14th basic algebra

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First isomorphism theorem. ①

Second isomorphism theorem ②

external direct sum. ③

internal direct sum ④

proposition: Let $L: V \to W$ be a linear map between vector spaces over $\mathbb{F}$, let $U_0 = \ker L$, let U be a vector subspace of V, contained in U_0, let $q: V \to V/U$ be the quotient map. Then there exists a linear map, $\bar{L}: V/U \to W$ such that $L = \bar{L} q$. It has the same image as L and $\ker \bar{L} = \{u_0 + U, / u_0 \in U_0\}$. ✓

$U \subseteq U_0$

$L(v) = \bar{L}(q(v))$

$$\begin{tikzcd} v \quad V \arrow[r, "L"] \arrow[d, "q"'] & W \quad \begin{matrix} L(v) \\ \bar{L}(q(v)) \end{matrix} \\ q(v) \quad V/U \arrow[ur, dashed, "\bar{L}"'] & \end{tikzcd}$$

$v + U$

L factors through V/U

descends V/U

proof: $\bar{L}(v+U) = L(v)$ $\quad \bar{L}$: linear map $\begin{cases} \bar{L}(v_1+U+v_2+U) \\ = L(v_1+v_2) \\ L(v_1+U) + \bar{L}(v_2+U) = L(v_1) + L(v_2) \end{cases}$

L is a linear map $= L(v_1+v_2)$

$\bar{L}$ is well-defined.

suppose $v_1 \sim v_2$ $\quad \bar{L}(v_1+U) = \bar{L}(v_2+U)$

$v_1 - v_2$ in U $\quad L(v_1) = L(v_2)$

$L(v_1 - v_2) = 0$ then $L(v_1) = L(v_1 - v_2) + L(v_2)$

$\ker L = u_0$, $q(u_0) = u_0 + U. = \ker \bar{L}$. $\quad = 0 + L(v_2)$

$= L(v_2)$ $\square$

Corollary: let $L: V \to W$ be a linear map between vector spaces over $\mathbb{F}$. and suppose that L is onto W, and has kernel U. Then.

Corollary: let $L: V \to W$ be a linear map between vector spaces over $\mathbb{F}$. and suppose that L is onto $\underline{W}$. and has kernel $\underline{U}$. Then.

V/U is canonically isomorphic to W.

proof: take $U = V_0$ in proposition form $\bar{L}: V/U \to W$.

$L = \bar{L} \cdot q$.

$\bar{L}$ is onto W

$\ker \bar{L} = \{ \underline{u_0} + \underline{U} \mid \underline{u_0 \in U} \} = \{0 + U\}$. $\underline{\text{trivial kernel}}$.

$\bar{L}$ is $\underline{\text{one-to-one}}$. $\square$.

Theorem. FIT let $\underline{L:} V \to W$ be a $\underline{\text{linear map}}$ between vector spaces over $\mathbb{F}$. and suppose that L is $\underline{\text{onto } W}$ and has kernel U.

Then the map $S \mapsto L(S)$ gives a correspondence between

(a) the vector $\underline{\text{subspaces}}$ S of V $\underline{\text{containing } U}$ and

(b) the vector subspaces of W.

$L: S \to T$ $L(S)$ $L^{-1}(T)$

direct images inverse images.

proof: (a) $\to$ (b): L $L(S)$ is a vector subspace of W.

(b) $\to$ (a) L^{-1}

For any $\underline{\text{vector subspace}}$ $\underline{T}$ of W. $L^{-1}(T)$ is a vector subspace of V.

if $v_1, v_2 \in L^{-1}(T)$

$L(v_1) = t_1$, $L(v_2) = t_2$. $t_1, t_2 \in T$.

$\begin{cases} L(\underline{v_1+v_2}) = \underline{t_1+t_2}. \\ L(\underline{cv_1}) = cL(v_1) = \underline{ct_1} \end{cases}$ $v_1+v_2 \in L^{-1}T$ $cv_1 \in L^{-1}T$. L is a function

$\underline{L(\underbrace{L^{-1}(T)}_{v_1}) = T}$.

$\underline{L^{-1}(T)}$ contains $L^{-1}(0) = \underline{U}$. $\underline{(b)} \to \underline{(a)}$

If S is a vector subspace of V containing U. (a)

$\underline{S = L^{-1}(L(S))}$ Assume $\underline{v \in L^{-1}(L(S))}$

$S = L^{-1}(L(S))$. Assume $v \in L^{-1}(L(S))$

$S \subseteq L^{-1}(L(S))$ $\quad$ $L(v)$ is in $L(S)$

$L(v) = L(s)$, $s \in S$.

