3rd abstract algebra

Saturday, 18. September 2021 13:25

Cosets

$\mathbb{Z}$

$\cancel{Ha}$ C_a odd → not subgroup

even He → subgroup

$a.b \in \cancel{Ha}\ C_a$. $a+b \in He \notin \cancel{Ha}\ C_a$

odd numbe $\underline{a}$ $a + He = \cancel{Ha}\ C_a$

Subset.

Definition: Let X be a set, An equivalence relation $\sim$ is a relation on X, which is

1) reflexive. For every $x \in X$, $x \sim x$.
2) Symmetric $\forall\ x, y \in X$, $x \sim y$ then $y \sim x$
3) transitive $\forall\ x, y, z \in X$, $x \sim y$, $y \sim z$. then $x \sim z$.

Example. $x \sim y$ iff x is odd. y is odd

1) reflexive. $x \sim x$ x is odd
2) Symmetric $x \sim y$ x, y are odd. $y \sim x$
3). transitive. $x \sim y$. $y \sim z$. x. y. z are odd $x \sim y$.

$\boxed{\text{relation}}$

(a_1, a_2) (b_1, b_2) (c_1, c_2)

$a_1 \sim a_2$

$b_1 \sim b_2$

$a_1 \nsim b_1$

$x \sim y$ iff $x = y$ ✱

1) $x \sim x$ ✓
2) $x \sim y$, $y \sim x$ ✓
3) $x \sim y$. $y \sim z \Rightarrow x \sim z$ ✓

$x \sim y$ iff $x . y$ have the same colour ✓ ✱

1) $x \sim x$ ✓
2) $x \sim y$ x, y same colour. $y \sim x$ ✓

2) $x \sim y$ x, y same colour. $y \sim x$ ✓

3) $x \sim y$, $y \sim z$. $x \sim z$ ✓

$x \sim y$ iff $x \geq y$

1) $x \sim x$ $x \geq x$ ✓

2) $x \sim y$ $y \sim x \Leftrightarrow x \geq y$ $y \geq x$ ⊗

16 8 $8 \geq 16$ ✗

3). $x \sim y$, $y \sim z \Leftrightarrow x \geq y$ $y \geq z$ $x \sim z$ $x \geq z$ ✓

$x \geq y \geq z \longrightarrow x \geq z$

Lemma: G be a group. H be a subgroup $\sim$ be a relation in G., $a, b \in G$.

$$a \sim b \text{ iff } \underline{b^{-1}a \in H}$$

$\sim$ is an equivalence relation. $(ab)^{-1} = b^{-1}a^{-1}$ ✓

1) reflexive. $a \sim a$ $a^{-1} \cdot a = e \in H$

2) Symmetry: $a, b \in G$. $\overset{a \sim b \Rightarrow}{b^{-1}a \in H}$

H is closed under taking inverses.

$$(b^{-1}a)^{-1} \in H$$
$$\underline{a^{-1}b \quad \in H} \rightarrow b \sim a$$

3) transtive: $a \sim b$ $b^{-1}a \in H$

$b \sim c$ $c^{-1}b \in H$

H is closed under multiplication.

$$(c^{-1}b) \cdot (b^{-1}a) = c^{-1}a \in H \Rightarrow a \sim c$$

□

Definition: Let $\sim$ be an equivalence relation on a set X.
Let $a \in X$ be an element of X

The equivalence class of a is $[a] = \{b \in X \mid b \sim a\}$.

$a \in [a]$. $b \sim a$

$b \in [b]$

$x \sim x$

$a, b \in [a]$

$a, b \in [b]$

Example: $x \sim y$ iff $x = y$

$0 \in [0]$ $1 \in [1]$ ....

$x \sim y$ iff x, y have the same colour. clothes

$\Rightarrow$ [white] $\Rightarrow$ [red] $\Rightarrow$ [yellow] ....

