6th abstract algebra

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K cycle. $\tau = (a_1, \cdots a_k)$ $\mathrm{Ord}(\tau) = k$

has order k

bijection $\in S_n$

$\tau \cdot \tau (a_1) = a_3$ $\tau^k(a_1) = a_1$

$a_1 \to a_2$ $a_2 \to a_3$

$a_2 \to a_3$ $a_3 \to a_4$ $\tau \cdot \tau(a_2) = a_4$ $\tau^k(a_2) = a_2$

$\star\ \tau^k = i$

Definition - Lemma: Let σ be any element of S_n.

Then σ may be expressed as a product of disjoint cycles. The factorisation is unique, ignoring 1-cycles, up to order.

The cycle type of σ is the length of corresponding cycles.

$\{a^0, a', a^2 \cdots a^k\}$ ($a^0 = e$, $a^k = e$)

proof: existence: $a_1 = 1$ $\{a_i \mid i \in \mathbb{N}\}$.

$a_{i+1} = \sigma(a_i)$ 1 to n: finite.

we must have repetitions. $a_i = a_j$ $i < j$.

smallest i, j for which this happens.

$i \neq 1$ $\sigma(a_{i-1}) = a_i = \sigma(a_{j-1}) = a_j$

(1,2,5,6,8,5)X

σ injective $a_{i-1} = a_{j-1}$ $i' = i-1$ $j' = j-1$ $a_{i'} = a_{j'}$

hence $i = 1$. $\tau^k = i$ τ^{k-1} contradiction.

$\tau = (a_1 \cdots a_j)$ k-cycle.

$\rho = \sigma\tau^{-1}$ fixes each element of the set $\{a_i, i \leq j\}$

$\sigma = \rho \cdot \tau$ (1 . 2, 5)

$\sigma = \rho \cdot \tau$ $(1, 2, 5)$ $\{1, 2, 5, 1\}$

By induction $k-1$ disjoint cycles. $\tau_1 \cdots \tau_{k-1}$ which fix this set.

$$\sigma = \rho\tau = \tau_1\tau_2\cdots\tau_k \quad \text{where } \tau = \tau_k.$$

uniqueness:

$\sigma = \sigma'$? $\sigma = \sigma_1 \cdots \sigma_k$, $\sigma' = \tau_1 \cdots \tau_k$ are two disjoint cycles.

$(1, 2, 5, 3)$

$\sigma_1(i) = j$ $\tau_p(i) = j$

$\sigma_1(j) = j_1$ $\tau_p(j) = j_1$ $\Rightarrow \sigma_1 = \tau_p$. □

Example: $\sigma = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{matrix}$ ~~(1 3)(2 4 5)~~

$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix}$ $(1,3)(3,2)$ transpositions.

$1.23 * 1.23 = 1.32$, $\begin{matrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{matrix} * \begin{matrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{matrix} = \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix}$

$(1,3)(2,4,5) = (2,4,5)(1,3) = (4,5,2)(1,3)$

cycle type: $(2, 3)$

Lemma: Let $\sigma \in S_n$ be a permutation, with cycle type $(k_1 \cdots k_l)$

The order of σ is the least common multiple of $k_1, k_2, \cdots k_l$.

Proof: Suppose $\mathrm{Ord}(\sigma) = k$ $\sigma = \underset{k_1}{\tau_1}\underset{k_2}{\tau_2}\cdots\underset{k_l}{\tau_l}$.

Pick an integer h. $\sigma^h = \tau_1^h\tau_2^h\cdots\tau_l^h$

identity if and only if τ_1^h identity, τ_2^h identity, $\vdots$, τ_l^h identity

in particular $\tau_i^k = e$ k_i divides k.

Least common multiple (LCM) m divides k $m = k$.

$\sigma^m = \tau_1^m \cdots \tau_l^m = e$ □

Lemma: The transpositions generate S_n.

Lemma: The transpositions (a_i, a_j) generate S_n.

Proof: It suffices to prove that every permutations is a product of transpositions.

① $\sigma = \tau_1 \cdots \tau_k$

we need cycle is a product of transpositions.

Consider the k-cycle $\sigma = (a_1 \cdots a_k)$. ↔ equal to

$\sigma = (a_1, a_k)(a_1, a_{k-1})(a_1, a_{k-2}) \cdots (a_1, a_2)$ ⇐

$(a_1, a_{i+1})(a_i, a_i)(a_1, a_{i-1})$

integer j, $\sigma(j)$ i). $j \neq a_i$, $i \in (1, k)$ nothing to check

ii) $j = a_i$ transposition $\sigma(j) = \sigma(a_i) = a_{i+1}$

$\forall j$. ↙ k-cycle: $\sigma(j) = \sigma(a_i) = a_{i+1}$

$$\begin{pmatrix} a_1 \cdots a_i \cdots a_k \\ a_k \; a_i \; a_1 \end{pmatrix}\begin{pmatrix} a_1 \; a_i \; a_{k-1} \; a_k \\ a_{k-1} \; a_i \; a_1 \; a_k \end{pmatrix}\begin{pmatrix} a_1 \cdots a_{k-2} \cdots \\ a_{k-2} \cdots a_1 \cdots \end{pmatrix} \cdots =$$

$\tau\sigma(i)$

$a_k \to a_1$

$a_{k-2} \to a_{k-1}$

Both sides have the same effect on j, hence they are equal □

② permutation: rearrangement of n numbers.

