QUANTUM LEFSCHETZ THEOREM REVISITED

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ABSTRACT. Let X be any smooth Deligne-Mumford stack with projective coarse moduli, and Y be a smooth complete intersection in X associated with a direct sum of semi-positive line bundles. For any point on the Givental's Lagrangian cone of X satisfying a rather mild condition called admissible series, we will show that a hyper-geometric modification of the point lies on the Lagrangian cone of Y. This confirms a prediction from Coates-Corti-Iritani-Tseng about the genus zero quantum Lefschetz theorem beyond convexity.

1. INTRODUCTION

Gromov-Witten (GW) invariants count the (virtual) number of stable maps from curves to a target variety/stack X with prescribed conditions. *Quantum Lefschetz* is one of the central topics in Gromov-Witten theory and it compares the GW invariants of a complete intersection and its ambient space. On the other hand, in Givental's formalism[Giv04, CG07, Tse10], all information of genus-zero Gromov-Witten invariants is encoded in an *overruled Lagrangian cone* $\mathcal{L}_X$ sitting inside an infinite dimensional symplectic vector space. Consequently, a natural approach to the quantum Lefschetz problem in genus zero is to find an explicit slice on Givental's Lagrangian cone of the complete intersection using a given slice on Givental's Lagrangian cone of the ambient space, This approach is often referred to as a *mirror theorem* in the literature. The earliest quantum Lefschetz theorem can be traced back to the verification of the famous mirror conjecture for quintic threefolds[CDLOGP91, Giv96, LLY99]. Since then, more cases related to quantum Lefschetz have been proved. In genus zero, Givental-style quantum Lefschetz theorems can be divided into two categories:

(1) The complete intersection Y is associated with a direct sum of *convex* line bundles over the ambient space X, see e.g.,[Giv98, CG07, CCIT19]. The reason for requiring the convexity condition is that we need to apply the so-called Quantum Lefschetz principle proved in [KKP03]. This principle expresses the Gromov-Witten invariants of the complete intersection as an Euler class of a certain bundle over the moduli stack of genus-zero stable maps to X. See also [CGI+12] for some examples where this principle can fail if we drop the convexity condition, where the examples are associated with semi-positive line bundles in the sense of this paper. We note that convexity is a very rare condition when X is a not a variety.

(2) When convexity fails, proving a quantum Lefschetz theorem is much more difficult. There has been significant progress on this case recently by solving the genus zero quasimap wall-crossing conjecture [Wan19, Zho22]. However, it's important to note that we require the ambient space to be a GIT quotient, which is a prerequisite for applying quasimap theory[CFKM14, CCFK15].

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The main objective of this paper is to prove a genus-zero[1] quantum Lefschetz theorem that goes beyond the scope of the two categories mentioned previously, where the ambient space can be a non-GIT target and the convexity condition may not hold. The proof relies on the *recursive relation* of a point on the Givental's Lagrangian cone discovered by the author in [Wan19], which was used to demonstrate the genus-zero quasimap wall-crossing conjecture for abelian GIT quotients. However, to apply this characterization, we need to carefully pick a space carried with a $\mathbb{C}^*-$action which could provide a recursive relation we want; in our case we will use a root-stack modification of the space of deformation to the normal cone. This is different from the spaces used in loc.cit and it has the advantage of generalizing the quantum Lefschetz theorem proved in loc.cit to non-GIT targets.

1.1. **Main theorem.** Let X be a smooth Deligne-Mumford stack over $\mathbb{C}$ with projective coarse moduli. Let $\bar{I}_\mu X$ be the rigidified (cyclotomic) inertia stack of X, where a $\mathbb{C}-$point of $\bar{I}_\mu X$ can be written as a pair (x, g) where x is a $\mathbb{C}$-point of X and g is an element in the isotropy group $Aut(x)$. Denote by C the finite set of connected components of $\bar{I}_\mu X$ with $\bar{I}_c X$ being the component corresponding to the element $c \in C$. The involution on $\bar{I}_\mu X$ by sending (x, g) to (x, g^{-1}) also induces an involution on C, we write c^{-1} to be image of c under the involution.

The *overruled Lagrangian cone* $\mathcal{L}_X$ introduced by A.Givental is comprised by the so-called (big) J-function:

$$J^X(q, \mathbf{t}(z), -z) := -z\mathbb{1}_X + \mathbf{t}(z) + \sum_{\beta \in \mathrm{Eff}(X)} \sum_{m \geqslant 0} \frac{q^\beta}{m!} \phi^\alpha \langle \mathbf{t}(\bar{\psi}_1), \cdots, \mathbf{t}(\bar{\psi}_m), \frac{\phi_\alpha}{-z - \bar{\psi}_\star} \rangle^X_{0,[m]\cup\star,\beta} \ , \tag{1.1}$$

where the input[2] $\mathbf{t}(z)$ is a point in $H^*_{CR}(X, \mathbb{C})[z]$, the notation q^β stands for the Novikov variable corresponding to the degree β in the cone $\mathrm{Eff}(X)$ of effective curve classes of X, and $\{\phi_\alpha\}$ is a basis of the Chen-Ruan cohomology $H^*_{CR}(X, \mathbb{C}) = H^*(\bar{I}_\mu X, \mathbb{C})$ with dual basis $\{\phi^\alpha\}$. We note that such a choice of the input $\mathbf{t}(z)$ usually lead to some (rather mild) convergence issue about $J-$function. One way to handle this is using formal geometry; let N be any positive integer, we will choose our input $\mathbf{t}$ in the *super-space*

$$(q, t_1, \cdots, t_N)H^*(\bar{I}_\mu X, \mathbb{C})[z][\![t_1, \cdots, t_N]\!][\![\mathrm{Eff}(X)]\!] \ ,$$

where each variable t_i is associated with a grading in $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. More precisely, $\mathbf{t}(z)$ can be written in the form of

$$\sum_{\substack{\beta \in \mathrm{Eff}(X) \\ \vec{k}=(k_1,\cdots,k_N)\in \mathbb{Z}^N_{\geqslant 0}}} q^\beta t_1^{k_1} \cdots, t_N^{k_N} f_{\beta,\vec{k}}(z) \ , \tag{1.2}$$

where $f_{\beta,\vec{k}}(z) \in H^*(\bar{I}_\mu X, \mathbb{C})[z]$, $f_{0,\vec{0}}(z) = 0$ and $t_1^{k_1} \cdots t_N^{k_N} f_{\beta,\vec{k}}(z)$ is of even grading. See §2 for more details.

Let $Y \subset X$ be a smooth complete intersection associated with a direct sum of semi-positive line bundles $\oplus_{j=1}^r L_j$, i.e., the pairing $\beta(L_j) := (c_1(L_j), \beta) \geqslant 0$ for any degree $\beta \in \mathrm{Eff}(X)$. Then the age function $age_g(L_j|_x)$ for each $(x, g) \in \bar{I}_c X$ is constant on each connected component $\bar{I}_c X$ of $\bar{I}_\mu X$ and we will denote the constant value to be $age_c(L_j)$. To state the main theorem in this paper, we will introduce a useful category for discussing the quantum Lefschetz known as *admissible series*. This concept summarizes a common feature of all previous proved $J-$functions

[1]Recently, there have been significant advance in the high genus case, where the failure of convexity provides one main obstacle to the computation of GW invariants, see e.g., [Zin08, GJR17, GJR18, CGL21, CJR22, LR22, CJR21] and their references therein.

[2]We will also use the notation $\mu(z)$ in this paper.

about quantum Lefschetz. Here we will describe the content about admissible series needed in the statement of the main theorem, see §2 for a more general definition of admissible series.

Endow each variable t_i and line bundle L_j with a weight $w_{ij} \in \mathbb{Q}_{\geqslant 0}$. We will call a tuple $(\beta, \vec{k} = (k_1, \cdots, k_N), c) \in \mathrm{Eff}(X) \times \mathbb{Z}^N_{\geqslant 0} \times C$ an *admissible pair* if

$$age_c(L_j) \equiv \beta(L_j) + \sum_{i=1}^{N} w_{ij} k_i \bmod \mathbb{Z}$$

for all line bundles L_j. Denote by Adm the set of all admissible pairs. Write each component $f_{\beta,\vec{k}}(\mathbf{t}, z)$ in 1.2 as a sum

$$f_{\beta,\vec{k}}(z) = \sum_{c \in C} f_{\beta,\vec{k},c}(z)$$

where $f_{\beta,\vec{k},c}(z)$ belongs to the space $H^*(\bar{I}_{c^{-1}}X, \mathbb{C})[z]$. Then we call the input $\mathbf{t}(z)$ an *admissible power series* if $\mathbf{t}(z)$ can be written as in 1.2 and $f_{\beta,\vec{k},c}(z) = 0$ whenever $(\beta, \vec{k}, c)$ is *not* an admissible pair.

Let $J^X(q, \mathbf{t}, -z)$ be a point on the Lagrangian cone $\mathcal{L}_X$ of X with $\mathbf{t}(z)$ being an admissible (power) series. Following the tradition in the literature, to prove a quantum Lefschetz theorem, it's more convenient to flip the sign of z in the function $J(q, \mathbf{t}, -z)$; in our case, $J^X(q, \mathbf{t}, z)$ will always have an asymptotic expansion in variables $q^\beta t_1^{k_1} \cdots t_N^{k_N}$ as:

$$z + \sum_{(\beta,\vec{k},c) \in \mathrm{Adm}} q^\beta t_1^{k_1} \cdots t_N^{k_N} J^X_{\beta,\vec{k},c}(\mathbf{t}, z) \ ,$$

where $J^X_{\beta,\vec{k},c}(\mathbf{t}, z) \in H^*(\bar{I}_{c^{-1}}X, \mathbb{C})[z, z^{-1}]]$. Define $J^{X,tw}(q, \mathbf{t}, z)$ to be the hyper-geometric modification of $J^X(q, \mathbf{t}, z)$:

$$\begin{aligned} J^{X,tw}(q, \mathbf{t}, z) &:= z + \sum_{(\beta,\vec{k},c) \in \mathrm{Adm}} q^\beta t_1^{k_1} \cdots t_N^{k_N} J^X_{\beta,\vec{k},c}(\mathbf{t}, z) \\ &\cdot \prod_{j=1}^{r} \prod_{0 \leqslant m < \beta(L_j) + \sum_i w_{ij} k_i} \big(c_1(L_j) + (\beta(L_j) + \sum_i w_{ij} k_i - m) z\big) \ . \end{aligned} \tag{1.3}$$

Our main theorem in this paper will be the following:

Theorem 1.1 (=Theorem 4.6)**.** *Let $i : \bar{I}_\mu Y \to \bar{I}_\mu X$ be the inclusion of rigidified inertia stacks, then the series $i^* J^{X,tw}(q, \mathbf{t}, -z)$ lies on the Lagrangian cone $\mathcal{L}_Y$ of Y.*

Here we are suppressing a change about the $J-$function defining the Lagrangian cone $\mathcal{L}_Y$ of Y by using Novikov variable from $\mathrm{Eff}(X)$ rather than $\mathrm{Eff}(Y)$, see §3.4 for more details.

Remark 1.2. Our choice of the variables t_i with the associated weights w_{ij} plays the same role as the Novikov variables q^β with the associated numbers $\beta(L_j)$, which resembles much similarity with the usage of extended degrees for S-extended $I-$functions of toric stacks in [CCIT15, CCIT19]. For this reason we will call the pair $(\beta, \vec{k})$ an *extended degree*.

1.2. **Relation to twisted Gromov-Witten invariants.** We will discuss a relationship of our quantum Lefschetz theorem to twisted Gromov-Witten invariants, in particular a conjecture of Coates-Corti-Iritani-Tseng.

Consider the $\mathbb{C}^*-$action on the vector bundle $E := \oplus_{j=1}^r L_j$ via scaling the fibers. The $\mathbb{C}^*-$equivariant Euler class of E can be written in term of the Chern roots $c_1(L_j)$ as

$$e^{\mathbb{C}^*}(E) := \prod_j (\kappa + c_1(L_j)) \ ,$$

where κ be the equivariant parameter corresponding to the standard representation of $\mathbb{C}^*$. Then one can define a *twisted Lagrangian cone* $\mathcal{L}_X^{tw}$ using $(e^{\mathbb{C}^*}, E)$-*twisted Gromov-Witten invariants* (See [CG07] and [Tse10] for more details).

The series $J^X(q, \mathbf{t}, z)$ in our main theorems are related to points on the twisted Lagrangian cone $\mathcal{L}_X^{tw}$ in the following way: we can associate J^X a series defined by

$$\begin{aligned} I^{X,tw} &:= z + \sum_{(\beta,\vec{k},c)\in \mathrm{Adm}} q^\beta t_1^{k_1}\cdots t_N^{k_N} J^X_{\beta,\vec{k},c}(\mathbf{t},z) \\ &\cdot \prod_{j=1}^{r} \prod_{0\leqslant m<\beta(L_j)+\sum_i w_{ij}k_i} \kappa + \left(c_1(L_j) + (\beta(L_j) + \sum_i w_{ij}k_i - m)z\right) . \end{aligned} \tag{1.4}$$

Then we have

$$\lim_{\kappa\to 0} I^{X,tw} = J^{X,tw} .$$

Moreover, using the same idea in [CCIT19, Theorem 22] where we view our t_i here as their x_i in loc.cit, one can prove that $I^{X,tw}$ lies on the twisted Lagrangian cone $\mathcal{L}_X^{tw}$. In general, Coates-Corti-Iritani-Tseng make the following conjecture (see [OP18, Conejcture 5.2]):

Conjecture 1.3. *Let X be a smooth proper Deligne-Mumford stack with projective coarse moduli and Y is a smooth complete intersection*[3] *of X. Let I^{tw} be a point on the twisted Lagrangian cone $\mathcal{L}_X^{tw}$ of X, if the limit $\lim_{\kappa\to 0} i^* I^{tw}$ exits, then $\lim_{\kappa\to 0} i^* I^{tw}$ is a point on the Lagrangian cone $\mathcal{L}_Y$ of Y.*

This conjecture is known to be false in general, where a counterexample is found in [SW22]. However we can still take this as a guiding principle towards genus zero quantum Lefschetz theorem in general case and we are able to verify this conjecture under mild hypotheses as in Theorem 4.6.

1.3. **Sketch of the main idea of the proof.** The proof will first reduce to the hypersurface case. Set $\mu^{X,tw} := [J^{X,tw}(q, \mathbf{t}, z) - z]_+$ to be the truncation in nonnegative $z-$powers. Note that the $J-$function $J^Y(q, \mathbf{t}, z)$ is of the form $z + \mathbf{t} + \mathcal{O}(z^{-1})$, we observe that, to prove the main theorem, we need to show that

$$J^Y(q, i^*\mu^{X,tw}, z) = i^* J^{X,tw}(q, \mathbf{t}, z) ,$$

for which we only need to consider their *negative $z-$powers*. We aim to demonstrate that both sides of the negative $z-$powers of the above equation satisfy the *same recursive relations* in each degree corresponding to $q^\beta t_1^{k_1}\cdots t_N^{k_N}$ except a special consideration in low degrees (see Lemma 4.7). First we will consider two auxiliary spaces carried with $\mathbb{C}^*-$actions, which are root-stack modifications of a certain orbi-$\mathbb{P}^1$ bundle over Y (see §3.1) and the space of deformation to the normal cone (see §4.1). We then apply virtual localization to express two auxiliary cycles (see (3.7) and (4.3)) corresponding to the two spaces in graph sums and extract λ^{-1} coefficient and λ^{-2} coefficient respectively (where λ is an equivariant parameter). Finally, the polynomiality of the two auxiliary cycles implies that the coefficients must vanish, leading to the same type of recursive relations (see also Proposition 3.4 and Theorem 4.2).

1.4. **Outline.** The rest of this paper is organized as follows. In §2, we collect some background on Gromov-Witten theory. In §3, we will give a recursive characterization of an admissible slice on the Lagrangian cone. In §4, we will prove our main theorem. In the Appendix, we carry out a detailed analysis about edge contributions needed in the localization formula.

[3]Note that the conjecture doesn't require the line bundles defining the complete intersection to be semi-positive.

1.5. **Notation and convention.** We work over the field $\mathbb{C}$. Let D be an effective divisor of a smooth stack X, we will denote the $\mathcal{O}(D)$ or $\mathcal{O}_X(D)$ to be the line bundle associated with D.

For any positive integer i, we will use the notation $\boldsymbol{\mu}_i$ to mean the finite cyclic subgroup of $\mathbb{C}^*$ of order i, and use the notation $[i]$ to mean the set $\{1, 2, \cdots, i\}$. For any rational number a, we will use the notation $\mathrm{e}^{\,a}$ for to mean the exponential $\exp(\frac{\sqrt{-1}a}{2\pi})$.

Let λ be the equivariant class in $H^2_{\mathbb{C}^*}(\{\mathrm{Spec}(\mathbb{C})\})$ corresponding to the stand representation of $\mathbb{C}^*$. For any rational number q, we will write $\mathbb{C}_{q\lambda}$ to be the trivial line bundle with a $\mathbb{C}^*$−action of weight q.

2. Background on Gromov-Witten theory

2.1. **Admissible series.** Let X be a smooth DM stack over $\mathbb{C}$ with projective coarse moduli and $\bar{I}_\mu X := \sqcup_{a\geqslant 1} \bar{I}_{\mu_a} X$ be the associated rigidified (cyclotomic) inertia stack. For our purpose, a finer decomposition of the rigidified inertia stack $\bar{I}_\mu X$ is needed. Given a set of line bundles $L_1, \cdots, L_r$ over X, we will choose a decomposition of the rigidified inertia stack into open-closed components

$$\bar{I}_\mu X := \bigsqcup_{c\in C} \bar{I}_c X$$

where C is a *finite* index set for the decomposition which satisfies the following assumption:

Assumption 2.1. *For each $c \in C$ and every $\mathbb{C}$-point $(x, g) \in \bar{I}_c X$, where x is a $\mathbb{C}$-point of X and g is an element in the isotropy group $Aut(x)$, we will assume that the age function*[4] *$age_g L_j|_x$ and the order function $ord(g)$ are all constant functions on $\bar{I}_c X$. Then we can define $age_c L_j := age_g L_j|_x$ and $a(c) := ord(g)$, which are well-defined as they are independent of the choice of the point $(x, g) \in \bar{I}_c X$.*

Moreover, we require that the index set C hass an involution which is compatible with the involution $\iota : \bar{I}_\mu X \to \bar{I}_\mu X$ sending (x, g) to (x, g^{-1}) for any $\mathbb{C}$-point (x, g) of $\bar{I}_\mu X$; namely, for each index $c \in C$, there exists a unique index in C and we will denote it to be c^{-1} such that $\iota(\bar{I}_c X) = \bar{I}_{c^{-1}} X$.

We will call an index set satisfying the above assumption a *good index set*.

Remark 2.2. We note that a good index set C always exists (but may not unique), e.g., we can decompose $\bar{I}_\mu X$ into connected components, which corresponds to the biggest good index set. When X is a quotient stack $[W/G]$ where W is a connected affine scheme, then we can choose the conjugacy class $\mathrm{Conj}(G)$ of G to be the index set C.

When Y is a substack of X, we will usually use C to index $\bar{I}_\mu Y$ as well, which is also a good index set with respect to the restriction of line bundles $L_1, \ldots, L_r$ to Y.

Let m be a nonnegative integer, for an (ordered) tuple $\vec{m} = \{c_1, \cdots, c_m\} \in C^m$, we will denote

$$\mathcal{K}_{g,\vec{m}}(X, \beta) := \mathcal{K}_{g,m}(X, \beta) \cap \bigcap_{i=1}^{m} ev_i^{-1}(\bar{I}_{c_i} X)\ .$$

Here $\mathcal{K}_{g,m}(X, \beta)$ is the moduli stack of genus g twisted stable maps to X with m (not necessarily trivialized) gerby-marked points of degree β as defined in [AGV08]. Then $\mathcal{K}_{g,m}(X, \beta)$ can be written as a disjoint union

$$\mathcal{K}_{g,m}(X, \beta) := \bigsqcup_{\vec{m}\in C^m} \mathcal{K}_{g,\vec{m}}(X, \beta)\ .$$

[4]Recall that if g acts on the fiber of L over x by multiplication by the number e^{q}, where $q \in \mathbb{Q}$, then we define the age $age_g(L_j|_x)$ to be $\langle q\rangle$, which is the fractional part of q.

Sometimes, we will use the notation $[m]\cup\star$ (resp. $\vec{m}\cup\star$) to mean $[m+1]$ (resp. $\overrightarrow{m+1}$) to distinguish the $m+1$-th marking, which we denote to be $\star$.

Now we introduce *admissible pairs*:

Definition 2.3. *Given a set of line bundles $L_1,\cdots,L_r$ over X, a positive integer N and r weights $\overrightarrow{w_j}=(w_{1j},\cdots,w_{Nj})\in\mathbb{Q}^N_{\geqslant 0}$ corresponding to each L_j. Let C be a good index set. Let $c\in C$, $\vec{k}=(k_1,\cdots,k_N)\in\mathbb{Z}^N_{\geqslant 0}$ and $\beta\in\mathrm{Eff}(X)$, where $\mathrm{Eff}(X)$ is the cone of effective curve classes of X. we will call $(\beta,\vec{k},c)$ an admissible pair if and only if*

$$age_c(L_j)\equiv\beta(L_j)+(\overrightarrow{w_j},\vec{k})\ \mathrm{mod}\ \mathbb{Z}$$

for all line bundles L_j, where $(\overrightarrow{w_j},\vec{k})=\sum_i w_{ij}k_i$. We will denote Adm *to be the set of all admissible pairs.*

Definition 2.4. *Let*

$$f(z)=\sum_{(\beta,\vec{k},c)\in\mathrm{Eff}(X)\times\mathbb{Z}^N_{\geqslant 0}\times C} q^\beta t_1^{k_1}\cdots t_N^{k_N} f_{\beta,\vec{k},c}(z)\ ,$$

be a formal series in $H^*(\bar{I}_\mu X,\mathbb{C})[z,z^{-1}]]⟦t_1,\cdots,t_N⟧⟦\mathrm{Eff}(X)⟧$ in which

$$f_{\beta,\vec{k},c}\in H^*(\bar{I}_{c^{-1}}X,\mathbb{C})[z,z^{-1}]]\ .$$

We further put a (not unique!) $\mathbb{Z}_2(=\mathbb{Z}/2\mathbb{Z})$-grading on each variable t_i, which we denote the grading to be $\bar{t}_i$. Note that we also require the super-commutativity:

$$t_it_j=(-1)^{\bar{t}_i\cdot\bar{t}_j}t_jt_i\ .$$

We will put a Z_2-grading on $H^*(\bar{I}_\mu X,\mathbb{C})$, which is induced from its ordinary cohomological grading, and put even grading on Novikov variables q^β and z.

We will call $f(z)$ an admissible series if $f(z)$ satisfies the following condition.

(1) *$f_{\beta,\vec{k},c}(z)=0$ whenever $(\beta,\overrightarrow{k},c)$ is not an admissible pair;*
(2) *$f_{0,\overrightarrow{0},c}(z)=0$ for all $c\in C$.*
(3) *$f_{\beta,\vec{k},c}(z)$ belongs to the space $H^{\sum_i k_i\bar{t}_i}(\bar{I}_{c^{-1}}X,\mathbb{C})[z]$, i.e., when $\sum_i k_i\bar{t}_i=0\in\mathbb{Z}_2$, $H^{\sum_i k_i\bar{t}_i}(\bar{I}_{c^{-1}}X,\mathbb{C})$ means the even degree part of the cohomology group of $\bar{I}_{c^{-1}}X$; when $\sum_i k_i\bar{t}_i=1\in\mathbb{Z}_2$, $H^{\sum_i k_i\bar{t}_i}(\bar{I}_{c^{-1}}X,\mathbb{C})$ means the odd degree part of the cohomology group of $\bar{I}_{c^{-1}}X$.*

Furthermore, let $g\in\mathbb{C}[z]$, we say a formal series $g+f(z)$ is an admissible series near g if $f(z)$ is an admissible series as above.

For any pair $(\beta,\overrightarrow{k})$ in $\mathrm{Eff}(X)\times C$, we will write $q^\beta\mathbf{t}^{\overrightarrow{k}}:=q^\beta t_1^{k_1}\cdots t_N^{k_N}$ and call $(\beta,\overrightarrow{k})$ an extended degree.

Remark 2.5. Examples of admissible series include (twisted) $J-$functions[Giv04] when X is a variety and $I-$functions in quasimap theory[CCFK15, CFK14, Web18, Web21, Wan19]. Therefore admissible series consists of a large class of objects studied in Gromov-Witten theory.

2.2. **Background on orbifold Gromov-Witten theory.** Now assume that X carries a algebraic torus T action (can be trivial), we can define the so-called *Chen-Ruan cohomology* of X, Given any two elements α_1,α_2 in the $T-$equivariant

$$H^*_{\mathrm{CR},T}(X,\mathbb{C}):=H^*_T(\bar{I}_\mu X,\mathbb{C})\ ,$$

We can define the Poincaré pairing in the *non-rigidified* inertia stack $I_\mu X$ of X:

$$\langle\alpha_1,\alpha_2\rangle_{\mathrm{orb}}:=\int_{\sum_{c\in C}a(c)^{-1}[\bar{I}_cX]}\alpha_1\cdot\iota^*\alpha_2\ .$$

Here ι is the involution of $\bar{I}_\mu X$ obtained from the inversion automorphisms of the band. Therefore, the diagonal class $[\Delta_{\bar{I}_c X}]$ obtained via push-forward of the fundamental class by $(\mathrm{id}, \iota) : \bar{I}_c X \to \bar{I}_c X \times \bar{I}_{c^{-1}} X$ can be written as

$$\sum_{c\in C} a(c)[\Delta_{\bar{I}_v X}] = \sum_\alpha \phi_\alpha \otimes \phi^\alpha \text{ in } H_T^*(\bar{I}_\mu X \times \bar{I}_\mu X, \mathbb{C}),$$

where $\{\phi_\alpha\}$ is a basis of $H^*_{\mathrm{CR},T}(X, \mathbb{C})$ with $\{\phi^\alpha\}$ the dual basis with respect to the Poincaré pairing defined above. Set $g_{\alpha\beta} = \langle\phi_\alpha, \phi_\beta\rangle_{orb}$ and $g^{\alpha\beta} = \langle\phi^\alpha, \phi^\beta\rangle_{orb}$.

Denote by $\bar{\psi}_i$ the first Chern class of the universal cotangent line whose fiber at $((C, q_1, ..., q_m), [x])$ is the cotangent space of the coarse moduli $\underline{C}$ of C at i-th marking $\underline{q}_i$. For non-negative integers a_i and classes $\alpha_i \in H_T^*(\bar{I}_\mu X, \mathbb{C})$, using the virtual cycle $[\mathcal{K}_{0,\vec{m}}(X, \beta)]^{\mathrm{vir}}$ defined in [LT98, BF97, AGV08], we write the Gromov-Witten invariant:

$$\langle\alpha_1\bar{\psi}^{a_1}, ..., \alpha_m\bar{\psi}^{a_m}\rangle^X_{g,\vec{m},\beta} := \int_{[\mathcal{K}_{g,\vec{m}}(X,\beta)]^{\mathrm{vir}}} \prod_i ev_i^*(\alpha_i)\bar{\psi}_i^{a_i} \ .$$

When the tuple $(g, \vec{m}, \beta)$ gives rise to an empty stack $\mathcal{K}_{g,\vec{m}}(X, \beta)$, we define the above integral is zero.

We will also need the stablemap Chen-Ruan classes

$$(\widetilde{ev_j})_* = \iota_*(\boldsymbol{r}_j(ev_j)_*), \tag{2.1}$$

where $\boldsymbol{r}_j$ is the order function of the band of the gerbe structure at the marking q_j. Define a class in $H^T_*(\bar{I}_\mu X) \cong H^*_T(\bar{I}_\mu X)$ by

$$\begin{aligned}\langle\alpha_1, ..., \alpha_m, -\rangle^X_{0,\beta} &:= (\widetilde{ev_{m+1}})_* \left((\prod ev_i^*\alpha_i) \cap [Q^\epsilon_{0,m}(X, \beta)]^{\mathrm{vir}}\right) \\ &= \sum_\alpha \phi^\alpha\langle\alpha_1, ..., \alpha_m, \phi_\alpha\rangle^X_{0,m+1,\beta} \ .\end{aligned}$$

3. A characterization about the Lagrangian cone

3.1. **A root-stack modification of the twisted graph space.** Let Y be any smooth Deligne-Mumford stack with projective coarse moduli. Let L be a semi-positive line bundle over Y, i.e., $\beta(L) \geqslant 0$ for any degree $\beta \in \mathrm{Eff}(Y)$. Denote $\mathbb{P}Y_{r,s}$ by the root stack of the projective bundle (also named twisted graph space[5]) $\mathbb{P}_Y(L^\vee \oplus \mathbb{C})$ over Y by taking $s-$th root of the zero section $D_0 := \mathbb{P}(0 \oplus \mathbb{C})$ and $r-$th root of the infinity section $D_\infty := \mathbb{P}(L^\vee \oplus 0)$. Let $\mathcal{D}_0$ and $\mathcal{D}_\infty$ be the corresponding root divisors of D_0 and D_∞ respectively, then the zero section $\mathcal{D}_0 \subset \mathbb{P}Y_{r,s}$ is isomorphic to the root stack $\sqrt[s]{L^\vee/Y}$ with normal bundle isomorphic to the root bundle $(L^\vee)^{\frac{1}{s}}$, and the infinity section $\mathcal{D}_\infty \subset \mathbb{P}Y_{r,s}$ is isomorphic to the root stack $\sqrt[r]{L/Y}$ with normal bundle isomorphic to the root bundle $L^{\frac{1}{r}}$.

