

9th abstract algebra

Friday, 26. November 2021 10:14

$$[a][b] = [ab] \checkmark \quad \text{Category theory.}$$

$$aH \cdot bH \neq abH.$$

$$\underline{\underline{a \cdot h \cdot b \cdot k}} = \underline{\underline{abk'}}$$

$$H \text{ normal. } b^{-1}Hb \subset H.$$

$$b^{-1}hb = l \in H$$

$$\underline{\underline{a \cdot b \cdot h \cdot k}} = \underline{\underline{abk'}} \quad \leftarrow H \text{ is normal subgroup}$$

$$H \text{ subgroup. } hk = h' \in H$$

Definition: Let G be a group and let H be a normal subgroup.

Theorem. Then the left cosets of H in G form a group. denoted $\overline{G/H}$.

G/H is called the quotient of G modulo H . The rule of multiplication in G is defined as.

$$(aH)(bH) = abH. \quad \triangle$$

Furthermore, there is a natural surjective homomorphism.

$$\phi: G \rightarrow G/H$$

$$\phi(g) = gH.$$

Moreover, the kernel of ϕ is H .

$$2\mathbb{Z} \quad \underline{2/2\mathbb{Z}} = \{0, 1\}.$$

$$H = \{2, 4, 6, 8, 10, \dots\}.$$

$$g \in \mathbb{Z} \quad g=1. \quad 1+2+(-1) \in H \\ gHg^{-1} \in H.$$

$$\text{Furthermore, } \underline{\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} = \{0, 1\}}$$

$$\begin{aligned} & \text{odd } + H = \{\text{odd}\}. \\ & \text{even } + H = \{H\}. \end{aligned}$$

$$\begin{aligned} & \text{EG}(2) \\ & \overline{\{ \text{odd} \}, \{H\}} \\ & \overline{\{0, 1\}} \end{aligned}$$

Furthermore, $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} = \{\underline{0}, \underline{1}\}$

$$\phi(g) \rightarrow gH$$

$$\begin{aligned}\phi(\text{odd}) &= \{\underline{0}\}, \\ \phi(\text{even}) &= \{\underline{1}\}, \\ \phi(H) &= \{H\}\end{aligned}$$

Proof: well-defined ✓ $aH \cdot bH = abH$

$$\begin{aligned}\text{Quotient group : } \textcircled{1} \quad & \underbrace{(aH \cdot bH)}_{\sim} \cdot cH = abH \cdot cH = abcH \\ &= \underline{a(bc)H}. \quad \begin{matrix} a, b, c \in G. \\ abc \text{ associativity.} \end{matrix} \\ &= aH \cdot \underline{(bc)H} \\ &= aH \cdot \underline{(bHcH)}\end{aligned}$$

$$\textcircled{2} \quad G: e \cdot \underline{eH} = H.$$

$$aH \cdot eH = \underline{ae}H = aH$$

$$eH \cdot aH = aH$$

$$\textcircled{3} \quad aH \cdot bH = eH \quad a^{-1}H \text{ is the inverse of } a.$$

$$abH = eH.$$

$$b = a^{-1}.$$

Thus G/H form a group.

$$\phi(g) = gH. \quad \forall \underline{aH} \Rightarrow \exists a \in G, \text{ such that } \phi(a) = aH.$$

$$\ker \phi = \underline{H}$$

$$\begin{array}{l} \phi(g) = eH = H \\ \ker \phi \\ \phi(g) = gH \end{array}$$

$$\begin{array}{l} \underline{gH} = H \\ \underline{g \in H} \end{array}$$

$$\phi(\underline{g \in H}) = H$$

$$\ker(g) = H. \quad \square$$

Category theory:

category $\xrightarrow{\quad}$ objects

\underline{C} $\xrightarrow{\quad}$ $x, y \in \text{objects. associate a collection of morphisms}$
 $\text{Hom}(x, y)$

rule: $x, y, z \in \text{objects. } f \in \text{Hom}(x, y) \quad g \in \text{Hom}(y, z).$

rule: $X, Y, Z \in \text{objects}$. $f \in \text{Hom}(X, Y)$ $g \in \text{Hom}(Y, Z)$.
of composition. $\underline{g \circ f} \in \text{Hom}(X, Z)$

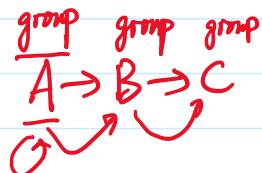
Axioms: 1) composition is associative. $f \in \text{Hom}(X, Y)$
 $g \in \text{Hom}(Y, Z)$.
 $h \in \text{Hom}(Z, W)$.
 $h \circ (g \circ f) = (h \circ g) \circ f$.

2). $\exists I_X \in \underline{\text{Hom}(X, X)}$ acts as an identity.
if $f \in \text{Hom}(X, Y)$ $g \in \text{Hom}(Y, X)$.
 $\Rightarrow \underline{f \circ I_X} = f$. $I_X \circ \underline{g} = g$.

$f \in \text{Hom}(X, Y)$, we say $g \in \text{Hom}(Y, X)$ is the inverse of f , if
 $f \circ g = \underline{I_Y}$ and $g \circ f = \underline{I_X}$.

we say f is an isomorphism. X, Y are isomorphic.

Category of groups. objects: all groups.
morphisms: homomorphism.
composition: functions.



Two groups are isomorphic iff they are isomorphic as objects of the category.

Category of sets. objects: sets
morphisms: function: $y = f(x)$

Category of topological spaces.