Definition: Let X be a set. A partition P of X is a collection of subsets $A_i, i \in I$ such that:

$\Rightarrow$ equivalence classes

1) $\bigcup_{i \in I} A_i = X$.

2) $A_i \cap A_j = \emptyset$ if $i \neq j$.

Lemma: Given an equivalence relation $\sim$ in X, there is a unique partition of X.

$\Rightarrow$ The elements of partition are the equivalence classes of $\sim$.

$[a], [b] \cdots$ $A_i, i \in I$ vice-verse.

$\Rightarrow$ the data of an equivalence relation $\overset{=}{\rightleftarrows}$ the data of partition.

$\longrightarrow$

Proof: Suppose $\sim$ is an equivalence relation. even equivalence classes

$x \in X$

$0 = [0]$

1) $x \in [x]$

$2 = [2]$

The equivalence classes cover X.

$4 = [4]$

2) If two equivalence classes intersect, then they are equal.

2) If two equivalence classes intersect, then they are equal.

$\boxed{\text{if } x \sim y \text{ then } [x] = [y]}$ need: $[x] \subset [y]$, $[y] \subset [x]$ $\Rightarrow$ $[x] = [y]$ ①

$y \sim x$ ☆

Suppose $a \in [x]$ $a \sim x$ $x \sim y$ $a \in [y]$.

transtivity $a \sim y$. definition e. classes

Suppose $x \in X$, $y \in X$ $z \in [x] \cap [y]$

$z \in [x]$ $z \sim x$

$z \in [y]$ $z \sim y$ ③ $\Big\}$ $[z] = [x] = [y]$

$[z] = [x]$, $[z] = [y]$

if two equivalence classes overlap, the they coincide we have a partition.

$\leftarrow$

Suppose we have $P = \{A_i \mid i \in I\}$

Define a relation $\sim$ on X

$x \sim y$ iff $x \in A_i$ and $y \in A_i$. i is same

then prove it is indeed equivalence relation.

1) $x \sim x$ $x \in A_i$ and $x \in A_i$

2) $x \sim y$. $y \sim x$ $x \in A_i$ $y \in A_i$

3) $x \sim y$, $y \sim z$ $x \in A_i$ $y \in A_i$ $z \in A_i$

$x \sim z$

□

Example: Let X be the set of integers.

$x \sim y$ iff $x - y$ is even. ☆

[odd] [even]

$0 \in [0]$ $1 \in [1]$ $2 \in [2]$ $3 \in [3]$ $4 \in [4]$ ....

$1-0$ odd $2-0$ even $3-0$ odd

$0 \in [0]$ $1 \in [1]$ $2 \in [2]$ $3 \in [3]$ $4 \in [4]$ ...

$1-0$ odd $2-0$ even $3-0$ odd

$2 \in [0]$ $3-1$ even

$2-1$ odd

Subgroup $n\mathbb{Z} := \{an \mid a \in \mathbb{Z}\}$ $\ast$

$\underline{n}$ 5

multiplication: $an + bn = (a+b)n \in n\mathbb{Z}$

inverses: $-(an) = (-a)n \in n\mathbb{Z}$

$a \sim b$ iff $a-b \in n\mathbb{Z}$ general case.

$[0], [1] \cdots [n-1]$.

$n=5$. $\uparrow$

$1, 6, 11, 16$

$6-1=5 \in n\mathbb{Z}$

$11-6=5 \in n\mathbb{Z}$

$11-1=10 \in n\mathbb{Z}$

$12-1=11 \notin n\mathbb{Z}$

$12 \in [2]$ $12-2=10 \in n\mathbb{Z}$.

Definition – Lemma: Let G be a group. H is a subgroup. $\sim$ be an equivalence relation.

$g \in G$. $[g] = gH = \{gh \mid h \in H\}$

H: even

g: integers

gH is called a left coset of H.

right: Hg.

$g +$ even coset H

proof: $\rightarrow$ Suppose $k \in [g]$ $g^{-1}k \in H$ (from lemma)

$k \sim g$

$h = g^{-1}k \in H$. $k = gh \in gH$ $[g] \subset gH$.

