8th abstract algebra

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proof: ① $\phi(e) = a$ identity: e : (G)

f (H)

$e \in G$ $a \in H$ $a^{-1} \in H$

$$a = \phi(e) = \phi(\underset{\substack{\uparrow \\ G}}{e \cdot e}) = \phi(e)\phi(e) = a \cdot a$$

$$a = a \cdot a \overset{a^{-1}}{\Rightarrow} a^{-1} \cdot a = f. = a^{-1} \cdot a \cdot a$$

$$a = f.$$

$\phi(e) = f.$

②. $b = a^{-1}$ $f = \phi(e) = \phi(a.b)$ $\phi(a), \phi(b), f \in H.$

$a, b \in G.$ $= \phi(a) \cdot \phi(b).$ $\phi(a) = [\phi(b)]^{-1}$

$f = \phi(e) = \phi(b \cdot a)$

$= \phi(b) \cdot \phi(a).$ $[\phi(a)]^{-1} = \phi(\underline{b}).$

$b = a^{-1} \Rightarrow [\phi(a)]^{-1} = \phi(\underline{a^{-1}})$

③. $\underline{K} \leq G.$ we need: X be a subgroup of H

$X = \phi(K)$

$e \subseteq K$ closed under multiplication and taking inverses

$f \subseteq X$ $\forall a \in K$ $\forall \phi(a) \in X$ (under taking inverses).

(identity) $\exists a^{-1} \in K$ $\exists [\phi(a)]^{-1} \in X$

$\phi(a) \in X$ $\phi(b) \in X.$ $\phi(a)\phi(b) \in X.$ (definition)

$ab \in K.$ $\phi(ab) \in X$ (under multiplication).

$\square$

Definition. Let $\phi: G \to H$ be a group homomorphism.

Lemma. The kernel of ϕ, denoted by $\mathrm{Ker}\,\phi$, is the inverse image of the identity.

Ker ϕ is a subgroup of G.

proof: $\phi(e) = f \Rightarrow$ Ker ϕ is non-empty.

Assume. $a, b \in \text{Ker}\,\phi. \Rightarrow \phi(a) = \phi(b) = f$

$$\underbrace{\phi(ab) = \phi(a)\cdot\phi(b)}_{\text{homomorphism definition}} = f\cdot f = f$$

$ab \in \text{Ker}\,\phi$. closed under multiplication.

Suppose: $\phi(a) = f$. $\quad \phi(a)^{-1} = f = \phi(\underline{a^{-1}})$ closed under taking inverses.

$a \in \text{Ker}\,\phi$. $\quad a^{-1} \in \text{Ker}\,\phi$.

Ker ϕ is a subgroup of G. $\square$

Lemma. Let $\phi: G \to H$ be a group homomorphism.

ϕ is injective iff $\text{Ker}\,\phi = \{e\}$.

$\phi(x) = \phi(y) \Rightarrow x = y$

proof: $\Rightarrow$ ϕ is injective. $f \in H$. $\exists$ unique element a: $\phi(a) = f$.

from lemma $\phi(e) = f$. $a = e$.

$\text{Ker}\,\phi = \{e\}$.

$\Leftarrow$ $\text{Ker}\,\phi = \{e\}$. $\underbrace{\phi(x) = \phi(y)}$

Let $g = x^{-1}y$. $g \in G$.

$\phi(g) = \phi(x^{-1}y) = \phi(x^{-1})\cdot\phi(y) = [\phi(x)]^{-1}\cdot\phi(y)$.

$= f$.

$\phi(g) = f \Rightarrow g \in \text{Ker}\,\phi. \Rightarrow g = e. \xRightarrow[g = x^{-1}y]{} x^{-1}y = e. \Rightarrow y = x.$

$x\cdot x^{-1}y = x\cdot e$

ϕ is injective. $\square$.

Definition: Let G be a group and let H be a subgroup of G.

We say that H is normal in G and write $H \lhd G$, $H \trianglelefteq G$.

if for every $g \in G$, $gHg^{-1} \subset H$.

$gHg^{-1} = H$ $\quad \phi(gHg^{-1})$

$\phi(g) \cdot \underbrace{\phi(H)}_{f} \cdot \phi(g^{-1}) = f$

Lemma: Let $\phi: G \to H$ be a group homomorphism. Then the kernel of ϕ is a normal subgroup of G.

$\operatorname{Ker} \phi \lhd G$.

proof: $\operatorname{Ker} \phi$ is a subgroup.

