6th basic algebra

2022年5月21日 星期六 上午10:52

permutation ①
row reduction / gaussian elimination. ✓ ②
homogeneous / inhomogeneous system ③
row-echelon form ④

proof: a) from definition $sgn(1) = +1$

b). $\sigma = \tau_1 \tau_2 \cdots \tau_k$

$sgn(\tau_1 \cdots \tau_k) = -1 \; sgn(\tau_1 \cdots \tau_{k-1})$ from lemma *

$= (-1)^2 sgn(\tau_1 \cdots \tau_{k-2})$

$\vdots$

$= (-1)^k sgn\, 1 = (-1)^k$ from a)

c) σ is the product of k transpositions.
τ is the product of l transpositions.
$\sigma\tau$ is the product of $k+l$ transpositions.
from b). we have c).

d). from c). $sgn(\sigma)\, sgn(\tau) = sgn(\sigma\tau)$

let $\tau = \sigma^{-1}$, $\underbrace{sgn(\sigma)}_{\pm}\, \underbrace{sgn(\sigma^{-1})}_{\pm} = sgn\, 1 \overset{a)}{=\!=} 1$ $\Rightarrow sgn(\sigma) = sgn(\sigma^{-1})$. □

S. $\{1, 2, 3, \cdots n\}$.

S'. $\{xxx, \quad xxx\}$

$\varphi: S \to S'$ one-to-one onto function.

σ: permutations of $\overset{n}{S}$.

$\varphi^{-1}\sigma\varphi$: permutation of S'.

$sgn_\varphi(\sigma) = sgn(\varphi^{-1}\sigma\varphi)$.

whether this sgn is indepent of φ.

$\psi: S \to S'$ is a second one-to-one onto function.

$$sgn_\psi(\sigma) = sgn(\psi^{-1}\sigma\psi) \qquad \varphi^{-1}\cdot\psi = \tau \text{ is a permutation of } \{1,2,\dots,n\}.$$

from c). d).

$$sgn_\psi(\sigma) = sgn(\psi^{-1}\sigma\psi) = sgn(\psi^{-1}\varphi\cdot\varphi^{-1}\sigma\varphi\varphi^{-1}\psi)$$

$$= sgn(\tau^{-1})\, sgn(\varphi^{-1}\sigma\varphi)\, sgn(\tau)$$

$$d) = sgn(\tau)\cdot sgn_\varphi(\sigma)\, sgn(\tau). = sgn_\varphi(\sigma).$$

the resulting signs of permutations are independent. in the way we enumerate. the set.

Row reduction: (algorithm used for solving simultaneous systems ⋯)

let $\mathbb{F}$ denote $\mathbb{Q}$. $\mathbb{R}$, $\mathbb{C}$. The members of $\mathbb{F}$ are called scalars.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \overset{+b_1'}{=} b_1$$

$$\vdots$$

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k$$

a_{ij}. b_i are know scalars.

x_j are the unknowns or variables.

row reduction repeated use of three operations.

each of which perserves the set of solutions:

(i). interchange two ~~equations.~~ rows.

(ii) multiply an ~~equation~~ row by a nonzero scalar.

(iii) replace an ~~equation~~ row by the sum of it and a multiple of some other ~~equation.~~ row.

elementary row operation.

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$b_1 - b_1'$

We can work with an array of the form

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ & & \ddots & & \\ a_{k1} & a_{k2} & \cdots & a_{kn} & b_k \end{array}\right) \quad \begin{array}{l} k \text{ rows} \\ m = n+1 \text{ columns.} \end{array}$$

b_1' $\quad b_1 - b_1'$

$b_1 = \cdots = b_k = 0$ homogeneous.

the individual scalars are called entries. a_{ij}. b_i.

An array with k rows and m columns is in <u>reduced row-echelon form</u> if it meets several conditions.

