11th basic algebra

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kernel. ①

Change-of-basis ②

isomorphic (isomorphism) ③

linear map: $\mathcal{L}: U \to V$ correspond to a matrix A.

$f_1 . f_2$

linear maps between column vectors:

$\mathcal{L}: \mathbb{F}^n \to \mathbb{F}^k$

$\mathcal{L} \leftrightarrow \left(\begin{matrix} & \mathcal{L} \\ \Delta & \Gamma \end{matrix} \right)$

the correspondence respects addition and scalar multiplication.

$\underbrace{f_1(u)}_{v_1} + \underbrace{f_2(u)}_{v_2} = \underbrace{f_1 + f_2(u)}_{v_1 + v_2}$

$\underbrace{A_1 \cdot u}_{0} + A_2 \cdot u = (\underbrace{A_1 + A_2}_{A_2}) \cdot (u)$

The vector subspace of the domain U with $\underbrace{\mathcal{L}(u) = 0}_{A \cdot u = 0}$. which is called the kernel of $\mathcal{L}$ ($\ker \mathcal{L}$.)

corresponds to the null space of A.

The linear map $\mathcal{L}$ is one-one if and only if $\ker \mathcal{L} = 0$.

Corrollary: If $\mathcal{L}: U \to V$ is a linear map between finite-dimensional vector spaces over $\mathbb{F}$, then.

$$\dim(\text{domain}(\mathcal{L})) = \dim(\text{kernel}(\mathcal{L})) + \dim(\text{image}(\mathcal{L})).$$

Theorem: let $\mathcal{L}: U \to V$ and $\mathcal{M}: V \to W$. be linear maps between finite-dimensional vector spaces. and let Γ, Δ and Ω

$e + e + \cdots + e \neq 0$ $\mathbb{F}_p$

finite-dimensional vector spaces. and let Γ, Δ and Ω

$e+e+\cdots+e\neq 0$ $\mathbb{F}_p$

be ordered basis of U, V, W. $\quad ML: U\to W$

Then ~~the~~ composition ML is linear. the corresponding matrix is given by:

$$\underbrace{\begin{pmatrix} ML \\ \Omega\Gamma \end{pmatrix}}_{ML} = \underbrace{\begin{pmatrix} M \\ \Omega\Delta \end{pmatrix}}_{M}\begin{pmatrix} L \\ \Delta\Gamma \end{pmatrix}$$

$v = A_1\cdot u$

$w = A_2\cdot v$

$w = A_2\cdot A_1\cdot u$

proof:

$$\begin{pmatrix} ML \\ \Omega\Gamma \end{pmatrix}\begin{pmatrix} u \\ \Gamma \end{pmatrix} \overset{\text{theorem}}{=\!=} \begin{pmatrix} ML(u) \\ \Omega \end{pmatrix} \overset{\text{theorem}}{=} \begin{pmatrix} M \\ \Omega\Delta \end{pmatrix}\begin{pmatrix} L(u) \\ \Delta \end{pmatrix}$$

$M(L(u))$

$$= \begin{pmatrix} M \\ \Omega\Delta \end{pmatrix}\left[\begin{pmatrix} L \\ \Delta\Gamma \end{pmatrix}\begin{pmatrix} u \\ \Gamma \end{pmatrix}\right] = \left[\begin{pmatrix} M \\ \Omega\Delta \end{pmatrix}\begin{pmatrix} L \\ \Delta\Gamma \end{pmatrix}\right]\begin{pmatrix} u \\ \Gamma \end{pmatrix}$$

→ arbitrary.

