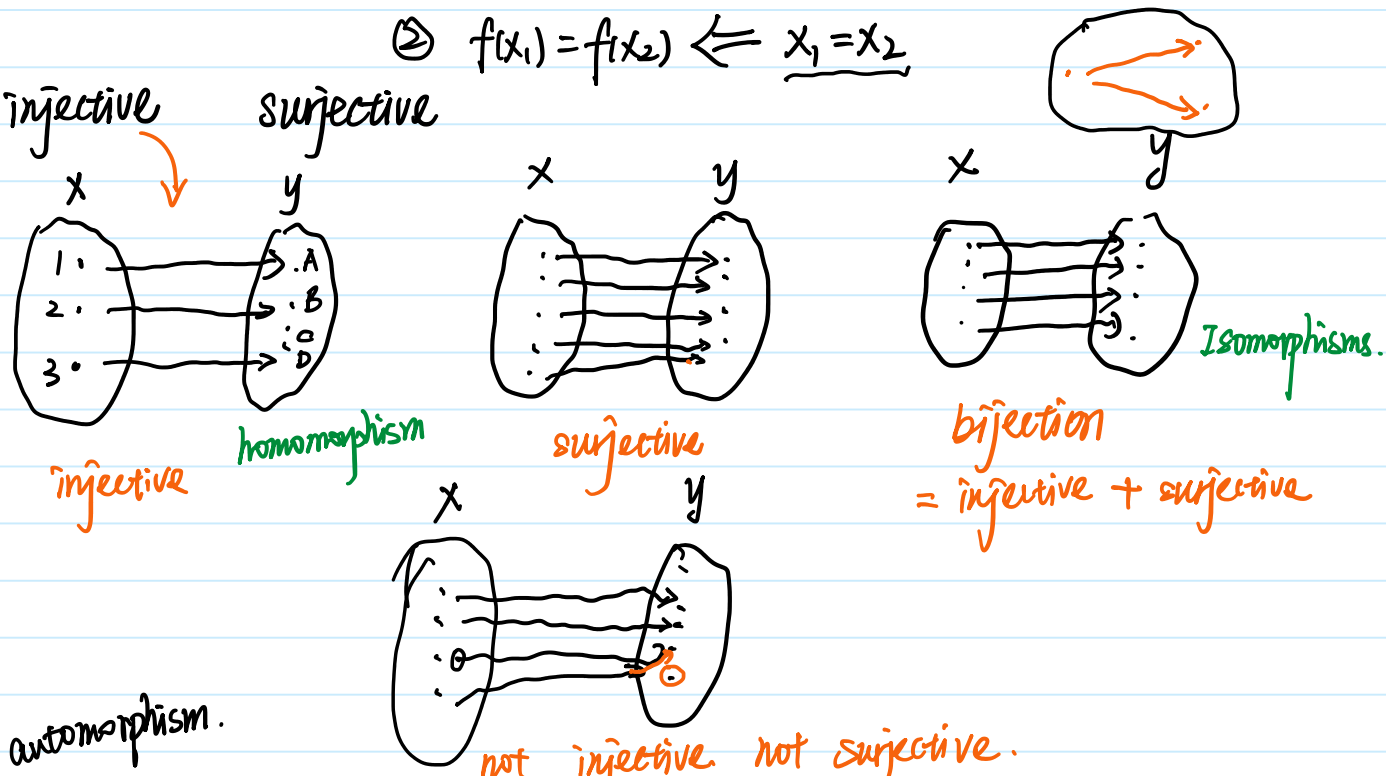
7th abstract algebra

Saturday, 30. October 2021 13:04

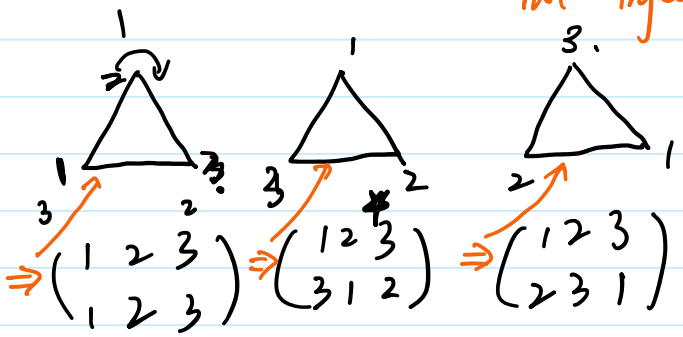
Isomorphisms.

function: x, y. ① $\forall x, \exists y$ s.t. $f(x)=y$ ✱

② $f(x_1)=f(x_2) \Leftarrow x_1=x_2$



automorphism.



