13th basic algebra

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contragredient ①

canonical map $V \to V''$ ②

Quotient of vector spaces ③

V' V

① $F(a+bi) = a$ $\mathbb{F}$

$L: U \to V$ is a linear map between finite-dimensional vector spaces.

$$(\overset{u'}{L^t(v')})(u) = v'(L(\overset{v}{u}))_{\mathbb{F}} \quad u \in U \quad v' \in V'.$$

defines a linear map $L^t: V' \to U'$ (contragredient of L).

$$\begin{tikzcd} U \arrow[r, "L"] \arrow[d, dashed] & V \arrow[d, dashed] \\ U' & V' \arrow[l, "L^t"] \end{tikzcd}$$

proposition: Let $L: U \to V$ be a linear map between finite-dimensional vector spaces. Let $L^t: V' \to U'$ be its contragredient. let Γ and Δ be the order bases of U and V. respectively. let Γ', Δ' be the order bases of U' and V', respectively.

Then $$\underbrace{\begin{pmatrix} L^t \\ \Gamma'\Delta' \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} L \\ \Delta\Gamma \end{pmatrix}}_{A}.$$

Proof: let $\Gamma = (u_1, \cdots u_n)$

$\Delta = (v_1, \cdots v_n)$

$\Gamma' = (u_1' \cdots u_n')$

$\Delta' = (v_1', \cdots v_n')$

$$L(u_j) = \sum_{i'=1}^{k} A_{i'j} v_{i'}. \qquad v'. \qquad A_{ij}.$$

$$L^t(v_i') = \sum_{j'=1}^{n} B_{j'i} u_{j'}'. \qquad u_j \qquad B_{ji}.$$

$$v_i'(L(u_j)) = v_i'\left(\sum_{i'=1}^{k} A_{i'j} v_{i'}\right) = A_{ij}.$$

$$L^t(v_i')(u_j) = \sum_{j'=1}^{n} B_{j'i} u_{j'}'(u_j) = B_{ji}.$$

$$B_{ji} = L^t(v_i')(u_j) \underset{def.}{=} v_i'(L(u_j)) = A_{ij} \qquad \square.$$

double dual $V'' = (V')'$.

$V = \mathbb{F}^n$. space of column vectors. ↙ dual
V': space of row vectors. ↙ dual.
V'': space of. column vectors.

$V \to V''$.

$\underline{\iota(v)}(v') = v'(v) \quad v \in V, \ v' \in V'$.

$\underline{\iota(v)}$ $\underline{\text{of } V''}$

ι: canonical map of V into V''.

proposition: If V is any finite-dimensional vector space over $\mathbb{F}$.
the $\underbrace{\iota}_{\text{canonical map}}$. $V \to V''$ is one-one onto.

infinite dimensional: one-one but not onto. Section 9.

proof. a linear map is one-one iff: $\ker \iota = 0$.

$\iota(\underline{v}) = 0 \qquad 0 = \iota(v)(v') = v'(v) \quad \forall \underline{\underline{v'}}$.

suppose $v \neq 0$ we can extend $\{v\}$. to a basis of V.

the linear functional v' that is 1 on v and is 0 on other members of. the basis. $\quad v'(v) = 1 \neq 0$. contradiction.

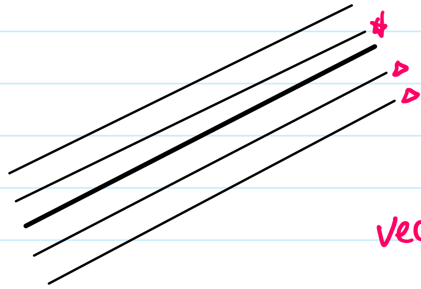
ι is one-one.

$$\dim V = \dim V' = \dim V''$$

ι carries any basis of V to a linearly independent set in V'', is a basis of V''. $\square$

Quotients of Vector Spaces.

$V = \mathbb{R}^2 \quad U$ be a line through the origin



The lines parallel to U are of the form
$\underline{v}+U=\{v+u \mid u\in U\}$.

