I-FUNCTIONS FOR TORIC STACK HYPERSURFACES

JUN WANG

This note begins with an example of *S-extended I-function* for *Toric stack hypersurface* considered in [CCIT14], we will also translate it into small I-functions using *Quasimap* language. In Section 2, we will compute small quantum product of a cubic hypersurface in $\mathbb{P}(1,1,1,2)$, which recovers Corti's geometrical calculation.

We start with the example as in [CCIT14, Example 9].

1. A sextic hypersurface in $\mathbb{P}(1,1,1,3,3)$

Let the orbifold Y be a smooth sextic hypersurface in $X = \mathbb{P}(1,1,1,3,3)$ defined by the polynomial $f := x_1^6 + x_2^6 + x_3^6 + x_4^2 + x_5^2$; this is a Fano 3-fold with canonical singularities. The ambient space X is the toric Deligne–Mumford stack associated to the stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$, where:

$$\rho = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathbb{Z}^5 \to N = \mathbb{Z}^4$$

and Σ is the complete fan in $N_{\mathbb{Q}} \cong \mathbb{Q}^4$ with rays given by the columns $\rho_1, \ldots, \rho_5$ of ρ. We identify $\mathrm{Box}(\mathbf{\Sigma})$ with the set $\left\{0, \frac{1}{3}, \frac{2}{3}\right\}$ via the map $\kappa\colon x \mapsto x(\rho_1 + \rho_2 + \rho_3)$. Consider the S-extended I-function where $S = \left\{0, \frac{1}{3}\right\}$ and $S \to N_{\Sigma}$ is the map κ. The S-extended fan map is:

$$\rho^S = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 & 0 & -1 \\ -3 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} : \mathbb{Z}^{5+2} \to N$$

so that $\mathbb{L}^S_{\mathbb{Q}} \cong \mathbb{Q}^3$ is identified as a subset of $\mathbb{Q}^{5+2}$ via the inclusion:

$$\begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix}$$

Date: Friday 1[st] November, 2019.

The S-extended Mori cone is the positive octant. We see that $\Lambda^S \subset \mathbb{L}^S_{\mathbb{Q}}$ is the sublattice of vectors:

$$\begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix} \quad \text{such that } l,\ k_0,\ k_1 \in \mathbb{Z}$$

and that the reduction function is:

$$v^S\colon \begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix} \mapsto \left\langle \frac{k_1 - l}{3} \right\rangle$$

Let's reformulate the above using the language of GIT. First, X admits the GIT representation (W, G, θ) with $W \cong \mathbb{C}^5$, $G \cong \mathbb{C}^*$ and $\theta = 6\mathrm{id}_{\mathbb{C}^*}$, where G acts from the right on W in the following way:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 3 \end{pmatrix} : \mathbb{Z} \to \mathbb{Z}^5 \ ,$$

where $\mathbb{Z}$ is identified with the character group $\chi(\mathbb{C}^*)$ and $\mathbb{Z}^5$ is identified with the character group $\chi((\mathbb{C}^*)^5)$, so that the action of G on W is induced from the above matrix.

Alternatively, X admits another GIT representation (W^S, G^S, θ^S) corresponding to the S-extended fan, where $W^S = \mathbb{C}^7$, $G^S = (\mathbb{C}^*)^3$ and $\theta^S = t^6 z_1 z_2^3 : G^S \to \mathbb{C}^*$ for $(t_1, z_1, z_2) \in G^S$. Here the GIT data (W^S, G^S, θ^S) is defined by the right G^S action on W^S in the following way:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \mathbb{Z}^3 \to \mathbb{Z}^7 \ ,$$

where $\mathbb{Z}^3$ is identified with the character group $\chi((\mathbb{C}^*)^3)$ and $\mathbb{Z}^7$ is identified with $\chi((\mathbb{C}^*)^7)$, so that the action of G^S on W^S is induced from 7 characters $\rho_1, \cdots, \rho_7$ of $(\mathbb{C}^*)^3$ using the rows of the above matrix. One can see that the semistable loci $(W^S)^{ss}(\theta^S)$ is equal to $W^{ss}(\theta) \times (\mathbb{C}^*)^2$.

