A MIRROR THEOREM FOR GROMOV-WITTEN THEORY WITHOUT CONVEXITY

JUN WANG

Abstract. We prove a genus zero Givental-style mirror theorem for all complete intersections in toric Deligne-Mumford stacks, which provides an explicit slice called big I-function on Givental's Lagrangian cone for such targets. In particular, we remove a technical assumption called convexity needed in the previous mirror theorem for such complete intersections. In the realm of quasimap theory, our mirror theorem can be viewed as solving the quasimap wall-crossing conjecture for big I-function [CFK16] for these targets. In the proof, we discover a new recursive characterization of the slice on Givental's Lagrangian cone, which may be of self-independent interests.

Contents

1. Introduction

In the past few decades, following predictions from string theory [CDLOGP91], a series of results known as mirror theorems has been proven; an incomplete list is [Giv96, CCIT15, CG07, Zin08, Giv98, CCLT09, LLY99, GJR17]. These theorems reveal elegant patterns and deep structures encoded in the collection of Gromov-Witten invariants of a given symplectic manifold or orbifold X. However, the scope of these results, and much of Gromov-Witten theory in general, is closely related to the world of toric geometry[1]; in all cases above, X is a toric variety/orbifold or certain complete intersection(See the discussion of convexity below) in a toric variety/orbifold. The essential reason for this is that one of most efficient way to compute Gromov-Witten invariants is to utilize the technique of the localization theorem [AB95, GP99], which requires the targets to be carried with a good torus action.

Smooth hypersurfaces (or complete intersections in general) in toric Deligne-Mumford stacks[2] are the next class of spaces to consider, but much less is known in this situation. The main difficulty comes from that a hypersurface in a toric stack doesn't have any nontrivial torus action in

Date: Friday 2$^{\text{nd}}$ June, 2023.

[1]By using the abelian-nonabelian correspondence, one can further extend the scope to include partial flag varieties [CFKS08, BCFK08] and other nonabelian GIT quotients like Nakajima quiver variety [Web23].

[2]We treat orbifold and Deligne-Mumford stack as synonyms.

general. Hence one can't directly apply localization theorem to compute the Gromov-Witten invariants of the toric hypersurface. Alternatively, the usual way to compute the Gromov-Witten invariants of a given hypersurface is to use *quantum Lefschetz principle* [KKP03], which relates the Euler-twisted virtual cycle of an ambient space X to the virtual cycle of its hypersurface Y which is the zero locus of a section of a given line bundle L on X. However, there is a technical assumption called *convexity* for the line bundle L to apply the *quantum Lefschetz principle*. The convexity says, for any stable map $f : C \to X$ of fixed genus and degree, one has

$$H^1(C, f^*L) = 0 \ ,$$

which holds, for example, when the ambient space X is a projective variety, the source curve C is of genus zero and L is a positive line bundle on X, and which doesn't hold, for example, when the ambient space X is a weighted projective space $\mathbb{P}(w_1, \cdots, w_n)$ and the line bundle $L \cong \mathcal{O}(d)$ satisfies that d is a positive integer which is not divided by all w_i. Hence, it's naturally to ask whether we can relax the condition from convexity to positivity to ensure the quantum Lefschetz principle to hold. Unfortunately, a counterexample was found in [CGI$^+$12] that *quantum Lefschetz principle* can fail for positive hypersurfaces in orbifolds. As a result, there are limited methods to compute the genus zero Gromov-Witten invariants of orbifold hypersurfaces where the convexity fails (see [Gué19] for a recent update for certain hypersurfaces in weighted projective spaces), and a genus zero mirror theorem[3] for these targets is lacking for a long time in the literature.

Quasimap theory, developed by Kim-Fontanine-Maulik and others, is a variation of Gromov-Witten theory and it's adapted to a wide class of GIT targets including complete intersection in toric orbifolds, Grassmanian and so on. Using quasimap theory, one can often calculate an explicit formula called big I-function, which is related to Gromov-Witten invariants by the so called genus zero quasimap wall-crossing conjecture [CFKM14, CCFK15, CFK16], which states the big I-function is a slice on the Lagrangian cone [Giv04]. Therefore we can use the big I-function to calculate Gromov-Witten invariants of toric complete intersections in the non-convex case once we solve the genus zero quasimap wall-crossing conjecture in such cases. The wall-crossing conjecture has been proved for GIT targets with a good torus action including toric orbifolds or complete intersections for which the convexity holds as a result of quantum Lefschetz principle and twisted version of the quasimap invariants. We will prove new cases of this conjecture to extend the validity to all toric complete intersections in this paper.

1.1. **Main results and Ideas of proof.**

1.1.1. *Big $\mathbb{I}-$function.* Let X be a *proper toric Deligne-Mumford stack* constructed by a GIT data $(W = \oplus_{\rho \in [n]} \mathbb{C}_\rho, G = (\mathbb{C}^*)^k, \theta)$, and $\iota : Y \subset X$ is a complete intersection with respect to a direct sum of line bundles $\oplus_{b=1}^c L_{\tau_b}$ on X (See §3 for more details). Now we introduce the following cohomology-valued series called big I-function (or I-function in short) of the toric

[3] In Givental's formalism, a mirror theorem usually means to construct an explicit slice on the Lagrangian cone.

stack complete intersections:

(1.1)

$$
\begin{aligned}
\mathbb{I}(q,t,z) = & \sum_{\beta\in \mathrm{Eff}(W,G,\theta)} q^\beta \exp\big(\frac{1}{z}\sum_{i=1}^{l} t_i u_i (c_1(L_{\pi_j})+\beta(L_{\pi_j})z)\big) \\
& \frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)<i<0}(D_\rho+(\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leqslant i<\beta(L_\rho)}(D_\rho+(\beta(L_\rho)-i)z)} \\
& \cdot \frac{\prod_{b:\beta(L_{\tau_b})>0}\prod_{i:0\leqslant i<\beta(L_{\tau_b})}(c_1(L_{\tau_b})+(\beta(L_{\tau_b})-i)z)}{\prod_{b:\beta(L_{\tau_b})<0}\prod_{i:\beta(L_{\tau_b})<i<0}(c_1(L_{\tau_b})+(\beta(L_{\tau_b})-i)z)} i_*(s^!_{E_\beta,loc}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)])) \ .
\end{aligned}
$$

We remark here $i_*(s^!_{E_\beta,loc}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]))$ and D_ρ are elements of the cohomology $H^*(\bar{I}_\mu Y,\mathbb{Q})$. $t=\sum_{i=1}^{l} t_i u_i(c_1(L_{\pi_j}))$ is an element in $H^*(Y,\mathbb{Q})[t_1,\cdots,t_l]$. See §3 for more details about the terminology appearing in $\mathbb{I}(q,t,z)$.

Now we state our main theorem:

Theorem 1.1 (Main Theorem). *$-z\mathbb{I}(q,t,-z)$ is a slice on Givental's Lagrangian cone of the toric complete intersection Y. More explicitly, let $\mu(z):=[z\mathbb{I}(q,t,z)-z\mathbb{1}_Y]_+$ be the* truncation in nonnegative z powers, *then we have the following identity:*

$$
\mathbb{I}(q,t,z)=J(q,\mu(y),z), \tag{1.2}
$$

where $J(q,\mu(y),z)$ is defined by the $J-$function

$$
\begin{aligned}
J(q,\mathbf{t},z) :=& \mathbb{1}_Y + \frac{\mathbf{t}(z)}{z} \\
& + \sum_{\beta\in\mathrm{Eff}(W,G,\theta)}\sum_{m\geqslant 0}\frac{q^\beta}{m!}\phi^\alpha\langle \mathbf{t}(-\bar{\psi}_1),\cdots,\mathbf{t}(-\bar{\psi}_m),\frac{\phi_\alpha}{z(z-\bar{\psi}_\star)}\rangle_{0,[m]\cup\star,\beta} \ .
\end{aligned}
$$

Here the input $\mathbf{t}$ is an element in $(q,t)H^*(\bar{I}_\mu Y,\mathbb{Q})[y][\![t_1,\cdots,t_l]\!][\![\mathrm{Eff}(W,G,\theta)]\!]$[4], and $\mathbf{t}(z)$ (resp. $\mathbf{t}(-\bar{\psi}_i)$) means that we replace the variable y in $\mathbf{t}$ by z (resp. $-\bar{\psi}_i$).

Note that here for any degree $\beta\in\mathrm{Eff}(W,G,\theta)$ of X (c.f. Definition 2.4), we will denote the Gromov-Witten invariant

$$
\phi^\alpha\langle \mathbf{t}(-\bar{\psi}_1),\cdots,\mathbf{t}(-\bar{\psi}_m),\frac{\phi_\alpha}{z(z-\bar{\psi}_\star)}\rangle_{0,[m]\cup\star,\beta}
$$

to be

$$
\sum_{\substack{d\in\mathrm{Eff}(AY,G,\theta)\\ i_*(d)=\beta}}\phi^\alpha\langle \mathbf{t}(-\bar{\psi}_1),\cdots,\mathbf{t}(-\bar{\psi}_m),\frac{\phi_\alpha}{z(z-\bar{\psi}_\star)}\rangle_{0,[m]\cup\star,d} \ ,
$$

where $\mathrm{Eff}(AY,G,\theta)$ is semigroup of degrees of Y.

Remark 1.2. The term $\mu(z)$ above is closely related to the procedure of *Birkohoff factorization* is the literature, from which we can get a closed-form $J-$function $J(q,\tau,z)$ with input $\tau\in H^*(\bar{I}_\mu Y,\mathbb{Q})[\![t_1,\ldots,t_l]\!][\![\mathrm{Eff}(W,G,\theta)]\!]$ having no $z-$terms, see e.g., [CCIT19] for more details. Actually the term τ, which is usually called a mirror map in the literature, is uniquely determined by $\mu(z)$ by the so-called Dijkgraaf–Witten formula [DW90].

The reader may also wonder how to apply this mirror theorem to calculate GW invariants(e.g. small quantum product); we present one example in §7, where we imitate the idea

[4]It means that $\mathbf{t}$ admits an expression as $\sum_{(\beta,i_1,\cdots,i_l)\neq 0} q^\beta t_1^{i_1}\cdots t_l^{i_l} f_{\beta,i_1,\cdots,i_l}$, where $f_{\beta,i_1,\cdots,i_l}\in H^*(\bar{I}_\mu Y,\mathbb{Q})[y]$. This choice of input $\mathbf{t}$ gives a much less general definition of Givental's $J-$function in the usual literature, but it suffices for the need in this paper.

used in [CCIT19] of computing GW invariants for toric stacks using extended variables from $S-$extended fan (Although the fan language for toric stacks is not used in this paper, we instead use the GIT setting. But these two approaches are equivalent. Further discussion of this equivalence can be found in [Wan]).

1.1.2. *Sketch of the proof of the main theorem.* Before sketching the proof of the main theorem, let's analyze the term $\mu(z)$ appearing in our main theorem. Write $z\mathbb{I}(q,t,z)$ as a formal Laurent series in variable z, z^{-1}:

$$\cdots + \mathbb{I}_{-1}(q,t)z^2 + \mathbb{I}_0(q,t)z + \mathbb{I}_1(q,t) + \mathcal{O}(z^{-1}),$$

then $\mu(z)$ can be expressed as:

$$\mu(z) := [z\mathbb{I}(q,t,z) - z\mathbb{1}_Y]_+ = \cdots + \mathbb{I}_{-1}(q,t)z^2 + (\mathbb{I}_0(q,t) - \mathbb{1}_Y)z + \mathbb{I}_1(q,t)\ .$$

By the definition of $\mathbb{I}(q,t,z)$, $z\mathbb{I}(q,t,z)$ admits an asymptotic expansion in q,t:

$$z\mathbb{I}(q,t,z) = z\mathbb{1}_Y + \mathcal{O}(q) + \mathcal{O}(t),$$

which implies that $\mu(z)$ belongs to the space $(q,t)H^*(\bar{I}_\mu Y,\mathbb{Q})[z][\![t_1,\cdots,t_l]\!][\![\mathrm{Eff}(W,G,\theta)]\!]$.

Let $\mathbb{I}(q,z) := \mathbb{I}(q,0,z)$, we can expand $\mathbb{I}(q,z)$ as

$$\mathbb{I}(q,z) = \sum_{\beta\in\mathrm{Eff}(W,G,\theta)} q^\beta \mathbb{I}_\beta(z)\ ,$$

where $\mathbb{I}_\beta(z) \in H^*(\bar{I}_\mu Y,\mathbb{Q})[z,z^{-1}]$. Then we can decompose $\mathbb{I}(q,t,z)$ as a formal sum

$$\mathbb{I}(q,t,z) = \sum_{\beta\in\mathrm{Eff}(W,G,\theta)} \sum_{p=0}^{\infty} q^\beta \frac{t^p}{p!z^p}\mathbb{I}_\beta(z)\ .$$

where $t = \sum_{i=1}^l t_i u_i(c_1(L_{\pi_j})+\beta(L_{\pi_j})z)$. For *nonzero pair* (β,p), set $\mu_{\beta,p}(z) := [\frac{t^p z\mathbb{I}_\beta(z)}{p!z^p}]_+$ as the truncation in nonnegative z powers. We note that $\mu_{\beta,p}(z)$ is a polynomial in $H^*(\bar{I}_\mu Y,\mathbb{Q})[t_0,\cdots,t_l,z]$ of homogeneous degree p in variables $t_1,\cdots,t_l$. Then we can write $\mu(z)$ as a sum

$$\mu(z) = \sum_{\beta\in\mathrm{Eff}(W,G,\theta)} \sum_{p\in\mathbb{Z}_{\geqslant 0}} q^\beta \mu_{\beta,p}(z)\ . \tag{1.3}$$

where $\mu_{(0,0)} = 0$, which we will also denote to be μ_0.

Multiply by z on both sides of (1.2), we observe that, to prove the main theorem, it suffices to prove that, for arbitrary pair $(\beta,p) \in \mathrm{Eff}(W,G,\beta)\times\mathbb{N}$ and any nonnegative integer c, one has

$$\begin{aligned}&[z\frac{t^p}{p!z^p}\mathbb{I}_\beta(z)]_{z^{-c-1}} := \\ &\sum_{m=0}^{\infty} \sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p}} \frac{1}{m!}\phi^\alpha\langle \mu_{\beta_1,p_1}(-\bar{\psi}_1),\cdots,\mu_{\beta_m,p_m}(-\bar{\psi}_m),\phi_\alpha\bar{\psi}_\star^c\rangle_{0,[m]\cup\star,\beta_\star}\ .\end{aligned} \tag{1.4}$$

The idea to prove (1.4) is to show that both sides of (1.4) satisfy the same recursive relations (see Theorem 6.4 and Theorem 6.6) by induction on the nonnegative integer $\beta(L_\theta)+p$. This is done by considering two master spaces carried with $\mathbb{C}^*-$actions (see §4.1 and §5.1), which are root-stack modifications of the twisted graph spaces. Then we apply virtual torus localization to express two auxiliary cycles (see (6.2) and (6.9)) corresponding to two master spaces in graph sums and extract λ^{-1} coefficients (λ is an equivariant parameter). Finally, the polynomiality of the two auxiliary cycles implies that the coefficients for λ^{-1} term must vanish, from which they yield the same type of recursive relations (see also Theorem 6.4 and Theorem 6.6) which finish the proof of the quasimap wall-crossing.

The quasimap wall-crossing conjecture for the big I-function was proven in [CFK16] for GIT targets possessing *a good torus action* or their complete intersections that fulfill convexity. Having a good torus action is described as having finite torus-fixed points and all one-dimensional torus-fixed orbits being isolated. The requirement of having a good torus action is essential in the previous proof of the big I-function since it allows for the characterization of a slice on the Lagrangian cone (or the twisted Lagrangian cone[5]). This characterization is established on the basis of having good torus action (c.f.[CFK14, Giv96, Bro13]). Consequently, it is natural to inquire whether it is possible to characterize a slice on the Lagrangian cone for targets lacking a good torus action. In this paper, we present one characterization (see Theorem 6.6) which can be adapted to general targets. This new result is expected to provide insights into other questions in Gromov-Witten theory as well.

The first version of this paper, available on arXiv, contains a section on explaining how to compute I-functions using quasimap theory, which was later realized by the author to be unnecessary in proving the mirror theorem. This highlights a unique aspect of the our method: we find a new recursive relation, detailed in Theorem 6.6, used to characterize the slice on the Lagrangian cone. To apply this new characterization, a suitable master space [6] together with a suitable auxiliary cycle is required to provide a recursive relation of the same type. From this, the explicit expression of the J-function can be obtained from a specific subgraph sum of the localization contribution. This naturally raises the question of whether other auxiliary master spaces can be found to prove a mirror theorem that was previously inaccessible. Further elaboration on this topic will be presented elsewhere[7]. For readers interested in the source of these I-functions, the first version of this paper on arXiv(which applies only to semi-positive hypersurfaces but can be extended to all complete intersections) or Rachel Webb's work [Web21] may be consulted. In her work, Webb obtains I-functions for all complete intersections in GIT quotients with possible non-abelian group actions, using the quasimap graph space directly and avoiding the p-fields method used in the author's first version.

During the preparation of this work, the author learns that Yang Zhou has used a totally different method to prove the quasimap wall-crossing conjecture for all GIT quotients and all genera [Zho22], which in particular implies the mirror theorem proved in this paper without exponential factor (but his formula is in less explicit form). The author also learns that Felix Janda, Nawaz Sultani and Zhou computed the (S-extended) I-function for some Calabi-Yau hypersurface in weighted projective spaces and use it to calculate Gromov-Witten invariants.

1.2. **Outline.** The rest of this paper is organized as follows. In §2, we will recall the quasimap theory, the author wants to draw readers' attention to the language of θ'-stable quasimaps (see Remark 2.3), where θ' can be a *rational character*, because it is more suitable than the language of ϵ-stable quasimaps for the later construction of the master space in §4. In §3, we collect some important facts about (rigidified) inertia stack of toric stack complete intersections, and compare them with rigidified inertia stack of toric stacks. Some special cycles in the inertia stack will be discussed as they will be appeared in our $\mathbb{I}-$function. In §4 and §5, we will construct two master spaces which carry $\mathbb{C}^*$-actions, a very explicit $\mathbb{C}^*$-localization computation which is based on localization computations [CJR17a, JPPZ17] will be presented, this part is technical, we encourage the reader to skip to go to §6 first and to refer back when needed. In §6, we will calculate two auxiliary cycles corresponding to the two master spaces via localization, they provide recursive relations to prove the genus zero quasimap wall-crossing conjecture for toric

[5]By leveraging the quantum Lefschetz principle, we can utilize the twisted analogue of the I-function quasimap wall-crossing to establish the I-function quasimap wall-crossing for complete intersections for which the convexity holds.

[6]In our case, this corresponds to the space constructed in §4.

[7]See the author's recent preprint [Wan23].

stack hypersurfaces. In §7, we calculate the small quantum product of a cubic hypersurface in $\mathbb{P}(1,1,1,2)$.

Notations: In this paper, we will always assume that all algebraic stacks and algebraic schemes are locally of finite type over the base field $\mathbb{C}$. Given a GIT target (W, G, θ), we will use symbols $\mathfrak{X}, \mathfrak{Y}$... to mean the quotient stack $[W/G]$, symbols X, Y... to mean the corresponding GIT stack quotient $[W^{ss}(\theta)/G]$, $I_\mu X$, $I_\mu Y$... to mean the corresponding inertia stacks, and $\bar{I}_\mu X$, $\bar{I}_\mu Y$...to mean the corresponding rigidified inertia stacks.

We will use the following construction a lot throughout this paper.

Definition 1.3 (**Borel Construction**). *Let G be a linear algebraic group and W be a variety. Fix a right G-action on the variety W. For any character ρ of G, we will denote L_ρ to be the line bundle on the quotient stack $[W/G]$ defined by*

$$W \times_G \mathbb{C}_\rho := [(W \times \mathbb{C}_\rho)/G] \ ,$$

where $\mathbb{C}_\rho$ is the 1-dimensional representation of G via ρ and the action is given by

$$(x, u) \cdot g = (x \cdot g, \rho(g)u) \in W \times \mathbb{C}_\rho$$

for all $(x, u) \in W \times \mathbb{C}_\rho$ and $g \in G$. For any linear algebraic group T, if we have a left T-action on W which commutes with the right action of G, we will lift the line bundle L_ρ defined above to be a T-equivariant line bundle, which is induced from the (left) T action on $W \times \mathbb{C}_\rho$ in the way that T acts on $\mathbb{C}_\rho$ trivially. By abusing notations, we will use the same notation L_ρ to mean the corresponding invertible sheaf (or $T-$equivariant invertible sheaf) over $[W/G]$ unless stated otherwise.

2. Background on quasimaps

We first recall the definition of a *quasimap* to a GIT target, our main reference is [CFKM14, CCFK15, CFK16]. By a GIT target, we mean a triple (W, G, θ), where W is an irreducible affine variety with locally complete intersection (l.c.i) singularity, G is a reductive group equipped with a right $G-$action on W and θ is an (integral) character of G. denote by $\mathfrak{X} := [W/G]$ the quotient stack. denote by W^{ss} (or $W^{ss}(\theta)$) the semistable locus in W, and by W^s (or $W^s(\theta)$) the stable locus. Throughout out this paper, for a GIT target (W, G, θ), we will always assume that $W^{ss}(\theta) = W^s(\theta)$ and the *GIT stack quotient*

$$X := [W^{ss}(\theta)/G]$$

is a smooth *Deligne-Mumford stack*, under which condition, X is always semi-projective, i.e. it's proper over the affine GIT quotient $\mathrm{Spec}(\mathbb{C}[W]^G)$ by the proj-construction of GIT quotient [CCFK15, §2.2][MFK94]:

$$\underline{X} = \mathbf{Proj} \oplus_{n=0}^{\infty} \Gamma(W, W \times \mathbb{C}_{n\theta})^G.$$

Let $\mathbf{e}$ be the least common multiple of the exponents $|\mathrm{Aut}(\bar{x})|$ of automorphism groups $\mathrm{Aut}(\bar{x})$ of all geometric points $\bar{x} \to X$ of X. Then, for any character ρ of G, the line bundle $L_\rho^{\otimes \mathbf{e}}$ is the pullback of a line bundle from the coarse moduli $\underline{X}$ of X, here the line bundle L_ρ is defined by the Borel (mixed) construction 1.3.

Definition 2.1. *Given a scheme S over Spec($\mathbb{C}$), $f = ((C, q_1, \cdots, q_m), P, x)$ is called a quasimap over S (alternatively θ-*quasimap *over S) of class (g, m, β) if it consists of the following data:*

1. *$(C, q_1, \cdots, q_m)$ is a flat family of genus g twisted curves over S* [AGV08, §4]*, and m gerbe marked sections $q_1, \cdots, q_m$ over S, here we don't require the gerbe sections to be trivialized;*
2. *P is a principal G-bundle on C;*

(3) *x is a section of the affine $W-$bundle $(P\times W)/G$ over C so that it determines a representable morphism $[x]: C\to\mathfrak{X}=[W/G]$ as the composition*

$$C \xrightarrow{x} (P\times W)/G \longrightarrow [W/G]\ .$$

We say that the quasimap f is of degree $\beta\in Hom_{\mathbb{Z}}(Pic(\mathfrak{X}),\mathbb{Q})$ if $\beta(L)=deg([x]^*L)$ for every line bundle $L\in Pic(\mathfrak{X})$;

(4) *The base locus of $[x]$ defined by $[x]^{-1}(\mathfrak{X}\backslash X)$ is purely of relative dimension zero over S.*

Sometimes we may also use the notation $f:(C,\boldsymbol{q}=(q_i))\to\mathfrak{X}$ to mean a quasimap (or θ-quasimap). A quasimap f is *prestable* (or *$\theta-$prestable*) if the base locus are away from nodes and markings.

Remark 2.2. We can extend the definition of θ-prestable quasimap to allow any *rational character* θ' such that θ'-prestable quasimap is same as $\alpha\theta'$-prestable quasimap for any $\alpha\in\mathbb{Q}_{>0}$.

Consider a prestable quasimap f, since the base point is away from nodes and marking points, for each $q\in C$, as in [CFKM14, Definition 7.1.1], we define the length function $l_\theta(q)$ as follows:

$$l_\theta(q)=min\{\frac{([x]^*s)_q}{n}|\ s\in\Gamma(W,W\times\mathbb{C}_{n\theta})^G,\ [x]^*s\neq 0,\ n\in\mathbb{Z}_{>0}\}\ , \tag{2.1}$$

where $([x]^*s)_q$ is the coefficient of the divisor $([x]^*s)$ at q. Note that here the length function l_θ depends on the *integral* character θ. We have the following important observation about the length function l_θ: choose $\alpha\in\mathbb{Q}_{>0}$ such that $\theta'=\frac{1}{\alpha}\theta$ is another integral character. Then

$$l_\theta=\alpha l_{\theta'}\ ,$$

then the length function l_θ can be defined for any *rational character* θ', i.e. choose $\alpha\in\mathbb{Q}_{>0}$ and an integral character θ such that $\theta'=\alpha\theta$, then we define

$$l_{\theta'}:=\alpha l_\theta$$

as in [CFK16, Definition 2.4], Note that the definition of $l_{\theta'}$ is independent of decomposition of θ' as a product of positive rational number α and an integral character θ by the above observation.

Fix a positive rational number $\epsilon\in\mathbb{Q}_{>0}$. Given a prestable quasimap f over $Spec(\mathbb{C})$, we say f is a *ϵ-stable* quasimap to X if f satisfies the following stability condition:

(1) the $\mathbb{Q}-$line bundle $(\phi_*([x]^*L_{\mathbf{e}\theta}))^{\frac{\epsilon}{\mathbf{e}}}\otimes\omega_{\underline{C}}^{log}$ on the coarse moduli curve $\underline{C}$ of C is ample. Here $\phi: C\to\underline{C}$ is the coarse moduli map. Note that the line bundle $[x]^*L_{\mathbf{e}\theta}$ on C a pullback of a line bundle on the coarse curve $\underline{C}$ by the choice of $\mathbf{e}$ and the prestable condition. Here $\omega_{\underline{C}}^{log}=\omega_{\underline{C}}(\sum_{i=1}^m\underline{q_i})$ is the log dualizing invertible sheaf of the coarse moduli $\underline{C}$;

(2) $\epsilon l_\theta(q)\leqslant 1$ for any $q\in C$.

Remark 2.3 (θ'*-quasimap*)**.** Using the above generalization of length function $l_{\theta'}$ for a rational character θ', we can give the definition of *θ'-stable quasimap*: given a θ'-prestable quasimap $f=((C,q_1,\cdots,q_m),[x])$, we say f is a *θ'-stable* quasimap to $\mathfrak{X}$ if

(1) the $\mathbb{Q}-$line bundle $(\phi_*([x]^*L_{\mathbf{be}\theta'}))^{\frac{1}{\mathbf{be}}}\otimes\omega_{\underline{C}}^{log}$ on the coarse moduli curve $\underline{C}$ of C is ample. Here $\phi: C\to\underline{C}$ is the coarse moduli map, and $\mathbf{b}$ is a positive integer which makes $\mathbf{b}\theta'$ an integral character. Note that the ampleness is independent of choice of the positive integer $\mathbf{b}$.

(2) $l_{\theta'}(q)\leqslant 1$ for any $q\in C$.

Given a GIT target (W, G, θ), following [CFK16, Propsition 2.7], an essentially equivalent definition about ϵ-stable quasimaps to X is, but from a different point of view, the concept of a $\epsilon\theta-$*stable quasimap* to $\mathfrak{X}$. The concept of θ'-stable quasimap will play an important role in the construction of master space in section 4. For a *rational character* θ' *of* G, we will use the notation $Q^{\theta'}_{g,m}(\mathfrak{X}, \beta)$ to mean the moduli stack of θ'-stable quasimaps to the quotient stack $\mathfrak{X}$ of class (g, m, β). If we choose $\theta' = \epsilon\theta$, then the space $Q^{\theta'}_{g,m}(\mathfrak{X}, \beta)$ is same as the space $Q^{\epsilon}_{0,m}([W^{ss}(\theta)/G], \beta)$ of ϵ-stable quasimaps we introduced before.

We call a prestable quasimap f over a scheme S is ϵ-*stable* if for every $\mathbb{C}$-point s of S, the restriction of f over s is ϵ-stable. We call f is $0+$*stable* if f is $\epsilon-$stable for every positive rational number $\epsilon \in \mathbb{Q}_{>0}$.

Definition 2.4. *A group homomorphism* $\beta \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Pic}\,\mathfrak{X}, \mathbb{Q})$ *is called* L_θ-effective *if it is realized as a finite sum of classes of some quasimaps to* X*. Such elements form a semigroup with identity* 0*, denoted by* $\mathrm{Eff}(W, G, \theta)$*.*

We will need the following lemma proved in [CCFK15, Lemma 2.3].

Lemma 2.5. *If* $((C, \boldsymbol{q}), [x])$ *is a quasimap of degree* β*, then* $\beta(L_\theta) \geqslant 0$*. Moreover,* $\beta(L_\theta) = 0$ *if and only if* $\beta = 0$*, if and only if the quasimap is constant (i.e.,* $[x]$ *is a map into* X*, factored through an inclusion* $\mathbb{B}\Gamma \subset X$ *of the classifying groupoid* $\mathbb{B}\Gamma$ *of a finite group* Γ*).*

In the following, we will give an explicit description of quasimaps to *toric Deligne-Mumford stacks.*

Example 2.6 (Quasimaps to toric stack)**.** Recall the construction of a (semi-projective) toric Deligne-Mumford stack (or toric stack in short) by a GIT data (W, G, θ). Let $G = (\mathbb{C}^*)^k$, and $W := \oplus_{i=1}^n \mathbb{C}_{\rho_i}$ be a direct sum of 1-dimensional representations of G given by the characters $\rho_i \in \chi(G)$ for $1 \leqslant i \leqslant n$. We will denote $[n]$ to be the tuple of (*not necessarily distinct*) characters ρ_i of G for $1 \leqslant i \leqslant n$. The toric stack X is defined to be the GIT stack quotient

$$[W^{ss}(\theta)/G].$$

Since we always assume that $W^{ss}(\theta) = W^s(\theta)$, then X is a *semi-projective Deligne-Mumford stack.* i.e., proper over an affine scheme.

Then in the definition of quasimaps to the toric stack X, we can replace the principal $G-$bundle P by k line bundles $(L_j : 1 \leqslant j \leqslant k)$ on C, and replace the section x in the definition of quasimap by n sections

$$\vec{x} = (x_i : 1 \leqslant i \leqslant n) \in \oplus_{\rho \in [n]} \Gamma(C, L_\rho)\ ,$$

where L_ρ is a line bundle on C defined by

$$L_\rho = \otimes_{j=1}^k L_j^{\otimes m_j}\ ,$$

where and the numbers $(m_j : 1 \leqslant j \leqslant k)$ are determined by the unique relation

$$\rho = \sum_{j=1}^k m_j \pi_j$$

in the character group $\chi(G)$ of G. Here $(\pi_j :\ 1 \leqslant j \leqslant k)$ are the standard characters of $G = (\mathbb{C}^*)^k$ by projecting to coordinates.

One novel application of $\theta'-$stable quasimap for a rational character θ' is the use of the notion of $(\theta', \boldsymbol{\varepsilon})-$stable quasimap introduced in [CFK16].

