Group action 1

2023年2月10日 星期五　　下午4:01

binary operation. ①

group axioms ②

cyclic group. ③

Dihedral group. ④

set. $\{1,2,3,4,5\}$. $+ - \cdot$　　$\{0,1,2,\cdots\}$.

$\{5,6,7\}$ △　　{Apple, banana, watermelon}. ⇈

Definition: A binary operation $*$ on a set S is a map.

$$*: S \times S \to S$$

$a * b$ for the image of (a,b) under $*$.

$x+(-x)=0$

a, c, 0　　a, c, 1

i). $+$ $-$ $\times$ $\div$　　$S = \mathbb{R}$.　　$(\mathbb{R}, +) \Rightarrow \mathbb{R}$.　　$(\mathbb{R}, \times)$　　$8-(6-7) \neq (8-6)-7$

✓ ✓ ✓ is not a binary operation.　$1 \div 0$ is undefined.　　$5-3 \neq 3-5$.

$x^{-1} = -x$　　$x \cdot y = 1$　$y = \frac{1}{x}$ for nonzero x.　　2, 3.

ii). cross product on $\mathbb{R}^3$.

$\times: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$.　　$\vec{u} = u_1 i + u_2 j + u_3 k = (u_1, u_2, u_3)$

$\vec{v} = v_1 i + v_2 j + v_3 k = (v_1, v_2, v_3)$

$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2) i + (u_3 v_1 - u_1 v_3) j + (u_1 v_2 - u_2 v_1) k$.

$= (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1)$　　not associative.

not. commutative

iii). min　$\mathbb{N}$　　min(a,b)　a. c. no identity.　　no identity

max　　　　a, c, 0　　$\max(0,5) = \max(5,0) = 5$.

iv. composition.　$Sym(S) = \{$ bijections of a set S to itself $\}$

I

$$Sym(\{5,6,7\}) = \left\{ \begin{matrix} 5 \to 5 \\ 6 \to 6 \\ 7 \to 7 \end{matrix} \quad \begin{matrix} 5 \to 6 \\ 6 \to 7 \\ 7 \to 5 \end{matrix} \quad \begin{matrix} 5 \to 6 \\ 6 \to 5 \\ 7 \to 7 \end{matrix} \quad \cdots \right\}$$

$$\mathrm{Sym}(\{5,6,7\}) = \left\{ \begin{array}{c} 5 \to 5 \\ 6 \to 6 \\ \underline{7 \to 7} \end{array}, \begin{array}{c} 5 \to 6 \\ 6 \to 7 \\ \underline{7 \to 5} \end{array}, \begin{array}{c} 5 \to 6 \\ 6 \to 5 \\ 7 \to 7 \end{array} \quad \cdots \right\}$$

$C_3^1 \cdot C_2^1 \cdot C_1^1 = 6.$

a, commutative ×

$$\Rightarrow \begin{array}{l} 5 \to 5 \to 6 \\ 6 \to 6 \to 7 \\ 7 \to 7 \to 5 \end{array} \qquad \begin{array}{l} 5 \to 6 \\ 6 \to 7 \\ 7 \to 5 \end{array} \in \mathrm{Sym}(S).$$

$\underline{m} * \underline{m^{-1}} = I$

$S * S \to S.$

v. matrix multiplication. $n \times n$ complex invertible ✓. matrix. (given n)

I_n: $\begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$. $\quad A \cdot A^{-1} = I_n$. 2×2

$\begin{pmatrix} 5+2i & 3-6i \\ i & 7 \end{pmatrix}$

a, commutative ×.

determinant $\neq 0$.

$S * S \to S.$

Definition: A binary operation $*$ on a set S is said to be associative if $\forall\, a, b, c \in S$.

$$(a * b) * c = a * (b * c)$$

Definition: commutative

if $\forall\, a, b \in S$.

$$a * b = b * a.$$

Definition: An element $e \in S$ is said to be an identity element for an operation $*$ on S if, $\forall\, a \in S$

$$\underline{e * a} = a = a * e.$$

Proposition: Let $*$ be a binary operation on a set S and let $a \in S$. If an identity e exists, then it is unique.

proof: Suppose that e_1, e_2 are two identities for $*$, then

$$\left.\begin{array}{l} e_1 * e_2 = e_1 \\ e_1 * e_2 = e_2 \end{array}\right\} \Rightarrow e_1 = e_2. \qquad \square.$$

$$e_1 * e_2 = e_1$$
$$e_1 * e_2 = e_2 \Rightarrow e_1 = e_2. \quad \square.$$

Definition: If an operation $*$ on a set S has an identity e and let $a \in S$, then we say that $b \in S$ is an inverse of a if.

$$a * b = e = b * a.$$

proposition: Let $*$ be an associate binary operation. on a set S with an identity e and let $a \in S$.