Then the monoid $\mathrm{Eff}(W^S, G^S, \theta^S) \subset Hom(Pic([W^S/G^S]), \mathbb{Q})$ is comprised of classes $\beta \in Hom(\chi(G^S), \mathbb{Q})$ such that $\beta(L_{z_1}) \in \mathbb{N}$, $\beta(L_{z_2}) \in \mathbb{N}$ and $\beta(L_{t_1^3 z_2}) \in \mathbb{N}$. Then we have

$$\mathrm{Eff}(W^S, G^S, \theta^S) \cong \mathrm{Eff}(W, G, \theta) \times \mathbb{N}^2 \ ,$$

and $\mathrm{Eff}(W^S, G^S, \theta^S)$ is generated by the basis $\beta_1, \beta_2, \beta_3$ defined by the following matrix

$$\begin{pmatrix} \beta_1(L_t) & \beta_1(L_{z_1}) & \beta_1(L_{z_2}) \\ \beta_2(L_t) & \beta_2(L_{z_1}) & \beta_2(L_{z_2}) \\ \beta_3(L_t) & \beta_3(L_{z_1}) & \beta_3(L_{z_2}) \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix} \ .$$

Here we can think that β_1 generates the semigroup of degrees of stable maps to Y.

Then an element $\beta = l\beta_1 + k_0\beta_2 + k_1\beta_3$ ($(l, k_0, k_1) \in \mathbb{N}^3$) in $\mathrm{Eff}(W^S, G^S, \theta^S)$ gives rise to the inertia component of $\bar{I}_\mu Y$ given by

$$g_\beta = (\mu_3^{l-k_1}, 1, 1) \in G^S$$

following the convention in [CCFK15, §5.3]. This coincides with the reduction map v^S above up to the minus sign (but it's will be offset when writing down the small I-function.). Here we identify the inertia component given by $\left\langle \frac{k_1-l}{3} \right\rangle$ using the convention in [CCIT14] with the inertia component given by $(\mu_3^{k_1-l}, 1, 1) \in G^S$ using the convention in [CCFK15].

Let $P \in H^2(X;\mathbb{Q})$ denote the (non-equivariant) first Chern class of $\mathcal{O}_X(1)$, and identify the Novikov ring $\mathbf{\Lambda}$ with $\mathbb{C}[\![Q]\!]$ via the map that sends $d \in H_2(X;\mathbb{Z})$ to $Q^{\int_d 3P}$. With notation as in [CCIT14, §4], we have $D_1 = D_2 = D_3 = P$, $D_4 = D_5 = 3P$, and so the non-equivariant limit of the S-extended I-function is:

$$I^S_{\mathrm{non}}(t, x, z) = z e^{(t_1+t_2+t_3+3t_4+3t_5)P/z}$$
$$\times \sum_{(l,k_0,k_1)\in\mathbb{N}^3} \frac{Q^l x_0^{k_0} x_1^{k_1} e^{(t_1+t_2+t_3+3t_4+3t_5)l}}{z^{k_0+k_1} k_0! k_1!} \frac{\prod_{\substack{\langle b\rangle = \left\langle \frac{l-k_1}{3}\right\rangle \\ b\leqslant 0}} (P+bz)^3}{\prod_{\substack{\langle b\rangle = \left\langle \frac{l-k_1}{3}\right\rangle \\ b\leqslant \frac{l-k_1}{3}}} (P+bz)^3} \frac{\mathbf{1}_{\left\langle \frac{k_1-l}{3}\right\rangle}}{\prod_{\substack{\langle b\rangle = 0 \\ 1\leqslant b\leqslant l}} (3P+bz)^2}$$

On the other hand, the small I-function of X using orbifold quasimap theory is

$$I(q, z) = \sum_{\substack{\beta = l\beta_1+k_0\beta_2+k_1\beta_3 \in \mathrm{Eff}(\mathrm{W^S},\mathrm{G^S},\theta^{\mathrm{S}}) \\ (l,k_0,k_1)\in\mathbb{N}^3}} Q^\beta \frac{\prod_{\beta(L_\rho)<0} \prod_{\beta(L_\rho)\leqslant i<0} (D_\rho + (\beta(L_\rho) - i)z)}{\prod_{\beta(L_\rho)>0} \prod_{0\leqslant i<\beta(L_\rho)} (D_\rho + (\beta(L_\rho) - i)z)} 1_{g_\beta^{-1}} .$$