$\leftarrow$ suppose $k \in gH$ $k = gh$ for some $h \in H$.

$h = g^{-1} \cdot k \in H$ (from lemma) $g \sim k$ $\square$

Example:

left cosets are

$[0] = \{an \mid a \in \mathbb{Z}\}$

$[1] = \{an+1 \mid a \in \mathbb{Z}\}$

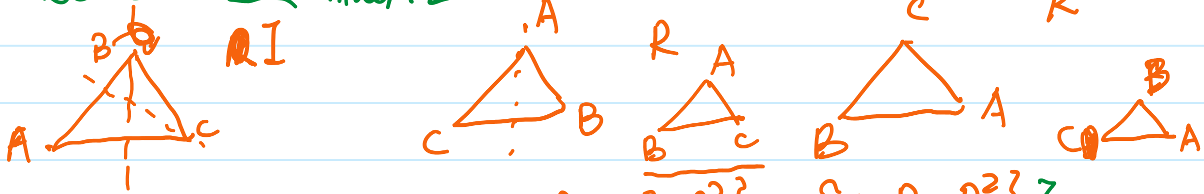
$[0] = \{an \mid a \in \mathbb{Z}\}$

$[1] = \{an+1 \mid a \in \mathbb{Z}\}$

$\vdots$

$[n-1] = \{an-1 \mid a \in \mathbb{Z}\}$

$H = \{I, R, R^2\}$ index: 2 $\qquad G = \{I, R, R^2, F_1, F_2, F_3\} = [I] \cup [F_1]$



$[I] = I \cdot \{I, R, R^2\} = \{I, R, R^2\}$

$[F_1] = F_1 \cdot H = \{F_1, F_3, F_2\}$ 3

partitions. $\boxed{\text{same size}}$

$H = \{I, F_1\}$. : index: 3.

$[I] = \{I, F_1\}$

$[R] = \{R, F_2\}$.

$[R^2] = \{R^2, F_3\}$.

partitions $\boxed{\text{same size}}$

Definition. Let G be a group and H be a subgroup. The index of H in G $[G:H]$ is the number of left cosets of H in G.

G: infinite. $(\mathbb{Z})$ index: (2) of even number in G.

Theorem. (Lagrange's theorem). Let G be a group.

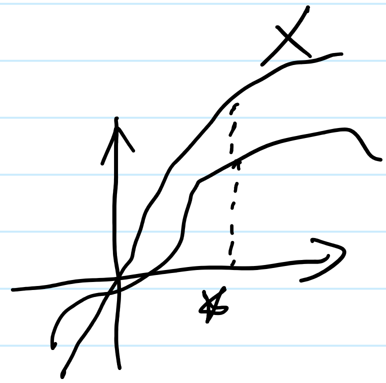
$$|H|\,[G:H] = |G|$$

G finite. the order of H divides the order of G.

proof: $|G| = \bigcup_{i \in I} A_i$ union of its left cosets (disjoint)

it suffices to prove the cardinality of each coset is equal to it suffices to prove the cardinality of each coset is equal to the cardinality of H. $|G| = i \cdot |H|$ $= |gH|$

$|G| = |H|[G:H]$



Suppose gH is a left coset of H in G

define map $A: H \to gH$

$A(a) = ga.$

define map $B: gH \to H$

$B(b) = g^{-1} \cdot b$

$\boxed{\text{well-defined}}$ $(a \to ga)$ A. B ✓

$a = b \Rightarrow ga = gb$

B is a inverse of A. identity map.

$B \circ A: H \to H$

$B \circ A(h) = B(A(h)) = B(gh) = g^{-1} \cdot gh = h$

$A \circ B: gH \to gH.$

$A \circ B(k) = A(B(k)) = A(g^{-1}k) = g \cdot g^{-1}k = k$

A. B bijection. $|H| = |gH|$ $\square$

(3).

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