Category of vector spaces. objects: vector spaces
morphism: linear maps.

Definition: Let

$$A \xrightarrow{a} B$$

objects: A, B, C, D .

Definition: Let

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ b \downarrow & & \downarrow c \\ C & \xrightarrow{d} & D \end{array}$$

objects: A, B, C, D.

morphisms: a, b, c, d.

We say that the diagram commutes if the two morphisms from A to D are same. that is: $c \circ a = d \circ b$.

morphisms from object to object.

Quotient group G/H .

Definition: Let G be a group. H be a subgroup

The categorical quotient of G by H is a group Q together with homomorphism:

$u: G \rightarrow Q$ such that

Absract the kernel of u is H.

which is universal amongst all such homomorphisms.

Suppose. $\phi: G \rightarrow G'$ is any homomorphism such that.
universal: $\ker \phi$ contains H

Then there is a unique induce homomorphism f.
which makes the diagram commute.

$$\begin{array}{ccc} G & \xrightarrow{\phi} & G' \\ u \downarrow & \nearrow f & \\ Q & & \end{array} \quad \phi = f \circ u.$$

Theorem: The category of groups admits categorical quotients.

Given a group G and a normal subgroup H.
there exists a categorical quotient group Q.

there exists ^v a categorical quotient group ^v \mathbb{Q} .

\mathbb{Q} is unique. up to a unique isomorphism.

Proof: existence. Assume quotient group exists. G/H .

$u: G \rightarrow G/H$ be the natural homomorphism

we need the above pair forms a categorical quotients.
 u is universal.

Suppose: $\phi: G \rightarrow G'$ kernel contains H . $g \mapsto \phi(g)$.

f. $G/H \rightarrow G'$ $\phi(H) = \text{identity}$.

$$\begin{array}{ccc}
 gH & \xrightarrow{\phi} & \phi(gH) \\
 \downarrow u & \nearrow f & \\
 G & \xrightarrow{\phi} & G' \\
 gH & & G/H
 \end{array}
 \quad \begin{array}{l}
 f(gH) = \phi(g) \\
 f(g_1H) = \phi(g_1)
 \end{array}
 \quad \text{diagram commutes.}$$

f: well-defined: suppose. $g_1H = g_2H$.

$$\text{need: } \phi(g_1) = \phi(g_2) \quad \checkmark$$

$$g_1h_1 = g_2h_2$$

$$g_1 = g_2 h_2 h_1^{-1}$$

for some $h \in H$.

$$\phi(g_1) = \phi(g_2h_2)$$

$$= \phi(g_2) \underline{\phi(h_2)}$$

$$= \phi(g_2) \cdot \text{identity} = \phi(g_2)$$

f: is a homomorphism:

$$x = g_1H \quad g_1, g_2 \in G.$$

$$y = g_2H$$

$$\begin{aligned}
 f(xy) &= f(g_1Hg_2H) = f(g_1g_2H) = \phi(g_1g_2) = \phi(g_1)\phi(g_2) \\
 &= f(g_1H)f(g_2H) \\
 &= f(x)f(y)
 \end{aligned}$$

commutes:

$$g \xrightarrow{\phi} \phi(g)$$

$$g \xrightarrow{gH} f(gH) = \phi(g)$$

G/H is a categorical quotient and it exists.

unique: (up to unique isomorphism). Q_1, Q_2 are categorical quotients.

$$u_2: G \rightarrow Q_2, \quad f: Q_1 \rightarrow Q_2$$

$$\begin{array}{ccc} G & \xrightarrow{u_2} & Q_2 \\ u_1 \downarrow & \nearrow f & \text{commutes.} \\ Q_1 & & \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{u_1} & Q_1 \\ u_2 \downarrow & \nearrow g & \\ Q_2 & & \end{array}$$

composition $f \circ g: Q_2 \rightarrow Q_2$.

$$\begin{array}{ccc} G & \xrightarrow{u_2} & Q_2 \\ u_2 \downarrow & \nearrow f \circ g & \text{make it commutes.} \\ Q_2 & & \end{array} \Rightarrow f \circ g = \text{identity.}$$

similarly: $g \circ f = \text{identity.}$ f, g are inverses of each other.
hence isomorphism.

f exists. for G/H

the quotient is unique. up to unique isomorphism. \square

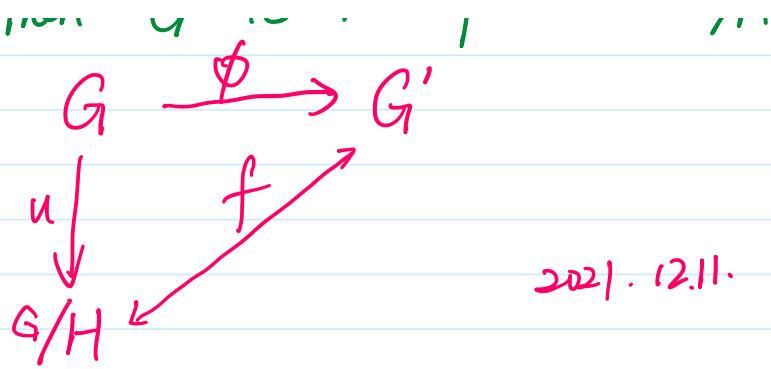
The isomorphism theorem.

First: $\phi: G \rightarrow G'$ be a homomorphism of groups.

Suppose: ϕ is surjective H be the kernel of ϕ .

Then G' is isomorphic to G/H

$$G \xrightarrow{\phi} G'$$



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