7th basic algebra

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matrix ①
matrix multiplications. ②
kronecker delta / identity matrix ③
invertible matrix. ④

Matrix operations.

A rectangular array of scalars (members of $\mathbb{F}$) with k rows and n columns is called a k-by-n matrix

A $\underline{k\text{-by-}n}$ matrix over $\mathbb{F}$ is a function from $\underline{\{1,\cdots k\}} \times \underline{\{1,\cdots n\}}$ to $\underline{\mathbb{F}}$.

size of the matrix

$(i,j) \Rightarrow \underline{a_{ij}}$

$\underline{a(i,j)}$: $(i,j)^{th}$ entry.

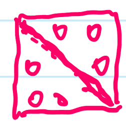
Two matrices are equal if they have the same size and their corresponding entries are equal.

$A = B$

$\cancel{|A| = |B|}$

A matrix is called square if its number of rows equals its number of columns.

A square matrix with all entries 0 for $i \neq j$ is called diagonal.
and the entries with $i=j$ are the diagonal entries



$\underset{\text{rows}}{2}$-by-$\underset{\text{columns.}}{3}$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

An n-dimensional row vector is a $\underline{1}$-by-$\underline{n}$ matrix,
column k-by-1 matrix

$\underline{\mathbb{F}^k}$
$\Uparrow$
$M_{k1}(\mathbb{F})$.

$(a_1, a_2, a_3). \Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

matrix A: $(i,j)^{th}$ entry $A_{ij} = [a_{ij}]_{\substack{i=1,\dots,k \\ j=1\dots n}} \Rightarrow [a_{ij}]$.

Let $M_{kn}(\mathbb{F})$ be the set of A k-by-n matrices with entries in $\mathbb{F}$.

Addition: whenever two matrices have the same size
it is defined entry by entry.
Thus if A and B are in $M_{kn}(\mathbb{F})$ then $A+B$ is in $M_{kn}(\mathbb{F})$

$$\boxed{(A+B)_{ij} = A_{ij} + B_{ij}}$$

$f(ab) = f(a)f(b)$

Scalar multiplication: is defined entry by entry

A is in $M_{kn}(\mathbb{F})$ c is in $\mathbb{F}$.
cA is the member of $M_{kn}(\mathbb{F})$
$$(cA)_{ij} = c \cdot A_{ij}$$

$(-1)\cdot A = -A$.

k-by-n matrix with 0 in each matrix is called zero matrix. simply denoted by 0.

Properties: $M_{kn}(\mathbb{F})$:

i). addition. A, B, C in $M_{kn}(\mathbb{F})$.

a) $A + (B+C) = (A+B)+C$ associative law
b) $A + 0 = 0 + A = A$
c) $A + (-A) = (-A) + A = 0$
d) $A + B = B + A$. commutative law

ii). scalar multiplication. $c, d, 1$ scalars.

a). $(cd)A = c(dA)$.
b) $1 \cdot A = A$.

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iii) two operation (distribute law)

a) $c\cdot(A+B) = cA + cB$

b) $(c+d)A = cA + dA$.

vector space.

Multiplication of matrices $\Leftarrow$ system of linear equation.

$$Ax = B$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}$$

A: k-by-m

B: m-by-n

$C = AB$: C: k-by-n.

$$\boxed{C_{ij} = \sum_{l=1}^{m} A_{il} B_{lj}}$$

$(i,j)^{th}$ entry of C is the product of i^{th} row of A and j^{th} column of B.

proposition: Matrix multiplication has the properties that.

a). associative. $(AB)\cdot C = A(BC)$ the size match correctly

$A: M_{km}(\mathbb{F})$ $B: M_{mn}(\mathbb{F})$ $C: M_{np}(\mathbb{F})$.

b). distributive over addition

$$A(B+C) = AB + AC. \checkmark$$
$$(B+C)D = BD + CD.$$

Matrix multiplication is not necessarily commutative. even for square matrix.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

proof: a). $((AB)C)_{ij} = \sum_{t=1}^{n} (AB)_{it} C_{tj} = \sum_{t=1}^{n}\sum_{s=1}^{m} A_{is} B_{st} C_{tj}$.

$(A(BC))_{ij} = \sum_{s=1}^{m} A_{is} \cdot (BC)_{sj} = \sum_{s=1}^{m}\sum_{t=1}^{n} A_{is} B_{st} C_{tj}$.

b). $(A(B+C))_{ij} = \sum A_{i\ldots}(B+C)_{\ldots}$ ...

$(A(BC))_{ij} = \sum_{s=1} A_{is} \cdot (BC)_{sj} = \sum_{s=1} \sum_{t=1} A_{is} B_{st} C_{tj}.$

b). $(A(B+C))_{ij} = \sum_{l} A_{il} (B+C)_{lj} \underset{\Delta}{=} \sum_{l} A_{il} (B_{lj} + C_{lj})$

$$= \sum_{l} A_{il} B_{lj} + \sum_{l} A_{il} C_{lj} = (AB)_{ij} + (AC)_{ij}.$$

the second identity is proved similarly. $\square$.

zero matrix: $0A = B \cdot 0 = 0$. if the sizes match correctly.

identity matrix: I or 1. $I_{ij} = \delta_{ij}$. Kronecker delta.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j. \end{cases} \qquad I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$I \cdot A = A$ $B \cdot I = B$ if the sizes match correctly.

