9th abstract algebra

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$[a][b] = [ab]$ ✓ Category theory.

$aH\, bH \neq abH.$

$\underbrace{a \cdot \underline{h} \cdot \underline{b \cdot k}} = ab\underline{h'}$

H normal. $b^{-1}Hb \subset H.$

$b^{-1}hb = l \in H$

$\underline{hb} = \underline{bl}$

$\underbrace{a \cdot b\,\underline{l}\,\underline{k} = ab}h'$ ← H is normal subgroup

H subgroup. $hk = h' \in H$

Definition: Let G be a group and let H be a normal subgroup.

Theorem. Then the left cosets of H in G form a group, denoted G/H.

G/H is called the quotient of G modulo H. The rule of multiplication in G is defined as.

$$(aH)(bH) = abH. \checkmark \qquad \triangle.$$

Furthermore, there is a natural surjective homomorphism.

$$\phi : G \to G/H$$

$$\phi(g) = gH.$$

Moreover, the kernel of ϕ is H.

$2\mathbb{Z}$ $\quad \mathbb{Z}/\underline{2\mathbb{Z}} = \{0,1\}.$ $\quad \in G(\mathbb{Z})$

$\underset{\sim}{\text{odd}} + H = \{\text{odd}\}.$

$H = \{2, 4, 6, 8, 10 \cdots\}.$ $\quad \underset{\sim}{\text{even}} + H = \{H\}.$

$g \in \mathbb{Z} \quad g = 1. \quad 1 + 2 + (-1) \in H$ $\quad \in G(\mathbb{Q})$

$gHg^{-1} \in H.$ $\quad \{\underline{\{\text{odd}\}}, \underline{\{H\}}\}$

Furthermore, $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} = \underline{\{0,1\}}$ $\quad \{0, 1\}$

Furthermore, $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ $\qquad \{0, 1\}$

$\phi(g) \to gH$

$\phi(\text{odd}) = \{\text{odd}\}.$

$\phi(\text{even}) = \{H\}.$

$\phi(H) = \{H\}$

$\{1\}$

Proof: well-defined ✓ $\quad aH \cdot bH = abH$

Quotient group: ① $(aH \cdot bH) \cdot cH = abH \cdot cH = abcH$ $\quad a.b.c \in G.$ abc associativity.

$= a(bc)H.$

$= aH \cdot (bcH)$

$= aH \cdot (bH cH)$

② $G: e. \quad eH = H.$ ✓

$aH \cdot eH = ae\,H = aH$

$eH \cdot aH = aH$

③ $aH \cdot bH = eH \qquad a^{-1}H$ is the inverse of a.

$abH = eH.$

$b = a^{-1}.$

Thus G/H form a group.

$\phi(g) = gH. \qquad \forall\, aH \Rightarrow \exists\, a \in G,$ such that $\phi(a) = aH.$

$\text{Ker}\,\phi = H \qquad \phi(g) = eH = H$

$\text{Ker}\,\phi$

$\phi(g) = gH$

$gH = H$

$g \in H.$

$\phi(g \in H) = H$

$\ker(g) = H.$ $\square$

Category theory:

category C → objects

→ $X, Y \in$ objects. associate a collection of morphisms $\text{Hom}(X, Y)$

rule: $X, Y, Z \in$ objects. $\quad f \in \text{Hom}(X, Y) \quad g \in \text{Hom}(Y, Z).$

rule of composition: $X, Y, Z \in$ objects. $f \in Hom(X,Y)$ $g \in Hom(Y,Z)$.

$$g \circ f \in Hom(X,Z)$$

Axioms: 1) composition is associative. $f \in Hom(X,Y)$, $g \in Hom(Y,Z)$, $h \in Hom(Z,W)$.

$$h\circ(g\circ f) = (h\circ g)\circ f.$$

2). $\exists I_X \in Hom(X,X)$ acts as an identity.

if $f \in Hom(X,Y)$ $g \in Hom(Y,X)$.

$$\Rightarrow f\circ I_X = f. \qquad I_X \circ g = g.$$

$f \in Hom(X,Y)$, we say $g \in Hom(Y,X)$ is the inverse of f, if

$$f\circ g = I_Y \text{ and } g\circ f = I_X.$$

we say f is an isomorphism., X, Y are isomorphic.

Category of groups. objects: all groups.
morphisms: homomorphism.
composition: functions.

$$\begin{tikzcd} \overset{\text{group}}{A} \arrow[loop left] \arrow[r] \arrow[r, bend right] & \overset{\text{group}}{B} \arrow[r] \arrow[r, bend right] & \overset{\text{group}}{C} \end{tikzcd}$$

Two groups are isomorphic iff they are isomorphic as objects of the category.