Definition 2.7. *[$(\theta',\boldsymbol{\varepsilon})$-stable quasimap] Given a tuple $\boldsymbol{\varepsilon}=(\varepsilon_1,\cdots,\varepsilon_p)\in(\mathbb{Q}_{>0}\cap(0,1])^p$, we will call prestable quasimap $\boldsymbol{f}:=(C,\boldsymbol{q},f:C\to[W/G]\times[\mathbb{C}/\mathbb{C}^*]^p)$ a $(\theta',\boldsymbol{\varepsilon})$-stable quasimap to $\mathfrak{X}$ of type (g,m,β) if $\boldsymbol{f}$ defines a $\theta'\oplus\bigoplus_{i=1}^p\varepsilon_i\mathrm{id}_{\mathbb{C}^*}$-stable quasimap to $[W/G]\times[\mathbb{C}/\mathbb{C}^*]^p$ of type $(g,m,(\beta,1,\cdots 1))$. We will denote $Q^{(\theta',\boldsymbol{\varepsilon})}_{g,m|p}(\mathfrak{X},\beta)$ to be the moduli stack of $(\theta',\boldsymbol{\varepsilon})$-stable quasimaps to $\mathfrak{X}$ of type (g,m,β). We call $\boldsymbol{f}$ is $(\theta',(0+)^p)$-stable if $\boldsymbol{f}$ is $(\theta',\boldsymbol{\varepsilon})$-stable for all $\boldsymbol{\varepsilon}\in\mathbb{Q}^p_{>0}$. And we will denote $Q^{\theta',0+}_{g,m|p}(\mathfrak{X},\beta)$ to be the moduli stack of $(\theta',(0+)^p)$-stable quasimaps to $\mathfrak{X}$ of type (g,m,β).*

Remark 2.8. It's shown in [CFK16] that a $(\theta',\boldsymbol{\varepsilon})$-stable map to $\mathfrak{X}$ is equivalent to a $\boldsymbol{\varepsilon}$-weighted θ-stable map to $\mathfrak{X}$, i.e. the source curve is allowed to be a Hassett-stable curve with additional p $\boldsymbol{\varepsilon}-$weighted markings. Thus the moduli stack $Q^{\theta',\boldsymbol{\varepsilon}}_{g,m|p}(\mathfrak{X},\beta)$ is equipped with p additional universal evaluation maps to $\mathfrak{X}$ (not only to X). We will denote them by

$$\hat{ev}_j:Q^{(\theta',\boldsymbol{\varepsilon})}_{g,m|p}(\mathfrak{X},\beta)\to\mathfrak{X},\quad 1\leqslant j\leqslant p\ .$$

2.1. **Quasimap invariants.** We define the quasimap invariants in this section following [CFK14, CFKM14, AGV08, CCFK15]. Consider an algebraic torus T action on W, which commutes with the given $G-$action on W, here T can be the identity group. Assume further that the $T-$fixed loci $\underline{X}_0^T$ of the affine quotient $\underline{X}_0=\mathrm{Spec}(\mathbb{C}[W]^G)$ is $0-$dimensional. We also denote $K:=\mathbb{Q}(\{\lambda_i\})$ by the rational localized T-equivariant cohomology of $\mathrm{Spec}\,\mathbb{C}$, with $\{\lambda_1,\dots,\lambda_{\mathrm{rank}(T)}\}$ corresponding to a basis for the characters of T. denote

$$\Lambda_K:=K[[\mathrm{Eff}(W,G,\theta)]]$$

to be the corresponding Novikov ring. We write q^β for the element corresponding to β in Λ_K so that Λ_K is the q-adic completion.

Given any two elements α_1,α_2 in the $T-$equivariant *Chen-Ruan cohomology* of X,

$$H^*_{\mathrm{CR},T}(X,\mathbb{Q}):=H^*_T(\bar{I}_\mu X,\mathbb{Q})\ ,$$

We can define the Poincaré pairing in the *non-rigidified* cyclotomic inertia stack $I_\mu X$ of X":

$$\langle\alpha_1,\alpha_2\rangle_{\mathrm{orb}}:=\int_{\sum_{r\in\mathbb{N}_{\geqslant 1}}r^{-1}[\bar{I}_{\mu_r}X]}\alpha_1\cdot\iota^*\alpha_2\ .$$

Here ι is the involution of $\bar{I}_\mu X$ obtained from the inversion automorphisms. Therefore, the diagonal class $[\Delta_{\bar{I}_{\mu_r}X}]$ obtained via push-forward of the fundamental class by $(\mathrm{id},\iota):\bar{I}_{\mu_r}X\to\bar{I}_{\mu_r}X\times\bar{I}_{\mu_r}X$ can be written as

$$\sum_{r=1}^\infty r[\Delta_{\bar{I}_{\mu_r}X}]=\sum_\alpha\phi_\alpha\otimes\phi^\alpha\text{ in }H^*(\bar{I}_\mu X\times\bar{I}_\mu X,\mathbb{Q}),$$

where $\{\phi_\alpha\}$ is a basis of $H^*_{\mathrm{CR},T}(X,\mathbb{Q})$ with $\{\phi^\alpha\}$ the dual basis with respect to the Poincaré pairing defined above.

denote by $\bar{\psi}_i$ the first Chern class of the universal cotangent line whose fiber at $((C,q_1,...,q_m),[x])$ is the cotangent space of the coarse moduli $\underline{C}$ of C at i-th marking $\underline{q}_i$. For non-negative integers a_i and classes $\alpha_i\in H^*_T(\bar{I}_\mu X,\mathbb{Q})$, $\delta_j\in H^*(\mathfrak{X},\mathbb{Q})$, we write

$$\langle\alpha_1\bar{\psi}^{a_1},...,\alpha_m\bar{\psi}^{a_m};\delta_1,\cdots,\delta_l\rangle^{\theta',\boldsymbol{\varepsilon}}_{0,m,\beta}:=\int_{[Q^{\theta',\boldsymbol{\varepsilon}}_{0,m|p}(\mathfrak{X},\beta)]^{\mathrm{vir}}}\prod_i ev_i^*(\alpha_i)\bar{\psi}_i^{a_i}\prod_j\hat{ev}_j^*(\delta_j)\ .$$

When ε is empty, $\theta'=\epsilon\theta$ for sufficiently large rational number ϵ, the above formula recovers the usual Gromov-Witten invariants, in which case, we will write this as

$$\langle\alpha_1\bar{\psi}^{a_1},...,\alpha_m\bar{\psi}^{a_m}\rangle\ .$$

We will also need the quasimap Chen-Ruan classes

$$(\widetilde{ev_j})_* = \iota_*(\boldsymbol{r}_j (ev_j)_*), \tag{2.2}$$

where $\boldsymbol{r}_j$ is the order function of the band of the gerbe structure at the marking q_j. Define a class in $H_*^T(\bar{I}_\mu X) \cong H_T^*(\bar{I}_\mu X)$ by

$$\begin{aligned}\langle \alpha_1, ..., \alpha_m, - \rangle_{0,\beta}^{\epsilon} &:= (\widetilde{ev_{m+1}})_* \left((\prod ev_i^* \alpha_i) \cap [Q_{0,m}^{\epsilon}(X, \beta)]^{\mathrm{vir}} \right) \\ &= \sum_{\alpha} \phi^{\alpha} \langle \alpha_1, ..., \alpha_m, \phi_\alpha \rangle_{0,m+1,\beta}^{\epsilon} .\end{aligned}$$

3. Geometry of complete intersections in toric Deligne-Mumford stacks

From now on, we will fix a GIT data $(W = \mathbb{C}^n, G = (\mathbb{C}^*)^k, \theta)$, which represents a *proper* toric Deligne-Mumford stack (or toric stack in short) $X := [W^{ss}(\theta)/G]$ as in example 2.6. We will also fix a vector bundle E over $\mathfrak{X} := [W/G]$ which is a direct sum of line bundles $\oplus_{b=1}^c L_{\tau_b}$ associated to characters $(\tau_b)_{b=1}^c$ of G. Let $s_b \in \Gamma(W, W \times \mathbb{C}_{\tau_b})^G$ be sections such that they cut off an irreducible complete intersection in W which is smooth in $W^{ss} := W^{ss}(\theta)$. denote by AY the zero loci of the section $s := \oplus_{b=1}^c s_b$ and by $AY^{ss} = AY^{ss}(\theta)$ the corresponding semistable loci, then (AY, G, θ) determines a GIT quotient $Y := [AY^{ss}(\theta)/G]$, which is a complete intersection in X. We will denote $\mathfrak{Y} := [AY/G]$ to be the quotient stack corresponding to Y. Note that AY^{ss} is equal to the intersection of W^{ss} and AY.

It's well known the rigidified inertia stacks of Y and X are

$$\bar{I}_\mu Y = \bigsqcup_{g \in G} [AY^{ss}(\theta)^g/(G/\langle g \rangle)], \ \bar{I}_\mu X = \bigsqcup_{g \in G} [W^{ss}(\theta)^g/(G/\langle g \rangle)] .$$

For each $g \in G$, denote by $\bar{I}_g Y := [AY^{ss}(\theta)^g/(G/\langle g \rangle)]$ and $\bar{I}_g X := [W^{ss}(\theta)^g/(G/\langle g \rangle)]$ the rigidified inertia components of X and Y respectively. We note that here that $\bar{I}_g Y$ or $\bar{I}_g X$ is nonempty only if g is torsion as Y(and X) are Deligne-Mumford stacks.

To describe the relationship between $\bar{I}_\mu X$ and $\bar{I}_\mu Y$, we will need the following lemma:

Lemma 3.1. *For any torsion element $g \in G$, the inclusion of $g-$fixed subspaces $AY^{ss}(\theta)^g \subset W^{ss}(\theta)^g$ is a complete intersection with respect to the sections $\{s_b | b : \tau_b(g) = 1\}$.*

Proof. For any point $p \in W^{ss}(\theta)^g$ such that s vanishes on p, we have the following short exact sequence of tangent spaces

$$0 \to T_p AY^{ss}(\theta) \to T_p W^{ss}(\theta) \to \oplus_{b=1}^c \mathbb{C}_{\tau_b} \to 0 ,$$

which is also exact as representations of the finite group generated by g. Taking the g-invariant subspace of the above exact sequence, we get

$$0 \to T_p AY^{ss}(\theta)^g \to T_p W^{ss}(\theta)^g \to \oplus_{b:\tau_b(g)=1} \mathbb{C}_{\tau_b} \to 0 ,$$

which imply the lemma. □

For any degree $\beta \in \mathrm{Eff}(W, G, \theta)$, we will define an element $g_\beta \in G$, and two special subvarieties $Y_\beta^{ss} \subset AY^{ss}$, $Z_\beta^{ss} \subset W^{ss}$ needed in the statement of the mirror theorem:

$$\begin{aligned} g_\beta &:= (\mathrm{e}^{2\pi\sqrt{-1}\beta(L_{\pi_1})}, \cdots, \mathrm{e}^{2\pi\sqrt{-1}\beta(L_{\pi_k})}) \in G = (\mathbb{C}^*)^k , \\ Y_\beta^{ss} &:= (AY^{ss})^{g_\beta} \cap \{(x_i) \in W | x_i = 0 \ \forall i : \beta(L_{\rho_i}) \in \mathbb{Z}_{<0}\} , \\ Z_\beta^{ss} &:= (W^{ss})^{g_\beta} \cap \{(x_i) \in W | x_i = 0 \ \forall i : \beta(L_{\rho_i}) \in \mathbb{Z}_{<0}\} . \end{aligned}$$

In the end of this section, we will prove a lemma 3.2 relating the geometry of Y_β^{ss} and Z_β^{ss}.

The geometrical significance of introducing Y_β^{ss} and Z_β^{ss} is that the quotient stacks $[Y_\beta^{ss}/G]$ and $[Z_\beta^{ss}/G]$ describe important classes in the stacky loop spaces for X and Y which we now describe.

First of all, let's recall the definition of stacky loop space into the toric stack X (c.f. [CCFK15]). Set $U = \mathbb{C}^2\backslash\{0\}$, for any positive integer a, denote $\mathbb{P}_{a,1}$ to be the quotient stack $[U/\mathbb{C}^*]$ defined by the $\mathbb{C}^*$-action on U with weights $[a, 1]$ so that $0 := [0 : 1]$ is a non-stacky point and $\infty := [1 : 0] \cong \mathbb{B}\mu_a$ is a stacky point. The stacky loop space into X

$$Q_{\mathbb{P}_{a,1}}(X,\beta) \subset Hom^{rep}_{\beta}(\mathbb{P}_{a,1},\mathfrak{X})$$

is defined to be the moduli stack of representable morphisms from $\mathbb{P}_{a,1}$ to $\mathfrak{X}$ of degree β such that the generic point of $\mathbb{P}_{a,1}$ is mapped into X. By [CCFK15, Lemma 4.6], for such a representable morphism to exist, a must be the order of the finite cyclic group generated by g_β. We note that a is also the minimal positive integer making $a\beta(L_\tau)$ an integer for all character τ of G. We can define the stacky loop space into Y in a similar manner, denote

$$Q_{\mathbb{P}_{a,1}}(Y,\beta) \subset Hom^{rep}_{\beta}(\mathbb{P}_{a,1},\mathfrak{Y})$$

by the moduli stack of representable morphisms from $\mathbb{P}_{a,1}$ to $\mathfrak{Y}$ of degree β such that the generic point of $\mathbb{P}_{a,1}$ is mapped into Y.

Let a be the integer associated to g_β. Let $\mathbb{C}[z_1, z_2]$ be the polynomial ring on variables z_1 and z_2 with weights a and 1 respectively. Consider the finite dimensional vector space

$$W_\beta := \bigoplus_{\rho\in[n]} \mathbb{C}[z_1, z_2]_{a\beta(L_\rho)}$$

with the G-action given by the direct sum of the diagonal G-actions where G on acts on the component $\mathbb{C}[z_1, z_2]_{a\beta(L_\rho)}$ by the character ρ, then $\mathbb{C}[z_1, z_2]_{a\beta(L_\rho)} \cong \bigoplus \mathbb{C}_\rho$. Given any element of W_β, we can naturally associate a morphism from $\mathbb{P}_{a,1}$ to $\mathfrak{X}$ of degree β. Then we have the equivalence of the following two stacks:

$$\mathrm{Hom}^{\mathrm{rep}}_{\beta}(\mathbb{P}_{a,1},\mathfrak{X}) \cong [W_\beta/G] \ ,$$

under which correspondence, we have

$$Q_{\mathbb{P}_{a,1}}(X,\beta) \cong [W^{ss}_\beta(\theta)/G] \ .$$

Consider the $\mathbb{C}^*-$action on $\mathbb{P}_{a,1}$ defined by

$$t(\zeta_1,\zeta_2) = (t\zeta_1,\zeta_2) \ ,$$

for all $(\zeta_1, \zeta_2) \in U$ and $t \in \mathbb{C}^*$. This induces a $\mathbb{C}^*-$action on $Q_{\mathbb{P}_{a,1}}(X, \beta)$ as well as on $Q_{\mathbb{P}_{a,1}}(Y, \beta)$. Denote $F_\beta(X)$ (resp. $F_\beta(Y)$) to be the subspace of $Q_{\mathbb{P}_{a,1}}(X, \beta)$(resp. $Q_{\mathbb{P}_{a,1}}(Y, \beta)$) which consists of representable morphisms $f : \mathbb{P}_{a,1} \to \mathfrak{X}$ (resp. $f : \mathbb{P}_{a,1} \to \mathfrak{Y}$) with $[0\!:\!1]$ as the only base point. More explicitly, $F_\beta(X)$ (resp. $F_\beta(Y)$) is comprised of the morphisms in the form

$$f : \mathbb{P}_{a,1} \to \mathfrak{X} \quad (\text{resp. } \mathfrak{Y}), \quad (\zeta_1,\zeta_2) \mapsto (a_\rho\zeta_1^{\beta(L_\rho)})_{\rho\in[n]} \ ,$$

where the coefficients (a_ρ) satisfy that $(a_\rho z_1^{\beta(L_\rho)} : \rho \in [n]) \in W^{ss}_\beta(\theta)$. Note that for such a map to be well-defined, a_ρ must be 0 when $\beta(L_\rho) \notin \mathbb{Z}_{\geqslant 0}$.

We can see that $F_\beta(X)$ is a component of the $\mathbb{C}^*-$fixed loci of $Q_{\mathbb{P}_{a,1}}(X, \beta)$, which we can describe more explicitly as follows. Define

$$Z_\beta := \bigoplus_{\rho\in[n],\beta(L_\rho)\in\mathbb{Z}_{\geqslant 0}} \mathbb{C}\cdot z_1^{\beta(L_\rho)} \subset W_\beta.$$

We have $Z^{ss}_\beta \cong Z_\beta \cap W^{ss}_\beta(\theta)$, and

$$F_\beta(X) \cong [Z^{ss}_\beta/G], \text{ and } F_\beta(Y) \cong [Y^{ss}_\beta/G] \ .$$

It's clear that Y^{ss}_β is cut off by the sections $\{s_b | b : \beta(L_{\tau_b}) \in \mathbb{Z}\}$ on Z^{ss}_β, but this may not be a complete intersection. Indeed, one can show the following.

Lemma 3.2. *For any b such that $\beta(L_{\tau_b}) \in \mathbb{Z}_{<0}$, the section s_b vanishes on Z^{ss}_β. Thus Y^{ss}_β is merely the vanishing loci of sections $\{s_b | b : \beta(L_{\tau_b}) \in \mathbb{Z}_{\geqslant 0}\}$ in Z^{ss}_β.*

Proof. For b with $\beta(L_{\tau_b}) \in \mathbb{Z}_{\leqslant 0}$, for any point $\vec{x} = (a_\rho)_{\rho\in[n]} \in Z^{ss}_\beta$, the corresponding morphism in $F_\beta(X)$ is in the form

$$[\vec{x}] : \mathbb{P}_{a,1} \to \mathfrak{X} : [\zeta_1, \zeta_2] \to (a_\rho \zeta_1^{\beta(L_\rho)})_{\rho\in[n]} \ .$$

Then the pullback of section s_b to $\mathbb{P}_{a,1}$ becomes $s_b(\vec{x}) z_1^{\beta(L_{\tau_b})}$. However as the pull-back line bundle $[\vec{x}]^* L_{\tau_b}$ is of degree $\beta(L_{\tau_b}) < 0$ on $\mathbb{P}_{a,1}$, hence there is no nonzero section in the line bundle $[\vec{x}]^* L$, which implies that $s_b(\vec{x}) = 0$. Now the lemma follows. □

Definition 3.3. *Denote $E_\beta := \oplus_{b:\beta(L_{\tau_b})\in\mathbb{Z}_{\geqslant 0}} L_{\tau_b}$ and $s_\beta = \oplus_{b:\beta(L_{\tau_b})\in\mathbb{Z}_{\geqslant 0}} s_b$. We will also use the notations E_β and s_β to mean the vector bundle and the section for $[Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]$ by descent. Using the above lemma, we have the following Cartesian diagram*

$$\begin{tikzcd}
{[Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]} \arrow[r, "i"] \arrow[d, "i"] & {[Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]} \arrow[d, "s_\beta"] \\
{[Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]} \arrow[r, "0"] & E_\beta \ ,
\end{tikzcd}$$

where the bottom arrow is the zero section, $i : [Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)] \to [Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]$ is the inclusion.

Then we have a Gysin pullback $0^! : A_*([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]) \to A_*([Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)])$, which is also denoted by $s^!_{E_\beta,loc}$ known as the localized top Chern class [Ful84, §14.1] *with respect to the vector bundle E_β over $[Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)]$ and the section s_β.*

Let $i : [Y^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)] \to \bar{I}_{g_\beta^{-1}} Y$ be the natural inclusion. Now we discuss two implications of the above lemma:

Corollary 3.4. *We have the following:*

(1) *If the set $\{b \mid \beta(L_{\tau_b}) \in \mathbb{Z}\}$ is exactly the set $\{b \mid \beta(L_{\tau_b}) \in \mathbb{Z}_{\geqslant 0}\}$, then we have*

$$i_*(s^!_{E_\beta,\mathrm{loc}}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)])) = (\prod_{\rho:\beta(L_\rho)\in\mathbb{Z}_{\leqslant 0}} D_\rho) \cdot \mathbb{1}_{g_\beta^{-1}}$$

in $A_*(\bar{I}_{g_\beta^{-1}} Y)$, where $\mathbb{1}_{g_\beta^{-1}}$ is the fundamental class of $\bar{I}_{g_\beta^{-1}} Y$, $D_\rho = c_1(L_\rho)$ is the class of the hyperplane given by $x_\rho = 0$.

(2) *If the set $\{b \mid \beta(L_{\tau_b}) \in \mathbb{Z}\}$ is empty, then we have $Y^{ss}_\beta = Z^{ss}_\beta$, and $\bar{I}_{g_\beta^{-1}} Y = \bar{I}_{g_\beta^{-1}} X$, and $s^!_{E_\beta,\mathrm{loc}}$ is the identity morphism. Thus*

$$i_*(s^!_{E_\beta,\mathrm{loc}}([Z^{ss}_\beta/(G/\langle g_\beta^{-1}\rangle)])) = (\prod_{\rho:\beta(L_\rho)\in\mathbb{Z}_{\leqslant 0}} D_\rho) \cdot \mathbb{1}_{g_\beta^{-1}}$$

in $A_*(\bar{I}_{g_\beta^{-1}} Y)$.

Recall that the twisted I-function [CCIT19] for toric stack X with respect to the vector bundle $\oplus_b L_{\tau_b}$ is

$$
\begin{aligned}
I_X^{tw} = & \sum_{\beta \in \mathrm{Eff}(W,G,\theta)} q^\beta \exp\big(\frac{1}{z}\sum_{i=1}^n t_i(c_1(L_{\rho_i}) + \beta(L_{\rho_i})z)\big) \\
& \frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)\leqslant i<0}(D_\rho + (\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leqslant i<\beta(L_\rho)}(D_\rho + (\beta(L_\rho)-i)z)} \\
& \cdot \frac{\prod_{b:\beta(L_{\tau_b})>0}\prod_{i:0\leqslant i<\beta(L_{\tau_b})}(\kappa + c_1(L_{\tau_b}) + (\beta(L_{\tau_b})-i)z)}{\prod_{b:\beta(L_{\tau_b})<0}\prod_{i:\beta(L_{\tau_b})\leqslant i<0}(\kappa + c_1(L_{\tau_b}) + (\beta(L_{\tau_b})-i)z)} \mathbb{1}_{g_\beta^{-1}} \ .
\end{aligned}
$$

Here we discard the factor z of the twisted I-funtion in [CCIT19].

We have the following relation between our big I-function and the twisted I-function.

Corollary 3.5. *Expand the twisted I-function I_X^{tw} in Novikov variables*

$$
I_X^{tw} = \sum_\beta q^\beta I_X^{\beta,tw} \ .
$$

Note that $I_X^{\beta,tw}$ belongs to $H^*(\bar{I}_{g_\beta^{-1}}X)[z^{-1},z][[t_1,\cdots,t_n]]$. Define $I_X^{tw}\prod_b\big(\kappa + c_1(L_{\tau_b})\big)$ to be

$$
\sum_\beta q^\beta I_X^{\beta,tw} \prod_{b:\beta(L_{\tau_b})\in\mathbb{Z}} \big(\kappa + c_1(L_{\tau_b})\big) \ .
$$

Note that $\prod_{b:\beta(L_{\tau_b})\in\mathbb{Z}} c_1(L_{\tau_b})$ is the Euler class of the normal bundle of inertia component $\bar{I}_{g_\beta^{-1}}Y$ in $\bar{I}_{g_\beta^{-1}}X$. Then $I_X^{tw}\prod_b\big(\kappa + c_1(L_{\tau_b})\big)$ has a limit as κ goes to zero, and it's equal to push-forward $\iota_*\mathbb{I}(q,t,z)|_{u_i\left(c_1(L_{\pi_j})\right)=c_1(L_{\rho_i})}$ along the inclusion $\iota : \bar{I}_\mu Y \to \bar{I}_\mu X$.

Proof. Using the fact(c.f. Corollary 3.4)

$$
\iota_*\big(i_*(s^!_{E_\beta,loc}([Z_\beta^{ss}/(G/\langle g_\beta^{-1}\rangle)]))\big) = \mathbb{1}_{g_\beta^{-1}} \cdot \prod_{b:\beta(L_{\tau_b})\in\mathbb{Z}_{\geqslant 0}} c_1(L_{\tau_b}) \cdot \prod_{\rho:\beta(L_\rho)\in\mathbb{Z}_{<0}} c_1(L_\rho) \ .
$$

□

3.1. **Two special cases of the mirror theorem.** Using corollary 3.4, we consider two interesting special cases of the I-function. The first case is when Y is a hypersurface with respect to a line bundle $L := L_\tau$ for some character τ, the mirror formula (1.1) becomes:

$$
\begin{aligned}
\mathbb{I}(q,t,z) = & \sum_{\substack{\beta\in \mathrm{Eff(W,G,\theta)}\\ \beta(L)\geqslant 0}} q^\beta \exp\cdot \frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)\leqslant i<0}(D_\rho+(\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leqslant i<\beta(L_\rho)}(D_\rho+(\beta(L_\rho)-i)z)}\\
&\times \prod_{0\leqslant i<\beta(L)}\big(c_1(L)+(\beta(L)-i)z\big)\mathbb{1}_{g_\beta^{-1}}\\
&+\sum_{\substack{\beta\in \mathrm{Eff(W,G,\theta)}\\ \beta(L)\in\mathbb{Z}_{<0}}} q^\beta \exp\cdot \frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)< i<0}(D_\rho+(\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leqslant i<\beta(L_\rho)}(D_\rho+(\beta(L_\rho)-i)z)}\\
&\times \prod_{\beta(L)<i<0}\frac{1}{\big(c_1(L)+(\beta(L)-i)z\big)}\big[[Y_\beta^{ss}/(G/\langle g_\beta^{-1}\rangle)]\big]\\
&+\sum_{\substack{\beta\in \mathrm{Eff(W,G,\theta)}\\ \beta(L)\in\mathbb{Q}_{<0}\backslash\mathbb{Z}_{<0}}} q^\beta \exp\cdot \frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)\leqslant i<0}(D_\rho+(\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leqslant i<\beta(L_\rho)}(D_\rho+(\beta(L_\rho)-i)z)}\\
&\times \prod_{\beta(L)<i<0}\frac{1}{\big(c_1(L)+(\beta(L)-i)z\big)}\mathbb{1}_{g_\beta^{-1}}\ .
\end{aligned}
\tag{3.1}
$$

Here exp is short for $\exp\big(\frac{1}{z}\sum_{i=1}^{l}t_iu_i(c_1(L_{\pi_j})+\beta(L_{\pi_j})z)\big)$ and $\big[[Y_\beta^{ss}/(G/\langle g_\beta^{-1}\rangle)]\big]$ is the fundamental class of $[Y_\beta^{ss}/(G/\langle g_\beta^{-1}\rangle)]$ in $H^*(\bar{I}_{g_\beta^{-1}}Y)$.

Remark 3.6. The reader may wonder whether we can express the cohomology class $\big[[Y_\beta^{ss}/(G/\langle g_\beta^{-1}\rangle)]\big]$ as the product of $\mathbb{1}_{g_\beta^{-1}}$ and D_ρ like in other cases. Note that this will in particular imply that $\big[[Y_\beta^{ss}/(G/\langle g_\beta^{-1}\rangle)]\big]$ is an ambient cohomology, i.e. a cohomology class pulled back from the Chen-Ruan cohomology $H^*(\bar{I}_\mu X)$ of the ambient toric stack. However $\big[[Y_\beta^{ss}/(G/\langle g_\beta^{-1}\rangle)]\big]$ is not an ambient cohomology class in general. For example, take $X=\mathbb{P}^3$, Y is quadratic hypersurface of X. We will choose a GIT presentation of X and degree β such that $\big[[Y_\beta^{ss}/(G/\langle g_\beta^{-1}\rangle)]\big]$ can be the line $\{[0,*,*,0]\in\mathbb{P}^3\}$. To achieve this, we choose a non-standard GIT presentation of $\mathbb{P}^3$: Let $W=\mathbb{C}^5$, $G=(\mathbb{C}^*)^2$ so that G acts on W via the right action

$$(x_1,x_2,x_3,x_4,x_5)\cdot(t_1,t_2)=(t_1x_1,t_1t_2x_2,t_1t_2x_3,t_1x_4,t_2x_5)\ ,$$

where $(x_1,x_2,x_3,x_4,x_5)\in W$ and $(t_1,t_2)\in G$. If we choose the stablity condition $\theta(t_1,t_2)=t_1t_2^2\in\chi(G)$, we have $W^{ss}(\theta)=(\mathbb{C}^4\backslash\{0\})\times\mathbb{C}^*$, let Y be the quadratic hypersurface cut off by the polynomial $x_1x_2-x_3x_4$ and we choose degree $\beta\in\mathrm{Eff}(W,G,\theta)$ defined by $\beta(L_{t_1})=-1$ and $\beta(L_{t_2})=1$. It's a very interesting question to use this to calculate the GW invariants with insertion of non-ambient cohomology classes and we will explain how to do it elsewhere.

The second case is when all the line bundles L_{τ_b} are all semi-positive, i.e. $\beta(L_{\tau_b})\geqslant 0$ for all $\beta\in\mathrm{Eff}(W,G,\theta)$ and b. Then the I-function specializes to:

$$
\begin{aligned}
\mathbb{I}(q,t,z) = & \sum_{\beta\in \mathrm{Eff(W,G,\theta)}} q^\beta \exp\big(\frac{1}{z}\sum_{i=1}^{l}t_iu_i(c_1(L_{\pi_j})+\beta(L_{\pi_j})z)\big)\\
&\frac{\prod_{\rho:\beta(L_\rho)<0}\prod_{\beta(L_\rho)\leqslant i<0}(D_\rho+(\beta(L_\rho)-i)z)}{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leqslant i<\beta(L_\rho)}(D_\rho+(\beta(L_\rho)-i)z)}\\
&\times\prod_b\prod_{0\leqslant i<\beta(L_{\tau_b})}\big(c_1(L_{\tau_b})+(\beta(L_{\tau_b})-i)z\big)\mathbb{1}_{g_\beta^{-1}}\ .
\end{aligned}
\tag{3.2}
$$

The above formulae match the formula for positive hypersurfaces in toric stacks for which the convexity holds [CCIT15, §5] and the formula for a ray divisor (given by a coordinate function corresponding to the ray) of a toric stack for which the convexity may fail [CCIT15, CCFK15]. See §7 for a non-positive example where the convexity fails.

4. Master space I

4.1. **Construction of master space I.** In this section, we will construct a master space which is a root stack modification of the twisted graph space considered in [CJR17a]. Let (AY, G, θ) be the GIT data which gives rise to a complete intersection in the toric stack $X = [W^{ss}(\theta)/G]$ as in previous sections. *Since a positive rational scaling of the stability character θ will not change the GIT quotient. Without loss of generality, let's assume that the line bundle L_θ on $Y = [AY^{ss}(\theta)/G]$ is the pullback of a positive line bundle on the coarse moduli space $\underline{Y}$ of Y.* First we will consider the following quotient stack

$$\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p} = [(AY \times \mathbb{C}^p \times \mathbb{C}^2)/(G \times (\mathbb{C}^*)^p \times \mathbb{C}^*)]$$

defined by the following (right) action

$$(\vec{x}, \vec{y}, z_1, z_2) \cdot (g, h, t) = (\vec{x} \cdot g, (h_j y_j)_{j=1}^p, \theta(g)^{-1}(\prod_{j=1}^p h_j^{-1}) t^r z_1, t z_2) \ ,$$

where $(g, h = (h_j)_{j=1}^p, t) \in G \times (\mathbb{C}^*)^p \times \mathbb{C}^*$ $(\vec{x}, \vec{y} = (y_j)_{j=1}^p, z_1, z_2) \in AY \times \mathbb{C}^p \times \mathbb{C}^2$. For simplicity, we will write $AY_p := AY \times \mathbb{C}^p$, and $G_p := G \times (\mathbb{C}^*)^p$. Let θ_p be the character of G_p defined by

$$\theta_p(g, h) = \theta(g) \prod_{j=1}^p h_j \text{ for all } (g, h) \in G_p \ .$$

Fix a positive rational number $\epsilon \in \mathbb{Q}_{>0} \cap (0, 1]$ and a tuple of positive rational numbers $\boldsymbol{\varepsilon} = (\epsilon, \cdots, \epsilon) \in (\mathbb{Q}_{>0})^p$, we consider the stability given by the rational character of $G_p \times \mathbb{C}^*$ defined by

$$\widetilde{\theta}(g, h, t) = \theta_p(g, h)^\epsilon t^{3r}$$

for $(g, h, t) \in G_p \times \mathbb{C}^*$. Then the GIT stack quotient $[(AY_p \times \mathbb{C}^2)^{ss}(\widetilde{\theta})/(G_p \times \mathbb{C}^*)]$ is the root stack of the $\mathbb{P}^1-$bundle $\mathbb{P}_Y(\mathcal{O}(-D_\theta) \oplus \mathcal{O})$ over Y by taking r-th root of the infinity divisor D_∞ given by $z_2 = 0$. We will denote the GIT stack quotient $[(AY_p \times \mathbb{C}^2)^{ss}(\widetilde{\theta})/(G_p \times \mathbb{C}^*)]$ to be $\mathbb{P}Y^{\frac{1}{r}}$, which is equipped with the infinity section $\mathcal{D}_\infty$ given by $z_2 = 0$ and the zero section $\mathcal{D}_0$ given by $z_1 = 0$. Note that this GIT quotient is independent of the integer p as the semistable(=stable) loci $(AY_p \times \mathbb{C}^2)^{ss}(\widetilde{\theta}) = AY^{ss}(\theta) \times (\mathbb{C}^*)^p \times (\mathbb{C}^2 \backslash \{0\})$. We will take $p = 0$ as our standard GIT quotient reference for $\mathbb{P}Y^{\frac{1}{r}}$, which will be canonically identified with other GIT quotients from $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ by choosing the embedding $AY \subset AY_p$ as $AY \cong AY_p \cap \{y_i = 1 | i = 1, \ldots, p\}$.

