线弹性力学模型–3月12日

1 弹性力学基本假定

1. 连续性假设

所有物理量均为物体所占空间的连续函数。

2. 均匀性假设假设弹性物体是由同一类型的均匀材料组成的，物体各个部分的物理性质都是相同的。

3. 各向同性假设

假定物体在各个不同的方向上具有相同的物理性质，物体的弹性常数不随坐标方向变化。

4. 完全弹性假设

应力和应变之间存在一一对应关系，与时间及变形历史无关，满足虎克定律。

5. 小变形假设

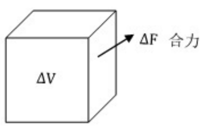
在弹性体的平衡问题讨论时，不考虑因变形所引起的几何尺寸变化，使用物体变形前的几何尺寸来替代变形后的尺寸。略去位移，应变和应力分量的高阶小量，使基本方程成为线性的偏微分方程。

2 弹性体的应力

2.1 分布力

单位体力（分布在物体内所有质点上的力）

$$F_b = \lim_{\Delta V \to 0} \frac{\Delta F}{\Delta V}$$



单位面力（作用在物体表面上的力）

$$F_s = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S}$$

2.2 集中力

集中力：ΔV或ΔS很小，但力很大。

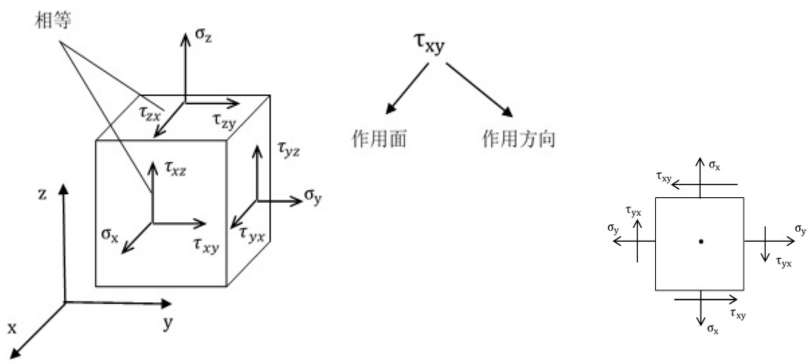
2.3 应力

应力是单位面力，是内力的平均，包括

- 正应力：$\sigma_x, \sigma_y, \sigma_z$.
- 剪应力分量6个：$\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}, \tau_{zx}, \tau_{zy}$

根据力矩平衡(参见下图右示意图，具体推导可参见弹性力学相关书籍)可以得到剪应力互等定理：

$$\begin{aligned}
\tau_{xy} &= \tau_{yx} \\
\tau_{yz} &= \tau_{zy} \\
\tau_{xz} &= \tau_{zx}
\end{aligned}$$



正应力和剪应力作为一个整体称为应力张量，写成

$$\sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

2.4 平衡方程

应力和体力满足如下平衡微分方程

$$
\begin{aligned}
\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+X=0\\
\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+Y=0\\
\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+Z=0
\end{aligned}
$$

3 弹性体的变形

3.1 应变

设u,v,w分别表示弹性体沿坐标轴x,y,z的位移分量。

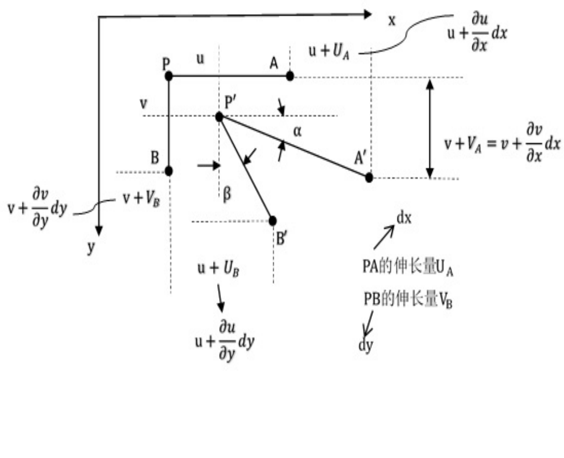
正应变（描述伸长）

$$\epsilon_x=\frac{\partial u}{\partial x},\epsilon_y=\frac{\partial v}{\partial y},\epsilon_z=\frac{\partial w}{\partial z}$$

剪应变（描述夹角变化）

$$\gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y},\gamma_{yz}=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z},\gamma_{zx}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}$$

正应变和切应变满足的上述方程称为几何方程，又称为柯西方程。下图是一个二维变形示意图。



3.2 应力与应变关系

各向同性材料应力和应变满足如下关系

$$\epsilon_x = \frac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)]$$
$$\epsilon_y = \frac{1}{E}[\sigma_y - \mu(\sigma_z + \sigma_x)]$$
$$\epsilon_z = \frac{1}{E}[\sigma_z - \mu(\sigma_x + \sigma_y)]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$
$$\gamma_{yz} = \frac{1}{G}\tau_{yz}$$
$$\gamma_{zx} = \frac{1}{G}\tau_{zx}$$