L linear: $L(v-s) = 0 \Rightarrow v-s \in \ker L = U \subseteq S$

$\Rightarrow v \in S \Rightarrow L^{-1}(L(S)) \subseteq S$. $\square$

V: vector space

V_1, V_2. vector subspace. $\quad V_1 + V_2 = \{v_1 + v_2 \mid v_1 \in V_1, v_2 \in V_2\}$. is a vector subspace of V.

sum of V_1 and V_2.

$V_1, \cdots V_n \quad (((V_1+V_2)+V_3)+\cdots V_n) \Rightarrow V_1+V_2+\cdots V_n$.

Theorem: Let M and N be vector subspaces of a vector space V over SIT. $\mathbb{F}$. Then the map: $n + (M \cap N) \to n + M$ is a well-defined canonical vector space isomorphism: $\subseteq N/M$.

$$N/(M \cap N) \cong (M+N)/M.$$

proof: $L(n + (M \cap N)) = n + M$. is well-defined.

$M \cap N \subseteq M$

L is linear: $L(n_1+n_2) = L(n_1) + L(n_2)$

$c \cdot L(n) = L(cn)$

domain of L. $\{n + (M \cap N) \mid n \in N\}$.

$\ker L$: n lies in M as well as in N.

$n \in M \cap N$

$\Rightarrow n + M \cap N = 0 + M \cap N$. kernel is 0 element of $N/(M \cap N)$.

L is one-to-one.

L is onto $(M+N)/M$, $\quad (m+n) + M \in (M+N)/M$.

M.

$L: n + (M \cap N) \to n + M = n + m + M$. $\quad L$. is onto. $\square$

$m + M$.

Corollary: Let M and N be finite-dimensional vector subspaces of a vector space V over $\mathbb{F}$. Then:

$$\dim(M+N) + \dim(M \cap N) = \dim M + \dim N.$$

vector space V over $\mathbb{F}$. Then:

$$\dim(M+N)+\dim(M\cap N)=\dim M+\dim N. \quad \triangle$$

proof. homework. $\square$

Direct sums and direct products of vector spaces.

V_1, V_2 vector spaces. external direct sum: $V_1 \oplus V_2$ is a vector space obtains as follows.

$$V_1 \times V_2 = \{(v_1, v_2) \mid v_1 \in V_1,\ v_2 \in V_2\}.$$

$$(\underline{v_1}, \underline{v_2}) + (\underline{u_1}, \underline{u_2}) = (\underline{v_1+u_1}, \underline{v_2+u_2}) \qquad V_1 \oplus V_2. \text{ is a vector space}$$

$$c\cdot(v_1, v_2) = (cv_1, cv_2).$$

If $\{a_i\}$ is the basis of V_1, $\{b_i\}$ is the basis of $\underline{V_2}$.

$$(v_1, v_2) = (v_1, 0) + (0, v_2)$$

$\{(a_i, 0)\} \cup \{(0, b_i)\}$ is the basis of $\underline{V_1 \oplus V_2}$.

$$\dim(V_1 \oplus V_2) = \dim V_1 + \dim V_2$$

Linear maps:

two projections $\quad p_1: V_1 \oplus V_2 \to V_1 \qquad p_1(v_1, v_2) = v_1$

$\quad p_2: V_1 \oplus V_2 \to V_2 \qquad p_2(v_1, v_2) = v_2$

two injections: $\quad i_1: V_1 \to V_1 \oplus V_2 \qquad i_1(\underline{v_1}) = (\underline{v_1, 0})$

$\quad i_2: V_2 \to V_1 \oplus V_2 \qquad i_2(\underline{v_2}) = (\underline{0, v_2})$

$$\Rightarrow. \qquad p_r i_s = \begin{cases} I & \text{on } V_s \text{ if } r=s. \\ 0 & \text{on } V_s \text{ if } r\neq s \end{cases}$$

$$i_1 p_1 + i_2 p_2 = I \quad \text{on } V_1 \oplus V_2.$$

$$\underline{i_1 p_1(v_1, v_2)} + i_2 p_2 \underline{(v_1, v_2)}$$

$$i_1(v_1) + i_2(v_2) = (v_1, 0) + (0, v_2) = \underline{(v_1, v_2)}$$

proposition: Let V be a vector space over $\mathbb{F}$ and let V_1 and V_2 be vector subspace of V. Then the following conditions are equivalent. (TFAE)

subspace of V. Then the following conditions are equivalent.
(TFAE)

(a) $\forall\ v \in V$ decomposes uniquely $v = v_1 + v_2$. $v_1 \in V_1$. $v_2 \in V_2$.

(b). $V_1 + V_2 = V$. $V_1 \cap V_2 = 0$ — V is the internal direct sum of V_1 and V_2. $V = V_1 \oplus V_2$

(c). $\perp$ (function from $V_1 \oplus V_2$ to V). $V \cong V_1 \oplus V_2$

$\perp: (v_1, v_2) \to v_1 + v_2$ is an isomorphism of vector spaces.