There are two morphisms

$$q_0, q_\infty : \mathbb{P}Y_{r,s} \to \mathbb{B}\mathbb{C}^* \ ,$$

associated to the line bundles $\mathcal{O}(\mathcal{D}_0)$ and $\mathcal{O}(\mathcal{D}_\infty)$ respectively such that $\mathcal{O}(\mathcal{D}_0) \cong q_0^*(\mathbb{L})$ and $\mathcal{O}(\mathcal{D}_\infty) \cong q_\infty^*(\mathbb{L})$, where $\mathbb{L}$ is the universal line bundle over $\mathbb{B}\mathbb{C}^*$. This will induce morphisms on their rigidified inertia stack counterparts:

$$q_0, q_\infty : \bar{I}_\mu\mathbb{P}Y_{r,s} \to \bar{I}_\mu\mathbb{B}\mathbb{C}^* \ ,$$

Note the rigidified inertia stack $\bar{I}_\mu\mathbb{B}\mathbb{C}^*$ of $\mathbb{B}\mathbb{C}^*$ can be written as the disjoint union

$$\bigsqcup_{c\in\mathbb{C}^*} \bar{I}_c\mathbb{B}\mathbb{C}^* \ .$$

[5]The terminology of "twisted graph space" is taken from [CJR17a, CJR17b], see loc.cits for some other applications of the twisted graph space in Gromov-Witten theory.

Let c be a complex number, any $\mathbb{C}$-point of $\bar{I}_c\mathbb{B}\mathbb{C}^*$ is isomorphic to the pair $(1_\mathbb{C}, c)$, where $1_\mathbb{C}$ is the trivial principal $\mathbb{C}$-bundle over $\mathbb{C}$ and c is an element in the automorphism group $\mathrm{Aut}_\mathbb{C}(1_\mathbb{C}) \cong \mathbb{C}^*$. Let (y, g) be a $\mathbb{C}-$point of $\bar{I}_\mu\mathbb{P}Y_{r,s}$, where $y \in Ob(\mathbb{P}Y_{r,s}(\mathbb{C}))$ and $g \in \mathrm{Aut}(y)$. Assume that $q_0((y,g)) = (1_\mathbb{C}, c)$, then g acts on the fiber $\mathcal{O}(\mathcal{D}_0)|_y$ via the multiplication by c. We have a similar relation for q_∞.

Let $\mathrm{pr}_{r,s} : \mathbb{P}Y_{r,s} \to Y$ be the projection to the base, which is a composition of deroot-stackification and projection from $\mathbb{P}_Y(L^\vee \oplus \mathbb{C})$ to Y.

Now choose a good index set C for $\bar{I}_\mu Y$ satisfying the assumption 2.1. For each $c \in C$, $c_0, c_\infty \in \mathbb{C}^*$, we will define the rigidified inertia component $\bar{I}_{(c,c_0,c_\infty)}\mathbb{P}Y_{r,s}$ to be

$$\mathrm{pr}_{r,s}^{-1}(\bar{I}_cY) \cap q_0^{-1}(\bar{I}_{c_0}\mathbb{B}\mathbb{C}^*) \cap q_\infty^{-1}(\bar{I}_{c_\infty}\mathbb{B}\mathbb{C}^*)\ .$$

We also denote $(c, c_0, c_\infty)^{-1} = (c^{-1}, c_0^{-1}, c_\infty^{-1})$. Notice that $\bar{I}_{(c,c_0,c_\infty)}\mathbb{P}Y_{r,s}$ is nonempty if and only if $s \cdot age_c(L) \equiv -c_0 \mod \mathbb{Z}$ and $r \cdot age_c(L) \equiv c_\infty \mod \mathbb{Z}$.

Definition 3.1. *For any degree $\beta \in \mathrm{Eff}(Y)$ and rational number $\delta \in \mathbb{Q}$, we say a stable map $f : C \to \mathbb{P}Y_{r,s}$ is of degree $(\beta, \frac{\delta}{r})$ if $(\mathrm{pr}_{r,s} \circ f)_*[C] = \beta$ and $deg(f^*\mathcal{O}(\mathcal{D}_\infty)) = \frac{\delta}{r}$. We will denote $\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ to be the corresponding moduli stack of stable maps to $\mathbb{P}Y_{r,s}$ of degree $(\beta, \frac{\delta}{r})$.*

3.2. **Localization analysis.** Consider the $\mathbb{C}^*-$action on the projection bundle $\mathbb{P}_Y(L^\vee \oplus \mathbb{C})$ by scaling the fiber such that the normal bundle of D_0 (resp. D_∞) in $\mathbb{P}_Y(L^\vee \oplus \mathbb{C})$ is of $\mathbb{C}^*-$weight 1 (resp. -1). This $\mathbb{C}^*-$action induces a $\mathbb{C}^*-$action on $\mathbb{P}Y_{r,s}$ such that the normal bundle of $\mathcal{D}_0$ (resp. $\mathcal{D}_\infty$) in $\mathbb{P}Y_{r,s}$ is of $\mathbb{C}^*-$weight $\frac{1}{s}$ (resp. $-\frac{1}{r}$). Then we have an induced $\mathbb{C}^*-$action on the moduli of twisted stable maps to $\mathbb{P}Y_{r,s}$. We will apply the virtual localization formula of Graber–Pandharipande [GP99] to $\mathcal{K}_{0,\overrightarrow{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$, for which introduce the notation of decorated graph so that we index the components of $\mathbb{C}^*-$fixed loci of $\mathcal{K}_{0,\overrightarrow{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ by decorated graphs (trees). First, a decorated graph Γ consists of vertexes, edges and legs with the following decorations:

- Each vertex v is associated with an index $j(v) \in \{0, \infty\}$, and a degree $\beta(v) \in \mathrm{Eff}(Y)$.
- Each edge e consists of a pair of half-edges $\{h_0, h_\infty\}$, and e is equipped with a degree $\delta(e) \in \mathbb{Q}_{>0}$. Here we call h_0 and h_∞ half edges, and h_0 (resp. h_∞) is attached to a vertex labeled by 0 (resp. ∞).
- Each half-edge h and each leg l has a multiplicity $m(h)$ or $m(l)$ in $C \times \mathbb{C}^* \times \mathbb{C}^*$.
- The legs are labeled with the numbers $\{1, \ldots, m\}$ and each leg is incident to a unique vertex.

By the "valence" of a vertex v, denoted $\mathrm{val}(v)$, we mean the total number of incident half-edges and legs.

For each $\mathbb{C}^*-$fixed stable map $f : (C; q_1, \ldots, q_m) \to \mathbb{P}Y_{r,s}$ in $\mathcal{K}_{0,\overrightarrow{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$, we can associate a decorated graph Γ in the following way.

- Each edge e corresponds to a genus-zero component C_e which maps constantly to the base Y and intersects $\mathcal{D}_0$ and $\mathcal{D}_\infty$ properly. The restriction $f|_{C_e}$ satisfies that $r \cdot deg(f|_{C_e}^*\mathcal{O}(\mathcal{D}_\infty)) = s \cdot deg(f|_{C_e}^*\mathcal{O}(\mathcal{D}_0)) = \delta(e)$.
- Each vertex v for which $j(v) = 0$ (with unstable exceptional cases noted below) corresponds to a maximal sub-curve C_v of C which maps totally into $\mathcal{D}_0$, then the restriction of f to C_v defines a twisted stable map in

$$\mathcal{K}_{0,val(v)}(\sqrt[s]{L^\vee/Y}, \beta(v))\ .$$

Each vertex v for which $j(v) = \infty$ (again with unstable exceptions) corresponds to a maximal sub-curve which maps totally into $\mathcal{D}_\infty$, then the restriction of f to C_v defines a twisted stable map in

$$\mathcal{K}_{0,val(v)}(\sqrt[r]{L/Y}, \beta(v))\ .$$

The label $\beta(v)$ denotes the degree coming from the restriction $\mathrm{pr}_{r,s} \circ f|_{C_v} : C_v \to Y$.

- A vertex v is *unstable* if stable twisted maps of the type described above do not exist. In this case, we have $\beta(v) = 0$ and v corresponds to a single point of the component C_e for each incident edge e, which may be a node at which C_e meets another edge curve $C_{e'}$, a marked point of C_e, or an unmarked point.
- Each leg l labeled by i corresponds to the ith marking q_i, and it's incident to the vertex v if q_i lies on C_v. The index $m(l)$ on a leg l indicates the rigidified inertia stack component $\bar{I}_{m(l)}\mathbb{P}Y_{r,s}$ of $\mathbb{P}Y_{r,s}$ on which the gerby marked point corresponding to the leg l is evaluated.
- A half-edge h of an edge e corresponds a ramification point $q \in C_e$, i.e., there are two distinguished points(we call them ramification points) q_0 and q_∞ on C_e satisfying that q_0 maps to $\mathcal{D}_0$ and q_∞ maps to $\mathcal{D}_\infty$, respectively. We associate half-edges h_0 and h_∞ to q_0 and q_∞ respectively. Then $m(h)$ indicates the rigidified inertia component $\bar{I}_{m(h)}\mathbb{P}Y_{r,s}$ of $\mathbb{P}Y_{r,s}$ on which the ramification point q associated with h is evaluated. We will write the edge e as $\{h_0, h_\infty\}$.

Moreover, the decorated graph from a $\mathbb{C}^*-$fixed stable map should satisfy the following:

(1) (Monodromy constraint for edge) For each edge $e = \{h_0, h_\infty\}$, we have that $m(h_0) = (c^{-1}, \mathrm{e}^{\frac{\delta(e)}{s}}, 1)$ and $m(h_\infty) = (c, 1, \mathrm{e}^{\frac{\delta(e)}{r}})$ for some $c \in C$. Moreover the choice of c should satisfy that

$$\delta(e) = age_c(L) \bmod \mathbb{Z} \ ,$$

which follows from the fact that the ages of line bundles $f|_{C_e}^*\mathcal{O}(s\mathcal{D}_0)$ and $f|_{C_e}^*(\mathcal{O}(r\mathcal{D}_\infty) \otimes \mathrm{pr}_{r,s}^* L^\vee)$ at q_∞ should be equal, as the two line bundles on C_e are isomorphic.

(2) (Monodromy constraint for vertex) For each vertex v, let $I_v \subset [m]$ be the set of incident legs to v, and H_v be the set of incident half-edges to v. For each $i \in I_v$, write $m(l_i) = (c_i, a_i, b_i)$, and for each $h \in H_v$, write $m(h) = (c_h, a_h, b_h)$. If v is labeled by 0, we have that all b_i and b_h are equal to 1, and

$$\mathrm{e}^{\frac{\beta(v)(L)}{s}} \times \prod_{i \in I_v} a_i \times \prod_{h \in H_v} a_h^{-1} = 1 \ . \tag{3.1}$$

If v is labeled by ∞, then we have all a_i and a_h are all equal to 1 and

$$\mathrm{e}^{-\frac{\beta(v)(L)}{r}} \times \prod_{i \in I_v} b_i \times \prod_{h \in E_v} b_h^{-1} = 1 \ . \tag{3.2}$$

In particular, we note that the decorations at each stable vertex v yield a tuple

$$\overrightarrow{val}(v) \in (C \times \mathbb{C}^* \times \mathbb{C}^*)^{\mathrm{val}(v)}$$

recording the multiplicities at every special point of C_v.[6] Then the restriction gives a stable map in

$$\mathcal{K}_{0,\overrightarrow{val}(v)}(\mathcal{D}_{j(v)}, \beta(v)) \ .$$

For each decorated graph Γ, we will associate each vertex v (resp. edge e) a moduli space $\mathcal{M}_v$(resp. $\mathcal{M}_e$) over which there is a family $\mathbb{C}^*-$stable map to $\mathbb{P}Y_{r,s}$ with decorated degree(there are exceptions for unstable vertexes v, see section 3.2.4 for more details).

Denote by F_Γ the space

$$\prod_{v: j(v)=0} \mathcal{M}_v \times_{\bar{I}_\mu \mathcal{D}_0} \prod_{e \in E} \mathcal{M}_e \times_{\bar{I}_\mu \mathcal{D}_\infty} \prod_{v: j(v)=\infty} \mathcal{M}_v \ ,$$

where the fiber product is taken by gluing the two branches at each node.

[6]For each node of C_v, let h be the incident half-edge, then we define the multiplicity at the branch of node at C_v to be $m(h)^{-1}$.

By virtual localization formula [GP99], we can write

$$[\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},\beta)]^{\mathrm{vir}} \ ,$$

in terms of contributions from each decorated graph Γ:

$$[\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},\beta)]^{\mathrm{vir}} = \sum_{\Gamma} \frac{1}{\mathbb{A}_\Gamma} \iota_{\Gamma*}\left(\frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})}\right) \ . \tag{3.3}$$

Here, for each graph Γ, $[F_\Gamma]^{\mathrm{vir}}$ is obtained via the $\mathbb{C}^*$-fixed part of the restriction to the fixed loci of the obstruction theory of $\mathcal{K}_{0,\vec{m}\cup\star}(\mathbb{P}Y_{r,s},\beta)$, and N_Γ^{vir} is the equivariant Euler class of the $\mathbb{C}^*$-moving part of this restriction. Besides, $\mathbb{A}_\Gamma$ is the automorphism factor for the graph Γ, which represents the degree of F_Γ into the corresponding open and closed $\mathbb{C}^*$-fixed substack $i_\Gamma(F_\Gamma)$ in $\mathcal{K}_{0,\vec{m}\cup\star}(\mathbb{P}Y_{r,s},\beta)$. In our case, $\mathbb{A}_\Gamma$ is the product of the size of the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ and the degrees from each edge moduli $\mathcal{M}_e$ into the corresponding fixed loci.

Assume that r,s are sufficiently large primes. We will do an explicit computation for the contributions of each graph Γ in the following. As for the contribution of a graph Γ to (3.3), one can first apply the normalization exact sequence to the obstruction theory, which decomposes the contribution from Γ to (3.3) into contributions from vertex, edge, and node factors.

3.2.1. *Vertex contribution.* Assume that v is a stable vertex. If the vertex is labeled by ∞, we define the vertex moduli $\mathcal{M}_v$ to be $\mathcal{K}_{0,\overrightarrow{val(v)}}(\mathcal{D}_\infty,\beta(v))$ and the fixed part of perfect obstruction theory gives rise to $[\mathcal{K}_{0,\overrightarrow{val(v)}}(\mathcal{D}_\infty,\beta(v))]^{\mathrm{vir}}$. Let $\pi:\mathcal{C}\to\mathcal{K}_{0,\overrightarrow{val(v)}}(\mathcal{D}_\infty,\beta(v))$ be the universal curve and $f:\mathcal{C}\to\mathcal{D}_\infty$ be the universal map, then the movable part of the perfect obstruction theory yields the *inverse of the Euler class* of the virtual normal bundle which is equal to

$$e^{\mathbb{C}^*}((-R^\bullet\pi_*f^*L^{\frac{1}{r}})\otimes\mathbb{C}_{-\frac{\lambda}{r}}) \ .$$

When r is a sufficiently large prime and the multiplicity $m(l)$ corresponding to each leg l incident to v is equal to $(g_l,1,\frac{\delta_l}{r})$ for some prefixed $\delta_l\in\mathbb{Q}_{>0}$(note this implies $\delta_l\ll r$) and $g_l\in G$, following a generalization of [JPPZ20] to the orbifold case, the above Euler class has a representation

$$\sum_{d\geqslant 0} c_d(-R^\bullet\pi_*f^*L^{\frac{1}{r}})(\frac{-\lambda}{r})^{|E(v)|-1-d} \ .$$

Here the virtual bundle $-R^\bullet\pi_*f^*L^{\frac{1}{r}}$ has virtual rank $|E(v)|-1$, where $|E(v)|$ is the number of edges incident to the vertex v.

If the vertex v is labeled by 0, we define the vertex moduli $\mathcal{M}_v$ to be $\mathcal{K}_{0,\overrightarrow{val(v)}}(\mathcal{D}_0,\beta(v))$ and the fixed part of perfect obstruction theory gives rise to $[\mathcal{K}_{0,\overrightarrow{val(v)}}(\mathcal{D}_0,\beta(v))]^{\mathrm{vir}}$. Let $\pi:\mathcal{C}\to\mathcal{K}_{0,\overrightarrow{val(v)}}(\mathcal{D}_0,\beta(v))$ be the universal curve and $f:\mathcal{C}\to\mathcal{D}_0$ be the universal map, then the movable part of the perfect obstruction theory yields the *inverse of the Euler class* of the virtual normal bundle which is equal to

$$e^{\mathbb{C}^*}((-R^\bullet\pi_*f^*(L^\vee)^{\frac{1}{s}})\otimes\mathbb{C}_{\frac{\lambda}{s}}) \ .$$

Now assume that there is only one edge incident to v and the multiplicity $m(l)$ corresponding to each leg l incident to v is equal to $(g_l,\frac{\delta_l}{s},1)$ for some prefixed $\delta_l\in\mathbb{Q}_{>0}$(note this implies $f_l\ll r$) and $g_l\in G$. By [Wan19, Lemma 5.2, Remark 5.3], the above Euler class is equal to 1 when $\beta(v)(L)>0$, we note here the semi-positivity of L is essentially used in the proof of[Wan19, Lemma 5.2]. On the other hand, when $\beta(v)(L)=0$, by the argument in the proof of [Wan19, Lemma 6.5], the above Euler class is still equal to 1 there is a special point (we will choose a node in our case) on C_v whose multiplicity m satisfies that $age_m(L^\vee)^{\frac{1}{s}}\neq 0$.

3.2.2. *Edge contribution.* Let $e = \{h_0, h_\infty\}$ be an edge in Γ with decorated degree $\delta(e) \in \mathbb{Q}$(we will write $\delta(e)$ as δ for simplicity if no confusion occurs) and decorated multiplicities $m(h_0)$ and $m(h_\infty)$. Assume that $m(h_\infty) = (c, 1, \mathrm{e}^{\frac{\delta(e)}{r}})$, denote $a_e := a(c)$ be the number as Assumption 2.1. We define the edge moduli $\mathcal{M}_e$ to be the root gerbe $\sqrt[a s\delta]{L^\vee/I_cY}$ over I_cY, then the virtual cycle $[\mathcal{M}_e]^{vir}$ coming from the fix part of the obstruction theory is equal to the fundamental class $[\mathcal{M}_e]$ of $\mathcal{M}_e$ and *inverse of the Euler class* of virtual normal bundle is given by 1 when r is a sufficiently large prime. We note that $\mathcal{M}_e$ allows a finite étale map into the corresponding fixed-loci in $\mathcal{K}_{0,2}(\mathbb{P}Y_{r,s}, (0, \frac{\delta(e)}{r}))$ of degree $\frac{1}{as}$. See appendix A.2 for more details.

3.2.3. *Node contributions.* The deformations in $\mathcal{K}_{0,\overrightarrow{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r})$ smoothing a node contribute to the Euler class of the virtual normal bundle as the first Chern class of the tensor product of the two cotangent line bundles at the branches of the node. For nodes at which a component C_e meets a component C_v over the vertex labeled by X_0, this contribution is

$$\frac{\lambda - c_1(L)}{a_e s\delta(e)} - \frac{\bar{\psi}_v}{a_e s}.$$

For nodes at which a component C_e meets a component C_v at the vertex over $\mathcal{D}_\infty$, this contribution is

$$\frac{-\lambda + c_1(L)}{a_e r\delta(e)} - \frac{\bar{\psi}_v}{a_e r}.$$

We will not need node contributions from other types of nodes as the above types suffice the need in this paper, see Lemma 4.3 for more details.

3.2.4. *Total contribution.* For any decorated graph Γ, we define F_Γ to be the fiber product

$$\prod_{v:j(v)=0} \mathcal{M}_v \times_{\bar{I}_\mu \mathcal{D}_0} \prod_{e\in E} \mathcal{M}_e \times_{\bar{I}_\mu \mathcal{D}_\infty} \prod_{v:j(v)=\infty} \mathcal{M}_v$$

of the following diagram:

$$\begin{tikzcd}
F_\Gamma \arrow[r] \arrow[d] & \prod_{v:j(v)=0} \mathcal{M}_v \times \prod_{e\in E} \mathcal{M}_e \times \prod_{v:j(v)=\infty} \mathcal{M}_v \arrow[d, "ev_{nodes}"] \\
\prod_E (\bar{I}_\mu \mathcal{D}_0) \times \bar{I}_\mu \mathcal{D}_\infty \arrow[r, "(\Delta^0\times\Delta^\infty)^{|E|}"] & \prod_E (\bar{I}_\mu \mathcal{D}_0)^2 \times (\bar{I}_\mu \mathcal{D}_\infty)^2 ,
\end{tikzcd}$$

where $\Delta^0 = (id, \iota)$(resp. $\Delta^\infty = (id, \iota)$) is the diagonal map of $\bar{I}_\mu \mathcal{D}_0$ (resp. $\bar{I}_\mu \mathcal{D}_\infty$). Here when v is a stable vertex, the vertex moduli $\mathcal{M}_v$ is described in 3.2.1; when v is an unstable vertex over 0 (resp. ∞), we treat $\mathcal{M}_v := \bar{I}_{m(h)^{-1}} \mathcal{D}_0$(resp. $\bar{I}_{m(h)^{-1}} \mathcal{D}_\infty$) with the identical virtual cycle, where h is the half-edge incident to v. The right-hand vertical map ev_{nodes} is the product of the evaluation maps at the two branches of each node. We note that there are two nodes corresponding to h_0 and h_∞ for each edge $e = \{h_0, h_\infty\}$.

We define $[F_\Gamma]^{\mathrm{vir}}$ to be:

$$\prod_{v:j(v)=0} [\mathcal{M}_v]^{\mathrm{vir}} \times_{\bar{I}_\mu \mathcal{D}_0} \prod_{e\in E} [\mathcal{M}_e]^{\mathrm{vir}} \times_{\bar{I}_\mu \mathcal{D}_\infty} \prod_{v:j(v)=\infty} [\mathcal{M}_v]^{\mathrm{vir}} .$$

Then the contribution of decorated graph Γ to the virtual localization is:

$$Cont_\Gamma = \frac{\prod_{e\in E} s a_e}{|\mathrm{Aut}(\Gamma)|} (\iota_\Gamma)_* \left(\frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})} \right) . \tag{3.4}$$

Here $\iota_F : F_\Gamma \to \mathcal{K}_{0,\overrightarrow{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ is a finite etale map of degree $\frac{|\mathrm{Aut}(\Gamma)|}{\prod_{e\in E} sa_e}$ into the corresponding $\mathbb{C}^*$-fixed loci. The virtual normal bundle $e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})$ is the product of virtual normal bundles from vertex contributions, edge contributions and node contributions.

3.3. **A characterization.** Take $L_1 = L$ and $r = 1$ in 2.4 and write $\vec{w} = \overrightarrow{w_1}$ to be the weight. Let $\mu(z) = \sum_{\beta,\vec{k},c} q^\beta \mathbf{t}^{\vec{k}} \mu_{\beta,\vec{k},c}(z)$ be an admissible (power) series as in 2.4. We further require that $\mu_{\beta,\vec{k},c} \in H^*(\bar{I}_{c^{-1}}X, \mathbb{C})[z]$ is a polynomial in z. For any admissible pair $(\beta, \vec{k}, c)$ in $\mathrm{Eff}(Y) \times \mathbb{Z}^N_{\geqslant 0} \times C$, define $J^Y_{\beta,\vec{k},c}(\mu, z)$ to be

$$\begin{aligned}&\mu_{\beta,\vec{k},c}(z) + \mathbf{Coeff}_{\mathbf{t}^{\vec{k}}}\Bigg[\sum_{m\geqslant 0}\sum_{\substack{(\beta_j,\overrightarrow{k_j},c_j)_{j=1}^m \in \mathrm{Adm}^m, \beta_\star\in \mathrm{Eff}(Y)\\ \beta_1+\cdots+\beta_m+\beta_\star=\beta\\ \overrightarrow{k_1}+\cdots+\overrightarrow{k_m}=\overrightarrow{k}}} \\ &\frac{1}{m!}\phi^\alpha \langle \mathbf{t}^{\overrightarrow{k_1}}\mu_{\beta_1,\overrightarrow{k_1},\mathbf{c_1}}(-\bar{\psi}_1), \cdots, \mathbf{t}^{\overrightarrow{k_m}}\mu_{\beta_\mathbf{m},\overrightarrow{k_m},\mathbf{c_m}}(-\bar{\psi}_m), \frac{\phi_\alpha}{z-\bar{\psi}_\star}\rangle^Y_{0,\overrightarrow{m}\cup\star,\beta_\star}\Bigg],\end{aligned} \tag{3.5}$$

where $\overrightarrow{m} \cup \star = (c_1^{-1} \cdots, c_m^{-1}, c) \in C^{m+1}$, $\overrightarrow{k_j} = (k_{j1}, \cdots, k_{jN})$ and $\overrightarrow{k_1} + \cdots + \overrightarrow{k_m} = \overrightarrow{k}$ means the component-wise addition in $\mathbb{Z}^N_{\geqslant 0}$.

Remark 3.2. We can show that when $(\beta, \vec{k}, c)$ is not an admissible pair and we define $J^Y_{\beta,\overrightarrow{k},c}(\mu, z)$ in the same way as above, we have $J^Y_{\beta,\overrightarrow{k},c}(\mu, z) = 0$. Moreover $J_{0,\vec{0},c} = 0$. This implies that $J-$function

$$J^Y(q, \mu, z) = z + \sum_{\beta,\overrightarrow{k},c} q^\beta \mathbf{t}^{\vec{k}} J^Y_{\beta,\overrightarrow{k},c}(\mu, z)$$

associated with the input $\mu(z)$ above is an admissible series near z.

Definition 3.3. *Let m, n be two nonnegative integers and $(\beta, \vec{k}, c)$ be an admissible pair. We denote $\Lambda_{\beta,\vec{k},c,m,n}$ to be the set of tuples*

$$\left(c, \beta_\star, ((\beta_1, \overrightarrow{k_1}, c_1), \cdots, (\beta_{m+n}, \overrightarrow{k_{m+n}}, c_{m+n}))\right) \in C \times \mathrm{Eff}(Y) \times \mathrm{Adm}^{m+n} ,$$

where we require that $\beta_\star + \sum_{i=1}^{m+n} \beta_i = \beta$, $\sum_{i=1}^{m+n} \overrightarrow{k_i} = \overrightarrow{k}$, $\beta_i(L) + (\vec{w}, \overrightarrow{k_i}) > 0$ for $1 \leqslant i \leqslant m$ and $\beta_i(L) + (\vec{w}, \overrightarrow{k_i}) = 0$ for $m + 1 \leqslant i \leqslant m + n$. We call an element of $\Lambda_{\beta,\vec{k},c,m,n}$ stable if $\beta_\star \neq 0$ or $m + n \geqslant 2$ when $\beta_\star = 0$.

We note that $\Lambda_{\beta,\vec{k},c,m,n}$ is a finite set as $\mathcal{K}_{0,m+n}(X, \beta)$ is finite type over $\mathbb{C}$, hence Noetherian.

For any admissible pair $(\beta, \vec{k}, c)$ with $\beta(L) + (\vec{w}, \vec{k}) > 0$. We have the following recursive characterization about $J^Y_{\beta,\overrightarrow{k},c}(\mu, z)$.

Proposition 3.4. *For any admissible pair $(\beta, \vec{k}, c)$ with $\beta(L) + (\vec{w}, \vec{k}) > 0$. Assume that r is a sufficiently large prime. Then for any nonnegative integer b, we have the following recursive* *relation:*

$$
\begin{aligned}
&[J^Y_{\beta,\vec{k},c}(\mu,z)]_{z^{-b-1}} \\
&=\Bigg[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{\substack{\Gamma\in\Lambda_{\beta,\vec{k},c,m,n}\\ \Gamma\ is\ stable}}\frac{1}{m!n!}(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*\big(c_d(-R^\bullet\pi_*f^*L^{\frac{1}{r}})(\frac{\lambda}{r})^{-1+m-d}(-1)^d \\
&\cap[\mathcal{K}_{0,\overrightarrow{m+n}\cup\star}(\sqrt[r]{L/Y},\beta_\star)]^{\mathrm{vir}}\big)\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}\mathbf{t}^{\overrightarrow{k_i}}J^Y_{\beta_i,\vec{k}_i,c_i}(\mu,z)|_{z=\frac{\lambda-c_1(L)}{\delta_i}})}{\frac{\lambda-ev_i^*c_1(L)}{r\delta_i}+\frac{\bar{\psi}_i}{r}} \\
&\cap\prod_{i=m+1}^{m+n}ev_i^*(\mathbf{t}^{\overrightarrow{k_i}}\mu_{\beta_i,\vec{k}_i,c_i}(-\bar{\psi}_i))\cap\bar{\psi}_\star^b\Bigg)\Bigg]_{\mathbf{t}^{\vec{k}}\lambda^{-1}}.
\end{aligned}
\tag{3.6}
$$

Here $\delta_i=\beta_i(L)+(\vec{w},\overrightarrow{k_i})$*, and* $\epsilon:\mathcal{K}_{0,\overrightarrow{m+n}\cup\star}(\sqrt[r]{L/Y},\beta_\star)\to\mathcal{K}_{0,\overrightarrow{m+n}\cup\star}(Y,\beta_\star)$ *is the natural structural morphism by forgetting the root structure of* $\sqrt[r]{L/Y}$ *(c.f.[TT16]), where we choose the tuple* $\overrightarrow{m+n}\cup\star$ *for* $\mathcal{K}_{0,\overrightarrow{m+n}\cup\star}(\sqrt[r]{L/Y},\beta_\star)$ *to be*

$$\big((c_1^{-1},1,\mathrm{e}^{\frac{-\delta_1}{r}}),\cdots,(c_{m+n}^{-1},1,\mathrm{e}^{\frac{-\delta_{m+n}}{r}}),(c,1,\mathrm{e}^{\frac{\delta}{r}})\big)\in(C\times\mathbb{C}^*\times\mathbb{C}^*)^{m+n+1}\ ,$$

and choose the tuple $\overrightarrow{m+n}\cup\star$ *for* $\mathcal{K}_{0,\overrightarrow{m+n}\cup\star}(Y,\beta_\star)$ *to be*

$$(c_1^{-1},\cdots,c_{m+n}^{-1},c)\in C^{m+n+1}\ .$$

The proof of the above proposition is based on applying virtual localization to the following integral.