Using the fact $D_{\rho_1} = D_{\rho_2} = D_{\rho_3} = P$, $D_{\rho_4} = D_{\rho_5} = 3P$ and $D_{\rho_6} = D_{\rho_7} = 0$ on X, after equaling Q^β with $Q^l x_0^{k_0} x_1^{k_1}$, this will identify $zI(q,z)$ with I^S_Y up to an exponential factor $e^{(t_1+t_2+t_3+3t_4+3t_5)P/z} e^{(t_1+t_2+t_3+3t_4+3t_5)l}$. But using the reconstruction theorem proved in [Giv15] or [CCFK15, Theorem 4.10]. We can show that the Givental's small I-function I^{giv} defined as in [CCFK15, Definition 4.1] coincides with the S-extend I-function of Y up to a z-factor after the Novikov-variables identification.

Now let's move to the mirror theorem for hypersurface Y defined by $f = x_1^6 + x_2^6 + x_3^6 + x_4^2 + x_5^2$ in X. Let $\mathcal{E} \to X$ be the line bundle corresponding to the element $\varepsilon \in (\mathbb{L}^S)^\vee$ given by:

$$\varepsilon\colon \begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix} \mapsto 2l$$

so that $\mathcal{E}=\mathcal{O}(6)$. The S-extended $(\boldsymbol{e},\mathcal{E})$-twisted I-function of X is:

$$I^S_{\boldsymbol{e},\mathcal{E}}(t,x,z)=ze^{(t_1+t_2+t_3+3t_4+3t_5)P/z}$$

$$\times\sum_{(l,k_0,k_1)\in\mathbb{N}^3}\frac{Q^l x_0^{k_0}x_1^{k_1}e^{(t_1+t_2+t_3+3t_4+3t_5)l}}{z^{k_0+k_1}k_0!k_1!}\frac{\prod_{\substack{\langle b\rangle=\langle\frac{l-k_1}{3}\rangle\\ b\leqslant 0}}(P+bz)^3}{\prod_{\substack{\langle b\rangle=\langle\frac{l-k_1}{3}\rangle\\ b\leqslant\frac{l-k_1}{3}}}(P+bz)^3}\frac{\prod_{\substack{\langle b\rangle=0\\ 1\leqslant b\leqslant 2l}}(\kappa+6P+bz)}{\prod_{\substack{\langle b\rangle=0\\ 1\leqslant b\leqslant l}}(3P+bz)^2}\mathbf{1}_{\langle\frac{k_1-l}{3}\rangle}$$

Then the S-extended I-function of Y defined in [CCIT14, §5] is given by

$$I^S_Y(t,x,z)=ze^{(t_1+t_2+t_3+3t_4+3t_5)P/z}$$

$$\times\sum_{(l,k_0,k_1)\in\mathbb{N}^3}\frac{Q^l x_0^{k_0}x_1^{k_1}e^{(t_1+t_2+t_3+3t_4+3t_5)l}}{z^{k_0+k_1}k_0!k_1!}\frac{\prod_{\substack{\langle b\rangle=\langle\frac{l-k_1}{3}\rangle\\ b\leqslant 0}}(P+bz)^3}{\prod_{\substack{\langle b\rangle=\langle\frac{l-k_1}{3}\rangle\\ b\leqslant\frac{l-k_1}{3}}}(P+bz)^3}\frac{\prod_{\substack{\langle b\rangle=0\\ 1\leqslant b\leqslant 2l}}(6P+bz)}{\prod_{\substack{\langle b\rangle=0\\ 1\leqslant b\leqslant l}}(3P+bz)^2}1_{\langle\frac{k_1-l}{3}\rangle}$$

Now let's also move the quasimap counterpart of the S-extended I-function above. There are two ways to get I-function of hypersurface as in quasimap: One way we can use the twisted quasimap invariants of the S-extended toric stack, which requires convexity to get the S-extended I-function for hypersurface; The second way is to use another GIT representation of Y as in the case of S-extended toric stack. We elaborate the second way here. Let the $AY\subset W$ be the zero loci of f. Let $f^S=x_1^6z_2^2+x_2^6z_2^2+x_3^6z_2+x_4^2+x_5^2$ and $AY^S=Z(f^S)$, note this is a positive hypersurface in $[W^S/G^S]$ associated to the positive line bundle[1] $L_{t^6z_2^2}$ in the sense of [Wanb]. Then (AY^S,G^S,θ^S) will be a GIT data with the same action described above (we need to check the G^S-action on W^S preserves AY^S) then the GIT stack quotient $[(AY^S)^{ss}(\theta^S)/G^S]$ is still isomorphic to Y. By the small $I-$function computed in [Wanb], we have the following formula:

$$I(q,z)=\sum_{\substack{\beta=l\beta_1+k_0\beta_2+k_1\beta_3\\(l,k_0,k_1)\in\mathbb{N}^3}}Q^\beta\frac{\prod_{\beta(L_\rho)<0}\prod_{\beta(L_\rho)\leqslant i<0}(D_\rho+(\beta(L_\rho)-i)z)}{\prod_{\beta(L_\rho)>0}\prod_{0\leqslant i<\beta(L_\rho)}(D_\rho+(\beta(L_\rho)-i)z)}$$
$$\times\prod_{i=1}^{2l}\Big(6P+iz\Big)1_{g_\beta^{-1}}\ .$$

[1]Recall that the definition of positive line bundles given in [Wanb, Definition 2.6]: given a GIT data (W,G,θ) which gives rise to a GIT stack quotient $X:=[W^{ss}(\theta)/G]$, let $L:=L_\tau$ be the line bundle associated to a character τ of G. Then we call L a positive line bundle on $\mathfrak{X}:=[W/G]$ if

$$\beta(L)\geqslant 0$$

whenever β is the degree of a quasimap to X. Sometimes if the GIT model of X and the character τ are clear in the context, we will also call the restricted line bundle $L|_X$ a positive line bundle on X. By abusing notations, we will denote the restricted line bundle on X to be L unless stated otherwise. Note the definition of a positive line bundle on X depends on the GIT model of X and the character τ. See an example in §2.2 when a degree 3 hypersurface in $\mathbb{P}(1,1,1,2)$ can be *non-positive* hypersurface by choosing a new GIT model of $\mathbb{P}(1,1,1,2)$

In [Wanb], one can show the above I-function is a slice on the Lagrangian cone of Y. After equaling Q^β with $Q^l x_0^{k_0} x_1^{k_1}$, this will identify $zI(q,z)$ with I_Y^S up to an exponential factor $e^{(t_1+t_2+t_3+3t_4+3t_5)P/z} e^{(t_1+t_2+t_3+3t_4+3t_5)l}$, but which can be offset by using the reconstruction theorem [Giv15].

The above discussion also applies to the case of a nonconvex hypersurface in a toric stack as shown below.

2. A quantum product computation

In this section, one recovers the quantum product computation by Corti for a cubic hypersurface Y which is cut off by the polynomial $x_1^3+x_2^3+x_3^3+x_4x_1$ in $\mathbb{P}(1,1,1,2)$. The following is the table for (small) quantum product of Y:

	1	p	p^2	$\mathbb{1}_{\frac{1}{2}}$
1	1	p	p^2	$\mathbb{1}_{\frac{1}{2}}$
p		$p^2+12r^2+3r\mathbb{1}_{\frac{1}{2}}$	$12r^2p$	rp
p^2			$108r^4+36r^3\mathbb{1}_{\frac{1}{2}}$	$12r^3$
$\mathbb{1}_{\frac{1}{2}}$				$\frac{1}{3}p^2-3r\mathbb{1}_{\frac{1}{2}}$

where $r=\frac{1}{2}q$ [1]. Due to the discussion in [MH14], the usual twisted $I-$function of $\mathbb{P}(1,1,1,2)$ only recovers the first two rows, and the rest two rows rely on Corti's key calculation

$$(\mathbb{1}_{\frac{1}{2}}\circ\mathbb{1}_{\frac{1}{2}},\mathbb{1}_{\frac{1}{2}})=\frac{-3}{2}r\ . \tag{2.1}$$

In the following, we will present two different GIT presentations which give two different (small) $I-$functions. We use the first $I-$function to verify some relations given by the table above and use the second $I-$function to recover Corti's key calculation.