When the integer r is prime to the orders of isotropy groups of all points for X, which happens, in particular, as r is a sufficiently large prime, the rigidified inertia stack $\bar{I}_\mu \mathbb{P}Y^{\frac{1}{r}}$ of $\mathbb{P}Y^{\frac{1}{r}}$ can be composed as the disjoint union

$$\underbrace{\mathbb{P}(\bar{I}_\mu Y)^{\frac{1}{r}}}_{1} \bigsqcup \underbrace{\sqcup_{j=1}^{r-1} \bar{I}_\mu Y}_{2} \ .$$

Let $(\vec{x}, (g, t))$ represents a $\mathbb{C}-$point of $\bar{I}_\mu \mathbb{P}Y^{\frac{1}{r}}$, if $(\vec{x}, (g, t))$ appears in the first factor of the decomposition above, then the element (g, t) in the subgroup $G \times \{1\} \subset G \times \mathbb{C}^*$, and the space $\mathbb{P}(\bar{I}_\mu Y)^{\frac{1}{r}}$ can be further decomposed as $\mathbb{P}(\bar{I}_\mu Y)^{\frac{1}{r}} = \sqcup_{g \in G} \mathbb{P}(\bar{I}_g Y)^{\frac{1}{r}}$, where $\mathbb{P}(\bar{I}_g Y)^{\frac{1}{r}}$ is defined as the quotient stack

$$\mathbb{P}(\bar{I}_g Y)^{\frac{1}{r}} := [(AY(\widetilde{\theta})^g \times (\mathbb{C} \backslash \{0\})^2)/((G/\langle g \rangle) \times \mathbb{C}^*)]$$

with the action similar to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},0}$ as above; if $(\vec{x},(g,t))$ occurs in the second factor of the decomposition above, the automorphism (g,t) lies in $G\times\{\mu_r^j : 1\leqslant j\leqslant r-1\}\subset G\times\boldsymbol{\mu}_r$, and the point $\vec{x}$ goes the infinity section $\mathcal{D}_\infty$ defined by $z_2=0$. Here $\mu_r=\exp(\frac{2\pi\sqrt{-1}}{r})\in\mathbb{C}^*$ and $\boldsymbol{\mu}_r$ is the cyclic group generated by μ_r.

For $(g,t)\in G\times\boldsymbol{\mu}_r$. we will use the notation $\bar{I}_{(g,t)}\mathbb{P}Y^{\frac{1}{r}}$ to mean the rigidified inertia stack component of $\bar{I}_\mu\mathbb{P}Y^{\frac{1}{r}}$ corresponding to the isotropy element (g,t).

Consider the moduli stack of $\widetilde{\theta}-$stable quasimaps to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$:

$$Q^{\widetilde{\theta}}_{0,m}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(d,1^p,\frac{\delta}{r}))\ .$$

More concretely,

$$Q^{\widetilde{\theta}}_{0,m}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(d,1^p,\frac{\delta}{r}))=\{(C;q_1,\dots,q_m;L_1,\cdots,L_{k+p},N;\vec{x},\vec{y},z_1,z_2)\},$$

where $(C;q_1,\dots,q_m)$ is a m-pointed prestable balanced orbifold curve of genus 0 with possible nontrivial isotropy only at special points, i.e. marked gerbes or nodes, the line bundles $(L_j : 1\leqslant j\leqslant k+p)$ and N are orbifold line bundles on C with

$$\deg([\vec{x}])=d\in Hom(Pic(\mathfrak{Y}),\mathbb{Q}),\quad \deg(N)=\frac{\delta}{r}\ , \tag{4.1}$$

$$\deg(L_{k+j})=1,\ \ 1\leqslant j\leqslant p\ , \tag{4.2}$$

and

$$(\vec{x},\vec{y},\vec{z}):=(x_1,\dots,x_n,y_1,\dots,y_p,z_1,z_2)\in\Gamma\left(\bigoplus_{i=1}^n L_{\rho_i}\oplus\bigoplus_{j=1}^p L_{k+j}\oplus(L_{-\theta_p}\otimes N^{\otimes r})\oplus N\right).$$

Here, for $1\leqslant i\leqslant n$, the line bundle L_{ρ_i} is equal to

$$\otimes_{j=1}^k L_j^{m_{ij}}\ ,$$

where (m_{ij}) $(1\leqslant i\leqslant n, 1\leqslant j\leqslant k+p)$ is given by the relation $\rho_i=\sum_{j=1}^k m_{ij}\pi_j$. The same construction applies to the line bundle $L_{-\theta_p}$ on C. Note that here δ is an integer when $Q^{\widetilde{\theta}}_{0,m}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(d,1^p,\frac{\delta}{r}))$ is nonempty as $N^{\otimes r}$ is the pull-back of some line bundle on the coarse moduli curve $\underline{C}$.

We require the the following conditions are satisfied for the above data:

- *Representability*: For every $q\in C$ with isotropy group G_q, the homomorphism $\mathbb{B}G_q\to\mathbb{B}(G_p\times\mathbb{C}^*)$ induced by the restriction of line bundles $(L_j : 1\leqslant j\leqslant k+p)$ and N to q is representable. Note that the image of the homomorphism lies in the subgroup $G\times\mathbb{C}^*\subset G_p\times\mathbb{C}^*$.
- *Nondegeneracy*: The sections z_1 and z_2 never simultaneously vanish. Furthermore, for each point q of C at which $z_2(q)\neq 0$, the stability condition 2.3
$$l_{\widetilde{\theta}}(q)\leqslant 1$$
for $\widetilde{\theta}$-stable map to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ becomes the stability condition
$$l_{\epsilon\theta_p}(q)\leqslant 1, \tag{4.3}$$
for the prestable quasimap $[\vec{x},\vec{y}] : C\to\mathfrak{Y}\times[\mathbb{C}/\mathbb{C}^*]^p$. For each point q of C at which $z_2(q)=0$, we have
$$\mathrm{ord}_q(\vec{x})=\mathrm{ord}_q(\vec{y})=0. \tag{4.4}$$
We note that this can be phrased as the length condition (2.1) bounding the order of contact of $(\vec{x},\vec{y},\vec{z})$ with the unstable loci of $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ as in [CFK16, §2.1].

- *Stability*: The $\mathbb{Q}-$line bundle
$$(\phi_*(L_\theta))^{\otimes\epsilon}\otimes\bigotimes_{j=1}^{p}\phi_*(L_{k+j})^{\otimes\epsilon}\otimes\phi_*(N^{\otimes 3r})\otimes\omega_{\underline{C}}^{log}$$
on the coarse curve $\underline{C}$ is ample. Here $\phi: C\to\underline{C}$ is the coarse moduli map. Note that here, the line bundles L_θ, $(L_{k+j})_{j=1}^{p}$ and $N^{\otimes 3r}$ are the pullback of line bundles on the coarse moduli of $\underline{C}$.
- *Vanishing*: The image of $[\vec{x}]: C\to\mathfrak{X}$ lies in $\mathfrak{Y}$.

Let $\vec{m}=(v_1,\cdots,v_m)\in(G\times\boldsymbol{\mu}_r)^m$, we will denote $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(d,1^p,\frac{\delta}{r}))$ to be:
$$Q^{\widetilde{\theta}}_{0,m}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(d,1^p,\frac{\delta}{r}))\cap ev_1^{-1}(\bar{I}_{v_1}\mathbb{P}Y^{\frac{1}{r}})\cap\cdots\cap ev_m^{-1}(\bar{I}_{v_m}\mathbb{P}Y^{\frac{1}{r}})\ ,$$
where
$$ev_i: Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))\to\bar{I}_\mu\mathbb{P}Y^{\frac{1}{r}}$$
are natural evaluation maps as before, by evaluating the sections $(\vec{x},\vec{z})$ at ith marking q_i. Evaluating the section $\vec{x}$ at the vanishing loci of the section y_j of the degree one line bundle L_{k+j} for $1\leqslant j\leqslant p$, which corresponds to a smooth non-orbifold point on C, one has another tuple of evaluation maps

$$\hat{ev}_j: Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))\to\mathfrak{Y}\ , \tag{4.5}$$

for $1\leqslant j\leqslant p$.

Remark 4.1. The above constructed master space is inspired by the twisted graph space used in [CJR17b, CJR17a], which they use to prove the high genus quasimap wall-crossing assuming the genus zero wall-crossing for quasimap $J-$function holds. So it may be surprising that certain modification of the twisted graph space can be used to prove the genus zero quasimap wall-crossing in this paper.

Because $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))$ is the moduli space of stable quasimaps to a proper lci GIT quotient, it is a proper Deligne-Mumford stack equipped with a natural perfect obstruction theory relative to the Artin stack $\mathfrak{M}^{tw}_{0,m}$ of prestable twisted curves by [CFKM14]. This relative perfect obstruction theory has the form

$$\mathbb{E}:=R^\bullet\pi_*(f^*\mathbb{T}_{\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}})\ . \tag{4.6}$$

Here, we denote the universal family over $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))$ by

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"] & \mathbb{P}\mathfrak{Y}^{\frac{1}{r},p} \\ Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))\ . & \end{tikzcd}$$

The obstruction theory (4.6) can written as cone of the morphism of complexes

$$R^\bullet\pi_*(\mathcal{O}_{\mathcal{C}}\otimes\mathfrak{g}_p)\to R^\bullet\pi_*\big(\mathcal{V}\oplus(\oplus_{j=1}^{p}\mathcal{L}_{k+j})\oplus(\mathcal{L}_{-\theta_p}\otimes\mathcal{N}^{\otimes r})\oplus\mathcal{N}\big)\ , \tag{4.7}$$

which is induced from applying $R^\bullet\pi_*$ to the Euler exact sequence of the tangent complex $\mathbb{T}_{\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}}$ of $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$
$$AY_{r,p}\times_{G_p}\mathfrak{g}_p\to AY_{r,p}\times_{G_p}\mathbb{T}_{AY_{r,p}}\to\mathbb{T}_{\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}}\ .$$

Here we use the GIT representation $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p} = [AY_{r,p}/G_p]$ constructed before, where[8] $AY_{r,p} = AY \times \mathbb{C}^p \times \mathbb{C}^2$. Here $\mathcal{L}_{\rho_i}$ $(1 \leqslant i \leqslant n)$, $\mathcal{L}_j$ $(1 \leqslant j \leqslant k)$ and $\mathcal{N}$ are the universal line bundles and

$$\mathcal{V} \subset \oplus_{i=1}^n \mathcal{L}_{\rho_i}$$

is the subsheaf of sections taking values in the affine cone of Y. Somewhat more explicitly, the sub-obstruction-theory $\mathbb{E}_{\mathrm{rel}} := R^\bullet\pi_*(\mathcal{V})$ comes from the deformations and obstructions of the sections $\vec{x}$, and $\mathbb{E}_{\mathrm{rel}}$ fits into the following distinguished triangle:

$$\mathbb{E}_{\mathrm{rel}} \longrightarrow R^\bullet\pi_*(\oplus_{i=1}^n \mathcal{L}_{\rho_i}) \xrightarrow{ds} R^\bullet\pi_*(\oplus_{b=1}^c \mathcal{L}_{\tau_b}) \xrightarrow{+1} . \tag{4.8}$$

Here $ds = \oplus_{b=1}^c ds_b$ where $ds_b : R^\bullet\pi_*(\oplus_{i=1}^n \mathcal{L}_{\rho_i}) \to R^\bullet\pi_*\mathcal{L}_{\tau_b}$ is included from the vector bundle map

$$\oplus_{i=1}^n \mathcal{L}_{\rho_i} \to \mathcal{L}_{\tau_b}$$

which sends $\vec{x} = (x_i)_{i=1}^n$ to $s_b(\vec{x})$. We note that we can interpret $R^\bullet\pi_*(\mathcal{O}_{\mathcal{C}} \otimes \mathfrak{g}_p)$ as the deformation theory of line bundles $(L_j)_{j=1}^{k+p}$ and N, and interpret the summand $R^\bullet\pi_*\big((\oplus_{j=1}^p \mathcal{L}_{k+j}) \oplus (\mathcal{L}_{-\theta_p} \otimes \mathcal{N}^{\otimes r}) \oplus \mathcal{N}\big)$ of $\mathbb{E}$ as the deformation theory of sections $\vec{y}$ and z_1, z_2.

4.2. **$\mathbb{C}^*$-action and fixed loci.** Consider the (left) $\mathbb{C}^*$-action on $AY_p \times \mathbb{C}^2$ defined by:

$$\lambda(\vec{x}, \vec{y}, z_1, z_2) = (\vec{x}, \vec{y}, \lambda z_1, z_2) ,$$

this action descents to be an action on $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$. We will denote λ to be the equivariant class corresponding to the $\mathbb{C}^*$-action of weight 1. Let's first state a criteria for a morphism to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ to be $\mathbb{C}^*$-equivariant (see also [CLLL16, §2.2]), which will be important in the analysis of localization computations.

Remark 4.2. (*Equivariant morphism to* $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$) Fix a stack S over $Spec(\mathbb{C})$ with a left $\mathbb{C}^*$-action, then a $\mathbb{C}^*$-equivariant morphism from S to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ is equivalent to the following data: there exists $k+p+1$ $\mathbb{C}^*$-equivariant line bundles on S

$$L_1, \cdots, L_{k+p}, N$$

together with $\mathbb{C}^*$-invariant sections

$$\begin{aligned}(\vec{x}, \vec{y}, \vec{z}) &:= (x_1, \ldots, x_n, y_{n+1}, \ldots, y_{n+p}, z_1, z_2) \\ &\in \Gamma\left(\oplus_{i=1}^n L_{\rho_i} \oplus (\oplus_{j=1}^p L_{k+j}) \oplus (L_{-\theta_p} \otimes N^{\otimes r} \otimes \mathbb{C}_\lambda) \oplus N\right)^{\mathbb{C}^*} .\end{aligned}$$

Here L_{ρ_i} $(1 \leqslant i \leqslant n)$ and $L_{-\theta_p}$ are constructed from $(L_j)_{1\leqslant j\leqslant k+p}$ as explained before, $\mathbb{C}_\lambda$ is the trivial line bundle over S with $\mathbb{C}^*-$linearization of weight 1. These sections should also satisfy the vanishing condition imposed by the cone of Y as above.

Fix a degree $\beta \in \mathrm{Eff}(W, G, \theta)$ and a tuple of nonnegative integers $(\delta_1, \cdots, \delta_m) \in \mathbb{N}^m$. Consider the tuple of multiplicities $\vec{m} = (v_1, \cdots, v_m) \in (G \times \boldsymbol{\mu}_r)^m$, where $v_i = (g_i, \mu_r^{\delta_i})$, we will denote $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, \frac{\delta}{r}))$ to be

$$\bigsqcup_{\substack{d \in \mathrm{Eff}(AY,G,\theta) \\ (i_{\mathfrak{Y}})_*(d)=\beta}} Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (d, 1^p, \frac{\delta}{r})) ,$$

where $i_{\mathfrak{Y}} : \mathfrak{Y} \to \mathfrak{X}$ is the inclusion morphism. Thus $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ inherits a $\mathbb{C}^*$-action from the $\mathbb{C}^*-$action on $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$.

We can index the components of $\mathbb{C}^*-$fixed loci of $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ by decorated graphs. A decorated graph Γ consists of vertices, edges, and m legs, and we decorate it as follows:

[8] We add the subscript r here to emphasis that the G_p-action on $AY_{r,p}$ depends on r.

- Each vertex v is associated with an index $j(v) \in \{0, \infty\}$, a degree $\beta(v) \in \mathrm{Eff}(W, G, \theta)$ and a subset $J_v \subset \{1, \cdots, p\}$.
- Each edge $e = \{h, h'\}$ consists of a pair of half edges and it is equipped with a degree $\beta(e) \in \mathrm{Eff}(W, G, \theta)$, a subset $J_e \subset \{1, \cdots, p\}$ and $\delta(e) \in \mathbb{Z}_{>0}$. Each half edge h (or h') is incident to a unique vertex.
- Each half-edge h and each leg l has an element (called multiplicity) $m(h)$ or $m(l)$ in $G \times \boldsymbol{\mu}_r$.
- The legs are labeled with the numbers $\{1, \ldots, m\}$, and each leg is incident to a unique vertex.

By the "valence" of a vertex v, denoted $\mathrm{val}(v)$, we mean the total number of incident half-edges, including legs.

For each $\mathbb{C}^*$−fixed stable map f in $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$, we can associate a decorated graph Γ in the following way.

- Each edge e corresponds to a genus-zero component C_e of C such that it maps constantly to the base Y with possible basepoints on C_e. We also require that $\deg(N|_{C_e}) = \frac{\delta(e)}{r}$, $deg(L_j|_{C_e}) = \beta(e)(L_{\pi_j})$ $(1 \leqslant j \leqslant k)$, and $deg(L_{k+j}|_{C_e}) = 1$ if and only if $j \in J_e$ and 0 otherwise. We denote 1^{J_e} to be the degree coming from the lines bundles $(L_{k+j} : 1 \leqslant j \leqslant p)$. There are two distinguished points q_0 and q_∞ on C_e such that q_∞ is the only point on C_e at which z_2 vanishes, and q_0 is the only point on C_e determined by the following conditions:
  - if C_e has base points, q_0 is the only base point on C_e;
  - if C_e does not have base points on it, q_0 is the only point on C_e at which z_1 vanishes.

  We will also call q_0, q_∞ the ramification points[9], and all of degree $(\beta(e), 1^{J_e})$ is concentrated at the ramification point q_0. That is,
  $$\text{when } x_i|_{C_e} \neq 0, \text{ we have } \mathrm{ord}_{q_0}(x_i) = \beta(e)(L_{\rho_i}), \text{ and } \mathrm{ord}_{q_0}(y_i) = 1 \text{ if } j \in J_e\ .$$
- Each vertex v for which $j(v) = 0$ (with unstable exceptional cases note thatd below) corresponds to a maximal sub-curve C_v of C over which $z_1 \equiv 0$, and each vertex v for which $j(v) = \infty$ (again with unstable exceptions) corresponds to a maximal sub-curve over which $z_2 \equiv 0$. The label $\beta(v)$ denotes the degree coming from the restriction map $[\vec{x}]|_{C_v}$, note that here we count the degree $\beta(v)$ in $\mathrm{Eff}(W, G, \theta)$, but not in $\mathrm{Eff}(AY, G, \theta)$. The subset J_v is equal to the set $\{j | deg(L_{k+j}|_{C_v}) = 1,\ 1 \leqslant j \leqslant p\}$. We denote 1^{J_v} to be the ordered tuple $(deg(L_{k+j}|_{C_v}))_{j=1}^p$.
- A vertex v is *unstable* if stable quasimap of the type described above do not exist (where, as always, we interpret legs as marked points and half-edges as half-nodes). In this case, v corresponds to a single point of the component C_e for each adjacent edge e, which may be a node at which C_e meets another edge curve $C_{e'}$, a marked point of C_e, an unmarked point, or a basepoint on C_e of order $\beta(v)$. Note that the base point only appears as a vertex v over 0 due to the nondegeneracy condition, in which case we have $\beta(v) = \beta(e)$ for the incident edge e to v.
- The index $m(l)$ on a leg l indicates the rigidified inertia stack component $\bar{I}_{m(l)}\mathbb{P}Y^{\frac{1}{r}}$ of $\mathbb{P}Y^{\frac{1}{r}}$ on which the marked point corresponding to the leg l is evaluated, this is determined by the multiplicity of $L_1, \cdots, L_k, N$ at the corresponding marked point.

[9]The definition of the ramification point here is different from the definition in [CJR17a, Page 13], where they claim that z_1 or z_2 each vanish at exactly one point on C_e. We find that there is a missing case when q_0 is a base point and $deg(L_1|_{C_e}) = deg L_2|_{C_e} = \delta(e)$ in their setting, then $z_1|_{C_e} \equiv 1$, which does not vanish anywhere on C_e. But the author finds this missing case does not affect their main result in [CJR17a].

- A half-edge h of an edge e corresponds a ramification point $q \in C_e$. If q is not a base point, then $m(h)$ indicates the rigidified inertia component $\bar{I}_{m(h)}\mathbb{P}Y^{\frac{1}{r}}$ of $\mathbb{P}Y^{\frac{1}{r}}$ on which the ramification point q associated with h is evaluated. If q is a base point, we take $m(h) = (1,1) \in G \times \boldsymbol{\mu}_r$.

In particular, we note that the decorations at each stable vertex v yield a tuple

$$\vec{m}(v) \in (G \times \boldsymbol{\mu}_r)^{\mathrm{val}(v)}$$

recording the multiplicities of $L_1, \cdots, L_k, N$ at every special point of C_v.[10] We have the following remarks:

Remark 4.3. The crucial observation, now, is the following. For a stable vertex v such that $j(v) = 0$, we have $z_1|_{C_v} \equiv 0$, so the stability condition (4.3) implies that $l_{\epsilon\theta_p}(q) \leqslant 1$ for each $q \in C_v$. That is, the restriction of $(C; q_1, \ldots, q_m; L_1, \cdots, L_{k+p}; \vec{x}, \vec{y})$ to C_v gives rise to a $\epsilon\theta_p$-stable quasimap to the quotient stack $\mathfrak{Y}_p := [AY/G] \times [\mathbb{C}/\mathbb{C}^*]^p$ (c.f. 2.7) in

$$Q^{\epsilon\theta_p}_{0,\vec{m}(v)}(\mathfrak{Y}_p, (\beta(v), 1^{J_v})) := \coprod_{\substack{d \in \mathrm{Eff}(AY,G,\theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta(v)}} Q^{\epsilon\theta_p}_{0,\vec{m}(v)}(\mathfrak{Y}_p, (d, 1^{J_v})) \ .$$

In this case, let $j \in J_v$, the evaluation map considered in (4.5) coincides with $\hat{ev}_j$ for $Q^{(\epsilon\theta, \epsilon^{|J_v|})}_{0,\vec{m}(v)|\,|J_v|}(\mathfrak{Y}, \beta(v))$ in Remark 2.8.[11] On the other hand, for a stable vertex v such that $j(v) = \infty$, we have $z_2|_{C_v} \equiv 0$, so the stability condition (4.4) implies that $\mathrm{ord}_q(\vec{x}) = \mathrm{ord}_q(\vec{y}) = 0$ for each $q \in C_v$. Thus, the restriction of $(C; q_1, \ldots, q_m; L_1, \cdots, L_k; \vec{x})$ to C_v gives rise to a usual twisted stable map in

$$\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, \beta(v)) := \coprod_{\substack{d \in \mathrm{Eff}(AY,G,\theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta(v)}} \mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y}, d) \ .$$

Here $\sqrt[r]{L_\theta/Y}$ is the root gerbe of Y by taking r-th root of L_θ.

Remark 4.4. For each edge e, the restriction of $(\vec{x}, \vec{y})$ to C_e defines a constant map to Y (possibly with an additional basepoint at the ramification point q_0). So if there is no basepoint on C_e, the restriction of $(\vec{x}, \vec{y}, \vec{z})$ to C_e defines a representable map

$$C_e \to \mathbb{B}G_y \times \mathbb{P}_{r,1}$$

where $y \in Y$ comes from $\vec{x}$, G_y is the isotropy group of $y \in Y$. Then we have $m(q_0) = (g^{-1}, 1)$ and $m(q_\infty) = (g, \mu_r^{\delta(e)})$ for some $g \in G_y$. Note that when r is a sufficiently large prime comparing to $\delta(e)$, assuming that the order of g is equal to a, we have $C_e \cong \mathbb{P}^1_{ar,a}$ and the ramification point q_∞ must be a special point. Here $\mathbb{P}^1_{ar,a}$ is the unique Deligne-Mumford stack with coarse moduli $\mathbb{P}^1$ with isotropy group $\boldsymbol{\mu}_a$ at $0 \in \mathbb{P}^1$, isotropy group $\boldsymbol{\mu}_{ar}$ at $\infty \in \mathbb{P}^1$, and generic trivial stabilizer.

If q_0 is a basepoint of degree $(\beta, 1^{J_e})$ (we write $\beta = \beta(e)$ for short), the ramification point q_0 can't be an orbifold point, thus $m(q_0) = (1,1) \in G \times \boldsymbol{\mu}_r$. When r is a sufficiently large prime. Assume $m(q_\infty) = (g, \mu_r^{\delta(e)})$, and a is the order of g, by the representable condition, we have $C_e \cong \mathbb{P}_{ar,1}$. Note that the restriction of $(\vec{x}, \vec{y})$ to C_e defines an element in the space $F_{(\beta, 1^{J_e})}(Y)$ of the stacky loop space $Q_{\mathbb{P}_{a,1}}(Y, (\beta, 1^{J_e}))$ (see §3). Then the restriction of $(\vec{x}, \vec{y}, \vec{z})$ to C_e defines a quasimap f which can be explicitly described as follows. Write $\mathbb{P}_{a,1}$ as the quotient stack

[10] For each node, let h be the incident half-edge and v be the incident vertex, then we define the multiplicity at the branch of the node at C_v to be $m(h)^{-1}$.

[11] Here we use a canonical bijection between the set $[|J_v|] := \{1, \cdots, |J_v|\}$ and the index set J_v using the natural order of elements in $J_v \subset [p]$.

$[U/C^*]$ where $U := \mathbb{C}^2 \setminus \{0\}$ and $\mathbb{C}^*$ acts on U with weights $[ar, 1]$. We define a map F from U to $AY_p \times U$ to be

$$\begin{aligned}&(x, y) \in U \mapsto \\ &\qquad \left((x_1 x^{\beta(L_{\rho_1})}, \cdots, x_n x^{\beta(L_{\rho_n})}), (x)_{j \in J_e}, x^{\delta(e)-\beta(L_\theta)-|J_e|}, y^{a\delta(e)} \right) \in AY_p \times U \ .\end{aligned}$$

Here $(x)_{j\in J_e}$ is an element belonging to $\mathbb{C}^p$ so that the $j-$th component is 1 if $j \notin J_e$ and all the other component is x. Notice that F is equivariant with respect to the group homomorphism

$$t \in \mathbb{C}^*_t \mapsto \left(t^{ar\beta(L_{\pi_1})}, \cdots, t^{ar\beta(L_{\pi_k})} \right), (t^{ar})_{j \in J_e}, t^{a\delta(e)} \Big) \in G_p \times \mathbb{C}^* \ .$$

Then F descents to be the desired morphism f from $\mathbb{P}_{ar,1}$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$. For F to exist, we must have $g = g_\beta$, and $(x_1, \cdots, x_n)$ must belong to the space Y^{ss}_β defined in §3, thus defining a unique point in the $F_{(\beta,1^{J_e})}(Y) \cong [Y^{ss}_\beta/G]$. Conversely, when given a point in $F_{(\beta,1^{J_e})}(Y)$, we can always construct a unique map in the above way up to $2-$isomorphisms.

Remark 4.5. If there is a basepoint on the edge curve C_e, then the degree $(\beta(e), 1^{J_e}, \frac{\delta(e)}{r})$ on C_e must satisfy the relation $\delta(e) \geqslant \beta(e)(L_\theta) + |J_e|$. Otherwise we have $z_1|_{C_e} \equiv 0$, given the fact z_2 vanishes at q_∞, this will violate the nondegeneracy condition for z_1 and z_2.

4.3. **Localization analysis.** Fix $\beta \in \mathrm{Eff}(W, G, \theta)$ and $\delta \in \mathbb{Z}_{\geqslant 0}$, we will consider the space $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$. The reason why we assume that the second degree is $\frac{\delta}{r}$ is that $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ corresponds to $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}, (\beta, \delta))$, here $\mathbb{P}\mathfrak{Y}$ is equal to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ for $r = 1$ and $p = 0$. In the remaining section, we will always assume that r is a *sufficiently large prime*.

For each decorated graph Γ, we will associate each vertex v (resp. edge e) a moduli space $\mathcal{M}_v$(resp. $\mathcal{M}_e$) over which there is a family $\mathbb{C}^*-$stable map to $\mathbb{P}Y_{r,s}$ with the decorated degree.

Denote by F_Γ the space

$$\prod_{v:j(v)=0} \mathcal{M}_v \times_{\bar{I}_\mu \mathcal{D}_0} \prod_{e \in E} \mathcal{M}_e \times_{\bar{I}_\mu \mathcal{D}_\infty} \prod_{v:j(v)=\infty} \mathcal{M}_v \ ,$$

where the fiber product is taken by gluing the two branches at each node.

By virtual localization formula of Graber–Pandharipande [GP99], we can write

$$[Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))]^{\mathrm{vir}} \ ,$$

in terms of contributions from each decorated graph Γ:

$$[Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))]^{\mathrm{vir}} = \sum_\Gamma \frac{1}{\mathbb{A}_\Gamma} \iota_{\Gamma *} \left(\frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}}_\Gamma)} \right) \ . \tag{4.9}$$

Here, for each graph Γ, $[F_\Gamma]^{\mathrm{vir}}$ is obtained via the $\mathbb{C}^*$-fixed part of the restriction to the fixed loci of the obstruction theory on $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$, and N^{vir}_Γ is the equivariant Euler class of the $\mathbb{C}^*$-moving part of this restriction. Besides, $\mathbb{A}_\Gamma$ is the automorphism factor for the graph Γ, which represents the degree of F_Γ into the corresponding open and closed $\mathbb{C}^*$-fixed substack $i_\Gamma(F_\Gamma)$ in $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$. Here our $\mathbb{A}_\Gamma$ is the product of the size of the automorphism group $Aut(\Gamma)$ of the graph Γ and degrees from each edge moduli $\mathcal{M}_e$ over the corresponding fixed loci.

We will do an explicit computation for the contributions of each graph Γ in the following. As for the contribution of a graph Γ to (4.9), one can first apply the normalization exact sequence to the relative obstruction theory (4.6) and (4.7), which decomposes the contribution from Γ to (4.9) into contributions from vertex, edge, and node factors. This includes all but the automorphisms and deformations within $\mathcal{M}^{tw}_{0,\vec{m}}$. The latter are distributed in the vertex, edge, and node factors as deformations of the vertex components, deformations of the edge components, and deformations of smoothing the nodes, respectively.

4.3.1. *Vertex contributions.* First of all, consider the stable vertex v over ∞, this vertex moduli $\mathcal{M}_v$ corresponds to the moduli stack $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y},\beta(v))$, which parameterizes twisted stable maps to the root gerbe $\sqrt[r]{L_\theta/Y}$ over Y.

Let

$$\pi:\mathcal{C}_\infty\to\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y},\beta(v))$$

be the universal curve over $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y},\beta(v))$. In this case, on $\mathcal{C}_\infty$, we have $\mathcal{L}_{-\theta}\otimes\mathcal{N}^{\otimes r}\otimes\mathbb{C}_\lambda\cong\mathcal{O}_{\mathcal{C}_\infty}$ as $z_1|_{\mathcal{C}_\infty}\equiv 1$, hence we have $\mathcal{N}\cong\mathcal{L}_\theta^{\frac{1}{r}}\otimes\mathbb{C}_{-\frac{\lambda}{r}}$, here $\mathcal{L}_\theta^{\frac{1}{r}}$ is the line bundle over $\mathcal{C}_\infty$ that is the pullback of the universal root bundle over $\sqrt[r]{L_\theta/Y}$ along the universal map $f:\mathcal{C}_\infty\to\sqrt[r]{L_\theta/Y}$. The movable part of the perfect obstruction theory comes from the deformation of z_2, thus the *inverse of Euler class* of the virtual normal bundle is equal to

$$e^{\mathbb{C}^*}((-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})\otimes\mathbb{C}_{-\frac{\lambda}{r}}).$$

When r is a sufficiently large prime and the multiplicity $m(l)$ corresponding to each leg l incident to v is equal to $(g_l,1,\mu_r^{f_l})$ for some prefixed number $f_l\in\mathbb{Z}_{\geqslant 0}$(note this implies $f_l\ll r$) and $g_l\in G$, following [JPPZ18] to the orbifold case, the above Euler class has a representation

$$\sum_{d\geqslant 0}c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{|E(v)|-1-d}\ .$$

Here the virtual bundle $-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}}$ has virtual rank $|E(v)|-1$, where $|E(v)|$ is the number of edges incident to the vertex v. The fixed part of the perfecct obstruction theory contributes to the virtual cycle

$$[\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y},\beta(v))]^{\mathrm{vir}}\ .$$

For the stable vertex v over 0, the vertex moduli $\mathcal{M}_v$ corresponds to the moduli space $Q^{\epsilon\theta_p}_{0,\vec{m}(v)}(\mathfrak{Y}_p,(\beta(v),1^{J_v}))$.