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{\substack{(\beta_j,\overrightarrow{k_i},c_i)_{i=1}^m\in\mathrm{Adm}^m,\beta_\star\in\mathrm{Eff}(Y)\\ \beta_\star+\sum_{i=1}^m\beta_i=\beta\\ \overrightarrow{k_1}+\cdots+\overrightarrow{k_m}=\vec{k}}}\frac{1}{m!}(\widetilde{EV}_\star)_*\Bigg(\prod_{i=1}^{m}ev_i^*\big(\mathrm{pr}_{r,s}^*(\mathbf{t}^{\overrightarrow{k_i}}\mu_{\beta_i,\vec{k}_i,c_i}(-\bar{\psi}_i))\big) \\
&\cap\bar{\psi}_\star^b\cap[\mathcal{K}_{0,\vec{m}\cup\star}(\mathbb{P}Y_{r,s},(\beta_\star,\frac{\delta}{r}))]^{\mathrm{vir}}\Bigg)\ .
\end{aligned}
\tag{3.7}
$$

Here an explanation of the notations is in order:

(1) For degrees $\beta_\star,\beta_1,\cdots,\beta_m$ in $\mathrm{Eff}(X)$ and tuples $\overrightarrow{k_1},\cdots,\overrightarrow{k_m}$ and with $\sum_{i=1}^m\beta_i+\beta_\star=\beta$ and $\overrightarrow{k_1}+\cdots+\overrightarrow{k_m}=\vec{k}$, let $(c_1,\cdots,c_m)\in C^m$ such that (β_i,c_i) are admissible pairs. Write $\delta_i=\beta_i(L)+(\vec{w},\overrightarrow{k_i})$ and $\delta=\beta(L)+(\vec{w},\vec{k})$, we define $\vec{m}\cup\star$ to be the $m+1$tuple

$$\big((c_1^{-1},\mathrm{e}^{\frac{\delta_i}{s}},1),\cdots,(c_m^{-1},\mathrm{e}^{\frac{\delta_m}{s}},1),(c,1,\mathrm{e}^{\frac{\delta}{r}})\big)\ .$$

(2) the morphism $EV_\star$ is a composition of the following maps:

$$\mathcal{K}_{0,\vec{m}\cup\star}(\mathbb{P}Y_{r,s},(\beta_\star,\tfrac{\delta}{r}))\xrightarrow{ev_\star}\bar{I}_\mu\mathbb{P}Y_{r,s}\xrightarrow{\mathrm{pr}_{r,s}}\bar{I}_\mu Y\ ,$$

and $(\widetilde{EV}_\star)_*$ is defined by

$$\iota_*(r_\star(EV_\star)_*)\ .$$

Note here $r_\star$ is the order of the band from the gerbe structure of $\bar{I}_\mu Y$ but not $\bar{I}_\mu\mathbb{P}Y_{r,s}$.

First we state a vanishing lemma regarding applying localization formula to 3.7.

Lemma 3.5. *Assume r,s is sufficiently large. If the localization graph Γ has more than one vertex labeled by ∞, then the corresponding fixed loci moduli F_Γ is empty, therefore it will contribute zero to (3.7).*

Proof. For any twisted stable map $f : C \to \mathbb{P}Y_{r,s}$ in $\mathcal{K}_{0,\vec{m}\cup\star}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$, denote $N := f^*(\mathcal{O}(-\mathcal{D}_\infty))$, first we show that $H^1(C,N)=0$. Indeed, using orbifold Riemann-Roch, we have

$$\chi(N) = 1 + deg(N) - age(N|_{q_\star}) = 0\ ,$$

as $deg(N) = -\frac{\delta}{r}$, and $age(N|_{q_\star}) = 1-\frac{\delta}{r}$, then showing $H^1(C,N)=0$ is equivalent to show $H^0(C,N)=0$. Now assume that f is $\mathbb{C}^*-$fixed, as the degree of N is negative on C, it remains to show that the degree of the restriction of the line bundle N to every irreducible component E of C is non-positive. Observe that the degree $deg(N|_E)$ is equal to intersection number $-([E],[\mathcal{D}_\infty])$. If the image of an irreducible component of C via f isn't contained in $\mathcal{D}_\infty$, the restricted degree of N to E is obviously non-positive. Otherwise, observe that N is isomorphic to $(L^{\frac{1}{r}})^\vee$ over $\mathcal{D}_\infty$. As L is semi-positive, the claim follows. This finishes the proof of the part $H^1(C,N)=0$.

Now assume by contradiction that the moduli of fixed-loci F_Γ is nonempty, by the connectedness of the graph Γ, there is at least one vertex of the graph Γ labeled by 0 with at least two edges attached. Suppose $f : C \to \mathbb{P}Y_{r,s}$ belongs to the moduli F_Γ of $\mathbb{C}^*-$fixed loci. Assume that $C_0 \cap C_1 \cap C_2$ is part of curve C, where C_0 is mapped by f to $\mathcal{D}_0$ and C_1, C_2 are edges meeting with C_0 at b_1 and b_2. Then in the normalization sequence for $R^\bullet\pi_*N$, it contains the part

$$\begin{aligned}
&H^0(C_0,N)\oplus H^0(C_1,N)\oplus H^0(C_2,N)\\
\to &H^0(b_1,N)\oplus H^0(b_2,N)\\
\to &H^1(C,N)\,.
\end{aligned}$$

Hence there is one of the weight-0 pieces in $H^0(b_1,N)\oplus H^0(b_2,N)$ that is canceled with a weight-0 piece of $H^0(C_0,N)$, and the other is mapped injectively into $H^1(C,N)$, but this contradicts that $H^1(C,N)=0$. So F_Γ is empty. □

Now we prove Proposition 3.4.

Proof of Proposition 3.4. The proof is almost identical to the proof in [Wan19, §6.2], we here sketch the main steps for the convenience of readers. First without loss of generality, we will

By Lemma 3.5, we only need to consider the decorated graph Γ that has only one vertex labeled by ∞. Note the marking $q_\star$ corresponding to $\star$ is incident to the vertex $v_\star$ due to the choice of multiplicity at the marking $q_\star$, then the vertex $v_\star$ can't be a node linking two edges. Such decorated graph is *star-shaped*, i.e., the vertex set V allows a decomposition

$$V = \{v_\star\} \bigsqcup V_0\ .$$

where the vertex $v_\star$ is labeled by ∞, and all the vertexes in V_0 are labeled by 0. Each vertex labeled by 0 is linked to $v_\star$ by a unique edge. There are three types of star-shaped decorated graphs:

(1) (Type I) The vertex $v_\star$ is unstable and there is only one edge incident to $v_\star$ in Γ, the unique vertex v labeled by 0 is unstable;
(2) (Type II) The vertex $v_\star$ is unstable and there is only one edge incident to $v_\star$ in Γ, the unique vertex v labeled by 0 is stable;
(3) (Type III) The vertex $v_\star$ is stable.

Denote by $\beta_\star$ the degree of the unique vertex $v_\star$ labeled by ∞. Now let's compute the localization contribution from the above three types of graphs:

(1) If the graph Γ is of type I, the vertex over 0 corresponds to a marked point with input $\mu_{\beta,\overrightarrow{k},c}$, then the graph Γ contributes

$$\mathbf{t}^{\overrightarrow{k}}\frac{\mu_{\beta,\vec{k},c}(\frac{\lambda-c_1(L)}{\delta})}{\delta}\cdot(\frac{\lambda-c_1(L)}{\delta})^b$$

to (4.3). Note the use the fact that the restriction of the psi-class $\bar{\psi}_\star$ to $\mathcal{M}_e$ is equal to $\frac{\lambda-c_1(L)}{\delta}$(see Remark A.4).

(2) If the graph Γ is of type II, Γ has the same description with type I except that the vertex over 0 is stable. Then this type of graphs contributes

$$\sum_{\substack{(\beta_j,\overrightarrow{k_j},c_j)_{j=1}^m\in\mathrm{Adm}^m,\beta_\star\in\mathrm{Eff}(Y)\\ \beta_1+\cdots+\beta_m+\beta_\star=\beta\\ \overrightarrow{k_1}+\cdots+\overrightarrow{k_m}=\overrightarrow{k}}}\frac{1}{m!}(\widetilde{ev_\star})_*\Bigg(\epsilon'_*\big([\mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L^\vee/Y},\beta_\star)]^{\mathrm{vir}}\big)$$
$$\cap\bigcap_{i=1}^m ev_i^*(\mathbf{t}^{\overrightarrow{k_i}}\mu_{\beta_i,\vec{k_i},c_i}(-\bar{\psi}_i))\cap\frac{\frac{1}{\delta}(\frac{\lambda-ev_\star^*c_1(L)}{\delta})^b}{\frac{\lambda-ev_\star^*c_1(L)}{s\delta}-\frac{\bar{\psi}_\star}{s}}\Bigg)$$

to (3.7), where $\vec{m}_s\cup\star=\big((c_1^{-1},e^{\frac{\delta_1}{s}},1),\cdots,(c_m^{-1},\mathrm{e}^{\frac{\delta_m}{s}},1),(c,e^{-\frac{\delta}{s}},1)\big)\in(C\times\mathbb{C}^*\times\mathbb{C}^*)^{m+1}$, $\vec{m}\cup\star=\big(c_1^{-1},\cdots,c_m^{-1},c\big)\in C^{m+1}$ and

$$\epsilon':\mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L^\vee/Y},\beta_\star)\to\mathcal{K}_{0,\vec{m}\cup\star}(Y,\beta_\star)$$

is the natural structure morphism by forgetting root of $\sqrt[s]{L^\vee/Y}$ (c.f.[TT16]). Note here we implicitly cancel out a_e factors from the node contribution and automorphic factor everywhere and use the fact that the Euler class of virtual normal bundle for the vertex over 0 is equal to 1. It's proved that

$$\epsilon'_*\big([\mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L^\vee/Y},\beta_\star)]^{\mathrm{vir}}\big)=\frac{1}{s}[\mathcal{K}_{0,\vec{m}\cup\star}(Y,\beta_\star)]^{\mathrm{vir}}\ ,$$

in [TT16], then the above formula is equal to

$$\sum_{m\geqslant 0}\sum_{\substack{(\beta_j,\overrightarrow{k_j},c_j)_{j=1}^m\in\mathrm{Adm}^m,\beta_\star\in\mathrm{Eff}(Y)\\ \beta_1+\cdots+\beta_m+\beta_\star=\beta\\ \overrightarrow{k_1}+\cdots+\overrightarrow{k_m}=\overrightarrow{k}}}\frac{1}{m!}\phi^\alpha\langle\mu_{\beta_1,\overrightarrow{k_1},c_1}(-\bar{\psi}_1),\cdots,\mu_{\beta_m,\overrightarrow{k_m},c_m}(-\bar{\psi}_m),\frac{\frac{1}{\delta}(\frac{\lambda-c_1(L)}{\delta})^b\phi_\alpha}{\frac{\lambda-c_1(L)}{\delta}-\bar{\psi}_\star}\rangle_{0,\vec{m}\cup\star,\beta_\star}\ .$$

(3) If the graph Γ is of type III, then $v_\star$ is incident to the distinguished leg corresponding to the marking $q_\star$ and m edges (m can be 0) and n legs. Each leg must be associated with the input $\mathbf{t}^{\overrightarrow{k'}}\mu_{\beta',\overrightarrow{k'},c'}(z)$ with $\beta'(L)+(\vec{w},\overrightarrow{k'})=0$, otherwise the support of the twisted sector $\bar{I}_{c'^{-1}}Y$ will avoid $\mathcal{D}_\infty$. Choose a labeling[7] of the m edges attached to the vertex $v_\star$ by $[m]:=\{1,\cdots,m\}$. Let $\frac{\delta_i}{r}$ be the decorated degree associated with the ith edge e_i. Let v_i be the vertex over 0 incident to e_i, then v_i can't be a unstable vertex of valence 1 as the corresponding ramification point q_0 is a stacky point, or it can't be a node linking two edges by Lemma 3.5. Therefore v_i corresponds to either a marking point or a stable vertex. Assume that there are l legs (l can be zero) incident to v. let's label the legs incident to v_i by $\{i1,\cdots,il\}\subset[\ |\{\text{Legs of }\Gamma\}|\ ]$. Note that when v_i is unstable, $l=1$.

Assume that the vertex v_i is decorated by the degree β_{i0}. Assume that the insertion at the marking q_{ij} on the curve[8] C_{v_i} corresponds to $\mathbf{t}^{\overrightarrow{k_{ij}}}\mu_{\beta_{ij},\overrightarrow{k_{ij}},c_{ij}}(-\bar{\psi}_{ij})$ in (3.7), let's

[7] Such a labeling is not unique, but we will divide $m!$ to offset the labeling in the end.

[8] When v is unstable, we just take v to be q_{i1}.

say the leg for q_{ij} has *virtual extended degree* $(\beta_{ij}, \overrightarrow{k_{ij}})$ contribution to the vertex v_i, denote $(\beta_i, \overrightarrow{k_i})$ to be summation of $(\beta_{i0}, 0)$ and all the virtual degrees from the legs incident to v_i. We call $(\beta_i, \overrightarrow{k_i})$ the *total extended degree* at the vertex v_i. From (3.7), one has

$$\beta_\star + \sum_{i=1}^{m} \beta_i = \beta, \quad \sum_{i=1}^{m} \overrightarrow{k_i} = \overrightarrow{k}\ .$$

Assume the multiplicity of the half edge incident to C_{v_i} is equal to $(c_i^{-1}, e^{\frac{\delta_i}{s}}, 1)$, here δ_i is the decorated degree associated with the edge e_i and $\delta(e) = age_c(L) \bmod \mathbb{Z}$ by 3.2. Observe that to ensure such a graph Γ exists, one must have

$$\beta_i(L) + (\vec{w}, \overrightarrow{k_i}) = \delta_i\ . \tag{3.8}$$

Indeed, by orbifold Riemann-Roch Theorem, one has

$$deg(f^*\mathcal{O}(\mathcal{D}_0)|_{C_{v_i}}) = -\frac{\beta_{i0}(L)}{s} = (1 - \frac{\delta_i}{s}) + \sum_{j=1}^{l} \frac{\beta_{ij}(L) + (\vec{w}, \overrightarrow{k_{ij}})}{s} \quad mod \quad \mathbb{Z}\ .$$

Here the first term on the right hand is the age of $f^*\mathcal{O}(D_0)$ at the node of C_{c_i}, and the second term on the right is the sum of the ages of $f^*\mathcal{O}(\mathcal{D}_0)$ at the marked points on C_{c_i}. As s is sufficiently large, one must have

$$\frac{\delta_i}{s} = \frac{\beta_{i0}(L)}{s} + \sum_{j=1}^{l} \frac{\beta_{ij}(L) + (\vec{w}, \overrightarrow{k_{ij}})}{s}\ ,$$

which implies that $\beta_i(L) + (\vec{w}, \overrightarrow{k_i}) = \delta_i$. Note that this implies that $(\beta_i, \overrightarrow{k_i}, c_i)$ is an admissible pair.

Now we can group the edge-labeled decorated graphs by the set $\Lambda_{\beta, \overrightarrow{k}, c, m, n}$ defined in 3.3. For any element of $\Lambda_{\beta, \overrightarrow{k}, c, m, n}$, we can naturally associate a group of edge-labeled star-shaped decorated graphs such that the vertex incident to the edge labeled by i has total extended degree $(\beta_i, \overrightarrow{k_i})$ and the multiplicity at the half-edge h_i incident to v_i over 0 is $m(h_i) := (c_i^{-1}, e^{\frac{\delta_i}{s}}, 1)$. We may call an element $\phi := (c, \beta_\star, ((\beta_1, \overrightarrow{k_1}, c_1), \cdots, (\beta_{m+n}, \overrightarrow{k_{m+n}}, c_{m+n})))$ of $\Lambda_{\beta, \overrightarrow{k}, c, m, n}$ a *meta graph*, and denote all the (star-shaped) edge-labeled decoration graph associated to ϕ by Γ_ϕ.

Now we use the localization formula in §3.2.4 to compute the contribution from Γ_ϕ to (3.7). Summing over the localization contribution of the vertex v_i together with branch of node h_i at v_i from all graphs in Γ_ϕ, and pushing forward to $\bar{I}_{c_i^{-1}} Y \cong \bar{I}_{m_{h_i}} \mathcal{D}_0$ along $\iota \circ (ev_{h_i})_*$(recall that ι is the inversion map on Chen-Ruan cohomology $H^*(\bar{I}_\mu Y)$), it yields

$$\mathbf{t}^{\overrightarrow{k_i}} \mu_{\beta_i, \vec{k}_i, c_i}(\frac{\lambda - c_1(L)}{\delta_i}) + \sum_{m=0}^{\infty} \sum_{\substack{(\beta_j, \overrightarrow{k_j}, c_j)_{j=1}^m \in \mathrm{Adm}^m, \beta_\star \in \mathrm{Eff}(Y) \\ \beta_\star + \sum_{j=1}^m \beta_j = \beta_i \\ \overrightarrow{k_1} + \cdots + \overrightarrow{k_m} = \overrightarrow{k}}} \frac{1}{m!}$$

$$(\widetilde{ev}_\star)_* \big(\epsilon'_* \big([\mathcal{K}_{0, \vec{m}_s \cup \star}(\sqrt[s]{L^\vee / Y}, \beta_\star)]^{\mathrm{vir}} \big) \cap \bigcap_{j=1}^{m} ev_j^* (\mathbf{t}^{\overrightarrow{k_j}} \mu_{\beta_j, \overrightarrow{k_j}, c_j}(-\bar{\psi}_j)) \cap \frac{1}{\frac{\lambda - ev_\star^* c_1(L)}{\delta_j s} - \frac{\bar{\psi}_\star}{s}} \big),$$

which is equal to $\mathbf{t}^{\overrightarrow{k_i}} J^Y_{\beta_i, \vec{k}_i, c_i}(\mu, z)|_{\frac{\lambda - c_1(L)}{\delta_i}}$. Note here we use the fact the Euler class of the virtual normal bundle for v_i is equal to 1 as the node has nontrivial isotropy action on the normal bundle $N_{\mathcal{D}_0/\mathbb{P}Y_{r,s}}$. Observe that all the edge-labeled decorated graphs in

Γ_ϕ have the same localization contributions from the unique vertex $v_\star$ labeled by ∞, the edge e_i and the node over ∞ incident to e_i. Moreover the localization formula for any graph in Γ_ϕ depends multi-linearly on the localization contributions of vertexes over 0, then the localization from all meta graphs in $\Lambda_{\beta,\overrightarrow{k},c,m,n}$ to (3.7) yields the summation:

$$
\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{\substack{\Gamma\in\Lambda_{\beta,\overrightarrow{k},c,m,n}\\ \Gamma \text{ is stable}}}\frac{1}{m!n!}(\widetilde{ev_\star})_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*\big(c_d(-R^\bullet\pi_*f^*L^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\\
&\cap[\mathcal{K}_{0,\overrightarrow{m+n}\cup\star}(\sqrt[r]{L/Y},\beta_\star)]^{\mathrm{vir}}\big)\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}\mathbf{t}^{\overrightarrow{k_i}}J_{\beta_i,\vec{k}_i,c_i}(\mu,z)|_{z=\frac{\lambda-c_1(L)}{\delta_i}})}{-\frac{\lambda-ev_i^*c_1(L)}{r\delta_i}-\frac{\bar\psi_i}{r}}\\
&\cap\prod_{i=m+1}^{m+n}ev_i^*(\mathbf{t}^{\overrightarrow{k_i}}\mu_{\beta_i,\vec{k}_i,c_i}(-\bar\psi_i))\cap\bar\psi_\star^b\Bigg)\,.
\end{aligned}\tag{3.9}
$$

By the discussion above, we can write (3.7) in the following way:

$$
\begin{aligned}
&\mathbf{t}^{\overrightarrow{k}}\frac{\mu_{\beta,\overrightarrow{k},c}(\frac{\lambda-c_1(L)}{\delta})}{\delta}\cdot(\frac{\lambda-c_1(L)}{\delta})^b+\sum_{m=0}^{\infty}\sum_{\substack{(\beta_j,\overrightarrow{k_j},c_j)_{j=1}^m\in\mathrm{Adm}^m,\beta_\star\in\mathrm{Eff}(Y)\\ \beta_1+\cdots+\beta_m+\beta_\star=\beta\\ \overrightarrow{k_1}+\cdots+\overrightarrow{k_m}=\overrightarrow{k}}}\frac{1}{m!}(\widetilde{ev_\star})_*\\
&\Bigg([\mathcal{K}_{0,\vec{m}\cup\star}(Y,\beta_\star)]^{\mathrm{vir}}\cap\bigcap_{i=1}^{m}ev_i^*(\mathbf{t}^{\overrightarrow{k_i}}\mu_{\beta_i,\vec{k}_i,c_i}(-\bar\psi_i))\bigcap\frac{\frac{1}{\delta}(\frac{\lambda-ev_\star^*c_1(L)}{\delta})^b}{\frac{\lambda-ev_\star^*c_1(L)}{\delta}-\bar\psi_\star}\Bigg)\\
&+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{\substack{\Gamma\in\Lambda_{\beta,\overrightarrow{k},c,m,n}\\ \Gamma \text{ is stable}}}\frac{1}{m!n!}(\widetilde{ev_\star})_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*\big(c_d(-R^\bullet\pi_*f^*L^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\\
&\cap[\mathcal{K}_{0,\overrightarrow{m+n}\cup\star}(\sqrt[r]{L/Y},\beta_\star)]^{\mathrm{vir}}\big)\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}\mathbf{t}^{\overrightarrow{k_i}}J_{\beta_i,\vec{k}_i,c_i}(\mu,z)|_{z=\frac{\lambda-c_1(L)}{\delta_i}})}{-\frac{\lambda-ev_i^*c_1(L)}{r\delta_i}-\frac{\bar\psi_i}{r}}\\
&\cap\prod_{i=m+1}^{m+n}ev_i^*(\mathbf{t}^{\overrightarrow{k_i}}\mu_{\beta_i,\vec{k}_i,c_i}(-\bar\psi_i))\cap\bar\psi_\star^b\Bigg)\,.
\end{aligned}\tag{3.10}
$$

As (3.7) lies in $\mathbf{t}^{\overrightarrow{k}}H^*(\bar I_\mu Y,\mathbb{Q})[\lambda]$, the coefficient of $\mathbf{t}^{\overrightarrow{k}}\lambda^{-1}$ term in (3.10) must vanish. Note that the coefficients before λ^{-1} in the first two terms in (3.10) yields

$$
\sum_{m=0}^{\infty}\sum_{\substack{(\beta_j,\overrightarrow{k_j},c_j)_{j=1}^m\in\mathrm{Adm}^m,\beta_\star\in\mathrm{Eff}(Y)\\ \beta_1+\cdots+\beta_m+\beta_\star=\beta\\ \overrightarrow{k_1}+\cdots+\overrightarrow{k_m}=\overrightarrow{k}}}\frac{1}{m!}\phi^\alpha\langle\mu_{\beta_1,\overrightarrow{k_1},c_1}(-\bar\psi_1),\cdots,\mu_{\beta_m,\overrightarrow{k_m},c_m}(-\bar\psi_m),\phi_\alpha\bar\psi_\star^b\rangle_{0,\vec{m}\cup\star,\beta_\star},
$$

which is the left hand side of equality in (4.4). Then we extract the coefficient of the $\mathbf{t}^{\overrightarrow{k}}\lambda^{-1}$ term in the third term in (3.10), this yields the term on the right hand side of (4.4) up to a minus sign. This completes the proof of (4.4). $\square$

3.4. **Specializing Novikov degrees.** When Y can be embedded into another smooth DM stack X with projective coarse moduli and L is a restriction of a line bundle of X, which we still denote to be l by an abuse of notation. For any degree $\beta\in\mathrm{Eff}(X)$, we will denote

$\mathcal{K}_{0,\vec{m}}(Y,\beta)$ to be the disjoint union[9]

$$\bigsqcup_{\substack{d\in \mathrm{Eff}(Y)\\ i_*(d)=\beta}} \mathcal{K}_{0,\vec{m}}(Y,d)\ .$$

The same rule applies to the notation when we define Gromov-Witten invariants: for any degree $\beta\in\mathrm{Eff}(X)$, we will denote the Gromov-witten invariants

$$\langle\cdots\rangle^Y_{0,\vec{m},\beta} := \sum_{d\in\mathrm{Eff}(Y):i_*(d)=\beta} \langle\cdots\rangle^Y_{0,\vec{m},d}\ .$$

Now we replace the effective cone $\mathrm{Eff}(Y)$ by $\mathrm{Eff}(X)$ in the definition of $\Lambda_{\beta,\overrightarrow{k},c,m,n}$ and the recursion relation 3.6 and apply the rule of specialization of degrees, we can also apply the same strategy of proving of proposition 3.4 to show the following variation of 3.4:

Proposition 3.6. *Let*

$$\mu(z)=\sum_{\beta,\vec{k},c} q^\beta \mathbf{t}^{\overrightarrow{k}}\mu_{\beta,\vec{k},c}(z)$$

be an admissible series in $H^*(\bar{I}_\mu Y,\mathbb{C})[z][\![t_1,\cdots,t_N]\!][\![\mathrm{Eff}(X)]\!]$. For any integer $b\geqslant 0$ and admissible pair $(\beta,\overrightarrow{k},c)$ with $\beta(L)+(\vec{w},\vec{k})>0$, we have the following recursive relation:

$$\begin{aligned}
&[J^Y_{\beta,\vec{k},c}(\mu,z)]_{z^{-b-1}}\\
&=\Bigg[\sum_{m=0}^\infty\sum_{n=0}^\infty \sum_{\substack{\Gamma\in\Lambda_{\beta,\overrightarrow{k},c,m,n}\\ \Gamma\ is\ stable}} \frac{1}{m!n!}(\widetilde{ev_\star})_*\Bigg(\sum_{d=0}^\infty \epsilon_*\big(c_d(-R^\bullet\pi_*f^*L^{\frac{1}{r}})(\frac{\lambda}{r})^{-1+m-d}(-1)^d\\
&\cap[\mathcal{K}_{0,\overrightarrow{m+n}\cup\star}(\sqrt[r]{L/Y},\beta_\star)]^{\mathrm{vir}}\big)\cap\prod_{i=1}^m \frac{ev_i^*(\frac{1}{\delta_i}\mathbf{t}^{\overrightarrow{k_i}}J^Y_{\beta_i,\vec{k}_i,c_i}(\mu,z)|_{z=\frac{\lambda-c_1(L)}{\delta_i}})}{\frac{\lambda-ev_i^*c_1(L)}{r\delta_i}+\frac{\bar{\psi}_i}{r}}\\
&\cap\prod_{i=m+1}^{m+n} ev_i^*(\mathbf{t}^{\overrightarrow{k_i}}\mu_{\beta_i,\vec{k}_i,c_i}(-\bar{\psi}_i))\cap\bar{\psi}_\star^b\Bigg)\Bigg]_{\mathbf{t}^{\overrightarrow{k}}\lambda^{-1}}\ .
\end{aligned}\tag{3.11}$$

4. Proof of the main theorem

In this section, we will first assume that X is a smooth Deligne-Mumford stack with projective moduli and $Y\subset X$ is a smooth *hypersurface* such that the line bundle $L:=\mathcal{O}_X(Y)$ is *semi-positive*, i.e., $\beta(L)\geqslant 0$ for any degree $\beta\in\mathrm{Eff}(X)$. Later in §4.4, Y will be assumed to be a complete intersection associated to a direct sum of semi-positive line bundles.

4.1. **A root stack modification of the space of the deformation to the normal cone.** Let $P_X:=\mathbb{P}_X(\mathbb{C}\oplus\mathbb{C})$ be the trivial $\mathbb{P}^1$ bundle over X. It has two section: one is the zero section $X_0:=\mathbb{P}(0\oplus\mathbb{C})\subset P_X$, and the other is the $\infty-$section $X_\infty:=\mathbb{P}(\mathbb{C}\oplus 0)$. We will introduce the notation P_Y to the mean the divisor $\mathbb{P}_Y(\mathbb{C}\oplus\mathbb{C})\subset P_X$.

Let $\mathfrak{Q}$ be the blow-up[10] of P_X along $Y_0:=X_0\cap P_Y$. More explicitly, $\mathfrak{Q}$ can be constructed as a hypersurface in the projective bundle $\pi:\mathfrak{F}:=\mathbb{P}_{P_X}(L^\vee\oplus\mathcal{O}_{P_X}(-1))\to P_X$ associated to the section $z_1\cdot\pi^{-1}(s_{P_Y})-z_2\cdot\pi^{-1}(s_{X_0})$ of the line bundle $\mathcal{O}_{\mathfrak{F}}(1)$, where s_Y and s_{X_0} are the defining equations of P_Y and X_0 in P_X, and z_1,z_2 are the tautological sections of line bundles $\pi^*(L^\vee)\otimes O_{\mathfrak{F}}(1)$ and $\pi^*\mathcal{O}_{P_X}(-1)\otimes\mathcal{O}_{\mathfrak{F}}(1)$ respectively. The space $\mathfrak{Q}$ has four special smooth divisors: 1, the strict transformation of P_Y, which we will still denote it to be P_Y; 2, the

[9]It's a finite disjoint union as $\mathcal{K}_{0,m}(X,\beta)$ is finite type over $\mathbb{C}$, hence Noetherian.

[10]This space is known as (a compactification of) the space of the deformation to the normal cone, c.f., [Ful84, ch5].

exceptional divisor, which we will denote to be E and it's isomorphic to the projective bundle $\mathbb{P}_Y(L^\vee \oplus \mathbb{C})$; 3, the strict transformation of the $0-$section X_0, we will still denote to be X_0 by an abuse of notation, and it's isomorphic to X with normal bundle equal to L, 4, the strict transformation of the $\infty-$section X_∞, which we still denote it to be X_∞.