$D_3 \cong S_3$.

Isomorphism.

$g: \circlearrowright_{60°}$ $h: \circlearrowright_{60°}$ $gh \circlearrowright_{120°}$

$\phi(g): \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$ $\phi(h) \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$ $\phi(gh) \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$

$\phi(gh)=\phi(g)\phi(h)$

Definition: Let G and H be two groups. We say that G and H are isomorphic if there is a bijective map:

$\phi: G \to H$. w.r.t. the group structure

For every g and h in G:

For every g and h in G:

$\phi(gh) = \phi(g)\phi(h)$. ← energyption

map ϕ is called an isomorphism. $\phi(g) \neq g$

Lemma: Let G and H be two cyclic groups of the same order.

$G \cong H$.

proof: Let a be a generator of G

Let b be a generator of H.

$\phi: G \to H$ $\forall g \in G$. $g = a^i$. $\phi(a^i) = b^i$.

$g' = b^i$

① well-defined.

G is infinite. H is infinite.

$\forall g \in G$. $\exists$ unique representation $\exists$ unique b^i.

a^i

G is finite. order is k. H

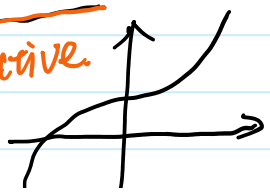
$g = a^j \Rightarrow b^j$ want. $a^i = a^j \Rightarrow b^i = b^j$

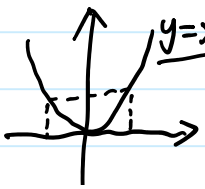
$g = a^i \Rightarrow b^i$.

$a^{i-j} = e$ k must divide $i-j$. $\Rightarrow b^{i-j} = e \Rightarrow b^i = b^j$.

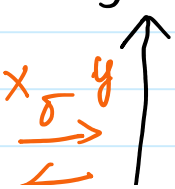
② bijection. $H \to G$ has inverse $\Leftrightarrow$ ϕ is a bijection.

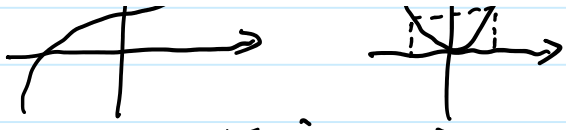
injective.





$\forall \phi(g) = g$





$\sigma \cdot \phi(g) = g$

$H \to G$: $\sigma(b^i) = a^i$ is the inverse of ϕ.

$\phi(a^i) = b^i \cdot \Leftrightarrow \phi$ is a bijection

③. $g = a^i$. $h = a^j$. $gh = a^{i+j}$.

$\phi(g) = b^i$ $\phi(h) = b^j$. $\phi(gh) = b^{i+j}$.

$\phi(g) \cdot \phi(h) = \phi(gh)$ □

Lemma: The group of real numbers under addition and positive real numbers under multiplication are isomorphic.

$+$ $-$ $\times$ $\div$ $\frac{10}{5} = \frac{2}{1}$

basic algebra.

ring: 不定. Field: 一定. { module over ring; vector space over field.

proof: G. H. $\phi: G \to H$. $\forall x \in G$.

$\phi(x) = e^x$.

① well-defined: obviously

② bijection.

③ $x, y \in G$

$\phi(x) = e^x$. $\phi(y) = e^y$ $\phi(x+y) = e^{x+y}$.

$e^x \cdot e^y = e^{x+y}$

$\phi(x+y) = \phi(x)\phi(y)$. □

Definition: $G \cong G$. automorphism.

Lemma: Let G be a group, $a \in G$ be an element of G.

$\phi: G \to G$.