PoT. product of transpositions, send the rearrangement back to the trivial one.

inverse of PoT. = permutation.

$\tau_k \cdots \tau_3 \tau_2 \tau_1 \cdot \sigma = e$ $\quad \tau_k^2 = e \quad \tau_k^{-1} = \tau_k$

$\tau_k \cdots \tau_3 \tau_2 \tau_1 \cdot \sigma = e$ $\qquad \tau_k^2 = e \quad \tau_k^{-1} = \tau_k$

$\sigma = \tau_1 \tau_2 \cdots \tau_k$

$i-1$ numbers are in right position $\quad i \xrightarrow{\text{is at}} j$.

i back to right position. $\quad j > i$

By induction we are done. $\quad (i, j)$ □

Conjugation 共轭

Definition: Let g, h be two elements of a group G.

The element ghg^{-1} is called the conjugate of h by g.

$ghg^{-1} = h \Rightarrow gh = hg$ (Abelian group)

it measures how far a group G is from abelian.

G is abelian if and only if (iff) the conjugate of every element by any other element is the same element.

Lemma: Let σ and τ be two elements of S_n.

Suppose $\sigma = (a_1 \cdots a_k)(b_1 \cdots b_l) \cdots$ is the cycle decomposition (bijection) of σ

Then $(\tau(a_1), \tau(a_2) \cdots \tau(a_k))(\tau(b_1), \tau(b_2) \cdots \tau(b_l)) \cdots$ is the cycle decomposition of $\tau\sigma\tau^{-1}$ (the conjugate of σ by τ).

$\sigma^\tau = \tau\sigma\tau^{-1}$

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Proof: Both are permutations. It suffices to check that both sides have the same effect on integer j from 1 to n.

τ is surjective. $j = \tau(i)$ for some i.

Assume $j = \tau(a_1)$

$\sigma(a_1) = a_2$. $\sigma^\tau(\tau(a_1)) = \tau(a_2)$

$\tau\sigma\tau^{-1}(\tau(a_1)) = \tau\sigma(a_1) = \tau(a_2)$ □

Translation. $\sigma = (3,7,4,2)(1,6,5)$ in S_8

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 5 & 1 & 8 & 7 & 6 & 4 \end{pmatrix}$$

$\tau\sigma\tau^{-1} = (5,6,1,2)(3,7,8)$

Definition - Lemma. Let G be a group.

We say that two elements a and b are conjugate.

if there is a third element $g \in G$. such that.

$b = gag^{-1}$ the conjugate of a by g.

The corresponding relation $\sim$. is an equivalence relation.

proof: $\sim$. reflexive. symmetric, transitive.

1. $e \cdot a e^{-1} = a$. $a \sim a$.
2. $gag^{-1} = b$ $a \sim b \Rightarrow b \sim a$.

$ga = bg$

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$a = g^{-1}bg$ let $h = g^{-1}$ $a = \underline{hbh^{-1}}$

3. $b = gag^{-1}$ $\quad a \sim b \quad b \sim c \Rightarrow a \sim c$

$c = hbh^{-1} = h(gag^{-1})h^{-1} = \underline{(hg)}a\underline{(g^{-1}h^{-1})} = kak^{-1}$

$k = hg. \quad k^{-1} = (hg)^{-1} = g^{-1}h^{-1}$ □

Definition: The equivalence class of the equivalence relation above are called conjugacy classes.

Lemma. Let G be a group. Then the conjugy classes all have exactly one element iff G is abelian.

proof: $hg = gh \quad h = ghg^{-1} \quad h \in [h]$

Assume $b \in [h]$. $b \neq h$. $gbg^{-1} = h$

$gb = hg = gh$ Abelian group.

$b = h$. contradiction. □.

Proposition: The equivalence classes of the symmetric group S_n are precisely given by cycle type.

That is. two permutation σ and σ' are conjugate iff they have the same cycle type.

proof: $\Rightarrow \quad \underline{\sigma^{\tau} = \tau\sigma\tau^{-1}}$ they have the same cycle type from previous lemma.

from previous lemma.

$\Leftarrow$ suppose σ. σ' have the same cycle type.

need. τ sends σ to σ'

cycles have the same length.

τ sends cycle of σ to cycle of σ'.

Pick an integer j. $j \in$ cycle of σ

correspondence j' $j' \in$ cycle of σ'.

$\tau(j) = j'$.

$\tau\sigma\tau^{-1}(j') = \tau\sigma(j) = \tau(j+1) = (j+1)'$.

$\sigma'(j') = (j+1)'$ □

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