Let $\mathbb{C}^*$ act on P_X by scaling the $\mathbb{P}^1-$fiber so that the weight of the $\mathbb{C}^*-$action on the normal bundle of X_∞ in P_X is -1. It induces a $\mathbb{C}^*-$action on $\mathfrak{Q}$ with the $\mathbb{C}^*-$fixed loci X_0, X_∞ and $D_0 := P_Y \cap E$.

We will consider the root stack $\mathfrak{R}$ of $\mathfrak{Q}$ by taking $s-$th root of X_0 and $r-$th root of P_Y. We will also assume that r, s are distinct primes. We will still use the notation X_0, X_∞, E and P_Y to mean their corresponding divisors in $\mathfrak{R}$ after taking roots. We note E is isomorphic to $\mathbb{P}Y_{r,s}$ so that $\mathcal{D}_0 := X_0 \cap E$ is isomorphic to the root stack $\sqrt[s]{L^\vee/Y}$ and $\mathcal{D}_\infty := P_Y \cap E$ is isomorphic to the root stack $\sqrt[r]{L/Y}$.

The $\mathbb{C}^*-$action on $\mathfrak{Q}$ induces a $\mathbb{C}^*-$action on $\mathfrak{R}$. We see that the normal bundle of X_0 in $\mathfrak{R}$ is ($\mathbb{C}^*-$equivariantly) isomorphic to $(L^\vee)^{\frac{1}{s}} \otimes \mathbb{C}_{\frac{1}{s}}$, the normal bundle of X_∞ in $\mathfrak{R}$ is isomorphic to $\mathbb{C}_{-\lambda}$, and the normal bundle of $\mathcal{D}_\infty$ in $\mathfrak{R}$ is isomorphic to $(L^{\frac{1}{r}} \otimes \mathbb{C}_{-\frac{1}{r}}) \oplus \mathbb{C}_\lambda$.

Let $\mathrm{pr}_{r,s} : \mathfrak{R} \to X$ be the morphism induced from composition of the following three maps: (1) a morphism $\mathfrak{R} \to \mathfrak{Q}$ forgetting root structure; (2) the blow-down from $\mathfrak{Q}$ to P_X; (3) and the projection from P_X to the base X. Note $\mathrm{pr_{r,s}}$ induces a morphism from the rigidified inertia stack $\bar{I}_\mu\mathfrak{R}$ to the rigidified inertia stack $\bar{I}_\mu X$.

There are two morphisms

$$q_0, q_\infty : \mathfrak{R} \to \mathbb{B}\mathbb{C}^* \ ,$$

associated to the line bundle $\mathcal{O}(X_0)$ and $\mathcal{O}(P_Y)$ respectively such that $\mathcal{O}(X_0) \cong q_0^*(\mathbb{L})$ and $\mathcal{O}(P_Y) \cong q_\infty^*(\mathbb{L})$, here $\mathbb{L}$ is the universal line bundle over $\mathbb{B}\mathbb{C}^*$. This will induce morphisms on their rigidified inertia stack counterparts:

$$q_0, q_\infty : \bar{I}_\mu\mathfrak{R} \to \bar{I}_\mu\mathbb{B}\mathbb{C}^* \ ,$$

Note the rigidified inertia stack $\bar{I}_\mu\mathbb{B}\mathbb{C}^*$ of $\mathbb{B}\mathbb{C}^*$ can be written as the disjoint union

$$\bigsqcup_{c\in\mathbb{C}^*} \bar{I}_c\mathbb{B}\mathbb{C}^* \ ,$$

Let c be a complex number, any $\mathbb{C}$-point of $\bar{I}_c\mathbb{B}\mathbb{C}^*$ is isomorphic to the pair $(1_\mathbb{C}, c)$, where $1_\mathbb{C}$ is the trivial principal $\mathbb{C}$-bundle over $\mathbb{C}$ and c is an element in the automorphism group $\mathrm{Aut}_\mathbb{C}(1_\mathbb{C}) \cong \mathbb{C}^*$. Let (y, g) be a $\mathbb{C}-$point of $\bar{I}_\mu\mathfrak{R}$, where $y \in Ob(\mathbb{P}Y_{r,s}(\mathfrak{R}))$ and $g \in \mathrm{Aut}(y)$. Assume that $q_0((y, g)) = (1_\mathbb{C}, c)$, then g acts on the fiber $\mathcal{O}(X_0)|_y$ via the multiplication by c. We have a similar relation for q_∞.

Let C be a good index set for rigidified inertia stack $\bar{I}_\mu X$ satisfying the assumption 2.1. Now we will use the index set $C \times \mathbb{C}^* \times \mathbb{C}^*$ to index the components of the rigidified inertia stack $\bar{I}_\mu\mathfrak{R}$: for each $c \in C$, $c_0, c_\infty \in \mathbb{C}^*$, we will define the rigidified inertia component $\bar{I}_{(c,c_0,c_\infty)}\mathfrak{R}$ to be

$$\mathrm{pr}_{r,s}^{-1}(\bar{I}_c X) \cap q_0^{-1}(\bar{I}_{c_0}\mathbb{B}\mathbb{C}^*) \cap q_\infty^{-1}(\bar{I}_{c_\infty}\mathbb{B}\mathbb{C}^*) \ .$$

Notice that $\bar{I}_{(c,c_0,c_\infty)}\mathfrak{R}$ is nonempty if and only if $s \cdot age_c(L) \equiv -c_0 \mod \mathbb{Z}$ and $r \cdot age_c(L) \equiv c_\infty \mod \mathbb{Z}$.

Definition 4.1. *For any degree $\beta \in \mathrm{Eff}(X)$ and $\delta, d \in \mathbb{Q}$, we say a stable map $f : C \to \mathfrak{R}$ is of degree $(\beta, \frac{\delta}{r}, d)$ if $(\mathrm{pr}_{r,s} \circ f)_*[C] = \beta$, $deg(f^*\mathcal{O}(P_Y)) = \frac{\delta}{r}$ and $deg(f^*\mathcal{O}(X_\infty)) = d$. We will denote $\mathcal{K}_{0,\vec{m}}(\mathfrak{R}, (\beta, \frac{\delta}{r}, d))$ to be the corresponding moduli stack of twisted stable maps to $\mathfrak{R}$ of degree $(\beta, \frac{\delta}{r}, d)$. We note $\mathcal{K}_{0,\vec{m}}(\mathfrak{R}, (\beta, \frac{\delta}{r}, d))$ is proper as the degree data uniquely determines a homology class in $H_2(\mathfrak{R}, \mathbb{Q})$. Indeed, dual to $H_2(\mathfrak{R}, \mathbb{Q})$, we have $H_2(\mathfrak{R}, \mathbb{Q})^\vee \cong H^2(\mathfrak{R}, \mathbb{Q}) \cong$*

$H^2(\mathfrak{Q},\mathbb{Q})$*, and, by the* blow-up *construction of* $\mathfrak{Q}$*,* $H^2(\mathfrak{Q},\mathbb{Q})$ *is isomorphic to the direct sum*

$$H^2(X,\mathbb{Q})\oplus\mathbb{Q}[X_\infty]\oplus\mathbb{Q}[P_Y]\ ,$$

where $[P_Y]$*,* $[X_\infty]$ *(and* $[E]$*) are the fundamental classes of* P_Y*,* X_∞ *(and* E*) respectively and the first summand is embedded into* $H^*(\mathfrak{Q},\mathbb{Q})$ *via the pullback* $\mathrm{pr}_{r,s}^*$*. Note that we have* $r[P_Y]+[E]=\mathrm{pr}_{r.s}^*([Y])$ *in* $H^2(\mathfrak{R},\mathbb{Q})$*.*

4.2. **Localization analysis.** The $\mathbb{C}^*-$action on $\mathfrak{R}$ induces a $\mathbb{C}^*-$action on the moduli of stable maps to $\mathfrak{R}$, we will use decorated graphs (trees) to index the components of $\mathbb{C}^*-$fixed loci of $\mathcal{K}_{0,\overrightarrow{m}}(\mathfrak{R},(\beta,\frac{\delta}{r},d))$ similar to 3.2. The decorations on a graph Γ is given by the following data:

- Each vertex v is associated with an index $j(v)\in\{X_0,\mathcal{D}_\infty,X_\infty\}$, and a degree $\beta(v)\in\mathrm{Eff}(X)$. In particular, we will also say the vertex v is labeled by 0 if $j(v)=X_0$ and is labeled by ∞ if $j(v)=\mathcal{D}_\infty$.
- Each edge e consists a pair of half-edges $\{h_j,h_{j'}\}$, and e is equipped with two numbers $\delta(e),d(e)\in\mathbb{Q}$. Here we call h_j and $h_{j'}$ half edges and the subscript j or j' is taken in the set $\{X_0,\mathcal{D}_\infty,X_\infty\}$. A half-edge h_j labeled by j is incident to a vertex labeled by j.
- Each half-edge h and each leg l has an element $m(h)$ or $m(l)$ in $C\times\mathbb{C}^*\times\mathbb{C}^*$.
- The legs are labeled with the numbers $\{1,\dots,m\}$ and each leg is incident to a unique vertex.

For $j\in\{X_0,\mathcal{D}_\infty,X_\infty\}$, we will use the symbol $\mathfrak{R}_j$ to mean the space j. By the "valence" of a vertex v, denoted val(v), we mean the total number of incident half-edges and legs. For each $\mathbb{C}^*-$fixed stable map $f:(C;q_1,\dots,q_m)\to\mathfrak{R}$ in $\mathcal{K}_{0,\overrightarrow{m}}(\mathfrak{R},(\beta_\star,\frac{\delta}{r},d))$, we can associate a decorated graph Γ in the following way.

- Each edge e corresponds to a genus-zero component C_e of restricted stale-map degree $(0,\frac{\delta(e)}{r},d(e))$. Thus the stable f maps C_e constantly to the base. There are two distinguished points(we will also call ramification points) q_j and $q_{j'}$ on C_e satisfying that q_j maps to $\mathfrak{R}_j$ and $q_{j'}$ maps to $\mathcal{R}_{j'}$, respectively. There are two corresponding half-edges h_j and $h_{j'}$ associated to q_j and $q_{j'}$ respectively.
- Each vertex v for which $j(v)$ (with unstable exceptional cases noted below) corresponds to a maximal sub-curve C_v of C which maps totally into $\mathcal{X}_0$, then the restriction of f to C_v defines a twisted stable map in
$$\mathcal{K}_{0,val(v)}(\mathfrak{R}_{j(v)},\beta(v))\ .$$
The label $\beta(v)$ denotes the degree coming from the composition $\mathrm{pr}_{r,s}\circ f|_{C_v}$ of the restriction and projection to the base.
- A vertex v is *unstable* if stable twisted maps of the type described above do not exist. In this case, v corresponds to a single point of the component C_e for each incident edge e, which may be a node at which C_e meets another edge curve $C_{e'}$, a marked point of C_e, or an unmarked point.
- The index $m(l)$ on a leg l indicates the rigidified inertia stack component $\bar{I}_{m(l)}\mathfrak{R}$ of $\mathfrak{R}$ on which the gerby marked point corresponding to the leg l is evaluated.
- A half-edge h of an edge e corresponds a ramification point $q\in C_e$ such that $f(q)\in\mathfrak{R}_j$ if h is labeled by j. Then $m(h)$ indicates the rigidified inertia component $\bar{I}_{m(h)}\mathfrak{R}$ of $\mathfrak{R}$ on which the ramification point q associated with h is evaluated.

Moreover, the decorated graph from a fixed stable map should satisfy the following:

(1) (Monodromy constraint for edge) For each edge $e=\{h_0:=h_{x_0},h_\infty:=h_{\mathcal{D}_\infty}\}$ linking X_0 and $\mathcal{D}_\infty$, we have that $m(h_0)=(c^{-1},\mathrm{e}^{\frac{\delta(e)}{s}},1)$ and $m(h_\infty)=(c,1,\mathrm{e}^{\frac{\delta(e)}{r}})$ for some $c\in C$. Moreover the choice of c should satisfy that
$$\delta(e)=age_c(L)\ \mathrm{mod}\ \mathbb{Z}\ ,$$

which follows from the fact that we have an isomorphism of line bundles $f|_{C_e}^*\mathcal{O}(sX_0) \cong f|_{C_e}^*(\mathcal{O}(rP_Y)\otimes \mathrm{pr}_{r,s}^* L^\vee)$ on C_e and then the ages of both line bundles at q_∞ should be equal.

(2) (Monodromy constraint for vertex) For each vertex v, let $I_v \subset [m]$ be the set of incident legs, and H_v be the set of incident half-edges. For each $i \in I_v$, write $m(l_i) = (c_i, a_i, b_i)$, and for each $h \in H_v$, write $m(h) = (c_h, a_h, b_h)$. If v is labeled by 0, we have that b_i, b_h are equal to 1, and

$$\mathrm{e}^{\frac{\beta(v)(L)}{s}} \times \prod_{i\in I_v} a_i \times \prod_{h\in H_v} a_h^{-1} = 1 \ .$$

If v is labeled by ∞, then we have a_i and a_h are all equal to 1 and

$$\mathrm{e}^{-\frac{\beta(v)(L)}{r}} \times \prod_{i\in I_v} b_i \times \prod_{h\in E_v} b_h^{-1} = 1 \ .$$

In particular, we note that the decorations at each stable vertex v yield a tuple

$$\overrightarrow{val}(v) \in (C\times \mathbb{C}^*\times\mathbb{C}^*)^{\mathrm{val}(v)}$$

recording the multiplicities at every special point of C_v. Then the restriction gives a stable map in

$$\mathcal{K}_{0,\overrightarrow{val}(v)}(\mathfrak{R}_{j(v)}, \beta(v)) \ .$$

Assume that r, s are sufficiently large primes. We will do a similar localization analysis as in §3.2.

4.2.1. *Vertex contribution.* Assume that v is a stable vertex, the localization contributions for X_0 and $\mathcal{D}_\infty$ repeat the discussions for $\mathcal{D}_0$ and $\mathcal{D}_\infty$ in 3.2.1 with the replacement of the words $\mathcal{D}_0$ and $\mathcal{D}_\infty$ by X_0 and $\mathcal{D}_\infty$ respectively, and one additional change that the *inverse of the Euler class* of the virtual normal bundle for $\mathcal{D}_\infty$ should be equal to

$$\frac{1}{\lambda}\cdot\sum_{d\geqslant 0} c_d(-R^\bullet\pi_* f^*(L)^{\frac{1}{r}})(\frac{-\lambda}{r})^{|E(v)|-1-d} \ .$$

If the vertex is labeled by X_∞, then the vertex moduli $\mathcal{M}_v$ is given by $\mathcal{K}_{0,\overrightarrow{val}(v)}(X_\infty, \beta(v))$ and the fixed part of perfect obstruction theory gives rise to $[\mathcal{K}_{0,\overrightarrow{val}(v)}(X_\infty,\beta(v))]^{\mathrm{vir}}$. The movable part of the perfect obstruction theory yields the *inverse of the Euler class* of the virtual normal bundle which is equal to $\frac{-1}{\lambda}$.

4.2.2. *Edge contribution.* Here we will only consider the type of edges linking the vertex labeled by X_0 and the vertex labeled by $\mathcal{D}_\infty$ as it is only the type of edge which appears in the localizaiton contribution of 4.3, see Lemma 4.3 for more details.

Let $e = \{h_0 := h_{X_0}, h_\infty := h_{\mathcal{D}_\infty}\}$ be an edge in Γ with decorated degree $\delta(e) \in \mathbb{Q}$ and decorated multiplicities $m(h_0)$ and $m(h_\infty)$. Write $m(h_\infty) = (c, 1, \mathrm{e}^{\frac{\delta(e)}{r}})$, let $a_e := a(c)$ be the number as Assumption 2.1, we will also write a as a_e for simplicity if there is no confusion. Then we define the edge moduli $\mathcal{M}_e$ to be the root stack $\sqrt[as\delta]{L^\vee/I_cY}$. Then the virtual cycle $[\mathcal{M}_e]^{vir}$ coming from the fix part of the obstruction theory is the fundamental class $[\mathcal{M}_e]$ and *the inverse of the Euler class* of the virtual normal bundle is $\prod_{i=1}^{-1-\lfloor-\delta(e)\rfloor}\big(\lambda + \frac{i}{\delta(e)}(c_1(L)-\lambda)\big)$.[11] In practice, we will use an equivalent form

$$\frac{1}{\lambda}\prod_{0\leqslant j<\delta(e)}\big(c_1(L) + (\delta(e)-j)\frac{\lambda - c_1(L)}{\delta(e)}\big) \ .$$

[11]Here, for any rational number q, we denote $\lfloor q\rfloor$ to be the largest integer no larger than q.

We note that $\mathcal{M}_e$ allows a finite étale map into the corresponding fixed-loci in $\mathcal{K}_{0,2}(\mathfrak{R},(0,\frac{\delta(e)}{r},0)$ of degree $\frac{1}{as}$. See appendix A.2 for more details.

4.2.3. *Node contributions.* The deformations in $\mathcal{K}_{0,\vec{m}}(\mathfrak{R},(\beta,\frac{\delta}{r},0))$ smoothing a node contribute to the Euler class of the virtual normal bundle as the first Chern class of the tensor product of the two cotangent line bundles at the branches of the node. For nodes at which a component C_e meets a component C_v over the vertex X_0, this contribution is

$$\frac{\lambda - c_1(L)}{a_e s\delta(e)} - \frac{\bar{\psi}_v}{a_e s}.$$

For nodes at which a component C_e meets a component C_v at the vertex over $\mathcal{D}_\infty$, this contribution is

$$\frac{-\lambda + c_1(L)}{a_e r\delta(e)} - \frac{\bar{\psi}_v}{a_e r}\ .$$

There is also one dimensional piece from deformations of maps (corresponding to the normal direction of E in $\mathfrak{R}$) at the node over $\mathcal{D}_\infty$, which contributes $\frac{1}{\lambda}$ as the Euler class of the virtual normal bundle to the localization.

We will not need node contributions from other types of nodes as the above types suffice the need in this paper, see Lemma 4.3 for more details.

4.2.4. *Total contribution.* Here we will only consider two types of graph: (1) all edges in Γ connects X_0 and $\mathcal{D}_\infty$; (2) Γ has only one vertex labeled by X_∞ and no edges. They are all we need in the later localization analysis(see Lemma 4.3 for more details).

The localization contribution from the decorated graph of type (2) is

$$-\frac{[\mathcal{M}_v]^{\mathrm{vir}}}{\lambda}\ .$$

For any decorated graph Γ of type (1), we define F_Γ to be

$$\prod_{v:j(v)=0}\mathcal{M}_v\times_{\bar{I}_\mu X_0}\prod_{e\in E}\mathcal{M}_e\times_{\bar{I}_\mu\mathcal{D}_\infty}\prod_{v:j(v)=\infty}\mathcal{M}_v$$

of the following diagram:

$$\begin{tikzcd}
F_\Gamma \arrow[r] \arrow[d] & \prod\limits_{v:j(v)=0}\mathcal{M}_v\times\prod\limits_{e\in E}\mathcal{M}_e\times\prod\limits_{v:j(v)=\infty}\mathcal{M}_v \arrow[d, "ev_{nodes}"] \\
\prod\limits_E(\bar{I}_\mu X_0\times\bar{I}_\mu\mathcal{D}_\infty) \arrow[r, "(\Delta^0\times\Delta^\infty)^{|E|}"] & \prod\limits_E(\bar{I}_\mu X_0)^2\times(\bar{I}_\mu\mathcal{D}_\infty)^2\ ,
\end{tikzcd}$$

where $\Delta^0=(id,\iota)$(resp. $\Delta^\infty=(id,\iota)$) is the diagonal map of $\bar{I}_\mu X_0$ (resp. $\bar{I}_\mu\mathcal{D}_\infty$). Here when v is a stable vertex, the vertex moduli $\mathcal{M}_v$ is described in 3.2.1; when v is an unstable vertex over 0 (resp. ∞), we treat $\mathcal{M}_v:=\bar{I}_{m(h)^{-1}}X_0$ (resp. $\mathcal{M}_v:=\bar{I}_{m(h)^{-1}}\mathcal{D}_\infty$) with the identical virtual cycle, where h is the half-edge incident to v, where h is the half-edge incident to v. The right-hand vertical map is the product of evaluation maps at the two branches of each gluing nodes.

We define $[F_\Gamma]^{\mathrm{vir}}$ to be:

$$\prod_{v:j(v)=0}[\mathcal{M}_v]^{\mathrm{vir}}\times_{\bar{I}_\mu X_0}\prod_{e\in E}[\mathcal{M}_e]^{\mathrm{vir}}\times_{\bar{I}_\mu\mathcal{D}_\infty}\prod_{v:j(v)=\infty}[\mathcal{M}_v]^{\mathrm{vir}}\ .$$

Then the contribution of decorated graph Γ to the virtual localization is:

$$Cont_\Gamma=\frac{\prod_{e\in E}sa_e}{|\mathrm{Aut}(\Gamma)|}(\iota_\Gamma)_*\left(\frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})}\right)\ . \tag{4.1}$$

Here $\iota_F : F_\Gamma \to \mathcal{K}_{0,\overrightarrow{m}}(\mathfrak{R}, (\beta, \frac{\delta}{r}, d))$ is a finite étale map of degree $\frac{|\mathrm{Aut}(\Gamma)|}{\prod_{e\in E} s a_e}$ into the corresponding $\mathbb{C}^*$-fixed loci. The virtual normal bundle $e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})$ is the product of virtual normal bundles from vertex contributions, edge contributions and node contributions.

4.3. **Auxiliary cycle.** Take $L_1 = L$ and $r = 1$ in 2.4 and write $\vec{w}$ to be the weight. Let $\mu^X(z) = \sum_{\beta,\vec{k},c} q^\beta \mathbf{t}^{\vec{k}} \mu^X_{\beta,\vec{k},c}(z)$ be an admissible series in $H^*_{CR}(X,\mathbb{C})[z][\![t_1,\cdots,t_N]\!][\![\mathrm{Eff}(X)]\!]$. For any triple $(\beta,\vec{k},c) \in \mathrm{Eff}(X)\times\mathbb{Z}^N_{\geqslant 0}\times C$, we define $J^{X,tw}_{\beta,\vec{k},c}(\mu^X,z)$ to be

(4.2)

$$\Bigg(\mu^X_{\beta,\vec{k},c}(z) + \mathbf{Coeff}_{\mathbf{t}^{\vec{k}}}\Bigg[\sum_{m\geqslant 0}\sum_{\substack{(\beta_j,\overrightarrow{k_j},c_j)_{j=1}^m\in\mathrm{Adm}^m\ \beta_\star\in\mathrm{Eff}(X)\\ \beta_1+\cdots+\beta_m+\beta_\star=\beta}} \frac{1}{m!}\phi^\alpha\langle \mathbf{t}^{\overrightarrow{k_1}}\mu^{\mathbf{X}}_{\beta_1,\overrightarrow{k_1},\mathbf{c_1}}(-\bar\psi_1),$$

$$\cdots, \mathbf{t}^{\overrightarrow{k_m}}\mu^{\mathbf{X}}_{\beta_m,\overrightarrow{k_m},\mathbf{c_m}}(-\bar\psi_m), \frac{\phi_\alpha}{z-\bar\psi_\star}\rangle^X_{0,\vec{m}\cup\star,\beta_\star}\Bigg]\Bigg) \prod_{0\leqslant j<\beta(L)+(\vec{w},\vec{k})}\Big(c_1(L)+(\beta(L)+(\vec{w},\vec{k})-j)z\Big)\ .$$

where $\overrightarrow{m}\cup\star = \big(c_1^{-1}\cdots, c_m^{-1}, c\big) \in C^{m+1}$. We note that $J^{X,tw}_{0,\vec{0},c} = 0$, then the formal series

$$J^{X,tw}(q,\mu^X,z) = z + \sum_{\beta,\vec{k},c} q^\beta \mathbf{t}^{\vec{k}} J^{X,tw}_{\beta,\vec{k},c}(\mu^X,z)$$

associated with the input $\mu^X(z)$ above is an admissible series and near z.

For any admissible pair $(\beta,\vec{k},c)$ with $\beta(L)+(\vec{w},\vec{k})>0$, denote $\delta = \beta(L)+(\vec{w},\vec{k})$. Assume that r, s are sufficiently large primes. For any nonnegative integer b, we will also compare (3.7) to the following auxiliary cycle:

$$\sum_{m=0}^{\infty}\sum_{\substack{(\beta_i,\overrightarrow{k_i},c_i)\in\mathrm{Adm}^m,\beta_\star\in\mathrm{Eff}(X)\\ \beta_\star+\sum_{i=1}^m\beta_i=\beta\\ \overrightarrow{k_1}+\cdots+\overrightarrow{k_m}=\vec{k}}}\frac{1}{m!}(\widetilde{EV_\star})_*\Bigg(\prod_{i=1}^m ev_i^*\big(\mathrm{pr}^*_{r,s}(\mathbf{t}^{\overrightarrow{k_i}}\mu^X_{\beta_i,\vec{k_i},c_i}(-\bar\psi_i))\big)$$
$$\cap\,\bar\psi_\star^b\cap[\mathcal{K}_{0,\overrightarrow{m}\cup\star}(\mathfrak{R},(\beta_\star,\frac{\delta}{r},0))]^{\mathrm{vir}}\Bigg) \tag{4.3}$$

Here an explanation of the notations is in order:

(1) For degrees $\beta_\star,\beta_1,\cdots,\beta_m$ in $\mathrm{Eff}(X)$ and tuples $\overrightarrow{k_1},\cdots,\overrightarrow{k_m}$ in $\mathbb{Z}^N_{\geqslant 0}$ with $\sum_{i=1}^m\beta_i+\beta_\star = \beta$ and $\overrightarrow{k_1}+\cdots+\overrightarrow{k_m} = \vec{k}$, let $(c_1,\cdots,c_m)\in C^m$ such that $(\beta_i,\overrightarrow{k_i},c_i)$ are admissible pairs. Write $\delta_i = \beta_i(L)+(\vec{w},\overrightarrow{k_i})$, we define $\vec{m}\cup\star$ to be the $m+1$tuple

$$\Big((c_1^{-1},\mathrm{e}^{\frac{\delta_i}{s}},1),\cdots,(c_m^{-1},\mathrm{e}^{\frac{\delta_m}{s}},1),(c,1,\mathrm{e}^{\frac{\delta}{r}})\Big)\ .$$

(2) Note $\bar I_{m_\star}\mathfrak{R}$ is canonically isomorphic to $\bar I_c Y\times\mathbb{P}^1$, it has a natural projection map $p:\bar I_{m_\star}\mathfrak{R}\to\bar I_c Y$. Let $EV_\star$ be the morphism which is a composition of the following maps:

$$\mathcal{K}_{0,\overrightarrow{m}\cup\star}(\mathfrak{R},(\beta_\star,\tfrac{\delta}{r},0)) \xrightarrow{ev_\star} \bar I_{m_\star}\mathfrak{R} \xrightarrow{\ p\ } \bar I_c Y\ .$$

Then we define $(\widetilde{EV_\star})_*$ to be

$$\iota_*(r_\star(EV_\star)_*)$$

as in 2.1. Note here $r_\star$ is the order of the band from the gerbe structure of $\bar I_\mu Y$ but not $\bar I_\mu\mathfrak{R}$.

Applying $\mathbb{C}^*-$localization to the 4.3, we will prove the following:

Proposition 4.2. *With the notation as above, for $b \geqslant 0$, we have the following recursive relation:*

$$
\begin{aligned}
&[i^* J^{X,tw}_{\beta,\vec{k},c}(\mu^X, z)]_{z^{-b-1}} \\
&= \Big[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\substack{\Gamma \in \Lambda_{\beta,\vec{k},c,m,n} \\ \Gamma \text{ is stable}}} \frac{1}{m!n!} (\widetilde{ev_\star})_* \Big(\sum_{d=0}^{\infty} \epsilon_* \big(c_d(-R^\bullet \pi_* f^* L^{\frac{1}{r}}) (\frac{\lambda}{r})^{-1+m-d} (-1)^d \\
&\cap [\mathcal{K}_{0,\overrightarrow{m+n}\cup\star}(\sqrt[r]{L/Y}, \beta_\star)]^{\mathrm{vir}} \big) \cap \prod_{i=1}^{m} \frac{ev_i^*(\frac{1}{\delta_i} i^* \mathbf{t}^{\overrightarrow{k_i}} J^{X,tw}_{\beta_i,\vec{k_i},c_i}(\mu^X, z)|_{z=\frac{\lambda - c_1(L)}{\delta_i}})}{\frac{\lambda - ev_i^* c_1(L)}{r\delta_i} + \frac{\bar{\psi}_i}{r}} \\
&\cap \prod_{i=m+1}^{m+n} ev_i^*(\mathbf{t}^{\overrightarrow{k_i}} \mu^X_{\beta_i,\vec{k_i},c_i}(-\bar{\psi}_i)) \cap \bar{\psi}_\star^b \Big) \Big]_{\mathbf{t}^{\vec{k}} \lambda^{-1}} .
\end{aligned}
\tag{4.4}
$$

Here $\delta_i = \beta_i(L) + (\vec{w}, \overrightarrow{k_i})$, and $\epsilon : \mathcal{K}_{0,\overrightarrow{m+n}\cup\star}(\sqrt[r]{L/Y}, \beta_\star) \to \mathcal{K}_{0,\overrightarrow{m+n}\cup\star}(Y, \beta_\star)$ is the natural structural morphism by forgetting root-gerbe structure of $\sqrt[r]{L/Y}$ (c.f.[TT16]), where we choose the tuple $\overrightarrow{m+n} \cup \star$ for $\mathcal{K}_{0,\overrightarrow{m+n}\cup\star}(\sqrt[r]{L/Y}, \beta_\star)$ to be

$$
\big((c_1^{-1}, 1, \mathrm{e}^{\frac{-\delta_1}{r}}), \cdots, (c_{m+n}^{-1}, 1, \mathrm{e}^{\frac{-\delta_{m+n}}{r}}), (c, 1, \mathrm{e}^{\frac{\delta}{r}}) \big) \in (C \times \mathbb{C}^* \times \mathbb{C}^*)^{m+n+1} ,
$$

and choose the tuple $\overrightarrow{m+n} \cup \star$ for $\mathcal{K}_{0,\overrightarrow{m+n}\cup\star}(Y, \beta_\star)$ to be

$$
(c_1^{-1}, \cdots, c_{m+n}^{-1}, c) \in C^{m+n+1} .
$$

Our main strategy to prove this is to apply $\mathbb{C}^*-$localization to the integral (4.3) and then extract the λ^{-2}-coefficient(or equivalently $\mathbf{t}^{\vec{k}} \lambda^{-2}$-coefficient). Using the polynomiality of (4.3), we have that the λ^{-2} coefficient must be zero, which yields the desired result. To simplify the localization calculation, we need the following two lemmas which limit the types of localization graphs one need to consider.