Suppose $g \in G$. we want $g \operatorname{Ker}\phi\, g^{-1} \subset \operatorname{Ker}\phi$.

suppose $\underline{h \in \operatorname{Ker}\phi}$ $\qquad g h g^{-1} \in \operatorname{Ker}\phi$. ☆

$$\phi(\underline{g}hg^{-1}) = \phi(g)\cdot\phi(h)\cdot\phi(g^{-1}) = \underline{\phi(g)}\cdot \underline{f}\,\underline{[\phi(g)]}^{-1} \in H.$$

$\phi(gh)\cdot\phi(g^{-1})$ $\qquad = \phi(g)\cdot[\phi(g)]^{-1}$

$ghg^{-1} \in \operatorname{Ker}\phi$. $\qquad = f$. $\quad \square$

Lemma. Let G be an abelian group and let H be any subgroup.

H is normal in G.

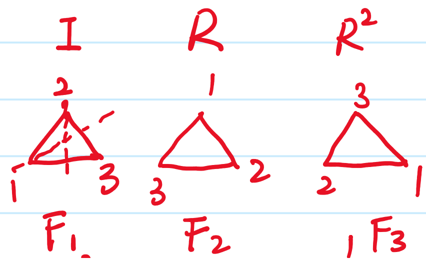
proof: $h \in H$. $\quad \underline{gh}g^{-1} = \underline{hg}\cdot g^{-1} = h\cdot e = h \in H$. $\quad \square$

$RF \neq FR$

$a \in H$. $\quad \underline{n}a = \underbrace{a+a+a+a}_{n}$

$n \in \mathbb{Z}$

not abelian: $G = D_3$. dihedral group.

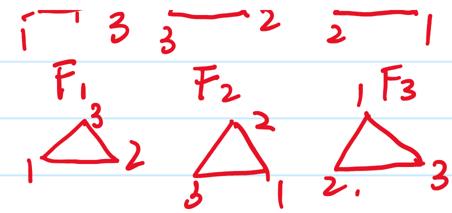


$H = \{I, R, R^2\}$ $\qquad F$

$G = D_3$. dihedral group

$H = \{I, R, R^2\}$. F

H is normal in G.



proof: $g \in D_3$. $gHg^{-1} \in H$. (want).

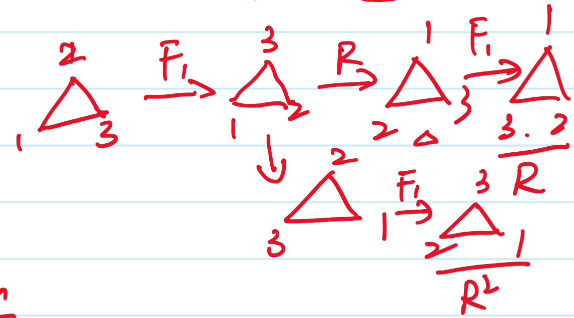
$\{I, R, R^2, F_1, F_2, F_3\}$.

① $g \in H$ $gHg^{-1} \in H$.

② $g \notin H$ $g = F_1$ $gIg^{-1} = I \in H$.

$gRg^{-1} = R \in H$

$gR^2g^{-1} = R^2 \in H$.



$ghg^{-1} \in H$ H is normal in G.

$R \cdot R^2 = I$

$H = \{I, F_1\}$. is not normal in G.

$h = F_1$, $g = R$. $ghg^{-1} = RF_1R^2 = F_2 \notin H$.

$gHg^{-1} \notin H$.

Lemma, Let H be a subgroup of a group G.

TFAE:

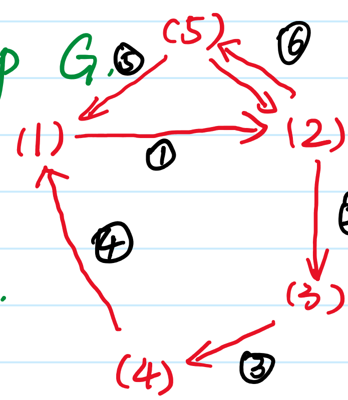
(1). H is normal in G.

(2). For every $g \in G$, $gHg^{-1} = H$.

(3). $Ha = aH$ for every $a \in G$.

(4). The set of left cosets is equal to the set of right cosets.

(5) H is a union of conjugacy classes.



proof. ① $gHg^{-1} \subset H$ $g \in G$. $g^{-1} \in G$.

$k = g^{-1} \in G$.

$(g^{-1})^{-1} H g^{-1} \subset H$

$k^{-1} H k \subset H$

$g^{-1}Hg \subset H$.