$\phi(x) = axa^{-1}$ is an automorphism of G.

Proof: ① well-defined.

$x_1 = x_2$

$a^{-1}ax_1a^{-1}a = a^{-1}ax_2a^{-1}a$

② bijection: $\psi(x) = a^{-1}xa$. $G \to G$.

$\psi(\phi(x)) = \psi(axa^{-1}) = a^{-1}axa^{-1}a$

$= x$.

③ $\phi(x)\phi(y) = \phi(xy)$.

$axa^{-1}aya^{-1} = axya^{-1}$ □

Theorem: Let G be a group.

Cayley's theorem

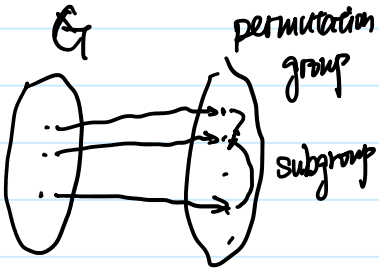
G is isomorphic to a subgroup of permutation group.

If G is finite, then so is the permutation group.

So that every finite group is a subgroup of S_n, for some n.

Proof: $G \to$ f.g.λ.

Let $H = A(G)$ the permutations of G.



Define a map. $\phi: G \to H$.

$a \in G$, $\phi(a) = \sigma$ $\sigma: G \to G$.

$\sigma(g) = ag$. $\forall g$.

σ is a bijection (permutation)

ϕ: isomorphism.

① well-defined.

② injection: $a, b \in G$. $\sigma, \tau \in A(G)$.

$\sigma = \tau$.

$a = ae = \sigma(e) = \tau(e) = be = b$.

$\phi(a) = \phi(b) \Rightarrow a = b$

surjection: h $a^{-1}h = g$

h $h \cdot g^{-1} = a$

σ is a bijection (permutation)

obvious.

③. $\phi(ab) = \phi(a)\phi(b)$

$\sigma = \phi(a)$ $\rho = \phi(ab)$.

$\tau = \phi(b)$.

$\sigma \cdot \tau = \rho$.

$\forall g \in G$.

$\sigma(\tau(g)) = \sigma(bg)$

$= abg$

$= \rho(g)$. □

G order 4.

One is cyclic of order 4.

Not cyclic.

*	e	a	b	c	
e	e	a	b	c ⇐	$a=a$ $b=b$ $c^2=e$.
a	a	e	c	b ⇐	$a^2=e$ $b^2=e$
b	b	c	e	a ⇐	$a^3=a$
c	c	b	a	e ⇖	

The subgroup of S_n. $n=4$. $4!$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

$\begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix}$ $\begin{pmatrix} e & a & b & c \\ a & e & c & b \end{pmatrix}$ $\begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix}$ $\begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix}$

e: 1. a: 2. b. 3. c: 4

$H = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$.

$G \cong H$. H is a subgroup of S_4.

Homomorphisms and kernels.

Definition: A map $\phi: G \to H$ between two groups is a homomorphism if for every g and h in G.

$$\phi(gh) = \phi(g)\phi(h).$$

$\phi: G \to H$ $\{0,1\}$. ϕ is a homomorphism.

$\mathbb{Z}$ $\mathbb{Z}/2\mathbb{Z}$

$$\phi(x) = \begin{cases} 0 & x \text{ is even.} \\ 1 & x \text{ is odd.} \end{cases}$$

	x, y	$\phi(x)$	$\phi(y)$	$\phi(x+y)$
even	even	0	0	0
odd	odd	1	1	0
even odd	even odd	0	1	1
odd even	odd even	1	0	1

Lemma: Let $\phi: G \to H$ be a homomorphism.

1. $\phi(e) = f$. ϕ maps the identity in G to the identity of H.
2. $\phi(a^{-1}) = \phi(a)^{-1}$. ϕ maps inverses to inverses.
3. K is a subgroup of G. $\phi(K)$ is a subgroup of H

③ K is a subgroup of G. $\phi(K)$ is a subgroup of H.

$K \leq G$. $\phi(K) \leq H$.

$K \trianglelefteq G$. normal subgroup

20.11.2021

20:00 (China).