Lemma 4.3. *Let Γ be a decorated graph for $\mathcal{K}_{0,\vec{m}\cup\star}(\mathfrak{R}, (\beta, \frac{\delta}{r}, 0))$. If the corresponding fixed loci F_Γ is non-empty, then there is no edge of Γ linking $\mathcal{D}_\infty$ and X_∞ and no edge of Γ linking X_0 and X_∞.*

Proof. Let $f : C \to \mathfrak{R}$ be a $\mathbb{C}^*-$fixed stable map in $\mathcal{K}_{0,\vec{m}\cup\star}(\mathfrak{R}, (\beta, \frac{\delta}{r}, 0))$, we have the pairing $(f_*([C]), [X_\infty]) = 0$. For each stable vertex v, obviously, we have the pairing $(f_*([C_v]), [X_\infty]) = 0$. If e is an edge, there three possibilities:

(1) the edge e links a vertex labeled by $\mathcal{D}_\infty$ and X_∞.
(2) the edge e links a vertex labeled by $\mathcal{D}_\infty$ and X_0.
(3) the edge e links a vertex labeled by X_0 and X_∞.

In the first two cases, we have the pairing $(f_*([C_e]), [X_\infty]) > 0$, while in the third case, the pairing $(f_*([C_e]), [X_\infty]) = 0$. As the total degree of $f^*\mathcal{O}(X_\infty)$ on C is zero, we see that the degree of the restriction of $f^*\mathcal{O}(X_\infty)$ to each component C_v or C_e must be zero, which implies that there is no edge linking $\mathcal{D}_\infty$ and X_∞ and no edge linking X_0 and X_∞. □

Remark 4.4. The above lemma also works for the case when $\delta = 0$, and the proof is verbatim.

By the above the discussion, they will only be two types of graphs which will contribute to the integral (4.3): (1) all edges in Γ connects X_0 and $\mathcal{D}_\infty$; (2) Γ has only one vertex labeled by X_∞ and no edges. Now let's assume the graph Γ is of type 1, we have the following lemma by using a similar argument as in Lemma 3.5 by using the line bundle $N := f^*(\mathcal{O}(-P_Y))$.

Lemma 4.5. *Assume r, s is sufficiently large. If localization graph Γ has more than one vertex labeled by ∞, then the corresponding $\mathbb{C}^*-$fixed loci moduli F_Γ is empty, therefore it will contribute zero to* (4.3).

Now we are ready to prove Proposition 4.2.

Proof. If a localization graph Γ has a vertex labeled by X_∞, by Lemma 4.3, there will be no edges in Γ. Then Γ is made of a single vertex. Such type of graph will contribute

$$\frac{-1}{\lambda}\sum_{m\geqslant 0}\sum_{\substack{(\beta_j,\vec{k_j},c_j)_{j=1}^m\in\mathrm{Adm}^m\\ \beta_\star\in\mathrm{Eff}(X)\\ \beta_1+\cdots+\beta_m+\beta_\star=\beta\\ \vec{k_1}+\cdots+\vec{k_m}=\vec{k}}}\frac{1}{m!}\phi^\alpha\langle\mathbf{t}^{\vec{k_1}}\mu^{\mathbf{X}}_{\beta_1,\vec{k_1},\mathbf{c_1}}(-\bar{\psi}_1),\cdots,\mathbf{t}^{\vec{k_m}}\mu^{\mathbf{X}}_{\beta_m,\vec{k_m},\mathbf{c_m}}(-\bar{\psi}_m),\frac{\phi_\alpha}{z-\bar{\psi}_\star}\rangle^{X_\infty}_{0,\vec{m}\cup\star,\beta_\star} \tag{4.5}$$

to 4.3.

The other type of graph which contributes to (4.4) must satisfy that all the vertexes are labeled by either 0 or ∞ and all the edges only link X_0 and $\mathcal{D}_\infty$. Using Lemma 4.5, we can apply the same strategy of the Proposition 3.6 to do the localization computation. Denote $v_\star$ to be the only vertex of Γ labeled by ∞, then the remaining types of localization graph is a star-shaped graph introduced in the proof of 3.6 and can be sorted as below:

(1) (Type I) The vertex $v_\star$ is unstable and there is only one edge incident to $v_\star$ in Γ, the vertex v labeled by 0 is unstable;
(2) (Type II) The vertex $v_\star$ is unstable and there is only one edge incident to $v_\star$ in Γ, the vertex v labeled by 0 is stable;
(3) (Type III) The vertex $v_\star$ is stable.

The graph of type I will contribute

$$\frac{1}{\delta\lambda}\mathbf{t}^{\vec{k}}\prod_{0\leqslant j<\delta}\left(c_1(L)+(\delta-j)\frac{\lambda-c_1(L)}{\delta}\right)i^*\mu^X_{\beta,\vec{k},c}\left(\frac{\lambda-c_1(L)}{\delta}\right)\cdot\left(\frac{\lambda-c_1(L)}{\delta}\right)^b \tag{4.6}$$

to 4.3.

The graph of type II will contribute

$$\begin{aligned}&\frac{1}{\delta\lambda}\prod_{0\leqslant j<\delta}\left(c_1(L)+(\delta-j)\frac{\lambda-c_1(L)}{\delta}\right)\sum_{m\geqslant 0}\sum_{\substack{(\beta_i,\vec{k_i},c_i)_{i=1}^m\in\mathrm{Adm}^m\\ \beta_\star\in\mathrm{Eff}(X),\ \sum_i\beta_i+\beta_\star=\beta\\ \sum_i\vec{k_i}=\vec{k}}}\frac{1}{m!}\\ &\cdot i^*\phi^\alpha\left(\int_{[\mathcal{K}_{0,\vec{m}\cup\star}(X,\beta_\star)]^{\mathrm{vir}}}\prod_{i=1}^m ev_i^*(\mathbf{t}^{\vec{k_i}}\mu^X_{\beta_i,\vec{k_i},c_i}(-\bar{\psi}_i))\cup\frac{ev_\star^*\phi_\alpha(\frac{\lambda-ev_\star^*c_1(L)}{\delta})^b}{\frac{\lambda-ev_\star^*c_1(L)}{\delta}-\bar{\psi}_\star}\right).\end{aligned} \tag{4.7}$$

to 4.3.

The graph of type III will contribute

$$\begin{aligned}&\frac{1}{\lambda}\sum_{m=0}^\infty\sum_{n=0}^\infty\sum_{\substack{\Gamma\in\Lambda_{\beta,\vec{k},c,m,n}\\ \Gamma\text{ is stable}}}\frac{1}{m!}\frac{1}{n!}(\widetilde{ev_\star})_*\left(\sum_{d=0}^\infty\epsilon_*\left(c_d(-R^\bullet\pi_*f^*L^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\right.\right.\\ &\cap[\mathcal{K}_{0,\overrightarrow{m+n}\cup\star}(\sqrt[r]{L/Y},\beta_\star)]^{\mathrm{vir}}\Big)\cap\prod_{i=1}^m\frac{ev_i^*(\frac{1}{\delta_i}i^*\mathbf{t}^{\vec{k_i}}J^{X,tw}_{\beta_i,\vec{k_i},c_i}(z)|_{z=\frac{\lambda-c_1(L)}{\delta_i}})}{-\frac{\lambda-ev_i^*c_1(L)}{r\delta_i}-\frac{\bar{\psi}_i}{r}}\\ &\cap\prod_{i=m+1}^{m+n}ev_i^*(\mathbf{t}^{\vec{k_i}}\mu^X_{\beta_i,\vec{k_i},c_i}(-\bar{\psi}_i))\cap\bar{\psi}_\star^b\Bigg).\end{aligned} \tag{4.8}$$

to 4.3.

Now our auxiliary cycle 4.3 is equal to the sum of 4.5, 4.6, 4.7 and 4.8. By the polynomiality of 4.3, the $\mathbf{t}^{\vec{k}}\lambda^{-2}-$coefficient of the auxiliary cycle is zero, which implies that the sum of $\mathbf{t}^{\vec{k}}\lambda^{-2}-$coefficient of 4.5, 4.6, 4.7 and 4.8 is zero, which proves the proposition 4.2. $\square$

4.4. **Statement of the main theorem and proof.** With the notations in this section, now we state our main theorem:

Theorem 4.6 (Main theorem)**.** *Let X be a smooth Deligne-Mumford stack with projective coarse moduli. Let $E := \oplus_{j=1}^{r} L_j$ be a direct sum of* semi-positive *line bundles over X with a regular section of E which cuts off a smooth complete intersection $Y \subset X$. Let $\mu^X(z) := \sum_{(\beta,\vec{k},c)\in \mathrm{Adm}} q^\beta \mathbf{t}^{\vec{k}} \mu^X_{\beta,\vec{k},c}(z)$ be an admissible series and $J^{X,tw}(q,\mu^X,z)$ be the hyper-geometric modification of $J^X(q,\mu^X,z)$ as in 1.3. Let $i: \bar{I}_\mu Y \to \bar{I}_\mu X$ be the inclusion, then the series $i^*J^{X,tw}(q,\mu^X,-z)$ is a point on the Lagrange cone of Y. More precisely, for any admissible pair $(\beta,\vec{k},c)$, if we define*

$$\mu^{X,tw}_{\beta,\vec{k},c}(z) := \Big[J^X_{\beta,\vec{k},c}(\mu^X,z)\prod_{j=1}^{r}\prod_{0\leqslant m<\beta(L_j)+(\vec{w_j},\vec{k})}\big(c_1(L_j)+(\beta(L_j)+(\vec{w_j},\vec{k})-m)z\big)\Big]_+$$

to be the truncation in nonnegative $z-$powers. Here $J^X_{\beta,\vec{k},c}$ is defined in 3.5. Write

$$\mu^{X,tw}(q,z) := \sum_{(\beta,\vec{k},c)\in\mathrm{Adm}} q^\beta \mathbf{t}^{\vec{k}} \mu^{X,tw}_{\beta,\vec{k},c}(z)\ .$$

Then for any integer $b\geqslant 0$, we have the following relation:

$$[J^Y_{\beta,\vec{k},c}(i^*\mu^{X,tw},z)]_{z^{-b-1}} = [i^*J^{X,tw}_{\beta,\vec{k},c}(\mu^X,z)]_{z^{-b-1}}\ . \tag{4.9}$$

As $\mu^{X,tw}$ is also an admissible series in $H^*(\bar{I}_\mu X,\mathbb{C})[z][\![t_1,\cdots,t_N]\!][\![\mathrm{Eff}(X)]\!]$, by induction on the number of line bundles, it's sufficient to prove the case when Y is a *hypersurface*. Now let's assume that Y is a hypersurface and use the notations in previous sections.

We will first prove 4.9 for any admissible pair $(\beta,\vec{k},c)$ with $\beta(L)+(\vec{w},\vec{k})=0$. Note in this case we have $\mu^{X,tw}_{\beta,\vec{k},c}(z)=\mu^X_{\beta,\vec{k},c}(z)$.

We will adopt the convention that for any substack Z of $\mathfrak{R}$ and $c\in C$, we will write $\bar{I}_c Z := \bar{I}_\mu Z \cap \bar{I}_{(c,1,1)}Z$. Now fix an admissible pair $(\beta,\vec{k},c)$ such that $\beta(L)+(\vec{w},\vec{k})=0$, we have $age_c L = 0$. Then $\bar{I}_c Y$ is proper hypersurface of $\bar{I}_c X$ with normal bundle L.[12] Recall that the divisor E of $\mathfrak{R}$ is isomorphic to $\mathbb{P}Y_{r,s}$, Denote by $\mathrm{pr}^E_{r,s}: E\to Y$ the projection to the base, which induces a morphism on the corresponding rigidified inertia stacks, which we still denote to be $\mathrm{pr}^E_{r,s}$. We have $\bar{I}_{c^{-1}}E \cong (\mathrm{pr}^E_{r,s})^{-1}(\bar{I}_{c^{-1}}Y)$ and $\bar{I}_{c^{-1}}E$ is a hypersurface of $\bar{I}_{c^{-1}}\mathfrak{R}$.

Denote by i_E the inclusion from $\bar{I}_{c^{-1}}E$ to $\bar{I}_{c^{-1}}\mathfrak{R}$. For any admissible pair $(\beta,\vec{k},c)$ with $\beta(L)+(\vec{w},\vec{k})=0$, consider the following class

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{\substack{(\beta_i,\vec{k_i},c_i)_{i=1}^m\in\mathrm{Adm}^m\\ \beta_\star+\sum_{i=1}^m\beta_i=\beta,\ \beta_\star\in\mathrm{Eff}(X)\\ \vec{k_1}+\cdots+\vec{k_m}=\vec{k}}}\frac{1}{m!}(\mathrm{pr}^E_{r,s})_*\circ i_E^*\circ(\widetilde{ev_\star})_*\\ &\Big(\mathcal{K}_{0,\vec{m}\cup\star}(\mathfrak{R},(\beta_\star,0,0))\cap\bar{\psi}_\star^b\cap\prod_{i=1}^m \mathrm{pr}^*_{r,s}(\mathbf{t}^{\vec{k_i}}\mu^X_{\beta_i,\vec{k_i},c_i}(-\bar{\psi}_i))\Big)\ .\end{aligned}\tag{4.10}$$

[12]Here the normal bundle is a pull-back of the line bundle L along the natural morphism from $\bar{I}_\mu X$ to X by sending (x,g) to x.

Here for any $\beta_\star, \beta_1, \cdots, \beta_m, \overrightarrow{k_1} \cdots, \overrightarrow{k_m}$ and $c_1, \cdots, c_m$ such that $(\beta_i, \overrightarrow{k_i}, c_i)_{i=1}^m \in \mathrm{Adm}^m$ with $\sum_{i=1}^m \beta_i + \beta_\star = \beta$ and $\overrightarrow{k_1} + \cdots + \overrightarrow{k_m} = \overrightarrow{k}$, we associated an element in $(C \times \mathbb{C}^* \times \mathbb{C}^*)^{m+1}$

$$\overrightarrow{m} \cup \star := \left((c_1^{-1}, 1, 1), \cdots, (c_m^{-1}, 1, 1,), (c, 1, 1)\right) .$$

i_E^* is the Gysin pull-back from the (equivariant) cohomology of $\bar{I}_{(c^{-1},1,1)}\mathfrak{R}$ to the (equivariant) cohomology of $\bar{I}_{(c^{-1},1,1)}E$. Note that $\beta_i(L) + (\vec{w}, \vec{k_i}) = 0$ for all $1 \leqslant i \leqslant m$,

Lemma 4.7. *For any admissible pair $(\beta, \overrightarrow{k}, c)$ with $\beta(L) + (\vec{w}, \vec{k}) = 0$, let b be an nonnegative integer, we have*

$$[J^Y_{\beta,\vec{k},c}(q, i^*\mu^X, z)]_{z^{-b-1}} = i^*[J^X_{\beta,\vec{k},c}(q, \mu^X, z)]_{z^{-b-1}} ,$$

where i^* is induced from the inclusion from $\bar{I}_\mu Y$ to $\bar{I}_\mu X$.

Proof. We will apply $\mathbb{C}^*-$localization to 4.10. First we claim there are only two types of localization graphs contributing to the integral (4.10): 1, the localization Γ_1 has only one vertex and it's labeled by X_0; 2, the localization graph Γ_2 has only one vertex and it's labeled by $\mathcal{D}_\infty$. Indeed, there is no vertex labeled by X_∞ as the marking $q_\star$ must go to the divisor E to make nonzero localization contribution to 4.10. Furthermore, as $deg(f^*\mathcal{O}(P_Y)) = 0$, we claim there is no edge linking X_0 and $\mathcal{D}_\infty$. Indeed, recall that we only need to consider edge curve C_e linking X_0 and $\mathcal{D}_\infty$ by Lemma 4.3 (see also Remark 4.4). then the claim comes from that the line bundle $\mathcal{O}(P_Y)$ is of nonnegative degree restricting to all vertex curves C_v and of positive degree restricting to all edge curves C_e. The localization contribution from Γ_1 is equal to

(4.11)

$$(\mathrm{pr}^E_{r,s})_* \circ i^*_E \circ (\iota_{X_0})_* \Bigg[\sum_{m=0}^\infty \sum_{\substack{(\beta_i,\overrightarrow{k_i},c_i)_{i=1}^m \in \mathrm{Adm}^m \\ \beta_\star + \sum_{i=1}^m \beta_i = \beta,\ \beta_\star \in \mathrm{Eff}(X) \\ \overrightarrow{k_1}+\cdots+\overrightarrow{k_m} = \overrightarrow{k}}} \frac{1}{m!} (\widetilde{ev}_\star)_* \Bigg(\sum_{d=0}^\infty \big(c_d(-R^\bullet\pi_* f^*(L^\vee)^{\frac{1}{s}})(\frac{\lambda}{s})^{-1-d}$$

$$\cap [\mathcal{K}_{0,\vec{m}\cup\star}(X_0, \beta_\star)]^{\mathrm{vir}}\big) \cap \prod_{i=1}^m ev_i^*(\mathrm{pr}^*_{r,s}\mathbf{t}^{\overrightarrow{k_i}} \mu^X_{\beta_i,\vec{k_i},c_i}(-\bar{\psi}_i)) \cap \bar{\psi}_\star^b \Bigg) \Bigg].$$

Here $\iota_{X_0} : \bar{I}_{c^{-1}} X_0 \to \bar{I}_{c^{-1}}\mathfrak{R}$ is the natural inclusion and $(\widetilde{ev}_\star)_*$ is a morphism from $H^*_{\mathbb{C}^*}(\mathcal{K}_{0,\vec{m}\cup\star}(X_0, \beta_\star))$ to $H^*_{\mathbb{C}^*}(\bar{I}_{c^{-1}}X_0)$ as defined in 2.1.

We have the following commutative diagram where the upper square and outer square are Cartesian:

$$\begin{tikzcd}
\bar{I}_{c^{-1}}\mathcal{D}_0 \arrow[r, "i_{\mathcal{D}_0}"] \arrow[d, "\iota_{\mathcal{D}_0}"] & \bar{I}_{c^{-1}}X_0 \arrow[d, "\iota_{X_0}"] \\
\bar{I}_{c^{-1}}E \arrow[r, "i_E"] \arrow[d, "\mathrm{pr}^E_{r,s}"] & \bar{I}_{c^{-1}}\mathfrak{R} \arrow[d, "\mathrm{pr}_{r,s}"] \\
\bar{I}_{c^{-1}}Y \arrow[r, "i"] & \bar{I}_{c^{-1}}X .
\end{tikzcd}$$

Here i is induced from the inclusion from $\mathcal{D}_0$ to X_0 and $\iota_{\mathcal{D}_0}$ is induced from the inclusion from $\mathcal{D}_0$ to E. As the age $age_c(L) = 0$, all rows are regular embeddings of codimension one. Then by the commutativity of the proper push-forward and Gysin pullback for the upper square, 4.11 is equal to

$$(\mathrm{pr}^E_{r,s} \circ \iota_{\mathcal{D}_0})_* i^*_E (\iota_{\mathcal{D}_0})_* [\cdots]. \tag{4.12}$$

Here the dots inside the big square bracket copies the stuff in the big square bracket of 4.11. Using the outer square, 4.12 is equal to

$$i_Y^*(\mathrm{pr}_{r,s}\circ\iota_{X_0})_*\big[\cdots\big]. \tag{4.13}$$

As the composition $\mathrm{pr}_{r,s}\circ\iota_{X_0}$ is induced from natural de-root stackfication of $X_0\cong\sqrt[s]{L^\vee/X}$, $(\mathrm{pr}_{r,s}\circ\iota_{X_0})_*$ is the identity map on cohomology groups as it induces an identity map on their coarse moduli spaces and the cohomology group of an orbifold is identified with the cohomology group of its coarse moduli space (see [AGV08] for more details). Therefore 4.13 is equal to

$$\begin{aligned} i^*\Bigg[\sum_{m=0}^{\infty}\sum_{\substack{(\beta_i,\vec{k_i},c_i)_{i=1}^m\in\mathrm{Adm}^m\\ \beta_\star+\sum_{i=1}^m\beta_i=\beta,\ \beta_\star\in\mathrm{Eff}(X)\\ \vec{k_1}+\cdots+\vec{k_m}=\vec{k}}}\frac{1}{m!}(\widetilde{ev_\star})_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*\big(c_d(-R^\bullet\pi_*f^*(L^\vee)^{\frac{1}{s}})(\frac{\lambda}{s})^{-1-d} \\ \cap[\mathcal{K}_{0,\vec{m}\cup\star}(X_0,\beta_\star)]^{\mathrm{vir}}\big)\cap\prod_{i=1}^m ev_i^*(\mathbf{t}^{\vec{k_i}}\mu^X_{\beta_i,\vec{k_i},c_i}(-\bar{\psi}_i))\cap\bar{\psi}_\star^b\Bigg)\Bigg], \end{aligned} \tag{4.14}$$

where $\epsilon:\mathcal{K}_{0,\vec{m}\cup\star}(X_0,\beta_\star)\to\mathcal{K}_{0,\vec{m}\cup\star}(X,\beta_\star)$ is induced from the natural structural map forgetting the root of $X_0\cong\sqrt[s]{L^\vee/X}$.

The localization contribution from Γ_2 is equal to

$$\begin{aligned} &\frac{1}{\lambda}(\mathrm{pr}^E_{r,s})_*\circ i_E^*\circ(i_{\mathcal{D}_\infty})_*\Bigg[\sum_{m=0}^{\infty}\sum_{\substack{(\beta_i,\vec{k_i},c_i)_{i=1}^m\in\mathrm{Adm}^m\\ \beta_\star\in\mathrm{Eff}(X),\ \beta_\star+\sum_{i=1}^m\beta_i=\beta\\ \vec{k_1}+\cdots+\vec{k_m}=\vec{k}}}\frac{1}{m!}(\widetilde{ev_\star})_*\Bigg(\sum_{d=0}^{\infty}\big(c_d(-R^\bullet\pi_*f^*L^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1-d} \\ &\cap[\mathcal{K}_{0,\vec{m}\cup\star}(\sqrt[r]{L/Y},\beta_\star)]^{\mathrm{vir}}\big)\cap\prod_{i=1}^m ev_i^*(\mathrm{pr}^*_{r,s}\mathbf{t}^{\vec{k_i}}\mu^X_{\beta_i,\vec{k_i},c_i}(-\bar{\psi}_i))\cap\bar{\psi}_\star^b\Bigg)\Bigg] \end{aligned} \tag{4.15}$$

Here $i_{\mathcal{D}_\infty}:\bar{I}_{c^{-1}}\mathcal{D}_\infty\to\bar{I}_{c^{-1}}\mathfrak{R}$ is the inclusion and $(\widetilde{ev_\star})_*$ is the morphism from $H^*(\mathcal{K}_{0,\vec{m}\cup\star}(\mathcal{D}_\infty,\beta_\star))$ to $H^*(\bar{I}_{c^{-1}}\mathcal{D}_\infty)$ as defined in 2.1. The inclusion $i_{\mathcal{D}_\infty}$ can be rewritten as the composition of two inclusions $\iota_{\mathcal{D}_\infty}:\bar{I}_{c^{-1}}\mathcal{D}_\infty\hookrightarrow\bar{I}_{c^{-1}}E$ and $i_E:\bar{I}_{c^{-1}}E\hookrightarrow\bar{I}_{c^{-1}}\mathfrak{R}$. Then we have

$$\begin{aligned} &\frac{1}{\lambda}(\mathrm{pr}^E_{r,s})_*\circ i_E^*\circ(i_{\mathcal{D}_\infty})_*\Big[\cdots\Big] \\ &=\frac{1}{\lambda}(\mathrm{pr}^E_{r,s})_*\circ i_E^*\circ(i_E)_*\circ(\iota_{\mathcal{D}_\infty})_*\Big[\cdots\Big] \\ &=(\mathrm{pr}^E_{r,s})_*(\iota_{\mathcal{D}_\infty})_*\Big[\cdots\Big] \\ &=\Big[\cdots\Big]. \end{aligned} \tag{4.16}$$

Here the dots inside the big square bracket copies the stuff in the big square bracket of 4.15. The second equality in 4.16 uses the fact that the morphism $i_E^*(i_E)_*:H^*_{\mathbb{C}^*}(\bar{I}_{c^{-1}}E)\to H^*_{\mathbb{C}^*}(\bar{I}_{c^{-1}}E)$ is equal to the multiplication map by $e^{\mathbb{C}^*}(N_{\bar{I}_{c^{-1}}Y/\bar{I}_{c^{-1}}\mathfrak{R}})$ using excess intersection formula, which implies that $i_E^*(i_E)_*$ acts on the space $(\iota_{\mathcal{D}_\infty})_*H^*_{\mathbb{C}^*}(\mathcal{D}_\infty)$ by multiplication by λ

as $e^{\mathbb{C}^*}(N_{\bar{I}_{c^{-1}}Y/\bar{I}_{c^{-1}}\mathfrak{R}})|_{\bar{I}_{c^{-1}}\mathcal{D}_\infty} = \lambda$. Finally 4.15 becomes

$$\begin{aligned}(4.17)\quad &\sum_{m=0}^{\infty} \sum_{\substack{(\beta_i,\vec{k}_i,c_i)_{i=1}^m\in \mathrm{Adm}^m, \beta_\star\in \mathrm{Eff}(X)\\ \beta_\star+\sum_{i=1}^m\beta_i=\beta\\ \vec{k}_1+\cdots+\vec{k}_m=\vec{k}}} \frac{1}{m!}(\widetilde{ev}_\star)_*\Big(\sum_{d=0}^{\infty}\epsilon_*\big(c_d(-R^\bullet\pi_*f^*L^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1-d}\\ &\cap[\mathcal{K}_{0,\vec{m}\cup\star}(\sqrt[r]{L/Y},\beta_\star)]^{\mathrm{vir}}\big)\cap\prod_{i=1}^m ev_i^*(i^*\mathbf{t}^{\vec{k}_i}\mu^X_{\beta_i,\vec{k}_i,c_i}(-\bar{\psi}_i))\cap\bar{\psi}_\star^b\Big)\end{aligned}$$

using the morphism $\epsilon : \mathcal{K}_{0,\vec{m}\cup\star}(\mathcal{D}_\infty,\beta_\star)\to\mathcal{K}_{0,\vec{m}\cup\star}(Y,\beta_\star)$ induced from forgetting the root of $\mathcal{D}_\infty\cong\sqrt[s]{L/Y}$.

Now taking λ^{-1} coefficients of the contributions 4.14 and 4.17 from Γ_1 and Γ_2, their sum is zero by polynomiality of 4.10, this finishes the proof of (4.9) with the help of the fact

$$\epsilon_*\big([\mathcal{K}_{0,\vec{m}\cup\{\star\}}(\sqrt[s]{L^\vee/X},\beta_\star)]^{\mathrm{vir}}\big)=\frac{1}{s}[\mathcal{K}_{0,\vec{m}\cup\{\star\}}(X,\beta_\star)]^{\mathrm{vir}}$$

$$\epsilon_*\big([\mathcal{K}_{0,\vec{m}\cup\{\star\}}(\sqrt[r]{L/Y},\beta_\star)]^{\mathrm{vir}}\big)=\frac{1}{r}[\mathcal{K}_{0,\vec{m}\cup\{\star\}}(Y,\beta_\star)]^{\mathrm{vir}}$$

proved in [TT16]. □

Now we are well prepared to prove the main theorem 4.6:

Proof of the main theorem 4.6. Since X has projective coarse moduli, there exists a *positive* line bundle M on X, i.e., $\beta(M)>0$ for all nonzero degree $\beta\in\mathrm{Eff}(X)$. Let's prove the relation 4.9 by induction on the nonnegative rational number $e:=\beta(M)+|\vec{k}|$.[13] When $e=0$, we have $(\beta,\vec{k})=(0,\vec{0})$, then $J^Y_{0,\vec{0},c}(i^*\mu^{X,tw},z)=J^{X,tw}_{0,\vec{0},c}(\mu^X,z)=0$. Now suppose the relation 4.9 holds for all admissible pair $(\beta',\vec{k'},c')$ with $\beta'(M)+|\vec{k'}|<e$ for some positive rational number e, it remains to show the relation holds for any admissible pair $(\beta,\vec{k},c)$ with $\beta(M)+|\vec{k}|=e$. If $\beta(L)+(\vec{w},\vec{k})=0$, by Proposition 4.7, the relation holds (note in this case, we have $J^{X,tw}_{\beta,\vec{k},c}(\mu,z)=J^X_{\beta,\vec{k},c}(\mu,z)$). Otherwise, we use the recursive relations 4.4 and 3.11; in which the terms corresponding to the element $\Gamma\in\Lambda_{\beta,\vec{k},c,m,n}$ with $\beta_\star\neq0$ are the same as their inputs are associated with the extended degrees $(\beta_i,\vec{k_i})$ with $\beta_i(M)+|\vec{k_i}|<e$, which are the same by induction assumption. We are left to show that the terms in 4.4 and 3.11 corresponding to the element $\Gamma\in\Lambda_{\beta,\vec{k},c,m,n}$ with $\beta_\star=0$ are the same. Note $m+n\geqslant2$ as Γ is stable and $J^Y_{0,\vec{0},c}(i^*\mu^{X,tw},z)=J^{X,tw}_{0,\vec{0},c}(\mu^X,z)=0$, then we can still use inductive assumption. This completes the proof. □

Remark 4.8. The above proof can also be extended to the equivariant case, where we assume that X has an algebraic torus $T-$action which preserves Y. Since we don't have a particular application in mind, we will leave the details to the interested readers.

