and  $\mathcal{D}_\infty$  repeat the discussions for  $\mathcal{D}_0$  and  $\mathcal{D}_\infty$  in 3.2.1 with the replacement of the words  $\mathcal{D}_0$  and  $\mathcal{D}_\infty$  by  $X_0$  and  $\mathcal{D}_\infty$  respectively, and one additional change that the *inverse of the Euler class* of the virtual normal bundle for  $\mathcal{D}_\infty$  should be equal to

$$\frac{1}{\lambda} \cdot \sum_{d \geq 0} c_d(-R^\bullet \pi_* f^*(L)^{\frac{1}{r}}) \left( \frac{-\lambda}{r} \right)^{|E(v)|-1-d} .$$

If  $v$  is a stable vertex labeled by  $X_\infty$ , then the vertex moduli  $\mathcal{M}_v$  is given by  $\mathcal{K}_{0, \overrightarrow{val}(v)}(X_\infty, \beta(v))$  and the fixed part of perfect obstruction theory gives rise to  $[\mathcal{K}_{0, \overrightarrow{val}(v)}(X_\infty, \beta(v))]^{\text{vir}}$ . The movable part of the perfect obstruction theory yields the *inverse of the Euler class* of the virtual normal bundle which is equal to  $\frac{1}{\lambda}$ .

When  $v$  is an unstable vertex over 0 (resp.  $\infty$ ), let  $h$  be the half-edge incident to  $v$ , the vertex moduli  $\mathcal{M}_v$  is defined to be  $\tilde{I}_{m(h)-1} \mathcal{D}_0$  (resp.  $\tilde{I}_{m(h)-1} \mathcal{D}_\infty$ ) with  $[\mathcal{M}_v]^{\text{vir}} = [\mathcal{M}_v]$  and Euler class of virtual normal bundle equal to 1 (resp.  $\frac{1}{\lambda}$ ).

**4.2.2. Edge contribution.** Here we will only consider the type of edges linking the vertex labeled by  $X_0$  and the vertex labeled by  $\mathcal{D}_\infty$  as it is only the type of edge which appears in the localization contribution of 4.3, see Lemma 4.3 for more details.

Let  $e = \{h_0 := h_{X_0}, h_\infty := h_{\mathcal{D}_\infty}\}$  be an edge in  $\Gamma$  with decorated degree  $\delta(e) \in \mathbb{Q}$  and decorated multiplicities  $m(h_0)$  and  $m(h_\infty)$ . Write  $m(h_\infty) = (c, 1, e^{\frac{\delta(e)}{r}})$ , let  $a_e := a(c)$  be the number as Assumption 2.1, we will also write  $a$  as  $a_e$  for simplicity if there is no confusion. Then we define the edge moduli  $\mathcal{M}_e$  to be the root stack  ${}^{as\delta(e)}\sqrt{L^\vee/I_e Y}$ . Then the virtual cycle  $[\mathcal{M}_e]^{\text{vir}}$  coming from the fix part of the obstruction theory is the fundamental class  $[\mathcal{M}_e]$  and the *inverse of the Euler class* of the virtual normal bundle is  $\prod_{i=1}^{-1-\lfloor -\delta(e) \rfloor} \left( \lambda + \frac{i}{\delta(e)} (c_1(L) - \lambda) \right)$ .<sup>11</sup> In practice, we will use an equivalent form

$$\frac{1}{\lambda} \prod_{0 \leq j < \delta(e)} \left( c_1(L) + (\delta(e) - j) \frac{\lambda - c_1(L)}{\delta(e)} \right) .$$

We note that  $\mathcal{M}_e$  allows a finite étale map into the corresponding fixed-loci in  $\mathcal{K}_{0,2}(\mathfrak{R}, (0, \frac{\delta(e)}{r}, 0))$  of degree  $\frac{1}{as}$ . See appendix A.2 for more details.

<sup>11</sup>Here, for any rational number  $q$ , we denote  $\lfloor q \rfloor$  to be the largest integer no larger than  $q$ .



4.2.3. *Node contributions.* The deformations in  $\mathcal{K}_{0,\vec{m}}(\mathfrak{R}, (\beta, \frac{\delta}{r}, 0))$  smoothing a node contribute to the Euler class of the virtual normal bundle as the first Chern class of the tensor product of the two cotangent line bundles at the branches of the node. For nodes at which a component  $C_e$  meets a component  $C_v$  over the vertex  $X_0$ , this contribution is

$$\frac{\lambda - c_1(L)}{a_e s \delta(e)} - \frac{\bar{\psi}_v}{a_e s}.$$

For nodes at which a component  $C_e$  meets a component  $C_v$  at the vertex over  $\mathcal{D}_\infty$ , this contribution is

$$\frac{-\lambda + c_1(L)}{a_e r \delta(e)} - \frac{\bar{\psi}_v}{a_e r}.$$

There is also one dimensional piece from deformations of maps (corresponding to the normal direction of  $E$  in  $\mathfrak{R}$ ) at the node over  $\mathcal{D}_\infty$ , which contributes  $\frac{1}{\lambda}$  as the Euler class of the virtual normal bundle to the localization.

We will not need node contributions from other types of nodes as the above types suffice the need in this paper, see Lemma 4.3 for more details.

4.2.4. *Total contribution.* Here we will only consider two types of graph: (1) all edges in  $\Gamma$  connects  $X_0$  and  $\mathcal{D}_\infty$ ; (2)  $\Gamma$  has only one vertex labeled by  $X_\infty$  and no edges. They are all we need in the later localization analysis (see Lemma 4.3 for more details).

The localization contribution from the decorated graph of type (2) is

$$-\frac{[\mathcal{M}_v]^{\text{vir}}}{\lambda}.$$

For any decorated graph  $\Gamma$  of type (1), we define  $F_\Gamma$  to be

$$\prod_{v:j(v)=0} \mathcal{M}_v \times \bar{I}_\mu X_0 \prod_{e \in E} \mathcal{M}_e \times \bar{I}_\mu \mathcal{D}_\infty \prod_{v:j(v)=\infty} \mathcal{M}_v$$

of the following diagram:

$$\begin{array}{ccc} F_\Gamma & \longrightarrow & \prod_{v:j(v)=0} \mathcal{M}_v \times \prod_{e \in E} \mathcal{M}_e \times \prod_{v:j(v)=\infty} \mathcal{M}_v \\ \downarrow & & \downarrow \text{ev}_{\text{nodes}} \\ \prod_E (\bar{I}_\mu X_0 \times \bar{I}_\mu \mathcal{D}_\infty) & \xrightarrow{(\Delta^0 \times \Delta^\infty)^{|E|}} & \prod_E (\bar{I}_\mu X_0)^2 \times (\bar{I}_\mu \mathcal{D}_\infty)^2, \end{array}$$

where  $\Delta^0 = (id, \iota)$  (resp.  $\Delta^\infty = (id, \iota)$ ) is the diagonal map of  $\bar{I}_\mu X_0$  (resp.  $\bar{I}_\mu \mathcal{D}_\infty$ ). Here the right-hand vertical map is the product of evaluation maps at the two branches of each gluing node.

We define  $[F_\Gamma]^{\text{vir}}$  to be:

$$\prod_{v:j(v)=0} [\mathcal{M}_v]^{\text{vir}} \times \bar{I}_\mu X_0 \prod_{e \in E} [\mathcal{M}_e]^{\text{vir}} \times \bar{I}_\mu \mathcal{D}_\infty \prod_{v:j(v)=\infty} [\mathcal{M}_v]^{\text{vir}}.$$

Then the contribution of decorated graph  $\Gamma$  to the virtual localization is:

$$(4.1) \quad \text{Cont}_\Gamma = \frac{\prod_{e \in E} sa_e}{|\text{Aut}(\Gamma)|} (\iota_\Gamma)_* \left( \frac{[F_\Gamma]^{\text{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\text{vir}})} \right).$$

Here  $\iota_F : F_\Gamma \rightarrow \mathcal{K}_{0,\vec{m}}(\mathfrak{R}, (\beta, \frac{\delta}{r}, d))$  is a finite étale map of degree  $\frac{|\text{Aut}(\Gamma)|}{\prod_{e \in E} sa_e}$  into the corresponding  $\mathbb{C}^*$ -fixed loci. The virtual normal bundle  $e^{\mathbb{C}^*}(N_\Gamma^{\text{vir}})$  is the product of virtual normal bundles from vertex contributions, edge contributions and node contributions.

**4.3. Auxiliary cycle.** Take  $L_1 = L$  and  $r = 1$  in 2.4 and write  $\vec{w}$  to be the weight. Let  $\mu^X(z) = \sum_{\beta, \vec{k}, c} q^\beta \mathbf{t}^{\vec{k}} \mu_{\beta, \vec{k}, c}^X(z)$  be an admissible series in  $H_{CR}^*(X, \mathbb{C})[z][[t_1, \dots, t_N]][\text{Eff}(X)]$ . For any triple  $(\beta, \vec{k}, c) \in \text{Eff}(X) \times \mathbb{Z}_{\geq 0}^N \times C$ , we define  $J_{\beta, \vec{k}, c}^{X, tw}(\mu^X, z)$  to be

$$(4.2) \quad \left( \mu_{\beta, \vec{k}, c}^X(z) + \text{Coeff}_{\mathbf{t}^{\vec{k}}} \left[ \sum_{m \geq 0} \sum_{\substack{(\beta_j, \vec{k}_j, c_j)_{j=1}^m \in \text{Adm}^m \\ \beta_1 + \dots + \beta_m + \beta_* = \beta}} \frac{1}{m!} \phi^\alpha \langle \mathbf{t}^{\vec{k}_1} \mu_{\beta_1, \vec{k}_1, c_1}^{\mathbf{X}}(-\bar{\psi}_1), \dots, \mathbf{t}^{\vec{k}_m} \mu_{\beta_m, \vec{k}_m, c_m}^{\mathbf{X}}(-\bar{\psi}_m), \frac{\phi_\alpha}{z - \bar{\psi}_*} \rangle_{0, \vec{m} \cup \star, \beta_*}^X \right] \right) \prod_{0 \leq j < \beta(L) + (\vec{w}, \vec{k})} (c_1(L) + (\beta(L) + (\vec{w}, \vec{k}) - j)z) .$$

where  $\vec{m} \cup \star = (c_1^{-1} \dots, c_m^{-1}, c) \in C^{m+1}$ . We note that  $J_{0, \vec{0}, c}^{X, tw} = 0$ , then the formal series

$$J^{X, tw}(q, \mu^X, z) = z + \sum_{\beta, \vec{k}, c} q^\beta \mathbf{t}^{\vec{k}} J_{\beta, \vec{k}, c}^{X, tw}(\mu^X, z)$$

associated with the input  $\mu^X(z)$  above is an admissible series and near  $z$ .

For any admissible pair  $(\beta, \vec{k}, c)$  with  $\beta(L) + (\vec{w}, \vec{k}) > 0$ , denote  $\delta = \beta(L) + (\vec{w}, \vec{k})$ . Assume that  $r, s$  are sufficiently large primes. For any nonnegative integer  $b$ , we will also compare (3.7) to the following auxiliary cycle:

$$(4.3) \quad \sum_{m=0}^{\infty} \sum_{\substack{(\beta_i, \vec{k}_i, c_i) \in \text{Adm}^m, \beta_* \in \text{Eff}(X) \\ \beta_* + \sum_{i=1}^m \beta_i = \beta \\ \vec{k}_1 + \dots + \vec{k}_m = \vec{k}}} \frac{1}{m!} (\widetilde{EV}_*)_* \left( \prod_{i=1}^m ev_i^*(\text{pr}_{r,s}^*(\mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}^X(-\bar{\psi}_i))) \cap \bar{\psi}_*^b \cap [\mathcal{K}_{0, \vec{m} \cup \star}(\mathfrak{R}, (\beta_*, \frac{\delta}{r}, 0))]^{\text{vir}} \right)$$

Here an explanation of the notations is in order:

- (1) For degrees  $\beta_*, \beta_1, \dots, \beta_m$  in  $\text{Eff}(X)$  and tuples  $\vec{k}_1, \dots, \vec{k}_m$  in  $\mathbb{Z}_{\geq 0}^N$  with  $\sum_{i=1}^m \beta_i + \beta_* = \beta$  and  $\vec{k}_1 + \dots + \vec{k}_m = \vec{k}$ , let  $(c_1, \dots, c_m) \in C^m$  such that  $(\beta_i, \vec{k}_i, c_i)$  are admissible pairs. Write  $\delta_i = \beta_i(L) + (\vec{w}, \vec{k}_i)$ , we define  $\vec{m} \cup \star$  to be the  $m+1$ tuple

$$((c_1^{-1}, e^{\frac{\delta_1}{s}}, 1), \dots, (c_m^{-1}, e^{\frac{\delta_m}{s}}, 1), (c, 1, e^{\frac{\delta}{r}})) .$$

- (2) Note  $\bar{I}_{m*} \mathfrak{R}$  is canonically isomorphic to  $\bar{I}_c Y \times \mathbb{P}^1$ , it has a natural projection map  $p : \bar{I}_{m*} \mathfrak{R} \rightarrow \bar{I}_c Y$ . Let  $EV_*$  be the morphism which is a composition of the following maps:

$$\mathcal{K}_{0, \vec{m} \cup \star}(\mathfrak{R}, (\beta_*, \frac{\delta}{r}, 0)) \xrightarrow{ev_*} \bar{I}_{m*} \mathfrak{R} \xrightarrow{p} \bar{I}_c Y .$$

Then we define  $(\widetilde{EV}_*)_*$  to be

$$\iota_*(r_*(EV_*)_*)$$

as in 2.1. Note here  $r_*$  is the order of the band from the gerbe structure of  $\bar{I}_\mu Y$  but not  $\bar{I}_\mu \mathfrak{R}$ .

Applying  $\mathbb{C}^*$ -localization to the 4.3, we will prove the following:

**Proposition 4.2.** *With the notation as above, for  $b \geq 0$ , we have the following recursive relation:*

$$\begin{aligned}
& [i^* J_{\beta, \vec{k}, c}^{X, tw}(\mu^X, z)]_{z^{-b-1}} \\
&= \left[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\substack{\Gamma \in \Lambda_{\beta, \vec{k}, c, m, n} \\ \Gamma \text{ is stable}}} \frac{1}{m!n!} (ev_*)_* \left( \sum_{d=0}^{\infty} \epsilon_* (c_d(-R^\bullet \pi_* f^* L^{\frac{1}{r}})) \left( \frac{\lambda}{r} \right)^{-1+m-d} (-1)^d \right. \right. \\
(4.4) \quad & \left. \cap [\mathcal{K}_{0, \overrightarrow{m+n} \cup \star}(\sqrt{L/Y}, \beta_*)]^{\text{vir}} \right) \cap \prod_{i=1}^m \frac{ev_i^* \left( \frac{1}{\delta_i} i^* \mathbf{t}^{\vec{k}_i} J_{\beta_i, \vec{k}_i, c_i}^{X, tw}(\mu^X, z) \right) \big|_{z = \frac{\lambda - c_1(L)}{\delta_i}}}{\frac{\lambda - ev_i^* c_1(L)}{r \delta_i} + \frac{\bar{\psi}_i}{r}} \\
& \left. \cap \prod_{i=m+1}^{m+n} ev_i^* \left( \mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}^X(-\bar{\psi}_i) \cap \bar{\psi}_*^b \right) \right]_{\mathbf{t}^{\vec{k}} \lambda^{-1}}.
\end{aligned}$$

Here  $\delta_i = \beta_i(L) + (\vec{w}, \vec{k}_i)$ , and  $\epsilon : \mathcal{K}_{0, \overrightarrow{m+n} \cup \star}(\sqrt{L/Y}, \beta_*) \rightarrow \mathcal{K}_{0, \overrightarrow{m+n} \cup \star}(Y, \beta_*)$  is the natural structural morphism by forgetting root-gerbe structure of  $\sqrt{L/Y}$  (c.f. [TT21]), where we choose the tuple  $\overrightarrow{m+n} \cup \star$  for  $\mathcal{K}_{0, \overrightarrow{m+n} \cup \star}(\sqrt{L/Y}, \beta_*)$  to be

$$((c_1^{-1}, 1, e^{\frac{-\delta_1}{r}}), \dots, (c_{m+n}^{-1}, 1, e^{\frac{-\delta_{m+n}}{r}}), (c, 1, e^{\frac{\delta}{r}})) \in (C \times \mathbb{C}^* \times \mathbb{C}^*)^{m+n+1},$$

and choose the tuple  $\overrightarrow{m+n} \cup \star$  for  $\mathcal{K}_{0, \overrightarrow{m+n} \cup \star}(Y, \beta_*)$  to be

$$(c_1^{-1}, \dots, c_{m+n}^{-1}, c) \in C^{m+n+1}.$$

Our main strategy to prove this is to apply  $\mathbb{C}^*$ -localization to the integral (4.3) and then extract the  $\lambda^{-2}$ -coefficient (or equivalently  $\mathbf{t}^{\vec{k}} \lambda^{-2}$ -coefficient). Using the polynomiality of (4.3), we have that the  $\lambda^{-2}$  coefficient must be zero, which yields the desired result. To simplify the localization calculation, we need the following two lemmas which limit the types of localization graphs one need to consider.