Let $\pi:\mathcal{C}_0\to Q^{\epsilon\theta_p}_{0,\vec{m}(v)}(\mathfrak{Y}_p,(\beta(v),1^{J_v}))$ be the universal curve over $Q^{\epsilon\theta_p}_{0,\vec{m}(v)}(\mathfrak{Y}_p,(\beta(v),1^{J_v}))$. In this case, the fixed part of the obstruction theory of the vertex moduli over 0 yields the virtual cycle

$$[Q^{\epsilon\theta_p}_{0,\vec{m}(v)}(\mathfrak{Y}_p,(\beta(v),1^{J_v}))]^{\mathrm{vir}}\ .$$

Note that $\mathcal{N}|_{\mathcal{C}_0}=\mathcal{O}_{\mathcal{C}_0}$ as $z_2|_{\mathcal{C}_0}\equiv 1$, therefore the virtual normal comes from the movable part of the infinitesimal deformations of the section z_1, which is a section of the line bundle $\mathcal{L}_{-\theta_p}$ over $\mathcal{C}_0$, whose Euler class is equal to

$$e^{\mathbb{C}^*}((R^\bullet\pi_*\mathcal{L}_{-\theta_p})\otimes\mathbb{C}_\lambda)\ .$$

4.3.2. *Edge contributions: basepoint case.* When there is a base point on the edge curve, it has degree $(\beta(e),1^{J_e},\frac{\delta(e)}{r})$ with $\beta(e)\neq 0$ and $\delta(e)\geqslant\beta(e)(L_\theta)+|J_e|$ by Remark 4.5, we will write $\beta(e)$ as β only in this subsection for simplicity unless stated otherwise. Then the multiplicity at $q_\infty\in C_e$ is equal to $(g,\mu_r^{\delta(e)})\in G\times\boldsymbol{\mu}_r$, where $g=g_\beta$ is defined in §3. Let a be the minimal positive integer associated to β as in §3, which is also the order of g_β. When r is a sufficiently large prime, due to Remark 4.4, C_e must be isomorphic to $\mathbb{P}^1_{ar,1}$ where the ramification point q_0 for which $z_1=0$ is an ordinary point, and the ramification point q_∞ for which $z_2=0$ must be a special point, which is isomorphic to $\mathbb{B}\boldsymbol{\mu}_{ar}$.

Recall that

$$F_\beta(Y)\cong[Y^{ss}_\beta/G]\cong[(Z^{ss}_\beta\cap AY)/G]$$

in §3. We now define the edge moduli $\mathcal{M}_e$ to be

$$\sqrt[a\delta(e)]{L_{-\theta}/[Y^{ss}_\beta/G]}\ ,$$

which is the root gerbe over the stack $[Y^{ss}_\beta/G]$ by taking $a\delta(e)$th root of the line bundle $L_{-\theta}$ on $[Y^{ss}_\beta/G]$.

The root gerbe $\sqrt[a\delta(e)]{L_{-\theta}/[Y^{ss}_\beta/G]}$ admits a representation as a quotient stack:

$$[(Y^{ss}_\beta\times\mathbb{C}^*)/(G\times\mathbb{C}^*_w)],$$

where the (right) action is defined by

$$(\vec{x},v)\cdot(g,w)=(\vec{x}\cdot g,\theta(g)vw^{a\delta(e)})\ ,$$

for all $(g,w)\in G\times\mathbb{C}^*_w$ and $(\vec{x},v)\in A(Y)^g\times\mathbb{C}^*$. Here $\vec{x}\cdot g$ is given by the action as in the definition of $[AY/G]$. For every character ρ of G, we can define a new character of $G\times\mathbb{C}^*_w$ by composing the projection map $\mathrm{pr}_G: G\times\mathbb{C}^*_w\to G$. By an abuse of notation, we will continue to use the notation ρ to name the new character of $G\times\mathbb{C}^*_w$. Then the new character ρ will determines a line bundle $L_\rho:=[(Y^{ss}_\beta\times\mathbb{C}^*\times\mathbb{C}_\rho)/(G\times\mathbb{C}^*_w)]$ on $\sqrt[a\delta(e)]{L_{-\theta}/[Y^{ss}_\beta/G]}$.

By virtue of its universal property of the root gerbe $\sqrt[a\delta(e)]{L_{-\theta}/[Y^{ss}_\beta/G]}$, there is a line bundle $\mathcal{R}$ called root bundle that is the $a\delta(e)$th root of line bundle $L_{-\theta}$ over the root gerbe. This root line bundle $\mathcal{R}$ can also constructed by the Borel construction, i.e. $\mathcal{R}$ is associated to the character p_2:

$$\mathrm{pr}_{\mathbb{C}^*_w}: G\times\mathbb{C}^*_w\to\mathbb{C}^*_w\quad (g,w)\in G\times\mathbb{C}^*_w\mapsto w\in\mathbb{C}^*_w\ .$$

We have the relation

$$L_{-\theta}=\mathcal{R}^{\otimes a\delta(e)}\ .$$

Then the coordinate function $(\vec{x},v)\in Y^{ss}_\beta\times\mathbb{C}^*$ descents to be a tautological sections of vector bundle $\bigoplus_{i=1}^n L_{\rho_i}\oplus(L_\theta\otimes\mathcal{R}^{\otimes a\delta(e)})$ on $\sqrt[a\delta(e)]{L_{-\theta}/[Y^{ss}_\beta/G]}$.

We will construct a universal family of $\mathbb{C}^*$−fixed quasimaps to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ of degree $(\beta,1^{J_e},\frac{\delta(e)}{r})$ over the edge moduli $\mathcal{M}_e$, which takes the form

$$\begin{tikzcd}
\mathcal{C}_e:=\mathbb{P}_{ar,1}(\mathcal{R}^{\otimes a}\oplus\mathcal{O}_{\mathcal{M}_e}) \arrow[r, "ev"] \arrow[d, "\pi"] & \mathbb{P}\mathfrak{Y}^{\frac{1}{r},p} \\
\mathcal{M}_e:=\sqrt[a\delta(e)]{L_{-\theta}/[Y^{ss}_\beta/G]}\ . &
\end{tikzcd}$$

The universal curve $\mathcal{C}_e$ over the edge moduli $\mathcal{M}_e$ is constructed as a quotient stack:

$$\mathcal{C}_e=[(Y^{ss}_\beta\times\mathbb{C}^*\times U)/(G\times\mathbb{C}^*_w\times\mathbb{C}^*_t)]\ ,$$

where the right action is defined by:

$$(\vec{x},v,x,y)\cdot(g,w,t)=(\vec{x}\cdot g,\theta(g)vw^{a\delta(e)},w^at^{ar}x,ty)\ ,$$

for all $(g,w,t)\in G\times\mathbb{C}^*_w\times\mathbb{C}^*_t$ and $(\vec{x},v,(x,y))=((x_1,\cdots,x_n),v,(x,y))\in Y^{ss}_\beta\times\mathbb{C}^*\times U$.

The universal map ev from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ can be presented as follows:

$$\widetilde{ev}: Y^{ss}_\beta\times\mathbb{C}^*\times U\to AY_p\times U\ ,$$

defined by:

(4.10)
$$\begin{aligned}&(\vec{x},v,(x,y))\in Y^{ss}_\beta\times\mathbb{C}^*\times U\mapsto\\ &\qquad\Big(\big(x_1x^{\beta(L_{\rho_1})},\cdots,x_nx^{\beta(L_{\rho_n})}\big),(x)_{j\in J_e},v^{-1}x^{\delta(e)-\beta(L_\theta)-|J_e|},y^{a\delta(e)}\Big)\in AY_p\times U\ .\end{aligned}$$

Here $(x)_{j\in J_e}$ is an element belonging to $\mathbb{C}^p$ so that the j−th component is 1 if $j\notin J_e$ and all the other components are x. Note that when $\beta(L_{\rho_i})\notin\mathbb{Z}_{\geqslant 0}$ for some i, we must have

$x_i = 0$ as $\vec{x} \in Y_\beta^{ss}$, so the $\tilde{ev}$ is well defined. Then $\tilde{ev}$ is equivariant with respect to the group homomorphism from $G\times\mathbb{C}^*_w\times\mathbb{C}^*_t$ to $G_p\times\mathbb{C}^*$ defined by:

$$(4.11)\quad \begin{aligned}&(g,w,t)\in G\times\mathbb{C}^*_w\times\mathbb{C}^*_t \mapsto \\ &\qquad \Big(g\cdot\big(t^{ar\beta(L_{\pi_1})}w^{a\beta(L_{\pi_1})},\cdots,t^{ar\beta(L_{\pi_k})}w^{a\beta(L_{\pi_k})}\big),(w^at^{ar})_{j\in J_e},t^{a\delta(e)}\Big)\in G_p\times\mathbb{C}^* \ .\end{aligned}$$

Here $(w^at^{ar})_{j\in J_e}$ is the element belonging to $(\mathbb{C}^*)^p$ so that the $j-$th component is 1 if $j\notin J_e$ and all the other components are w^at^{ar}. This gives the universal morphism f from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ by descent.

There is a tautological line bundle $\mathcal{O}_{C_e}(1)$ on C_e associated to the character $\mathrm{pr}_{\mathbb{C}^*_t}$ of $G\times\mathbb{C}^*_w\times\mathbb{C}^*_t$ by the Borel construction. Here $\mathrm{pr}_{\mathbb{C}^*_t}$ is the projection map from $G\times\mathbb{C}^*_w\times\mathbb{C}^*_t$ to $\mathbb{C}^*_t$.

We will define a (quasi[12]-left) $\mathbb{C}^*-$action on $\mathcal{C}_e$ such that the map ev constructed above is $\mathbb{C}^*-$equivariant. Define a (left) $\mathbb{C}^*-$action on $\mathcal{C}_e$ which is induced from the $\mathbb{C}^*-$action on $Y_\beta^{ss}\times\mathbb{C}^*\times U$:

$$m:\mathbb{C}^*\times Y_\beta^{ss}\times\mathbb{C}^*\times U\to Y_\beta^{ss}\times\mathbb{C}^*\times U\ ,$$
$$t\cdot(x,v,(x,y))=(x,v,(x,t^{\frac{-1}{ar\delta(e)}}y))\ .$$

Note that the morphism π is also $\mathbb{C}^*$-equivariant, where $\mathcal{M}_e$ is equipped with the trivial $\mathbb{C}^*$-action. By the universal property of the projectivized bundle $\mathcal{C}_e$ over $\mathcal{M}_e$, the line bundle $\mathcal{O}_{\mathcal{C}_e}(1)$ is equipped with a tautological section

$$(x,y)\in H^0\big(\big(\mathcal{O}_{\mathcal{C}_e}(ar)\otimes\pi^*\mathcal{R}^{\otimes a}\big)\oplus(\mathcal{O}_{\mathcal{C}_e}(1)\otimes\mathbb{C}_{\frac{-1}{ar\delta(e)}})\big)\ ,$$

which is also a $\mathbb{C}^*-$invariant section. Here $\mathcal{O}_{\mathcal{C}_e}(1)$ is the standard $\mathbb{C}^*$-equivariant line bundle on $\mathcal{C}_e$ by the Borel construction.

Now we can check that ev is a $\mathbb{C}^*-$equivariant morphism from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ with respect to the $\mathbb{C}^*-$actions for $\mathcal{C}_e$ and $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$. According to Remark 4.2, ev is equivalent to the following data:

(1) $k+p+1$ $\mathbb{C}^*$-equivariant line bundles on $\mathcal{C}_e$:

$$\mathcal{L}_j:=\pi^*L_{\pi_j}\otimes\mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\pi_j}))\otimes\pi^*\mathcal{R}^{\otimes a\beta(L_{\pi_j})},1\leqslant j\leqslant k\ ,$$
$$\mathcal{L}_{k+j}:=\pi^*\mathcal{R}^{\otimes a}\otimes\mathcal{O}_{\mathcal{C}_e}(ar),j\in J_e,\text{ and }\mathcal{L}_{k+j}:=\mathbb{C},\ j\notin J_e$$

and

$$\mathcal{N}:=\mathcal{O}_{\mathcal{C}_e}(a\delta(e))\otimes\mathbb{C}_{\frac{-\lambda}{r}}\ ,$$

where the line bundles L_{π_j}, $\mathcal{R}$ are the standard $\mathbb{C}^*$-equivariant line bundle on $\mathcal{M}_e$ by the Borel construction;

(2) a universal section

$$(4.12)\quad \begin{aligned}\big(\vec{x},\vec{y},(\zeta_1,\zeta_2)\big):=&\big((x_1x^{\beta(L_{\rho_1})},\cdots,x_nx^{\beta(L_{\rho_n})}),(x)_{J_e},(v^{-1}x^{\delta(e)-\beta(L_\theta)-|J_e|},y^{a\delta(e)})\big)\\ &\in H^0\big(\mathcal{C}_e,(\oplus_{i=1}^n\mathcal{L}_{\rho_i})\oplus(\oplus_{j=1}^p\mathcal{L}_{k+j})\oplus(\mathcal{L}_{-\theta_p}\otimes\mathcal{N}^{\otimes r}\otimes\mathbb{C}_\lambda)\oplus\mathcal{N}\big)^{\mathbb{C}^*}\ ,\end{aligned}$$

where the line bundles $\mathcal{L}_{-\theta_p}$ and $\mathcal{L}_{\rho_i}$ are constructed from line bundles $\mathcal{L}_j$ as before.

From the description of $\mathcal{M}_e$ with the associated family map $\tilde{ev}$, we see that $\mathcal{M}_e$ allows a finite étale map of degree[13] $\frac{1}{a}$ into the corresponding fixed loci in $Q_{0,1}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta(e),1^{J_e},\frac{\delta(e)}{r}))$. Then we can use $\mathcal{M}_e$ to do the edge localization contribution analysis.

---

[12]This means we allow $\mathbb{C}^*-$action on C_e with fractional weight. See a similar discussion in [CLLL16, §2.2].

[13]This can be seen by comparing the order of the isotropy group of a $\mathbb{C}-$point x of $\mathcal{M}_e$ with the order of the isotropy group of the corresponding point in $Q_{0,1}^{\tilde{\theta}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta(e),1^{J_e},\frac{\delta(e)}{r}))$. The former is equal to the product of the number $a\delta(e)$ and the order of the isotropy group of the corresponding point in $[Y_\beta^{ss}/G]$, while the later is equal to the product of the number $\delta(e)$(as it represents the order of the group of cyclic coverings of $\mathbb{P}_{ar,1}$ of degree $\delta(e)$, see Remark 4.4), and the order of isotropy group of the corresponding point in $[Y_\beta^{ss}/G]$.

Equipped with these notations, now we compute the localization contribution from $\mathcal{M}_e$. Based on the perfect obstruction theory for quasimaps in $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))$, the restriction of the prefect obstruction theory to $\mathcal{M}_e$ decomposes into three parts: (1) the deformation theory of source curve $\mathcal{C}_e$; (2) the deformation theory of the lines bundles $(\mathcal{L}_j)_{1\leqslant j\leqslant k+p}$ and $\mathcal{N}$; (3) the deformation theory for the section

$$(\vec{x},\vec{y},(\zeta_1,\zeta_2))\in\Gamma\left(\oplus_{i=1}^n\mathcal{L}_{\rho_i}\oplus(\oplus_{j=1}^p\mathcal{L}_{k+j})\oplus(\mathcal{L}_{-\theta_p}\otimes\mathcal{N}^{\otimes r}\otimes\mathbb{C}_\lambda)\oplus\mathcal{N}\right)\ .$$

The virtual normal bundle comes from the movable part of the three parts, and the fixed part will contribute to the virtual cycle of $\mathcal{M}_e$. First every fiber curve C_e in $\mathcal{C}_e$ is isomorphic to $\mathbb{P}_{ar,1}$, which is rational. Then the infinitesimal deformations/obstructions of C_e and the line bundles $L_j:=\mathcal{L}_j|_{C_e},\ N:=\mathcal{N}|_{C_e}$ are zero. Hence their contribution to the perfect obstruction theory solely comes from infinitesimal automorphisms. The infinitesimal automorphisms of C_e come from the space of vector field on C_e that vanishes on special points. Thus the $\mathbb{C}^*-$fixed part of the infinitesimal automorphisms of C_e comes from the $1-$dimensional subspace of vector fields on C_e which vanish on the two ramification points, which, together with the infinitesimal automorphisms of line bundle N, will be canceled with the fixed part of infinitesimal deformation of sections $(z_1,z_2):=(\zeta_1,\zeta_2)|_{C_e}$. The movable part of infinitesimal automorphisms of C_e is nonzero only if at least one of ramification points on C_e is not a special point. By Remark 4.4, the ramification q_∞ must be a special point since it has nontrivial stacky structure when r is sufficiently large, and the ramification point q_0 is not a special point. Then the movable part of infinitesimal automorphisms of C_e contributes

$$\frac{\delta(e)}{\lambda-D_\theta}$$

to the virtual normal bundle.

Now let's turn to the localization contribution from sections. As for the deformations of z_2, we continue to use the tautological section (x,y) in (4.3.2). Sections of N is spanned by monomials $(x^my^n)|_{C_e}$ with $arm+n=a\delta(e)$ and $m,n\in\mathbb{Z}_{\geqslant 0}$. Note that x^my^n may not be a global section of $\mathcal{N}$ but always a global section of the line bundle $R^{\otimes am}\otimes\mathcal{N}\otimes\mathbb{C}_{\frac{m}{\delta(e)}\lambda}$. Then $R^\bullet\pi_*\mathcal{N}$ will decompose as a direct sum of line bundles, each corresponds to the monomial x^my^n, whose first chern class is

$$c_1(\mathcal{R}^{\otimes -am}\bigotimes\mathbb{C}_{\frac{-m}{\delta(e)}\lambda})=\frac{m}{\delta(e)}(D_\theta-\lambda)\ .$$

So the total contribution is equal to

$$\prod_{m=0}^{\lfloor\frac{\delta(e)}{r}\rfloor}\left(\frac{m}{\delta(e)}(D_\theta-\lambda)\right)\ .$$

The term corresponding to $m=0$ in the above product is the $\mathbb{C}^*-$invariant part of $R^\bullet\pi_*\mathcal{N}$, it will contribute to the virtual cycle of $\mathcal{M}_e$. The rest contributes to the virtual normal bundle as

$$\prod_{m=1}^{\lfloor\frac{\delta(e)}{r}\rfloor}\left(\frac{m}{\delta(e)}(D_\theta-\lambda)\right)\ .$$

Note that when r is sufficiently large, the above product becomes 1.

For the deformation of z_1, arguing in the same way as z_2, the Euler class of $R^\bullet\pi_*(\mathcal{L}_{-\theta_p}\otimes\mathcal{N}^{\otimes r}\otimes\mathbb{C}_\lambda)$ is equal to

$$\prod_{m=0}^{\delta(e)-\beta(L_\theta)-|J_e|}\left(\frac{m}{\delta(e)}(-D_\theta+\lambda)\right)\ .$$

The factor for $m=0$ appearing in the above product is the $\mathbb{C}^*-$fixed part of $R^\bullet\pi_*(\mathcal{L}_{-\theta}\otimes\mathcal{N}^{\otimes r}\otimes\mathbb{C}_\lambda)$, it will contribute to the virtual cycle of $\mathcal{M}_e$. The rest contributes to the virtual normal bundle as

$$\prod_{m=1}^{\delta(e)-\beta(L_\theta)-|J_e|}\left(\frac{m}{\delta(e)}(-D_\theta+\lambda)\right).$$

Finally, let's turn to the localization contribution from the sections $\vec{x}$ and $\vec{y}$. Before that, using the same argument above, one can prove the following lemma:

Lemma 4.6. *When $n\in\mathbb{Z}_{\geqslant 0}$, we have*

$$e^{\mathbb{C}^*}\left(R^\bullet\pi_*(\mathcal{O}_{\mathcal{C}_e}(n))\right)=\prod_{m=0}^{\lfloor\frac{n}{ar}\rfloor}\left(\frac{m}{\delta(e)}(D_\theta-\lambda)+\frac{n}{ar\delta(e)}\lambda\right).$$

When $n\in\mathbb{Z}_{<0}$, we have

$$e^{\mathbb{C}^*}\left(R^\bullet\pi_*(\mathcal{O}_{\mathcal{C}_e}(n))\right)=\prod_{\frac{n}{ar}<m<0}\frac{1}{\frac{m}{\delta(e)}(D_\theta-\lambda)+\frac{n}{ar\delta(e)}\lambda}.$$

Using the above lemma, we have the following description of $e^{\mathbb{C}^*}(R^\bullet\pi_*\mathcal{L}_{\rho_i})$ for $1\leqslant i\leqslant n$. Then for each ρ_i, we have:

(1) If $\beta(L_{\rho_i})\in\mathbb{Q}_{\geqslant 0}$, one has

$$\begin{aligned}
e^{\mathbb{C}^*}\left(R^\bullet\pi_*(\mathcal{L}_{\rho_i})\right)&=e^{\mathbb{C}^*}\left(R^\bullet\pi_*(\pi^*(L_{\rho_i})\otimes\mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i}))\otimes\pi^*(\mathcal{R}^{\otimes a\beta(L_{\rho_i})}))\right)\\
&=e^{\mathbb{C}^*}\left(L_{\rho_i}\otimes\mathcal{R}^{\otimes a\beta(L_{\rho_i})}\otimes R^0\pi_*(\mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i})))\right)\\
&=\prod_{m=0}^{\lfloor\beta(L_{\rho_i})\rfloor}\left(D_{\rho_i}+\frac{\beta(L_{\rho_i})(-D_\theta)}{\delta(e)}+\frac{m}{\delta(e)}(D_\theta-\lambda)+\frac{\beta(L_{\rho_i})}{\delta(e)}\lambda\right)\\
&=\prod_{m=0}^{\lfloor\beta(L_{\rho_i})\rfloor}\left(D_{\rho_i}+\frac{\beta(L_{\rho_i})-m}{\delta(e)}(\lambda-D_\theta)\right).
\end{aligned}$$

Hence we have

$$e^{\mathbb{C}^*}((R^\bullet\pi_*\mathcal{L}_{\rho_i})^{\mathrm{mov}})=\prod_{0\leqslant m<\beta(L_{\rho_i})}\left(D_{\rho_i}+\frac{\beta(L_{\rho_i})-m}{\delta(e)}(\lambda-D_\theta)\right).$$

Note that the invariant part of $R^\bullet\pi_*\mathcal{L}_{\rho_i}$ is nonzero only when $\beta(L_{\rho_i})\in\mathbb{Z}_{\geqslant 0}$.

(2) If $\beta(L_{\rho_i})\in\mathbb{Q}_{<0}$, one has

$$\begin{aligned}
e^{\mathbb{C}^*}\left(R^\bullet\pi_*\mathcal{L}_{\rho_i}\right)&=e^{\mathbb{C}^*}\left(R^\bullet\pi_*\left(\pi^*L_{\rho_i}\otimes\mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i}))\otimes\pi^*\mathcal{R}^{\otimes a\beta(L_{\rho_i})}\right)\right)\\
&=\frac{1}{e^{\mathbb{C}^*}\left(L_{\rho_i}\otimes\mathcal{R}^{\otimes a\beta(L_{\rho_i})}\otimes R^1\pi_*(\mathcal{O}_{\mathcal{C}_e}(ar\beta(L_{\rho_i})))\right)}\\
&=\prod_{\beta(L_{\rho_i})<m<0}\frac{1}{D_{\rho_i}+\frac{\beta(L_{\rho_i})(-D_\theta)}{\delta(e)}+\frac{m}{\delta(e)}(D_\theta-\lambda)+\frac{\beta(L_{\rho_i})}{\delta(e)}\lambda}\\
&=\prod_{\beta(L_{\rho_i})<m<0}\frac{1}{D_{\rho_i}+\frac{\beta(L_{\rho_i})-m}{\delta(e)}(\lambda-D_\theta)},
\end{aligned}$$

which implies that

$$e^{\mathbb{C}^*}\left((R^\bullet\pi_*\mathcal{L}_{\rho_i})^{\mathrm{mov}}\right) = e^{\mathbb{C}^*}\left(R^\bullet\pi_*\mathcal{L}_{\rho_i}\right) = \prod_{\beta(L_{\rho_i})<m<0} \frac{1}{D_{\rho_i} + \frac{\beta(L_{\rho_i})-m}{\delta(e)}(\lambda - D_\theta)} \, .$$

The movable part of deformation of $\vec{y}$ contributes

$$e^{\mathbb{C}^*}\left(\oplus_{j=1}^p R^\bullet\pi_*(\mathcal{L}_{k+j})^{\mathrm{mov}}\right) = (\frac{\lambda - D_\theta}{\delta(e)})^{|J_e|}$$

to the virtual normal bundle and the fixed part of the deformation of $\vec{y}$ will be canceled with the automorphisms of line bundles $(L_{k+j} : 1 \leqslant j \leqslant p)$.

Recall that the complete intersection Y is cut off by the section $s := \oplus_{b=1}^c s_b$ of the direct sum of the line bundles $E = \oplus_{b=1}^c L_{\tau_b}$ on X associated to the characters τ_b. There is also an obstruction corresponding to the infinitesimal deformations of $\vec{x}$ being moved away from $[AY^{ss}(\theta)/G] \subset [W^{ss}(\theta)/G]$, which contributes to the virtual normal bundle as the movable part of

$$\begin{aligned} e^{\mathbb{C}^*}\left(-(\oplus_b R^\bullet\pi_*\mathcal{L}_{\tau_b})\right) &= \frac{e^{\mathbb{C}^*}\left(R^1\pi_* \oplus_{b:\beta(L_{\tau_b})<0} \mathcal{L}_{\tau_b}\right)}{e^{\mathbb{C}^*}\left(R^0\pi_* \oplus_{b:\beta(L_{\tau_b})\geqslant 0} \mathcal{L}_{\tau_b}\right)} \\ &= \frac{\prod_{b:\beta(L_{\tau_b})<0}\prod_{\beta(L_{\tau_b})<m<0}\left(c_1(L_{\tau_b}) + \frac{\beta(L_{\tau_b})-m}{\delta(e)}(\lambda - D_\theta)\right)}{\prod_{b:\beta(L_{\tau_b})\geqslant 0}\prod_{0\leqslant m\leqslant\beta(L_{\tau_b})}\left(c_1(L_{\tau_b}) + \frac{\beta(L_{\tau_b})-m}{\delta(e)}(\lambda - D_\theta)\right)} \, . \end{aligned}$$

Here m are all integers.

One can see that the fixed part only comes from the summand corresponding to the terms b with $\beta(L_{\tau_b}) \in \mathbb{Z}_{\geqslant 0}$, for which there is one dimensional $\mathbb{C}^*-$fixed piece to each $-R^\bullet\pi_*\mathcal{L}_{\tau_b}$, which contributes to the virtual cycle of $\mathcal{M}_e$.

Now let's move to the virtual cycle of $\mathcal{M}_e$ coming from the $\mathbb{C}^*-$fixed part of the restriction of perfect obstruction theory. Let $E_\beta := \oplus_{b:\beta(L_{\tau_b})\in\mathbb{Z}_{\geqslant 0}} L_{\tau_b}$ be the vector bundle over $[Z_\beta^{ss}/G]$ and $s_\beta := \oplus_{b:\beta(L_{\tau_b})\in\mathbb{Z}_{\geqslant 0}} s_b$ be the section inside E_β. Using Lemma 3.2, we can define the Gysin morphism

$$s^!_{E_\beta,loc} : A_*([Z_\beta^{ss}/G]) \to A_*([Y_\beta^{ss}/G])$$

as the localized top Chern class [Ful84, §14.1]. This Gysin morphism commutes with the one defined in 3.3 by the flat pullback $A^*([Y_\beta^{ss}/(G/\langle g_\beta^{-1}\rangle)]) \to A^*([Y_\beta^{ss}/G])$ on the target and the flat pullback $A^*([Z_\beta^{ss}/(G/\langle g_\beta^{-1}\rangle)]) \to A^*([Z_\beta^{ss}/G])$ on the source.

Lemma 4.7. *We have the following:*

$$[\mathcal{M}_e]^{\mathrm{vir}} = i^*_{\mathcal{M}_e}\left(s^!_{E_\beta,loc}([Z_\beta^{ss}/G])\right) \, .$$

Here $i_{\mathcal{M}_e} : \mathcal{M}_e \to [Y_\beta^{ss}/G]$ is the natural étale morphism by forgetting root structure.

Proof. By the previous discussion, the perfect obstruction theory of $\mathcal{M}_e$ solely comes automorphisms of line bundles $(\mathcal{L}_j)_{j=1}^k$, the fixed part of deformations/obstructions of the section $\vec{x}$. Using the distinguished triangle 4.8 and 4.7 in §4.2, the $\mathbb{C}^*-$fixed part of the obstruction complex $\mathbb{E}^{\mathrm{fix}}$ over $\mathcal{M}_e$ is quasi-isomorphic to the complex

$$\mathbb{T}_{[Z_\beta^{ss}/G]}|_{\mathcal{M}_e} \xrightarrow{ds_\beta} E_\beta$$

with the first term sitting in degree 0 and the second term sitting in degree 1, which also fits into the following distinguished triangle (from cone construction)

$$\mathbb{E}^{\mathrm{fix}} \longrightarrow \mathbb{T}_{[Z_\beta^{ss}/G]}|_{\mathcal{M}_e} \xrightarrow{ds_\beta} E_\beta \ .$$

Here ds_β is the differential induced the section s_β (c.f. 4.8) and $\mathbb{T}_{[Z_\beta^{ss}/G]}|_{\mathcal{M}_e}$ is the pullback of the tangent bundle $\mathbb{T}_{[Z_\beta^{ss}/G]}$ along the composition of morphisms

$$\mathcal{M}_e \to \sqrt[a\delta(e)]{L_{-\theta}/[Z_\beta^{ss}/G]} \to [Z_\beta^{ss}/G] \ ,$$

where the first arrow is the inclusion and the second arrow is the natural *é*tale morphism by forgetting root.

When we replace Y by X, repeat the same localization analysis as above, we see the fixed part of the restriction of the obstruction theory to the edge moduli $\mathcal{M}_e(X) := \sqrt[a\delta(e)]{L_{-\theta}/[Z_\beta^{ss}/G]}$ of X is equal to the tangent complex of $\mathcal{M}_e(X)$, which is a locally free sheaf sitting in degree zero as $\mathcal{M}_e(X)$ is a smooth Deligne-Mumford stack. Then we can view $\mathcal{M}_e$ as the zero loci of the section s_β of the vector bundle E_β over $\mathcal{M}_e(X)$ by Lemma 3.2, one has the following Cartesian diagram:

$$\begin{tikzcd} \mathcal{M}_e \arrow[r, "i"] \arrow[d, "i"] & \mathcal{M}_e(X) \arrow[d, "s_\beta"] \\ \mathcal{M}_e(X) \arrow[r, "0"] & E_\beta \ , \end{tikzcd}$$

where the bottom arrow is the zero section. Then we have a morphism of two distinguished triangles in $D^b_{coh}(\mathcal{M}_e)$ with all terms in the first low are perfect complexes with amplitude in $[-1, 0]$

$$\begin{tikzcd} \mathbb{T}^\vee_{[Z_\beta^{ss}/G]}|_{\mathcal{M}_e} \arrow[r] \arrow[d, equal] & (\mathbb{E}^{\mathrm{fix}})^\vee \arrow[r] \arrow[d] & E^\vee_{\geqslant 0}[1] \arrow[r, "ds_\beta^\vee"] \arrow[d, "i^*"] & \mathbb{T}^\vee_{[Z_\beta^{ss}/G]}|_{\mathcal{M}_e}[1] \arrow[d, equal] \\ \Omega_{\mathcal{M}_e(X)}|_{\mathcal{M}_e} \arrow[r] & t_{\geqslant -1}\mathbb{L}_{\mathcal{M}_e} \arrow[r] & \mathcal{I}_{\mathcal{M}_e/\mathcal{M}_e(X)}/\mathcal{I}^2_{\mathcal{M}_e/\mathcal{M}_e(X)}[1] \arrow[r, "d"] & \Omega_{\mathcal{M}_e(X)}|_{\mathcal{M}_e} \ . \end{tikzcd}$$

Here the first and the second vertical maps are the perfect (dual) obstruction theory for $\mathcal{M}_e(X)$ and $\mathcal{M}_e$ (both restricted to $\mathcal{M}_e$) respectively, while the third vertical map is an obstruction theory for $\mathbb{C}^*$−fixed quasimaps in $\mathcal{M}_e$ with the section $\vec{x}$ moving away from Y into X, and a standard deformation theory argument (c.f. [CL11, Proposition 2.5]) shows the third vertical map i^* is induced from the pullback of the conormal sheafs for the horizontal arrows in the above Cartesian square along the left arrow i. Then virtual cycle $[\mathcal{M}_e]^{vir}$ with respect to the (dual) perfect obstruction theory $(\mathbb{E}^{\mathrm{fix}})^\vee \to t_{\geqslant -1}\mathbb{L}_{\mathcal{M}_e}$ can be obtained by Manolache's virtual pull-back [Man11, Construction 3.6], which is also identical to Gysin pullback $0^! = s^!_{E_\beta, loc}$ (by the very of definition of localized top Chern class). Now the Lemma is immediate by flat pull-back along $i_{\mathcal{M}_e}$. □

We have the expression of virtual normal bundle from the movable part of curves, line bundles and sections as follows:

$$e^{\mathbb{C}^*}(N^{\mathrm{vir}}) = \frac{\prod_{\rho:\beta(L_\rho)>0}\prod_{0\leqslant i<\beta(L_\rho)}(D_\rho + (\beta(L_\rho)-i)\frac{\lambda-D_\theta}{\delta(e)})}{\prod_{\rho:\beta(L_\rho)<0}\prod_{\lfloor\beta(L_\rho)+1\rfloor\leqslant i<0}(D_\rho + (\beta(L_\rho)-i)\frac{\lambda-D_\theta}{\delta(e)})}\cdot(\frac{\lambda-D_\theta}{\delta(e)})^{|J_e|}\frac{\delta(e)}{\lambda-D_\theta}$$
$$\cdot\frac{\prod_{b:\beta(L_{\tau_b})<0}\prod_{\beta(L_{\tau_b})<m<0}\left(c_1(L_{\tau_b})+\frac{\beta(L_{\tau_b})-m}{\delta(e)}(\lambda-D_\theta)\right)}{\prod_{b:\beta(L_{\tau_b})\geqslant 0}\prod_{0\leqslant m<\beta(L_{\tau_b})}\left(c_1(L_{\tau_b})+\frac{\beta(L_{\tau_b})-m}{\delta(e)}(\lambda-D_\theta)\right)}\cdot\prod_{m=1}^{\delta(e)-\beta(L_\theta)-|J_e|}\left(\frac{m}{\delta(e)}(-D_\theta+\lambda)\right).$$

We observe that, after taking the push-forward along the morphism $ft:\mathcal{M}_e\to I_{g_\beta}Y$ which is the composition of the map of forgetting root structure of $\mathcal{M}_e$ first and the map of taking inclusion $[Y^{ss}_\beta/G]\to[AY^{ss}(\theta)^{g_\beta}/G]\cong I_{g_\beta}Y$ afterwards, the localization contribution from the edge moduli with basepoints yields:

Lemma 4.8.

$$ft_*(Cont_{\mathcal{M}_e}) = ft_*\left(\frac{[\mathcal{M}_e]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N^{\mathrm{vir}})}\right) = \frac{1}{a\delta(e)}\iota_*\left(\frac{\left(\frac{z}{z^{|J_e|}}\mathbb{I}_\beta(z)\right)|_{z=\frac{\lambda-D_\theta}{\delta(e)}}}{\prod_{m=1}^{\delta(e)-\beta(L_\theta)-|J_e|}\left(\frac{m}{\delta(e)}(-D_\theta+\lambda)\right)}\right),$$

where ι is the involution of $\bar{I}_\mu Y$ obtained from taking the inverse of the band, and $\mathbb{I}_\beta(z)$ is the coefficient of q^β of $\mathbb{I}(q,0,z)$ defined in the introduction 1.1.2.