Let $\{\phi_\alpha\}$ be a *graded* basis of $H^*(\bar{I}_\mu X,\mathbb{C})$ such that $\phi_\alpha\in H^*(\bar{I}_cX)$ for some $c\in C$. Let $\{t_\alpha\}$ be a coordinate system of $H^*(\bar{I}_\mu X,\mathbb{C})$ corresponding to the basis $\{\phi_\alpha\}$. We also put $\mathbb{Z}_2$-grading on $\{t_\alpha\}$ according to ϕ_α. Then we have the following:

Corollary 4.9. *Let $i:\bar{I}_\mu Y\to\bar{I}_\mu X$ be the inclusion of rigidified inertia stacks and $\mathbf{t}=\sum_\alpha t_\alpha\phi_\alpha$. Then the series $i^*J^{X,tw}(q,\mathbf{t},-z)$ lies on the Lagrangian cone $\mathcal{L}_Y$ of Y.*

[13] Here $|\vec{k}|=\sum_i k_i$.

Appendix A. Edge contribution

In the proof of our main theorem, we need to apply torus localization formula to the moduli of stable maps to $\mathbb{P}Y_{r,s}$ (or to $\mathfrak{R}$ which factors through E), which is an orbi-$\mathbb{P}^1$ bundle over a stack, for which we don't find a suitable reference of discussing the explicit localization contribution from the torus-fixed 1-dimensional orbits (named edge contribution). In this appendix, we will explain how to compute the edge contributions in 3.2.2 and 4.2.2. To achieve this, we will construct a space called $\mathcal{M}_e$ with a family of $\mathbb{C}^*-$fixed stable maps to $\mathbb{P}Y_{r,s}$ corresponding to a decorated edge. This space has the property that it allows a finite étale map to the corresponding substack of $\mathbb{C}^*-$fixed loci in $\mathcal{K}_{0,2}(\mathbb{P}Y_{r,s},(0,\frac{\delta}{r}))$. Then we can use $\mathcal{M}_e$ as a substitute for $\mathbb{C}^*-$fixed loci in the edge contribution and the explicit description of the family map helps us carry out the localization analysis concretely. To begin with, we will prove a classification result (see Lemma A.1) about $\mathbb{C}^*-$fixed stable maps to $\mathbb{P}Y_{r,s}$ which map to a fiber of $\mathbb{P}Y_{r,s}$ over Y using the relation between fundamental groups and $\mathbb{C}^*-$fixed stable maps to an orbi-$\mathbb{P}^1$ curve from [LS20].

A.1. **Local picture.** Write $\mathfrak{P} := \mathbb{P}Y_{r,s}$ for simplicity. For any $\mathbb{C}-$point $y \in Y$, let G_y be the residual isotropy group of y in Y. Let $\rho : G_y \to \mathbb{C}^* = Aut(L|_y)$ be the associated representation. Denote $\mathfrak{P}_y$ by the fiber product in the following square

$$\begin{tikzcd} \mathfrak{P}_y \arrow[r] \arrow[d] & \mathfrak{P} \arrow[d, "\mathrm{pr}_{r,s}"] \\ \mathbb{B}G_y \arrow[r, "i"] & Y \end{tikzcd},$$

where $i : \mathbb{B}G_y \to Y$ is the inclusion. Let $U := \mathbb{C}^2\backslash\{0\}$, the fiber curve $\mathfrak{P}_y$ can be represented as the quotient stack

$$[\mathbb{C}^* \times U/(G_y \times (\mathbb{C}^*)^2)]$$

via the action

$$(m, x_1, x_2)(g, t_1, t_2) = (\rho^{-1}(g)t_1^{-s}t_2^{r}m, t_1x_1, t_2x_2)\ ,$$

where $(m, x_1, x_2) \in \mathbb{C}^* \times U$, $(g, t_1, t_2) \in G_y \times (\mathbb{C}^*)^2$.

The $\mathbb{C}^*-$action on $\mathfrak{P}$ defined in 3.1 restricts to be a $\mathbb{C}^*-$action on $\mathfrak{P}_y$ by scaling x_1 with weight one. There are two $\mathbb{C}^*-$fixed points of $\mathfrak{P}_y$: (1) The point 0 corresponding to $x_1 = 0$; (2) the point ∞ corresponding to $x_2 = 0$. Now let's describe the isotropy groups of points 0 and ∞, for which we need the following definition. Let G be an arbitrary finite group and η be a character of G. Assume that image of η is a finite cyclic group $\boldsymbol{\mu_t} \subset \mathbb{C}^*$ of order t. In other words, we have the following exact sequence from the action

$$1 \longrightarrow \ker(\eta) \longrightarrow G \xrightarrow{\eta} \boldsymbol{\mu}_t \longrightarrow 1\ .$$

For any positive integer i, we define the central extension $G(\eta, i)$ of G by the cyclic group $\boldsymbol{\mu_i}$:

$$G(\eta, i) := \{(g, b) \in G \times \mathbb{C}^* | \eta(g) = b^i\}\ .$$

We note $G(\eta, i)$ fits in the following Cartesian digram with all rows and all columns exact:

$$\begin{tikzcd}
 & & \boldsymbol{\mu}_i \arrow[r, "="] \arrow[d, hook] & \boldsymbol{\mu}_i \arrow[d, hook] & \\
1 \arrow[r] & \ker(\eta) \arrow[r] \arrow[d, "="] & G(\eta, i) \arrow[r, "\eta_i"] \arrow[d] & \boldsymbol{\mu}_{ti} \arrow[r] \arrow[d, "a \mapsto a^i"] & 1 \\
1 \arrow[r] & \ker(\eta) \arrow[r] & G \arrow[r, "\eta"] & \boldsymbol{\mu}_t \arrow[r] & 1 ,
\end{tikzcd}$$

where the morphism η_i is induced from the projection from $G \times \mathbb{C}^*$ to the second factor. Then the isotropy group G_0(resp. G_∞) of 0(resp. ∞) is isomorphic to $G_y(\rho^{-1}, s)$(resp. $G_y(\rho, r)$). Denote

$$\mathfrak{U}_0 := \mathfrak{P}_y \backslash \{\infty\} \cong [\mathbb{C}/G_y(\rho^{-1}, s)], \text{ and } \mathfrak{U}_\infty := \mathfrak{P}_y \backslash \{0\} \cong [\mathbb{C}/G_y(\rho, r)] \ ,$$

and

$$\mathfrak{o}_e := \mathfrak{P}_y \backslash \{0, \infty\} \ .$$

Now we quote the result from [LS20, §6](see also [NB06]). Assume that the image of ρ is $\boldsymbol{\mu}_t$. Then the curve $\mathfrak{P}_y$ is a $G_e := \mathrm{Ker}(\rho)$-gerbe over $\mathbb{P}^1_{tr,ts}$. More explicitly, note $\mathfrak{o}_e$ is isomorphic to the quotient stack $[\mathbb{C}^*/G_y]$[14] then the morphism $\rho : G_y \to \boldsymbol{\mu}_t$ induces that $\mathfrak{o}_e$ is a G_e-gerbe over its coarse moduli $o_e \cong [\mathbb{C}^*/\boldsymbol{\mu}_t] = \mathbb{C}^*$. Then the morphism ρ_s^{-1} induces a morphism from $\mathfrak{U}_0 \cong [\mathbb{C}/G_y(\rho^{-1}, s)]$ (resp.$\mathfrak{U}_\infty \cong [\mathbb{C}/G_y(\rho, r)]$) to $[\mathbb{C}/\boldsymbol{\mu}_{ts}]$(resp. $[\mathbb{C}/\boldsymbol{\mu}_{tr}]$). The above discussion fits into the following commutative digram

$$\begin{tikzcd}
\mathfrak{U}_0 \arrow[r] & {[\mathbb{C}/\boldsymbol{\mu}_{ts}]} \\
\mathfrak{o}_e \arrow[r] \arrow[u] \arrow[d] & o_e \cong [\mathbb{C}^*/\boldsymbol{\mu}_{ts}] \cong [\mathbb{C}^*/\boldsymbol{\mu}_{t}] \cong [\mathbb{C}^*/\boldsymbol{\mu}_{tr}] \arrow[u] \arrow[d] \\
\mathfrak{U}_\infty \arrow[r] & {[\mathbb{C}/\boldsymbol{\mu}_{tr}]} \ .
\end{tikzcd}$$

Let $\mathfrak{p}_e \cong \mathbb{B}G_e$ be a point of $\mathfrak{o}_e$. Define

$$H_e := \pi_1(\mathfrak{o}_e, \mathfrak{p}_e) \ ,$$

then the coarsification map $\mathfrak{o}_e \to o_e$ (with the corresponding base points $\mathfrak{p}_e \to p_e$) induces a surjection

$$\phi_e : H_e = \pi_1(\mathfrak{o}_e, \mathfrak{p}_e) \to \pi_1(o_e, p_e) \cong \mathbb{Z} \ ,$$

whose kernel is isomorphic to G_e as shown in [LS20, §6].

Choose a path from $\mathfrak{p}_e$ to 0 and a path from $\mathfrak{p}_e$ to ∞, the open embeddings $\mathfrak{o}_e \hookrightarrow \mathfrak{U}_0$ and $\mathfrak{o}_e \hookrightarrow \mathfrak{U}_\infty$ induce surjective group homomorphisms

$$H_e = \pi_1(\mathfrak{o}_e, \mathfrak{p}_e) \xrightarrow{\pi_{e,0}} \pi_1(\mathfrak{U}_0, 0) \cong G_y(\rho^{-1}, s), \quad H_e = \pi_1(\mathfrak{o}_e, \mathfrak{p}_e) \xrightarrow{\pi_{e,\infty}} \pi_1(\mathfrak{U}_\infty, \infty) \cong G_y(\rho, r) \ .$$

[14]One can check that $\mathfrak{o}_e$ can be also represented as the quotient stack $[\mathbb{C}^*/G_y(\rho^{-1}, s)]$ or $[\mathbb{C}^*/G_y(\rho, r)]$ via ρ_s^{-1} (resp. ρ_r).

Applying π_1, the above discussion will be summarized into the following commutative diagram with all rows exact:

(A.1)
$$\begin{tikzcd}
1 \arrow[r] & G_e \arrow[r] \arrow[d, equal] & \pi_1(\mathfrak{U}_0) \cong G_y(\rho^{-1}, s) \arrow[r, "\rho_s^{-1}"] & \boldsymbol{\mu}_{ts} \arrow[r] & 1 \\
1 \arrow[r] & G_e \arrow[r] \arrow[d, equal] & \pi_1(\mathfrak{o}_e) \cong H_e \arrow[r, "\phi_e"] \arrow[u, "\pi_{e,0}"] \arrow[d, "\pi_{e,\infty}"'] & \mathbb{Z} = \pi_1(o_e) \arrow[r] \arrow[u] \arrow[d] & 1 \\
1 \arrow[r] & G_e \arrow[r] & \pi_1(\mathfrak{U}_\infty) \cong G_y(\rho, r) \arrow[r, "\rho_r"] & \boldsymbol{\mu}_{tr} \arrow[r] & 1 \ ,
\end{tikzcd}$$

where in the third column $\mathbb{Z} \to \boldsymbol{\mu}_{tr}$ and $\mathbb{Z} \to \boldsymbol{\mu}_{ts}$ are given by $d \mapsto e^{\frac{d}{rt}}$ and $d \mapsto e^{\frac{d}{st}}$ respectively. By chasing diagram, we can show that H_e is isomorphic to the fiber product $G_y(\rho^{-1}, s) \times_{\boldsymbol{\mu}_{ts}} \mathbb{Z}$ (or $G_y(\rho, r) \times_{\boldsymbol{\mu}_{tr}} \mathbb{Z}$ as well) using the above diagram. Furthermore using the fact $G_y(\rho^{-1}, s)$ isomorphic to the fiber product $G_y \times_{\boldsymbol{\mu}_t} \boldsymbol{\mu}_{ts}$, we have H_e is also isomorphic to the fiber product $G_y \times_{\boldsymbol{\mu}_t} \mathbb{Z}$.

Let $\mathbb{P}^1_{m,n}$ be the unique smooth DM stack with coarse moduli $\mathbb{P}^1$ and trivial generic stablizer and only two special points $q_\infty \cong \mathbb{B}\boldsymbol{\mu}_m$ and $q_0 \cong \mathbb{B}\boldsymbol{\mu}_n$. Here we can choose a $\mathbb{C}^*-$action on $\mathbb{P}^1_{m,n}$ such that q_0, q_∞ are the only $\mathbb{C}^*-$fixed points and the $\mathbb{C}^*-$weight of the normal space N_{q_0}(resp. N_{q_∞}) is $\frac{1}{m}$ (resp. $\frac{-1}{n}$). Let $f : \mathbb{P}^1_{m,n} \to \mathfrak{P}$ be a $\mathbb{C}^*-$fixed stable map of degree $(0, \frac{\delta}{r})$ which factors through $\mathfrak{P}_y$. Assume that $f(q_0) = 0 \in \mathfrak{P}_y$ and $f(q_\infty) = \infty \in \mathfrak{P}_y$. Moreover, if the multiplicity[15] at q_0 is $([g^{-1}], \mathrm{e}^{\frac{\delta}{s}}, 1) \in \mathrm{Conj}(G_y) \times \mathbb{C}^* \times \mathbb{C}^*$ such that $(x, [g]) \in \bar{I}_c Y$ where $g \in G_y$, then the multiplicity at q_∞ is $([g], 1, \mathrm{e}^{\frac{\delta}{r}}) \in \mathrm{Conj}(G_y) \times \mathbb{C}^* \times \mathbb{C}^*$, and vice versa. Let a be the order of g, when r, s are sufficiently large primes, as f is representable, we have $m = ar, n = as$. Assume that $age_g(L|_y) = \frac{h}{a}$ for some integer h with $0 \leqslant h < a$. As $\delta = r \cdot deg(f^*\mathcal{O}(\mathcal{D}_\infty))$, using orbifold Riemann-Roch for the $f^*\mathcal{O}(r\mathcal{D}_\infty)$, the condition

$$\delta - \frac{h}{a} \in \mathbb{Z} \ ,$$

is a necessary condition to ensure the map f to exist. Then $a\delta$ must be an integer.

Let $T_{ar,as}$ be the subgroup of $(\mathbb{C}^*)^2$ defined by the equation $t_1^{as} = t_2^{ar}$. Let $F : U \to \mathbb{C}^* \times U$ be the morphism sending (x, y) to $(1, x^{a\delta}, y^{a\delta})$. Then F is equivariant with respect to the group homomorphism from $T_{ar,as}$ to $G_y \times (\mathbb{C}^*)^2$:

$$(t_1, t_2) \mapsto (\tau(t_1^{-s} t_2^r), t_1^{a\delta}, t_2^{a\delta}) \ ,$$

where $\tau : \boldsymbol{\mu}_a \to G_y$ is the group morphism sending μ_a to g. We can check F descents to be a morphism $\widetilde{F}$ from $\mathbb{P}^1_{ar,as}$ to $\mathfrak{P}_y$ of degree $(0, \frac{\delta}{r})$[16] with the multiplicity decoration $([g], 1, \mathrm{e}^{\frac{\delta}{r}})$ at q_∞. Moreover, if we change g to any conjugate of g in G_y, the descended morphisms are all isomorphic to each other. Conversely, we will show the following.

Lemma A.1. *With the notations as above, let f be a $\mathbb{C}^*-$fixed morphism of degree $(0, \frac{\delta}{r})$ to $\mathfrak{P}$ factor through $\mathfrak{P}_y$ such that the multiplicity is given by $([g], 1, \mathrm{e}^{\frac{\delta}{s}})$ at q_∞. Then the morphism f must be isomorphic to $\widetilde{F}$ (up to a unique $2-$isomorphism) constructed as above.*

Proof. Let f_1, f_2 be two stable maps with same associated decorated graph as above, if we assume they are isomorphic when restricting to $O_e := \mathbb{P}^1_{ar,as} \backslash \{q_0, q_\infty\}$, then f_1 is isomorphic to f_2 up to a unique $2-$isomorphism(see e.g.,[CCFK15, Lemma 2.5] for a proof). Thus we only need to show the restriction $f|_{\mathcal{O}_e}$ is uniquely determined by the degree data and multiplicity

[15]Here we can replace the index set C for $\bar{I}_\mu \mathfrak{P}_y$ by the set $\mathrm{Conj}(G_y)$ of conjugacy classes of G_y.

[16]We use the degree notation by viewing $\widetilde{F}$ as a morphism to $\mathfrak{P}$.

data at q_∞. Note that $f|_{O_e} : O_e \to \mathfrak{o}_e$ is an étale covering map. Indeed, as O_e and the coarse moduli o_e of $\mathfrak{o}_e$ are both isomorphic to $\mathbb{C}^*$ and the coarsing map from $\mathfrak{o}_e$ to o_e is étale, then the étaleness of $f|_{O_e}$ follows from the étaleness of the composition

$$O_e \to \mathfrak{o}_e \to o_e$$

as any $\mathbb{C}^*-$fixed morphism from $\mathbb{C}^*$ to $\mathbb{C}^*$ must be a cyclic covering. By [Noo05, Theorem 18.19], we have a bijection between conjugacy classes of subgroup $(f|_{O_e})_*(\pi_1(O_e, P_e))$ (P_e is the inverse image of the point $\mathfrak{p}_e$ under $O_e \to \mathfrak{o}_e$) of $\pi_1(\mathfrak{o}_e, \mathfrak{p}_e)$ and isomorphism classes of covering map of $\mathfrak{o}_e$ (not necessarily base point preserving), it's enough to show the conjugacy class of subgroup $(f|_{O_e})_*(\pi_1(O_e, P_e))$ is uniquely determined by the degree data and the monodromy data at q_∞.

We will drop the base points in the notation of fundmental group in the sequel if it's clear in the context. Let

$$U_0 := \mathbb{P}^1_{ar,as}\backslash\{q_\infty\} \cong [\mathbb{C}/\boldsymbol{\mu}_{as}], \quad \text{and} \quad U_\infty := \mathbb{P}^1_{ar,as}\backslash\{q_0\} \cong [\mathbb{C}/\boldsymbol{\mu}_{ar}] \ .$$

Choose a path from P_e to q_0 in U_0 and a path from P_e to q_∞ in U_∞, then the open embeddings $O_e \hookrightarrow U_0$ and $O_e \hookrightarrow U_\infty$ induces surjective group homomorphisms

$$\mathbb{Z} = \pi_1(O_e) \xrightarrow{\pi_{q_0}} \pi_1(U_0) \cong \boldsymbol{\mu}_{\boldsymbol{as}} \qquad \mathbb{Z} = \pi_1(O_e) \xrightarrow{\pi_{q_\infty}} \pi_1(U_\infty) \cong \boldsymbol{\mu}_{\boldsymbol{ar}} \ .$$

Here we choose the identification between $\pi_1(O_e)$ and $\mathbb{Z}$ such that $\pi_{q_0}(1) = e^{\frac{-1}{as}}$ and $\pi_{q_\infty}(1) = e^{\frac{1}{ar}}$.[17] Then the morphism

$$\pi_1(O_e) \longrightarrow \pi_1(U_\infty) \xrightarrow{(f|_{U_\infty})_*} \pi_*(\mathfrak{U}_\infty)$$

sends the generator 1 of $\pi_1(O_e)$ to the conjugacy class of $(g, e^{\frac{\delta}{r}}) \in G_y(\rho, r) \cong \pi_1(\mathfrak{U}_\infty)$ by the multiplicity data at q_∞. We have a commutative digram with all vertical lines are inclusions:

$$\begin{tikzcd}
O_e \arrow[r, "f|_{O_e}"] \arrow[d] & \mathfrak{o}_e \arrow[r] \arrow[d] & o_e \arrow[d] \\
U_\infty \arrow[r] & \mathfrak{U}_\infty \arrow[r] & {[\mathbb{C}/\boldsymbol{\mu}_{tr}]} \ .
\end{tikzcd}$$

Using A.1, the above diagram induces the following commutative diagram

(A.2)
$$\begin{tikzcd}
\mathbb{Z} = \pi_1(O_e) \arrow[r, "(f|_{O_e})_*"] \arrow[d] & \pi_1(\mathfrak{o}_e) \cong H_e \arrow[r, two heads, "\phi_e"] \arrow[d, "\pi_{e,\infty}"] & \pi_1(o_e) \cong \mathbb{Z} \arrow[d] \\
\boldsymbol{\mu}_{ar} = \pi_1(U_\infty) \arrow[r] & \pi_1(\mathfrak{U}_\infty) \cong G_y(\rho, r) \arrow[r, two heads, "\rho_r"] & \boldsymbol{\mu}_{tr} \ ,
\end{tikzcd}$$

where in the third column $\mathbb{Z} \to \boldsymbol{\mu}_{tr}$ is given by $d \mapsto \mathrm{e}^{\frac{d}{tr}}$. Then we see that the composition

$$\pi_1(O_e) \cong \mathbb{Z} \xrightarrow{(f|_{O_e})_*} \pi_1(\mathfrak{o}_e) = H_e \xrightarrow{\phi_e} \pi_1(o_e) \cong \mathbb{Z} \ .$$

from $\mathbb{Z}$ to $\mathbb{Z}$ is multiplication by $t\delta$ as the generator 1 of $\pi_1(O_e)$ sends to $e^{\frac{\delta}{r}}$ in the group $\boldsymbol{\mu}_{tr}$ in whatever path from the up-left corner to the bottom-right corner in the above commutative diagram and r is sufficiently large.

Let $\gamma_e \in H_e$ be the image of generator $1 \in \pi_1(O_e)$ in H_e. By the previous discussion, γ_e maps to the conjugacy class of $(g, e^{\frac{\delta}{r}})$ via $\pi_{e,\infty}$, and maps to $t\delta$ in $\pi_1(o_e)$. Conversely, using the fact H_e is isomorphic to fiber product $G_y(\rho, r) \times_{\boldsymbol{\mu}_{tr}} \mathbb{Z}$ using the right square of A.2 , we

[17] Different choices of path from P_e to q_0(resp. q_∞) will only affect π_{q_0}(resp. π_{q_∞}) up to conjugacy, then the image $\pi_{q_0}(1)$ (resp. $\pi_{q_\infty}(1)$) doesn't depend on the choice of the path.

see that the conjugacy class of γ_e, is uniquely determinded by the conjugacy class of the image of γ_e in $\pi_1(\mathfrak{U}_\infty)$ and the image of γ_e in $\pi_1(o_e)$, which are determined by the decorated data. Hence the conjugacy class of the subgroup $(f|_{O_e})_*(\pi_1(O_e))$ is also uniquely determined by the decorated data. This completes the proof. $\square$

Next we describe the automorphism group $Aut(f)$ of f.

Proposition A.2. *Use the notation in the above proof, we have $Aut(f) \cong C_{H_e}(\gamma_e)/\langle\gamma_e\rangle$, and the order of automorphism group is equal to $|Aut(f)| = |C_{H_e}(\gamma_e)/\langle\gamma_e\rangle| = \delta|C_{G_y}(g)|$. Here $C_{H_e}(\gamma)$ (resp. $C_{G_y}(g)$) is the centralizer of γ_e(resp. g) in the group H_e(resp. G_y).*

Proof. The automorphism group of f is isomorphic to the automorphism group of the restriction $f|_{O_e}$, then the first claim follows from standard fact about covering deck transformation(not necessarily basepoint preserving) as γ_e generates the subgroup $(f|_{O_e})_*(\pi_1(O_e))$ in $\pi_1(\mathfrak{o}_e)$.

Assume that the image $\phi_e(C_{H_e}(\gamma_e))$ in $\mathbb{Z}$ is generated by the positive integer u, as H_e is isomorphic to the fiber product $G_y \times_{\boldsymbol{\mu}_t} \mathbb{Z}$.[18] More precisely,

$$H_e \cong \{(g, d) \in G_y \times \mathbb{Z} | \rho(g) = \mathrm{e}^{\frac{d}{t}}\} \ .$$

Then $\gamma_e = (g, t\delta) \in H_e$. We can show that t and $t\delta$ are both divided by u. Thus the image of $C_{G_y}(g)$ under ρ is generated by $\mathrm{e}^{\frac{u}{t}}$.

The group $C_{H_e}(\gamma_e)$ can be represented as

$$\{(h, d) \in C_{G_y}(g) \times \mathbb{Z} | \rho(h) = \mathrm{e}^{\frac{d}{t}}\}$$

in H_e. Then the claim about $|Aut(f)|$ follows from the following two exact sequences:

$$1 \longrightarrow G_e \cap C_{G_y}(g) \xrightarrow{\theta} C_{H_e}(\gamma_e)/\langle\gamma_e\rangle \longrightarrow u\mathbb{Z}/t\delta\mathbb{Z} \longrightarrow 1 \ ,$$

and

$$1 \longrightarrow G_e \cap C_{G_y}(g) \longrightarrow C_{G_y}(g) \xrightarrow{\rho} \boldsymbol{\mu}_{\frac{t}{u}} \longrightarrow 1 \ .$$

Here $u\mathbb{Z}$(resp. $t\delta\mathbb{Z}$) is the subgroup of $\mathbb{Z}$ generated by the integer u(resp. $t\delta$), the morphism θ is the induced from the restriction to $G_e \cap C_{G_y}(g)$ of the inclusion from G_e to H_e. $\square$

We can make the above proposition more concretely. Let $(h, d) \in C_{H_e}(\gamma_e)$ be an element, we associate (h, d) an element in the automorphism group $Aut(f)$ as follows: define

$$\theta_{h,d} : U \to U : \quad (x, y) \mapsto (x, e^{\frac{d}{art\delta}} y) \ ,$$

which descents to be an isomorphism of $\mathbb{P}^1_{ar,as}$ which we still denote to be $\theta_{h,d}$. Recall that F is a morphism from $U \to \mathbb{C}^* \times U$ defined as in Lemma A.1, then $F \circ \theta_{h,d}$ differ from F by the action of the element $(h, 1, e^{\frac{d}{rt}})$ on the target via the action of $G_y \times (\mathbb{C}^*)^2$ on $\mathbb{C}^* \times U$ which also commutes the action of $T_{ar,as}$ and $G_y \times (\mathbb{C}^*)^2$ on the source and target respectively. Then by descent, this defines an $2-$isomorphism $\alpha_{h,d} : F \to F \circ \theta_{h,d}$. Thus the pair $(\theta_{h,d}, \alpha_{h,d})$ defines an automorphism in $Aut(f)$.

For two elements (h_1, d_1) and (h_2, d_2) in $C_{H_2}(\gamma_e)$, the pairs $(\theta_{h_1,d_1}, \alpha_{h_1,d_1})$ and $(\theta_{h_2,d_2}, \alpha_{h_2,d_2})$ are isomorphic to each other if there exists some integer n such that $h_2 = h_1 g^n$ and $d_2 = d_1 + nt\delta$. Indeed, let $m_{(1, e^{\frac{nt\delta}{art\delta}})}$ be the automorphism of U acted upon by $(1, e^{\frac{nt\delta}{art\delta}})$, we have $\theta_{h_2,d_2} = m_{(1, e^{\frac{nt\delta}{art\delta}})} \circ \theta_{h_1,d_1}$, which defines a two-isomorphism $\beta_n : \theta_{h_1,d_1} \to \theta_{h_2,d_2}$ in $\mathrm{Hom}(\mathbb{P}^1_{ar,as}, \mathbb{P}^1_{ar,as})$, then the composition $f \circ \beta_n$ defines a two-isomorphism from $F \circ \theta_{h_1,d_1}$ to $F \circ \theta_{h_2,d_2}$ such that $\alpha_{h_2,d_2} = f \circ \beta_n \circ \alpha_{h_1,d_1}$.

[18]We have already seen that H_e is isomorphic to $G_y(\rho^{-1}, s) \times_{\boldsymbol{\mu}_{st}} \mathbb{Z}$, then we use the isomorphism $G_y(\rho^{-1}, s) \cong G_y \times_{\boldsymbol{\mu}_t} \boldsymbol{\mu}_{st}$

The morphism $f : \mathbb{P}^1_{ar,as} \to \mathfrak{P}_y$ defines a $\mathbb{C}-$point of $\mathcal{K}_{0,2}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r}))$, where we denote by q_0(resp. q_∞) by the first (resp. second) marking, then the evaluation map

$$ev_{q_\infty} : \mathcal{K}_{0,2}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r})) \to \bar{I}_\mu \mathbb{P}Y_{r,s} ,$$

induces a morphism from the isotropy group $Aut(f)$ to the isotropy group $Aut((y, ([g], 1, e^{\frac{\delta}{r}})))$, which, from the concrete description of $Aut(f)$ above, coincides the morphism from $C_{H_e}(\gamma_e)/\langle\gamma_e\rangle$ to $C_{G_y}(g)/\langle g\rangle$ induced from the projection of $G_y \times \mathbb{Z}$ to the first factor. From this concrete description, we also know the kernel of the morphism of isotropy groups:

Proposition A.3. *The kernel of the morphism* $Aut(f) \to Aut\big((y, ([g], 1, e^{\frac{\delta}{r}}))\big)$ *is isomorphic to* $\mathbb{Z}/a\delta\mathbb{Z}$.