**Lemma 4.3.** *Let  $\Gamma$  be a decorated graph for  $\mathcal{K}_{0, \overrightarrow{m} \cup \star}(\mathfrak{R}, (\beta, \frac{\delta}{r}, 0))$ . If the corresponding fixed loci  $F_\Gamma$  is non-empty, then there is no edge of  $\Gamma$  linking  $\mathcal{D}_\infty$  and  $X_\infty$  and no edge of  $\Gamma$  linking  $X_0$  and  $X_\infty$ .*

*Proof.* Let  $f : C \rightarrow \mathfrak{R}$  be a  $\mathbb{C}^*$ -fixed stable map in  $\mathcal{K}_{0, \overrightarrow{m} \cup \star}(\mathfrak{R}, (\beta, \frac{\delta}{r}, 0))$ , we have the pairing  $(f_*([C]), [X_\infty]) = 0$ . For each stable vertex  $v$ , obviously, we have the pairing  $(f_*([C_v]), [X_\infty]) = 0$ . If  $e$  is an edge, there three possibilities:

- (1) The edge  $e$  links a vertex labeled by  $\mathcal{D}_\infty$  and  $X_\infty$ .
- (2) The edge  $e$  links a vertex labeled by  $\mathcal{D}_\infty$  and  $X_0$ .
- (3) The edge  $e$  links a vertex labeled by  $X_0$  and  $X_\infty$ .

In the first two cases, we have the pairing  $(f_*([C_e]), [X_\infty]) > 0$ , while in the third case, the pairing  $(f_*([C_e]), [X_\infty]) = 0$ . As the total degree of  $f^* \mathcal{O}(X_\infty)$  on  $C$  is zero, we see that the degree of the restriction of  $f^* \mathcal{O}(X_\infty)$  to each component  $C_v$  or  $C_e$  must be zero, which implies that there is no edge linking  $\mathcal{D}_\infty$  and  $X_\infty$  and no edge linking  $X_0$  and  $X_\infty$ .  $\square$

**Remark 4.4.** The above lemma also works for the case when  $\delta = 0$ , and the proof is verbatim.

By the above the discussion, they will only be two types of graphs which will contribute to the integral (4.3): (1) all edges in  $\Gamma$  connects  $X_0$  and  $\mathcal{D}_\infty$ ; (2)  $\Gamma$  has only one vertex labeled by  $X_\infty$  and no edges. Now let's assume the graph  $\Gamma$  is of type 1, we have the following lemma by using a similar argument as in Lemma 3.5 by using the line bundle  $N := f^*(\mathcal{O}(-P_Y))$ .

**Lemma 4.5.** *Assume  $r, s$  is sufficiently large. If localization graph  $\Gamma$  has more than one vertex labeled by  $\mathcal{D}_\infty$ , then the corresponding  $C^*$ -fixed loci moduli  $F_\Gamma$  is empty, therefore it will contribute zero to (4.3).*

Now we are ready to prove Proposition 4.2.

*Proof.* If a localization graph  $\Gamma$  has a vertex labeled by  $X_\infty$ , by Lemma 4.3, there will be no edges in  $\Gamma$ . Then  $\Gamma$  is made of a single vertex. Such type of graph will contribute

$$(4.5) \quad \frac{-1}{\lambda} \sum_{m \geq 0} \sum_{\substack{(\beta_j, \vec{k}_j, c_j)_{j=1}^m \in \text{Adm}^m \\ \beta_\star \in \text{Eff}(X) \\ \beta_1 + \dots + \beta_m + \beta_\star = \beta \\ \vec{k}_1 + \dots + \vec{k}_m = \vec{k}}} \frac{1}{m!} \phi^\alpha(\mathbf{t}^{\vec{k}_1} \mu_{\beta_1, \vec{k}_1, c_1}^{\mathbf{X}}(-\bar{\psi}_1), \dots, \mathbf{t}^{\vec{k}_m} \mu_{\beta_m, \vec{k}_m, c_m}^{\mathbf{X}}(-\bar{\psi}_m), \frac{\phi_\alpha}{z - \bar{\psi}_\star})_{0, \vec{m} \cup \star, \beta_\star}^{X_\infty}$$

to 4.3.

The other type of graph which contributes to (4.4) must satisfy that all the vertexes are labeled by either 0 or  $\infty$  and all the edges only link  $X_0$  and  $\mathcal{D}_\infty$ . Using Lemma 4.5, we can apply the same strategy of the Proposition 3.6 to do the localization computation. Denote  $v_\star$  to be the only vertex of  $\Gamma$  labeled by  $\infty$ , then the remaining types of localization graph is a star-shaped graph introduced in the proof of 3.6 and can be sorted as below:

- (1) (Type I) The vertex  $v_\star$  is unstable and there is only one edge incident to  $v_\star$  in  $\Gamma$ , the vertex  $v$  labeled by 0 is unstable;
- (2) (Type II) The vertex  $v_\star$  is unstable and there is only one edge incident to  $v_\star$  in  $\Gamma$ , the vertex  $v$  labeled by 0 is stable;
- (3) (Type III) The vertex  $v_\star$  is stable.

The graph of type I will contribute

$$(4.6) \quad \frac{1}{\delta \lambda} \mathbf{t}^{\vec{k}} \cdot \left( \frac{\lambda - c_1(L)}{\delta} \right)^b \cdot \prod_{0 \leq j < \delta} (c_1(L) + (\delta - j) \frac{\lambda - c_1(L)}{\delta}) i^* \mu_{\beta, \vec{k}, c}^{\mathbf{X}} \left( \frac{\lambda - c_1(L)}{\delta} \right)$$

to 4.3.

The graph of type II will contribute

$$(4.7) \quad \frac{1}{\delta \lambda} \prod_{0 \leq j < \delta} \left( \frac{\lambda - c_1(L)}{\delta} \right)^b \cdot (c_1(L) + (\delta - j) \frac{\lambda - c_1(L)}{\delta}) \sum_{m \geq 0} \sum_{\substack{(\beta_i, \vec{k}_i, c_i)_{i=1}^m \in \text{Adm}^m \\ \beta_\star \in \text{Eff}(X), \sum_i \beta_i + \beta_\star = \beta \\ \sum_i \vec{k}_i = \vec{k}}} \frac{1}{m!} \\ \cdot i^* \phi^\alpha \left( \int_{[\mathcal{K}_{0, \vec{m} \cup \star}(X, \beta_\star)]^{\text{vir}}} \prod_{i=1}^m ev_i^*(\mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}^{\mathbf{X}}(-\bar{\psi}_i)) \cup \frac{ev_\star^* \phi_\alpha}{\frac{\lambda - ev_\star^* c_1(L)}{\delta} - \bar{\psi}_\star} \right).$$

to 4.3.

The graph of type III will contribute

$$(4.8) \quad \frac{1}{\lambda} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\substack{\Gamma \in \Lambda_{\beta, \vec{k}, c, m, n} \\ \Gamma \text{ is stable}}} \frac{1}{m!} \frac{1}{n!} (ev_\star)_* \left( \sum_{d=0}^{\infty} \epsilon_* (c_d(-R^\bullet \pi_* f^* L^{\frac{1}{r}})) \left( \frac{-\lambda}{r} \right)^{-1+m-d} \right. \\ \cap [\mathcal{K}_{0, \vec{m}+n \cup \star}(\sqrt[r]{L/Y}, \beta_\star)]^{\text{vir}} \cap \prod_{i=1}^m \frac{ev_i^*(\frac{1}{\delta_i} i^* \mathbf{t}^{\vec{k}_i} J_{\beta_i, \vec{k}_i, c_i}^{X, tw}(z))|_{z=\frac{\lambda - c_1(L)}{\delta_i}}}{-\frac{\lambda - ev_i^* c_1(L)}{r \delta_i} - \frac{\bar{\psi}_i}{r}} \\ \left. \cap \prod_{i=m+1}^{m+n} ev_i^*(\mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}^{\mathbf{X}}(-\bar{\psi}_i)) \cap \bar{\psi}_\star^b \right).$$

to 4.3.

Note that the sum of 4.6 and 4.7 is equal to

$$\frac{1}{\delta\lambda} i^* \mathbf{t}^{\vec{k}} (z^b \cdot J_{\beta, \vec{k}, c}^{X, tw}(z)) \Big|_{z=\frac{\lambda-c_1(L)}{\delta}} ,$$

of which the  $\mathbf{t}^{\vec{k}} \lambda^{-2}$ -coefficient is equal to

$$[i^* J_{\beta, \vec{k}, c}^{X, tw}(\mu^X, z)]_{z^{-b-1}} .$$

Indeed, expand  $z^b \cdot J_{\beta, \vec{k}, c}^{X, tw}(z)$  into Laurent series in  $z$ :

$$J_{\beta, \vec{k}, c}^{X, tw}(\mu^X, z) = \cdots + zJ_0 + J_1 + z^{-1}J_2 + \cdots$$

we see that the  $z^{-b-1}$ -coefficient of  $J_{\beta, \vec{k}, c}^{X, tw}(z)$  and  $\lambda^{-1}$ -coefficient of

$$\frac{1}{\delta\lambda} i^* (z^b \cdot J_{\beta, \vec{k}, c}^{X, tw}(z)) \Big|_{z=\frac{\lambda-c_1(L)}{\delta}}$$

are both equal to  $J_{b+1}$ .

Now our auxiliary cycle 4.3 is equal to the sum of 4.5, 4.6, 4.7 and 4.8. By the polynomiality of 4.3, the  $\mathbf{t}^{\vec{k}} \lambda^{-2}$ -coefficient of the auxiliary cycle is zero, which implies that the sum of  $\mathbf{t}^{\vec{k}} \lambda^{-2}$ -coefficient of 4.5, 4.6, 4.7 and 4.8 is zero, which proves the proposition 4.2.  $\square$

**4.4. Statement of the main theorem and proof.** With the notations in this section, now we state our main theorem:

**Theorem 4.6** (Main theorem). *Let  $X$  be a smooth Deligne-Mumford stack with projective coarse moduli. Let  $E := \bigoplus_{j=1}^r L_j$  be a direct sum of semi-positive line bundles over  $X$  with a regular section of  $E$  which cuts off a smooth complete intersection  $Y \subset X$ . Let  $\mu^X(z) := \sum_{(\beta, \vec{k}, c) \in \text{Adm}} q^\beta \mathbf{t}^{\vec{k}} \mu_{\beta, \vec{k}, c}^X(z)$  be an admissible series and  $J^{X, tw}(q, \mu^X, z)$  be the hypergeometric modification of  $J^X(q, \mu^X, z)$  as in 1.3. Let  $i : \bar{I}_\mu Y \rightarrow \bar{I}_\mu X$  be the inclusion, then the series  $i^* J^{X, tw}(q, \mu^X, -z)$  is a point on the Lagrange cone of  $Y$ . More precisely, for any admissible pair  $(\beta, \vec{k}, c)$ , if we define*

$$\mu_{\beta, \vec{k}, c}^{X, tw}(z) := [J_{\beta, \vec{k}, c}^X(\mu^X, z) \prod_{j=1}^r \prod_{0 \leq m < \beta(L_j) + (\vec{w}_j, \vec{k})} (c_1(L_j) + (\beta(L_j) + (\vec{w}_j, \vec{k}) - m)z)]_+$$

to be the truncation in nonnegative  $z$ -powers. Here  $J_{\beta, \vec{k}, c}^X$  is defined in 3.5. Write

$$\mu^{X, tw}(q, z) := \sum_{(\beta, \vec{k}, c) \in \text{Adm}} q^\beta \mathbf{t}^{\vec{k}} \mu_{\beta, \vec{k}, c}^{X, tw}(z) .$$

Then for any integer  $b \geq 0$ , we have the following relation:

$$(4.9) \quad [J_{\beta, \vec{k}, c}^Y(i^* \mu^{X, tw}, z)]_{z^{-b-1}} = [i^* J_{\beta, \vec{k}, c}^{X, tw}(\mu^X, z)]_{z^{-b-1}} .$$

As  $\mu^{X, tw}$  is also an admissible series in  $H^*(\bar{I}_\mu X, \mathbb{C})[z][[t_1, \dots, t_N]]\llbracket \text{Eff}(X) \rrbracket$ , by induction on the number of line bundles, it's sufficient to prove the case when  $Y$  is a *hypersurface*. Now let's assume that  $Y$  is a hypersurface and use the notations in previous sections.

We will first prove 4.9 for any admissible pair  $(\beta, \vec{k}, c)$  with  $\beta(L) + (\vec{w}, \vec{k}) = 0$ . Note in this case we have  $\mu_{\beta, \vec{k}, c}^{X, tw}(z) = \mu_{\beta, \vec{k}, c}^X(z)$ .

We will adopt the convention that for any substack  $Z$  of  $\mathfrak{X}$  and  $c \in C$ , we will write  $\bar{I}_c Z := \bar{I}_\mu Z \cap \bar{I}_{(c, 1, 1)} Z$ . Now fix an admissible pair  $(\beta, \vec{k}, c)$  such that  $\beta(L) + (\vec{w}, \vec{k}) = 0$ , we

have  $\text{age}_c L = 0$ . Then  $\bar{I}_c Y$  is proper hypersurface of  $\bar{I}_c X$  with normal bundle  $L$ .<sup>12</sup> Recall that the divisor  $E$  of  $\mathfrak{R}$  is isomorphic to  $\mathbb{P}Y_{r,s}$ . Denote by  $\text{pr}_{r,s}^E : E \rightarrow Y$  the projection to the base, which induces a morphism on the corresponding rigidified inertia stacks, which we still denote to be  $\text{pr}_{r,s}^E$ . We have  $\bar{I}_{c-1} E \cong (\text{pr}_{r,s}^E)^{-1}(\bar{I}_{c-1} Y)$  and  $\bar{I}_{c-1} E$  is a hypersurface of  $\bar{I}_{c-1} \mathfrak{R}$ .

Denote by  $i_E$  the inclusion from  $\bar{I}_{c-1} E$  to  $\bar{I}_{c-1} \mathfrak{R}$ . For any admissible pair  $(\beta, \vec{k}, c)$  with  $\beta(L) + (\vec{w}, \vec{k}) = 0$ , consider the following class

$$(4.10) \quad \sum_{m=0}^{\infty} \sum_{\substack{(\beta_i, \vec{k}_i, c_i)_{i=1}^m \in \text{Adm}^m \\ \beta_* + \sum_{i=1}^m \beta_i = \beta, \beta_* \in \text{Eff}(X) \\ \vec{k}_1 + \dots + \vec{k}_m = \vec{k}}} \frac{1}{m!} (\text{pr}_{r,s}^E)_* \circ i_E^* \circ (\widetilde{ev}_*)_* \left( \mathcal{K}_{0, \vec{m} \cup \star}(\mathfrak{R}, (\beta_*, 0, 0)) \cap \bar{\psi}_*^b \cap \prod_{i=1}^m \text{pr}_{r,s}^* (\mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}^X (-\bar{\psi}_i)) \right).$$

Here for any  $\beta_*, \beta_1, \dots, \beta_m, \vec{k}_1, \dots, \vec{k}_m$  and  $c_1, \dots, c_m$  such that  $(\beta_i, \vec{k}_i, c_i)_{i=1}^m \in \text{Adm}^m$  with  $\sum_{i=1}^m \beta_i + \beta_* = \beta$  and  $\vec{k}_1 + \dots + \vec{k}_m = \vec{k}$ , we associated an element in  $(C \times \mathbb{C}^* \times \mathbb{C}^*)^{m+1}$

$$\vec{m} \cup \star := ((c_1^{-1}, 1, 1), \dots, (c_m^{-1}, 1, 1), (c, 1, 1)).$$

$i_E^*$  is the Gysin pull-back from the (equivariant) cohomology of  $\bar{I}_{(c-1, 1, 1)} \mathfrak{R}$  to the (equivariant) cohomology of  $\bar{I}_{(c-1, 1, 1)} E$ . Note that  $\beta_i(L) + (\vec{w}, \vec{k}_i) = 0$  for all  $1 \leq i \leq m$ ,

**Lemma 4.7.** *For any admissible pair  $(\beta, \vec{k}, c)$  with  $\beta(L) + (\vec{w}, \vec{k}) = 0$ , let  $b$  be an nonnegative integer, we have*

$$[J_{\beta, \vec{k}, c}^Y(q, i^* \mu^X, z)]_{z^{-b-1}} = i^* [J_{\beta, \vec{k}, c}^X(q, \mu^X, z)]_{z^{-b-1}},$$

where  $i^*$  is induced from the inclusion from  $\bar{I}_\mu Y$  to  $\bar{I}_\mu X$ .

*Proof.* We will apply  $\mathbb{C}^*$ -localization to (4.10). First we claim there are only two types of localization graphs contributing to the integral (4.10): 1, the localization  $\Gamma_1$  has only one vertex and it's labeled by  $X_0$ ; 2, the localization graph  $\Gamma_2$  has only one vertex and it's labeled by  $\mathcal{D}_\infty$ . Indeed, there is no vertex labeled by  $X_\infty$  as the marking  $q_*$  must go to the divisor  $E$  to make nonzero localization contribution to (4.10). Furthermore, as  $\deg(f^* \mathcal{O}(P_Y)) = 0$ , we claim there is no edge linking  $X_0$  and  $\mathcal{D}_\infty$ . Indeed, recall that we only need to consider edge curve  $C_e$  linking  $X_0$  and  $\mathcal{D}_\infty$  by Lemma 4.3 (see also Remark 4.4). then the claim comes from that the line bundle  $\mathcal{O}(P_Y)$  is of nonnegative degree restricting to all vertex curves  $C_v$  and of positive degree restricting to all edge curves  $C_e$ . The localization contribution from  $\Gamma_1$  is equal to

$$(4.11) \quad (\text{pr}_{r,s}^E)_* \circ i_E^* \circ (\iota_{X_0})_* \left[ \sum_{m=0}^{\infty} \sum_{\substack{(\beta_i, \vec{k}_i, c_i)_{i=1}^m \in \text{Adm}^m \\ \beta_* + \sum_{i=1}^m \beta_i = \beta, \beta_* \in \text{Eff}(X) \\ \vec{k}_1 + \dots + \vec{k}_m = \vec{k}}} \frac{1}{m!} (\widetilde{ev}_*)_* \left( \sum_{d=0}^{\infty} (c_d (-R^\bullet \pi_* f^* (L^\vee)^{\frac{1}{s}})) \left( \frac{\lambda}{s} \right)^{-1-d} \right. \right. \\ \left. \left. \cap [\mathcal{K}_{0, \vec{m} \cup \star}(X_0, \beta_*)]^{\text{vir}} \right) \cap \prod_{i=1}^m ev_i^* (\text{pr}_{r,s}^* (\mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}^X (-\bar{\psi}_i)) \cap \bar{\psi}_*^b) \right].$$

Here  $\iota_{X_0} : \bar{I}_{c-1} X_0 \rightarrow \bar{I}_{c-1} \mathfrak{R}$  is the natural inclusion and  $(\widetilde{ev}_*)_*$  is a morphism from  $H_{\mathbb{C}^*}^*(\mathcal{K}_{0, \vec{m} \cup \star}(X_0, \beta_*))$  to  $H_{\mathbb{C}^*}^*(\bar{I}_{c-1} X_0)$  as defined in 2.1.