We make a set of these lines into a vector space: $(v_1+U)+(v_2+U)=(v_1+v_2)+U$
$c\cdot(v+U)=c\cdot v+U$

Proposition: Let V be a vector space over $\mathbb{F}$. and let U be a vector subspace.
The relation defined by saying that $v_1 \sim v_2$ if v_1-v_2 is in U.
is an equivalence relation, and the equivalence classes are all sets of form $v+U$ with $v\in V$.
The set of equivalent classes V/U is a vector space under the definitions:

$$(v_1+U)+(v_2+U)=(v_1+v_2)+U$$
$$c\cdot(v+U)=cv+U$$

$V: (v_1,\cdots v_n)$
$U: (v_1,v_2,v_3,0,0\cdots0)$
$V/U: (0\ 0\ 0,v_4\cdots v_n)$

and the function $q(v)=v+U$ is linear from V onto V/U with kernel U.

V/U is the quotient space of V by U.

linear map $q(v)$ is called the quotient map of V onto V/U.

(v_1,v_2,v_3,v_4,v_5)
$U: (v_1,v_2,v_3,0,0)$
$\times$ $(0,1,0,1,0)$
$(0,0,1,0,1)$
$(0,1,1,1,1) \notin U$

$(0,1,0,1,0)$
$(0,0,1,1,0)$
$(0,1,1,0,0)\in U$

proof: reflexitive $x\sim x$
symmetry $x\sim y \Rightarrow y\sim x$
transitive $x\sim y,\ y\sim z \Rightarrow x\sim z$

$v_1\sim v_1$. v_1-v_1 is in U. 0 is in U ✓

$v_1\sim v_2 \Rightarrow v_2\sim v_1$. v_1-v_2 is in U
$\Rightarrow v_2-v_1$ is in U ✓
$-(v_1-v_2)\in V$.

$v_1\sim v_2,\ v_2\sim v_3 \Rightarrow v_1\sim v_3$. v_1-v_2 is in V, v_2-v_3 is in V. $v_1-v_2+v_2-v_3$ is in V. ✓
v_1-v_3

The class of v_1 consists all of v_2 s.t. v_2-v_1 is in U.
$v_2=v_1+U$

Equivalence classes are the sets $\underset{\sim\sim}{v+U}$

Addition. scalar multiplication: well defined.

$v_1 \sim w_1$ $v_1 - w_1$
$v_2 \sim w_2$ $v_2 - w_2$ are in $\underline{U}$ vector space. $(v_1+v_2)-(w_1+w_2)$ is in U.

$v_1+v_2 \sim w_1+w_2$ ✓

$v \sim w$ $v-w$ is in $\underline{U}$. $c\cdot(v-w) = cv - cw$ is in U

$c\cdot v \sim cw$ ✓

Association $((v_1+U)+(v_2+U))+(v_3+U) = ((v_1+v_2)+U)+(v_3+U)$

$(a+b)+c = a+(b+c)$ $= ((v_1+v_2)+v_3)+U)$

$= (v_1+(v_2+v_3))+U) = (v_1+U)+((v_2+v_3)+U)$

$= (v_1+U)+((v_2+U)+(v_3+U))$

$q: V \to V/U$ given by $q(v) = v+U$ is linear.

$q(v_1)+q(v_2) = q(v_1+v_2)$
$v_1+U + v_2+U = (v_1+v_2)+U$

kernel is $\{v \mid v+U = 0+U\} \Rightarrow \underline{\{v \mid v \in U\}}$.

V/U $v+U = q(v)$ onto

$\square$

Corollary: If V is a vector space over $\mathbb{F}$. and U is a vector subspace. then a). $\dim U = \dim V - \dim(V/U)$ infinite ✓ $\dim V = +\infty$ sum of $+\infty = +\infty$

b) $\underline{\text{the subspace } U}$ is the kernel of some linear map defined on V.

proof. a). homework.

b) q: quotient map.

$\square$.