Proof. This follows from by applying the snake lemma to the first two exact rows of the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z} \cong \langle(1, at\delta)\rangle \arrow[r] \arrow[d, hook] & \mathbb{Z} \cong \langle\gamma_e\rangle \arrow[r, "pr_1"] \arrow[d, hook] & \mathbb{Z}/a\mathbb{Z} \cong \langle g\rangle \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathbb{Z} \cong \langle(1, t)\rangle \arrow[r] \arrow[d] & C_{H_e}(\gamma_e) \arrow[r, "pr_1"] \arrow[d] & C_{G_y}(g) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{Z}/a\delta\mathbb{Z} \arrow[r] & C_{H_e}(\gamma_e)/\langle\gamma_e\rangle \arrow[r, "pr_1"] \arrow[d, equal] & C_{G_y}(g)/\langle g\rangle \arrow[r] \arrow[d, equal] & 1 \\
 & & Aut(f) & Aut\big((y, ([g], 1, e^{\frac{\delta}{r}}))\big) ,&
\end{tikzcd}$$

where pr_1 are induced from the projection from $G_y \times \mathbb{Z}$ to the first factor. □

A.2. **Construction of family stable map.** Let e be an edge with decorated degree $\frac{\delta}{r}$ and decorated multiplicity $(c, 1, \mathrm{e}^{\frac{\delta}{r}})$ at the half-edge h_∞ as in §3.2. Let $a := a(c)$ be the integer associated to c as in the Assumption 2.1. Define the space $\mathcal{M}_e$ to be the root gerbe $\sqrt[as\delta]{L^\vee/I_cY}$. We will generalize the construction in Lemma A.1 to a family version; we will construct a family of $\mathbb{C}^*-$fixed stable map over $\mathcal{M}_e$ to $\mathbb{P}Y_{r,s}$ with the associated decorated degree. Then we get a morphism

$$g : \mathcal{M}_e \to \mathcal{K}_{0,2}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r})) .$$

We will show the image of g is a closed and open substack of the $\mathbb{C}^*-$fixed part of $\mathcal{K}_{0,2}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r}))$, and g is a finite étale map into the image of g. Then we can use $\mathcal{M}_e$ to do an explicit computation of the edge contribution in the virtual localization formula.

By an abuse of notation, denote $L^\vee$ to be the line bundle over $\mathcal{M}_e$ via a pull-back of the line bundle $L^\vee$ over Y along the natural map $\mathcal{M}_e \to Y$, which is a composition of maps $\mathcal{M}_e \to \bar{I}_cY \to Y$. Denote by $(L^\vee)^0$ the open set of $L^\vee$ with zero section removed.

Let $U := \mathbb{C}^2\backslash\{0\}$. Let $\mathcal{C}_e$ be the quotient stack $[(L^\vee)^0 \times U/\mathbb{C}^* \times T_{ar,as}]$ defined by the (right) action

$$(l, x, y)(t, t_1, t_2) = (t^{-as\delta}l, tt_1x, t_2y) ,$$

where $(l, x, y) \in (L^\vee)^0 \times U$ and $(t, t_1, t_2) \in \mathbb{C}^* \times T_{ar,as}$. Here $T_{ar,as}$ is the subgroup of $(\mathbb{C}^*)^2$ defined by$\{(t_1, t_2) \in (\mathbb{C}^*)^2|\ t_1^{as} = t_2^{ar}\}$. Then $\mathcal{C}_e$ is a family of orbi-$\mathbb{P}^1$ bundle over $\mathcal{M}_e$ such that all fibers are isomorphic to $\mathbb{P}^1_{ar,as}$.

Remark A.4. Let $\bar{\psi}_\star$ be the psi-class associated with the gerbe marking corresponding to the zero loci of y of $\mathcal{C}_e$. We see that $\bar{\psi}_\star$ is equal to $\frac{\lambda - c_1(L)}{\delta}$.

We will define a family map f using the categorical description of $\mathbb{P}Y_{r,s}$ and C_e:

$$\begin{tikzcd} \mathcal{C}_e \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}Y_{r,s} \\ \mathcal{M}_e := \sqrt[as\delta]{L^\vee/I_cY}. & \end{tikzcd}$$

A.2.1. *A categorical description about stacks $\mathbb{P}Y_{r,s}$ and $\mathcal{C}_e$.* The space $\mathbb{P}Y_{r,s}$ can be represented as the quotient $[(L^\vee)^0 \times U/(\mathbb{C}^*)^2]$ defined by the action

$$(l,x,y)(t_1,t_2) = (t_1^{-s}t_2^r l, t_1 x, t_2 y)\ .$$

We see that, for any $\mathbb{C}-$scheme S, an object in $\mathbb{P}Y_{r,s}(S)$ is given by a tuple $(w, M_1, M_2, \rho, s_1, s_2)$ (c.f. [Ols16, §10.2.7,§10.2.8]), where

(1) $w : S \to Y$ is an object[19] in $Y(S)$,
(2) M_1, M_2 are line bundles over S,
(3) $\rho : M_1^{\otimes s} \otimes M_2^{-\otimes r} \cong w^* L^\vee$ is an isomorphism,
(4) $s_i \in \Gamma(S, M_i), i \in \{1,2\}$ are sections such that s_1, s_2 don't vanish simultaneously.

Let $(w', M_1', M_2', \rho', s_1', s_2')$ be another object in $\mathbb{P}Y_{r,s}(S')$, given a morphism $g : S' \to S$, then a morphism from $(w', M_1', M_2', \rho', s_1', s_2')$ to $(w, M_1, M_2, \rho, s_1, s_2)$ over g is a tuple $(g^b, \alpha_1, \alpha_2)$. Here $g^b : w' \to w$ is a morphism over g, $\alpha_i : g^* M_i \to M_i'$ is an isomorphism which maps $g^* s_i$ to s_i',. Furthermore, we require a commutative square

$$\begin{tikzcd} g^* M_1^{\otimes s} \otimes g^* M_2^{-\otimes r} \arrow[r, "\alpha_1^{\otimes s} \otimes \alpha_2^{-\otimes r}"] \arrow[d, "g^*\rho"'] & M_1'^{\otimes s} \otimes M_2'^{-\otimes r} \arrow[d, "\rho'"] \\ g^* w^* L^\vee \arrow[r, "g^b"] & (w')^* L^\vee\ , \end{tikzcd}$$

and α_i should satisfy obvious commutative laws for composition of two maps (c.f. [Ols16, §9.1.10(i)]).

Remark A.5. Let $w : S \to Y$ be any object in $Y(S)$, denote by $\mathbb{P}_{r,s}(w^* L^\vee \oplus \mathbb{C})$ the stack quotient

$$[(w^* L^\vee)^0 \times U/(\mathbb{C}^*)^2]\ ,$$

where the action is defined in the same way as $\mathbb{P}Y_{r,s}$. One can verify that we have a Cartesian diagram:

$$\begin{tikzcd} \mathbb{P}_{r,s}(w^* L^\vee \oplus \mathbb{C}) \arrow[r] \arrow[d] & \mathbb{P}Y_{r,s} \arrow[d, "\mathrm{pr_{r,s}}"] \\ S \arrow[r, "w"] & Y\ . \end{tikzcd}$$

This property can also lead to our categorical description of $\mathbb{P}Y_{r,s}$, see a closed-related discussion in [Ols16, §10.2.8].

[19] Here we view the morphism $S \to Y$ as an equivalent definition of objects in $Y(S)$ using the 2-Yoneda Lemma.

We also give a categorical description of $\mathcal{C}_e$. Recall that $\mathcal{C}_e$ can be represented as the quotient stack $[(L^\vee)^0 \times U/(T_{ar,as} \times \mathbb{C}^*)]$ given by the action

$$(l,x,y)(t_1,t_2,t_3) = (t_3^{-as\delta} l, t_3 t_1 x, t_2 y)$$

where $(l,x,y) \in (L^\vee)^0 \times U$ and $(t_1,t_2,t_3) \in T_{ar,as} \times \mathbb{C}^*$. For any scheme S over $\mathrm{Spec}(\mathbb{C})$, an object in $\mathcal{C}_e(S)$ is given by a tuple $\tau := (w, \theta, M_1, M_2, M_3, \rho_1, \rho_2, s_1, s_2)$, where

1. w is a object in $Y(S)$, θ is an automorphism in $Aut_S(w)$ such that $(w,\theta) \in I_c Y$,
2. M_1, M_2, M_3 are line bundles over S,
3. $\rho_1 : M_3^{\otimes as\delta} \cong w^* L^\vee$ and $\rho_2 : M_1^{\otimes as} \otimes M_2^{-\otimes ar} \cong \mathcal{O}_S$ are isomorphisms,
4. $s_1 \in \Gamma(S, M_1 \otimes M_3)$ and $s_2 \in \Gamma(S, M_2)$ are sections such that s_1, s_2 don't vanish simultaneously.

Given another $\mathbb{C}-$scheme S', let $\tau' = (w', \theta', M_1', M_2', M_3', \rho_1', \rho_2', s_1', s_2')$ be an object in $\mathcal{C}_e(S')$, given a morphism $g : S' \to S$, then a morphism from $(w', \theta', M_1', M_2', M_3', \rho_1', \rho_2, s_1', s_2')$ to $(w, \theta, M_1, M_2, M_3, \rho_1, \rho_2, s_1, s_2)$ over g is a tuple $(g^b, \alpha_1, \alpha_2, \alpha_3)$, where $g^b : (w', \theta') \to (w, \theta)$ is a morphism in $I_c Y$ over the morphism g, $\alpha_i : g^* M_i \to M_i'$ are isomorphisms such that $\alpha_1 \otimes \alpha_3 : g^* M_1 \otimes g^* M_3 \to M_1' \otimes M_3'$ maps $g^* s_1$ to s_1', and $\alpha_2 : g^* M_2 \cong M_2'$ maps $g^* s_2$ to s_2'. Furthermore, these data are require to satisfy the following conditions:

1. We have two commutative squares

$$\begin{tikzcd}
g^* M_3^{\otimes as\delta} \arrow[r, "\alpha_3^{\otimes as\delta}"] \arrow[d, "g^*\rho_1"'] & M_3'^{\otimes as\delta} \arrow[d, "\rho_1'"] \\
g^* w^* L^\vee \arrow[r, "g^b"] & (w')^* L^\vee
\end{tikzcd}$$

   and

$$\begin{tikzcd}
g^* M_1^{\otimes as} \otimes g^* M_2^{-\otimes ar} \arrow[r, "\alpha_1^{\otimes as} \otimes \alpha_2^{-\otimes ar}"] \arrow[d, "g^*\rho_2"'] & M_1'^{\otimes as} \otimes M_2'^{-\otimes ar} \arrow[d, "\rho_2'"] \\
g^* \mathcal{O}_S \arrow[r, "\cong"] & \mathcal{O}_{S'} .
\end{tikzcd}$$

2. α_1, α_2 should satisfy commutative laws for composition of maps.

For each object $\tau := (w, \theta, M_1, M_2, M_3, \rho_1, \rho_2, s_1, s_2) \in \mathcal{C}_e(S)$, we give the following definition:

Definition A.6. *The collection of line bundles $M_1^{-\otimes s} \otimes M_2^{\otimes r}$ and morphisms $\alpha_1^{\otimes s} \otimes \alpha_2^{-\otimes r}$ defines a line bundle on $\mathcal{C}_e$, which we denote to be $\mathcal{L}$.*[20] *Using ρ_2, we see that the line bundle $M_1^{-\otimes s} \otimes M_2^{\otimes r}$ represents a torsion element in $\mathrm{Pic}(S)$ of which the order is divided by a. This defines a morphism $\alpha : \mathcal{C}_e \to \mathbb{B}\boldsymbol{\mu}_a$ satisfying that: let $\mathbb{L} = [\mathbb{C}/\boldsymbol{\mu}_a]$ be the line bundle over $\mathbb{B}\boldsymbol{\mu}_a$ associated with the standard representation of $\boldsymbol{\mu}_a$, we have $\alpha^* \mathbb{L} \cong \mathcal{L}$. Then we can naturally associate each object $\tau : S \to \mathcal{C}_e$ a morphism $\bar{f}_1 := \alpha \circ \tau$ and an isomorphism $\alpha_\tau : M_1^{-\otimes s} \otimes M_2^{\otimes r} \cong \bar{f}_1^* \mathbb{L}$. Given another object $\tau' \in \mathcal{C}_e(S')$ and a morphism $u := (g^b, \alpha_1, \alpha_2, \alpha_3)$ in $Hom_{\mathcal{C}_e}(\tau', \tau)$ over $g : S' \to S$, we have*

$$\begin{tikzcd}
g^*(M_1^{\otimes s} \otimes M_2^{\otimes -r}) \arrow[r, "\alpha_1^{-\otimes s} \otimes \alpha_2^{\otimes r}"] \arrow[d, "g^*\alpha_\tau"] & M_1'^{-\otimes s} \otimes M_2'^{\otimes r} \arrow[d, "\alpha_{\tau'}"] \\
g^* \bar{f}_1^* \mathbb{L} \arrow[r, "\alpha(u)"] & (\bar{f}_1')^* \mathbb{L} .
\end{tikzcd} \tag{A.3}$$

[20] This line bundle is also isomorphic to $L_{-s\chi_1} \otimes L_{r\chi_2}$ defined in §A.3.

Note $\bar{f}_1$ induces a $S-$morphism $f_1 : S \to S \times \mathbb{B}\boldsymbol{\mu}_a$ such that $\mathrm{pr}_2 \circ f_1 = \bar{f}_1$, where pr_2 is the projection from $S \times \mathbb{B}\boldsymbol{\mu}_a$ to the second factor.

Recall that there is an isomorphism of the stack $\mathrm{RepHom}(\mathbb{B}\mu_a, Y)$ *with the $I_{\mu_a}Y$ from* [AGV08, §3.2]*, the data (w,θ) naturally defines a morphism $f_2 : S \times \mathbb{B}\boldsymbol{\mu}_a \to Y$ such that the generator of the isotropy group $Aut_S(i_S) = (\boldsymbol{\mu_a})_S$ sends to the automorphism $\theta \in Aut_S(w)$, where i_S is a $S-$morphism $i_S : S \to S \times \mathbb{B}\boldsymbol{\mu}_a$ corresponding to the trivial $\boldsymbol{\mu}_a$ principal bundle over S. Moreover we have $w = f_2 \circ i_S$.*

Let a be the integer associated to c as in 2.1. Write $age_c(L) = \frac{h}{a}$ for an integer h with $0 \leqslant h < a$. Let $\mathbb{L}_{a,-h}$ be the line bundle over $\mathbb{B}\boldsymbol{\mu}_a$ associated to the character of $\boldsymbol{\mu}_a$ by sending μ_a^i to μ_a^{-ih}. Then $\mathbb{L}_{a,-h} = \mathbb{L}^{-\otimes h}$ is defined in A.6.

Proposition A.7. *We have a natural isomorphism $\epsilon_{f_2} : w^*L^\vee \boxtimes \mathbb{L}_{a,-h} \to f_2^*L$, and we have a natural isomorphism $\bar{\rho}_\tau : w^*L^\vee \otimes (M_1^{\otimes sh} \otimes M_2^{\otimes -rh}) \to (f_2 \circ f_1)^*L^\vee$.*

Proof. We will prove ϵ_{f_2} by étale descent. First note $i_S : S \to S \times \mathbb{B}\mu_a$ is an étale covering with Galois group $\boldsymbol{\mu}_a$, then f_2 can be recovered form the descent data given by the pair (w,θ)(c.f., [Vis05]). Then the $f_2^*L^\vee$ can be recovered descent data given by the pair $(w^*L^\vee, \widetilde{\theta} : w^*L^\vee \to w^*L^\vee)$, where $\widetilde{\theta}$ is an automorphism of $w^*L^\vee$ induced by θ. As $age_c(L) = \frac{h}{a}$, $\widetilde{\theta}$ is nothing else but the scalar multiplication by $\mathrm{e}^{\frac{-h}{a}}$. By descent, we see that the line bundle $f_1^*L^\vee$ is isomorphic to $w^*L^\vee \boxtimes \mathbb{L}_{a,-h}$.

Finally the claim about $(f_2 \circ f_1)^*L^\vee$ follows the next lemma. □

Lemma A.8. *Let W, U_1, U_2 be stacks and L_1 and L_2 be two line bundles over U_1 and U_2 respectively. Let $t_1 : W \to U_1$ and $t_2 : W \to U_2$ be two morphisms and $t_1 \times t_2 : W \to U_1 \times U_2$ is the induced morphism using the fiber product. Then we have that a natural isomorphism $(t_1 \times t_2)^*(L_1 \boxtimes L_2) \cong t_1^*L_1 \otimes t_2^*L_2$.*

Definition A.9. *Recall that we have $a\delta \equiv h \bmod a$, then $\rho_2^{\otimes \frac{a\delta - h}{a}}$ induces a natural isomorphism $w^*L^\vee \otimes (M_1^{\otimes as\delta} \otimes M_2^{\otimes -ar\delta}) \cong M_1^{\otimes sh} \otimes M_2^{\otimes -rh}$ which we still denote to be $\rho_2^{\otimes \frac{a\delta-h}{a}}$. We construct an natural isomorphism*

$$\rho_\tau : (M_1 \otimes M_3)^{\otimes as\delta} \otimes M_2^{-\otimes ar\delta} \to (f_2 \circ f_1)^*L^\vee$$

as the composition

$$\begin{aligned} &(M_1 \otimes M_3)^{\otimes as\delta} \otimes M_2^{-\otimes ar\delta} \xrightarrow{\cong} M_3^{\otimes as\delta} \otimes (M_1^{\otimes as\delta} \otimes M_2^{-\otimes ar\delta}) \\ &\xrightarrow{\rho_1 \otimes \rho_2^{\otimes \frac{a\delta-h}{a}}} w^*L^\vee \otimes (M_1^{\otimes sh} \otimes M_2^{\otimes -rh}) \xrightarrow{\bar{\rho}_\tau} (f_2 \circ f_1)^*L^\vee\ . \end{aligned} \tag{A.4}$$

A.2.2. *Defining the family map.* We first collect some facts about line bundles over stacks. By a line bundle $\mathcal{L}$ over a stack $\mathcal{X}$, we adopt the following definition: for any scheme S over $\mathbb{C}$ and any object x of $\mathcal{X}(S)$, we associate a line bundle over S and we denote it by $x^*\mathcal{L}$, this notation coincides with the notation of pullback of line bundles when we view an object of $\mathcal{X}$ as a morphism $x : S \to \mathcal{X}$ by the $2-$Yoneda lemma. Let S' be another scheme and $x' \in \mathcal{X}(S')$. Let $g^b : x' \to x$ be a morphism over $g : S' \to S$. Then there is an isomorphism $g^*x^*\mathcal{L} \to x^{'*}\mathcal{L}$, which we denote to be g^b as well, the collection of line bundles and morphisms between them should satisfy obvious compatibility with composition of morphisms in $\mathcal{X}$.

Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism between stacks, then the pullback $f^*\mathcal{L}$ of the line bundle $\mathcal{L}$ along f can be defined as follows: to each object $y \in Ob(\mathcal{Y})$, we associate the line bundle $f(y)^*\mathcal{L}$; the morphisms between line bundles are defined in an obvious way. Let $f' : \mathcal{Y} \to \mathcal{X}$ be another morphism and α be a 2-isomorphism from f' to f, then we can define an isomorphism from $f^*\mathcal{L}$ to $f^{'*}\mathcal{L}$ as follows, view $\mathcal{X}$ and $\mathcal{Y}$ as its underlying category, then α is a natural transformation from f' to f, which are viewed as functors. For any object $y \in Ob(\mathcal{Y})$ over a scheme S, there is an associated morphism $\alpha_y : f^{'}(y) \to f(y)$ over S, which determines a morphism $\alpha_y : f(y)^*\mathcal{L} \to f^{'}(y)^*\mathcal{L}$ of line bundles over S. This defines the isomorphism $f^*\mathcal{L} \to f^{'*}\mathcal{L}$, which we will still denote by α by an abuse of notation. Let $g : \mathcal{Z} \to \mathcal{Y}$ and $h : \mathcal{X} \to \mathcal{W}$ be two morphisms between stacks, we will write $\alpha \circ g$ to mean the 2$-$isomorphism from $f' \circ g$ to $f \circ g$ and write $h \circ \alpha$ to mean the 2$-$isomorphism from $h \circ f'$ to $h \circ f$.

Now we define the family map $f : \mathcal{C}_e \to \mathbb{P}Y_{r,s}$ over $\mathcal{M}_e$. To any object

$$\tau = (w, \theta, M_1, M_2, M_3, \rho_1, \rho_2, s_1, s_2) \in \mathcal{C}_e(S)\ ,$$

we associate an object $\phi = (f_2 \circ f_1, (M_1 \otimes M_3)^{\otimes a\delta}, M_2^{\otimes a\delta}, \rho_\tau, s_1^{a\delta}, s_2^{a\delta}) \in \mathbb{P}Y_{r,s}(S)$, where f_1, f_2, ρ_τ (and $\bar{f}_1$, α_τ, ϵ_{f_2}, $\bar{\rho}_\tau$) are defined in Definition A.6, Definition A.9 (and Proposition A.7). Given another scheme S' and another object $\tau' \in \mathcal{C}_e(S')$, let ϕ' be the associated object in $\mathbb{P}Y_{r,s}$, similarly for $f_1', f_2', \rho_{\tau'}$ (and $\bar{f}'_1$, $\alpha_{\tau'}$, $\epsilon_{f_2'}$, $\bar{\rho}_{\tau'}$). Let $u := (g^b, \alpha_1, \alpha_2, \alpha_3)$ be a morphism in $Hom_{\mathcal{C}_e}(\tau', \tau)$ over $g : S' \to S$, , we need some preparation before we define a morphism from ϕ' to ϕ over g.

Using the notations in A.6, by 2-Yoneda lemma, the morphism $\alpha(u)$ induces a unique 2-isomorphism $\bar{\eta}_1 : \bar{f}_1' \to \bar{f}_1 \circ g$ marking the following triangle commute:

$$\begin{tikzcd} S' \arrow[rr, "g"] \arrow[dr, "\bar{f}_1'"'] & & S \arrow[dl, "\bar{f}_1"] \\ & \mathbb{B}\boldsymbol{\mu}_{\boldsymbol{a}} & \end{tikzcd} .$$

Let $\bar{\eta}_1 : (\bar{f}_1 \circ g)^*\mathbb{L} \to (\bar{f}_1')^*\mathbb{L}$ be the isomorphism induced from the 2-isomorphism $\bar{\eta}_1$. Then we have the following commutative triangle

(A.5)
$$\begin{tikzcd} g^*\bar{f}_1^*\mathbb{L} \arrow[d, "\cong"] \arrow[dr, "\alpha(u)"] & \\ (\bar{f}_1 \circ g)^*\mathbb{L} \arrow[r, "\bar{\eta}_1"] & (\bar{f}_1')^*\mathbb{L}\ . \end{tikzcd}$$

In the following, to simplify the presentation, we will treat $g^*\bar{f}_1^*\mathbb{L}$ and $(\bar{f}_1 \circ g)^*\mathbb{L}$ are the same via the vertical isomorphism in A.5 defined in an obvious way, thus $\bar{\eta}_1 = \alpha(u)$ above, and $\bar{\eta}_1$ can be also viewed as a map from $g^*\bar{f}_1^*\mathbb{L}$. The same rule applies to other pullbacks of a line bundle along a composition of maps.

Note that $\bar{\eta}_1$ induces a 2-isomorphism $\eta_1 : (g \times id) \circ f_1' \to f_1 \circ g$ satisfying that $\bar{\eta}_1 = \mathrm{pr}_2 \circ \eta_1$ and making the following square commute:

$$\begin{tikzcd} S' \arrow[r, "g"] \arrow[d, "f_1'"] & S \arrow[d, "f_1"] \\ S' \times \mathbb{B}\boldsymbol{\mu}_{\boldsymbol{a}} \arrow[r, "g\times id"] & S \times \mathbb{B}\boldsymbol{\mu}_{\boldsymbol{a}}\ . \end{tikzcd}$$

For any line bundle M over S and any integer n, the 2$-$isomorphism η_1 induces an isomorphism $\eta_1 : (f_1 \circ g)^*(M \boxtimes \mathbb{L}^{\otimes n}) \to ((g \times id) \circ f_1')^*(M \boxtimes \mathbb{L}^{\otimes n})$ such that we have the following commutative square using A.3 and Lemma A.8

(A.6)
$$\begin{tikzcd}
g^*M \otimes g^*(M_1^{\otimes sn} \otimes M_2^{\otimes -rn}) \arrow[r, "id\otimes(\alpha_1^{-\otimes sn}\otimes\alpha_2^{\otimes rn})"] \arrow[d, "\cong"] & g^*M \otimes (M_1^{'-\otimes sn} \otimes M_2^{'\otimes rn}) \arrow[d, "\cong"'] \\
g^*f_1^*(M \boxtimes \mathbb{L}^{\otimes n}) \arrow[r, "\eta_1"] & ((g\times id)\circ f_1')^*(M \boxtimes \mathbb{L}^{\otimes n}) \ .
\end{tikzcd}$$

Here the left vertical map is induced from Lemma A.8 by taking $t_1 \times t_2 = f_1$ and pullback along the map g, while the right vertical map is induced from Lemma A.8 by taking $t_1 \times t_2 = (g \times id) \circ f_1'$.

Using the equivalence of the stack $\mathrm{RepHom}(\mathbb{B}\mu_a, Y)$ with the $I_{\mu_a}Y$ from [AGV08, §3.2], the morphism g^b between $(w', \theta') \in I_{\mu_a}Y(S')$ and $(w, \theta) \in I_{\mu_a}Y(S)$ induces a 2-isomorphism $\eta_2 : f_2' \to f_2 \circ (g \times id)$ marking the following triangle commute

$$\begin{tikzcd}
S' \times \mathbb{B}\boldsymbol{\mu_a} \arrow[rr, "g\times id"] \arrow[dr, "f_2'"'] & & S \times \mathbb{B}\boldsymbol{\mu_a} \arrow[dl, "f_2"] \\
& Y & .
\end{tikzcd}$$

The 2-isomorphism η_2 induces a morphism from $(f_2\circ(g \times id))^*L^\vee$ to m,$(f_2')^*L^\vee$ which makes the following diagram commutes:

(A.7)
$$\begin{tikzcd}
g^*w^*L^\vee \boxtimes \mathbb{L}_{a,-h} \arrow[r, "g^b\boxtimes id"] \arrow[d, "(g\times id)^*\epsilon_{f_2}"'] & (w')^*L^\vee \boxtimes \mathbb{L}_{a,-h} \arrow[d, "\epsilon_{f_2'}"] \\
(f_2\circ(g\times id))^*L^\vee \arrow[r, "\eta_2"] & (f_2')^*L^\vee \ .
\end{tikzcd}$$

Indeed, using the notations in Proposition A.7, the 2-isomorphism $i_S \circ \eta_2$ induces a morphism $g^*w^*L^\vee \to (w')^*L^\vee$, which is exactly induced by g^b. As g^b can be view a morphism from w' to w which commutes with θ' and θ, then by descent, we get the above diagram.

Then we make the composition $\eta_2 \star \eta_1 := (\eta_1 \circ f_2) \circ (f_1' \circ \eta_2)$,[21] which is a 2-isomorphism from $f_2' \circ f_1'$ to $f_2 \circ f_1 \circ g$. When we view $f_2' \circ f_1'$ (resp. $f_2 \circ f_1$) as a point in $Y(S')$ (resp. $Y(S)$), then $\eta_2 * \eta_1$ can be viewed as a morphism over g in Y from the point $f_2' \circ f_1'$ to the point $f_2 \circ f_1$.

Now we can define the family map:

Definition A.10 (**Family map**)**.** *With the notations as above, for any object*

$$\tau = (w, \theta, M_1, M_2, M_3, \rho_1, \rho_2, s_1, s_2) \in \mathcal{C}_e(S) \ ,$$

we define the image $f(\tau)$ under the family map f to be

$$\phi = (f_2 \circ f_1, (M_1 \otimes M_3)^{\otimes a\delta}, M_2^{\otimes a\delta}, \rho_\tau, s_1^{a\delta}, s_2^{a\delta}) \in \mathbb{P}Y_{r,s}(S) \ .$$

Given a morphism $u : \tau' \to \tau$ in $\mathcal{C}_e$, we define the morphism $f(u) : \phi' \to \phi$ to be $(\eta_2 \star \eta_1, (\alpha_1 \otimes \alpha_3)^{\otimes a\delta}, (\alpha_2)^{\otimes a\delta})$.

[21] This is the so-called Godement product.