<sup>12</sup>Here the normal bundle is a pull-back of the line bundle  $L$  along the natural morphism from  $\bar{I}_\mu X$  to  $X$  by sending  $(x, g)$  to  $x$ .

We have the following commutative diagram where the upper square and outer square are Cartersian:

$$\begin{array}{ccc}
 \bar{I}_{c-1}\mathcal{D}_0 & \xrightarrow{i_{\mathcal{D}_0}} & \bar{I}_{c-1}X_0 \\
 \downarrow \iota_{\mathcal{D}_0} & & \downarrow \iota_{X_0} \\
 \bar{I}_{c-1}E & \xrightarrow{i_E} & \bar{I}_{c-1}\mathfrak{R} \\
 \downarrow \text{pr}_{r,s}^E & & \downarrow \text{pr}_{r,s} \\
 \bar{I}_{c-1}Y & \xrightarrow{i} & \bar{I}_{c-1}X.
 \end{array}$$

Here  $i$  is induced from the inclusion from  $\mathcal{D}_0$  to  $X_0$  and  $\iota_{\mathcal{D}_0}$  is induced from the inclusion from  $\mathcal{D}_0$  to  $E$ . As the age  $\text{age}_c(L) = 0$ , all rows are regular embeddings of codimension one. Then by the commutativity of the proper push-forward and Gysin pullback for the upper square, 4.11 is equal to

$$(4.12) \quad (\text{pr}_{r,s}^E \circ \iota_{\mathcal{D}_0})_* i_E^* (\iota_{\mathcal{D}_0})_* [\cdots].$$

Here the dots inside the big square bracket copies the stuff in the big square bracket of 4.11. Using the outer square, 4.12 is equal to

$$(4.13) \quad i_Y^* (\text{pr}_{r,s} \circ \iota_{X_0})_* [\cdots].$$

As the composition  $\text{pr}_{r,s} \circ \iota_{X_0}$  is induced from natural de-root stackfication of  $X_0 \cong \sqrt[s]{L^\vee/X}$ ,  $(\text{pr}_{r,s} \circ \iota_{X_0})_*$  is the identity map on cohomology groups as it induces an identity map on their coarse moduli spaces and the cohomology group of an orbifold is identified with the cohomology group of its coarse moduli space (see [AGV08] for more details). Therefore 4.13 is equal to

$$(4.14) \quad i^* \left[ \sum_{m=0}^{\infty} \sum_{\substack{(\beta_i, \vec{k}_i, c_i)_{i=1}^m \in \text{Adm}^m \\ \beta_* + \sum_{i=1}^m \beta_i = \beta, \beta_* \in \text{Eff}(X) \\ \vec{k}_1 + \cdots + \vec{k}_m = \vec{k}}} \frac{1}{m!} (\widetilde{ev}_*)_* \left( \sum_{d=0}^{\infty} \epsilon_* (c_d(-R^\bullet \pi_* f^*(L^\vee)^{\frac{1}{s}}) (\frac{\lambda}{s})^{-1-d} \right. \right. \\
 \left. \left. \cap [\mathcal{K}_{0, \vec{m} \cup \star}(X_0, \beta_*)]^{\text{vir}} \right) \cap \prod_{i=1}^m ev_i^* (\mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}^X (-\bar{\psi}_i)) \cap \bar{\psi}_*^b \right],$$

where  $\epsilon : \mathcal{K}_{0, \vec{m} \cup \star}(X_0, \beta_*) \rightarrow \mathcal{K}_{0, \vec{m} \cup \star}(X, \beta_*)$  is induced from the natural structural map forgetting the root of  $X_0 \cong \sqrt[s]{L^\vee/X}$ .

The localization contribution from  $\Gamma_2$  is equal to

$$(4.15) \quad \frac{1}{\lambda} (\text{pr}_{r,s}^E)_* \circ i_E^* \circ (i_{\mathcal{D}_\infty})_* \left[ \sum_{m=0}^{\infty} \sum_{\substack{(\beta_i, \vec{k}_i, c_i)_{i=1}^m \in \text{Adm}^m \\ \beta_* \in \text{Eff}(X), \beta_* + \sum_{i=1}^m \beta_i = \beta \\ \vec{k}_1 + \cdots + \vec{k}_m = \vec{k}}} \frac{1}{m!} (\widetilde{ev}_*)_* \left( \sum_{d=0}^{\infty} (c_d(-R^\bullet \pi_* f^* L^{\frac{1}{r}}) (\frac{-\lambda}{r})^{-1-d} \right. \right. \\
 \left. \left. \cap [\mathcal{K}_{0, \vec{m} \cup \star}(\sqrt[r]{L/Y}, \beta_*)]^{\text{vir}} \right) \cap \prod_{i=1}^m ev_i^* (\text{pr}_{r,s}^* \mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}^X (-\bar{\psi}_i)) \cap \bar{\psi}_*^b \right]$$

Here  $i_{\mathcal{D}_\infty} : \bar{I}_{c-1}\mathcal{D}_\infty \rightarrow \bar{I}_{c-1}\mathfrak{R}$  is the inclusion and  $(\widetilde{ev}_*)_*$  is the morphism from  $H^*(\mathcal{K}_{0, \vec{m} \cup \star}(\mathcal{D}_\infty, \beta_*))$  to  $H^*(\bar{I}_{c-1}\mathcal{D}_\infty)$  as defined in 2.1. The inclusion  $i_{\mathcal{D}_\infty}$  can be rewritten as the composition of



two inclusions  $\iota_{\mathcal{D}_\infty} : \bar{I}_{c-1}\mathcal{D}_\infty \hookrightarrow \bar{I}_{c-1}E$  and  $i_E : \bar{I}_{c-1}E \hookrightarrow \bar{I}_{c-1}\mathfrak{R}$ . Then we have

$$\begin{aligned}
 & \frac{1}{\lambda}(\mathrm{pr}_{r,s}^E)_* \circ i_E^* \circ (i_{\mathcal{D}_\infty})_* \left[ \cdots \right] \\
 &= \frac{1}{\lambda}(\mathrm{pr}_{r,s}^E)_* \circ i_E^* \circ (i_E)_* \circ (\iota_{\mathcal{D}_\infty})_* \left[ \cdots \right] \\
 (4.16) \quad &= (\mathrm{pr}_{r,s}^E)_* (\iota_{\mathcal{D}_\infty})_* \left[ \cdots \right] \\
 &= \left[ \cdots \right].
 \end{aligned}$$

Here the dots inside the big square bracket copies the stuff in the big square bracket of 4.15. The second equality in 4.16 uses the fact that the morphism  $i_E^*(i_E)_* : H_{\mathbb{C}^*}^*(\bar{I}_{c-1}E) \rightarrow H_{\mathbb{C}^*}^*(\bar{I}_{c-1}E)$  is equal to the multiplication map by  $e^{\mathbb{C}^*}(N_{\bar{I}_{c-1}Y/\bar{I}_{c-1}\mathfrak{R}})$  using excess intersection formula, which implies that  $i_E^*(i_E)_*$  acts on the space  $(\iota_{\mathcal{D}_\infty})_* H_{\mathbb{C}^*}^*(\mathcal{D}_\infty)$  by multiplication by  $\lambda$  as  $e^{\mathbb{C}^*}(N_{\bar{I}_{c-1}Y/\bar{I}_{c-1}\mathfrak{R}})|_{\bar{I}_{c-1}\mathcal{D}_\infty} = \lambda$ . Finally 4.15 becomes

$$\begin{aligned}
 (4.17) \quad & \sum_{m=0}^{\infty} \sum_{\substack{(\beta_i, \vec{k}_i, c_i)_{i=1}^m \in \mathrm{Adm}^m, \beta_* \in \mathrm{Eff}(X) \\ \beta_* + \sum_{i=1}^m \beta_i = \beta \\ \vec{k}_1 + \cdots + \vec{k}_m = \vec{k}}} \frac{1}{m!} (\widetilde{ev}_*)_* \left( \sum_{d=0}^{\infty} \epsilon_* (c_d(-R^\bullet \pi_* f^* L^{\frac{1}{r}})) \left( \frac{-\lambda}{r} \right)^{-1-d} \right. \\
 & \left. \cap [\mathcal{K}_{0, \vec{m} \cup \star}(\sqrt[rs]{L/Y}, \beta_*)]^{\mathrm{vir}} \right) \cap \prod_{i=1}^m ev_i^*(i^* \mathbf{t}^{\vec{k}_i} \mu_{\beta_i, \vec{k}_i, c_i}^X(-\bar{\psi}_i)) \cap \bar{\psi}_*^b
 \end{aligned}$$

using the morphism  $\epsilon : \mathcal{K}_{0, \vec{m} \cup \star}(\mathcal{D}_\infty, \beta_*) \rightarrow \mathcal{K}_{0, \vec{m} \cup \star}(Y, \beta_*)$  induced from forgetting the root of  $\mathcal{D}_\infty \cong \sqrt[rs]{L/Y}$ .

Now taking  $\lambda^{-1}$  coefficients of the contributions 4.14 and 4.17 from  $\Gamma_1$  and  $\Gamma_2$ , their sum is zero by polynomiality of 4.10, this finishes the proof of (4.9) with the help of the fact

$$\begin{aligned}
 \epsilon_*([\mathcal{K}_{0, \vec{m} \cup \star}(\sqrt[rs]{L/Y}, \beta_*)]^{\mathrm{vir}}) &= \frac{1}{s} [\mathcal{K}_{0, \vec{m} \cup \star}(X, \beta_*)]^{\mathrm{vir}} \\
 \epsilon_*([\mathcal{K}_{0, \vec{m} \cup \star}(\sqrt[rs]{L/Y}, \beta_*)]^{\mathrm{vir}}) &= \frac{1}{r} [\mathcal{K}_{0, \vec{m} \cup \star}(Y, \beta_*)]^{\mathrm{vir}}
 \end{aligned}$$

proved in [TT21]. □

Now we are well prepared to prove the main theorem 4.6:

*Proof of the main theorem 4.6.* Since  $X$  has projective coarse moduli, there exists a *positive* line bundle  $M$  on  $X$ , i.e.,  $\beta(M) > 0$  for all nonzero degree  $\beta \in \mathrm{Eff}(X)$ . Let's prove the relation 4.9 by induction on the nonnegative rational number  $e := \beta(M) + |\vec{k}|$ .<sup>13</sup> When  $e = 0$ , we have  $(\beta, \vec{k}) = (0, \vec{0})$ , then  $J_{0, \vec{0}, c}^Y(i^* \mu^{X, tw}, z) = J_{0, \vec{0}, c}^{X, tw}(\mu^X, z) = 0$ . Now suppose the relation 4.9 holds for all admissible pair  $(\beta', \vec{k}', c')$  with  $\beta'(M) + |\vec{k}'| < e$  for some positive rational number  $e$ , it remains to show the relation holds for any admissible pair  $(\beta, \vec{k}, c)$  with  $\beta(M) + |\vec{k}| = e$ . If  $\beta(L) + (\vec{w}, \vec{k}) = 0$ , by Proposition 4.7, the relation holds (note in this case, we have  $J_{\beta, \vec{k}, c}^{X, tw}(\mu, z) = J_{\beta, \vec{k}, c}^X(\mu, z)$ ). Otherwise, we use the recursive relations 4.4 and 3.11; in which the terms corresponding to the element  $\Gamma \in \Lambda_{\beta, \vec{k}, c, m, n}$  with  $\beta_* \neq 0$  are the same as their inputs are associated with the extended degrees  $(\beta_i, \vec{k}_i)$  with  $\beta_i(M) + |\vec{k}_i| < e$ , which are the same by induction assumption. We are left to show that the terms in 4.4 and 3.11 corresponding

<sup>13</sup>Here  $|\vec{k}| = \sum_i k_i$ .

to the element  $\Gamma \in \Lambda_{\beta, \vec{k}, c, m, n}$  with  $\beta_* = 0$  are the same. Note  $m + n \geq 2$  as  $\Gamma$  is stable and  $J_{0, \vec{0}, c}^Y(i^* \mu^{X, tw}, z) = J_{0, \vec{0}, c}^{X, tw}(\mu^X, z) = 0$ , then we can still use inductive assumption. This completes the proof.  $\square$

**Remark 4.8.** The above proof can also be extended to the equivariant case, where we assume that  $X$  has an algebraic torus  $T$ -action which preserves  $Y$ . Since we don't have a particular application in mind, we will leave the details to the interested readers.

Let  $\{\phi_\alpha\}$  be a *graded* basis of  $H^*(\bar{I}_\mu X, \mathbb{C})$  such that  $\phi_\alpha \in H^*(\bar{I}_c X)$  for some  $c \in C$ . Let  $\{t_\alpha\}$  be a coordinate system of  $H^*(\bar{I}_\mu X, \mathbb{C})$  corresponding to the basis  $\{\phi_\alpha\}$ . We also put  $\mathbb{Z}_2$ -grading on  $\{t_\alpha\}$  according to  $\phi_\alpha$ . Then we have the following:

**Corollary 4.9.** *Let  $i : \bar{I}_\mu Y \rightarrow \bar{I}_\mu X$  be the inclusion of rigidified inertia stacks and  $\mathbf{t} = \sum_\alpha t_\alpha \phi_\alpha$ . Then the series  $i^* J^{X, tw}(q, \mathbf{t}, -z)$  lies on the Lagrangian cone  $\mathcal{L}_Y$  of  $Y$ .*

## APPENDIX A. EDGE CONTRIBUTION

In the proof of our main theorem, we need to apply torus localization formula to the moduli of stable maps to  $\mathbb{P}Y_{r,s}$  (or to  $\mathfrak{R}$  which factors through  $E$ ), which is an orbi- $\mathbb{P}^1$  bundle over a stack, for which we don't find a suitable reference of fitting in the discussion of the explicit localization contribution from the torus-fixed 1-dimensional orbits (named edge contribution) in our case. In this appendix, we will explain how to compute the edge contributions in 3.2.2 and 4.2.2. To achieve this, we will construct a space called  $\mathcal{M}_e$  with a family of  $\mathbb{C}^*$ -fixed stable maps to  $\mathbb{P}Y_{r,s}$  corresponding to a decorated edge. This space has the property that it allows a finite étale map to the corresponding substack of  $\mathbb{C}^*$ -fixed loci in  $\mathcal{K}_{0,2}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r}))$  (or  $\mathcal{K}_{0,2}(\mathfrak{R}, (0, \frac{\delta}{r}, 0))$ ). Then we can use  $\mathcal{M}_e$  as a substitute for  $\mathbb{C}^*$ -fixed loci in the edge contribution and the explicit description of the family map helps us carry out the localization analysis concretely. To begin with, we will prove a classification result (see Lemma A.1) about  $\mathbb{C}^*$ -fixed stable maps to  $\mathbb{P}Y_{r,s}$  which map to a fiber of  $\mathbb{P}Y_{r,s}$  over  $Y$  using the relation between fundamental groups and  $\mathbb{C}^*$ -fixed stable maps to an orbi- $\mathbb{P}^1$  curve from [LS20]. Note that we will always assume that  $r, s$  are sufficiently large primes in this appendix.

**A.1. Local picture.** Write  $\mathfrak{P} := \mathbb{P}Y_{r,s}$  for simplicity. For any  $\mathbb{C}$ -point  $y \in Y$ , let  $G_y$  be the residual isotropy group of  $y$  in  $Y$ . Let  $\rho : G_y \rightarrow \mathbb{C}^* = \text{Aut}(L|_y)$  be the associated representation. Denote  $\mathfrak{P}_y$  by the fiber product in the following square

$$\begin{array}{ccc} \mathfrak{P}_y & \longrightarrow & \mathfrak{P} \\ \downarrow & & \downarrow \text{pr}_{r,s} \\ \mathbb{B}G_y & \xrightarrow{i} & Y \end{array},$$

where  $i : \mathbb{B}G_y \rightarrow Y$  is the inclusion. Let  $U := \mathbb{C}^2 \setminus \{0\}$ , the fiber curve  $\mathfrak{P}_y$  can be represented as the quotient stack

$$[\mathbb{C}^* \times U / (G_y \times (\mathbb{C}^*)^2)]$$

via the action

$$(m, x_1, x_2)(g, t_1, t_2) = (\rho^{-1}(g)t_1^{-s}t_2^r m, t_1 x_1, t_2 x_2),$$

where  $(m, x_1, x_2) \in \mathbb{C}^* \times U$ ,  $(g, t_1, t_2) \in G_y \times (\mathbb{C}^*)^2$ .