2.1. **Example 1: positive case.** Choose the matrix

$$\rho=\begin{pmatrix}1&1&1&2&0&0\\0&1&1&1&1&0\\0&0&0&0&-1&1\end{pmatrix}:\mathbb{Z}^3\to\mathbb{Z}^6\ ,$$

which gives the action of $G:=\mathbb{C}_t^*\times\mathbb{C}_{t_1}^*\times\mathbb{C}_{t_2}^*$ on $W:=\mathbb{C}^6$ so that the GIT (stack) quotient is still $\mathbb{P}(1,1,1,2)$ (with choice of stability condition $\theta=tt_1^2t_2$). Consider the polynomial $z_1^3z_2^3x_1^3+x_2^3+x_3^3+x_4x_1z_1^2z_2^2$ which still cuts off Y in the new GIT stack quotient corresponding to the line bundle $L_{t^3t_1^3}$. The monoid of effective curve class

[1]In [MH14], they use $r=\frac{1}{2}q^{\frac{1}{2}}$, their $q^{\frac{1}{2}}$ corresponds our q here.

$\mathrm{Eff}(W,G,\theta)$ is generated by $\beta_1,\beta_2,\beta_3\in Hom(\chi(G),\mathbb{Q})$ such that

$$\begin{pmatrix}\beta_1(L_t) & \beta_1(L_{t_1}) & \beta_1(L_{t_2})\\ \beta_2(L_t) & \beta_2(L_{t_1}) & \beta_2(L_{t_2})\\ \beta_3(L_t) & \beta_3(L_{t_1}) & \beta_3(L_{t_2})\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & 0 & 0\\ -1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}\,.$$

Then the hypersurface Y comes from the positive line bundle $L_{t^3t_1^3}$ on $\mathbb{P}^1(1,1,1,2)$ as $\beta_i(L_{t^3t_1^3})\geqslant 0$ for all $i=1,2,3$. Then we can think $q:=q^{\beta_1}$ generates the semigroup of degrees of *stable maps* to the hypersurface Y, and we can treat $x:=q^{\beta_2}$ and $y:=q^{\beta_3}$ as formal variables.

Then the $I-$function for Y using this *GIT* model of $\mathbb{P}(1,1,1,2)$ is equal to

$$\begin{aligned}I(q,x,y,z)=&\sum_{(l,m,n)\in\mathbb{N}^3}\frac{q^lx^my^n}{z^{m+n}m!n!}\frac{\prod_{i<0}(p+(\frac{l}{2}-m-i)z)}{\prod_{i<\frac{l}{2}-m}(p+(\frac{l}{2}-m-i)z)}\\
&\frac{\prod_{i<0}(p+(\frac{l}{2}+n-i)z)^2}{\prod_{i<\frac{l}{2}+n}(p+(\frac{l}{2}+n-i)z)^2}\frac{\prod_{i<0}(2p+(l-m+n-i)z)}{\prod_{i<l-m+n}(2p+(l-m+n-i)z)}\\
&\prod_{0\leqslant i<\frac{3l}{2}+3n}\left(3p+(\frac{3l}{2}+3n-i)z\right)\mathbb{1}_{\frac{l}{2}}\,.\end{aligned}\tag{2.2}$$

Here $\mathbb{1}_{\frac{l}{2}}=\mathbb{1}_{\frac{1}{2}}$ if l is odd, otherwise $\mathbb{1}_{\frac{l}{2}}=\mathbb{1}$.

Let $L_{l,m,n}$ be the coefficient of $q^lx^my^n$ in $zI(q,x,y,z)$. Then we can show that

$L_{l,m,0}$	q^0	q^1	q^2	q^3
x^0	$z\mathbb{1}$	$\mathcal{O}(z^{-1})$	$\mathcal{O}(z^{-1})$	$\mathcal{O}(z^{-1})$
x^1	$2p^2$	$3\mathbb{1}_{\frac{1}{2}}$	$6\mathbb{1}+\frac{9p}{z}$	$\mathcal{O}(z^{-2})$
x^2	p^2z	0	$3p+\frac{21p^2}{2z}$	$\frac{105\mathbb{1}_{\frac{1}{2}}}{4z}$

Let $\mu(q,x,z)=[zI(q,x,0,z)-z]_+$ be the truncation of $zI(q,x,y,z)-z$ in positive z powers, thus we have that

$$\mu(q,x,z)=2xp+3qx\mathbb{1}_{\frac{1}{2}}+6q^2x\mathbb{1}+x^2p^2z+3x^2q^2p+\mathcal{O}(x^3)+\mathcal{O}(q^3)\,.$$