Then an inverse of a, a^{-1} if it exists, is unique.

proof: Suppose b_1, b_2 are inverses of a

$$b_1 * (a * b_2) = b_1 * e = b_1$$
$$(b_1 * a) * b_2 = e * b_2 = b_2 \quad \Rightarrow b_1 = b_2 \quad \square.$$

$$b_1 * (a * b_2) = (b_1 * a) * b_2 \Rightarrow \text{associative}.$$

$*$ on S. $T \subseteq S$.

$*: T \times T \to S$. $\quad *: T \times T \to T$ $\quad T$ is said to be closed under. $*$.

$S \times S \to S$. $\quad$ (subgroup).

$+$ is a binary operation on $\mathbb{Z}$. (integers.)

(i) Let $m \in \mathbb{Z}$. The set $m\mathbb{Z} = \{mn, n \in \mathbb{Z}\}$ is closed with a identity 0 and inverses.

5 $\quad \{0, -5, 5, -10, 10 \cdots\}$.

6 $\quad \{0, -6, 6, -12, 12 \cdots\}$.

(ii). $\{n \in \mathbb{Z} : n \geq 1\}$ closed under $+$ but has no identity nor inverses.

$\{1, 2, 3 \cdots\}$.

Definition: A group. $(G, *)$ consists of a set G and a binary operation.

$*$ on G $\quad$ closure. $a * b \in G$.

$*$ on G

Closure.
$a*b \in G$.

$*: G\times G \to G, \quad (a,b)\mapsto a*b \quad \forall a,b\in G.$

such that

(i) $a*(b*c)=(a*b)*c \quad \forall a.b.c \in G.$ ✓

(ii). $\forall a\in G.\ \exists e$ such that $e*a=a=a*e.$ ✓

(iii) $\forall a\in G\ \exists a^{-1}$ such that $a*a^{-1}=e=a^{-1}*a.$ ✓

$(G.*)$ is a group $\Rightarrow$ G is a group.

$a*b \Rightarrow ab.$

$$a^n=\begin{cases}\underbrace{aaaaa\cdots a}_{n.} & n>0\\ e & n=0\\ \underbrace{a^{-1}a^{-1}\cdots a^{-1}}_{n.} & n<0\end{cases}$$

Proposition: Let G be a group $x.y.z\in G.$ $m,n\in\mathbb{Z}.$ Then

(a) $(xy)^{-1}=y^{-1}x^{-1}$

(b) $(x^n)^{-1}=x^{-n}.$

(c) $x^mx^n=x^{m+n}$ △

(d) $(x^m)^n=x^{mn}.$

(e). If $xy=xz$ then $y=z$ ← $(x^{-1}.x)y=(x^{-1}.x)z.\Rightarrow y=z.$

(f) If $xy=zy$ then $x=z.$

proof: (a) $(xy)^{-1}.(xy)=e.$

ab=ba X.

$((xy)^{-1}.x).y=e$

$((xy)^{-1}.x).\underbrace{y.y^{-1}}_{e}=e.y^{-1}$

$(xy)^{-1}.x=y^{-1}$ $\Rightarrow x^{-1}(xy)^{-1}.x=x^{-1}.y^{-1}$

$(xy)^{-1}\underbrace{(x.x^{-1})}_{e}=y^{-1}.x^{-1}$

$(xy)^{-1}=y^{-1}.x^{-1}$

□

Niels Abel (1802–1829).

Definition: We say that a group G is abelian. if the group operation $*$ is commutative

Definition: We say that a group G is abelian. If the group operation $*$ is commutative

$$g_1 g_2 = g_2 g_1 \quad \forall g_1, g_2 \in G.$$

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. $+$ abelian group $e=0$. $x^{-1}=-x$

$\mathbb{Q}/\{0\}$. $\mathbb{R}/\{0\}$, $\mathbb{C}/\{0\}$. $\times$ abelian group. $e=1$ $x^{-1}=\frac{1}{x}$.

$\mathbb{Q}^*$, $\mathbb{R}^*$, $\mathbb{C}^*$.

$(\mathbb{F},+,\times)$ is a field $(\mathbb{F},+)$, $(\mathbb{F}/\{0\},\times)$ are both abelian groups.

positive real numbers $(0,\infty)$ $\times$ $e=1$ $x^{-1}=\frac{1}{x}$.