The  $\mathbb{C}^*$ -action on  $\mathfrak{P}$  defined in 3.1 restricts to be a  $\mathbb{C}^*$ -action on  $\mathfrak{P}_y$  by scaling  $x_1$  with weight one. There are two  $\mathbb{C}^*$ -fixed points of  $\mathfrak{P}_y$ : (1) The point 0 corresponding to  $x_1 = 0$ ; (2) the point  $\infty$  corresponding to  $x_2 = 0$ . Now let's describe the isotropy groups of points 0 and  $\infty$ , for which we need the following definition. Let  $G$  be an arbitrary finite group and  $\eta$  be

a character of  $G$ . Assume that image of  $\eta$  is a finite cyclic group  $\mu_t \subset \mathbb{C}^*$  of order  $t$ . In other words, we have the following exact sequence from the action

$$1 \longrightarrow \ker(\eta) \longrightarrow G \xrightarrow{\eta} \mu_t \longrightarrow 1 .$$

For any positive integer  $i$ , we define the central extension  $G(\eta, i)$  of  $G$  by the cyclic group  $\mu_i$ :

$$G(\eta, i) := \{(g, b) \in G \times \mathbb{C}^* | \eta(g) = b^i\} .$$

We note  $G(\eta, i)$  fits in the following Cartesian diagram with all rows and all columns exact:

$$\begin{array}{ccccccc} & & \mu_i & \xrightarrow{=} & \mu_i & & \\ & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \ker(\eta) & \longrightarrow & G(\eta, i) & \xrightarrow{\eta_i} & \mu_{ti} \longrightarrow 1 \\ & & \downarrow = & & \downarrow & & \downarrow a \mapsto a^i \\ 1 & \longrightarrow & \ker(\eta) & \longrightarrow & G & \xrightarrow{\eta} & \mu_t \longrightarrow 1 , \end{array}$$

where the morphism  $\eta_i$  is induced from the projection from  $G \times \mathbb{C}^*$  to the second factor. Then the isotropy group  $G_0$ (resp.  $G_\infty$ ) of 0(resp.  $\infty$ ) is isomorphic to  $G_y(\rho^{-1}, s)$ (resp.  $G_y(\rho, r)$ ). Denote

$$\mathfrak{U}_0 := \mathfrak{P}_y \setminus \{\infty\} \cong [\mathbb{C}/G_y(\rho^{-1}, s)], \text{ and } \mathfrak{U}_\infty := \mathfrak{P}_y \setminus \{0\} \cong [\mathbb{C}/G_y(\rho, r)] ,$$

and

$$\mathfrak{o}_e := \mathfrak{P}_y \setminus \{0, \infty\} .$$

Now we quote the result from [LS20, §6](see also [NB06]). Assume that the image of  $\rho$  is  $\mu_t$ . Then the curve  $\mathfrak{P}_y$  is a  $G_e := \text{Ker}(\rho)$ -gerbe over  $\mathbb{P}_{tr, ts}^1$ . More explicitly, note  $\mathfrak{o}_e$  is isomorphic to the quotient stack  $[\mathbb{C}^*/G_y]$ <sup>14</sup> then the morphism  $\rho : G_y \rightarrow \mu_t$  induces that  $\mathfrak{o}_e$  is a  $G_e$ -gerbe over its coarse moduli  $o_e \cong [\mathbb{C}^*/\mu_t] = \mathbb{C}^*$ . Then the morphism  $\rho_s^{-1}$  induces a morphism from  $\mathfrak{U}_0 \cong [\mathbb{C}/G_y(\rho^{-1}, s)]$  (resp.  $\mathfrak{U}_\infty \cong [\mathbb{C}/G_y(\rho, r)]$ ) to  $[\mathbb{C}/\mu_{ts}]$ (resp.  $[\mathbb{C}/\mu_{tr}]$ ). The above discussion fits into the following commutative diagram

$$\begin{array}{ccc} \mathfrak{U}_0 & \longrightarrow & [\mathbb{C}/\mu_{ts}] \\ \uparrow & & \uparrow \\ \mathfrak{o}_e & \longrightarrow & o_e \cong [\mathbb{C}^*/\mu_{ts}] \cong [\mathbb{C}^*/\mu_t] \cong [\mathbb{C}^*/\mu_{tr}] \\ \downarrow & & \downarrow \\ \mathfrak{U}_\infty & \longrightarrow & [\mathbb{C}/\mu_{tr}] . \end{array}$$

Let  $\mathfrak{p}_e \cong \mathbb{B}G_e$  be a point of  $\mathfrak{o}_e$ . Define

$$H_e := \pi_1(\mathfrak{o}_e, \mathfrak{p}_e) ,$$

then the coarsification map  $\mathfrak{o}_e \rightarrow o_e$  (with the corresponding base points  $\mathfrak{p}_e \rightarrow p_e$ ) induces a surjection

$$\phi_e : H_e = \pi_1(\mathfrak{o}_e, \mathfrak{p}_e) \rightarrow \pi_1(o_e, p_e) \cong \mathbb{Z} ,$$

whose kernel is isomorphic to  $G_e$  as shown in [LS20, §6].

Choose a path from  $\mathfrak{p}_e$  to 0 and a path from  $\mathfrak{p}_e$  to  $\infty$ , the open embeddings  $\mathfrak{o}_e \hookrightarrow \mathfrak{U}_0$  and  $\mathfrak{o}_e \hookrightarrow \mathfrak{U}_\infty$  induce surjective group homomorphisms

<sup>14</sup>One can check that  $\mathfrak{o}_e$  can be also represented as the quotient stack  $[\mathbb{C}^*/G_y(\rho^{-1}, s)]$  or  $[\mathbb{C}^*/G_y(\rho, r)]$  via  $\rho_s^{-1}$  (resp.  $\rho_r$ ).

$$H_e = \pi_1(\mathfrak{o}_e, \mathfrak{p}_e) \xrightarrow{\pi_{e,0}} \pi_1(\mathfrak{U}_0, 0) \cong G_y(\rho^{-1}, s), \quad H_e = \pi_1(\mathfrak{o}_e, \mathfrak{p}_e) \xrightarrow{\pi_{e,\infty}} \pi_1(\mathfrak{U}_\infty, \infty) \cong G_y(\rho, r).$$

Applying  $\pi_1$ , the above discussion will be summarized into the following commutative diagram with all rows exact:

$$(A.1) \quad \begin{array}{ccccccc} 1 & \longrightarrow & G_e & \longrightarrow & \pi_1(\mathfrak{U}_0) \cong G_y(\rho^{-1}, s) & \xrightarrow{\rho_s^{-1}} & \mu_{ts} \longrightarrow 1 \\ & & \parallel & & \uparrow \pi_{e,0} & & \uparrow \\ 1 & \longrightarrow & G_e & \longrightarrow & \pi_1(\mathfrak{o}_e) \cong H_e & \xrightarrow{\phi_e} & \mathbb{Z} = \pi_1(o_e) \longrightarrow 1 \\ & & \parallel & & \downarrow \pi_{e,\infty} & & \downarrow \\ 1 & \longrightarrow & G_e & \longrightarrow & \pi_1(\mathfrak{U}_\infty) \cong G_y(\rho, r) & \xrightarrow{\rho_r} & \mu_{tr} \longrightarrow 1, \end{array}$$

where in the third column  $\mathbb{Z} \rightarrow \mu_{tr}$  and  $\mathbb{Z} \rightarrow \mu_{ts}$  are given by  $d \mapsto e^{\frac{d}{r}}_t$  and  $d \mapsto e^{\frac{d}{s}}_t$  respectively. By chasing diagram, we can show that  $H_e$  is isomorphic to the fiber product  $G_y(\rho^{-1}, s) \times_{\mu_{ts}} \mathbb{Z}$  (or  $G_y(\rho, r) \times_{\mu_{tr}} \mathbb{Z}$  as well) using the above diagram. Furthermore using the fact  $G_y(\rho^{-1}, s)$  is isomorphic to the fiber product  $G_y \times_{\mu_t} \mu_{ts}$ , we have  $H_e$  is also isomorphic to the fiber product  $G_y \times_{\mu_t} \mathbb{Z}$ .

Let  $\mathbb{P}_{m,n}^1$  be the unique smooth DM stack with coarse moduli  $\mathbb{P}^1$  and trivial generic stablizer and only two special points  $q_\infty \cong \mathbb{B}\mu_m$  and  $q_0 \cong \mathbb{B}\mu_n$ . Here we can choose a  $\mathbb{C}^*$ -action on  $\mathbb{P}_{m,n}^1$  such that  $q_0, q_\infty$  are the only  $\mathbb{C}^*$ -fixed points and the  $\mathbb{C}^*$ -weight of the normal space  $N_{q_0}$  (resp.  $N_{q_\infty}$ ) is  $\frac{1}{m}$  (resp.  $\frac{-1}{n}$ ). Let  $f : \mathbb{P}_{m,n}^1 \rightarrow \mathfrak{P}$  be a  $\mathbb{C}^*$ -fixed stable map of degree  $(0, \frac{\delta}{r})$  which factors through  $\mathfrak{P}_y$ . Assume that  $f(q_0) = 0 \in \mathfrak{P}_y$  and  $f(q_\infty) = \infty \in \mathfrak{P}_y$ . Moreover, if the multiplicity<sup>15</sup> at  $q_0$  is  $([g^{-1}], e^{\frac{\delta}{s}}, 1) \in \text{Conj}(G_y) \times \mathbb{C}^* \times \mathbb{C}^*$  such that  $(x, [g]) \in \bar{I}_c Y$  where  $g \in G_y$ , then the multiplicity at  $q_\infty$  is  $([g], 1, e^{\frac{\delta}{r}}) \in \text{Conj}(G_y) \times \mathbb{C}^* \times \mathbb{C}^*$ , and vice versa. Let  $a$  be the order of  $g$ , when  $r, s$  are sufficiently large primes, as  $f$  is representable, we have  $m = ar, n = as$ . Assume that  $\text{age}_g(L|_y) = \frac{h}{a}$  for some integer  $h$  with  $0 \leq h < a$ . As  $\delta = r \cdot \deg(f^* \mathcal{O}(\mathcal{D}_\infty))$ , using orbifold Riemann-Roch for the  $f^* \mathcal{O}(r\mathcal{D}_\infty)$ , the condition

$$\delta - \frac{h}{a} \in \mathbb{Z},$$

is a necessary condition to ensure the map  $f$  to exist. Then  $a\delta$  must be an integer.

Let  $T_{ar,as}$  be the subgroup of  $(\mathbb{C}^*)^2$  defined by the equation  $t_1^{as} = t_2^{ar}$ . Let  $F : U \rightarrow \mathbb{C}^* \times U$  be the morphism sending  $(x, y)$  to  $(1, x^{a\delta}, y^{a\delta})$ . Then  $F$  is equivariant with respect to the group homomorphism from  $T_{ar,as}$  to  $G_y \times (\mathbb{C}^*)^2$ :

$$(t_1, t_2) \mapsto (\tau(t_1^{-s} t_2^r), t_1^{a\delta}, t_2^{a\delta}),$$

where  $\tau : \mu_a \rightarrow G_y$  is the group morphism sending  $\mu_a$  to  $g$ . We can check  $F$  descends to be a morphism  $\tilde{F}$  from  $\mathbb{P}_{ar,as}^1$  to  $\mathfrak{P}_y$  of degree  $(0, \frac{\delta}{r})$ <sup>16</sup> with the multiplicity decoration  $([g], 1, e^{\frac{\delta}{r}})$  at  $q_\infty$ . Moreover, if we change  $g$  to any conjugate of  $g$  in  $G_y$ , the descended morphisms are all isomorphic to each other. Conversely, we will show the following.

**Lemma A.1.** *With the notations as above, let  $f$  be a  $\mathbb{C}^*$ -fixed morphism of degree  $(0, \frac{\delta}{r})$  to  $\mathfrak{P}$  factor through  $\mathfrak{P}_y$  such that the multiplicity is given by  $([g], 1, e^{\frac{\delta}{s}})$  at  $q_\infty$ . Then the morphism  $f$  must be isomorphic to  $\tilde{F}$  (up to a unique 2-isomorphism) constructed as above.*

<sup>15</sup>Here we can replace the index set  $C$  for  $\bar{I}_\mu \mathfrak{P}_y$  by the set  $\text{Conj}(G_y)$  of conjugacy classes of  $G_y$ .

<sup>16</sup>We use the degree notation by viewing  $\tilde{F}$  as a morphism to  $\mathfrak{P}$ .

*Proof.* Let  $f_1, f_2$  be two stable maps with same associated decorated graph as above, if we assume they are isomorphic when restricting to  $O_e := \mathbb{P}_{ar,as}^1 \setminus \{q_0, q_\infty\}$ , then  $f_1$  is isomorphic to  $f_2$  up to a unique 2-isomorphism (see e.g., [CCFK15, Lemma 2.5] for a proof). Thus we only need to show the restriction  $f|_{O_e}$  is uniquely determined by the degree data and multiplicity data at  $q_\infty$ . Note that  $f|_{O_e} : O_e \rightarrow \mathfrak{o}_e$  is an étale covering map. Indeed, as  $O_e$  and the coarse moduli  $o_e$  of  $\mathfrak{o}_e$  are both isomorphic to  $\mathbb{C}^*$  and the coarsening map from  $\mathfrak{o}_e$  to  $o_e$  is étale, then the étaleness of  $f|_{O_e}$  follows from the étaleness of the composition

$$O_e \rightarrow \mathfrak{o}_e \rightarrow o_e$$

as any  $\mathbb{C}^*$ -fixed morphism from  $\mathbb{C}^*$  to  $\mathbb{C}^*$  must be a cyclic covering. By [Noo05, Theorem 18.19], we have a bijection between conjugacy classes of subgroup  $(f|_{O_e})_*(\pi_1(O_e, P_e))$  ( $P_e$  is the inverse image of the point  $\mathfrak{p}_e$  under  $O_e \rightarrow \mathfrak{o}_e$ ) of  $\pi_1(\mathfrak{o}_e, \mathfrak{p}_e)$  and isomorphism classes of covering map of  $\mathfrak{o}_e$  (not necessarily base point preserving), it's enough to show the conjugacy class of subgroup  $(f|_{O_e})_*(\pi_1(O_e, P_e))$  is uniquely determined by the degree data and the monodromy data at  $q_\infty$ .

We will drop the base points in the notation of fundamental group in the sequel if it's clear in the context. Let

$$U_0 := \mathbb{P}_{ar,as}^1 \setminus \{q_\infty\} \cong [\mathbb{C}/\mu_{as}], \quad \text{and} \quad U_\infty := \mathbb{P}_{ar,as}^1 \setminus \{q_0\} \cong [\mathbb{C}/\mu_{ar}].$$

Choose a path from  $P_e$  to  $q_0$  in  $U_0$  and a path from  $P_e$  to  $q_\infty$  in  $U_\infty$ , then the open embeddings  $O_e \hookrightarrow U_0$  and  $O_e \hookrightarrow U_\infty$  induces surjective group homomorphisms

$$\mathbb{Z} = \pi_1(O_e) \xrightarrow{\pi_{q_0}} \pi_1(U_0) \cong \mu_{as} \quad \mathbb{Z} = \pi_1(O_e) \xrightarrow{\pi_{q_\infty}} \pi_1(U_\infty) \cong \mu_{ar}.$$

Here we choose the identification between  $\pi_1(O_e)$  and  $\mathbb{Z}$  such that  $\pi_{q_0}(1) = e^{\frac{-1}{as}}$  and  $\pi_{q_\infty}(1) = e^{\frac{1}{ar}}$ .<sup>17</sup> Then the morphism

$$\pi_1(O_e) \longrightarrow \pi_1(U_\infty) \xrightarrow{(f|_{U_\infty})_*} \pi_*(\mathfrak{U}_\infty)$$

sends the generator 1 of  $\pi_1(O_e)$  to the conjugacy class of  $(g, e^{\frac{\delta}{r}}) \in G_y(\rho, r) \cong \pi_1(\mathfrak{U}_\infty)$  by the multiplicity data at  $q_\infty$ . We have a commutative diagram with all vertical lines are inclusions:

$$\begin{array}{ccccc} O_e & \xrightarrow{f|_{O_e}} & \mathfrak{o}_e & \longrightarrow & o_e \\ \downarrow & & \downarrow & & \downarrow \\ U_\infty & \longrightarrow & \mathfrak{U}_\infty & \longrightarrow & [\mathbb{C}/\mu_{tr}]. \end{array}$$

Using A.1, the above diagram induces the following commutative diagram

$$(A.2) \quad \begin{array}{ccccccc} \mathbb{Z} = \pi_1(O_e) & \xrightarrow{(f|_{O_e})_*} & \pi_1(\mathfrak{o}_e) \cong H_e & \xrightarrow{\phi_e} & \pi_1(o_e) \cong \mathbb{Z} \\ \downarrow & & \downarrow \pi_{e,\infty} & & \downarrow \\ \mu_{ar} = \pi_1(U_\infty) & \longrightarrow & \pi_1(\mathfrak{U}_\infty) \cong G_y(\rho, r) & \xrightarrow{\rho_r} & \mu_{tr}, \end{array}$$

where in the third column  $\mathbb{Z} \rightarrow \mu_{tr}$  is given by  $d \mapsto e^{\frac{d}{tr}}$ . Then we see that the composition

$$\pi_1(O_e) \cong \mathbb{Z} \xrightarrow{(f|_{O_e})_*} \pi_1(\mathfrak{o}_e) = H_e \xrightarrow{\phi_e} \pi_1(o_e) \cong \mathbb{Z}.$$

<sup>17</sup>Different choices of path from  $P_e$  to  $q_0$  (resp.  $q_\infty$ ) will only affect  $\pi_{q_0}$  (resp.  $\pi_{q_\infty}$ ) up to conjugacy, then the image  $\pi_{q_0}(1)$  (resp.  $\pi_{q_\infty}(1)$ ) doesn't depend on the choice of the path.

from  $\mathbb{Z}$  to  $\mathbb{Z}$  is multiplication by  $t\delta$  as the generator 1 of  $\pi_1(O_e)$  sends to  $e^{\frac{\delta}{r}}$  in the group  $\mu_{tr}$  in whatever path from the up-left corner to the bottom-right corner in the above commutative diagram and  $r$  is sufficiently large.