$\mathbb{F}_2$

$(0,1,1)$ $(1,1,1)$

$(1,0,0)$ $(0,1,0)$ $(1,0,0)$

$(0,1,0)$ $(0,0,1)$ $(0,1,0)$

u

let u be the j^{th} member of Γ.

j^{th} column of $\begin{pmatrix} ML \\ \Omega\Gamma \end{pmatrix}$ = j^{th} column of $\begin{pmatrix} M \\ \Omega\Delta \end{pmatrix}\begin{pmatrix} L \\ \Delta\Gamma \end{pmatrix}$. $\square$

Change-of-basis

$$\begin{pmatrix} I \\ \Delta\Gamma \end{pmatrix}\begin{pmatrix} L \\ \Gamma\Gamma \end{pmatrix} = \begin{pmatrix} L \\ \Delta\Gamma \end{pmatrix}$$

$I: V\to U$ $\quad L: U\to U$ $\quad IL: V\to U$.

basis → basis

$\boxed{\begin{matrix}(0,1)\to(1,0)\\(1,0)\to(0,1)\end{matrix}}$

$y''(t) = y(t)$

L: d/dt as a linear map

$((c_1,c_2) = c_1(1,0) + c_2(0,1)$

$\Gamma = (e^t, e^{-t})$

$c_1e^t + c_2e^{-t}$,

$(1,0)$

$(0,1)$

$\Delta = (\cosh t, \sinh t)$

$\boxed{u = c_1e^t + c_2e^{-t}}$.

$v = L(u) = c_1e^t - c_2e^{-t}$. $c_1(1,0) + c_2(0,-1)$

(e_1, e_2)

$(0,1)\to e_1$

$(1,0)\to e_2$

$$\begin{pmatrix} d/dt \\ \Gamma\Gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$e^t = \cosh t + \sinh t$.

$$\begin{pmatrix} I \\ \Delta\Gamma \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$e^t = \cosh t + \sinh t$ $\quad \begin{pmatrix} I \\ \Delta P \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$e^{-t} = \cosh t - \sinh t$

$$\begin{pmatrix} d/dt \\ \Delta P \end{pmatrix} = \underline{\begin{pmatrix} I \\ \Delta P \end{pmatrix} \cdot \begin{pmatrix} d/dt \\ PP \end{pmatrix}} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{change-of-basis}$$

$$I = \begin{pmatrix} I \\ \Gamma\Gamma \end{pmatrix} = \underline{\begin{pmatrix} I \\ \Delta\Gamma \end{pmatrix}} \begin{pmatrix} I \\ \Gamma\Delta \end{pmatrix} \Leftarrow$$

If we are working on a linear map from a space of column vectors to itself: one ordered basis is the standard ordered basis $\underline{\Omega}$.

Another ordered basis Δ might be determined by linear map.

$\begin{pmatrix} I \\ \Sigma\Delta \end{pmatrix}$ $\quad$ eigenvector.

$$\Sigma : \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \qquad \begin{pmatrix} L \\ \Sigma\Sigma \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\Delta : \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \Leftarrow \qquad \begin{pmatrix} I \\ \Sigma\Delta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Leftarrow$$

$(1,0) = \underline{1}(1,0) + \underline{0}(0,1)$

$(1,1) = \underline{1}(1,0) + \underline{1}(0,1)$

$$\begin{pmatrix} L \\ \Delta\Delta \end{pmatrix} = \begin{pmatrix} I \\ \Delta\Sigma \end{pmatrix} \begin{pmatrix} L \\ \Sigma\Sigma \end{pmatrix} \begin{pmatrix} I \\ \Sigma\Delta \end{pmatrix} \Leftarrow.$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Leftarrow$$

proposition: Let $L: U \to V$ be a linear map on a finite-dimensional vector space, and let A be the matrix of L relative to an ordered basis Γ (in domain and range).

Then if any other ordered basis Δ, the matrix of L is of the form ordered basis Γ (in domain and range).

Then in any other ordered basis Δ, the matrix of L is of the form

$C^{-1} \cdot A \cdot C$. for some invertible matrix C. depending on Δ.

If A is a square matrix. the any square matrix of the form

$C^{-1}AC$ is said to be similar. is to: equivalence class.

1). $x \sim x$.

1) $C^{-1}AC$ is similar to $C^{-1}AC$.

2) $x \sim y$. $y \sim x$.

3). $x \sim y$. $y \sim z$. $\exists$. $x \sim z$

Two vector space U and V are said to be isomorphic if there is a one to one linear map of U onto V.

isomorphism.

$U \cong V$.

allowed to be infinite-dimensional.

16. October 2022.
3 pm (+8)