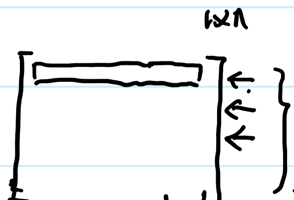
16th basic algebra

2022年11月6日 星期日 下午12:12

proof. (continue).

expansion in cofactors.

proof: (continue): Fix an n-by-n matrix B.



$(v_1, \cdots v_n) \mapsto f_0(v_1 B, \cdots v_n B)$. since f_0 is alternating n-multilinear.

$v_1\ (0, 0, 1, 0 \cdots)$

$v_2\ (0, 1, 0, 0, \cdots)$

The vector space of alternating n-linear functionals has dimension 1.

$f_0(v_1 B, \cdots v_n B) = c(B) \cdot f_0(v_1, \cdots v_n)$ for some scalar $c(B)$.

In notation $\det(AB) = c(B) \cdot \det(A)$

put $A = I$. $\Rightarrow \det A = \det I = 1$ $\Rightarrow \det(B) = c(B) \cdot \det I = c(B)$.

$\Rightarrow$. $\det(AB) = \det(B) \cdot \det(A)$. (a)

(b). $\det(I) = 1$ $\quad \begin{cases} v_1\ (1, 0, 0 \cdots 0) \\ v_2\ (0, 1, 0 \cdots 0) \\ \vdots \\ v_n\ (0, 0, 0, \cdots 1) \end{cases}$ $\quad (v_1, \cdots v_n) \mapsto f_0(B_1, \cdots B_n) = c(B)$ [$\det(B)$.]

$\Uparrow$. $\quad \mapsto c(B) \cdot f_0(v_1, \cdots v_n) = 1$.

$I \mapsto 1$.

(d) definition of $\det A$.

(c). If A^{-1} exists, $\det(A) \cdot \det(A^{-1}) \overset{(a)}{\Longrightarrow} \det(A \cdot A^{-1}) = \det I = 1$

$\det(A) \neq 0$

If A^{-1} does not exist. the reduced row-echelon form R of A has a row of 0's.

A has a row of 0's.

det (A) is the product of det (R) and a non-zero scalar.

det (R) = 0 $\Rightarrow$ det (A) = 0. $\square$

i). interchange two rows. multiplies the determinant by -1.

$f(v,w)+f(w,v)=0$

alternating.

ii). multiply a row by a nonzero scalar c. multiplies the determinant by c.

linearity.

iii) Replace i^{th} row by the sum of it and a multiple of the j^{th} row for $j \neq i$.

i^{th} row change to j^{th} row. has determinat 0. (alternating).

i^{th} replaced by i^{th} row. unchanged.

$$B: \begin{array}{l} i^{th} \\ j^{th} \end{array}\left[\;\right] \Rightarrow \det\begin{bmatrix} i^{th}+nj^{th} \\ j^{th} \end{bmatrix} = \det\begin{bmatrix} i^{th} \\ j^{th} \end{bmatrix} + n \underbrace{\det\begin{bmatrix} j^{th} \\ j^{th} \end{bmatrix}}_{0}, \quad \det\begin{bmatrix} i^{th} \\ j^{th} \end{bmatrix}$$

alternating: $f(v_1, \cdots v_1) = 0$.

compute the determinant requires. $\leq C \cdot n^3$ steps in the n-by-n case.

full-fledged. arrange for the reduced matrix to be 0 below the diagonal

determinant = product of the diagonal entries. ? (d)

diagonal.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix} \qquad \det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix} \overset{(iii)}{=} \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{pmatrix} \overset{(ii)}{=} -3 \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -11 \end{pmatrix} \overset{(iii)}{=}$$

$$\Rightarrow -3\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = -3 \times (1 \times 1 \times 1) = -3.$$

proposition: $\det A^t = \det A$. (A is an n-by-n matrix).

proposition: $\det A^t = \det A$. (A is an n-by-n matrix).

proof: A: row space = column space.