We should check that $f(u)$ is indeed a well-defined morphism, for which we only need to check the following diagram commutes

(A.8)

$$\begin{tikzcd} g^*(M_1\otimes M_3)^{\otimes as\delta}\otimes g^*M_2^{-\otimes ar\delta} \arrow[r, "(\alpha_1\otimes\alpha_3)^{\otimes as\delta}\otimes\alpha_2^{-\otimes ar\delta}"] \arrow[d, "g^*\rho_\tau"'] & (M_1'\otimes M_3')^{\otimes as\delta}\otimes M_2'^{-\otimes ar\delta} \arrow[d, "\rho_{\tau'}"] \\ g^*(f_2\circ f_1)^*L^\vee \arrow[r, "\eta_2\star\eta_1"] & (f_2'\circ f_1')^*L^\vee \ . \end{tikzcd}$$

Note that we have the following diagram

$$\begin{tikzcd} g^*(M_1\otimes M_3)^{\otimes as\delta}\otimes g^*M_2^{-\otimes ar\delta} \arrow[r, "(\alpha_1\otimes\alpha_3)^{\otimes as\delta}\otimes\alpha_2^{-\otimes ar\delta}"] \arrow[d, "\cong"'] & (M_1'\otimes M_3')^{\otimes as\delta}\otimes M_2^{'-\otimes ar\delta} \arrow[d, "\cong"] \\ g^*M_3^{\otimes as\delta}\otimes g^*(M_1^{\otimes as\delta}\otimes M_2^{-\otimes ar\delta}) \arrow[r, "(\alpha_3)^{\otimes as\delta}\otimes\alpha_1^{as\delta}\otimes\alpha_2^{-\otimes ar\delta}"] \arrow[d, "g^*\rho_1\otimes g^*\rho_2^{\frac{a\delta-h}{a}}"'] & M_3^{'\otimes as\delta}\otimes(M_1^{'\otimes as\delta}\otimes M_2^{'-\otimes ar\delta}) \arrow[d, "\rho_1'\otimes(\rho_2')^{\frac{a\delta-h}{a}}"] \\ g^*w^*L^\vee\otimes g^*(M_1^{\otimes sh}\otimes M_2^{\otimes -rh}) \arrow[r, "g^b\otimes(\alpha_1^{\otimes sh}\otimes\alpha_2^{-\otimes rh})"] & (w')^*L^\vee\otimes(M_1^{'\otimes sh}\otimes M_2^{'\otimes -rh})\ , \end{tikzcd}$$

by the definition of ρ_τ, we see that showing that A.8 is a commutative diagram is equivalent to show that the following diagram commutes:

(A.9)

$$\begin{tikzcd} g^*w^*L^\vee\otimes g^*(M_1^{\otimes sh}\otimes M_2^{\otimes -rh}) \arrow[r, "g^b\otimes(\alpha_1^{\otimes sh}\otimes\alpha_2^{-\otimes rh})"] \arrow[d, "g^*\bar{\rho}_\tau"'] & (w')^*L^\vee\otimes(M_1^{'\otimes sh}\otimes M_2^{'\otimes -rh}) \arrow[d, "\bar{\rho}_{\tau'}"] \\ g^*(f_2\circ f_1)^*L^\vee \arrow[r, "\eta_2\star\eta_1"] & (f_2'\circ f_1')^*L^\vee \ . \end{tikzcd}$$

Denote $\bar{\rho}_{\tau,\tau'} := f_1^*(g\times id)^*\epsilon_{f_2}$, then $\bar{\rho}_{\tau,\tau'}$ induces an isomorphism from $g^*w^*L^\vee\otimes(M_1^{'\otimes sh}\otimes M_2^{'\otimes -rh})$ to $(f_2\circ(g\times id)\circ f_1')^*L^\vee$. The 2-isomorphism $\eta_1\circ f_2$ induces a morphism $\eta_1\circ f_2 : g^*(f_2\circ f_1)^*L^\vee\to(f_2\circ(g\times id)\circ f_1')^*L^\vee$. Take $M=w^*L^\vee$, $n=-h$ in A.6, using Proposition A.7, we get the following commutative diagram

$$\begin{tikzcd} g^*w^*L^\vee\otimes g^*(M_1^{\otimes sh}\otimes M_2^{\otimes -rh}) \arrow[r, "id\otimes(\alpha_1^{\otimes sh}\otimes\alpha_2^{-\otimes rh})"] \arrow[d, "g^*\bar{\rho}_{\tau'}"] & g^*w^*L^\vee\otimes(M_1^{'\otimes sh}\otimes M_2^{'\otimes -rh}) \arrow[d, "\bar{\rho}_{\tau',\tau}"'] \\ g^*(f_2\circ f_1)^*L^\vee \arrow[r, "\eta_1\circ f_2"] & (f_2\circ(g\times id)\circ f_1')^*L^\vee \ . \end{tikzcd}$$

The 2-isomorphism $f_1'\circ\eta_2$ induces a morphism $f_1'\circ\eta_2 : (f_2\circ(g\times id)\circ f_1')^*L^\vee\to(f_2'\circ f_1')^*L^\vee$. Pull back the square A.7 via $f_1^{'*}$, we get the commutative square:

$$\begin{tikzcd} g^*w^*L^\vee\otimes(M_1^{'\otimes sh}\otimes M_2^{'\otimes -rh}) \arrow[r, "g^b\otimes id"] \arrow[d, "\bar{\rho}_{\tau',\tau}"'] & (w')^*L^\vee\otimes(M_1^{'\otimes sh}\otimes M_2^{'\otimes -rh}) \arrow[d, "\bar{\rho}_{\tau'}"] \\ (f_2\circ(g\times id)\circ f_1')^*L^\vee \arrow[r, "f_1'\circ\eta_2"] & (f_2'\circ f_1')^*L^\vee \ . \end{tikzcd}$$

Join the above two squares horizontally, this verifies the commutativity of A.9.

Remark A.11. When Y is a complete intersection in a toric stack, our construction of the family map is equivalent to the construction in [Wan19, §5.3.2].

A.3. **Computation of edge contribution.** Recall that in §3, we define a $\mathbb{C}^*-$action on $\mathbb{P}Y_{r,s}$ which is induced from the $\mathbb{C}^*-$action on $(L^\vee)^0 \times U$:

$$t \cdot (l, x, y) = (tl, x, y) \ .$$

In the language of functor of points, for any $t \in \mathbb{C}^*$ and any scheme S, this action will change any object $(w, M_1, M_2, \rho, s_1, s_2)$ of $Ob(\mathbb{P}Y_{r,s}(S))$ to $(w, M_1, M_2, t \cdot \rho, s_1, s_2)$ of $Ob(\mathbb{P}Y_{r,s}(S))$. where $t \cdot \rho$ means that we compose ρ by the automorphism of $w^* L^\vee$ obtained by scaling by t.

Now we define a $\mathbb{C}^*-$action on $\mathcal{C}_e$ such that the morphism f is $\mathbb{C}^*-$equivariant. Recall that $\mathcal{C}_e$ can be represented as $[(L^\vee)^0 \times U / T_{ar,as} \times \mathbb{C}^*]$ given by the action

$$(l, x, y)(t_1, t_2, t_3) = (t_3^{-as\delta} l, t_3 t_1 x, t_2 y) \ .$$

We define the $\mathbb{C}^*-$action on $(L^\vee)^0 \times U$ in the way that

$$t \cdot (l, x, y) = (t \cdot l, x, y) \ .$$

One can see this action descends to be $\mathbb{C}^*-$action on $\mathcal{C}_e$. In the language of functor of points, for any $t \in \mathbb{C}^*$ and any scheme S this $\mathbb{C}^*-$action will change any object $(w, \theta, M_1, M_2, M_3, \rho_1, \rho_2, s_1, s_2) \in \mathcal{C}_e(S)$ to $(w, \theta, M_1, M_2, M_3, t \cdot \rho_1, \rho_2, s_1, s_2) \in \mathcal{C}_e(S)$, where $t \cdot \rho_1$ means that we compose ρ_1 by the automorphism of $w^* L^\vee$ obtained by scaling by t. Then this action will change the associated ρ_τ by $t\rho_\tau$, which implies that f is $\mathbb{C}^*-$equivariant.

Let χ_1 (resp. χ_2) be the character associated to $T_{ar,as}$ by sending $(t_1, t_2) \in T_{ar,as}$ to t_1 (resp. t_2). Then we can construct the line bundle L_{χ_1}(resp. L_{χ_2}) over $\mathcal{C}_e$ as the stack quotient $[(L^\vee)^0 \times U \times \mathbb{C} / T_{ar,as} \times \mathbb{C}^*]$ using the Borel construction:

$$(l, x, y, z) \sim (t_3^{-as\delta} l, t_3 t_1 x, t_2 y, t_1 z) \text{ (resp. } (t_3^{-as\delta} l, t_3 t_1 x, t_2 y, t_2 z))$$

for any $(l, x, y, z) \in (L^\vee)^0 \times U \times \mathbb{C}$ and $(t_1, t_2, t_3) \in T_{ar,as} \times \mathbb{C}^*$. Then for any scheme S and any object $\tau := (w, \theta, M_1, M_2, M_3, \rho_1, \rho_2, s_1, s_2)$ of $Ob(\mathcal{C}_e(S))$, we have $\tau^* L_{\chi_1} = M_1$ and $\tau^* L_{\chi_2} = M_2$. We also note M_3 is isomorphic to the line bundle $\tau^* \pi^* L^\vee$. Using the stack quotient representation, the line bundles L_{χ_1} and L_{χ_2} also carry a $\mathbb{C}^*-$action such that the fiber of L_{χ_1} over $\infty-$section(corresponding to the coordinate $y = 0$) is of weight $\frac{-1}{as\delta}$, the fiber of L_{χ_2} over D_∞ is of weight 0. Then we have the following:

Proposition A.12. *The coordinate x of $(L^\vee)^0 \times U$ descents to be a $\mathbb{C}^*-$equivariant section of the $\mathbb{C}^*-$equivariant line bundle $L_{\chi_1} \otimes (\pi^* L^\vee)^{\frac{1}{as\delta}} \otimes \mathbb{C}_{\frac{\lambda}{as\delta}}$ over $\mathcal{C}_e$, and the coordinate y of $(L^\vee)^0 \times U$ descents to be a $\mathbb{C}^*-$equivariant section of the $\mathbb{C}^*-$equivariant line bundle L_{χ_2} over $\mathcal{C}_e$.*

Fix the edge $e = \{h_0, h_\infty\}$ with the decoration as above, we will denote

$$\mathcal{K}_{0,h_0 \sqcup h_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r})) := \mathcal{K}_{0,2}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r})) \cap ev_1^{-1}(\bar{I}_{(c^{-1}, e^{\frac{\delta}{s}}, 1)} \mathbb{P}Y_{r,s}) \cap ev_2^{-1}(\bar{I}_{(c, 1, e^{\frac{\delta}{r}})} \mathbb{P}Y_{r,s}) \ .$$

Here we label the first marking by h_0 and the second marking by h_∞. Let $\mathcal{K}_{0,h_0 \sqcup h_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r}))^{\mathbb{C}^*}$ be the $\mathbb{C}^*-$fixed loci of $\mathcal{K}_{0,h_0 \sqcup h_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r}))$, write $\mathcal{K} := \mathcal{K}_{0,h_0 \sqcup h_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r}))^{\mathbb{C}^*}$ for short, we will first show the following:

Proposition A.13. *Let $[\mathcal{K}]^{vir}$ be the virtual cycle associated to the $\mathbb{C}^*-$fixed part of the perfect obstruction theory of $\mathcal{K}_{0,h_0 \sqcup h_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta}{r}))$ to $\mathcal{K}$. We have that $[\mathcal{K}]^{\mathrm{vir}} = [\mathcal{K}]$ in $A_*(\mathcal{K})$ and the evaluation map*

$$ev_{h_\infty} : \mathcal{K} \to \bar{I}_{(c, 1, e^{\frac{\delta}{r}})} \mathbb{P}Y_{r,s}$$

is étale and finite of degree $\frac{1}{a\delta}$.

Proof. Let $\mathbb{E}$ be the restriction of the perfect obstruction theory[22] for $\mathcal{K}_{0,h_0\sqcup h_\infty}(\mathbb{P}Y_{r,s},(0,\frac{\delta}{r}))$ to $\mathcal{K}$, then the $\mathbb{C}^*-$fixed part of $\mathbb{E}$, which we denote to be $\mathbb{E}^{fix}$, is a perfect obstruction theory for $\mathcal{K}$ by [GP99]. We will show that $\mathbb{E}=\mathbb{E}^{fix}$ and it's isomorphic to the locally free sheaf $ev_{h_\infty}^*\mathbb{T}_{\bar{I}_{(c,1,e^{\frac{\delta}{r}})}\mathbb{P}Y_{r,s}}$. Note this implies that $\mathbb{E}^{fix}\cong\mathbb{T}_{\mathcal{K}}\cong ev_{h_\infty}^*\mathbb{T}_{\bar{I}_{(c,1,e^{\frac{\delta}{r}})}\mathbb{P}Y_{r,s}}$. This will implies the étaleness part, while the finiteness part will follow the properness of ev_{h_∞}, Lemma A.1 and Proposition A.3.

It suffices to check $\mathbb{E}=\mathbb{E}^{fix}\cong ev_{h_\infty}^*\mathbb{T}_{\bar{I}_{(c,1,e^{\frac{\delta}{r}})}\mathbb{P}Y_{r,s}}$ locally on every $\mathbb{C}$-point. For any $\mathbb{C}-$point $[f]$ of $\mathcal{K}$ represented by a twisted stable map f, the tangent space $T_{[f]}$ and the obstruction space $Ob_{[f]}$ of the space $\mathcal{K}$ at the point $[f]$ fit into the following exact sequence of $\mathbb{C}^*-$presentations[23]

$$\begin{aligned}&0\to H^0(C,\mathbb{T}_C(-[q_0]-[q_\infty]))\to H^0(C,f^*\mathbb{T}_{\mathbb{P}Y_{r,s}})\to T_{[f]}\\ &\to H^1(C,\mathbb{T}_C(-[q_0]-[q_\infty]))\to H^1(C,f^*\mathbb{T}_{\mathbb{P}Y_{r,s}})\to Ob_{[f]}\to 0\ .\end{aligned}\tag{A.10}$$

We have the following two exact sequences of vector bundles over $\mathbb{P}Y_{r,s}$ (which are also $\mathbb{C}^*$-equivariant)

$$0\longrightarrow\mathbb{T}_{\mathbb{P}Y_{r,s}/Y}\longrightarrow\mathbb{T}_{\mathbb{P}Y_{r,s}}\longrightarrow\mathrm{pr}_{r,s}^*\mathbb{T}_Y\longrightarrow 0\ ,\tag{A.11}$$

and the Euler exact sequence

$$0\longrightarrow\mathbb{C}\longrightarrow\mathcal{O}(\mathcal{D}_0)\oplus\mathcal{O}(\mathcal{D}_\infty)\longrightarrow\mathbb{T}_{\mathbb{P}Y_{r,s}/Y}\longrightarrow 0\ .\tag{A.12}$$

Then we have the following two right exact sequences

$$\begin{aligned}&0\to H^0(C,f^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y})\to H^0(C,f^*\mathbb{T}_{\mathbb{P}Y_{r,s}})\to H^0(C,f^*\mathrm{pr}_{r,s}^*\mathbb{T}_Y)\\ &\to H^1(C,f^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y})\to H^1(C,f^*\mathbb{T}_{\mathbb{P}Y_{r,s}})\to H^1(C,f^*\mathrm{pr}_{r,s}^*\mathbb{T}_Y)\to 0\ .\end{aligned}\tag{A.13}$$

and

$$\begin{aligned}&0\to H^0(C,\mathcal{O})\to H^0(C,f^*\mathcal{O}(\mathcal{D}_0)\oplus f^*\mathcal{O}(\mathcal{D}_\infty))\to H^0(C,f^*T_{\mathbb{P}Y_{r,s}/Y})\\ &\to H^1(C,\mathcal{O})\to H^1(C,f^*\mathcal{O}(\mathcal{D}_0)\oplus f^*\mathcal{O}(\mathcal{D}_\infty))\to H^1(C,f^*T_{\mathbb{P}Y_{r,s}/Y})\to 0\ .\end{aligned}\tag{A.14}$$

By Lemma A.1, we have

$$f^*\mathcal{O}(\mathcal{D}_0)\oplus f^*\mathcal{O}(\mathcal{D}_\infty)\cong\mathcal{O}(a\delta[q_0]+a\delta[q_\infty])$$

over $C\cong\mathbb{P}^1_{ar,as}$, then we have

$$H^1(C,f^*\mathcal{O}(\mathcal{D}_0)\oplus f^*\mathcal{O}(\mathcal{D}_\infty))=0\ ,$$

which implies that $H^1(C,f^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y})=0$ by A.14. As $H^1(C,\mathbb{T}_C(-[q_0]-[q_\infty]))=0$, we have

$$Ob_{[f]}\cong H^1(C,f^*\mathrm{pr}_{r,s}^*\mathbb{T}_Y)$$

and

$$T_{[f]}\cong H^0(C,f^*\mathbb{T}_{\mathbb{P}Y_{r,s}})/H^0(C,\mathbb{T}_C(-[q_0]-[q_\infty]))$$

by A.10 and A.13.

Moreover, the exact sequence A.14 becomes

$$0\to\mathbb{C}\to\mathbb{C}\oplus\mathbb{C}\to H^0(C,f^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y})\to 0\to 0\to 0\to 0\ ,\tag{A.15}$$

which implies that $H^0(C,f^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y})\cong\mathbb{C}$(of weight 0). Furthermore, the composition

$$H^0(C,T_C(-[q_0]-[q_\infty]))\to H^0(C,f^*\mathbb{T}_{\mathbb{P}Y_{r,s}})\to H^0(C,f^*\mathrm{pr}_{r,s}^*\mathbb{T}_Y)$$

[22]Here a perfect obstruction is a morphism $\mathbb{T}\to\mathbb{E}$ in [BF97], where $\mathbb{T}=\mathbb{L}^\vee$ is the derived dual of the cotangent complex.

[23]All the arrows in this proof are $\mathbb{C}^*-$equivaraint unless otherwise stated.

where the arrows are coming from A.10 and A.13, is equal to zero as the composition is induced from the composition of sheaves

$$\mathbb{T}_C(-[q_0]-[q_\infty]) \to f^*\mathbb{T}_{\mathbb{P}Y_{r,s}} \to f^*\mathrm{pr}_{r,s}^*\mathbb{T}_Y \ ,$$

which is zero. Note $H^0(C, \mathbb{T}_C(-[q_0]-[q_\infty]))$ is one dimensional, which corresponds to the space of tangent vector field of orbi-$\mathbb{P}^1$ vanishing on the two markings, then the space $H^0(C, \mathbb{T}_C(-[q_0]-[q_\infty]))$ maps isomorphically into the subspace $H^0(C, f^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y}) \subset H^0(C, f^*\mathbb{T}_{\mathbb{P}Y_{r,s}})$ via A.13, which implies that

$$\begin{aligned} T_{[f]} &\cong H^0(C, f^*\mathbb{T}_{\mathbb{P}Y_{r,s}})/H^0(C, \mathbb{T}_C(-[q_0]-[q_\infty])) \\ &\cong H^0(C, f^*\mathbb{T}_{\mathbb{P}Y_{r,s}})/H^0(C, f^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y}) \cong H^0(C, f^*\mathrm{pr}_{r,s}^*\mathbb{T}_{\mathbb{P}Y_{r,s}}) \ . \end{aligned}$$

Let $\pi : C \to \mathrm{Spec}(\mathbb{C})$ be the projection to a point, next we show that

$$R^\bullet\pi_* f^*\mathrm{pr}_{r,s}^* T_{\mathbb{P}Y_{r,s}} \cong (ev_{q_\infty}^*\mathbb{T}_{\bar{I}_{(g,1,e^{\frac{\delta}{r}})}\mathbb{P}Y_{r,s}})|_{[f]} \ ,$$

which will imply that

$$T_{[f]} \cong (ev_{q_\infty}^*\mathbb{T}_{\bar{I}_{(g,1,e^{\frac{\delta}{r}})}\mathbb{P}Y_{r,s}})|_{[f]} \ , \quad Ob_{[f]} = 0 \ ,$$

in particular $T_{[f]}$ is $\mathbb{C}^*-$fixed. Applying $R^\bullet\pi_* f^*(-)$ to the first sequence A.11, we get a distinguished triangle

$$R^\bullet\pi_* f^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y} \longrightarrow R^\bullet\pi_* f^*\mathbb{T}_{\mathbb{P}Y_{r,s}} \longrightarrow R^\bullet\pi_* f^*\mathrm{pr}_{r,s}^*\mathbb{T}_Y \xrightarrow{+1} R^\bullet\pi_* f^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y}[1] \ .$$

Let $b_\infty : q_\infty \hookrightarrow C$ be the gerbe section of C corresponding to the half-edge h_∞. Using the divisor sequence for the section b_∞

$$0 \to \mathcal{O}(-q_\infty) \to \mathcal{O}_C \to \mathcal{O}_{q_\infty} \to 0 \ ,$$

tensor with $f^*\mathrm{pr}_{r,s}^*\mathbb{T}_Y$ and taking $R^\bullet\pi_*$, we have

$$R^\bullet\pi_* f^*\mathrm{pr}_{r,s}^*\mathbb{T}_Y = R^\bullet(\pi\circ b_\infty)_*(f\circ b_\infty)^*(\mathrm{pr}_{r,s}^*\mathbb{T}_Y) \ ,$$

where we use the fact that $R^\bullet\pi_* f^*\mathrm{pr}_{r,s}^*\mathbb{T}_Y(-q_\infty) = 0$. Next we show that

$$R^\bullet(\pi\circ b_\infty)_*(f\circ b_\infty)^*(\mathrm{pr}_{r,s}^*\mathbb{T}_Y) \cong (ev_{q_\infty}^*\mathbb{T}_{\bar{I}_{(c,1,e^{\frac{\delta}{r}})}\mathbb{P}Y_{r,s}})|_{[f]} \ ,$$

First by the tangent bundle lemma in [AGV08, §3.6.1], we have

$$(\pi\circ b_\infty)_*(f\circ b_\infty)^*(\mathbb{T}_{\mathbb{P}Y_{r,s}}) \cong (ev_{q_\infty}^*\mathbb{T}_{\bar{I}_{(c,1,e^{\frac{\delta}{r}})}\mathbb{P}Y_{r,s}})|_{[f]} \ ,$$

thus we only need to show that

$$R^\bullet(\pi\circ b_\infty)_*(f\circ b_\infty)^*(\mathrm{pr}_{r,s}^*\mathbb{T}_Y) \cong (\pi\circ b_\infty)_*(f\circ b_\infty)^*\mathbb{T}_{\mathbb{P}Y_{r,s}} \ .$$

As $(\pi\circ b_\infty)_*$ is exact on coherent sheaf, we have $R^\bullet(\pi\circ b_\infty)_* = (\pi\circ b_\infty)_*$ and the following exact sequence of vector spaces:

(A.16)

$$0 \to (\pi\circ b_\infty)_*(f\circ b_\infty)^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y} \to (\pi\circ b_\infty)_*(f\circ b_\infty)^*\mathbb{T}_{\mathbb{P}Y_{r,s}} \to (\pi\circ b_\infty)_*(f\circ b_\infty)^*\mathrm{pr}_{r,s}^*\mathbb{T}_Y \to 0 \ ,$$

we are left to show that $(\pi\circ b_\infty)_*(f\circ b_\infty)^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y} = 0$, which can be checked by the next lemma A.14 by pulling back to every $\mathbb{C}$-point using the idea of proving the tangent bundle lemma [AGV08, §3.6.1]. □

Lemma A.14. *Let $x := [f]$ be a $\mathbb{C}$-point of $\mathcal{K}$ represented by a twisted stable map $f : C \to \mathbb{P}Y_{r,s}$ as in Lemma A.1. Use the notation in the proof of Proposition A.13, we have*

$$(\pi\circ b_\infty)_*(f\circ b_\infty)^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y} = 0 \ .$$

Proof. The gerbe marking q_∞ of C is canonically isomorphic to the classifying stack $\mathbb{B}\boldsymbol{\mu_{ar}}$, which gives a unique lift of the $\mathbb{C}-$point x in the gerbe marking q_∞, which we denote to be z. The isotropy group $Aut(z)$ of z in $\mathbb{B}\boldsymbol{\mu_{ar}}$ is canonically isomorphic to the cyclic group $\boldsymbol{\mu}_{ar}$, and the generator of $\boldsymbol{\mu}_{ar}$ acts on the vector space $f^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y}|_z$ via the multiplication by $e^{\frac{\delta}{r}}$ due to the very choice of the multiplicity associated to the half-edge h_∞. Note the space $(\pi\circ b_\infty)_*(f\circ b_\infty)^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y}$ is given by the $\boldsymbol{\mu}_{ar}-$invariant space $(f^*\mathbb{T}_{\mathbb{P}Y_{r,s}/Y}|_z)^{\boldsymbol{\mu}_{ar}}$, then it is equal to zero. □

The family map f constructed in Section A.2.2 induces a morphism

$$g : \mathcal{M}_e \to \mathcal{K}$$

by the universal property of moduli stack $\mathcal{K}$. Then g is surjective by Lemma A.1. Furthermore,. we have:

Proposition A.15. *The morphism $g : \mathcal{M}_e \to \mathcal{K}$ is finite étale of degree $\frac{1}{as}$. The evaluation map ev_{h_∞} from $\mathcal{K}$ to $\bar{I}_{(c,1,\frac{\delta}{r})}\mathbb{P}Y_{r,s}$ is finite étale of degree $\frac{1}{a\delta}$.*

Proof. Let $\mathrm{pr}_{r,s} : \bar{I}_{(c,1,\frac{\delta}{r})}\mathbb{P}Y_{r,s} \to \bar{I}_c Y$ be the morphism induced from the projection of $\mathbb{P}Y_{r,s}$ to the base Y, we see $\mathrm{pr}_{r,s}$ is an isomorphism. Then the composition $\mathrm{pr}_{r,s}\circ ev_{h_\infty}\circ g$ is finite étale of degree $\frac{1}{a^2 s\delta}$ as the composition can be also obtained by first forgetting root construction of $\mathcal{M}_e$ and taking rigidification afterwards. By two-of-there property for étale morphisms[Sta19, Lemmma 100.35.6], the étaleness of g comes from that ev_{h_∞} is finite étale, which is proved in Proposition A.13. Finally as the composition $ev_{h_\infty} \circ g$ is of degree $\frac{1}{a^2 s\delta}$ and ev_{h_∞} is of degree $\frac{1}{a\delta}$ by Proposition A.2, we see that g is of degree $\frac{1}{as}$. □

Lemma A.16. *We have the following:*

(1) *When the edge moduli $\mathcal{M}_e$ arises from the localization analysis in 3.2.2, we have that $[\mathcal{M}_e]^{vir} = [\mathcal{M}_e]$ and the Euler class of the virtual normal bundle is equal to 1.*
(2) *When the edge moduli $\mathcal{M}_e$ arises from the localization analysis in 4.2.2, we have $[\mathcal{M}_e]^{vir} = [\mathcal{M}_e]$ and the inverse of Euler class of the virtual normal bundle is equal to*

$$\prod_{i=1}^{-1-\lfloor -\delta \rfloor} \left(\lambda + \frac{i}{\delta}(c_1(L) - \lambda)\right) .$$

Proof. The first case follows from Proposition A.13. Now we calculate the edge contribution from the second case. Recall the divisor E of $\mathfrak{R}$ introduced in Section 4.1 is isomorphic to $\mathbb{P}Y_{r,s}$. The normal bundle $N_{E/\mathfrak{R}}$ is isomorphic to $\mathcal{O}(-s\mathcal{D}_0)$, and the fiber of the normal bundle $N_{E/\mathfrak{R}}$ over $\mathcal{D}_\infty$ has $\mathbb{C}^*-$weight 1, thus we have $f^*N_{E/\mathfrak{R}} = L^\vee_{as\delta\chi_1} \otimes \pi^*L$ as $\mathbb{C}^*-$equivariant line bundles over $\mathcal{C}_e$. Then we see that

$$-R^*\pi_*f^*N_{E/\mathfrak{R}} = R^1\pi_*(\pi^*L \otimes L^\vee_{as\delta\chi_1})$$

in the $K-$group $K^*(\mathcal{M}_e)$.

Write $\mathfrak{P} := \mathbb{P}Y_{r,s}$ for simplicity. Let $\mathbb{E}_\mathfrak{R}$ (resp. $\mathbb{E}_\mathfrak{P}$) be the pull-back of the perfect obstruction theory of $\mathcal{K}_{0,2}(\mathfrak{R}, (0, \frac{\delta}{r}, 0))$ (resp. $\mathcal{K}_{0,2}(\mathfrak{P}, (0, \frac{\delta}{r})))$ to $\mathcal{M}_e$. As $\mathcal{M}_e$ is étale over the corresponding $\mathbb{C}^*$-fixed loci in $\mathcal{K}_{0,2}(\mathfrak{P}, (0, \frac{\delta}{r}))$ (or $\mathcal{K}_{0,2}(\mathfrak{R}, (0, \frac{\delta}{r}, 0)))$ by Proposition A.15, we see that the $\mathbb{C}^*-$fixed parts $\mathbb{E}^{fix}_\mathfrak{P}$ and $\mathbb{E}^{fix}_\mathfrak{R}$ are perfect obstruction theory for $\mathcal{M}_e$. Note we have the distinguished triangle:

$$\mathbb{E}_\mathfrak{P} \longrightarrow \mathbb{E}_\mathfrak{R} \longrightarrow R^\bullet\pi_*f^*N_{E/\mathfrak{R}} \xrightarrow{+1} \mathbb{E}_\mathfrak{P}[1] . \tag{A.17}$$

As shown in Lemma A.13, $\mathbb{E}_\mathfrak{P}$ is $\mathbb{C}^*-$fixed, then the movable part of $\mathbb{E}_\mathfrak{R}$ is equal to the movable part of $R^\bullet\pi_*f^*N_{E/\mathfrak{R}}$, which we will calculate below.

The restriction of sections $x^{-sa\delta+asi}y^{ari}(1\leqslant i\leqslant -1-\lfloor -\delta\rfloor)$ to each fiber curve C_e of $\mathcal{C}_e$ over $\mathcal{M}_e$ is a basis of the vector space $H^1(C_e, f|_{C_e}^* N_{E/\mathfrak{R}})$. However they may not be sections of the vector bundle $R^1\pi_*(f^*N_{E/\mathfrak{R}})$. Instead, by Proposition A.12, one can see that $x^{-sa\delta+asi}y^{ari}$ is a section of vector bundle $R^1\pi_*(f^*N_{E/\mathfrak{R}})\otimes L^{-\otimes\frac{i}{\delta}}\otimes\mathbb{C}_{-\frac{\delta-i}{\delta}\lambda}$. Therefore $R^1\pi_*f^*N_{E/\mathfrak{R}}$ is isomorphic to the direct sum of line bundles $\bigoplus_{i=1}^{-1-\lfloor-\delta\rfloor}L^{\otimes\frac{i}{\delta}}\otimes\mathbb{C}_{\frac{\delta-i}{\delta}\lambda}$. Then the inverse of the Euler class of the virtual normal bundle form $\mathbb{E}_{\mathfrak{R}}$ is equal to $\prod_{i=1}^{-1-\lfloor-\delta\rfloor}\big(\lambda+\frac{i}{\delta}(c_1(L)-\lambda)\big)$.

The calculation above also shows that the $\mathbb{C}^*-$fixed part of $\mathbb{E}_{\mathfrak{R}}$ is the equal to $\mathbb{E}_{\mathfrak{P}}$, then we have $[\mathcal{M}_e]^{vir}=[\mathcal{M}_e]$ in Section 4.2.2. ☐

Remark A.17. When Y is of zero dimensional, our computation of the Euler class of the virtual normal bundle coincides the computation from [LS20, §6].

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