- Each member of the first l of the rows, for some l with $0 \le l \le k$, has at least one nonzero entry, and other rows have all entries 0.
- Each of the nonzero rows has 1 as its first nonzero entry.
  we say that the i^{th} nonzero row has this 1 in its $j(i)^{th}$ entry.
- The integers $j(i)$ are to be strictly increasing as a function of i.
- the only entry in the $j(i)^{th}$ column that is nonzero is to be the one in the i^{th} row.

$$\begin{pmatrix} 0 & 0 & 2 & 7 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & -4 & 5 \\ -2 & 2 & -5 & 4 \end{pmatrix} \xrightarrow{(i)} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 2 & 7 \\ -1 & 1 & -4 & 5 \\ -2 & 2 & -5 & 4 \end{pmatrix} \xrightarrow{(iii)} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & -3 & 6 \end{pmatrix} \xrightarrow{(ii)} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & \frac{7}{2} \\ 0 & 0 & -3 & 6 \\ 0 & 0 & -3 & 6 \end{pmatrix}$$

$$\xrightarrow{(iii)} \begin{pmatrix} 1 & -1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{7}{2} \\ 0 & 0 & 0 & \frac{33}{2} \\ 0 & 0 & 0 & \frac{33}{2} \end{pmatrix} \xrightarrow{(ii)} \begin{pmatrix} 1 & -1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{7}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{33}{2} \end{pmatrix} \xrightarrow{(iii)} \begin{pmatrix} 1 & -1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{7}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

reduced row-echelon form

$j(i)$. $\quad j(1) = 1 \quad j(2) = 3 \quad j(3) = 4 \qquad l = 3.$

1 column that is nonzero $\to$ 1
3 $\to$ 2
4 $\to$ 3

$3 \to 2$
$4 \to 3$

rows are understood to be horizonal.
columns vertical.

Proposition: Any array with k rows and m columns can be transfered. into reduced row-echelon form by a succession of steps of types (i), (ii), (iii).

The example makes clear what the algorithm is that proves. proposition.

The corner variables are those x_j's for which $j \le n$ and is some $j(i)$ in the definition of "reduced row-echelon form".

The independent variables are other x_js. with $j \le n$.

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \text{ or } 0 \end{array}\right)$$

4 rows
5 columns.

If the lower right entry is 1. there are no solutions.
$0 = 1$ which announces a contradiction.

If the lower right entry is 0,
Restore the reduced array to the system of equations.

$$\begin{aligned} x_1 - x_2 &= 1 \\ x_3 &= 2 \\ x_4 &= 3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 1 + x_2 \\ x_3 &= 2 \\ x_4 &= 3. \end{aligned}$$

we collect everything in a tidy fashion as.

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

independent variables are allowed to take arbitrary values.

The method in the example works completely generally.

proposition: In the solution process of a system of k linear equations in n variables with the vertical line in place.

(a) the sum of the number of corner variables and the number of independent variables is n.

(b). the number of corner variables equals the number of nonzero rows on the left side of the vertical line and hence is $\leq k$.

(c) when solutions exist, they are of the form.

$$\text{column} + \underset{x_j}{\text{in. va.}} \times \underset{j=1}{\underline{\text{column}}} + \dots + \underset{x_j}{\text{in. va.}} \times \text{column}. \quad j=0$$

in such a way that each independent variables x_j is a free parameters in $\mathbb{F}$.

the column multipling x_j has a 1 in its j^{th} entry. and the other columns have a 0 in that entry.

(d). a homogeneous system: one with all right sides equal to 0.

has a nonzero solution if the number k of equation is < the number n of variables.

(e) the solution of an inhomogeneous system. ---- are not necessarily all 0.

are all given by the sum of any one particular solution and an arbitrary solution of the corresponding homogenous system.

an arbitrary solution of the corresponding homogeneous system.

proof. a). b). c). follow immediately by inspection of the solution method.

d). no contradiction equation can arise. when the right side are 0. in addition, there must be at least one in. va. by (a) since (b). shows co. va. $\le k < n$.

e). from c). since first column in the solution written in c) is a column of 0's in homogeneous case. $\square$

Proposition: For an array with k rows and n columns in reduced row-echelon form:

a) co. va. + in. va. $= n$.

b). the number of corner variables equals the number of nonzero rows and hence $\le k$.

c). $k = n$. either the array is of the form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \end{pmatrix}$$ or else it has a row of 0's.

proof: c). failure of the reduced row-echelon form to be some. non-corner variable. so that the number of corner variables is $< n = k$. from b). the number of nonzero rows $< n$. hence there is a row of 0's. $\square$

Cramer's rule. $x_1 = \frac{|D_1|}{|D|}$, $x_2 = \frac{|D_2|}{|D|}$, $\cdots$ $x_n = \frac{|D_n|}{|D|}$ ← determinants.

will be discussed.

n large. more lengthy process.

the number of steps for solving a system via row reduction: multiple of n^3

a system via row reduction: multiple of n^3.

via Cramer's rule: multiple of n^4.

28. May. 4 pm.