Let  $\gamma_e \in H_e$  be the image of generator 1  $\in \pi_1(O_e)$  in  $H_e$ . By the previous discussion,  $\gamma_e$  maps to the conjugacy class of  $(g, e^{\frac{\delta}{r}})$  via  $\pi_{e,\infty}$ , and maps to  $t\delta$  in  $\pi_1(o_e)$ . Conversely, using the fact  $H_e$  is isomorphic to fiber product  $G_y(\rho, r) \times_{\mu_{tr}} \mathbb{Z}$  using the right square of A.2, we see that the conjugacy class of  $\gamma_e$ , is uniquely determined by the conjugacy class of the image of  $\gamma_e$  in  $\pi_1(\mathcal{U}_\infty)$  and the image of  $\gamma_e$  in  $\pi_1(o_e)$ , which are determined by the decorated data. Hence the conjugacy class of the subgroup  $(f|_{O_e})_*(\pi_1(O_e))$  is also uniquely determined by the decorated data. This completes the proof.  $\square$

Next we describe the automorphism group  $\text{Aut}(f)$  of  $f$ .

**Proposition A.2.** *Use the notation in the above proof, we have  $\text{Aut}(f) \cong C_{H_e}(\gamma_e)/\langle \gamma_e \rangle$ , and the order of automorphism group is equal to  $|\text{Aut}(f)| = |C_{H_e}(\gamma_e)/\langle \gamma_e \rangle| = \delta |C_{G_y}(g)|$ . Here  $C_{H_e}(\gamma)$  (resp.  $C_{G_y}(g)$ ) is the centralizer of  $\gamma_e$  (resp.  $g$ ) in the group  $H_e$  (resp.  $G_y$ ).*

*Proof.* The automorphism group of  $f$  is isomorphic to the automorphism group of the restriction  $f|_{O_e}$ , then the first claim follows from standard fact about covering deck transformation (not necessarily basepoint preserving) as  $\gamma_e$  generates the subgroup  $(f|_{O_e})_*(\pi_1(O_e))$  in  $\pi_1(o_e)$ .

Assume that the image  $\phi_e(C_{H_e}(\gamma_e))$  in  $\mathbb{Z}$  is generated by the positive integer  $u$ , as  $H_e$  is isomorphic to the fiber product  $G_y \times_{\mu_t} \mathbb{Z}$ .<sup>18</sup> More precisely,

$$H_e \cong \{(g, d) \in G_y \times \mathbb{Z} \mid \rho(g) = e^{\frac{d}{t}}\}.$$

Then  $\gamma_e = (g, t\delta) \in H_e$ . We can show that  $t$  and  $t\delta$  are both divided by  $u$ . Thus the image of  $C_{G_y}(g)$  under  $\rho$  is generated by  $e^{\frac{u}{t}}$ .

The group  $C_{H_e}(\gamma_e)$  can be represented as

$$\{(h, d) \in C_{G_y}(g) \times \mathbb{Z} \mid \rho(h) = e^{\frac{d}{t}}\}$$

in  $H_e$ . Then the claim about  $|\text{Aut}(f)|$  follows from the following two exact sequences:

$$1 \longrightarrow G_e \cap C_{G_y}(g) \xrightarrow{\theta} C_{H_e}(\gamma_e)/\langle \gamma_e \rangle \longrightarrow u\mathbb{Z}/t\delta\mathbb{Z} \longrightarrow 1,$$

and

$$1 \longrightarrow G_e \cap C_{G_y}(g) \longrightarrow C_{G_y}(g) \xrightarrow{\rho} \mu_{\frac{t}{u}} \longrightarrow 1.$$

Here  $u\mathbb{Z}$  (resp.  $t\delta\mathbb{Z}$ ) is the subgroup of  $\mathbb{Z}$  generated by the integer  $u$  (resp.  $t\delta$ ), the morphism  $\theta$  is the induced from the restriction to  $G_e \cap C_{G_y}(g)$  of the inclusion from  $G_e$  to  $H_e$ .  $\square$

We can make the above proposition more concretely. Let  $(h, d) \in C_{H_e}(\gamma_e)$  be an element, we associate  $(h, d)$  an element in the automorphism group  $\text{Aut}(f)$  as follows: define

$$\theta_{h,d} : U \rightarrow U : (x, y) \mapsto (x, e^{\frac{d}{ar\delta}} y),$$

which descends to be an isomorphism of  $\mathbb{P}_{ar,as}^1$  which we still denote to be  $\theta_{h,d}$ . Recall that  $F$  is a morphism from  $U \rightarrow \mathbb{C}^* \times U$  defined as in Lemma A.1, then  $F \circ \theta_{h,d}$  differ from  $F$  by the action of the element  $(h, 1, e^{\frac{d}{rt}})$  on the target via the action of  $G_y \times (\mathbb{C}^*)^2$  on  $\mathbb{C}^* \times U$  which also commutes the action of  $T_{ar,as}$  and  $G_y \times (\mathbb{C}^*)^2$  on the source and target respectively. Then by descent, this defines an 2-isomorphism  $\alpha_{h,d} : F \rightarrow F \circ \theta_{h,d}$ . Thus the pair  $(\theta_{h,d}, \alpha_{h,d})$  defines an automorphism in  $\text{Aut}(f)$ .

For two elements  $(h_1, d_1)$  and  $(h_2, d_2)$  in  $C_{H_e}(\gamma_e)$ , the pairs  $(\theta_{h_1,d_1}, \alpha_{h_1,d_1})$  and  $(\theta_{h_2,d_2}, \alpha_{h_2,d_2})$  are isomorphic to each other if there exists some integer  $n$  such that  $h_2 = h_1 g^n$  and  $d_2 = d_1 + nt\delta$ .

<sup>18</sup>We have already seen that  $H_e$  is isomorphic to  $G_y(\rho^{-1}, s) \times_{\mu_{st}} \mathbb{Z}$ , then we use the isomorphism  $G_y(\rho^{-1}, s) \cong G_y \times_{\mu_t} \mu_{st}$

Indeed, let  $m_{(1, e^{\frac{nt\delta}{art\delta}})}$  be the automorphism of  $U$  acted upon by  $(1, e^{\frac{nt\delta}{art\delta}})$ , we have  $\theta_{h_2, d_2} = m_{(1, e^{\frac{nt\delta}{art\delta}})} \circ \theta_{h_1, d_1}$ , which defines a two-isomorphism  $\beta_n : \theta_{h_1, d_1} \rightarrow \theta_{h_2, d_2}$  in  $\text{Hom}(\mathbb{P}_{ar, as}^1, \mathbb{P}_{ar, as}^1)$ , then the composition  $f \circ \beta_n$  defines a two-isomorphism from  $F \circ \theta_{h_1, d_1}$  to  $F \circ \theta_{h_2, d_2}$  such that  $\alpha_{h_2, d_2} = f \circ \beta_n \circ \alpha_{h_1, d_1}$ .

The morphism  $f : \mathbb{P}_{ar, as}^1 \rightarrow \mathfrak{P}_y$  defines a  $\mathbb{C}$ -point of  $\mathcal{K}_{0,2}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r}))$ , where we denote by  $q_0$  (resp.  $q_\infty$ ) by the first (resp. second) marking, then the evaluation map

$$ev_{q_\infty} : \mathcal{K}_{0,2}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r})) \rightarrow \bar{I}_\mu \mathbb{P}Y_{r,s} ,$$

induces a morphism from the isotropy group  $\text{Aut}(f)$  to the isotropy group  $\text{Aut}((y, ([g], 1, e^{\frac{\delta}{r}})))$ , which, from the concrete description of  $\text{Aut}(f)$  above, coincides the morphism from  $C_{H_e}(\gamma_e)/\langle \gamma_e \rangle$  to  $C_{G_y}(g)/\langle g \rangle$  induced from the projection of  $G_y \times \mathbb{Z}$  to the first factor. From this concrete description, we also know the kernel of the morphism of isotropy groups:

**Proposition A.3.** *The kernel of the morphism  $\text{Aut}(f) \rightarrow \text{Aut}((y, ([g], 1, e^{\frac{\delta}{r}})))$  is isomorphic to  $\mathbb{Z}/a\delta\mathbb{Z}$ .*

*Proof.* This follows from by applying the snake lemma to the first two exact rows of the following diagram:

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathbb{Z} \cong \langle (1, at\delta) \rangle & \longrightarrow & \mathbb{Z} \cong \langle \gamma_e \rangle & \xrightarrow{pr_1} & \mathbb{Z}/a\mathbb{Z} \cong \langle g \rangle \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \mathbb{Z} \cong \langle (1, t) \rangle & \longrightarrow & C_{H_e}(\gamma_e) & \xrightarrow{pr_1} & C_{G_y}(g) \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \mathbb{Z}/a\delta\mathbb{Z} & \longrightarrow & C_{H_e}(\gamma_e)/\langle \gamma_e \rangle & \xrightarrow{pr_1} & C_{G_y}(g)/\langle g \rangle \longrightarrow 1 \\
& & & & \parallel & & \parallel \\
& & & & \text{Aut}(f) & & \text{Aut}((y, ([g], 1, e^{\frac{\delta}{r}}))) ,
\end{array}$$

where  $pr_1$  are induced from the projection from  $G_y \times \mathbb{Z}$  to the first factor.  $\square$

**A.2. Construction of family stable map.** Let  $e$  be an edge with decorated degree  $\frac{\delta}{r}$  and decorated multiplicity  $(c, 1, e^{\frac{\delta}{r}})$  at the half-edge  $h_\infty$  as in §3.2. Let  $a := a(c)$  be the integer associated to  $c$  as in the Assumption 2.1. Define the space  $\mathcal{M}_e$  to be the root gerbe  ${}^{as\delta}\sqrt{L^\vee/I_c Y}$ . We will generalize the construction in Lemma A.1 to a family version; we will construct a family of  $\mathbb{C}^*$ -fixed stable map over  $\mathcal{M}_e$  to  $\mathbb{P}Y_{r,s}$  with the associated decorated degree. Then we get a morphism

$$g : \mathcal{M}_e \rightarrow \mathcal{K}_{0,2}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r})) .$$

We will show the image of  $g$  is a closed and open substack of the  $\mathbb{C}^*$ -fixed part of  $\mathcal{K}_{0,2}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r}))$ , and  $g$  is a finite étale map into the image of  $g$ . Then we can use  $\mathcal{M}_e$  to do an explicit computation of the edge contribution in the virtual localization formula.

By an abuse of notation, denote  $L^\vee$  to be the line bundle over  $\mathcal{M}_e$  via a pull-back of the line bundle  $L^\vee$  over  $Y$  along the natural map  $\mathcal{M}_e \rightarrow Y$ , which is a composition of maps  $\mathcal{M}_e \rightarrow \bar{I}_c Y \rightarrow Y$ . Denote by  $(L^\vee)^0$  the open set of  $L^\vee$  with zero section removed.

Let  $U := \mathbb{C}^2 \setminus \{0\}$ . Let  $\mathcal{C}_e$  be the quotient stack  $[(L^\vee)^0 \times U/\mathbb{C}^* \times T_{ar, as}]$  defined by the (right) action

$$(l, x, y)(t, t_1, t_2) = (t^{-as\delta}l, tt_1x, t_2y) ,$$



where  $(l, x, y) \in (L^\vee)^0 \times U$  and  $(t, t_1, t_2) \in \mathbb{C}^* \times T_{ar,as}$ . Here  $T_{ar,as}$  is the subgroup of  $(\mathbb{C}^*)^2$  defined by  $\{(t_1, t_2) \in (\mathbb{C}^*)^2 \mid t_1^{as} = t_2^{ar}\}$ . Then  $\mathcal{C}_e$  is a family of orbi- $\mathbb{P}^1$  bundle over  $\mathcal{M}_e$  such that all fibers are isomorphic to  $\mathbb{P}_{ar,as}^1$ .

**Remark A.4.** Let  $\bar{\psi}_*$  be the psi-class associated with the gerbe marking corresponding to the zero loci of  $y$  of  $\mathcal{C}_e$ . We see that  $\bar{\psi}_*$  is equal to  $\frac{\lambda - c_1(L)}{\delta}$ .

We will define a family map  $f$  using the categorical description of  $\mathbb{P}Y_{r,s}$  and  $\mathcal{C}_e$ :

$$\begin{array}{ccc} \mathcal{C}_e & \xrightarrow{f} & \mathbb{P}Y_{r,s} \\ \pi \downarrow & & \\ \mathcal{M}_e := \sqrt[\delta]{L^\vee / I_c Y} & & \end{array}$$

A.2.1. *A categorical description about stacks  $\mathbb{P}Y_{r,s}$  and  $\mathcal{C}_e$ .* The space  $\mathbb{P}Y_{r,s}$  can be represented as the quotient  $[(L^\vee)^0 \times U / (\mathbb{C}^*)^2]$  defined by the action

$$(l, x, y)(t_1, t_2) = (t_1^{-s} t_2^r l, t_1 x, t_2 y) .$$

We see that, for any  $\mathbb{C}$ -scheme  $S$ , an object in  $\mathbb{P}Y_{r,s}(S)$  is given by a tuple  $(w, M_1, M_2, \rho, s_1, s_2)$  (c.f. [Ols16, §10.2.7, §10.2.8]), where

- (1)  $w : S \rightarrow Y$  is an object<sup>19</sup> in  $Y(S)$ ,
- (2)  $M_1, M_2$  are line bundles over  $S$ ,
- (3)  $\rho : M_1^{\otimes s} \otimes M_2^{-\otimes r} \cong w^* L^\vee$  is an isomorphism,
- (4)  $s_i \in \Gamma(S, M_i), i \in \{1, 2\}$  are sections such that  $s_1, s_2$  don't vanish simultaneously.

Let  $(w', M'_1, M'_2, \rho', s'_1, s'_2)$  be another object in  $\mathbb{P}Y_{r,s}(S')$ , given a morphism  $g : S' \rightarrow S$ , then a morphism from  $(w', M'_1, M'_2, \rho', s'_1, s'_2)$  to  $(w, M_1, M_2, \rho, s_1, s_2)$  over  $g$  is a tuple  $(g^b, \alpha_1, \alpha_2)$ . Here  $g^b : w' \rightarrow w$  is a morphism over  $g$ ,  $\alpha_i : g^* M'_i \rightarrow M'_i$  is an isomorphism which maps  $g^* s'_i$  to  $s'_i$ . Furthermore, we require a commutative square

$$\begin{array}{ccc} g^* M_1^{\otimes s} \otimes g^* M_2^{-\otimes r} & \xrightarrow{\alpha_1^{\otimes s} \otimes \alpha_2^{-\otimes r}} & M_1'^{\otimes s} \otimes M_2'^{-\otimes r} \\ g^* \rho \downarrow & & \downarrow \rho' \\ g^* w^* L^\vee & \xrightarrow{g^b} & (w')^* L^\vee , \end{array}$$

and  $\alpha_i$  should satisfy obvious commutative laws for composition of two maps (c.f. [Ols16, §9.1.10(i)]).

**Remark A.5.** Let  $w : S \rightarrow Y$  be any object in  $Y(S)$ , denote by  $\mathbb{P}_{r,s}(w^* L^\vee \oplus \mathbb{C})$  the stack quotient

$$[(w^* L^\vee)^0 \times U / (\mathbb{C}^*)^2] ,$$

where the action is defined in the same way as  $\mathbb{P}Y_{r,s}$ . One can verify that we have a Cartesian diagram:

$$\begin{array}{ccc} \mathbb{P}_{r,s}(w^* L^\vee \oplus \mathbb{C}) & \longrightarrow & \mathbb{P}Y_{r,s} \\ \downarrow & & \downarrow \text{pr}_{r,s} \\ S & \xrightarrow{w} & Y . \end{array}$$

This property can also lead to our categorical description of  $\mathbb{P}Y_{r,s}$ , see a closed-related discussion in [Ols16, §10.2.8].

<sup>19</sup>Here we view the morphism  $S \rightarrow Y$  as an equivalent definition of objects in  $Y(S)$  using the 2-Yoneda Lemma.

We also give a categorical description of  $\mathcal{C}_e$ . Recall that  $\mathcal{C}_e$  can be represented as the quotient stack  $[(L^\vee)^0 \times U / (T_{ar,as} \times \mathbb{C}^*)]$  given by the action

$$(l, x, y)(t_1, t_2, t_3) = (t_3^{-as\delta} l, t_3 t_1 x, t_2 y)$$

where  $(l, x, y) \in (L^\vee)^0 \times U$  and  $(t_1, t_2, t_3) \in T_{ar,as} \times \mathbb{C}^*$ . For any scheme  $S$  over  $\text{Spec}(\mathbb{C})$ , an object in  $\mathcal{C}_e(S)$  is given by a tuple  $\tau := (w, \theta, M_1, M_2, M_3, \rho_1, \rho_2, s_1, s_2)$ , where

- (1)  $w$  is a object in  $Y(S)$ ,  $\theta$  is an automorphism in  $\text{Aut}_S(w)$  such that  $(w, \theta) \in I_c Y$ ,
- (2)  $M_1, M_2, M_3$  are line bundles over  $S$ ,
- (3)  $\rho_1 : M_3^{\otimes as\delta} \cong w^* L^\vee$  and  $\rho_2 : M_1^{\otimes as} \otimes M_2^{-\otimes ar} \cong \mathcal{O}_S$  are isomorphisms,
- (4)  $s_1 \in \Gamma(S, M_1 \otimes M_3)$  and  $s_2 \in \Gamma(S, M_2)$  are sections such that  $s_1, s_2$  don't vanish simultaneously.