Then the mirror theorem proved in [Wanb] shows the following:

$$\begin{aligned}I(q,x,y,z)=&\mathbb{1}_Y+\frac{\mu(q,x,z)}{z}\\
&+\sum_{l\geqslant 0}\sum_{k\geqslant 0}\frac{q^l}{k!}\phi^\alpha\langle\mu(q,x,y,-\bar{\psi}_1),\cdots,\mu(q,x,y,-\bar{\psi}_k),\frac{\phi_\alpha}{z(z-\bar{\psi}_\star)}\rangle^\infty_{0,[k]\cup\star,l\beta_1}\,.\end{aligned}\tag{2.3}$$

Extract the coefficients of $q^3x^2z^{-2}$ term on both sides of the above equality, we have the following:

$$\begin{aligned}\frac{105\mathbb{1}_{\frac{1}{2}}}{4} &= \langle 3\mathbb{1}_{\frac{1}{2}}, 6\mathbb{1}, -\rangle_{q^0} + \frac{1}{2}\langle 3\mathbb{1}_{\frac{1}{2}}, 3\mathbb{1}_{\frac{1}{2}}, -\rangle_q \\ &+ \langle 2p^2, 6\mathbb{1}, -\rangle_q + \langle 3p, -\rangle_{q^1} + \langle 2p^2, 3\mathbb{1}_{\frac{1}{2}}, -\rangle_{q^2} \\ &+ \langle -p^2\bar{\psi}_1, -\rangle_{q^3} + \frac{1}{2}\langle 2p^2, 2p^2, -\rangle_{q^3}\end{aligned} \tag{2.4}$$

Here given $\alpha_i \in H^*(\bar{I}_\mu Y, \mathbb{Q})$ with $1 \leqslant i \leqslant k$

$$\langle \alpha_1\bar{\psi}_1^{m_1}, ..., \alpha_k\bar{\psi}_k^{m_k}, -\rangle_{q^l} = (\widetilde{ev_{m+1}})_*\left((\prod ev_i^*\alpha_i\bar{\psi}_i^{m_i}) \cap [\mathcal{K}_{0,k}(Y,\beta_1)]^{\mathrm{vir}}\right) .$$

When $k = 2$,$\langle \alpha_1, \alpha_2, -\rangle_{q^l}$ coincides with the coefficient of q^l in the small quantum product $\alpha_1 \circ \alpha_2$.

Now using the table above, the RHS of (2.4) is equal to

$$18\mathbb{1}_{\frac{1}{2}} - \frac{27}{4}\mathbb{1}_{\frac{1}{2}} + 0 + 9\mathbb{1}_{\frac{1}{2}} + 0 - 3\mathbb{1}_{\frac{1}{2}} + 9\mathbb{1}_{\frac{1}{2}} \tag{2.5}$$

Here, for the fourth term on the RHS of (2.4), we use divisor equation, and for the sixth term of the RHS of (2.4), first we use the divisor equation and topological equation to show that

$$\begin{aligned}\langle -p^2\bar{\psi}_1, -\rangle_{q^3} &= -\frac{2}{3}\langle p^2\bar{\psi}_1, p, -\rangle_{q^3} \\ &= -\frac{2}{3}(\langle p, \langle p^2, -\rangle_{q^0}, -\rangle_{q^3} + \langle p, \langle p^2, -\rangle_{q^1}, -\rangle_{q^2} \\ &+ \langle p, \langle p^2, -\rangle_{q^2}, -\rangle_{q^1} + \langle p, \langle p^2, -\rangle_{q^3}, -\rangle_{q^0}) ,\end{aligned} \tag{2.6}$$

then we use divisor equation again to show

$$\frac{l}{2}\langle p^2, -\rangle_{q^l} = \langle p^2, p, -\rangle_{q^l}$$

with $0 \leqslant l \leqslant 3$, using the table again, (2.6) becomes:

$$-3\mathbb{1}_{\frac{1}{2}} .$$

Now one can see the sum of (2.5) is equal to the LHS of (2.4). One would hope that only the first two rows of the table and equation (2.4) would recover Corti's calculation. However, the equation (2.4) also contains other unknown term such as $\langle 2p^2, 2p^2, -\rangle_{q^3}$ from the rest two rows of the table, one can't isolate $\frac{1}{2}\langle 3\mathbb{1}_{\frac{1}{2}}, 3\mathbb{1}_{\frac{1}{2}}, -\rangle_q$ from all known terms. We will recover Corti's calculation in the next example.

2.2. **Example 2: non-positive example.** In the following, we will recover Corti's key calculation using the $I-$function by choosing a different GIT presentation of $\mathbb{P}(1,1,1,2)$.