A: $n \times n$.

We say that A is invertible and has n-by-n matrix B as inverse if

$$AB = BA = I.$$

If B and C are n-by-n matrices. with $AB = I$ $CA = I$.

$$B = IB = (CA)B = C \cdot (AB) = C \cdot I = C. \Rightarrow B = C$$

an inverse for A is unique if it exists A^{-1} if A is invertible. A^{-1} is invertible $(A^{-1})^{-1} = A$.

$\boxed{\text{if } A \text{ and } D \text{ are invertible: } AD \text{ is invertible}}$ $(AD)^{-1} = D^{-1} \cdot A^{-1}$.

computing the inverse of a matrix: row reduction suggests a way.

suppose A is a square matrix to be inverted. we are seeking B s.t.

$$\underline{A}\,\underline{B} = \underline{I}.$$

product of A and the first column of B = first column of I.

example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix} \quad \text{we set up} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 10 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{row reduction}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \end{array}\right).$$

$A = \begin{pmatrix} \cdots \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$ we set up $\left(\begin{array}{ccc|ccc} \cdots \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 10 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \end{array}\right)$

$\underset{I}{\uparrow} \quad \underset{\underline{B} = \underline{B}}{\uparrow}$ inverse of A.

<u>Does A^{-1} exist?</u>

proposition: Each elementary row operation is given by left multiplication by an invertible matrix.

The inverse matrix is the matrix of another elementary row operation.

The square matrices giving these left multiplications are called <u>elementary matrices</u>.

Proof: interchange of i and j.
$$\begin{array}{c|cc} & i & j \\ i & 0 & 1 \\ j & 1 & 0 \end{array}$$
This matrix is its own <u>inverse</u>.

multiplication of i^{th} row by a nonzero scalar c.
$$i \cdots \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & c & \\ 0 & & & \ddots \end{pmatrix}$$
the inverse of this matrix is of this form with c^{-1} in place of c.

replacement of i^{th} row by the sum of j^{th} row and the product of a times j^{th} row.
$$\begin{array}{c|cc} & i & j \\ i & 1 & a \\ j & 0 & 1 \end{array} \qquad \text{inverse: } \underline{\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}}$$
$\square$.

$\boxed{\text{Theorem}}$: TFAE. on n-by-n square matrix A.

a) the reduced row-echelon form of A is the identity.

b). A is the product of elementary matrices

c) A has an inverse

d) the system of equations $AX = 0$ with $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ has only the solution $X = 0$.

Proof: $a \Rightarrow b$.

choose a sequence of elementary row operations that reduce A to the identity.

choose a sequence of elementary row operations that reduce A to the identity.

let $E_1, \cdots E_r$ be the corresponding elementary matrices given by proposition.

$$E_r \cdot E_{r-1} \cdots E_1 A = I$$

is invertible.

$$A = \underline{E_1^{-1} \cdots E_r^{-1}}$$

and the inverse is also an elementary matrix.

hence b).

$b \Rightarrow c$

elementary matrices are invertible.

the product of elementary matrices is invertible. hence c)

$c \Rightarrow d$.

$AX = 0$

$$X = IX = (A^{-1} \cdot A) \cdot X = A^{-1} \cdot (\underline{AX}) = A^{-1} \cdot 0 = 0. \quad \text{hence d).}$$

$d \Rightarrow a$.

independent variables in the row reduction of A is 0.

corner variables is n. $\quad c.v. + i.v. = n$.

proposition shows that the reduced row-echelon form of A is I.

Thus. a). $\square$

Corollary: If the solution procedure for finding the inverse of a square matrix A leads from $(A|I)$ to $(I|X)$, then A is invertible and its inverse is X.

Conversely if the solution procedure leads to $(R|Y)$ and R has a row of 0's, then A is not invertible.

Proof. we apply the equivalence of (a) and (c) in proposition.

A is invertible.

If A^{-1} exists. X must be A^{-1}. $\square$

Corollary: Let A be a square matrix. If B is a square matrix s.t.

Corollary: Let A be a square matrix. If B is a square matrix s.t. $B \cdot A = I$.
then A is invertible and B is its inverse.
If C is a square matrix s.t.
$A \cdot C = I$
then A is invertible and C is its inverse.

proof: Suppose $BA = I$. let x be a column vector with $A \cdot x = 0$.

$$x = Ix = (BA)x = B(Ax) = B \cdot 0 = 0.$$

(d) implies (c) in theorem, A is invertible.

Suppose $AC = I$ similarly. A is invertible. $\square$.

4. June. 4pm