$n\times n$ invertible real matrices. form a group under matrix multiplication.

$GL(n,\mathbb{R})$ nth $\boxed{\text{general linear group}}$ $\quad n\times n.$ $A\cdot x$ $[\ \ ][\ \ ]$ $n\times 1$.

$GL(n,\mathbb{R})$ is non-abelian when $n>1$

i) A invertible. B invertible. AB is invertible. $(A\cdot B)B^{-1}\cdot A^{-1}=I$.

$(AB)^{-1}=B^{-1}\cdot A^{-1}$

ii). $(A\cdot B)C = A\cdot(B\cdot C)$.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix} \qquad \begin{pmatrix} b_{11}c_{11}+b_{12}c_{21} & 0 \\ b_{21}c_{11}+b_{22}c_{21} & 0\end{pmatrix}$$

$$\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{pmatrix}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix} = \frac{a_{11}b_{11}c_{11}+a_{12}b_{21}c_{11}+a_{11}b_{12}c_{21}+a_{12}b_{22}c_{21}.}{a_{11}b_{11}c_{11}+a_{11}b_{12}c_{21}+a_{12}b_{21}c_{11}+a_{12}b_{22}c_{21}}$$

iii). $A\cdot I_n = I_n\cdot A = A \quad \forall A.$ I_n is invertible. $\begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots \end{bmatrix}$

iv). $A\in GL(n,\mathbb{R})$.

A^{-1} is invertible. $A^{-1}\cdot A = I_n$.

$(A^{-1})^{-1}=A$. $A^{-1}\in GL(n,\mathbb{R})$.

$$A=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2}\end{pmatrix}. \qquad B=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-2}\end{pmatrix}.$$

$(AB)_{11} = 2 \neq (BA)_{11} = 1$ $AB \neq BA$. non-abelian.

The real invertible $n \times n$ matrices with determinant 1 form a group.

$SL(n, \mathbb{R})$ $\boxed{\text{special linear group}}$ under matrix multiplication which is non-abelian for $n \geq 2$.

$\det(A) = 1$. $\det(B) = 1$. $\Rightarrow \det(AB) = 1$.

$AB \in SL(n, \mathbb{R})$.

PGL.

APL.

Definition: A group G is called cyclic if $\exists\, g \in G$. s.t.

$$G = \{g^k : k \in \mathbb{Z}\}.$$

such a g is called a generator. $g^i g^j = g^{i+j} = g^j g^i$ cyclic groups are abelian.

$\mathbb{Z}$ is cyclic. generators. $1, -1$.

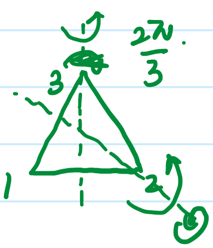
$1 \to 1 - 1 - 1 - 1$ $\quad 1 + 1 + 1 + 1$

$-1 \to -1 + (-1) + (-1)$ $\quad -1 - (-1) - (-1)$

$n \geq 1$. nth cyclic group C_n is a group with elements $\{e, g, g^2, \dots, g^{n-1}\}$. $g^n = e$.

$$g^i * g^j = \begin{cases} g^{i+j} & 0 \leq i+j < n \\ g^{i+j-n} & n \leq i+j \leq 2n-2 \end{cases}$$

Dihedral group $n = 3$ D_6.



$\frac{2\pi}{3}$

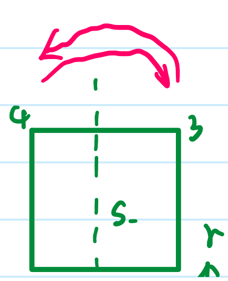
rotation

$$\begin{array}{ccc} e & r & r^2 \\ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \\ s & rs & r^2 s \\ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \end{array}$$

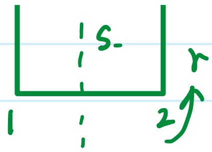
reflection

$rs = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$n = 4$ D_8



$$\begin{array}{cccc} e & r & r^2 & r^3 \\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \end{array}$$



$\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$ $\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}$ $\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$ $\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}$

s. $\quad rs \quad r^2 s \quad r^3 s$.

$\begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix}$ X.

proposition: Let P be a regular n-sided polygon P in the plane with r denoting anticlockwise rotation through $\frac{2\pi}{n}$ about P's centre and s denoting reflection in an axis of P, Then the symmetries of P are:

$$e, r, r^2, \dots r^{n-1}, s, rs, r^2 s, \dots r^{n-1} s.$$

18. Feb, 2023.
16:00 pm.