Choose the matrix

$$\rho = \begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^5 ,$$

which gives the action of $G := \mathbb{C}_t^* \times \mathbb{C}_z^*$ on $W := \mathbb{C}^5$ so that the GIT (stack) quotient is still $\mathbb{P}(1,1,1,2)$ (with the choice of stability condition $\theta = t^2z^3$, this also corresponds to the S-extended data $S=\{\frac{1}{2}\}$ in the sense of [CCIT14, CCIT15]). Consider the polynomial $zx_1^3+zx_2^3+zx_3^3+x_4x_1$, then it cuts off a hypersurface in the new GIT stack quotient $[W^{ss}(\theta)/G]$, which is isomorphic to Y. The monoid $\mathrm{Eff}(W,G,\theta)$ is generated by $\beta_1,\beta_2\in Hom(\chi(G),\mathbb{Q})$ such that

$$\begin{pmatrix}\beta_1(L_t) & \beta_1(L_z)\\ \beta_2(L_t) & \beta_2(L_z)\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & 0\\ -\frac{1}{2} & 1\end{pmatrix}\ .$$

Then we can think $q:=q^{\beta_1}$ generates the semigroup of degrees of *stable maps* to the hypersurface Y and $x:=q^{\beta_2}$ is a formal variable. Note this hypersurface doesn't come from a positive line bundle on $[W/G]$. Indeed in this case Y comes from the line bundle L_{t^3z} on $[W/G]$, so we have $\beta_2(L_{t^3z})<0$.

In this forthcoming work [Wana], we will generalize our mirror theorem to all hypersurfaces in toric stacks rather than positive hypersurfaces in [Wanb].As a result, the $I-$function of Y using this new GIT presentation of $\mathbb{P}(1,1,1,2)$ is

$$\begin{aligned}
I(q,x,z)=&\sum_{\substack{(l,k)\in\mathbb{N}^2\\ \frac{3l-k}{2}\geqslant 0}}\frac{q^lx^k}{z^kk!}\frac{\prod_{i<0}(p+(\frac{l-k}{2}-i)z)^3}{\prod_{i<\frac{l-k}{2}}(p+(\frac{l-k}{2}-i)z)^3}\\
&\frac{1}{\prod_{0\leqslant i<l}(2p+(l-i)z)}\prod_{0\leqslant i<\frac{3l-k}{2}}\left(3p+(\frac{3l-k}{2}-i)z\right)\mathbb{1}_{\frac{k-l}{2}}\\
&+\sum_{\substack{(l,k)\in\mathbb{N}^2\\ \frac{3l-k}{2}\in\mathbb{Z}_{<0}}}\frac{q^lx^k}{z^kk!}\frac{\prod_{\frac{l-k}{2}<i<0}(p+(\frac{l-k}{2}-i)z)^3}{\prod_{0\leqslant i<l}(2p+(l-i)z)}\\
&\frac{1}{\prod_{\frac{3l-k}{2}<i<0}\left(3p+(\frac{3l-k}{2}-i)z\right)}\frac{1}{3}p^2\\
&+\sum_{\substack{(l,k)\in\mathbb{N}^2\\ \frac{3l-k}{2}\in Q_{<0}\setminus\mathbb{Z}_{<0}}}\frac{q^lx^k}{z^kk!}\frac{\prod_{\frac{l-k}{2}<i<0}(p+(\frac{l-k}{2}-i)z)^3}{\prod_{0\leqslant i<l}(2p+(l-i)z)}\\
&\frac{1}{\prod_{\frac{3l-k}{2}<i<0}\left(3p+(\frac{3l-k}{2}-i)z\right)}\mathbb{1}_{\frac{k-l}{2}}\ .
\end{aligned}\tag{2.7}$$