4.3.3. *Edge contributions: without basepoint case.* The contribution from an edge without basepoint will not appear in the later analysis in §6. However we include the discussion for this case here for completeness. The reader is encouraged to skip this part in the first reading. In this case, J_e is empty. Assume that the multiplicity at $q_\infty\in C_e$ is equal to $(g,\mu_r^{\delta(e)})\in G\times\boldsymbol{\mu}_r$ and a_e (or a for simplicity) is the order of g. When r is sufficiently large, due to Remark 4.4, C_e must be isomorphic to $\mathbb{P}^1_{ar,a}$ where the ramification point q_0 for which $z_1=0$ is isomorphic to $\mathbb{B}\boldsymbol{\mu}_a$, and the ramification point q_∞ for which $z_2=0$ must be a special point and is isomorphic to $\mathbb{B}\boldsymbol{\mu}_{ar}$. The restriction of degree $(\beta,\frac{\delta}{r})$ from C to C_e is equal to $(0,\frac{\delta(e)}{r})$, which is equivalent to:

$$deg(L_j|_{C_e})=0\quad for\ 1\leqslant j\leqslant k,\quad deg(N|_{C_e})=\frac{\delta(e)}{r}\ .$$

Recall that the inertia stack component I_gY of $I_\mu Y$ is isomorphic to the quotient stack

$$[AY^{ss}(\theta)^g/G]\ .$$

We construct the edge moduli $\mathcal{M}_e$ as

$$\mathcal{M}_e := \sqrt[a\delta(e)]{L_{-\theta}/I_gY}\ ,$$

which is the root gerbe over the stack I_gY by taking the $a\delta(e)$th root of the line bundle $L_{-\theta}$.

The root gerbe $\sqrt[a\delta(e)]{L_{-\theta}/I_gY}$ admits a representation as a quotient stack:

$$[(AY^{ss}(\theta)^g\times\mathbb{C}^*)/(G\times\mathbb{C}^*_w)]\ , \tag{4.13}$$

where the (right) action is defined by:

$$(\vec{x},v)\cdot(g,w)=(\vec{x}\cdot g,\theta(g)vw^{a\delta(e)})\ ,$$

for all $(g,w)\in G\times\mathbb{C}^*_w$ and $(\vec{x},v)\in AY^{ss}(\theta)^g\times\mathbb{C}^*$. Here $\vec{x}\cdot g$ is given by the action as in the definition of $[AY/G]$, the torus $\mathbb{C}^*_w$ is isomorphic to $\mathbb{C}^*$ with variable w. For any character ρ of G, define a new character of $G\times\mathbb{C}^*_w$ by composing the projection map $\mathrm{pr}_G: G\times\mathbb{C}^*_w\to G$. By an abuse of notation, we will continue to use the notation ρ to mean the new character of $G\times\mathbb{C}^*_w$. Then ρ will determines a line bundle $L_\rho := [(AY^{ss}(\theta)^g\times\mathbb{C}^*\times\mathbb{C}_\rho)/(G\times\mathbb{C}^*_w)]$ on $\sqrt[a\delta(e)]{L_{-\theta}/I_gY}$ by the Borel construction.

By virtue of the universal property of root gerbe, on $\mathcal{M}_e = \sqrt[a\delta(e)]{L_{-\theta}/I_gY}$, there is a universal line bundle $\mathcal{R}$ that is the $a\delta(e)$th root of the line bundle $L_{-\theta}$. The root bundle $\mathcal{R}$ is associated to the character

$$\mathrm{pr}_{\mathbb{C}^*} : G\times\mathbb{C}^*_w \to \mathbb{C}^*_w, \quad (g,w) \in G\times\mathbb{C}^*_w \mapsto w \in \mathbb{C}^*_w$$

by the Borel construction. We have the relation

$$L_{-\theta} = \mathcal{R}^{a\delta(e)} .$$

The coordinate functions $\vec{x}$ and v of $AY^{ss}(\theta)^g \times \mathbb{C}^*$ descents to be universal sections of line bundles $\oplus_{\rho\in[n]}L_\rho$ and $L_\theta \otimes \mathcal{R}^{\otimes a\delta(e)}$ over $\mathcal{M}_e$, respectively.

We will construct a universal family of $\mathbb{C}^*-$fixed quasimaps to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ of degree $(0, 1^\emptyset, \frac{\delta(e)}{r})$ over $\mathcal{M}_e$:

$$\begin{tikzcd} \mathcal{C}_e := \mathbb{P}_{ar,a}(\mathcal{R}\oplus\mathcal{O}_{\mathcal{M}_e}) \arrow[r, "f"] \arrow[d, "\pi"'] & \mathbb{P}\mathfrak{Y}^{\frac{1}{r},p} \\ \mathcal{M}_e := \sqrt[a\delta(e)]{L_{-\theta}/I_gY}. & \end{tikzcd}$$

Then the universal curve $\mathcal{C}_e$ over $\mathcal{M}_e$ can be represented as a quotient stack:

$$\mathcal{C}_e = [(AY^{ss}(\theta)^g\times\mathbb{C}^*\times U)/(G\times\mathbb{C}^*_w\times T)] ,$$

where $T = \{(t_1,t_2) \in (\mathbb{C}^*)^2 | \quad t_1^a = t_2^{ar}\}$. The (right) action is defined by:

$$(\vec{x}, v, x, y)\cdot(g, w, (t_1,t_2)) = (\vec{x}\cdot g, \theta(g)vw^{a\delta(e)}, wt_1x, t_2y) ,$$

for all $(g, w, (t_1, t_2)) \in G\times\mathbb{C}^*_w\times T$ and $(\vec{x}, v, (x, y)) \in AY^{ss}(\theta)^g\times\mathbb{C}^*\times U$. Then $\mathcal{C}_e$ is a family of orbifold $\mathbb{P}_{ar,a}$ parameterized by $\mathcal{M}_e$.

There are two standard characters χ_1 and χ_2 of T:

$$\chi_1 : (t_1, t_2) \in T \mapsto t_1 \in \mathbb{C}^*, \quad \chi_2 : (t_1, t_2) \in T \mapsto t_2 \in \mathbb{C}^* .$$

We can lift them to be new characters of $G\times\mathbb{C}^*_w\times T$ by composing the projection map $\mathrm{pr}_T : G\times\mathbb{C}^*_w\times T \to T$. By an abuse of notation, we continue to use χ_1, χ_2 to denote the new characters. Then χ_1, χ_2 defines two line bundles

$$M_1 := (AY^{ss}(\theta)^g\times\mathbb{C}^*\times U) \times_{G\times\mathbb{C}^*_w\times T} \mathbb{C}_{\chi_1}$$

and

$$M_2 := (AY^{ss}(\theta)^g\times\mathbb{C}^*\times U) \times_{G\times\mathbb{C}^*_w\times T} \mathbb{C}_{\chi_2}$$

on $\mathcal{C}_e$ by the Borel construction, respectively. We have the relation $M_1^{\otimes a} = M_2^{\otimes ar}$ on $\mathcal{C}_e$. The universal map f from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ can be constructed as follows: let

$$\tilde{f} : AY^{ss}(\theta)^g\times\mathbb{C}^*\times U \to AY\times U$$

be the morphism defined by:

$$\begin{aligned} (\vec{x}, v, x, y) \in AY^{ss}(\theta)^g\times\mathbb{C}^*\times U \mapsto& \\ &((x_1,\cdots,x_n), v^{-1}x^{a\delta(e)}, y^{a\delta(e)}) \in AY\times U \end{aligned} \tag{4.14}$$

Then $\tilde{f}$ is equivariant with respect to the group homomorphism from $G\times\mathbb{C}^*_w\times T$ to $G\times\mathbb{C}^*$ defined by:

$$\begin{aligned} (g, w, (t_1,t_2)) \in G\times\mathbb{C}^*_w\times T \mapsto& \\ &(g\cdot((t_1^{-1}t_2^r)^{p_1},\cdots,(t_1^{-1}t_2^r)^{p_k}), t_2^{a\delta(e)}) \in G\times\mathbb{C}^* , \end{aligned} \tag{4.15}$$

where the tuple $(p_1, \cdots, p_k) \in \mathbb{N}^k$ satisfies that $g = (\mu_a^{p_1}, \cdots, \mu_a^{p_k}) \in G$. Note that $\tilde{f}$ is well defined for $\chi_1^{-1}\chi_2^r$ is a torsion character of T of order a. The above construction gives the universal morphism f from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ by descent.

Now we define a (quasi left) $\mathbb{C}^*-$action on $\mathcal{C}_e$ such that f is $\mathbb{C}^*-$equivariant. The $\mathbb{C}^*$-action on $\mathcal{C}_e$ is induced by the $\mathbb{C}^*-$action on $AY^{ss}(\theta)^g \times \mathbb{C}^* \times U$:

$$m : \mathbb{C}^* \times AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \to AY^{ss}(\theta) \times \mathbb{C}^* \times U \ ,$$
$$t \cdot (\vec{x}, v, (x, y)) = (\vec{x}, v, (x, t^{\frac{-1}{ar\delta(e)}} y)) \ .$$

Note that then π is $\mathbb{C}^*$-equivariant map, where $\mathcal{M}_e$ is equipped with the trivial $\mathbb{C}^*$-action. By the universal property of the projectivized bundle $\mathcal{C}_e$ over $\mathcal{M}_e$, one has a tautological section

$$(x, y) \in H^0(\mathcal{C}_e, (M_1 \otimes \pi^*\mathcal{R}) \oplus (M_2 \otimes \mathbb{C}_{\frac{-1}{ar\delta(e)}})) \ ,$$

which is also a $\mathbb{C}^*-$invariant section.

Now we can check that f is a $\mathbb{C}^*-$equivariant morphism from $\mathcal{C}_e$ to $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ with respect to the $\mathbb{C}^*-$actions for $\mathcal{C}_e$ and $\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$. Using Remark 4.2, f is given by the following data:

(1) $k + p + 1$ $\mathbb{C}^*-$equivariant line bundles $\mathcal{C}_e$:

$$\mathcal{L}_j := \pi^* L_{\pi_j} \otimes (M_1^\vee \otimes M_2^{\otimes r})^{p_j}, 1 \leqslant j \leqslant k \ ,$$
$$\mathcal{L}_{k+j} := \mathbb{C}, \ 1 \leqslant j \leqslant p$$

and

$$\mathcal{N} := M_2^{a\delta(e)} \otimes \mathbb{C}_{\frac{-\lambda}{r}} \ ,$$

where $(L_{\pi_j})_{1\leqslant j\leqslant k}$ are the standard $\mathbb{C}^*$-equivariant line bundles on $\mathcal{M}_e$ by the Borel contribution, M_1, M_2 are the standard $\mathbb{C}^*$-equivariant line bundles on C_e by the Borel construction;

(2) a universal section

$$\begin{aligned}(\vec{x}, \vec{y}, (\zeta_1, \zeta_2)) :=& ((x_1, \cdots, x_n), 1^p, (v^{-1}x^{a\delta(e)}, y^{a\delta(e)})) \\ &\in H^0(\mathcal{C}_e, \oplus_{i=1}^n \mathcal{L}_{\rho_j} \oplus (\oplus_{j=1}^p \mathcal{L}_{k+j}) \oplus (\mathcal{L}_{-\theta_p} \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda) \oplus \mathcal{N})^{\mathbb{C}^*} \ .\end{aligned} \tag{4.16}$$

Use the similar analysis as previous subsection, we have that $[\mathcal{M}_e]^{vir} = [\mathcal{M}_e]$ and the Euler class of virtual normal bundle from the sections is equal to

$$e^{\mathbb{C}^*}(N^{\mathrm{vir}}) = \prod_{m=1}^{\delta(e)} \left(\frac{m}{\delta(e)} (-D_\theta + \lambda) \right) .$$

when r is a sufficiently large prime. Besides the movable part of infinitesimal automorphisms of C_e contributes

$$\frac{\delta(e)}{\lambda - D_\theta}$$

to the Euler class of virtual normal bundle when $a = 1$.

4.3.4. *Node contributions.* The deformations in $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ smoothing a node contribute to the Euler class of the virtual normal bundle as the first Chern class of the tensor product of the two cotangent line bundles at the branches of the node. For nodes at which a component C_e meets a component C_v over the vertex 0, this contribution is

$$\frac{\lambda - D_\theta}{a\delta(e)} - \frac{\bar{\psi}_v}{a} \ ;$$

for nodes at which a component C_e meets a component C_v over the vertex ∞, this contribution is

$$\frac{-\lambda + D_\theta}{ar\delta(e)} - \frac{\bar{\psi}_v}{ar} \ ;$$

for nodes at which two edge component C_e meets with a vertex v over 0, the node-smoothing contribution is

$$\frac{\lambda - D_\theta}{a\delta(e)} + \frac{\lambda - D_\theta}{a\delta(e')} .$$

The nodes at which two edge component C_e meets with a vertex v over ∞ will not occur using a similar argument in [JPPZ17, Lemma 6] when r is sufficiently large.

As for the node contributions from the normalization exact sequence of relative obstruction theory (4.6), each node q (specified by a vertex v) contributes the inverse of Euler class of

$$(R^0\pi_*(\mathcal{L}_\theta^\vee \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda)|_q)^{\mathrm{mov}} \oplus (R^0\pi_*\mathcal{N}|_q)^{\mathrm{mov}} \tag{4.17}$$

to the Euler class of the virtual normal bundle. Note that here we use the fact that the node can't be a base point, which implies that $\mathcal{L}_{\theta_p}|_q = \mathcal{L}_\theta|_q$.

In the case where $j(v) = 0$, $z_2|_q = 1$ gives a trivialization of $\mathcal{N}$ at q. Thus, the second factor in (4.17) is trivial, while the inverse of the Euler class of the first factor equals

$$\frac{1}{\lambda - D_\theta} .$$

In the case where $j(v) = \infty$, $z_1|_q = 1$ gives a trivialization of the fiber $(\mathcal{L}_\theta^\vee \otimes \mathcal{N}^{\otimes r} \otimes \mathbb{C}_\lambda)|_q$. Hence we have $\mathcal{N}|_q \cong \mathcal{L}_\theta^{\frac{1}{r}}|_q \otimes \mathbb{C}_{-\frac{\lambda}{r}}$, this implies that it $R^0\pi_*(\mathcal{N}|_q) = 0$ because of the nontrivial stacky structure when r is sufficiently large. Thus there is no localization contribution from the normalization sequence at the node over ∞.

4.4. **Total localization contributions.** For each decorated graph Γ, denote the moduli F_Γ to be the fiber product

$$\prod_{v:j(v)=0} \mathcal{M}_v \times_{\bar{I}_\mu Y} \prod_{e\in E} \mathcal{M}_e \times_{\bar{I}_\mu \sqrt[r]{L_\theta/Y}} \prod_{v:j(v)=\infty} \mathcal{M}_v$$

of the following diagram:

$$\begin{tikzcd}
F_\Gamma \arrow[r] \arrow[d] & \prod_{v:j(v)=0} \mathcal{M}_v \times \prod_{e\in E} \mathcal{M}_e \times \prod_{v:j(v)=\infty} \mathcal{M}_v \arrow[d, "ev_{nodes}"] \\
\prod_E (\bar{I}_\mu Y \times \bar{I}_\mu \sqrt[r]{L_\theta/Y}) \arrow[r, "(\Delta\times\Delta^{\frac{1}{r}})^{|E|}"] & \prod_E (\bar{I}_\mu Y)^2 \times (\bar{I}_\mu \sqrt[r]{L_\theta/Y})^2
\end{tikzcd} ,$$

where $\Delta = (id, \iota)$(resp. $\Delta^{\frac{1}{r}} = (id, \iota)$) is the diagonal map of $\bar{I}_\mu Y$ (resp. $\bar{I}_\mu \sqrt[r]{L_\theta/Y}$). Here when v is a stable vertex, the vertex moduli $\mathcal{M}_v$ is described in 4.3.1; when v is an unstable vertex, we treat $\mathcal{M}_v := \bar{I}_{m(h)^{-1}}\mathcal{D}_{j(v)}$ with the virtual cycle given by the fundamental class of $\mathcal{M}_v$ and zero virtual normal bundle, where h is the half-edge incident to v. The right-hand vertical map ev_{nodes} is the product of the evaluation maps at the two branches of each gluing node.

We define $[F_\Gamma]^{\mathrm{vir}}$ to be the fiber product:

$$\prod_{v:j(v)=0} [\mathcal{M}_v]^{\mathrm{vir}} \times_{\bar{I}_\mu Y} \prod_{e\in E} [\mathcal{M}_e]^{\mathrm{vir}} \times_{\bar{I}_\mu \sqrt[r]{L_\theta/Y}} \prod_{v:j(v)=\infty} [\mathcal{M}_v]^{\mathrm{vir}} .$$

Then the contribution of decorated graph Γ to the virtual localization is:

$$Cont_\Gamma = \frac{\prod_{e\in E} a_e}{|\mathrm{Aut}(\Gamma)|} (\iota_\Gamma)_* \left(\frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})} \right) . \tag{4.18}$$

Here $\iota_F : F_\Gamma \to Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$ is a finite etale map of degree $\frac{|\mathrm{Aut}(\Gamma)|}{\prod_{e\in E} a_e}$ into the corresponding $\mathbb{C}^*$-fixed loci. The virtual normal bundle $e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})$ is the product of virtual normal bundles from vertex contributions, edge contributions and node contributions.

Remark 4.9. Let u be a polynomial on $c_1(L_{\pi_1}), \cdots, c_1(L_{\pi_k})$. In the contribution from the graph Γ, assume that $j \in J_e$ for some edge e, then $\hat{ev}_j|_{F_\Gamma}$ factors through the projection from F_Γ to $\mathcal{M}_e$. By abusing notations, we denote $\hat{ev}_j : \mathcal{M}_e \to \mathfrak{Y}$. Thus when we want to apply virtual localization to $\prod_{j=1}^p \hat{ev}_j^*(u(c_1(L_{\pi_k})))$, we can replace $\mathbb{I}_\beta(q,z)$ in Lemma 4.8 by $u(c_1(L_{\pi_k}) + \beta(L_{\pi_k})z)^{|J_e|}\mathbb{I}_\beta(q,z)$. Indeed, use the setting in §4.3.2, denote $\underline{ev} := \mathrm{pr}_{r,p} \circ ev : \mathcal{C}_e \to \mathfrak{Y}$, where $\mathrm{pr}_{r,p} : \mathbb{P}\mathfrak{Y}^{\frac{1}{r},p} \to \mathfrak{Y}$ is the natural projection map. Then we have

$$\underline{ev}^*(L_\tau) = \pi^*(L_\tau \otimes \mathcal{R}^{a\beta(L_\tau)}) \otimes \mathcal{O}_{\mathcal{C}_e}(ar\beta(L_\tau))$$

for any character τ of G. Let D_0 be the zero section of $\mathcal{C}_e$ over $\mathcal{M}_e$ given by $x = 0$. Then $\hat{ev}_j = \underline{ev}|_{D_0}$. Note that $\mathcal{O}_{\mathcal{C}_e}(1)|_{D_0} = \mathbb{C}_{\frac{\lambda}{ar\delta(e)}}$. Using the fact $\mathcal{R}^{a\delta(e)} = L_{-\theta}$, we have $c_1(\hat{ev}^*(L_\tau)) = c_1(L_\tau) + \frac{\beta(L_\tau)(\lambda - D_\theta)}{\delta(e)}$.

5. Master space II

5.1. **Construction of master space II.** Fix two different primes $r, s \in \mathbb{N}$, let θ be as in the previous section, let $\mathbb{P}Y_{r,s}$ be the root stack of the $\mathbb{P}^1$ bundle $\mathbb{P}_Y(\mathcal{O}(-D_\theta) \oplus \mathcal{O})$ over Y by taking the s-th root of the zero section ($z_1 = 0$) and r-th root of the infinity section ($z_2 = 0$). Then the zero section $\mathcal{D}_0 \subset \mathbb{P}Y_{r,s}$ is isomorphic to the root stack $\sqrt[s]{L_{-\theta}/Y}$, and the infinity section $\mathcal{D}_\infty \subset \mathbb{P}Y_{r,s}$ is isomorphic to the root stack $\sqrt[r]{L_\theta/Y}$.

We give a more concrete presentation of $\mathbb{P}Y_{r,s}$ as a quotient stack:

$$\mathbb{P}Y_{r,s} = [(\mathbb{C}^* \times AY^{ss}(\theta) \times U)/(G \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t)] \ ,$$

where the (right) $G \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t$-action on $\mathbb{C}^* \times AY^{ss}(\theta) \times U$ is given by:

$$(u, \vec{x}, z_1, z_2) \cdot (g, \alpha, t) = (\alpha^{-s}\theta(g)^{-1}t^r u, \vec{x}g, \alpha z_1, t z_2) \ ,$$

for $(g, \alpha, t) \in G \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t$, and $(u, \vec{x}, z_1, z_2) \in \mathbb{C}^* \times AY^{ss}(\theta) \times U$. Here $U = \mathbb{C}^2 \backslash \{0\}$. This quotient stack presentation of $\mathbb{P}Y_{r,s}$ comes from the root stack construction in [AGV08, Appendix B] after some simplification.

When the integer r is prime to the orders of isotropy groups of all points of X, which happens, in particular, as r is a sufficiently large prime, the rigidified inertia stack $\bar{I}_\mu \mathbb{P}Y_{r,s}$ of $\mathbb{P}Y_{r,s}$ is isomorphic to the disjoint union

$$\underbrace{\mathbb{P}(\bar{I}_\mu Y)_{r,s}}_{1} \sqcup \underbrace{\bigsqcup_{i=1}^{s-1} \bar{I}_\mu Y}_{2} \sqcup \underbrace{\bigsqcup_{j=1}^{r-1} \bar{I}_\mu Y}_{3} \ .$$

Let $(\vec{x}, (g, \alpha, t))$ be a $\mathbb{C}-$point of the rigidified inertia stack $\bar{I}_\mu \mathbb{P}Y_{r,s}$, if the point $(\vec{x}, (g, \alpha, t))$ appears in the first factor of the decomposition above, then the automorphism $\mu = (g, \alpha, t)$ lies in $G \times \{1\} \times \{1\}$, and the space $\mathbb{P}(\bar{I}_\mu Y)_{r,s}$ can be further decomposed as the disjoint union $\bigsqcup_{g \in G} \mathbb{P}(\bar{I}_g Y)_{r,s}$, where $\mathbb{P}(\bar{I}_g Y)_{r,s}$ is defined as the quotient stack

$$\mathbb{P}(\bar{I}_g Y)_{r,s} := [(\mathbb{C}^* \times AY^{ss}(\theta)^g \times U)/((G/\langle g \rangle) \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t)] \ ,$$

with the action similar to $\mathbb{P}Y_{r,s}$ as above, Note that this action is well-defined as the character θ is trivial on the subgroup $\langle g \rangle$ of G; if the point $(\vec{x}, (g, \alpha, t))$ occurs in the second factor of the decomposition above, then the automorphism (g, α, t) lies in $G \times \{\mu_s^i : 1 \leqslant i \leqslant s-1\} \times \{1\} \subset G \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t$, and the point $\vec{x}$ is in the zero section $\mathcal{D}_0$ defined by $z_1 = 0$; finally if the point $(\vec{x}, (g, \alpha, t))$ belongs to the third factor of the decomposition above, then the automorphism (g, α, t) lies in $G \times \{1\} \times \{\mu_r^j : 1 \leqslant i \leqslant r-1\} \subset G \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t$, and $\vec{x}$ is in the infinity section $\mathcal{D}_\infty$ defined by $z_2 = 0$. Here $\mu_r = \exp(\frac{2\pi\sqrt{-1}}{r}) \in \mathbb{C}^*$ and $\mu_s = \exp(\frac{2\pi\sqrt{-1}}{s}) \in \mathbb{C}^*$.

Fix $(g,\alpha,t)\in G\times\boldsymbol{\mu}_s\times\boldsymbol{\mu}_r$, we will use the notation $\bar{I}_{(g,\alpha,t)}\mathbb{P}Y_{r,s}$ to mean the rigidified inertia stack component of $\bar{I}_\mu\mathbb{P}Y_{r,s}$ which has automorphism (g,α,t). Note that if α and t are not equal to 1 simultaneously, then the corresponding rigidified inertia stack component is empty.

Let $\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))$ be the moduli stack of m-Pointed twisted stable maps to $\mathbb{P}Y_{r,s}$ of degree $(d,\frac{\delta}{r})$. More concretely,

$$\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))=\{(C;q_1,\dots,q_m;L_1,\cdots,L_k,N_1,N_2;u,\vec{x}:=(x_1,\dots,x_m),z_1,z_2)\},$$

where $(C;q_1,\dots,q_m)$ is a m-pointed prestable balanced twisted curve of genus 0 with nontrivial isotropy only at special points, $(L_j:1\leqslant j\leqslant k)$ and N_1,N_2 are orbifold line bundles on C with

$$deg([\vec{x}])=d\in Hom(Pic(\mathfrak{Y}),\mathbb{Q}),\quad \deg(N_2)=\frac{\delta}{r},$$

and

$$(u,(\vec{x},\vec{z})):=(u,x_1,\dots,x_n,z_1,z_2)\in\Gamma\left(\left((N_1^\vee)^{\otimes s}\otimes L_{-\theta}\otimes N_2^{\otimes r}\right)\oplus\bigoplus_{i=1}^n L_{\rho_i}\oplus N_1\oplus N_2\right).$$

Here, for $1\leqslant i\leqslant n$, the line bundle L_{ρ_i} is equal to

$$\otimes_{j=1}^k L_j^{m_{ij}}\ ,$$

where $(m_{ij})_{1\leqslant i\leqslant n,\ 1\leqslant j\leqslant k}$ is given by the relation $\rho_i=\sum_{j=1}^k m_{ij}\pi_j$. The same construction applies to the line bundle $L_{-\theta}$ on C. Note that here δ is an integer when $\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))$ is nonempty as $N_2^{\otimes r}$ is the pullback of some line bundle on the coarse moduli curve $\underline{C}$.

We require this data to satisfy the following conditions:

- *Representability*: For every $q\in C$ with isotropy group G_q, the homomorphism $\mathbb{B}G_q\to\mathbb{B}(G\times\mathbb{C}^*_\alpha\times\mathbb{C}^*_t)$ given by the restriction of line bundles $(L_j:1\leqslant j\leqslant k)$ and N_1,N_2 on q is representable.
- *Nondegeneracy*: The sections z_1 and z_2 never simultaneously vanish, and we have

$$\operatorname{ord}_q(\vec{x})=0. \tag{5.1}$$

  for all $q\in C$. Furthermore, the section u never vanish, so we have $(N_1^\vee)^{\otimes s}\otimes L_{-\theta}\otimes N_2^{\otimes r}\cong\mathcal{O}_C$.
- *Stability*: the map $[u,\vec{x},\vec{z}]:(C,q_1,\cdots,q_m)\to\mathbb{P}Y_{r,s}$ satisfies the usual stability condition defined by a twisted stable map;
- *Vanishing*: The image of $[\vec{x}]:C\to\mathfrak{X}$ lies in $\mathfrak{Y}$.

Let $\vec{m}=(v_1,\cdots,v_m)\in(G\times\boldsymbol{\mu}_s\times\boldsymbol{\mu}_r)^m$, we will denote $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))$ to be:

$$\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))\cap ev_1^{-1}(\bar{I}_{v_1}\mathbb{P}Y_{r,s})\cap\cdots\cap ev_m^{-1}(\bar{I}_{v_m}\mathbb{P}Y_{r,s})\ ,$$

where

$$ev_i:\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))\to\bar{I}_\mu\mathbb{P}Y_{r,s},$$

are natural evaluation maps as before, by evaluating the sections $(u,\vec{x},\vec{z})$ at q_i.

5.2. **$\mathbb{C}^*$-action and fixed loci.** Define a (left) $\mathbb{C}^*$-action on $\mathbb{C}^*\times AY^{ss}(\theta)\times U$ given by

$$t\cdot(u,\vec{x},(z_1,z_2))=(tu,\vec{x},(z_1,z_2))\ .$$

This action descends to be a (left) $\mathbb{C}^*$-action on $\mathbb{P}Y_{r,s}$, which induces a $\mathbb{C}^*$-action on $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(d,\frac{\delta}{r}))$. The reason why we define this action is that this definition lifts the $\mathbb{C}^*$-action on $\mathbb{P}Y$ defined in §4.1 along the canonical structure map $\pi_{r,s}:\mathbb{P}Y_{r,s}\to\mathbb{P}Y$. We will denote λ to be equivariant parameter corresponding to the action of weight 1. In this remaining subsection, r,s will be always assumed to be sufficiently large primes.

We will describe the virtual localization for $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$ similar to $Q^{\widetilde{\theta}}_{0,\vec{m}}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p},(\beta,1^p,\frac{\delta}{r}))$, but the edge contribution is easier to analyze as there is no basepoint occurring for twisted stable maps.

We index the components of $\mathbb{C}^*-$fixed loci of $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$ by decorated graphs. A decorated graph Γ consists of vertices, edges, and m legs with the following decorations it:

- Each vertex v is associated with an index $j(v)\in\{0,\infty\}$, and a degree $\beta(v)\in \mathrm{Eff}(W,G,\theta)$.
- Each edge $e=\{h,h'\}$ is equipped with a degree $\delta(e)\in\mathbb{N}$, here we call h and h' half edges and each edge is incident to a unique vertex.
- Each half-edge h and each leg l has an element $m(h)$ or $m(l)$ in $G\times\boldsymbol{\mu}_s\times\boldsymbol{\mu}_r$.
- The legs are labeled with the numbers $\{1,\ldots,m\}$, and each leg is incident to a unique vertex.

By the "valence" of a vertex v, denoted $\mathrm{val}(v)$, we mean the total number of incident half-edges and legs.

For each $\mathbb{C}^*-$fixed stable map f in $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$, we can associate a decorated graph Γ in the following way.