Given another  $\mathbb{C}$ -scheme  $S'$ , let  $\tau' = (w', \theta', M'_1, M'_2, M'_3, \rho'_1, \rho'_2, s'_1, s'_2)$  be an object in  $\mathcal{C}_e(S')$ , given a morphism  $g : S' \rightarrow S$ , then a morphism from  $(w', \theta', M'_1, M'_2, M'_3, \rho'_1, \rho'_2, s'_1, s'_2)$  to  $(w, \theta, M_1, M_2, M_3, \rho_1, \rho_2, s_1, s_2)$  over  $g$  is a tuple  $(g^b, \alpha_1, \alpha_2, \alpha_3)$ , where  $g^b : (w', \theta') \rightarrow (w, \theta)$  is a morphism in  $I_c Y$  over the morphism  $g$ ,  $\alpha_i : g^* M_i \rightarrow M'_i$  are isomorphisms such that  $\alpha_1 \otimes \alpha_3 : g^* M_1 \otimes g^* M_3 \rightarrow M'_1 \otimes M'_3$  maps  $g^* s_1$  to  $s'_1$ , and  $\alpha_2 : g^* M_2 \cong M'_2$  maps  $g^* s_2$  to  $s'_2$ . Furthermore, these data are require to satisfy the following conditions:

- (1) We have two commutative squares

$$\begin{array}{ccc} g^* M_3^{\otimes as\delta} & \xrightarrow{\alpha_3^{\otimes as\delta}} & M'_3{}^{\otimes as\delta} \\ g^* \rho_1 \downarrow & & \downarrow \rho'_1 \\ g^* w^* L^\vee & \xrightarrow{g^b} & (w')^* L^\vee \end{array}$$

and

$$\begin{array}{ccc} g^* M_1^{\otimes as} \otimes g^* M_2^{-\otimes ar} & \xrightarrow{\alpha_1^{\otimes as} \otimes \alpha_2^{-\otimes ar}} & M'_1{}^{\otimes as} \otimes M'_2{}^{-\otimes ar} \\ g^* \rho_2 \downarrow & & \downarrow \rho'_2 \\ g^* \mathcal{O}_S & \xrightarrow{\cong} & \mathcal{O}_{S'} \end{array}$$

- (2)  $\alpha_1, \alpha_2$  should satisfy commutative laws for composition of maps.

For each object  $\tau := (w, \theta, M_1, M_2, M_3, \rho_1, \rho_2, s_1, s_2) \in \mathcal{C}_e(S)$ , we give the following definition:

**Definition A.6.** The collection of line bundles  $M_1^{-\otimes s} \otimes M_2^{\otimes r}$  and morphisms  $\alpha_1^{\otimes s} \otimes \alpha_2^{-\otimes r}$  defines a line bundle on  $\mathcal{C}_e$ , which we denote to be  $\mathcal{L}$ .<sup>20</sup> Using  $\rho_2$ , we see that the line bundle  $M_1^{-\otimes s} \otimes M_2^{\otimes r}$  represents a torsion element in  $\text{Pic}(S)$  of which the order is divided by  $a$ . This defines a morphism  $\alpha : \mathcal{C}_e \rightarrow \mathbb{B}\mu_a$  satisfying that: let  $\mathbb{L} = [\mathbb{C}/\mu_a]$  be the line bundle over  $\mathbb{B}\mu_a$  associated with the standard representation of  $\mu_a$ , we have  $\alpha^* \mathbb{L} \cong \mathcal{L}$ . Then we can naturally associate each object  $\tau : S \rightarrow \mathcal{C}_e$  a morphism  $\bar{f}_1 := \alpha \circ \tau$  and an isomorphism  $\alpha_\tau : M_1^{-\otimes s} \otimes M_2^{\otimes r} \cong \bar{f}_1^* \mathbb{L}$ . Given another object  $\tau' \in \mathcal{C}_e(S')$  and a morphism  $u := (g^b, \alpha_1, \alpha_2, \alpha_3)$  in  $\text{Hom}_{\mathcal{C}_e}(\tau', \tau)$  over  $g : S' \rightarrow S$ , we have

$$(A.3) \quad \begin{array}{ccc} g^*(M_1^{\otimes s} \otimes M_2^{-\otimes r}) & \xrightarrow{\alpha_1^{-\otimes s} \otimes \alpha_2^{\otimes r}} & M'_1{}^{-\otimes s} \otimes M'_2{}^{\otimes r} \\ \downarrow g^* \alpha_\tau & & \downarrow \alpha_{\tau'} \\ g^* \bar{f}_1^* \mathbb{L} & \xrightarrow{\alpha(u)} & (\bar{f}'_1)^* \mathbb{L} \end{array}$$

<sup>20</sup>This line bundle is also isomorphic to  $L_{-s\chi_1} \otimes L_{r\chi_2}$  defined in §A.3.

Note  $\bar{f}_1$  induces a  $S$ -morphism  $f_1 : S \rightarrow S \times \mathbb{B}\mu_a$  such that  $\text{pr}_2 \circ f_1 = \bar{f}_1$ , where  $\text{pr}_2$  is the projection from  $S \times \mathbb{B}\mu_a$  to the second factor.

Recall that there is an isomorphism of the stack  $\text{RepHom}(\mathbb{B}\mu_a, Y)$  with the  $I_{\mu_a}Y$  from [AGV08, §3.2], the data  $(w, \theta)$  naturally defines a morphism  $f_2 : S \times \mathbb{B}\mu_a \rightarrow Y$  such that the generator of the isotropy group  $\text{Aut}_S(i_S) = (\mu_a)_S$  sends to the automorphism  $\theta \in \text{Aut}_S(w)$ , where  $i_S$  is a  $S$ -morphism  $i_S : S \rightarrow S \times \mathbb{B}\mu_a$  corresponding to the trivial  $\mu_a$  principal bundle over  $S$ . Moreover we have  $w = f_2 \circ i_S$ .

Let  $a$  be the integer associated to  $c$  as in 2.1. Write  $\text{age}_c(L) = \frac{h}{a}$  for an integer  $h$  with  $0 \leq h < a$ . Let  $\mathbb{L}_{a,-h}$  be the line bundle over  $\mathbb{B}\mu_a$  associated to the character of  $\mu_a$  by sending  $\mu_a^i$  to  $\mu_a^{-ih}$ . Then  $\mathbb{L}_{a,-h} = \mathbb{L}^{-\otimes h}$  is defined in A.6.

**Proposition A.7.** *We have a natural isomorphism  $\epsilon_{f_2} : w^*L^\vee \boxtimes \mathbb{L}_{a,-h} \rightarrow f_2^*L$ , and we have a natural isomorphism  $\bar{\rho}_\tau : w^*L^\vee \otimes (M_1^{\otimes sh} \otimes M_2^{\otimes -rh}) \rightarrow (f_2 \circ f_1)^*L^\vee$ .*

*Proof.* We will prove  $\epsilon_{f_2}$  by étale descent. First note  $i_S : S \rightarrow S \times \mathbb{B}\mu_a$  is an étale covering with Galois group  $\mu_a$ , then  $f_2$  can be recovered from the descent data given by the pair  $(w, \theta)$  (c.f., [Vis05]). Then the  $f_2^*L^\vee$  can be recovered from the descent data given by the pair  $(w^*L^\vee, \tilde{\theta} : w^*L^\vee \rightarrow w^*L^\vee)$ , where  $\tilde{\theta}$  is an automorphism of  $w^*L^\vee$  induced by  $\theta$ . As  $\text{age}_c(L) = \frac{h}{a}$ ,  $\tilde{\theta}$  is nothing else but the scalar multiplication by  $e^{-\frac{h}{a}}$ . By descent, we see that the line bundle  $f_1^*L^\vee$  is isomorphic to  $w^*L^\vee \boxtimes \mathbb{L}_{a,-h}$ .

Finally the claim about  $(f_2 \circ f_1)^*L^\vee$  follows the next lemma.  $\square$

**Lemma A.8.** *Let  $W, U_1, U_2$  be stacks and  $L_1$  and  $L_2$  be two line bundles over  $U_1$  and  $U_2$  respectively. Let  $t_1 : W \rightarrow U_1$  and  $t_2 : W \rightarrow U_2$  be two morphisms and  $t_1 \times t_2 : W \rightarrow U_1 \times U_2$  is the induced morphism using the fiber product. Then we have that a natural isomorphism  $(t_1 \times t_2)^*(L_1 \boxtimes L_2) \cong t_1^*L_1 \otimes t_2^*L_2$ .*

**Definition A.9.** *Recall that we have  $a\delta \equiv h \pmod{a}$ , then  $\rho_2^{\otimes \frac{a\delta-h}{a}}$  induces a natural isomorphism  $w^*L^\vee \otimes (M_1^{\otimes as\delta} \otimes M_2^{\otimes -ar\delta}) \cong M_1^{\otimes sh} \otimes M_2^{\otimes -rh}$  which we still denote to be  $\rho_2^{\otimes \frac{a\delta-h}{a}}$ . We construct an natural isomorphism*

$$\rho_\tau : (M_1 \otimes M_3)^{\otimes as\delta} \otimes M_2^{\otimes -ar\delta} \rightarrow (f_2 \circ f_1)^*L^\vee$$

as the composition

$$(A.4) \quad \begin{aligned} & (M_1 \otimes M_3)^{\otimes as\delta} \otimes M_2^{\otimes -ar\delta} \xrightarrow{\cong} M_3^{\otimes as\delta} \otimes (M_1^{\otimes as\delta} \otimes M_2^{\otimes -ar\delta}) \\ & \xrightarrow{\rho_1 \otimes \rho_2^{\otimes \frac{a\delta-h}{a}}} w^*L^\vee \otimes (M_1^{\otimes sh} \otimes M_2^{\otimes -rh}) \xrightarrow{\bar{\rho}_\tau} (f_2 \circ f_1)^*L^\vee. \end{aligned}$$

**A.2.2. Defining the family map.** We first collect some facts about line bundles over stacks. By a line bundle  $\mathcal{L}$  over a stack  $\mathcal{X}$ , we adopt the following definition: for any scheme  $S$  over  $\mathbb{C}$  and any object  $x$  of  $\mathcal{X}(S)$ , we associate a line bundle over  $S$  and we denote it by  $x^*\mathcal{L}$ , this notation coincides with the notation of pullback of line bundles when we view an object of  $\mathcal{X}$  as a morphism  $x : S \rightarrow \mathcal{X}$  by the 2-Yoneda lemma. Let  $S'$  be another scheme and  $x' \in \mathcal{X}(S')$ . Let  $g^b : x' \rightarrow x$  be a morphism over  $g : S' \rightarrow S$ . Then there is an isomorphism  $g^*x^*\mathcal{L} \rightarrow x'^*\mathcal{L}$ , which we denote to be  $g^b$  as well, the collection of line bundles and morphisms between them should satisfy obvious compatibility with composition of morphisms in  $\mathcal{X}$ .

Let  $f : \mathcal{Y} \rightarrow \mathcal{X}$  be a morphism between stacks, then the pullback  $f^*\mathcal{L}$  of the line bundle  $\mathcal{L}$  along  $f$  can be defined as follows: to each object  $y \in \text{Ob}(\mathcal{Y})$ , we associate the line bundle  $f(y)^*\mathcal{L}$ ; the morphisms between line bundles are defined in an obvious way. Let  $f' : \mathcal{Y} \rightarrow \mathcal{X}$  be another morphism and  $\alpha$  be a 2-isomorphism from  $f'$  to  $f$ , then we can define an isomorphism from  $f^*\mathcal{L}$  to  $f'^*\mathcal{L}$  as follows, view  $\mathcal{X}$  and  $\mathcal{Y}$  as its underlying category, then  $\alpha$  is a natural

transformation from  $f'$  to  $f$ , which are viewed as functors. For any object  $y \in \text{Ob}(\mathcal{Y})$  over a scheme  $S$ , there is an associated morphism  $\alpha_y : f'(y) \rightarrow f(y)$  over  $S$ , which determines a morphism  $\alpha_y : f(y)^*\mathcal{L} \rightarrow f'(y)^*\mathcal{L}$  of line bundles over  $S$ . This defines the isomorphism  $f^*\mathcal{L} \rightarrow f'^*\mathcal{L}$ , which we will still denote by  $\alpha$  by an abuse of notation. Let  $g : \mathcal{Z} \rightarrow \mathcal{Y}$  and  $h : \mathcal{X} \rightarrow \mathcal{W}$  be two morphisms between stacks, we will write  $\alpha \circ g$  to mean the 2-isomorphism from  $f' \circ g$  to  $f \circ g$  and write  $h \circ \alpha$  to mean the 2-isomorphism from  $h \circ f'$  to  $h \circ f$ .

Now we define the family map  $f : \mathcal{C}_e \rightarrow \mathbb{P}Y_{r,s}$  over  $\mathcal{M}_e$ . To any object

$$\tau = (w, \theta, M_1, M_2, M_3, \rho_1, \rho_2, s_1, s_2) \in \mathcal{C}_e(S),$$

we associate an object  $\phi = (f_2 \circ f_1, (M_1 \otimes M_3)^{\otimes a\delta}, M_2^{\otimes a\delta}, \rho_\tau, s_1^{a\delta}, s_2^{a\delta}) \in \mathbb{P}Y_{r,s}(S)$ , where  $f_1, f_2, \rho_\tau$  (and  $\bar{f}_1, \alpha_\tau, \epsilon_{f_2}, \bar{\rho}_\tau$ ) are defined in Definition A.6, Definition A.9 (and Proposition A.7). Given another scheme  $S'$  and another object  $\tau' \in \mathcal{C}_e(S')$ , let  $\phi'$  be the associated object in  $\mathbb{P}Y_{r,s}$ , similarly for  $f'_1, f'_2, \rho_{\tau'}$  (and  $\bar{f}'_1, \alpha_{\tau'}, \epsilon_{f'_2}, \bar{\rho}_{\tau'}$ ). Let  $u := (g^b, \alpha_1, \alpha_2, \alpha_3)$  be a morphism in  $\text{Hom}_{\mathcal{C}_e}(\tau', \tau)$  over  $g : S' \rightarrow S$ , we need some preparation before we define a morphism from  $\phi'$  to  $\phi$  over  $g$ .

Using the notations in A.6, by 2-Yoneda lemma, the morphism  $\alpha(u)$  induces a unique 2-isomorphism  $\bar{\eta}_1 : \bar{f}'_1 \rightarrow \bar{f}_1 \circ g$  marking the following triangle commute:

$$\begin{array}{ccc} S' & \xrightarrow{g} & S \\ & \searrow \bar{f}'_1 & \swarrow \bar{f}_1 \\ & \mathbb{B}\mu_a & \end{array}.$$

Let  $\bar{\eta}_1 : (\bar{f}_1 \circ g)^*\mathbb{L} \rightarrow (\bar{f}'_1)^*\mathbb{L}$  be the isomorphism induced from the 2-isomorphism  $\bar{\eta}_1$ . Then we have the following commutative triangle

$$(A.5) \quad \begin{array}{ccc} g^*\bar{f}'_1^*\mathbb{L} & & \\ \downarrow \cong & \searrow \alpha(u) & \\ (\bar{f}_1 \circ g)^*\mathbb{L} & \xrightarrow{\bar{\eta}_1} & (\bar{f}'_1)^*\mathbb{L}. \end{array}$$

In the following, to simplify the presentation, we will treat  $g^*\bar{f}'_1^*\mathbb{L}$  and  $(\bar{f}_1 \circ g)^*\mathbb{L}$  are the same via the vertical isomorphism in A.5 defined in an obvious way, thus  $\bar{\eta}_1 = \alpha(u)$  above, and  $\bar{\eta}_1$  can be also viewed as a map from  $g^*\bar{f}'_1^*\mathbb{L}$ . The same rule applies to other pullbacks of a line bundle along a composition of maps.

Note that  $\bar{\eta}_1$  induces a 2-isomorphism  $\eta_1 : (g \times id) \circ f'_1 \rightarrow f_1 \circ g$  satisfying that  $\bar{\eta}_1 = \text{pr}_2 \circ \eta_1$  and making the following square commute:

$$\begin{array}{ccc} S' & \xrightarrow{g} & S \\ \downarrow f'_1 & & \downarrow f_1 \\ S' \times \mathbb{B}\mu_a & \xrightarrow{g \times id} & S \times \mathbb{B}\mu_a. \end{array}$$

For any line bundle  $M$  over  $S$  and any integer  $n$ , the 2-isomorphism  $\eta_1$  induces an isomorphism  $\eta_1 : (f_1 \circ g)^*(M \boxtimes \mathbb{L}^{\otimes n}) \rightarrow ((g \times id) \circ f'_1)^*(M \boxtimes \mathbb{L}^{\otimes n})$  such that we have the following commutative square using A.3 and Lemma A.8

(A.6)

$$\begin{array}{ccc}
g^*M \otimes g^*(M_1^{\otimes sn} \otimes M_2^{\otimes -rn}) & \xrightarrow{id \otimes (\alpha_1^{-\otimes sn} \otimes \alpha_2^{\otimes rn})} & g^*M \otimes (M_1'^{\otimes -sn} \otimes M_2'^{\otimes rn}) \\
\downarrow \cong & & \downarrow \cong \\
g^*f_1^*(M \boxtimes \mathbb{L}^{\otimes n}) & \xrightarrow{\eta_1} & ((g \times id) \circ f_1')^*(M \boxtimes \mathbb{L}^{\otimes n}) .
\end{array}$$

Here the left vertical map is induced from Lemma A.8 by taking  $t_1 \times t_2 = f_1$  and pullback along the map  $g$ , while the right vertical map is induced from Lemma A.8 by taking  $t_1 \times t_2 = (g \times id) \circ f_1'$ .

Using the equivalence of the stack  $\text{RepHom}(\mathbb{B}\mu_a, Y)$  with the  $I_{\mu_a}Y$  from [AGV08, §3.2], the morphism  $g^b$  between  $(w', \theta') \in I_{\mu_a}Y(S')$  and  $(w, \theta) \in I_{\mu_a}Y(S)$  induces a 2-isomorphism  $\eta_2 : f_2' \rightarrow f_2 \circ (g \times id)$  marking the following triangle commute

$$\begin{array}{ccc}
S' \times \mathbb{B}\mu_a & \xrightarrow{g \times id} & S \times \mathbb{B}\mu_a \\
& \searrow f_2' & \swarrow f_2 \\
& Y & .
\end{array}$$

The 2-isomorphism  $\eta_2$  induces a morphism from  $(f_2 \circ (g \times id))^*L^\vee$  to  $m_*(f_2')^*L^\vee$  which makes the following diagram commutes:

(A.7)

$$\begin{array}{ccc}
g^*w^*L^\vee \boxtimes \mathbb{L}_{a, -h} & \xrightarrow{g^b \boxtimes id} & (w')^*L^\vee \boxtimes \mathbb{L}_{a, -h} \\
\downarrow (g \times id)^* \epsilon_{f_2} & & \downarrow \epsilon_{f_2'} \\
(f_2 \circ (g \times id))^*L^\vee & \xrightarrow{\eta_2} & (f_2')^*L^\vee .
\end{array}$$

Indeed, using the notations in Proposition A.7, the 2-isomorphism  $i_S \circ \eta_2$  induces a morphism  $g^*w^*L^\vee \rightarrow (w')^*L^\vee$ , which is exactly induced by  $g^b$ . As  $g^b$  can be view a morphism from  $w'$  to  $w$  which commutes with  $\theta'$  and  $\theta$ , then by descent, we get the above diagram.