$\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$

Proof: (a) $\to$ (b) $V = V_1 + V_2 \Rightarrow V_1 + V_2 = V$.

If $v \in V_1 \cap V_2$. $0 = v + (-v)$ is a decomposition
$0 \in V$ $\qquad 0 = v_1 + v_2 \Rightarrow v = 0$
unique.

$V_1 \cap V_2 = 0$

it has.

(b) $\to$ (c). $v \in V$. $V_1 + V_2 = V \Rightarrow v = v_1 + v_2$. (c): $\perp$ is onto.

suppose $v_1 + v_2 = 0$ $\quad v_1 = -v_2$ $\quad v_1 \in V_1 \cap V_2 \overset{(b)}{\Rightarrow} V_1 \cap V_2 = 0$. $v_1 = 0$.
$v_1 \in V_2$ $\qquad v_1 = v_2 = 0$.
ker $\perp$. trivial. $\perp$ is one-to-one.

(c) $\to$ (a) $\perp$ is onto V. existence of the decomposition
uniqueness: suppose: $v_1 + v_2 = u_1 + u_2$ $\quad u_1, v_1 \in V_1$. $u_2, v_2 \in V_2$.
$\perp(v_1, v_2) = \perp(u_1, u_2)$
$\perp$ is one-to-one. $(v_1, v_2) = (u_1, u_2)$ $\square$

(d): There exist linear maps: $p_1: V \to V$. $p_2: V \to V$
i_1: image $p_1 \to V$ $\quad i_2$: image $p_2 \to V$. s.t.

i) $p_r i_s p_s = \begin{cases} p_r & \text{if } r = s \\ 0 & \text{if } r \neq s. \end{cases}$

universal mapping property. of $V_1 \oplus V_2$.

ii) $i_1 p_1 + i_2 p_2 = I$.

iii): $V_1 =$ image $i_1 p_1$. $\quad V_2 =$ image $i_2 p_2$

U is a vector space over $\mathbb{F}$:

$\perp_1: U \to V_1$

linear ... $\perp: U \to V$

U is a vector space over $\mathbb{F}$.

linear. $L_1: U \to V_1$ $\quad$ $L: U \to V$

$L_2: U \to V_2$

$L = i_1 \underline{L_1} + i_2 \underline{L_2}$

$L(u) = (\underline{i_1 L_1 + i_2 L_2}) \cdot (u) = (L_1(u), L_2(u))$.

$\Rightarrow L_1 = p_1 L. \quad L_2 = p_2 L.$

W is a vector space over $\mathbb{F}$.

linear: $M_1: V_1 \to W$ $\quad$ $M: V \to W$.

$M_2: V_2 \to W$. $\quad$ $M = M_1 p_1 + M_2 p_2$

$M(v_1, v_2) = M_1(v_1) + M_2(v_2)$.

$\Rightarrow M_1 = M i_1. \quad M_2 = M i_2.$

The external direct sum of $V_1 \oplus \cdots \oplus V_n$. is the set of ordered pairs.

$(v_1, \cdots v_n)$, $v_j \in V_j$, $j = 1, \cdots n$ $\quad$ coordinate by coordinate

$\dim(V_1 \oplus \cdots \oplus V_n) = \dim V_1 + \cdots + \dim V_n$.

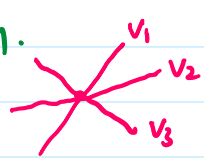
proposition: Let V be a vector space over $\mathbb{F}$. let $V_1, \cdots V_n$ be vector subspaces of V. TFAE:

(a). $v \in V$. $\exists$ uniquely decomposition: $v = v_1 + \cdots v_n$. $v_j \in V_j$ for $1 \le j \le n$.

(b). $V_1 + \cdots V_n = V$.

$V_j \cap (V_1 + \cdots V_{j-1} + V_{j+1} + \cdots + V_n) = 0$. for $1 \le j \le n$.

$\neq$. $V_i \cap V_j = 0$ for $i \neq j$. $\overset{\text{非}}{\Leftarrow}$



(c). $L: V_1 \oplus \cdots \oplus V_n \to V$.

$(v_1, \cdots v_n) \mapsto v_1 + \cdots v_n$. is an isomorphism of vector spaces.

V is internal direct sum of $V_1, \cdots V_n$ if any of ~~(a)~~. (b). (c). hold.

proof: similiar. $\square$.

$P_1 \cdots P_n$.

$i_1, \cdots i_n$.

universal mapping property.

back and forth.

Infinite many vector spaces. (different).

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4pm China.