- Each edge e corresponds to a genus-zero component C_e on which $\deg(N_2)=\frac{\delta(e)}{r}$ for some integer $\delta(e)\in\mathbb{Z}_{>0}$, where there are two distinguished points q_0 and q_∞ on C_e satisfying that $z_2|_{q_\infty}=0$ and $z_1|_{q_0}=0$, respectively. We call them the "ramification points". Note that we have $deg(L_j|_{C_e})=0$ for all $1\leqslant j\leqslant k$.
- Each vertex v for which $j(v)=0$ (with unstable exceptional cases noted below) corresponds to a maximal sub-curve C_v of C over which $z_1\equiv 0$, then the restriction of $(C;q_1,\ldots,q_m;L_1,\cdots,L_k;\vec{x})$ to C_v defines a twisted stable map in
$$\mathcal{K}_{0,val(v)}(\sqrt[s]{L_{-\theta}/Y},\beta(v)):=\coprod_{\substack{d\in\mathrm{Eff}(AY,G,\theta)\\(i_{\mathfrak{Y}})_*(d)=\beta(v)}}\mathcal{K}_{0,val(v)}(\sqrt[s]{L_{-\theta}/Y},d)\ .$$
Each vertex v for which $j(v)=\infty$ (again with unstable exceptions) corresponds to a maximal sub-curve for which $z_2\equiv 0$, then the restriction of $(C;q_1,\ldots,q_m;L_1,\cdots,L_k;\vec{x})$ to C_v defines a twisted stable map in
$$\mathcal{K}_{0,val(v)}(\sqrt[r]{L_{\theta}/Y},\beta(v)):=\coprod_{\substack{d\in\mathrm{Eff}(AY,G,\theta)\\(i_{\mathfrak{Y}})_*(d)=\beta(v)}}\mathcal{K}_{0,val(v)}(\sqrt[r]{L_{\theta}/Y},d)\ .$$
The label $\beta(v)$ denotes the degree coming from the restriction $[x]|_{C_v}:C_v\to\mathfrak{X}$. Note that here we count the degree $\beta(v)$ in $\mathrm{Eff}(W,G,\theta)$, but not in $\mathrm{Eff}(AY,G,\theta)$.
- A vertex v is *unstable* if stable twisted maps of the type described above do not exist (where, as always, we interpret legs as marked points and half-edges as half-nodes). In this case, we have $\beta(v)=0$ and v corresponds to a single point of the component C_e for each adjacent edge e, which may be a node at which C_e meets another edge curve $C_{e'}$, a marked point of C_e, or an unmarked point.
- The index $m(l)$ on a leg l indicates the rigidified inertia stack component $\bar{I}_{m(l)}\mathbb{P}Y_{r,s}$ of $\mathbb{P}Y_{r,s}$ on which the marked point corresponding to the leg l is evaluated, this is determined by the multiplicity of $L_1,\cdots,L_k,N_1,N_2$ at the corresponding marked points.
- A half-edge h of an edge e corresponds a ramification point $q\in C_e$. Then $m(h)$ indicates the rigidified inertia component $\bar{I}_{m(h)}\mathbb{P}Y_{r,s}$ of $\mathbb{P}Y_{r,s}$ on which the ramification point q associated with h is evaluated.

In particular, we note that the decorations at each stable vertex v yield a vector
$$\vec{m}(v)\in(G\times\boldsymbol{\mu}_s\times\boldsymbol{\mu}_r)^{\mathrm{val}(v)}$$
recording the multiplicities of $L_1,\cdots,L_k,N_1,N_2$ at every special point of C_v

Remark 5.1. For each edge e, the restriction of $\vec{x}$ to C_e defines a constant map to Y. So the restriction of $(u,\vec{x},\vec{z})$ to C_e defines a representable map

$$f : C_e \to \mathbb{B}G_y \times \mathbb{P}^1_{r,s}$$

where $y \in Y$ comes from $\vec{x}$ and G_y is the isotropy group of $y \in Y$. Then we have $m(q_0) = (g^{-1}, \mu_s^{\delta(e)}, 1)$ and $m(q_\infty) = (g, 1, \mu_r^{\delta(e)})$ for some $g \in G_y$. denote a to be the order of element $g \in G$. Note that when r and s are sufficiently large primes comparing to $\delta(e)$, we must have $C_e \cong \mathbb{P}^1_{ar,as}$ and q_0 and q_∞ are special points as they are nontrivial stacky points. Here $\mathbb{P}^1_{ar,as}$ is the unique Deligne-Mumford stack with coarse moduli $\mathbb{P}^1$, isotropy group $\boldsymbol{\mu}_{as}$ at $0 \in \mathbb{P}^1$, isotropy group $\boldsymbol{\mu}_{ar}$ at $\infty \in \mathbb{P}^1$, and generic trivial stabilizer. We can write down the morphism f more precisely. First C_e can be represented as the quotient stack:

$$[U/T_{ar,as}] \ ,$$

where $U = \mathbb{C}^2\backslash\{0\}$, $T_{ar,as}$ is a subtorus of $(\mathbb{C}^*)^2$ defined by the equation $t_1^{as} = t_2^{ar}$, and $T_{ar,as}$ acts on U in the standard way as $(\mathbb{C}^*)^2$ does. Then f can be constructed explicitly from descent data $(\widetilde{f}, \widetilde{\beta})$: let $\widetilde{f}$ be the morphism

$$\widetilde{f} : U \to \mathbb{C}^* \times U \ ; (x,y) \to (1, x^{\delta(e)}, y^{\delta(e)}) \ ,$$

which is equivariant with respect to the group homomorphism

$$\widetilde{\beta} : T_{ar,as} \to G_y \times T_{r,s}; \quad (t_1,t_2) \to (\tau(t_1^{-s}t_2^{r}), t_1^{a\delta(e)}, t_2^{a\delta(e)}) \ ,$$

where τ is the morphism from the cyclic group $\boldsymbol{\mu}_a$ to G_y which sends the generator μ_a to g.

5.3. **Localization analysis.** Fix $\beta \in \mathrm{Eff(W,G,\theta)}, \delta \in \mathbb{Z}_{\geqslant 0}$ and $\vec{m} = (v_1,\cdots,v_m) \in (G\times\boldsymbol{\mu}_s\times\boldsymbol{\mu}_r)^m$, we will consider the space $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$. The reason why we assume that the second degree is $\frac{\delta}{r}$ is that $\mathcal{K}_{0,m}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$ admits a natural morphism to $\mathcal{K}_{0,m}(\mathbb{P}Y,(\beta,\delta))$(c.f.[AJT15, TT16]). Here $\mathbb{P}Y$ is equal to $\mathbb{P}Y_{r,s}$ for $r=s=1$. In this section, we will always assume that r and s are *sufficiently large primes*.

Now we analyze the $\mathbb{C}^*-$localization contribution for $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s},(\beta,\frac{\delta}{r}))$ as in §4.3.

5.3.1. *Vertex contributions.* The analysis of localization contribution for the stable vertex v is similar to the analysis in §4.3.1.

For each stable vertex v over ∞, the vertex moduli $\mathcal{M}_v$ corresponds to the moduli stack$\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y},\beta(v))$, which parameterizes twisted stable maps to the root gerbe $\sqrt[r]{L_\theta/Y}$ over Y.

Let

$$\pi : \mathcal{C}_\infty \to \mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y},\beta(v))$$

be the universal curve over $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y},\beta(v))$. Follow the same discussion in §4.3.1, the *inverse of the Euler class* of the virtual normal bundle for the vertex moduli $\mathcal{M}_v$ over ∞ is equal to

$$e^{\mathbb{C}^*}((-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})\otimes\mathbb{C}_{-\frac{\lambda}{r}}) \ .$$

When r is a sufficiently large prime and the multiplicity $m(l)$ corresponding to each leg l incident to v is equal to $(g_l, 1, \mu_r^{f_l})$ for some prefixed $f_l \in \mathbb{Z}_{\geqslant 0}$(note this implies $f_l \ll r$) and $g_l \in G$, following a generalization of [JPPZ18] to the orbifold case. The above Euler class has a representation

$$\sum_{d\geqslant 0} c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{|E(v)|-1-d} \ .$$

Here the virtual bundle $-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}}$ has virtual rank $|E(v)|-1$, where $|E(v)|$ is the number of edges incident to the vertex v. The fixed part of the obstruction theory contributes to the virtual cycle

$$[\mathcal{K}_{0,\vec{m}(v)}(\sqrt[r]{L_\theta/Y},\beta(v))]^{\mathrm{vir}}\ .$$

For the stable vertex v over 0, the vertex moduli $\mathcal{M}_v$ corresponds to the moduli space $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[s]{L_{-\theta}/Y},\beta(v))$,

Let

$$\pi:\mathcal{C}_0\to\mathcal{K}_{0,\vec{m}(v)}(\sqrt[s]{L_{-\theta}/Y},\beta(v))$$

be the universal curve over $\mathcal{K}_{0,\vec{m}(v)}(\sqrt[s]{L_{-\theta}/Y},\beta(v))$, and $f:\mathcal{C}_0\to\sqrt[s]{L_{-\theta}/Y}$ be the universal map. In this case, the fixed part of the perfect obstruction theory for the vertex moduli over 0 yields the virtual cycle

$$[\mathcal{K}_{0,\vec{m}(v)}(\sqrt[s]{L_{-\theta}/Y},\beta(v))]^{\mathrm{vir}}\ .$$

Note that $\mathcal{N}_2|_{\mathcal{C}_0}\cong\mathcal{O}_{\mathcal{C}_0}$ as $z_2|_{\mathcal{C}_0}\equiv 1$, the virtual normal bundle comes from the movable part of the infinitesimal deformations of z_1, which is a section of the line bundle $\mathcal{L}^{\frac{1}{s}}_{-\theta}$ over $\mathcal{C}_0$, which is the pullback of the universal $s-$th root line bundle on $\sqrt[s]{L_{-\theta}/Y}$ via the universal map f. Then the *inverse of the Euler class* of the virtual normal bundle is equal to

$$e^{\mathbb{C}^*}((-R^\bullet\pi_*\mathcal{L}^{\frac{1}{s}}_{-\theta})\otimes\mathbb{C}_{\frac{\lambda}{s}})\ .$$

We will simplify the above presentation when $\beta(v)\neq 0$. First, we will state a simple vanishing lemma regarding a line bundle of negative degree on a genus zero twisted curve, of which the proof is proceeded by induction on the number of irreducible components.

Lemma 5.2. *Let L be a line bundle of negative degree on a genus zero twisted curve C. Assume that the degree of the restriction of the line bundle $L|_{C_i}$ to every irreducible component C_i is non-positive. Then we have $H^0(C,L)=0$.*

Remark 5.3. For every fiber curve C_0 of the universal curve $\mathcal{C}_0$ over $\mathcal{M}_v$. The degree of the restricted line bundle $\mathcal{L}^{\frac{1}{s}}_{-\theta}|_{C_0}$ to C_0 is non-positive. Indeed, $\mathcal{L}^{\frac{1}{s}}_{-\theta}$ is the pullback of the s-th root of the line bundle $L_{-\theta}$ on $\sqrt[s]{L_{-\theta}/Y}$, where $L_{-\theta}$ is the pullback of an *anti-ample* line bundle from the coarse moduli of $\sqrt[s]{L_{-\theta}/Y}$. Now assuming $\beta(v)\neq 0$, we have the degree of the restricted line bundle $\mathcal{L}^{\frac{1}{s}}_{-\theta}|_{C_0}$ is negative by Lemma 2.5. By the above lemma, one has

$$R^0\pi_*\mathcal{L}^{\frac{1}{s}}_{-\theta}=0\ .$$

Then we have

$$-R^\bullet\pi_*\mathcal{L}^{\frac{1}{s}}_{-\theta}=R^1\pi_*\mathcal{L}^{\frac{1}{s}}_{-\theta}\ ,$$

which implies that $R^1\pi_*\mathcal{L}^{\frac{1}{s}}_{-\theta}$ is a vector bundle. When s is sufficiently large, and the multiplicity $m(l)$ corresponding to each leg l incident to v is equal to $(g_l,\mu_s^{f_l},1)$ for some prefixed number $f_l\in\mathbb{Z}_{\geqslant 0}$(note this implies $f_l<<s$) and $g_l\in G$, it has rank $|E(v)|-1$ where $|E(v)|$ is the number of edges incident to the vertex v. Especially when $|E(v)|=1$, it has rank 0, thus the Euler class becomes 1, this case will be important in the later simplification of the localization contribution in §6.2.

5.3.2. *Edge contributions.* Assume that the multiplicity at $q_\infty\in C_e$ is equal to $(g,1,\mu_r^{\delta(e)})$ and a (or a_e) is the order of $g\in G$. When r,s are sufficiently large primes, due to the Remark 5.1, C_e must be isomorphic to $\mathbb{P}^1_{ar,as}$ where the ramification point q_0 for which $z_1=0$ is isomorphic to $\mathbb{B}\mu_{as}$, and the ramification point q_∞ for which $z_2 = 0$ is isomorphic to $\mathbb{B}\mu_{ar}$. The restriction of the degree $(\beta, \frac{\delta}{r})$ from C to C_e is equal to $(0, \frac{\delta(e)}{r})$, which is equivalent to:

$$deg(L_j|_{C_e}) = 0, \quad for \quad 1 \leqslant j \leqslant k, \quad deg(N_2|_{C_e}) = \frac{\delta(e)}{r} \ .$$

When we fix the multiplicity $(g, 1, \mu_r^{\delta(e)})$ at q_∞, due to the Remark 5.1,[14] the evaluation map

$$ev_{q_\infty} : \mathcal{K}_{q_0 \sqcup q_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta(e)}{r}))^{\mathbb{C}^*} \to \bar{I}_{(g,1,\mu_r^{\delta(e)})}\mathbb{P}Y_{r,s} \cong \bar{I}_g Y$$

coming from the moduli $\mathcal{K}^{\mathbb{C}^*} := \mathcal{K}_{0,q_0 \sqcup q_\infty}(\mathbb{P}Y_{r,s}, (0, \frac{\delta(e)}{r}))^{\mathbb{C}^*}$ of $\mathbb{C}^*-$fixed maps of degree $(0, \frac{\delta(e)}{r})$ with the decorations at two markings as above induces the identity on the their coarse moduli. Moreover it's finite étale of degree $\frac{1}{a\delta(e)}$. To compute the edge contribution, which is topological in nature, it suffices to do a localization analysis over a finite étale cover of $\mathcal{K}^{\mathbb{C}^*}$. In the following, we will construct a space called $\mathcal{M}_e$ which is finite étale over $\mathcal{K}^{\mathbb{C}^*}$ of degree $\frac{1}{as}$ and carries a family of $\mathbb{C}^*-$fixed morphisms.

Recall that the inertia stack component $I_g Y$ of $I_\mu Y$ is isomorphic to

$$[AY^{ss}(\theta)^g/G] \ .$$

We define the edge moduli $\mathcal{M}_e$ to be

$$\sqrt[as\delta(e)]{L_{-\theta}/I_g Y} = \sqrt[as\delta(e)]{L_{-\theta}/[AY^{ss}(\theta)^g/G]} \ ,$$

which is the $as\delta(e)$th root gerbe over the inertia stack component $I_g Y$ of $I_\mu Y$ by taking the $as\delta(e)$th root of the line bundle $L_{-\theta}$.

The root gerbe $\sqrt[as\delta(e)]{L_{-\theta}/I_g Y}$ admits a representation as a quotient stack:

$$[AY^{ss}(\theta)^g \times \mathbb{C}^* / (G \times \mathbb{C}_w^*)],$$

where the (right) action is defined by:

$$(\vec{x}, v) \cdot (g, w) = (\vec{x}g, \theta(g)^{-1} v w^{-as\delta(e)}) \ ,$$

for all $(g, w) \in G \times \mathbb{C}_w^*$ and $(\vec{x}, v) \in AY^{ss}(\theta)^g \times \mathbb{C}^*$. For every character ρ of G, we can define a new character of $G \times \mathbb{C}_w^*$ by composing the projection map $\mathrm{pr}_G : G \times \mathbb{C}_w^* \to G$, we will still use ρ to name the new character of $G \times \mathbb{C}_w^*$ by an abuse of notation. Then ρ will determines a line bundle $L_\rho := [(AY^{ss}(\theta)^g \times \mathbb{C}^* \times \mathbb{C}_\rho)/(G \times \mathbb{C}_w^*)]$ on $\sqrt[as\delta(e)]{L_{-\theta}/I_g Y}$ by the Borel construction.

By virtue of the universal property of root gerbe, on $\mathcal{M}_e = \sqrt[as\delta(e)]{L_{-\theta}/I_g Y}$, there is a universal line bundle $\mathcal{R}$ that is the $as\delta(e)$th root of the line bundle $L_{-\theta}$. The root bundle $\mathcal{R}$ is determined by the character $\mathrm{pr}_{\mathbb{C}^*}$:

$$\mathrm{pr}_{\mathbb{C}^*} : G \times \mathbb{C}_w^* \to \mathbb{C}_w^* \quad (g, w) \in G \times \mathbb{C}_w^* \mapsto w \in \mathbb{C}_w^* \ .$$

We have the relation

$$L_{-\theta} = \mathcal{R}^{as\delta(e)} \ .$$

The coordinate functions $\vec{x}$ and v of $AY^{ss}(\theta)^g \times \mathbb{C}^*$ descents to be universal sections of line bundles $\oplus_{\rho \in [n]} L_\rho$ and $L_{-\theta} \otimes \mathcal{R}^{-\otimes as\delta(e)}$ over $\mathcal{M}_e$, respectively.

[14]This will imply the multiplicity at q_0 is $(g, \mu_s^{\delta(e)}, 1)$

We will construct a universal family of $\mathbb{C}^*-$fixed twisted stable maps to $\mathbb{P}Y_{r,s}$ of degree $(0, \frac{\delta(e)}{r})$ over $\mathcal{M}_e$:

$$\begin{array}{ccc} \mathcal{C}_e := \mathbb{P}_{ar,as}(\mathcal{R} \oplus \mathcal{O}_{\mathcal{M}_e}) & \xrightarrow{f} & \mathbb{P}Y_{r,s} \\ \downarrow{\scriptstyle \pi} & & \\ \mathcal{M}_e := \sqrt[as\delta(e)]{L_{-\theta}/I_g Y}. & & \end{array}$$

Then the universal curve $\mathcal{C}_e$ over $\sqrt[as\delta(e)]{L_{-\theta}/I_g Y}$ can be represented as a quotient stack:

$$\mathcal{C}_e = [(AY^{ss}(\theta)^g \times \mathbb{C}^* \times U)/(G \times \mathbb{C}^*_w \times T)] \ ,$$

where $T = \{(t_1, t_2) \in (\mathbb{C}^*)^2 | \quad t_1^{as} = t_2^{ar}\}$. The right action is defined by:

$$(\vec{x}, v, x, y) \cdot (g, w, (t_1, t_2)) = (\vec{x}g, \theta(g)^{-1} v w^{-as\delta(e)}, w t_1 x, t_2 y) \ ,$$

for all $(g, w, (t_1, t_2)) \in G \times \mathbb{C}^*_w \times T$ and $(\vec{x}, v, (x, y)) \in AY^{ss}(\theta)^g \times \mathbb{C}^* \times U$. Then $\mathcal{C}_e$ is a family of orbifold curves parameterized by $\mathcal{M}_e$ with all fibers isomorphic to $\mathbb{P}_{ar,as}$.

There are two standard characters of T

$$\chi_1 : (t_1, t_2) \in T \mapsto t_1 \in \mathbb{C}^* \quad \chi_2 : (t_1, t_2) \in T \mapsto t_2 \in \mathbb{C}^* \ ,$$

and we can lift them to be characters of $G \times \mathbb{C}^*_w \times T$ by composing the projection map $\mathrm{pr}_T : G \times \mathbb{C}^*_w \times T \to T$. By an abuse of notation, we continue to use χ_1, χ_2 to denote the new characters. These two new characters defines two line bundles

$$M_1 := (AY^{ss}(\theta)^g \times \mathbb{C}^* \times U) \times_{G \times \mathbb{C}^*_w \times T} \mathbb{C}_{\chi_1}$$

and

$$M_2 := (AY^{ss}(\theta)^g \times \mathbb{C}^* \times U) \times_{G \times \mathbb{C}^*_w \times T} \mathbb{C}_{\chi_2}$$

on $\mathcal{C}_e$ by the Borel construction, respectively. We have the relation $M_1^{\otimes as} = M_2^{\otimes ar}$ over $\mathcal{C}_e$. The universal map f from $\mathcal{C}_e$ to $\mathbb{P}Y_{r,s}$ can be described as follows: Let

$$\tilde{f} : AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \to \mathbb{C}^* \times AY^{ss}(\theta) \times U$$

be the morphism defined by:

$$\begin{aligned} (5.2) \qquad & (\vec{x}, v, x, y) \in AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \mapsto \\ & \qquad (v, (x_1, \cdots, x_n), x^{a\delta(e)}, y^{a\delta(e)}) \in \mathbb{C}^* \times AY^{ss}(\theta) \times U \ . \end{aligned}$$

Then $\tilde{f}$ is equivariant with respect to the group homomorphism from $G \times \mathbb{C}^*_w \times T$ to $G \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t$ defined by:

$$\begin{aligned} (5.3) \qquad & (g, w, (t_1, t_2)) \in G \times \mathbb{C}^*_w \times T \mapsto \\ & \qquad \big(g \cdot \big((t_1^{-s} t_2^r)^{p_1}, \cdots, (t_1^{-s} t_2^r)^{p_k}\big), (w t_1)^{a\delta(e)}, t_2^{a\delta(e)}\big) \in G \times \mathbb{C}^*_\alpha \times \mathbb{C}^*_t \ , \end{aligned}$$

where the tuple $(p_1, \cdots, p_k) \in \mathbb{N}^k$ satisfies that $g = (\mu_a^{p_1}, \cdots, \mu_a^{p_k}) \in G$. Note that $\tilde{f}$ is well-defined for $\chi_1^{-s}\chi_2^r$ is a torsion character of T of order a. The above construction gives the universal morphism f from $\mathcal{C}_e$ to $\mathbb{P}Y_{r,s}$ by descent.

We will define a (quasi left) $\mathbb{C}^*-$action on $\mathcal{C}_e$ such that the map f constructed above is $\mathbb{C}^*-$equivariant. Define a $\mathbb{C}^*$-action on $\mathcal{C}_e$ induced by the $\mathbb{C}^*-$action on $AY^{ss}(\theta)^g \times \mathbb{C}^* \times U$:

$$m : \mathbb{C}^* \times AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \to AY^{ss}(\theta)^g \times \mathbb{C}^* \times U \ ,$$

$$t \cdot (\vec{x}, v, (x, y)) = (\vec{x}, v, (x, t^{\frac{-1}{ar\delta(e)}} y)) \ .$$

note that the morphism π is also $\mathbb{C}^*$-equivariant, where $\mathcal{M}_e$ is equipped with trivial $\mathbb{C}^*$-action. By the universal property of the projectivized bundle $\mathcal{C}_e$ over $\mathcal{M}_e$, one has a tautological section

$$(5.4) \qquad (x, y) \in H^0\big((M_1 \otimes \pi^*\mathcal{R}) \oplus (M_2 \otimes \mathbb{C}_{\frac{-\lambda}{ar\delta(e)}})\big) \ ,$$

which is also a $\mathbb{C}^*$−invariant section.

Now we can check that f is a $\mathbb{C}^*$−equivariant morphism from $\mathcal{C}_e$ to $\mathbb{P}Y_{r,s}$ with respect to the $\mathbb{C}^*$−actions for $\mathcal{C}_e$ and $\mathbb{P}Y_{r,s}$. Similar to 4.2, f is equivalent to the following data:

(1) $k+2$ $\mathbb{C}^*$-equivariant line bundles on $\mathcal{C}_e$:

$$\mathcal{L}_j := \pi^* L_{\pi_j} \otimes (M_1^{-\otimes s} \otimes M_2^{\otimes r})^{p_j}, 1 \leqslant j \leqslant k$$

and

$$\mathcal{N}_1 := (M_1 \otimes \pi^* \mathcal{R})^{\otimes a\delta(e)} \quad \mathcal{N}_2 := M_2^{a\delta(e)} \otimes \mathbb{C}_{\frac{-\lambda}{r}} .$$

Where L_{π_j} are the standard $\mathbb{C}^*$-equivariant line bundles on $\mathcal{M}_e$ by the Borel construction, M_1, M_2 are the standard $\mathbb{C}^*$-equivariant line bundles on C_e by the Borel construction.

(2) a universal section

$$\begin{aligned}(u, \vec{x}, (\zeta_1, \zeta_2)) :=& (v, x_1, \cdots, x_n, (x^{a\delta(e)}, y^{a\delta(e)})) \\ &\in \Gamma\Big(\big((\mathcal{N}_1^\vee)^{\otimes s} \otimes \mathcal{L}_{-\theta} \otimes \mathcal{N}_2^{\otimes r} \otimes \mathbb{C}_\lambda\big) \oplus \bigoplus_{1 \leqslant i \leqslant n} \mathcal{L}_{\rho_i} \oplus \mathcal{N}_1 \oplus \mathcal{N}_2\Big)^{\mathbb{C}^*} .\end{aligned} \tag{5.5}$$

Here one only need to check $v \in \Gamma((\mathcal{N}_1^\vee)^{\otimes s} \otimes \mathcal{L}_{-\theta} \otimes \mathcal{N}_2^{\otimes r} \otimes \mathbb{C}_\lambda)$, which is easy to be verified.

Now we compute the localization contribution from $\mathcal{M}_e$. Based on the perfect obstruction theory for stable maps in $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$, the restriction of the perfect obstruction theory to $\mathcal{M}_e$ decomposes into three parts: (1) the deformation theory of source curve $\mathcal{C}_e$; (2) the deformation theory of the lines bundles $(\mathcal{L}_i)_{1 \leqslant j \leqslant k}$ and $\mathcal{N}$; (3) the deformation theory for the section

$$(u, \vec{x}, (\zeta_1, \zeta_2)) \in \Gamma\big(\big(\mathcal{N}_1^{-\otimes s} \otimes \mathcal{L}_{-\theta} \otimes \mathcal{N}_2^{\otimes r} \otimes \mathbb{C}_\lambda\big) \oplus \bigoplus_{1 \leqslant i \leqslant n} \mathcal{L}_{\rho_i} \oplus \mathcal{N}_1 \oplus \mathcal{N}_2\big) .$$

The $\mathbb{C}^*$−fixed part of three parts above will contribute to the virtual cycle of $\mathcal{M}_e$, we will show that $[\mathcal{M}_e]^{\mathrm{vir}} = [\mathcal{M}_e]$. The virtual normal bundle comes from the $\mathbb{C}^*$−moving part of the above three parts.

First every fiber curve C_e in $\mathcal{C}_e$ over a geometrical point in $\mathcal{M}_e$ is isomorphic to $\mathbb{P}_{ar,as}$, which is rational. There are no infinitesimal deformations/obstructions for C_e, line bundles $L_j := \mathcal{L}_j|_{C_e}$, $N_1 := \mathcal{N}_1|_{C_e}$ and $N_2 := \mathcal{N}_2|_{C_e}$. Hence their contribution to the perfect obstruction theory comes from infinitesimal automorphisms. The infinitesimal automorphisms of C_e come from the space of vector fields on C_e that vanish on special points. Thus the $\mathbb{C}^*$−fixed part of infinitesimal automorphisms of C_e comes from the 1−dimensional subspace of vector fields on C_e which vanish on the two ramification points. The movable part of infinitisimial automorphisms of C_e is nonzero only if one of ramification points on C_e is not a special point. by Remark 5.1, the ramifications on C_e are both nontrivial stacky points when r and s are sufficiently large, hence they must be special points. So there is no movable part for infinitesimal automorphisms of C_e.

Now let's turn to the localizations from sections. First the infinitesimal deformations of sections $(u, \vec{x})$ are fixed, which, together with fixed part of infinitesimal automorphisms of C_e and line bundles L_j, N_1, N_2, as well as fixed parts of infinitesimal deformations of sections $(z_1, z_2) := (\zeta_1, \zeta_2)|_{C_e}$, contribute to the virtual cycle $[\mathcal{M}_e]^{\mathrm{vir}}$, which is equal to the fundamental class of $\mathcal{M}_e$. The localization contribution from the infinitesimal deformations of sections (z_1, z_2) to the virtual normal bundle is:

$$(R^\bullet \pi_*(\mathcal{N}_1 \oplus \mathcal{N}_2))^{\mathrm{mov}} .$$

We first come to the deformations of z_2, we continue to use the tautological section (x, y) as in (5.4). For each fiber C_e, sections of N_2 is spanned by monomials $(x^{asm} y^n)|_{C_e}$ with

$arm + n = a\delta(e)$ and $m, n \in \mathbb{Z}_{\geqslant 0}$. Note that $x^{asm}y^n$ may not be a global section of $\mathcal{N}_2$ but always a global section of $\mathcal{R}^{\otimes asm} \otimes \mathcal{N}_2 \otimes \mathbb{C}_{\frac{m}{\delta(e)}\lambda}$. Then $R^\bullet\pi_*\mathcal{N}_2$ will decompose as a direct sum of line bundles, each corresponds to the monomial $x^{asm}y^n$, whose first chern class is

$$c_1(\mathcal{R}^{\otimes -asm} \bigotimes \mathbb{C}_{\frac{-m}{\delta(e)}\lambda}) = \frac{m}{\delta(e)}(D_\theta - \lambda)\ .$$

So the total contribution is equal to

$$\prod_{m=0}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left(\frac{m}{\delta(e)}(D_\theta - \lambda) \right) .$$

The factor for $m = 0$ appearing in the above product is the $\mathbb{C}^*-$fixed part of $R^\bullet\pi_*\mathcal{N}_2$, it will contribute to the virtual cycle of $\mathcal{M}_e$. The rest contributes to the virtual normal bundle as

$$\prod_{m=1}^{\lfloor \frac{\delta(e)}{r} \rfloor} \left(\frac{m}{\delta(e)}(D_\theta - \lambda) \right) .$$

Note that when r is sufficiently large, the above product becomes 1.

For the deformations of z_1, arguing in the same way as z_2, the Euler class of $R^\bullet\pi_*\mathcal{N}_1$ is equal to

$$\prod_{n=0}^{\lfloor \frac{\delta(e)}{s} \rfloor} \left(\frac{n}{\delta(e)}(-D_\theta + \lambda) \right) .$$

The factor for $m = 0$ appearing in the above product is the $\mathbb{C}^*-$fixed part of $R^\bullet\pi_*\mathcal{N}_1$, it will contribute to the virtual cycle of $\mathcal{M}_e$. The Euler class of virtual normal bundle of $\mathcal{M}_e$ comes from the movable part of deformations of section z_1 is:

$$\prod_{n=1}^{\lfloor \frac{\delta(e)}{s} \rfloor} \left(\frac{n}{\delta(e)}(-D_\theta + \lambda) \right) .$$

note that when s is sufficiently large, the above product becomes 1.

In summary, when r, s are sufficiently large primes, we have $[\mathcal{M}_e]^{vir} = [\mathcal{M}_e]$ and $e^{\mathbb{C}^*}(N^{vir}) = 1$.

5.3.3. *Node contributions.* The deformations in $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ smoothing a node contribute to the Euler class of the virtual normal bundle as the first Chern class of the tensor product of the two cotangent line bundles at the branches of the node. For nodes at which a component C_e meets a component C_v over the vertex 0, this contribution is

$$\frac{\lambda - D_\theta}{as\delta(e)} - \frac{\bar{\psi}_v}{as}.$$

For nodes at which a component C_e meets a component C_v at the vertex over ∞, this contribution is

$$\frac{-\lambda + D_\theta}{ar\delta(e)} - \frac{\bar{\psi}_v}{ar}.$$

The type of node at which two edge component C_e meets with a vertex v over 0 or ∞ will not occur using a similar argument in [JPPZ17, Lemma 6].

As for the node contributions from the normalization exact sequence, each node q (specified by a vertex v) contributes the Euler class of

$$\left(R^0\pi_*\mathcal{N}_1|_q\right)^{\mathrm{mov}} \oplus \left(R^0\pi_*\mathcal{N}_2|_q\right)^{\mathrm{mov}} \tag{5.6}$$

to the virtual normal bundle. In the case where $j(v) = 0$, $z_2|_q \equiv 1$ gives a trivialization of the fiber $\mathcal{N}_2|_q$, note that $(\mathcal{N}_1^\vee)^{\otimes s} \otimes \mathcal{L}_{-\theta} \otimes \mathcal{N}_2^{\otimes r} \otimes \mathbb{C}_\lambda \cong \mathbb{C}$ we have $\mathcal{N}_2|_q \cong \mathbb{C}$ and $\mathcal{N}_1|_q \cong L_{-\theta}^{\frac{1}{s}} \otimes \mathbb{C}_{\frac{\lambda}{s}}$, this implies that $(R^0\pi_*\mathcal{N}_2|_q)^{\mathrm{mov}} = 0$ and $R^0\pi_*\mathcal{N}_1|_q = 0$. The later vanishes because of the nontrivial stacky structure of the line bundle $\mathcal{N}_1$ at q when s is sufficiently large. Hence there is no localization contribution from the normalization at the node q over 0. Similarly, for each node q incident to a vertex v with $j(v) = \infty$, there is no localization contribution from the normalization at the node over ∞.