Then we make the composition  $\eta_2 \star \eta_1 := (\eta_1 \circ f_2) \circ (f_1' \circ \eta_2)$ ,<sup>21</sup> which is a 2-isomorphism from  $f_2' \circ f_1'$  to  $f_2 \circ f_1 \circ g$ . When we view  $f_2' \circ f_1'$  (resp.  $f_2 \circ f_1$ ) as a point in  $Y(S')$  (resp.  $Y(S)$ ), then  $\eta_2 \star \eta_1$  can be viewed as a morphism over  $g$  in  $Y$  from the point  $f_2' \circ f_1'$  to the point  $f_2 \circ f_1$ .

Now we can define the family map:

**Definition A.10 (Family map).** *With the notations as above, for any object*

$$\tau = (w, \theta, M_1, M_2, M_3, \rho_1, \rho_2, s_1, s_2) \in \mathcal{C}_e(S) ,$$

*we define the image  $f(\tau)$  under the family map  $f$  to be*

$$\phi = (f_2 \circ f_1, (M_1 \otimes M_3)^{\otimes a\delta}, M_2^{\otimes a\delta}, \rho_\tau, s_1^{a\delta}, s_2^{a\delta}) \in \mathbb{P}Y_{\tau, s}(S) .$$

*Given a morphism  $u : \tau' \rightarrow \tau$  in  $\mathcal{C}_e$ , we define the morphism  $f(u) : \phi' \rightarrow \phi$  to be  $(\eta_2 \star \eta_1, (\alpha_1 \otimes \alpha_3)^{\otimes a\delta}, (\alpha_2)^{\otimes a\delta})$ .*

<sup>21</sup>This is the so-called Godement product.

We should check that  $f(u)$  is indeed a well-defined morphism, for which we only need to check the following diagram commutes

(A.8)

$$\begin{array}{ccc} g^*(M_1 \otimes M_3)^{\otimes as\delta} \otimes g^*M_2^{-\otimes ar\delta} & \xrightarrow{(\alpha_1 \otimes \alpha_3)^{\otimes as\delta} \otimes \alpha_2^{-\otimes ar\delta}} & (M'_1 \otimes M'_3)^{\otimes as\delta} \otimes M'_2^{-\otimes ar\delta} \\ \downarrow g^*\rho_\tau & & \downarrow \rho_{\tau'} \\ g^*(f_2 \circ f_1)^*L^\vee & \xrightarrow{\eta_2 \star \eta_1} & (f'_2 \circ f'_1)^*L^\vee. \end{array}$$

Note that we have the following diagram

$$\begin{array}{ccc} g^*(M_1 \otimes M_3)^{\otimes as\delta} \otimes g^*M_2^{-\otimes ar\delta} & \xrightarrow{(\alpha_1 \otimes \alpha_3)^{\otimes as\delta} \otimes \alpha_2^{-\otimes ar\delta}} & (M'_1 \otimes M'_3)^{\otimes as\delta} \otimes M'_2^{-\otimes ar\delta} \\ \cong \downarrow & & \downarrow \cong \\ g^*M_3^{\otimes as\delta} \otimes g^*(M_1^{\otimes as\delta} \otimes M_2^{-\otimes ar\delta}) & \xrightarrow{(\alpha_3)^{\otimes as\delta} \otimes \alpha_1^{\otimes as\delta} \otimes \alpha_2^{-\otimes ar\delta}} & M'_3^{\otimes as\delta} \otimes (M'_1^{\otimes as\delta} \otimes M'_2^{-\otimes ar\delta}) \\ \downarrow g^*\rho_1 \otimes g^*\rho_2 \frac{a\delta-h}{a} & & \downarrow \rho'_1 \otimes (\rho'_2) \frac{a\delta-h}{a} \\ g^*w^*L^\vee \otimes g^*(M_1^{\otimes sh} \otimes M_2^{-\otimes rh}) & \xrightarrow{g^b \otimes (\alpha_1^{\otimes sh} \otimes \alpha_2^{-\otimes rh})} & (w')^*L^\vee \otimes (M'_1^{\otimes sh} \otimes M'_2^{-\otimes rh}), \end{array}$$

by the definition of  $\rho_\tau$ , we see that showing that A.8 is a commutative diagram is equivalent to show that the following diagram commutes:

(A.9)

$$\begin{array}{ccc} g^*w^*L^\vee \otimes g^*(M_1^{\otimes sh} \otimes M_2^{-\otimes rh}) & \xrightarrow{g^b \otimes (\alpha_1^{\otimes sh} \otimes \alpha_2^{-\otimes rh})} & (w')^*L^\vee \otimes (M'_1^{\otimes sh} \otimes M'_2^{-\otimes rh}) \\ \downarrow g^*\bar{\rho}_\tau & & \downarrow \bar{\rho}_{\tau'} \\ g^*(f_2 \circ f_1)^*L^\vee & \xrightarrow{\eta_2 \star \eta_1} & (f'_2 \circ f'_1)^*L^\vee. \end{array}$$

Denote  $\bar{\rho}_{\tau,\tau'} := f_1^*(g \times id)^* \epsilon_{f_2}$ , then  $\bar{\rho}_{\tau,\tau'}$  induces an isomorphism from  $g^*w^*L^\vee \otimes (M_1^{\otimes sh} \otimes M_2^{-\otimes rh})$  to  $(f_2 \circ (g \times id) \circ f_1')^*L^\vee$ . The 2-isomorphism  $\eta_1 \circ f_2$  induces a morphism  $\eta_1 \circ f_2 : g^*(f_2 \circ f_1)^*L^\vee \rightarrow (f_2 \circ (g \times id) \circ f_1')^*L^\vee$ . Take  $M = w^*L^\vee$ ,  $n = -h$  in A.6, using Proposition A.7, we get the following commutative diagram

$$\begin{array}{ccc} g^*w^*L^\vee \otimes g^*(M_1^{\otimes sh} \otimes M_2^{-\otimes rh}) & \xrightarrow{id \otimes (\alpha_1^{\otimes sh} \otimes \alpha_2^{-\otimes rh})} & g^*w^*L^\vee \otimes (M'_1^{\otimes sh} \otimes M'_2^{-\otimes rh}) \\ \downarrow g^*\bar{\rho}_{\tau'} & & \downarrow \bar{\rho}_{\tau',\tau} \\ g^*(f_2 \circ f_1)^*L^\vee & \xrightarrow{\eta_1 \circ f_2} & (f_2 \circ (g \times id) \circ f'_1)^*L^\vee. \end{array}$$

The 2-isomorphism  $f'_1 \circ \eta_2$  induces a morphism  $f'_1 \circ \eta_2 : (f_2 \circ (g \times id) \circ f'_1)^*L^\vee \rightarrow (f'_2 \circ f'_1)^*L^\vee$ . Pull back the square A.7 via  $f_1^*$ , we get the commutative square:

$$\begin{array}{ccc} g^*w^*L^\vee \otimes (M'_1^{\otimes sh} \otimes M'_2^{-\otimes rh}) & \xrightarrow{g^b \otimes id} & (w')^*L^\vee \otimes (M'_1^{\otimes sh} \otimes M'_2^{-\otimes rh}) \\ \downarrow \bar{\rho}_{\tau',\tau} & & \downarrow \bar{\rho}_{\tau'} \\ (f_2 \circ (g \times id) \circ f'_1)^*L^\vee & \xrightarrow{f'_1 \circ \eta_2} & (f'_2 \circ f'_1)^*L^\vee. \end{array}$$

Join the above two squares horizontally, this verifies the commutativity of A.9.

**Remark A.11.** When  $Y$  is a complete intersection in a toric stack, our construction of the family map is equivalent to the construction in [Wan19, §5.3.2].

**A.3. Computation of edge contribution.** Recall that in §3, we define a  $\mathbb{C}^*$ -action on  $\mathbb{P}Y_{r,s}$ . This can be induced from the  $\mathbb{C}^*$ -action on  $(L^\vee)^0 \times U$ :

$$t \cdot (l, x, y) = (tl, x, y) .$$

In the language of functor of points, for any  $t \in \mathbb{C}^*$  and any scheme  $S$ , this action will change any object  $(w, M_1, M_2, \rho, s_1, s_2)$  of  $Ob(\mathbb{P}Y_{r,s}(S))$  to  $(w, M_1, M_2, t \cdot \rho, s_1, s_2)$  of  $Ob(\mathbb{P}Y_{r,s}(S))$ . where  $t \cdot \rho$  means that we compose  $\rho$  by the automorphism of  $w^*L^\vee$  obtained by scaling by  $t$ .

Now we define a  $\mathbb{C}^*$ -action on  $\mathcal{C}_e$  such that the morphism  $f$  is  $\mathbb{C}^*$ -equivariant. Recall that  $\mathcal{C}_e$  can be represented as  $[(L^\vee)^0 \times U / T_{ar,as} \times \mathbb{C}^*]$  given by the action

$$(l, x, y)(t_1, t_2, t_3) = (t_3^{-as\delta} l, t_3 t_1 x, t_2 y) .$$

We define the  $\mathbb{C}^*$ -action on  $(L^\vee)^0 \times U$  in the way that

$$t \cdot (l, x, y) = (t \cdot l, x, y) .$$

One can see this action descends to be  $\mathbb{C}^*$ -action on  $\mathcal{C}_e$ . In the language of functor of points, for any  $t \in \mathbb{C}^*$  and any scheme  $S$  this  $\mathbb{C}^*$ -action will change any object  $(w, \theta, M_1, M_2, M_3, \rho_1, \rho_2, s_1, s_2) \in \mathcal{C}_e(S)$  to  $(w, \theta, M_1, M_2, M_3, t \cdot \rho_1, \rho_2, s_1, s_2) \in \mathcal{C}_e(S)$ , where  $t \cdot \rho_1$  means that we compose  $\rho_1$  by the automorphism of  $w^*L^\vee$  obtained by scaling by  $t$ . Then this action will change the associated  $\rho_\tau$  by  $t\rho_\tau$ , which implies that  $f$  is  $\mathbb{C}^*$ -equivariant.

Let  $\chi_1$  (resp.  $\chi_2$ ) be the character associated to  $T_{ar,as}$  by sending  $(t_1, t_2) \in T_{ar,as}$  to  $t_1$  (resp.  $t_2$ ). Then we can construct the line bundle  $L_{\chi_1}$  (resp.  $L_{\chi_2}$ ) over  $\mathcal{C}_e$  as the stack quotient  $[(L^\vee)^0 \times U \times \mathbb{C} / T_{ar,as} \times \mathbb{C}^*]$  using the Borel construction:

$$(l, x, y, z) \sim (t_3^{-as\delta} l, t_3 t_1 x, t_2 y, t_1 z) \text{ (resp. } (t_3^{-as\delta} l, t_3 t_1 x, t_2 y, t_2 z))$$

for any  $(l, x, y, z) \in (L^\vee)^0 \times U \times \mathbb{C}$  and  $(t_1, t_2, t_3) \in T_{ar,as} \times \mathbb{C}^*$ . Then for any scheme  $S$  and any object  $\tau := (w, \theta, M_1, M_2, M_3, \rho_1, \rho_2, s_1, s_2)$  of  $Ob(\mathcal{C}_e(S))$ , we have  $\tau^* L_{\chi_1} = M_1$  and  $\tau^* L_{\chi_2} = M_2$ . Using the stack quotient representation, the line bundles  $L_{\chi_1}$  and  $L_{\chi_2}$  also carry a  $\mathbb{C}^*$ -action such that the fiber of  $L_{\chi_1}$  over  $\infty$ -section (corresponding to the coordinate  $y = 0$ ) is of weight  $\frac{-1}{as\delta}$ , the fiber of  $L_{\chi_2}$  over  $D_\infty$  is of weight  $\frac{-1}{ar\delta}$ . We also note  $M_3$  is ( $\mathbb{C}^*$ -equivariantly) isomorphic to the line bundle  $\tau^* \pi^* (L^\vee)^{\frac{1}{as\delta}} \otimes \mathbb{C}_{\frac{\lambda}{as\delta}}$ . Then we have the following:

**Proposition A.12.** *The coordinate  $x$  of  $(L^\vee)^0 \times U$  descends to be a  $\mathbb{C}^*$ -equivariant section of the  $\mathbb{C}^*$ -equivariant line bundle  $L_{\chi_1} \otimes (\pi^* L^\vee)^{\frac{1}{as\delta}} \otimes \mathbb{C}_{\frac{\lambda}{as\delta}}$  over  $\mathcal{C}_e$ , and the coordinate  $y$  of  $(L^\vee)^0 \times U$  descends to be a  $\mathbb{C}^*$ -equivariant section of the  $\mathbb{C}^*$ -equivariant line bundle  $L_{\chi_2}$  over  $\mathcal{C}_e$ .*

Fix the edge  $e = \{h_0, h_\infty\}$  with the decoration as above, we will denote

$$\mathcal{K}_{0,h_0 \sqcup h_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r})) := \mathcal{K}_{0,2}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r})) \cap ev_1^{-1}(\bar{I}_{(c^{-1}, e^{\frac{\delta}{s}}, 1)} \mathbb{P}Y_{r,s}) \cap ev_2^{-1}(\bar{I}_{(c, 1, e^{\frac{\delta}{r}})} \mathbb{P}Y_{r,s}) .$$

Here we label the first marking by  $h_0$  and the second marking by  $h_\infty$ . Let  $\mathcal{K}_{0,h_0 \sqcup h_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r}))^{\mathbb{C}^*}$  be the  $\mathbb{C}^*$ -fixed loci of  $\mathcal{K}_{0,h_0 \sqcup h_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r}))$ , write  $\mathcal{K} := \mathcal{K}_{0,h_0 \sqcup h_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r}))^{\mathbb{C}^*}$  for short, we will first show the following:

**Proposition A.13.** *Let  $[\mathcal{K}]^{vir}$  be the virtual cycle associated to the  $\mathbb{C}^*$ -fixed part of the perfect obstruction theory of  $\mathcal{K}_{0,h_0 \sqcup h_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r}))$  to  $\mathcal{K}$ . We have that  $[\mathcal{K}]^{vir} = [\mathcal{K}]$  in  $A_*(\mathcal{K})$  and the evaluation map*

$$ev_{h_\infty} : \mathcal{K} \rightarrow \bar{I}_{(c, 1, e^{\frac{\delta}{r}})} \mathbb{P}Y_{r,s}$$

*is étale and finite of degree  $\frac{1}{a\delta}$ .*

*Proof.* Let  $\mathbb{E}$  be the restriction of the perfect obstruction theory<sup>22</sup> for  $\mathcal{K}_{0,h_0 \sqcup h_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r}))$  to  $\mathcal{K}$ , then the  $\mathbb{C}^*$ -fixed part of  $\mathbb{E}$ , which we denote to be  $\mathbb{E}^{fix}$ , is a perfect obstruction theory for  $\mathcal{K}$  by [GP99]. We will show that  $\mathbb{E} = \mathbb{E}^{fix}$  and it's isomorphic to the locally free sheaf  $ev_{h_\infty}^* \mathbb{T}_{\bar{I}_{(c,1,e\frac{\delta}{r})}} \mathbb{P}Y_{r,s}$ . Note this implies that  $\mathbb{E}^{fix} \cong \mathbb{T}_{\mathcal{K}} \cong ev_{h_\infty}^* \mathbb{T}_{\bar{I}_{(c,1,e\frac{\delta}{r})}} \mathbb{P}Y_{r,s}$ . This will implies the étaleness part, while the finiteness part will follow the properness of  $ev_{h_\infty}$ , Lemma A.1 and Proposition A.3.