其中μ称为泊松比，表示横向正应变与纵向正应变的绝对值的比值，满足$0 < \mu < \frac{1}{2}$，与材料性质有关。$G = \frac{E}{2(1+\mu)}$表示剪切模量，是剪应力与剪应变的比值。可改写成如下形式，即虎克定律

$$\sigma_x = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}(\epsilon_x + \frac{\mu}{1-\mu}\epsilon_y + \frac{\mu}{1-\mu}\epsilon_z)$$
$$\sigma_y = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}(\frac{\mu}{1-\mu}\epsilon_x + \epsilon_y + \frac{\mu}{1-\mu}\epsilon_z)$$
$$\sigma_z = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}(\frac{\mu}{1-\mu}\epsilon_x + \frac{\mu}{1-\mu}\epsilon_y + \epsilon_z)$$
$$\tau_{xy} = \frac{E}{2(1+\mu)}\gamma_{xy}$$
$$\tau_{yz} = \frac{E}{2(1+\mu)}\gamma_{yz}$$
$$\tau_{zx} = \frac{E}{2(1+\mu)}\gamma_{zx}$$

3.3 边界条件

由平衡方程、几何方程和虎克定律可以建立弹性力学的基本方程。解这些方程还须给出边界条件，归结为如下三种类型

- 已知边界s上的位移$\bar{u}, \bar{v}, \bar{w}$，给出位移边界条件，

  $$u = \bar{u}, v = \bar{v}, w = \bar{w}(\text{在}s\text{上})$$

- 已知边界s上面力f，给出力边界条件

  $$\sigma \cdot n = f(\text{在}s\text{上})$$

  其中n是s的法向。

- 混合边界条件，部分给定面力，部分给定位移。

3.4 变形协调方程

$$
\begin{aligned}
&\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\
&\frac{\partial^2 \epsilon_z}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial z^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\
&\frac{\partial^2 \epsilon_x}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial x^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \\
&\frac{\partial}{\partial x}(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}) = 2\frac{\partial^2 \epsilon_x}{\partial y \partial z} \\
&\frac{\partial}{\partial y}(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}) = 2\frac{\partial^2 \epsilon_y}{\partial z \partial x} \\
&\frac{\partial}{\partial z}(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}) = 2\frac{\partial^2 \epsilon_z}{\partial x \partial y}
\end{aligned}
$$

4 平面问题

4.1 平面应变问题

满足

$$u = u(x, y), v = v(x, y), w = 0$$

根据几何方程有

$$\epsilon_x = \frac{\partial u}{\partial x} = \phi_1(x, y),\ \epsilon_y = \frac{\partial v}{\partial y} = \phi_2(x, y),\ \epsilon_z = \frac{\partial w}{\partial z} = 0,$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \phi_3(x, y),\ \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0,\ \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$$

物理方程

$$\tau_{yz} = \tau_{zx} = 0, \epsilon_z = \frac{1}{E}[\sigma_z - \mu(\sigma_x + \sigma_y)]$$

推出$\sigma_z = \mu(\sigma_x + \sigma_y)$。由此得到

$$
\begin{aligned}
\epsilon_x &= \frac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)] \\
\epsilon_y &= \frac{1}{E}[\sigma_y - \mu(\sigma_z + \sigma_x)] \\
\sigma_z &= \mu(\sigma_x + \sigma_y)
\end{aligned}
$$

等价于

$$
\begin{aligned}
\epsilon_x &= \frac{1+\mu}{E}[(1-\mu)\sigma_x - \mu\sigma_y] \\
\epsilon_y &= \frac{1+\mu}{E}[(1-\mu)\sigma_y - \mu\sigma_x] \\
\gamma_{xy} &= \frac{1}{G}\tau_{xy} = \frac{2(1+\mu)}{E}\tau_{xy}
\end{aligned}
$$

即

$$\sigma_x = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}(\epsilon_x + \frac{\mu}{1-\mu}\epsilon_y)$$
$$\sigma_y = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}(\frac{\mu}{1-\mu}\epsilon_x + \epsilon_y)$$
$$\tau_{xy} = \frac{E}{2(1+\mu)}\gamma_{xy} = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \cdot \frac{1-2\mu}{2(1-\mu)}\gamma_{xy}$$

4.2 平面应力

一个方向的尺寸远小于另外两个方向，如当板很薄时，满足

$$\sigma_z = \tau_{zx} = \tau_{zy} = 0$$

由$\sigma_z = 0$推出$\epsilon_z = \frac{1}{E}[\sigma_z - \mu(\sigma_x + \sigma_y)] = -\frac{\mu}{E}\mu(\sigma_x + \sigma_y)$。而且有

$$\gamma_{yz} = \frac{1}{G}\tau_{yz} = 0, \gamma_{zx} = \frac{1}{G}\tau_{zx} = 0$$

5 作业

证明变形协调方程的第一式和第四式

$$\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$
$$\frac{\partial}{\partial x}(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}) = 2\frac{\partial^2 \epsilon_x}{\partial y \partial z}$$