5.4. **Total localization contributions.** For each decorated graph Γ, denote F_Γ to be the fiber product

$$\prod_{v:j(v)=0} \mathcal{M}_v \times_{\bar{I}_\mu \sqrt[s]{L_{-\theta}/Y}} \prod_{e\in E} \mathcal{M}_e \times_{\bar{I}_\mu \sqrt[r]{L_\theta/Y}} \prod_{v:j(v)=\infty} \mathcal{M}_v$$

of the following diagram:

$$\begin{tikzcd} F_\Gamma \arrow[r] \arrow[d] & \prod_{v:j(v)=0} \mathcal{M}_v \times \prod_{e\in E} \mathcal{M}_e \times \prod_{v:j(v)=\infty} \mathcal{M}_v \arrow[d, "ev_{nodes}"] \\ \prod_E \bar{I}_\mu \sqrt[s]{L_{-\theta}/Y} \times \bar{I}_\mu \sqrt[r]{L_\theta/Y} \arrow[r, "(\Delta^{\frac{1}{s}} \times \Delta^{\frac{1}{r}})^E"] & \prod_E \left((\bar{I}_\mu \sqrt[s]{L_{-\theta}/Y})^2 \times (\bar{I}_\mu \sqrt[r]{L_\theta/Y})^2 \right) \end{tikzcd},$$

where $\Delta^{\frac{1}{s}} = (id, \iota)$(resp. $\Delta^{\frac{1}{r}} = (id, \iota)$) is the diagonal map of $\bar{I}_\mu \sqrt[s]{L_{-\theta}/Y}$ (resp. $\bar{I}_\mu \sqrt[r]{L_\theta/Y}$). Here when v is a stable vertex, the vertex moduli $\mathcal{M}_v$ is described in 5.3.1; when v is an unstable vertex over 0, we treat $\mathcal{M}_v := \iota(\bar{I}_{m(h)}\mathcal{D}_{j(v)})$ with $[\mathcal{M}_e]^{\mathrm{vir}} = [\mathcal{M}_e]$ and zero virtual normal bundle., where $m(h)$ is the multiplicity of the half-edge incident to v. The right-hand vertical map is the product of the evaluation maps of the two branches at the gluing nodes for each edge.

We define that $[F_\Gamma]^{\mathrm{vir}}$ to be:

$$\prod_{v:j(v)=0} [\mathcal{M}_v]^{\mathrm{vir}} \times_{\bar{I}_\mu \sqrt[s]{L_{-\theta}/Y}} \prod_{e\in E} [\mathcal{M}_e]^{\mathrm{vir}} \times_{\bar{I}_\mu \sqrt[r]{L_\theta/Y}} \prod_{v:j(v)=\infty} [\mathcal{M}_v]^{\mathrm{vir}} .$$

Then the contribution of decorated graph Γ to the virtual localization is is:

$$Cont_\Gamma = \frac{\prod_{e\in E} sa_e}{|\mathrm{Aut}(\Gamma)|} (\iota_\Gamma)_* \left(\frac{[F_\Gamma]^{\mathrm{vir}}}{e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})} \right) . \tag{5.7}$$

Here $\iota_F : F_\Gamma \to \mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$ is a finite etale map of degree $\frac{|\mathrm{Aut}(\Gamma)|}{\prod_{e\in E} sa_e}$ into the corresponding $\mathbb{C}^*$-fixed loci in $\mathcal{K}_{0,\vec{m}}(\mathbb{P}Y_{r,s}, (\beta, \frac{\delta}{r}))$. The virtual normal bundle $e^{\mathbb{C}^*}(N_\Gamma^{\mathrm{vir}})$ is the product of virtual normal bundles from vertex contributions, edge contributions and node contributions.

6. Recursion relations from auxiliary cycles

Let's first fix some notations in this section. For any $\beta \in \mathrm{Eff}(W, G, \theta)$, for simplicity, we will denote

$$\mathcal{K}_{0,\vec{m}}(\bullet, \beta) := \coprod_{\substack{d\in \mathrm{Eff}(\bullet) \\ (i_\bullet)_*(d)=\beta}} \mathcal{K}_{0,\vec{m}}(\bullet, d),$$

where $\bullet$ can be $Y, \sqrt[r]{L_\theta/Y}$ and $\sqrt[s]{L_{-\theta}/Y}$, and $i_\bullet$ is the natural structure map from $\bullet$ to $\mathfrak{X}$ which factors through the inclusion $i_{\mathfrak{Y}} : \mathfrak{Y} \to \mathfrak{X}$.

For any $\beta_\star, \beta_1, \cdots, \beta_m$ in $\mathrm{Eff}(W, G, \theta)$ and $p_1, \cdots, p_m$ in $\mathbb{Z}_{\geqslant 0}$, write $\beta = \beta_\star + \sum_{i=1}^m \beta_i$ and $p = \sum_i p_i$. We will denote $\vec{m}_s \cup \star$ to be

$$\left((g_{\beta_1}^{-1}, \mu_s^{\beta_1(L_\theta)+p_1}), \cdots, (g_{\beta_m}^{-1}, \mu_s^{\beta_m(L_\theta)+p_m}), (g_\beta, \mu_s^{-\beta(L_\theta)-p})\right) \in (G\times \boldsymbol{\mu}_s)^{m+1} ,$$

and define $\overrightarrow{m}_r \cup \star$ to be

$$\big((g_{\beta_1}^{-1}, \mu_r^{-\beta_1(L_\theta)-p_1}), \cdots, (g_{\beta_m}^{-1}, \mu_r^{-\beta_m(L_\theta)-p_m}), (g_\beta, \mu_r^{\beta(L_\theta)+p})\big) \in (G \times \boldsymbol{\mu}_r)^{m+1} .$$

Then we have two natural structural morphisms

$$\epsilon : \mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y}, \beta_\star) \to \mathcal{K}_{0,\vec{m}\cup\star}(Y, \beta_\star)$$

and

$$\epsilon : \mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L_{-\theta}/Y}, \beta_\star) \to \mathcal{K}_{0,\vec{m}\cup\star}(Y, \beta_\star)$$

induced from the morphisms from $\sqrt[r]{L_\theta/Y}$ and $\sqrt[s]{L_{-\theta}/Y}$ to Y by forgetting roots. Here the tuple $\overrightarrow{m} \cup \star$ for $\mathcal{K}_{0,\overrightarrow{m}\cup\star}(Y, \beta_\star)$ is

$$(g_{\beta_1}^{-1}, \cdots, g_{\beta_m}^{-1}, g_\beta) \in G^{m+1} .$$

We note that the right hand side of 1.4 can be written as

$$\sum_{m=0}^{\infty} \sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta \\ p_1+\cdots+p_m=p}} \frac{1}{m!} \phi^\alpha \langle \mu_{\beta_1,p_1}(-\bar{\psi}_1), \cdots, \mu_{\beta_m,p_m}(-\bar{\psi}_m), \phi_\alpha \bar{\psi}_\star^c \rangle_{0,\vec{m}\cup\star,\beta_\star}$$

as $\mu_{\beta_i,p_i}(z) \in H^*(\bar{I}_{g_{\beta_i}^{-1}} Y.\mathbb{Q})$ for $1 \leqslant i \leqslant m$.

We will also need the two following definitions.

Definition 6.1. *Let m, p be two nonnegative integers, β be a degree in $\mathrm{Eff}(W, G.\theta)$, we denote $\Lambda_{\beta,p,m}$ to the set of tuples*

$$\big(\beta_\star, ((\beta_1, p_1), \cdots, (\beta_m, p_m))\big) \in \mathrm{Eff}(W, G, \theta) \times (\mathrm{Eff} \times \mathbb{Z}_{\geqslant 0})^m ,$$

where we require that $\beta_\star + \sum_{i=1}^m \beta_i = \beta$, $\sum_i p_i = p$ and $\beta_i(L_\theta) + p > 0$ for $1 \leqslant i \leqslant m$. We call an element of $\Lambda_{\beta,p,m}$ stable if $\beta_\star \neq 0$ or $m \geqslant 2$ when $\beta_\star = 0$.

We note that $\Lambda_{\beta,p,m}$ is a finite set as $\mathcal{K}_{0,m}(X, \beta)$ is finite type over $\mathbb{C}$, hence Noetherian.

Definition 6.2. *For any degree β and nonnegative integers c and p, we define the function*

$$G_{\beta,p,c} : \bigoplus_{\substack{\beta'\in\mathrm{Eff}(W,G,\theta), p'\in\mathbb{Z}_{\geqslant 0} \\ \beta'(L_\theta)+p'<\beta(L_\theta)+p}} \oplus H^*(\bar{I}_\mu Y)[z, z^{-1}]] \to H^*(\bar{I}_{g_\beta^{-1}} Y, \mathbb{Q}) ,$$

which sends

$$\big(f_{(\beta',p')}(z) : \beta'(L_\theta) + p' < \beta(L_\theta) + p\big)$$

to

$$\begin{aligned} \Big[\sum_{m=0}^{\infty} \sum_{\substack{\Gamma\in\Lambda_{\beta,p,m} \\ \Gamma \text{ is stable}}} \frac{1}{m!} (\widetilde{ev}_\star)_* \Big(\sum_{d=0}^{\infty} \epsilon_* \big(c_d(-R^\bullet \pi_* \mathcal{L}_\theta^{\frac{1}{r}})(\frac{\lambda}{r})^{-1+m-d}(-1)^d \\ \cap [\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y}, \beta_\star)]^{\mathrm{vir}}\big) \cap \prod_{i=1}^m \frac{ev_i^*(\frac{1}{\delta_i}(f_{\beta_i,p_i}(z))|_{z=\frac{\lambda-D_\theta}{\delta_i}})}{\frac{\lambda-ev_i^*D_\theta}{r\delta_i} + \frac{\bar{\psi}_i}{r}} \cap \bar{\psi}_\star^c \Big) \Big]_{\lambda^{-1}} . \end{aligned} \tag{6.1}$$

Here $\delta_i = \beta_i(L_\theta) + p_i$ for $1 \leqslant i \leqslant m$, r is a sufficient large prime. We will write $\big(f_{(\beta',p')}(z) : \beta'(L_\theta) + p' < \beta(L_\theta) + p\big)$ as $f_{<(\beta,p)}(z)$ for short.

6.1. **Auxiliary cycle I.** We will use the notations from §4 in this subsection. Fix a nonzero pair $(\beta, p) \in \mathrm{Eff}(\mathrm{W}, \mathrm{G}, \theta) \times \mathbb{Z}_{\geqslant 0}$ and a positive rational number ϵ and the tuple $\boldsymbol{\varepsilon} = (\epsilon, \cdots, \epsilon) \in (\mathbb{Q}_{>0})^p$ such that $\epsilon\beta(L_\theta) + p\epsilon \leqslant 1$. Set $\delta = \beta(L_\theta) + p$. For simplicity, we will denote

$$Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r})) := \bigsqcup_{\substack{d \in \mathrm{Eff}(AY,G,\theta) \\ (i_{\mathfrak{Y}})_*(d) = \beta}} Q^{\widetilde{\theta}}_{0,1}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (d, 1^p, \frac{\delta}{r})) \cap ev_1^{-1}(\bar{I}_{(g_\beta, \mu^{\delta}_{r})}\mathbb{P}Y^{\frac{1}{r}}) \ .$$

Recall that $g_\beta \in G$ is defined in §3. We will always assume that r is a sufficiently large prime in this subsection.

For any nonnegative integer c, we will first consider the following auxiliary cycle:

$$\frac{1}{p!}(\widetilde{EV_\star})_* \left(\bar{\psi}^c_\star \cap \prod_{j=1}^{p} \hat{ev}^*_j(\hat{t}) \cap [Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))]^{\mathrm{vir}} \right). \tag{6.2}$$

Here an explanation of the notations is in order:

(1) The morphism $EV_\star$ is a composition of the following maps:

$$Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \tfrac{\delta}{r})) \xrightarrow{ev_\star} \bar{I}_\mu \mathbb{P}Y^{\frac{1}{r}} \xrightarrow{\mathrm{pr}_r} \bar{I}_\mu Y \ ,$$

where $\mathrm{pr}_r : \bar{I}_\mu \mathbb{P}Y^{\frac{1}{r}} \to \bar{I}_\mu Y$ is the morphism induced from the map from $\mathbb{P}Y^{\frac{1}{r}}$ to Y forgetting z_1, z_2. $(\widetilde{EV_\star})_*$ is defined by

$$\iota_*(r_\star(EV_\star)_*)$$

as in (2.2). Note that here $r_\star$ is the order of the band from the gerbe structure of $\bar{I}_\mu Y$ but not $\bar{I}_\mu \mathbb{P}Y^{\frac{1}{r}}$.

(2) Recall that $\hat{ev}_j$ is defined in (4.5). The cohomology class $\hat{t} \in H^*(\mathfrak{Y}, \mathbb{Q})[t_1, \cdots, t_l]$ is of the form $\sum_{i=1}^{l} t_i u_i(c_1(L_{\pi_j}))$, where t_i are formal variables and u_i are (arbitrary) polynomials in the first chern class of the line bundles L_{π_j} associated to the standard characters π_j of $G = (\mathbb{C}^*)^k$ defined in 2.6.

Apply virtual localization to $Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$, we first prove the following vanishing result, where the idea is borrowed from [JPT].

Lemma 6.3. *Assume r is a sufficiently large prime. If localization graph Γ has more than one vertex labeled by ∞, then the corresponding fixed loci moduli F_Γ is empty, therefore it will contribute zero to* (6.2).

Proof. First we show that for any quasimap $f : C \to \mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ in $Q^{\widetilde{\theta}}_{0,\star}(\mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}, (\beta, 1^p, \frac{\delta}{r}))$, we have $H^1(C, N^\vee) = 0$ (recall that the line bundle N is introduced in the definition of $\widetilde{\theta}-$stable quasimap in §4.1). Indeed, using orbifold Riemann-Roch, we have

$$\chi(N^\vee) = 1 + deg(N^\vee) - age(N^\vee|_{q_\star}) = 0 \ ,$$

as $deg(N^\vee) = -\frac{\beta(L_\theta)+p}{r}$, and $age(N^\vee|_{q_\star}) = 1 - \frac{\beta(L_\theta)+p}{r}$, then showing $H^1(C, N^\vee) = 0$ is equivalent to show $H^0(C, N^\vee) = 0$. By Lemma 5.2, it remains to show that the degree of the restriction of the line bundle $N^\vee$ to every irreducible component E of C is non-positive. Observe that $N^\vee$ is equal to the line bundle $f^*\mathcal{O}(-\mathcal{D}_\infty)$, so the degree is equal to the intersection number of $[E]$ and the divisor $-[\mathcal{D}_\infty]$. If the image of an irreducible component of C via f isn't contained in $\mathcal{D}_\infty$, the restricted degree is obviously non-positive. If the image of an irreducible component of C under f is contained in $\mathcal{D}_\infty$, observe that $\mathcal{O}(-\mathcal{D}_\infty)$ is isomorphic to $(L_\theta^{\frac{1}{r}})^\vee$ over

$$\mathcal{D}_\infty \cong \sqrt[r]{L_\theta/Y}$$

then the $\mathcal{O}(-r\mathcal{D}_\infty)$ is a line bundle pull-back of an anti-ample line bundle over Y, thus the degree is also non-positive. This finishes the proof that $H^1(C, N^\vee) = 0$.

Now assume by contradiction that the moduli of fixed-loci F_Γ is nonempty, by the connectedness of the graph Γ, there is at least one vertex of the graph Γ labeled by 0 with at least two edges attached. Suppose $f : C \to \mathbb{P}\mathfrak{Y}^{\frac{1}{r},p}$ belongs to the $\mathbb{C}^*$−fixed loci F_Γ. Assume that $C_0 \cap C_1 \cap C_2$ is part of curve C, where C_0 is mapped by f to $\mathcal{D}_0$ (given by $z_1 = 0$) and C_1, C_2 are edges meeting with C_0 at b_1 and b_2. Then in the normalization sequence for $R^\bullet\pi_*N^\vee$, it contains the part

$$\begin{aligned} &H^0\left(C_0, N^\vee\right) \oplus H^0\left(C_1, N^\vee\right) \oplus H^0\left(C_2, N^\vee\right) \\ \to &H^0\left(b_1, N^\vee\right) \oplus H^0\left(b_2, N^\vee\right) \\ \to &H^1\left(C, N^\vee\right). \end{aligned}$$

Hence there is one of the weight-0 pieces in $H^0\left(b_1, N^\vee\right) \oplus H^0\left(b_2, N^\vee\right)$ that is canceled with a weight-0 piece of $H^0\left(C_0, N^\vee\right)$, and the other is mapped injectively into $H^1\left(C, N^\vee\right)$, but this contradicts that $H^1(C, N^\vee) = 0$. So F_Γ is empty. □

Recall that we can write $\mathbb{I}(q, t, z) = \sum_{\beta,p} q^\beta \mathbb{I}_{\beta,p}$ as in §1.1.2, where $\mathbb{I}_{\beta,p} := \frac{\boldsymbol{t}^p}{p! z^p}\mathbb{I}_\beta(z)$ is a Laurent polynomial in z, z^{-1} with coefficients in the homogeneous degree p part of $H^*(\bar{I}_\mu Y, \mathbb{Q})[t_0, \cdots, t_l]$. We will prove the following recursion relation by applying localization to (6.9).

Theorem 6.4. *For any nonnegative integer c, $[z\mathbb{I}_{\beta,p}]_{z^{-c-1}}$ satisfies the following relation:*

$$[z\mathbb{I}_{\beta,p}]_{z^{-c-1}} = G_{\beta,p,c}\big(z\mathbb{I}_{<(\beta,p)}(z)\big)\ . \tag{6.3}$$

where $G_{\beta,p,c}$ is defined in 6.2.

Proof. By Lemma 6.3, only decorated graph Γ which has only one vertex labeled by ∞, may have nonzero localization contribution to the (6.2). We will denote the vertex labeled by ∞ to be $v_\star$. Note that the marking $q_\star$ can only be incident to the vertex $v_\star$ due to the choice of the multiplicity at $q_\star$. Furthermore, for such graph Γ, we claim there is no stable vertex labeled by 0. Indeed, for any vertex v over 0, its decorated degree $(\beta(v), 1^{J_v})$ satisfies that $\beta(v)(L_\theta) + |J_v| \leqslant \beta(L_\theta) + p \leqslant \frac{1}{\epsilon}$, and it has valence 1 as no legs can attach to it and at most one edge is incident to it by Lemma 6.3, then the vertex v must be unstable. So the decorated graph Γ has only one vertex over ∞ with possible several edges (can be empty) attached, and each vertex labeled by 0 corresponds to an edge in the graph Γ and appears as an unmarked point (actually a base point as we will see). In the following, we analyze the localization contribution to (6.2) from the graph Γ described just before. We have two cases which depends on whether the vertex $v_\star$ on the graph Γ is stable or unstable.

(1) If the only vertex $v_\star$ over ∞ is unstable, then it's a vertex with valence 2, i.e, it's incident to a leg and an edge. In this case the degree $(\beta, 1^p, \frac{\delta}{r})$ is concentrated on the ramification point over 0 on the edge as a base point. Then it contributes

$$\frac{1}{\delta}(z\mathbb{I}_{\beta,p}(z))|_{z=\frac{\lambda-D_\theta}{\delta}} \cdot (\frac{\lambda - D_\theta}{\delta})^c$$

to (6.2). Here we use the fact that the restriction of $\bar{\psi}_\star$ to $\mathcal{M}_e$ is equal to $\frac{\lambda-D_\theta}{\delta}$.

(2) If the vertex $v_\star$ is stable, then $v_\star$ is incident to only one leg and possible several edges (can be none). We assume that the vertex $v_\star$ has degree $(\beta_*, \frac{\delta_*}{r})$ with $\delta_* = \beta_*(L_\theta)$. If there is no edges in the graph Γ, which happens if and only if $\beta_\star = \beta$ and $p = 0$, the corresponding graph has contribution

$$(\widetilde{ev}_\star)_*\bigg(\sum_{d=0}^{\infty} \epsilon_*(c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1-d} \cap [\mathcal{K}_{0,\star}(\sqrt[r]{L_\theta/Y}, \beta_\star)]^{\mathrm{vir}}) \cap \bar{\psi}_\star^c\bigg). \tag{6.4}$$

to the (6.2). Otherwise we label all the edges attached to the vertex $v_\star$ from 1 to m such that the edge e_i corresponding to the index i has degree $(\beta_i, 1^{J_{e_i}}, \frac{\delta_i}{r})$. Note that the index is not unique, we will divided by $m!$ to offset the labeling. is Since we assume that the total degree is $(\beta, 1^p, \frac{\delta}{r}) = (\beta, 1^p, \frac{\beta(L_\theta)+p}{r})$, and the degree on every edge satisfies the relation $\delta_i \geqslant \beta_i(L_\theta) + p_i$ by Remark 4.5, where $p_i = |J_{e_i}|$, then we must have $\delta_i = \beta_i(L_\theta) + p_i$ for every edge e_i. It follows that all the edge has a base point and (β_i, p_i) is nonzero.

Equipped with these notations, by Remark 4.4, the vertex moduli $\mathcal{M}_{v_\star}$ over ∞ is $\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y}, \beta_\star)$. Using the localization analysis in §4.3, the localization contribution of the graph Γ to (6.2) is equal to

$$\begin{aligned}\frac{1}{Aut(\Gamma)}(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*\big(c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\cap[\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y}, \beta_\star)]^{\mathrm{vir}}\big)\\ \cap\prod_{i=1}^{m}\frac{ev_i^*\big(\frac{1}{\delta_i}(z\frac{t^{p_i}}{z^{p_i}}\mathbb{I}_{\beta_i}(q,z))|_{z=\frac{\lambda-D_\theta}{\delta_i}}\big)}{-\frac{\lambda-ev_i^*D_\theta}{r\delta_i}-\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\Bigg),\end{aligned}\tag{6.5}$$

where $t = \sum t_i u_i(c_1(L_{\tau_{ij}}) + \beta(L_{\tau_{ij}})z)$ and $\epsilon : \mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y}, \beta_\star) \to \mathcal{K}_{0,\vec{m}\cup\star}(Y, \beta_\star)$ is the natural structure map.

Now varying over all $\beta_*, \beta_1, \cdots, \beta_m$ and $p_1, \cdots, p_m$ and m, and labeling of edges. The sum of (6.5) coming from all possible decorated graphs which has stable $\infty-$vertex $v_\star$ yields:

$$\begin{aligned}&\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p\\ (\beta_i,p_i)\neq 0 \text{ for } 1\leqslant i\leqslant m}}\frac{1}{m!}(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*\big(c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\\ &\qquad\cap[\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y}, \beta_\star)]^{\mathrm{vir}}\big)\cap\prod_{i=1}^{m}\frac{ev_i^*\big(\frac{1}{\delta_i}(z\mathbb{I}_{\beta_i,p_i}(z))|_{z=\frac{\lambda-D_\theta}{\delta_i}}\big)}{-\frac{\lambda-ev_i^*D_\theta}{r\delta_i}-\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\Bigg).\end{aligned}\tag{6.6}$$

In summary, the auxiliary cycle (6.2) is equal to:

$$\begin{aligned}&\frac{1}{\delta}(z\mathbb{I}_{\beta,p}(z))|_{z=\frac{\lambda-D_\theta}{\delta}}\cdot(\frac{\lambda-D_\theta}{\delta})^c\\ &+\sum_{m=0}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p\\ (\beta_i,p_i)\neq 0 \text{ for } 1\leqslant i\leqslant m}}\frac{1}{m!}(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*\big(c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\\ &\cap[\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y}, \beta_\star)]^{\mathrm{vir}}\big)\cap\prod_{i=1}^{m}\frac{ev_i^*\big(\frac{1}{\delta_i}(z\mathbb{I}_{\beta_i,p_i}(z))|_{z=\frac{\lambda-D_\theta}{\delta_i}}\big)}{-\frac{\lambda-ev_i^*D_\theta}{r\delta_i}-\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\Bigg)\ .\end{aligned}\tag{6.7}$$

Observe that (6.2) does not have negative λ powers, then the λ^{-1} coefficient in the equation (6.7) is equal to zero. Note that the λ^{-1} coefficient in (6.7) is equal to

$$[z\mathbb{I}_{\beta,p}(z)]_{z^{-c-1}} - G_{\beta,p,c}(z\mathbb{I}_{<(\beta,p)}(z))\ .\tag{6.8}$$

Now (6.8) immediately implies the formula (6.3).

□

6.2. **Auxiliary cycle II.** We will use the notations from §5 in this subsection. Let $\mu(z) = \sum_{\beta,p} q^\beta \mu_{\beta,p}(z)$ as in the introduction 1.1.2. For any nonzero pair (β,p), denote $\delta = \beta(L_\theta) + p$. Assume that r,s are sufficiently large primes, we will also compare (6.2) to the following auxiliary cycle:

(6.9)
$$\sum_{m=0}^{\infty} \sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta \\ p_1+\cdots+p_m=p}} \frac{1}{m!}(\widetilde{EV_\star})_* \Big(\prod_{i=1}^m ev_i^*\big(\mathrm{pr}_{r,s}^*(\mu_{\beta_i,p_i}(-\bar{\psi}_i))\big) \cap \bar{\psi}_\star^c \cap [\mathcal{K}_{0,\vec{m}\cup\star}(\mathbb{P}Y_{r,s},(\beta_\star,\frac{\delta}{r}))]^{\mathrm{vir}} \Big)$$

Here an explanation of the notations is in order:

(1) For any nonnegative integers $p_1,\cdots,p_m$, and degrees $\beta_\star,\beta_1,\cdots,\beta_m$ in $\mathrm{Eff}(W,G,\theta)$, we denote the tuple of multiplicities $\vec{m}\cup\star$ to be
$$\big((g_{\beta_1}^{-1},\mu_s^{\beta_1(L_\theta)+p_i},1),\cdots,(g_{\beta_m}^{-1},\mu_s^{\beta_m(L_\theta)+p_m},1),(g_\beta,1,\mu_r^\delta)\big)$$
to define $\mathcal{K}_{0,\vec{m}\cup\star}(\mathbb{P}Y_{r,s},(\beta_\star,\frac{\delta}{r}))$.

(2) The morphism $EV_\star$ is a composition of the following maps:
$$\mathcal{K}_{0,\vec{m}\cup\star}(\mathbb{P}Y_{r,s},(\beta_\star,\tfrac{\delta}{r})) \xrightarrow{ev_\star} \bar{I}_\mu\mathbb{P}Y_{r,s} \xrightarrow{\mathrm{pr}_{r,s}} \bar{I}_\mu Y \ ,$$
where $\mathrm{pr}_{r,s}: \bar{I}_\mu\mathbb{P}Y_{r,s} \to \bar{I}_\mu Y$ is the morphism induced from the natural structure map from $\mathbb{P}Y_{r,s}$ to Y forgetting u and z_1,z_2, and $(\widetilde{EV_\star})_*$ is defined by
$$\iota_*(r_\star(EV_\star)_*)$$
as in 2.2. Note that here $r_\star$ is the order of the band from the gerbe structure of $\bar{I}_\mu Y$ but not $\bar{I}_\mu\mathbb{P}Y_{r,s}$.

First we have a similar vanishing result as Lemma 6.3 by an analogous argument.

Lemma 6.5. *Assume r is sufficiently large. If the localization graph Γ has more than one vertex labeled by ∞, then the corresponding fixed loci moduli F_Γ is empty, therefore it will contribute zero to* (6.9).

For any pair $(\beta,p) \in \mathrm{Eff}(W,G,\theta)\times\mathbb{Z}_{\geqslant 0}$, we define $J_{\beta,p}(z)$ in (6.11) to be:

(6.10)
$$J_{\beta,p}(z) := \mu_{\beta,p}(z) + \sum_{m=0}^{\infty} \sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta \\ p_1+\cdots+p_m=p}} \frac{1}{m!} (\widetilde{ev_\star})_*\Big([\mathcal{K}_{0,\vec{m}\cup\star}(Y,\beta_\star)]^{\mathrm{vir}} \cap \bigcap_{j=1}^m ev_j^*(\mu_{\beta_j,p_j}(-\bar{\psi}_j)) \cap \frac{1}{z-\bar{\psi}_\star}\Big) \ .$$

We will prove the following recursion relation by applying localization to (6.9).

Theorem 6.6. *For any nonnegative integer c, we have the following relation:*

$$[J_{\beta,p}]_{z^{-c-1}} = G_{\beta,p,c}\big(J_{<(\beta,p)}(z)\big) \ . \tag{6.11}$$

where $G_{\beta,p,c}$ is defined in 6.2.

Proof. By Lemma 6.5, only decorated graph Γ that has only one vertex labeled by ∞ may have nonzero localization contribution to the (6.9). Let's denote the unique vertex over ∞ by $v_\star$ with decorated degree $\beta_\star$. Note that the leg $\star$ must be incident to the vertex $v_\star$ due to the choice of multiplicity at the leg $\star$. Thus the vertex $v_\star$ can't be a node linking two edges. Note that we can assume that all the other legs should be incident with the vertexes labeled by 0 due to the choice of multiplicity on the other legs and the fact $\mu_0 = 0$. Then there are only two types of graph Γ depending on whether $v_\star$ is stable or unstable.

(1) If the vertex $v_\star$ in Γ is unstable. In the case, v is of valence 2, i.e. it's incident to an edge and an leg corresponding to the marking $q_\star$. Then Γ has only one edge with decarated degree δ, and has only one vertex over 0, which is incident to the edge. The vertex over 0 can be stable or unstable. If the vertex over 0 is unstable, it must be a marked point with input $\mu_{\beta,p}$, then the graph Γ contributes

$$\frac{\mu_{\beta,p}(\frac{\lambda-D_\theta}{\delta})}{\delta}\cdot(\frac{\lambda-D_\theta}{\delta})^c$$

to (6.9). Otherwise, this type of graphs contributes

$$\sum_{m=0}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p}}\frac{1}{m!}(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^{\infty}\epsilon'_*\big(c_d(-R^\bullet\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}})(\frac{\lambda}{s})^{-d}$$
$$\cap\,[\mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L_{-\theta}/Y},\beta_\star)]^{\mathrm{vir}}\big)\cap\bigcap_{i=1}^{m}ev_i^*(\mu_{\beta_i,p_i}(-\bar{\psi}_i))\cap\frac{\frac{1}{\delta}(\frac{\lambda-ev_\star^*D_\theta}{\delta})^c}{\frac{\lambda-ev_\star^*D_\theta}{s\delta}-\frac{\bar{\psi}_\star}{s}}\Bigg)$$

to (6.9). By Lemma 6.7 proved below, the above formula is equal to

$$\sum_{m=0}^{\infty}\sum_{\substack{\beta_\star+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p}}\frac{1}{m!}\phi^\alpha\langle\mu_{\beta_1,p_1}(-\bar{\psi}_1),\cdots,\mu_{\beta_m\cdot p_m}(-\bar{\psi}_m),\frac{\frac{1}{\delta}(\frac{\lambda-D_\theta}{\delta})^c\phi_\alpha}{\frac{\lambda-D_\theta}{\delta}-\bar{\psi}_\star}\rangle_{0,\vec{m}\cup\star,\beta_\star}\,.$$

In summary, the localization contribution from the decorated graphs of which the vertex $v_\star$ is unstable contributes

$$\begin{aligned}&\mu_{\beta,p}(\frac{\lambda-D_\theta}{\delta})\cdot(\frac{\lambda-D_\theta}{\delta})^c\\ &+\sum_{m=0}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p}}\frac{1}{m!}\phi^\alpha\langle\mu_{\beta_1,p_1}(-\bar{\psi}_1),\cdots,\mu_{\beta_m,p_m}(-\bar{\psi}_m),\frac{\frac{1}{\delta}(\frac{\lambda-D_\theta}{\delta})^c\phi_\alpha}{\frac{\lambda-D_\theta}{\delta}-\bar{\psi}_\star}\rangle_{0,\vec{m}\cup\star,\beta_\star}\end{aligned}\tag{6.12}$$

to the (6.9).

(2) If the vertex $v_\star$ in Γ is stable, $v_\star$ is incident to only one leg (corresponding to the marking $q_\star$) and m edges (m can be 0). Let's assume that the vertex $v_\star$ is decorated by the degree $\beta_\star$. If there is no edges in the graph Γ, which happens if and only if $\beta_\star=\beta$ and $p=0$. Then this has contribution:

$$(\widetilde{ev}_\star)_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*\big(c_d(-R^\bullet\pi_*(f^*\mathcal{L}_\theta^{\frac{1}{r}}))(\frac{-\lambda}{r})^{-1-d}\cap[\mathcal{K}_{0,\star}(\sqrt[r]{L_\theta/Y},\beta)]^{\mathrm{vir}}\big)\cap\bar{\psi}_\star^c\Bigg)\tag{6.13}$$

to (6.9). Otherwise, there are m ($m\geqslant 1$) edges attached to the vertex v, let's index them by $[m]:=\{1,\cdots,m\}$. Let δ_i be the degree associated with the ith edge e_i. On each edge e_i there is exactly one vertex v_i over 0 incident to it, which can't be a unstable vertex of valence 1 (see Remark 5.1) or a node linking two edges by Lemma 6.5. So v_i corresponds to either a marking or a stable vertex. There are possible l marked points (l can be zero) on it, let's label the legs incident to v_i by $\{i1,\cdots,il\}\subset[n]$ (n is the total number of legs on Γ). Note that when v_i is unstable, $l=1$.