It suffices to check  $\mathbb{E} = \mathbb{E}^{fix} \cong ev_{h_\infty}^* \mathbb{T}_{\bar{I}_{(c,1,e\frac{\delta}{r})}} \mathbb{P}Y_{r,s}$  locally on every  $\mathbb{C}$ -point  $[f]$  of  $\mathcal{K}$  represented by a twisted stable map  $f$ , the tangent space  $T_{[f]}$  and the obstruction space  $Ob_{[f]}$  of the space  $\mathcal{K}$  at the point  $[f]$  fit into the following exact sequence of  $\mathbb{C}^*$ -presentations<sup>23</sup>

$$(A.10) \quad \begin{aligned} 0 &\rightarrow H^0(C, \mathbb{T}_C(-[q_0] - [q_\infty])) \rightarrow H^0(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}}) \rightarrow T_{[f]} \\ &\rightarrow H^1(C, \mathbb{T}_C(-[q_0] - [q_\infty])) \rightarrow H^1(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}}) \rightarrow Ob_{[f]} \rightarrow 0. \end{aligned}$$

We have the following two exact sequences of vector bundles over  $\mathbb{P}Y_{r,s}$  (which are also  $\mathbb{C}^*$ -equivariant)

$$(A.11) \quad 0 \longrightarrow \mathbb{T}_{\mathbb{P}Y_{r,s}/Y} \longrightarrow \mathbb{T}_{\mathbb{P}Y_{r,s}} \longrightarrow \text{pr}_{r,s}^* \mathbb{T}_Y \longrightarrow 0,$$

and the Euler exact sequence

$$(A.12) \quad 0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}(\mathcal{D}_0) \oplus \mathcal{O}(\mathcal{D}_\infty) \longrightarrow \mathbb{T}_{\mathbb{P}Y_{r,s}/Y} \longrightarrow 0.$$

Then we have the following two right exact sequences

$$(A.13) \quad \begin{aligned} 0 &\rightarrow H^0(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}/Y}) \rightarrow H^0(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}}) \rightarrow H^0(C, f^* \text{pr}_{r,s}^* \mathbb{T}_Y) \\ &\rightarrow H^1(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}/Y}) \rightarrow H^1(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}}) \rightarrow H^1(C, f^* \text{pr}_{r,s}^* \mathbb{T}_Y) \rightarrow 0. \end{aligned}$$

and

$$(A.14) \quad \begin{aligned} 0 &\rightarrow H^0(C, \mathcal{O}) \rightarrow H^0(C, f^* \mathcal{O}(\mathcal{D}_0) \oplus f^* \mathcal{O}(\mathcal{D}_\infty)) \rightarrow H^0(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}/Y}) \\ &\rightarrow H^1(C, \mathcal{O}) \rightarrow H^1(C, f^* \mathcal{O}(\mathcal{D}_0) \oplus f^* \mathcal{O}(\mathcal{D}_\infty)) \rightarrow H^1(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}/Y}) \rightarrow 0. \end{aligned}$$

By Lemma A.1, we have

$$f^* \mathcal{O}(\mathcal{D}_0) \oplus f^* \mathcal{O}(\mathcal{D}_\infty) \cong \mathcal{O}(a\delta[q_0] + a\delta[q_\infty])$$

over  $C \cong \mathbb{P}_{ar,as}^1$ , as  $r, s$  are sufficiently large primes, then we have

$$H^1(C, f^* \mathcal{O}(\mathcal{D}_0) \oplus f^* \mathcal{O}(\mathcal{D}_\infty)) = 0,$$

which implies that  $H^1(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}/Y}) = 0$  by A.14. As  $H^1(C, \mathbb{T}_C(-[q_0] - [q_\infty])) = 0$ , we have

$$Ob_{[f]} \cong H^1(C, f^* \text{pr}_{r,s}^* \mathbb{T}_Y)$$

and

$$T_{[f]} \cong H^0(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}}) / H^0(C, \mathbb{T}_C(-[q_0] - [q_\infty]))$$

by A.10 and A.13.

Moreover, the exact sequence A.14 becomes

$$(A.15) \quad 0 \rightarrow \mathbb{C} \rightarrow \mathbb{C} \oplus \mathbb{C} \rightarrow H^0(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}/Y}) \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow 0,$$

which implies that  $H^0(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}/Y}) \cong \mathbb{C}$  (of weight 0). Furthermore, the composition

$$H^0(C, \mathbb{T}_C(-[q_0] - [q_\infty])) \rightarrow H^0(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}}) \rightarrow H^0(C, f^* \text{pr}_{r,s}^* \mathbb{T}_Y)$$

<sup>22</sup>Here a perfect obstruction is a morphism  $\mathbb{T} \rightarrow \mathbb{E}$  in [BF97], where  $\mathbb{T} = \mathbb{L}^\vee$  is the derived dual of the cotangent complex.

<sup>23</sup>All the arrows in this proof are  $\mathbb{C}^*$ -equivariant unless otherwise stated.



where the arrows are coming from A.10 and A.13, is equal to zero as the composition is induced from the composition of sheaves

$$\mathbb{T}_C(-[q_0] - [q_\infty]) \rightarrow f^* \mathbb{T}_{\mathbb{P}Y_{r,s}} \rightarrow f^* \mathrm{pr}_{r,s}^* \mathbb{T}_Y ,$$

which is zero. Note  $H^0(C, \mathbb{T}_C(-[q_0] - [q_\infty]))$  is one dimensional, which corresponds to the space of tangent vector field of  $\mathrm{orbi}\text{-}\mathbb{P}^1$  vanishing on the two markings, then the space  $H^0(C, \mathbb{T}_C(-[q_0] - [q_\infty]))$  maps isomorphically into the subspace  $H^0(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}/Y}) \subset H^0(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}})$  via A.13, which implies that

$$\begin{aligned} T_{[f]} &\cong H^0(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}}) / H^0(C, \mathbb{T}_C(-[q_0] - [q_\infty])) \\ &\cong H^0(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}}) / H^0(C, f^* \mathbb{T}_{\mathbb{P}Y_{r,s}/Y}) \cong H^0(C, f^* \mathrm{pr}_{r,s}^* \mathbb{T}_{\mathbb{P}Y_{r,s}}) . \end{aligned}$$

Let  $\pi : C \rightarrow \mathrm{Spec}(\mathbb{C})$  be the projection to a point, next we show that

$$R^\bullet \pi_* f^* \mathrm{pr}_{r,s}^* \mathbb{T}_{\mathbb{P}Y_{r,s}} \cong (ev_{q_\infty}^* \mathbb{T}_{\tilde{I}_{(g,1,e\frac{\delta}{r})}} \mathbb{P}Y_{r,s})|_{[f]} ,$$

which will imply that

$$T_{[f]} \cong (ev_{q_\infty}^* \mathbb{T}_{\tilde{I}_{(g,1,e\frac{\delta}{r})}} \mathbb{P}Y_{r,s})|_{[f]} , \quad Ob_{[f]} = 0 ,$$

in particular  $T_{[f]}$  is  $\mathbb{C}^*$ -fixed. Applying  $R^\bullet \pi_* f^*(-)$  to the first sequence A.11, we get a distinguished triangle

$$R^\bullet \pi_* f^* \mathbb{T}_{\mathbb{P}Y_{r,s}/Y} \longrightarrow R^\bullet \pi_* f^* \mathbb{T}_{\mathbb{P}Y_{r,s}} \longrightarrow R^\bullet \pi_* f^* \mathrm{pr}_{r,s}^* \mathbb{T}_Y \xrightarrow{+1} R^\bullet \pi_* f^* \mathbb{T}_{\mathbb{P}Y_{r,s}/Y}[1] .$$

Let  $b_\infty : q_\infty \hookrightarrow C$  be the gerbe section of  $C$  corresponding to the half-edge  $h_\infty$ . Using the divisor sequence for the section  $b_\infty$

$$0 \rightarrow \mathcal{O}(-q_\infty) \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{q_\infty} \rightarrow 0 ,$$

tensor with  $f^* \mathrm{pr}_{r,s}^* \mathbb{T}_Y$  and taking  $R^\bullet \pi_*$ , we have

$$R^\bullet \pi_* f^* \mathrm{pr}_{r,s}^* \mathbb{T}_Y = R^\bullet (\pi \circ b_\infty)_* (f \circ b_\infty)^* (\mathrm{pr}_{r,s}^* \mathbb{T}_Y) ,$$

where we use the fact that  $R^\bullet \pi_* f^* \mathrm{pr}_{r,s}^* \mathbb{T}_Y(-q_\infty) = 0$ . Next we show that

$$R^\bullet (\pi \circ b_\infty)_* (f \circ b_\infty)^* (\mathrm{pr}_{r,s}^* \mathbb{T}_Y) \cong (ev_{q_\infty}^* \mathbb{T}_{\tilde{I}_{(c,1,e\frac{\delta}{r})}} \mathbb{P}Y_{r,s})|_{[f]} ,$$

First by the tangent bundle lemma in [AGV08, §3.6.1], we have

$$(\pi \circ b_\infty)_* (f \circ b_\infty)^* (\mathbb{T}_{\mathbb{P}Y_{r,s}}) \cong (ev_{q_\infty}^* \mathbb{T}_{\tilde{I}_{(c,1,e\frac{\delta}{r})}} \mathbb{P}Y_{r,s})|_{[f]} ,$$

thus we only need to show that

$$R^\bullet (\pi \circ b_\infty)_* (f \circ b_\infty)^* (\mathrm{pr}_{r,s}^* \mathbb{T}_Y) \cong (\pi \circ b_\infty)_* (f \circ b_\infty)^* \mathbb{T}_{\mathbb{P}Y_{r,s}} .$$

As  $(\pi \circ b_\infty)_*$  is exact on coherent sheaf, we have  $R^\bullet (\pi \circ b_\infty)_* = (\pi \circ b_\infty)_*$  and the following exact sequence of vector spaces:

(A.16)

$$0 \rightarrow (\pi \circ b_\infty)_* (f \circ b_\infty)^* \mathbb{T}_{\mathbb{P}Y_{r,s}/Y} \rightarrow (\pi \circ b_\infty)_* (f \circ b_\infty)^* \mathbb{T}_{\mathbb{P}Y_{r,s}} \rightarrow (\pi \circ b_\infty)_* (f \circ b_\infty)^* \mathrm{pr}_{r,s}^* \mathbb{T}_Y \rightarrow 0 ,$$

we are left to show that  $(\pi \circ b_\infty)_* (f \circ b_\infty)^* \mathbb{T}_{\mathbb{P}Y_{r,s}/Y} = 0$ , which can be checked by the next lemma A.14 by pulling back to every  $\mathbb{C}$ -point using the idea of proving the tangent bundle lemma [AGV08, §3.6.1].  $\square$

**Lemma A.14.** *Let  $x := [f]$  be a  $\mathbb{C}$ -point of  $\mathcal{K}$  represented by a twisted stable map  $f : C \rightarrow \mathbb{P}Y_{r,s}$  as in Lemma A.1. Use the notation in the proof of Proposition A.13, we have*

$$(\pi \circ b_\infty)_* (f \circ b_\infty)^* \mathbb{T}_{\mathbb{P}Y_{r,s}/Y} = 0 .$$

*Proof.* The gerbe marking  $q_\infty$  of  $C$  is canonically isomorphic to the classifying stack  $\mathbb{B}\mu_{ar}$ , which gives a unique lift of the  $\mathbb{C}$ -point  $x$  in the gerbe marking  $q_\infty$ , which we denote to be  $z$ . The isotropy group  $\text{Aut}(z)$  of  $z$  in  $\mathbb{B}\mu_{ar}$  is canonically isomorphic to the cyclic group  $\mu_{ar}$ , and the generator of  $\mu_{ar}$  acts on the vector space  $f^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y}|_z$  via the multiplication by  $e^{\frac{\delta}{r}}$  due to the very choice of the multiplicity associated to the half-edge  $h_\infty$ . Note the space  $(\pi \circ b_\infty)_*(f \circ b_\infty)^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y}$  is given by the  $\mu_{ar}$ -invariant space  $(f^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y}|_z)^{\mu_{ar}}$ , then it is equal to zero.  $\square$

The family map  $f$  constructed in Section A.2.2 induces a morphism

$$g : \mathcal{M}_e \rightarrow \mathcal{K}$$

by the universal property of moduli stack  $\mathcal{K}$ . Then  $g$  is surjective by Lemma A.1. Furthermore, we have:

**Proposition A.15.** *The morphism  $g : \mathcal{M}_e \rightarrow \mathcal{K}$  is finite étale of degree  $\frac{1}{as}$ . The evaluation map  $ev_{h_\infty}$  from  $\mathcal{K}$  to  $\bar{I}_{(c,1,\frac{\delta}{r})}\mathbb{P}Y_{r,s}$  is finite étale of degree  $\frac{1}{a\delta}$ .*

*Proof.* Let  $\text{pr}_{r,s} : \bar{I}_{(c,1,\frac{\delta}{r})}\mathbb{P}Y_{r,s} \rightarrow \bar{I}_c Y$  be the morphism induced from the projection of  $\mathbb{P}Y_{r,s}$  to the base  $Y$ , we see  $\text{pr}_{r,s}$  is an isomorphism. Then the composition  $\text{pr}_{r,s} \circ ev_{h_\infty} \circ g$  is finite étale of degree  $\frac{1}{a^2 s \delta}$  as the composition can be also obtained by first forgetting root construction of  $\mathcal{M}_e$  and taking rigidification afterwards. By two-of-there property for étale morphisms [Sta19, Lemmma 100.35.6], the étaleness of  $g$  comes from that  $ev_{h_\infty}$  is finite étale, which is proved in Proposition A.13. Finally as the composition  $ev_{h_\infty} \circ g$  is of degree  $\frac{1}{a^2 s \delta}$  and  $ev_{h_\infty}$  is of degree  $\frac{1}{a\delta}$  by Proposition A.2, we see that  $g$  is of degree  $\frac{1}{as}$ .  $\square$

**Lemma A.16.** *We have the following:*

- (1) *When the edge moduli  $\mathcal{M}_e$  arises from the localization analysis in 3.2.2, we have that  $[\mathcal{M}_e]^{vir} = [\mathcal{M}_e]$  and the Euler class of the virtual normal bundle is equal to 1.*
- (2) *When the edge moduli  $\mathcal{M}_e$  arises from the localization analysis in 4.2.2, we have  $[\mathcal{M}_e]^{vir} = [\mathcal{M}_e]$  and the inverse of Euler class of the virtual normal bundle is equal to*

$$\prod_{i=1}^{-1-\lfloor -\delta \rfloor} \left( \lambda + \frac{i}{\delta} (c_1(L) - \lambda) \right).$$

*Proof.* The first case follows from Proposition A.13. Now we calculate the edge contribution from the second case. Recall the divisor  $E$  of  $\mathfrak{R}$  introduced in Section 4.1 is isomorphic to  $\mathbb{P}Y_{r,s}$ . The normal bundle  $N_{E/\mathfrak{R}}$  is isomorphic to  $\mathcal{O}(-s\mathcal{D}_0)$ , and the fiber of the normal bundle  $N_{E/\mathfrak{R}}$  over  $\mathcal{D}_\infty$  has  $\mathbb{C}^*$ -weight 1, thus we have  $f^*N_{E/\mathfrak{R}} = L_{as\delta\chi_1}^\vee \otimes \pi^*L$  as  $\mathbb{C}^*$ -equivariant line bundles over  $\mathcal{C}_e$ . Then we see that

$$-R^*\pi_*f^*N_{E/\mathfrak{R}} = R^1\pi_*(\pi^*L \otimes L_{as\delta\chi_1}^\vee)$$

in the  $K$ -group  $K^*(\mathcal{M}_e)$ .

Write  $\mathfrak{P} := \mathbb{P}Y_{r,s}$  for simplicity. Let  $\mathbb{E}_{\mathfrak{R}}$  (resp.  $\mathbb{E}_{\mathfrak{P}}$ ) be the pull-back of the perfect obstruction theory of  $\mathcal{K}_{0,2}(\mathfrak{R}, (0, \frac{\delta}{r}, 0))$  (resp.  $\mathcal{K}_{0,2}(\mathfrak{P}, (0, \frac{\delta}{r}))$ ) to  $\mathcal{M}_e$ . As  $\mathcal{M}_e$  is étale over the corresponding  $\mathbb{C}^*$ -fixed loci in  $\mathcal{K}_{0,2}(\mathfrak{P}, (0, \frac{\delta}{r}))$  (or  $\mathcal{K}_{0,2}(\mathfrak{R}, (0, \frac{\delta}{r}, 0))$ ) by Proposition A.15, we see that the  $\mathbb{C}^*$ -fixed parts  $\mathbb{E}_{\mathfrak{P}}^{fix}$  and  $\mathbb{E}_{\mathfrak{R}}^{fix}$  are perfect obstruction theory for  $\mathcal{M}_e$ . Note we have the distinguished triangle:

$$(A.17) \quad \mathbb{E}_{\mathfrak{P}} \longrightarrow \mathbb{E}_{\mathfrak{R}} \longrightarrow R^\bullet\pi_*f^*N_{E/\mathfrak{R}} \xrightarrow{+1} \mathbb{E}_{\mathfrak{P}}[1].$$

As shown in Lemma A.13,  $\mathbb{E}_{\mathfrak{P}}$  is  $\mathbb{C}^*$ -fixed, then the movable part of  $\mathbb{E}_{\mathfrak{R}}$  is equal to the movable part of  $R^\bullet\pi_*f^*N_{E/\mathfrak{R}}$ , which we will calculate below.

The restriction of sections  $x^{-sa\delta+asi}y^{-ari}(1 \leq i \leq -1-[-\delta])$  to each fiber curve  $C_e$  of  $\mathcal{C}_e$  over  $\mathcal{M}_e$  is a basis of the vector space  $H^1(C_e, f|_{C_e}^* N_{E/\mathfrak{R}})$ . However they may not be sections of the vector bundle  $R^1\pi_*(f^* N_{E/\mathfrak{R}})$ . Instead, by Proposition A.12, one can see that  $x^{-sa\delta+asi}y^{-ari}$  is a section of vector bundle  $R^1\pi_*(f^* N_{E/\mathfrak{R}}) \otimes L^{-\otimes \frac{i}{\delta}} \otimes \mathbb{C}_{-\frac{\delta-i}{\delta}\lambda}$ . Therefore  $R^1\pi_* f^* N_{E/\mathfrak{R}}$  is isomorphic to the direct sum of line bundles  $\bigoplus_{i=1}^{-1-[-\delta]} L^{\otimes \frac{i}{\delta}} \otimes \mathbb{C}_{\frac{\delta-i}{\delta}\lambda}$ . Then the inverse of the Euler class of the virtual normal bundle form  $\mathbb{E}_{\mathfrak{R}}$  is equal to  $\prod_{i=1}^{-1-[-\delta]} (\lambda + \frac{i}{\delta}(c_1(L) - \lambda))$ .

The calculation above also shows that the  $\mathbb{C}^*$ -fixed part of  $\mathbb{E}_{\mathfrak{R}}$  is the equal to  $\mathbb{E}_{\mathfrak{R}}$ , then we have  $[\mathcal{M}_e]^{vir} = [\mathcal{M}_e]$  in Section 4.2.2.  $\square$

**Remark A.17.** When  $Y$  is of zero dimensional, our computation of the Euler class of the virtual normal bundle coincides the computation from [LS20, §6].

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