Category of sets. objects: sets
morphisms: function: $y = f(x)$

Category of topological spaces.

Category of vector spaces. objects: vector spaces
morphism: linear maps.

Definition: Let $A \xrightarrow{a} B$ objects: A, B, C, D.

Definition: Let

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ C & \xrightarrow[d]{} & D \end{array}$$

objects: A, B, C, D.
morphisms: a, b, c, d.

We say that the diagram commutes if the two morphisms form A to D are same. that is: $c \circ a = d \circ b$.

morphisms from object to object.

Quotient group G/H.

Definition: Let G be a group. H be a subgroup

The categorical quotient of G by H is a group Q together with homomorphism:

$u: G \to Q$. such that

Absract the kernel of u is H.

which is universal amongst all such homomorphisms.

universal. Suppose. $\phi: G \to G'$ is any homomorphism such that. Ker ϕ contains H

Then there is a unique induce homomorphism f. which makes the diagram commute.

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "u"'] & G' \\ Q \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

$\phi = f \circ u$.

Theorem: The category of groups admits categorical quotients.

Given a group G and a normal subgroup H. there exists a categorical quotient group Q.

there exists a categorical quotient group Q.

Q is unique. up to a unique isomorphism.

Proof: existence: Assume quotient group exists. G/H.

$u: G \to G/H$ be the natural homomorphism

we need the above pair forms a categorical quotients.

u is universal.

Suppose: $\phi: G \to G'$ kernel contains H. $g \to \phi(g)$.

$f.\ G/H \to G'$ $\phi(H) = \text{identity}$.

$gH \to \phi(g)$ $f(gH) = \phi(g)$

$f(g_1H) = \phi(g_1)$

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "u"'] & G' \\ G/H \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

$g \to \phi(g)$, gH

diagram commutes.

f: well-defined: suppose. $g_1H = g_2H$.

need: $\phi(g_1) = \phi(g_2)$ ✓

$g_1h_1 = g_2h_2$

$g_1 = g_2 \underline{h_2 h_1^{-1}}$

$g_1 = g_2\, h$ for some $h \in H$.

$$\begin{aligned}\phi(g_1) &= \phi(g_2h) \\ &= \phi(g_2)\phi(h) \\ &= \phi(g_2) \cdot \text{identity} = \phi(g_2)\end{aligned}$$

f: is a homomorphism.

$x = g_1H$ $g_1, g_2 \in G$.

$y = g_2H$

$$\begin{aligned}f(xy) = f(g_1H g_2H) = f(g_1g_2H) = \phi(g_1g_2) &= \phi(g_1)\phi(g_2) \\ &= f(g_1H)f(g_2H) \\ &= f(x)f(y)\end{aligned}$$

commutes:
$$\begin{tikzcd} & \phi(g) \\ g \arrow[ur, mapsto] \arrow[r, mapsto] & gH \arrow[r, mapsto] & f(gH) = \phi(g) \end{tikzcd}$$

G/H is a categorical quotient and it exists.

unique: (up to unique isomorphism). Q_1. Q_2 are categorical quotients.

$u_2 : G \to Q_2$. $f: Q_1 \to Q_2$

$$\begin{tikzcd} G \arrow[r, "u_2"] \arrow[d, "u_1"'] & Q_2 \\ Q_1 \arrow[ur, "f"'] \end{tikzcd}$$
commutes.
$$\begin{tikzcd} G \arrow[r, "u_1"] \arrow[d, "u_2"'] & Q_1 \\ Q_2 \arrow[ur, "g"'] \end{tikzcd}$$

composition $f \circ g: Q_2 \to Q_2$.

$$\begin{tikzcd} G \arrow[r, "u_2"] \arrow[d, "u_2"'] & Q_2 \\ Q_2 \arrow[ur, "f\circ g"'] \end{tikzcd}$$
make it commutes. $\Rightarrow$ $f \circ g =$ identity.

similarly: $g \circ f =$ identity. f. g are inverses of each other. hence isomorphism.

f exists. for G/H

the quotient is unique. up to unique isomorphism. $\square$.

The isomorphism theorem.

First: $\phi: G \to G'$ be a homomorphism of groups.

Suppose: ϕ is surjective H be the kernel of ϕ.

Then G' is isomorphic to G/H

$$G \xrightarrow{\phi} G'$$

$$
\begin{tikzcd}
G \arrow[r, "\phi"] \arrow[d, "u"'] & G' \\
G/H \arrow[ur, leftrightarrow, "f"'] &
\end{tikzcd}
$$

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