$gg^{-1}Hg \subset gH$

$H = gHg^{-1}$

$H \subset gHg^{-1}$.

$gg^{-1}Hg \subset gH\ldots$

$H \subset gHg^{-1}.$

② $aHa^{-1} = H.$ $\quad a \in G.$

$aH = Ha.$

③ obviously.

④. Let $g \in G$. $g \in gH$ $\quad$ If the set of left cosets is

$g \in Hg.$ $\quad$ equal to the set of right cosets.

$Hg = gH.$

$H \cdot gg^{-1} = gHg^{-1} \Rightarrow H = gHg^{-1} \Rightarrow gHg^{-1} \subset H.$ $\quad$ hence (1)

⑤ $H = \bigcup A_i.$ A_i are conjugacy classes.

$$gHg^{-1} = \bigcup g\underbrace{A_i}_{aba^{-1}}g^{-1}.$$

$$= \bigcup A_i$$

$$= H.$$ hence H is normal.

⑥ $gHg^{-1} = H.$ Suppose $a \in H$. $A = [a]$. conjugacy class.

$\forall\, b \in \underbrace{[a]}_{A}.$ $\qquad A \subset H.$

$\exists g.$ $gag^{-1} = b$ $\qquad b \in gHg^{-1} = H.$

Then H is a union of conjugacy classes. $\square$

S_3: $H = \{e, (1,2,3), (1,3,2)\}$. union of conjugacy classes.

conjugacy classes of S_n are entirely determined by cycle type

H is normal.

$H = \{e, \underbrace{(1,2)}_{(1,2)(1,2)=e}\}$. let $g = (2,3)$

$gHg^{-1} = \{\underline{geg^{-1}}, \underbrace{g(1,2)g^{-1}}_{(1,2,3)(2,3) = (1,3)}\} = \{e, (1,3)\} \not\subset H$.

H is normal.

$G = S_4$, $H = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$.

H is a subgroup of G.

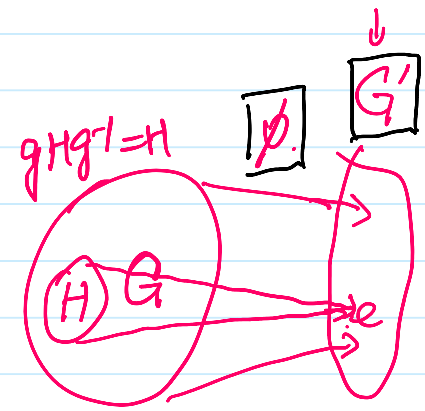
H is a union of conjugacy classes. 2, 2

H is normal in G.

9.

Quotient Group.

Ker of ϕ.



Given a group G and a subgroup H.

under what circumstances. $\phi: G \to \underline{G'}$ such that. H is the kernel of ϕ.

necessary condition: H is normal in G.

sufficient condition: H is normal in G

Idea:

Put X be a collection of left cosets of H in G.

$$\phi: G \to \underline{X}.$$

$(g) \quad [g] = gH$

$$\phi: G \to \underline{X}.$$
$$\phi(g) = [g] = gH.$$

Is X a group?

$[a] = aH$. multiplication: $[a][b] = (ab)H$.

$[b] = bH$. $= \underline{[ab]}$.

well-defined?

absurd. mapping.

Y all the people

$\sim$. have the same color eyes.

$Y/\sim$. all possible color of people's eyes.

$f: Y/\sim \to R$ (height. age).

[black eyes.] $\to 1.8$, $\to 1.7$

[yellow eyes] $\to 1.8$, $\to 1.7$

$aH = a'H$ $[a] = [a']$

$bH = b'H$ $[b] = [b']$

$a'H \cdot b'H = a'b'H. \Rightarrow [a'b'] = [ab].$

$aH \cdot bH = abH$

$a'b' = abH.$

$\underline{a'b' = ab \cdot h'}$ $h' \in H.$

$a' = ah$ $h \in H.$

$b' = bk$ $k \in H.$

$\underline{a'b' = ah\,bk}.$

$\underline{ahbk = abh'}$ X going to fail.

$b^{-1}Hb \subset H.$

$b^{-1}hb = l \in H. \Rightarrow$

$\underline{hb} = \underline{bl}.$

$hbk = bh'$

$\underline{b^{-1}hbk} = h'.$

$l\ k = h'.$

only if $\underline{aH = Ha}. \Rightarrow H$ is normal.

such that $\ker\phi$ is H. $\square$.

$(aH)(bH) = (ab)H.$

2021.11.27. 16:00 (china)