Assume that the vertex v_i is decorated by the degree β_{i0}. Since the insertion at the marking q_{ij} on the curve[15] C_{v_i} corresponding to v_i is of the form $\mu_{\beta_{ij},p_{ij}}(-\bar{\psi}_{ij})$ in (6.9), let's say the leg for q_{ij} has *virtual degree* (β_{ij},p_{ij}) contribution to the vertex v_i, denote β_i to be summation of β_{i0} and the degrees β_{ij} from the markings on C_{v_i}, and p_i to be

[15] When v is unstable, we just take v to be q_{i1}.

the summation of p_{ij} from the markings on c_{v_i}. We call (β_i, p_i) the *total degree* at the vertex v_i. From the (6.9), one has

$$\beta_\star + \beta_1 + \cdots + \beta_m = \beta, \;\; p_1 + \cdots + p_m = p \; .$$

Note that to ensure such a graph Γ exists, one must have

$$\beta_i(L_\theta) + p_i = \delta_i \; . \tag{6.14}$$

Indeed, by Riemann-Roch Theorem, one has

$$deg(N_1|_{C_{v_i}}) = -\frac{\beta_{i0}(L_\theta)}{s} = (1 - \frac{\delta_i}{s}) + \sum_{j=1}^{l} \frac{\beta_{ij}(L_\theta) + p_{ij}}{s} \quad mod \quad \mathbb{Z} \; .$$

Here the first term on the right hand is the age of N_1 at the node of C_{v_i}, and the second term on the right is the sum of the ages of N_1 at the marked points on C_{v_i}. As s is sufficiently large, one must have

$$\frac{\delta_i}{s} = \frac{\beta_{i0}(L_\theta)}{s} + \sum_{j=1}^{l} \frac{\beta_{ij}(L_\theta) + p_{ij}}{s} \; ,$$

which implies that $\beta_i(L_\theta) + p_i = \delta_i$.

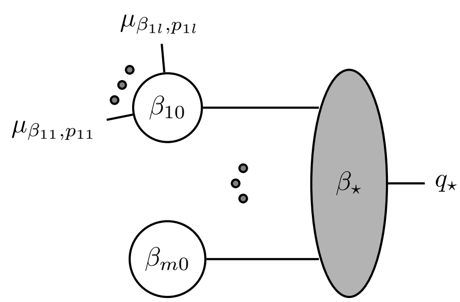

FIGURE 1. The ellipse dubbed gray on the right means the vertex labeled by ∞ with a leg attached, and the two big circles on the left mean vertexes labeled by 0. The text inside the vertex means the decorated degree for this vertex. On the upper left vertex, texts near the legs mean the insertion terms. On the bottom left vertex, we assume that there is no legs attached to it. The three grey dots in the middle mean the other edges (together with its incident vertexes and legs on them) besides edges indexed by 1 and m.

Now we can group the decorated graphs by elements of $\Lambda_{\beta,p,m}$. For each element $(m, \beta_\star, ((\beta_1, p_1), \cdots, (\beta_m, p_m)))$ in $\Lambda_{\beta,p,m}$. denoted by $\Lambda_{(m,\beta_\star,((\beta_1,p_1),\cdots,(\beta_m,p_m)))}$ the collection of all the edge-labeled decorated graphs such that the vertex incident to the edge labeled by i has total degree (β_i, p_i) and the decorated data for each vertex and incident half-edge over 0 satisfies (6.14). Note that our definition of the edge-labeled decorated graph has more decorations than the decorated graph introduced in Section 5 as we also label the edges. Then the automorphism group of an admissible decorated graph Γ is identity, which is usually smaller than the automorphism group of the corresponding decorated graph without labeling the edges. If we want to use admissible decorated graphs to compute the localization contribution, we need to divide $m!$ to offset the labeling as shown below.

Now we use the localization formula in §5.4 to compute the contribution from $\Lambda_{(m,\beta_\star,((\beta_1,p_1),\cdots,(\beta_m,p_m))}$ to (6.9). Summing over the contribution of the vertex v_i together with node h_i at v_i from all graphs in $\Lambda_{(m,\beta_\star,((\beta_1,p_1),\cdots,(\beta_m,p_m))}$, and pushing forward to $\bar{I}_{g_\beta^{-1}}Y \cong \bar{I}_{(g_\beta^{-1},\mathrm{e}^{\frac{\delta_i}{s}})}Y$ along $\iota\circ(ev_{h_i})_*$, it yields

$$\mu_{\beta_i,p_i}(\frac{\lambda-D_\theta}{\delta_i})+\sum_{l=0}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_l=\beta_i\\ p_1+\cdots+p_l=p_i}}\frac{1}{l!}(\widetilde{ev_\star})_*\Bigg(\sum_{d=0}^{\infty}\epsilon'_*\big(c_d(-R^\bullet\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}})(\frac{\lambda}{s})^{-d}$$

$$\cap[\mathcal{K}_{0,\vec{l}\cup\{0\}}(\sqrt[s]{L_{-\theta}/Y},\beta_\star)]^{\mathrm{vir}}\big)\cap\bigcap_{j=1}^{l}ev_j^*(\mu_{(\beta_j,p_j)}(-\bar{\psi}_j))\cap\frac{1}{\frac{\lambda-ev_\star^*D_\theta}{\delta_i s}-\frac{\bar{\psi}_0}{s}}\Bigg),$$

which, by Lemma 6.7 below, is equal to $J_{\beta_i,p_i}(z)|_{\frac{\lambda-D_\theta}{\delta_i}}$.

Note that all decorated graphs Γ in $\Lambda_{(m,\beta_\star,((\beta_1,p_1),\cdots,(\beta_m,p_m))}$ have the same localization contribution for the unique vertex $v_\star$ labeled by ∞, the edge e_i and the node over ∞ incident to e_i. As the localization formula for any graph in $\Lambda_{(m,\beta_\star,((\beta_1,p_1),\cdots,(\beta_m,p_m))}$ depends multi-linearly on the contributions of vertexes over 0. Now go over all possible triples $(m,\beta_\star,((\beta_1,p_1),\cdots,(\beta_m,p_m)))$, it yields the summation:

(6.15)
$$\sum_{m=1}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p}}\frac{1}{m!}(\widetilde{ev_\star})_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*\big(c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}\cap[\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y},\beta_\star)]^{\mathrm{vir}}\big)$$

$$\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}(f_{\beta_i,p_i}(z)|_{z=\frac{\lambda-D_\theta}{\delta_i}})}{-\frac{\lambda-ev_i^*D_\theta}{r\delta_i}-\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\Bigg).$$

Combing 6.16 and 6.15, we can write (6.9) as the following:

$$\frac{\mu_{\beta,p}(\frac{\lambda-D_\theta}{\delta})}{\delta}\cdot(\frac{\lambda-D_\theta}{\delta})^c+\sum_{m=0}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p}}\frac{1}{m!}(\widetilde{ev_\star})_*$$

$$\Bigg([\mathcal{K}_{0,\vec{m}\cup\star}(Y,\beta_\star)]^{\mathrm{vir}}\cap\bigcap_{i=1}^{m}ev_i^*(\mu_{\beta_i,p_i}(-\bar{\psi}_i))\bigcap\frac{\frac{1}{\delta}(\frac{\lambda-ev_\star^*D_\theta}{\delta})^c}{\frac{\lambda-ev_\star^*D_\theta}{\delta}-\bar{\psi}_\star}\Bigg)$$

(6.16)
$$+\sum_{m=0}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m\\ p_1+\cdots+p_m=p\\ (\beta_i,p_i)\neq 0\text{ for all }1\leqslant i\leqslant m}}\frac{1}{m!}(\widetilde{ev_\star})_*\Bigg(\sum_{d=0}^{\infty}\epsilon_*(c_d(-R^\bullet\pi_*\mathcal{L}_\theta^{\frac{1}{r}})(\frac{-\lambda}{r})^{-1+m-d}$$

$$\cap[\mathcal{K}_{0,\vec{m}_r\cup\star}(\sqrt[r]{L_\theta/Y},\beta_\star)]^{\mathrm{vir}})\cap\prod_{i=1}^{m}\frac{ev_i^*(\frac{1}{\delta_i}(J_{\beta_i,p_i}(z)|_{z=\frac{\lambda-D_\theta}{\delta_i}})}{-\frac{\lambda-ev_i^*D_\theta}{r\delta_i}-\frac{\bar{\psi}_i}{r}}\cap\bar{\psi}_\star^c\Bigg).$$

As (6.9) lies in $H^*(\bar{I}_\mu Y,\mathbb{Q})[\lambda][t_1,\cdots,t_l]$, the coefficient of λ^{-1} term in (6.16) must vanish. Note that the coefficients before λ^{-1} in the first two terms in (6.16) yields (after replacing the index 0 by $\star$)

$$\sum_{m=0}^{\infty}\sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta\\ p_1+\cdots+p_m=p}}\frac{1}{m!}\phi^\alpha\langle\mu_{\beta_1,p_1}(-\bar{\psi}_1),\cdots,\mu_{\beta_m,p_m}(-\bar{\psi}_m),\phi_\alpha\bar{\psi}_\star^c\rangle_{0,\vec{m}\cup\star,\beta_\star},$$

which is the left hand side of equality in (6.11). Then we extract the coefficient of the λ^{-1} term in the third term in (6.16), this yields the term on the right hand side of (6.11) up to a minus sign, where we note if $(\beta_i, p_i) \neq 0$, then $\beta_i(L_\theta) + p_i < \beta(L_\theta) + p$. This completes the proof of (6.11). ☐

Lemma 6.7. *For any $\beta_\star, \beta_1, \cdots, \beta_m$ in $\mathrm{Eff}(W, G, \theta)$ and $p_1, \cdots, p_m$ in $\mathbb{Z}_{\geqslant 0}$, write $\beta = \beta_\star + \sum_{i=1}^m \beta_i$ and $p = \sum_i p_i$. When s is sufficiently large, one has*

$$\epsilon_*\Big(\sum_{d=0}^{\infty} c_d(-R^\bullet\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}})(\frac{\lambda}{s})^{-d} \cap [\mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L_{-\theta}/Y}, \beta_\star)]^{\mathrm{vir}}\Big) = \frac{1}{s}\big([\mathcal{K}_{0,\vec{m}\cup\{\star\}}(Y, \beta_\star)]^{\mathrm{vir}}\ , \tag{6.17}$$

Here $\epsilon : \mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L_{-\theta}/Y}, \beta_\star) \to \mathcal{K}_{0,\vec{m}\cup\star}(Y, \beta_\star)$ is the natural structure map.

Proof. We will first show that $R^0\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}} = 0$ on $\mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L_{-\theta}/Y}, \beta_\star)$, which implies that $R^1\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}} = 0$ as $R^\bullet\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}}$ has virtual rank 0 when s is sufficiently large. By Remark 5.3, when $\beta_\star \neq 0$, we have $R^0\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}} = 0$. So it remains to prove the case when $\beta_\star = 0$. Assume now that $\beta_\star = 0$, as the corresponding moduli is stable, we have $m \geqslant 2$. Let $f : C \to \sqrt[s]{L_{-\theta}/Y}$ be a stable map in $\mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L_{-\theta}/Y}, \beta_\star)$. Assume q_i is one of the marked points with insertion μ_{β_i,p_i}. Without loss of generality, we can assume $(\beta_i, p_i) \neq 0$ for all i as $\mu_0(z) = 0$ by the very definition. Note that we have

$$age_{q_i}((\mathcal{L}_{-\theta}^{\frac{1}{s}})|_C) = \frac{\beta_i(L_\theta) + p_i}{s} \neq 0\ ,$$

then the restricted line bundle $L_{-\theta}^{\frac{1}{s}} := (\mathcal{L}_{-\theta}^{\frac{1}{s}})|_C$ can't have any nonzero section on C. Indeed the degree of the restriction of $L_{-\theta}^{\frac{1}{s}}$ to every irreducible component is zero by Lemma 2.5 as the total degree $\beta_\star$ is zero, then a nonzero section of $L_{-\theta}^{\frac{1}{s}}$ will trivialize the line bundle $L_{-\theta}^{\frac{1}{s}}$, this contradicts the fact that $L_{-\theta}^{\frac{1}{s}}$ has nontrivial stacky structure at q_i.

Now as $-R^\bullet\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}} = R^1\pi_*\mathcal{L}_{-\theta}^{\frac{1}{s}} = 0$, (6.17) follows immediately from the identity

$$\epsilon'_*([\mathcal{K}_{0,\vec{m}_s\cup\star}(\sqrt[s]{L_{-\theta}}/Y, \beta_\star)]^{\mathrm{vir}}) = \frac{1}{s}[\mathcal{K}_{0,\vec{m}\cup\star}(Y, \beta_\star)]^{\mathrm{vir}}\ ,$$

which is proved in [TT16, Theorem 5.16]. ☐

6.3. **Proof of Main Theorem.** Using the notation in the introduction, now we prove the main theorem 1.1:

Proof. According to the analysis in the introduction, it suffices to prove the following:

$$\begin{aligned}&[z\mathbb{I}_{\beta,p}(z)]_{z^{-c-1}} = \\ &\sum_{m=0}^{\infty} \sum_{\substack{\beta_\star+\beta_1+\cdots+\beta_m=\beta \\ p_1+\cdots+p_m=p}} \frac{1}{m!}\phi^\alpha\langle\mu_{\beta_1,p_1}(-\bar{\psi}_1), \cdots, \mu_{\beta_m,p_m}(-\bar{\psi}_m), \phi_\alpha\bar{\psi}_\star^c\rangle_{0,\vec{m}\cup\star,\beta_\star}\ ,\end{aligned} \tag{6.18}$$

for any nonnegative integer c and nonzero pair (β, p). Now (6.18) immediately follows from Theorem 6.4 and 6.6. ☐

Remark 6.8. The proof of the mirror theorem here is quite robust; the main geometrical construction including twisted graph space and root stack construction, and recursive relations can be directly generalized to all proper GIT targets considered in quasimap theory. Hence we expect the method developed here can be used to prove the genus zero quasimap wall-crossing conjecture for all proper GIT targets considered in quasimap theory.

7. An example

In this section, we will recover the quantum product computation by Corti for a cubic hypersurface Y which is cut off by the polynomial $x_1^3 + x_2^3 + x_3^3 + x_4x_1$ in $\mathbb{P}(1,1,1,2)$. The following is the table for (small) quantum product of Y obtained by Corti (see [MH14]):

	$\mathbb{1}$	p	p^2	$\mathbb{1}_{\frac{1}{2}}$
$\mathbb{1}$	$\mathbb{1}$	p	p^2	$\mathbb{1}_{\frac{1}{2}}$
p		$p^2 + 12r^2 + 3r\mathbb{1}_{\frac{1}{2}}$	$12r^2p$	rp
p^2			$108r^4 + 36r^3\mathbb{1}_{\frac{1}{2}}$	$12r^3$
$\mathbb{1}_{\frac{1}{2}}$				$\frac{1}{3}p^2 - 3r\mathbb{1}_{\frac{1}{2}}$

Here $r = \frac{1}{2}q$ [16], p is the hyperplane class of Y and $\mathbb{1}_{\frac{1}{2}}$ is the fundamental class of the unique non-trivial twisted sector of $H^*(\bar{I}_\mu Y, \mathbb{Q})$. Due to the discussion in [MH14], the usual $(\mathcal{O}(3), Euler)$-twisted I-function of $\mathbb{P}(1,1,1,2)$ only recovers the first two rows, and the rest two rows rely on Corti's key calculation

$$(\mathbb{1}_{\frac{1}{2}} \circ \mathbb{1}_{\frac{1}{2}}, \mathbb{1}_{\frac{1}{2}}) = \frac{-3}{2}r\ . \tag{7.1}$$

In the following, we will recover Corti's key calculation using the I-function by choosing a different GIT presentation of $\mathbb{P}(1,1,1,2)$.

Choose the matrix

$$\rho = \begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^5\ ,$$

which gives the action of $G := \mathbb{C}_t^* \times \mathbb{C}_z^*$ on $W := \mathbb{C}^5$ so that the GIT (stack) quotient is still $\mathbb{P}(1,1,1,2)$ (with the choice of stability condition $\theta = t^2z^3$, this also corresponds to the S-extended data $S = \{\frac{1}{2}\}$ in the sense of [CCIT19, CCIT15]). Consider the polynomial $zx_1^3 + zx_2^3 + zx_3^3 + x_4x_1$, then it cuts off a hypersurface in the new GIT stack quotient $[W^{ss}(\theta)/G]$. Note that Y comes from the line bundle L_{t^3z} on $[W/G]$, which is not semi-positive as the following table shows.

The semigroup $\mathrm{Eff}(W, G, \theta)$ is generated by $\beta_1, \beta_2 \in Hom(\chi(G), \mathbb{Q})$ such that

$$\begin{pmatrix} \beta_1(L_t) & \beta_1(L_z) \\ \beta_2(L_t) & \beta_2(L_z) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}\ .$$

Then we can think $q := q^{\beta_1}$ generates the semigroup of degrees of *stable maps* to the hypersurface Y and $x := q^{\beta_2}$ is a formal variable.

By §3.1, the small I-function of Y using this new GIT presentation of $\mathbb{P}(1,1,1,2)$ is

[16] In [MH14], they use $r = \frac{1}{2}q^{\frac{1}{2}}$, their $q^{\frac{1}{2}}$ corresponds our q here.

$$
\begin{aligned}
I(q,x,z) = & \sum_{\substack{(l,k)\in\mathbb{N}^2\\ \frac{3l-k}{2}\geqslant 0}} \frac{q^l x^k}{z^k k!} \frac{\prod_{i<0}(p+(\frac{l-k}{2}-i)z)^3}{\prod_{i<\frac{l-k}{2}}(p+(\frac{l-k}{2}-i)z)^3} \\
& \frac{1}{\prod_{0\leqslant i<l}(2p+(l-i)z)} \prod_{0\leqslant i<\frac{3l-k}{2}} \left(3p+(\frac{3l-k}{2}-i)z\right)\mathbb{1}_{\frac{k-l}{2}} \\
& + \sum_{\substack{(l,k)\in\mathbb{N}^2\\ \frac{3l-k}{2}\in\mathbb{Z}_{<0}}} \frac{q^l x^k}{z^k k!} \frac{\prod_{\frac{l-k}{2}<i<0}(p+(\frac{l-k}{2}-i)z)^3}{\prod_{0\leqslant i<l}(2p+(l-i)z)} \\
& \frac{1}{\prod_{\frac{3l-k}{2}<i<0}\left(3p+(\frac{3l-k}{2}-i)z\right)}\frac{1}{3}p^2 \\
& + \sum_{\substack{(l,k)\in\mathbb{N}^2\\ \frac{3l-k}{2}\in Q_{<0}\setminus\mathbb{Z}_{<0}}} \frac{q^l x^k}{z^k k!} \frac{\prod_{\frac{l-k}{2}<i<0}(p+(\frac{l-k}{2}-i)z)^3}{\prod_{0\leqslant i<l}(2p+(l-i)z)} \\
& \frac{1}{\prod_{\frac{3l-k}{2}<i<0}\left(3p+(\frac{3l-k}{2}-i)z\right)}\mathbb{1}_{\frac{k-l}{2}} \ .
\end{aligned} \tag{7.2}
$$

where $\mathbb{1}_{\frac{k-l}{2}}=\mathbb{1}_{\frac{1}{2}}$ if $k-l$ is odd, otherwise $\mathbb{1}_{\frac{k-l}{2}}=\mathbb{1}$. We can show the following fact about $I(q,x,z)$:

$$
I(q,x,z)=\mathbb{1}+\frac{x\mathbb{1}_{\frac{1}{2}}+qx\mathbb{1}}{z}+\mathcal{O}(x^3)+\mathcal{O}(\frac{1}{z^2}) \ , \tag{7.3}
$$

$$
\frac{\partial I(q,x,z)}{\partial x}=\frac{\mathbb{1}_{\frac{1}{2}}+q\mathbb{1}}{z}+\frac{x(q^2\mathbb{1}+\frac{1}{3}p^2+\frac{q\mathbb{1}_{\frac{1}{2}}}{2})}{z^2}+\mathcal{O}(x^2)+\mathcal{O}(\frac{1}{z^3}) \ , \tag{7.4}
$$

and

$$
\frac{\partial^2 I(q,x,z)}{\partial^2 x}=\frac{q^2\mathbb{1}+\frac{1}{3}p^2+\frac{q\mathbb{1}_{\frac{1}{2}}}{2}}{z^2}+\mathcal{O}(x)+\mathcal{O}(\frac{1}{z^3}) \ . \tag{7.5}
$$

Since $-ze^{\frac{qx}{z}}I(q,x,-z)$ is a slice on the Givental's cone by string flow and have the asympotic following expansion

$$
ze^{\frac{-qx\mathbb{1}}{z}}I(q,x,z)=z\mathbb{1}+x\mathbb{1}_{\frac{1}{2}}+\mathcal{O}(x^3)+\mathcal{O}(\frac{1}{z}) \ . \tag{7.6}
$$

Then

$$
ze^{\frac{-qx\mathbb{1}}{z}}I(q,x,z)=J^{Giv}(q,x\mathbb{1}_{\frac{1}{2}},z)+\mathcal{O}(x^3) \ , \tag{7.7}
$$

where $J^{Giv}(q,t,z)$ is Givental's $J-$function which has an asymptotic expansion

$$
z\mathbb{1}+t+\mathcal{O}(\frac{1}{z}) \ , \tag{7.8}
$$

and $t=\sum t^\alpha\phi_\alpha\in H^*(\bar{I}_\mu Y,\mathbb{Q})$. We have the following standard fact about Givental's $J-$function (c.f. [Giv04]):

$$
z\frac{\partial}{\partial t_\alpha}\frac{\partial}{\partial t_\beta}J^{Giv}(q,t,z)=\phi_\alpha\star_t\phi_\beta+\mathcal{O}(z^{-1}) \ . \tag{7.9}
$$

Now consider the function

$$z\frac{\partial^2}{\partial^2 x}\left(ze^{\frac{-qx\mathbb{1}}{z}}I(q,x,z)\right)\ , \tag{7.10}$$

a direction computation using product rule yields:

$$q^2e^{\frac{-qx\mathbb{1}}{z}}I(q,x,z)-2zqe^{\frac{-qx\mathbb{1}}{z}}\frac{\partial}{\partial x}I(q,x,z)+z^2e^{\frac{-qx\mathbb{1}}{z}}\frac{\partial^2}{\partial^2 x}I(q,x,z)\ . \tag{7.11}$$

Apply (7.3), (7.4), (7.5) to the first term, second term and third term in (7.11), respectively, we have the following asymptotic expansion of (7.10):

$$q^2\mathbb{1}-2q(\mathbb{1}_{\frac{1}{2}}+q\mathbb{1})+(q^2\mathbb{1}+\frac{1}{3}p^2+\frac{q\mathbb{1}_{\frac{1}{2}}}{2})+\mathcal{O}(x)+\mathcal{O}(z^{-1})\ . \tag{7.12}$$

On the other hand, using equation (7.7), (7.9), one has another asymptotic expansion about (7.10):

$$\mathbb{1}_{\frac{1}{2}}\star_x\mathbb{1}_{\frac{1}{2}}+\mathcal{O}(x)+\mathcal{O}(z^{-1})\ . \tag{7.13}$$

Compare (7.12) and (7.13), after evaluating $x=0$ and ignoring all negative z powers, we have

$$\mathbb{1}_{\frac{1}{2}}\circ\mathbb{1}_{\frac{1}{2}}=\frac{1}{3}p^2-\frac{3}{2}q\mathbb{1}_{\frac{1}{2}}\ ,$$

which recovers Corti's calculation[17] (7.1)!

Acknowledgments: First, I want to express my deep gratitude to my advisor Hsian-Hua Tseng for his support and guidance. I want to especially thank his excellent talk on relative Gromov-Witten invariants and orbifold Gromov-Witten invariants at OSU, which draws my attention to double ramification cycles, which eventually leads to the birth of this paper. I would like to thank the organizers of the 2017 FRG workshop "Crossing the Walls in Enumerative Geometry" at Columbia, which exposed me to the most recent process in the subject as well as many inspiring ideas. Part of the paper is based on the author's thesis at OSU. I also want to thank Yang Zhou and Cristina Manolache on discussions on quasimap theory and thank Honglu Fan and Fenglong You on discussions on Givental's formalism. I want to thank Felix Janda on the presentation of the big I-function. Finally, I want to thank an anonymous referee for the valuable comments which greatly improves this paper.

References

[AGV08] Dan Abramovich, Tom Graber, and Angelo Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*, American Journal of Mathematics **130** (2008), no. 5, 1337–1398. ↑6, 9, 33

[AJT15] Elena Andreini, Yunfeng Jiang, and Hsian-Hua Tseng, *Gromov-Witten theory of root Gerbes I: Structure of genus 0 moduli spaces*, Journal of Differential Geometry **99** (2015Jan), no. 1, 1–45. ↑36

[AB95] M. F. Atiyah and R. Bott, *The Moment Map and Equivariant Cohomology*, Raoul Bott Collected Papers (1995), 241–268. ↑1

[BCFK08] Aaron Bertram, Ionuţ Ciocan-Fontanine, and Bumsig Kim, *Gromov-witten invariants for abelian and nonabelian quotients*, Journal of Algebraic Geometry **17** (2008), no. 2, 275–294. ↑1

[Bro13] Jeff Brown, *Gromov–Witten Invariants of Toric Fibrations*, International Mathematics Research Notices **2014** (2013Jul), no. 19, 5437–5482. ↑5

[CDLOGP91] Philip Candelas, Xenia C. De La Ossa, Paul S. Green, and Linda Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nuclear Physics B **359** (1991Jul), no. 1, 21–74. ↑1

[CL11] H.-L. Chang and J. Li, *Gromov-Witten Invariants of Stable Maps With Fields*, International Mathematics Research Notices **2012** (2011), no. 18, 4163–4217. ↑28

[17] Here the small quantum product $\circ$ is defined by the specialization of the big quantum product $\star_t$ to $t=0$.

[CLLL16] Huai-Liang Chang, Jun Li, Wei-Ping Li, and Chiu-Chu Melissa Liu, *An Effective Theory of Gw and Fjrw Invariants of Quintics Calabi-Yau Manifolds*, ArXiv e-prints (2016), available at 1603.06184. ↑18, 24

[CCFK15] Daewoong Cheong, Ionuţ Ciocan-Fontanine, and Bumsig Kim, *Orbifold quasimap theory*, Mathematische Annalen **363** (2015Mar), no. 3-4, 777–816. ↑2, 6, 8, 9, 11, 15

[CFK14] Ionut Ciocan-Fontanine and Bumsig Kim, *Wall-Crossing in Genus Zero Quasimap Theory and Mirror Maps*, Algebraic Geometry **1** (2014), no. 4, 400–448. ↑5, 9

[CFKS08] Ionuţ Ciocan-Fontanine, Bumsig Kim, and Claude Sabbah, *The abelian/nonabelian correspondence and Frobenius manifolds*, Inventiones mathematicae **171** (2008), no. 2, 301–343. ↑1

[CFK16] Ionuţ Ciocan-Fontanine and Bumsig Kim, *Big I-functions*, Development of Moduli Theory — Kyoto 2013, 2016, pp. 323–347. ↑1, 2, 5, 6, 7, 8, 9, 16

[CFKM14] Ionuţ Ciocan-Fontanine, Bumsig Kim, and Davesh Maulik, *Stable quasimaps to GIT quotients*, Journal of Geometry and Physics **75** (2014Jan), 17–47. ↑2, 6, 7, 9, 17

[CJR17a] Emily Clader, Felix Janda, and Yongbin Ruan, *Higher-Genus Quasimap Wall-Crossing Via Localization*, ArXiv e-prints (2017), available at 1702.03427v1. ↑5, 15, 17, 19

[CJR17b] ———, *Higher-Genus Wall-Crossing in the Gauged Linear Sigma Model*, CoRR (2017), available at 1706.05038. ↑17

[CG07] Thomas Coates and Alexander Givental, *Quantum Riemann-Roch, Lefschetz and Serre*, Annals of Mathematics **165** (2007), no. 1, 15–53. ↑1

[CCIT15] Tom Coates, Alessio Corti, Hiroshi Iritani, and Hsian-Hua Tseng, *A mirror theorem for toric stacks*, Compositio Mathematica **151** (2015Jun), no. 10, 1878–1912. ↑1, 15, 52

[CCIT19] ———, *Some applications of the mirror theorem for toric stacks*, Advances in Theoretical and Mathematical Physics **23** (2019), no. 3, 767–802. ↑3, 4, 13, 52

[CCLT09] Tom Coates, Alessio Corti, Yuan-Pin Lee, and Hsian-Hua Tseng, *The quantum orbifold cohomology of weighted projective spaces*, Acta Mathematica **202** (2009), no. 2, 139–193. ↑1

[CGI+12] Tom Coates, Amin Gholampour, Hiroshi Iritani, Yunfeng Jiang, Paul Johnson, and Cristina Manolache, *The quantum Lefschetz hyperplane principle can fail for positive orbifold hypersurfaces*, Mathematical Research Letters **19** (2012), no. 5, 997–1005. ↑2

[DW90] Robbert Dijkgraaf and Edward Witten, *Mean field theory, topological field theory, and multi-matrix models*, Nuclear Physics B **342** (1990), no. 3, 486–522. ↑3

[Ful84] William Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 2, Springer-Verlag, Berlin, 1984. MR732620 ↑12, 27

[Giv96] Alexander Givental, *Equivariant Gromov-Witten invariants*, International Mathematics Research Notices **1996** (1996), no. 13, 613. ↑1, 5

[Giv98] ———, *A Mirror Theorem for Toric Complete Intersections*, Topological Field Theory, Primitive Forms and Related Topics (1998), 141–175. ↑1

[Giv04] Alexander B. Givental, *Symplectic geometry of Frobenius structures*, Aspects of Mathematics (2004), 91–112. ↑2, 53

[GP99] T. Graber and R. Pandharipande, *Localization of virtual classes*, Inventiones Mathematicae **135** (1999Jan), no. 2, 487–518. ↑1, 21

[GJR17] Shuai Guo, Felix Janda, and Yongbin Ruan, *A Mirror Theorem for Genus Two Gromov-Witten Invariants of Quintic Threefolds*, ArXiv e-prints (2017), available at 1709.07392. ↑1

[Gué19] Jérémy Guéré, *Hodge–Gromov–Witten theory* (2019), available at 1908.11409. ↑2

[JPPZ17] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine, *Double ramification cycles on the moduli spaces of curves*, Publications mathématiques de l'IHÉS **125** (2017Jun), no. 1, 221–266. ↑5, 32, 41

[JPPZ18] ———, *Double Ramification Cycles With Target Varieties*, ArXiv e-prints (2018), available at 1812.10136. ↑22, 36

[JPT] P. Johnson, R. Pandharipande, and H.-H. Tseng, *Notes on local $\mathbb{P}^1-$orbifolds*. Available at https://people.math.ethz.ch/ rahul/lPab.ps. ↑44

[KKP03] Bumsig Kim, Andrew Kresch, and Tony Pantev, *Functoriality in Intersection Theory and a Conjecture of Cox, Katz, and Lee*, Journal of Pure and Applied Algebra **179** (2003), no. 1-2, 127–136. ↑2

[LLY99] Bong H. Lian, Kefeng Liu, and S. T. Yau, *Mirror Principle I*, Surveys in Differential Geometry **5** (1999), no. 1, 405–454. ↑1

[Man11] Cristina Manolache, *Virtual Pull-Backs*, Journal of Algebraic Geometry **21** (2011), no. 2, 201–245. ↑28

[MH14] Guest Martin and Sakai Hironori, *Orbifold quantum D-modules associated to weighted projective spaces*, Commentarii Mathematici Helvetici **89** (2014), no. 2, 273–297 (English). ↑52

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, Third, Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR1304906 ↑6

[TT16] Xiang Tang and Hsian-Hua Tseng, *A Quantum Leray-Hirsch Theorem for Banded Gerbes*, ArXiv e-prints (2016), available at 1602.03564. ↑36, 51

[Wan] Jun Wang, *I-functions for toric stack hypersurfaces.* Available at https://junmathwang.github.io/notes. ↑4

[Wan23] ———, *Quantum Lefschetz Theorem Revisited*, CoRR (2023), available at 2305.04906. ↑5

[Web21] Rachel Webb, *Abelianization and Quantum Lefschetz for Orbifold Quasimap I-Functions*, CoRR (2021), available at 2109.12223. ↑5

[Web23] ———, *The abelian–nonabelian correspondence for I-functions*, International Mathematics Research Notices **2023** (2023), no. 3, 2592–2648. ↑1

[Zho22] Yang Zhou, *Quasimap wall-crossing for GIT quotients*, Inventiones mathematicae **227** (2022), no. 2, 581–660. ↑5

[Zin08] Aleksey Zinger, *The Reduced Genus* 1 *Gromov-Witten Invariants of Calabi-Yau Hypersurfaces*, Journal of the American Mathematical Society **22** (2008), no. 3, 691–737. ↑1

Beijing International Center for Mathematical Research, Peking University, Beijing,100091, China

E-mail address: wangjun@bicmr.pku.edu.cn