Here $\mathbb{1}_{\frac{k-l}{2}}=\mathbb{1}_{\frac{1}{2}}$ if $k-l$ is odd, otherwise $\mathbb{1}_{\frac{k-l}{2}}=\mathbb{1}$. We can show the following fact about $I(q,x,z)$:

$$I(q,x,z)=\mathbb{1}+\frac{x\mathbb{1}_{\frac{1}{2}}+qx\mathbb{1}}{z}+\mathcal{O}(x^3)+\mathcal{O}(\frac{1}{z^2})\ ,\tag{2.8}$$

$$\frac{\partial I(q,x,z)}{\partial x}=\frac{\mathbb{1}_{\frac{1}{2}}+q\mathbb{1}}{z}+\frac{x(q^2\mathbb{1}+\frac{1}{3}p^2+\frac{q\mathbb{1}_{\frac{1}{2}}}{2})}{z^2}+\mathcal{O}(x^2)+\mathcal{O}(\frac{1}{z^3})\ ,\tag{2.9}$$

and

$$\frac{\partial^2 I(q,x,z)}{\partial^2 x} = \frac{q^2\mathbb{1} + \frac{1}{3}p^2 + \frac{q\mathbb{1}_{\frac{1}{2}}}{2}}{z^2} + \mathcal{O}(x) + \mathcal{O}(\frac{1}{z^3}) \ . \tag{2.10}$$

In [Wana], we will also show that $-ze^{\frac{qx}{z}} I(q,x,-z)$ is a slice on the Givental's cone. Note we have the asympotic expansion

$$ze^{\frac{-qx\mathbb{1}}{z}} I(q,x,z) = z\mathbb{1} + x\mathbb{1}_{\frac{1}{2}} + \mathcal{O}(x^3) + \mathcal{O}(\frac{1}{z}) \ . \tag{2.11}$$

Then

$$ze^{\frac{-qx\mathbb{1}}{z}} I(q,x,z) = J^{Giv}(q, x\mathbb{1}_{\frac{1}{2}}, z) + \mathcal{O}(x^3) \ , \tag{2.12}$$

where $J^{Giv}(q,\mathbf{t},z)$ is Givental's $J-$function which has an asymptotic expansion

$$z\mathbb{1} + \mathbf{t} + \mathcal{O}(\frac{1}{z}) \ , \tag{2.13}$$

and $\mathbf{t} = \sum t^\alpha \phi_\alpha \in H^*(\bar{I}_\mu Y, \mathbb{Q})$. We have the following standard fact about Givental's $J-$function (c.f. [Giv04]):

$$z\frac{\partial}{\partial t_\alpha}\frac{\partial}{\partial t_\beta} J^{Giv}(q,\mathbf{t},z) = \phi_\alpha \star_{\mathbf{t}} \phi_\beta + \mathcal{O}(z^{-1}) \ . \tag{2.14}$$

Now consider the function

$$z\frac{\partial^2}{\partial^2 x}\left(ze^{\frac{-qx\mathbb{1}}{z}} I(q,x,z)\right) \ , \tag{2.15}$$

a direction computation using product rule yields:

$$q^2 e^{\frac{-qx\mathbb{1}}{z}} I(q,x,z) - 2zqe^{\frac{-qx\mathbb{1}}{z}} \frac{\partial}{\partial x} I(q,x,z) + z^2 e^{\frac{-qx\mathbb{1}}{z}} \frac{\partial^2}{\partial^2 x} I(q,x,z) \ . \tag{2.16}$$

Apply (2.8), (2.9), (2.10) to the first term, second term and third term in (2.16), respectively, we have the following asymptotic expansion of (2.15):

$$q^2\mathbb{1} - 2q(\mathbb{1}_{\frac{1}{2}} + q\mathbb{1}) + (q^2\mathbb{1} + \frac{1}{3}p^2 + \frac{q\mathbb{1}_{\frac{1}{2}}}{2}) + \mathcal{O}(x) + \mathcal{O}(z^{-1}) \ . \tag{2.17}$$

On the other hand, using equation (2.12), (2.14), one has another asymptotic expansion about (2.15):

$$\mathbb{1}_{\frac{1}{2}} \star_x \mathbb{1}_{\frac{1}{2}} + \mathcal{O}(x) + \mathcal{O}(z^{-1}) \ . \tag{2.18}$$

Compare (2.17) and (2.18), after evaluating $x=0$ and ignoring all negative z powers, we have

$$\mathbb{1}_{\frac{1}{2}} \circ \mathbb{1}_{\frac{1}{2}} = \frac{1}{3}p^2 - \frac{3}{2}q\mathbb{1}_{\frac{1}{2}} \ ,$$

which recovers Corti's calculation[1] (2.1)!

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[1]Here the small quantum product $\circ$ is defined by the specialization of the big quantum product $\star_{\mathbf{t}}$ to $\mathbf{t} = 0$.

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Department of Mathematics, The Ohio State Universty, Columbus, OH 43210, USA
E-mail address: `wang.6410@osu.edu`