A is invertible iff the row space has dimension n. $\Leftarrow$.

$\Rightarrow$ A is invertible iff A^t is invertible.

$\Rightarrow$ $\det A = 0$. iff $\det A^t = 0$

suppose $\det A \neq 0$. $\det A^t \neq 0$ $\quad A^t = E_1^t \cdots E_r^t$.

$A = E_1 \cdots E_r$. E_j. elementary matrix. (7th class)

$\det(A) = \det(E_1 \cdots E_r) = \prod_j \det E_j$. $\quad \det(A) = \det(A^T)$

$\det(A^T) = \det(E_1^T \cdots E_r^T) = \prod_j \det E_j^T$.

it is enough to prove $\det E_j = \det E_j^T$ for each j.

i), ii): $E_j = E_j^T$ $\quad$ iii: $\det E_j = \det E_j^T$. $\square$.

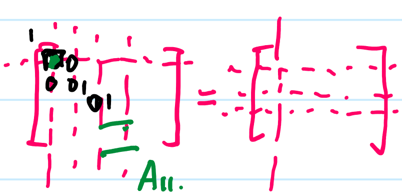
proposition: expansion in cofactors.

Let A be n-by-n matrix. let $\hat{A}_{ij}$ be the square matrix of size $n-1$ by deleting the i^{th} row and j^{th} column.

$1 \cdot 1 + 0 \cdot 1$

a). $\forall j$. $\det A = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det \hat{A}_{ij}$

b). $\forall i$ $\det A = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} . \det \hat{A}_{ij}$



remark: $C \cdot n \cdot (n-1) \cdots = C \cdot n!$ theoretical application

proof: it is enough to prove (a). since $\det A = \det A^T$.

$A = I$. the right side is 1. $\det A = 1$

we need to prove the right side is alternating and n-multilinear.

Each of the term. is n-multilinear. hence the sum is n-multilinear.

▷ k^{th} row and l^{th} row are equal. $k < l$. we want $\det A = 0$

i is not equal k. or l $\quad$ k^{th} and l^{th} row are both present in $\hat{A}_{ij}$

▷ k^{th} row and l^{th} row are equal. $k < l$. we want $\det A = 0$

i is not equal k or l k^{th} and l^{th} row are both present in $\hat{A}_{ij}$

thus each $\det \hat{A}_{ij} = 0$. for i not equal to k or l.

we want. $(-1)^{k+j} A_{kj} \det \underbrace{\hat{A}_{kj}}_{\text{delete } k^{th} \text{ row.}} + (-1)^{l+j} A_{lj} \det \underbrace{\hat{A}_{lj}}_{\text{delete } l^{th} \text{ row.}} = 0.$

$\hat{A}_{kj}$: $1, \cdots k-1, \ \triangle \ k+1, \cdots l-1, \underset{10}{l}, l+1, \cdots n.$ different orders.

$\hat{A}_{lj}$: $1, \cdots k-1, \underset{9}{k}, k+1, \cdots l-1, \ \triangle \ l+1, \cdots n.$

transposing the index for row l to the left one step at a time. until. it gets to k^{th} position. $\underline{l-k-1}$ steps.

$$\det \hat{A}_{lj} = (-1)^{l-k-1} \hat{A}_{kj}.$$

$$(-1)^{k+j} A_{kj} \det \hat{A}_{kj} + (-1)^{l+j} A_{lj} \det \hat{A}_{lj}$$

$$= \left((-1)^{k+j} A_{kj} + (-1)^{2l-k-1+j} A_{lj}\right) \det \hat{A}_{kj} = 0.$$

$A_{kj} = A_{lj}$. alternating. $\square$.

$k+j$

$2l-k-1+j$

$2l-(k+1)+j$

Vandermonde matrix. and determinant.

20. Nov. 4pm.