16th abstract algebra

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Prime and Maximal Ideals.

I is an ideal of R. $I \neq R$

$R/I \Rightarrow$ domain
field.

Definition: Let R be a ring and let I be an ideal of R
Lemma
We say that I is prime if whenever $ab \in I$ then either $a \in I$ or $b \in I$.

zero-divisors

R/I is a domain if and only if I is prime.

$\mathbb{Z}$ $\{7 \quad 14 \quad 21\} = I$

$7 \Rightarrow 7\cdot 1.\ 7\cdot 2,\ 7\cdot 3$ prime. $\mathbb{Z}/I$ is a domain.

$8 \notin I$ $\quad 2 \in I$, $4 \in I$

proof: $\Leftarrow$ I is prime.

$\begin{cases} x \in R/I \Rightarrow x = a + I \\ y \in R/I \Rightarrow y = b + I \end{cases}$ suppose $x \cdot y = 0$

$(a+I)(b+I) = ab + \underbrace{\overbrace{\underbrace{aI}_{\in I} + \underbrace{bI}_{\in I} + \underbrace{I^2}_{\in I}}^{\in I}}_{\text{ideal}} = ab + I = 0$

additive subgroup

I is an ideal $0 \in I$ $\quad ab + I \in I$

$\{\forall a \in R.\ a \cdot I \in I\}$ $\quad ab \in I$ I is prime. suppose $a \notin I.\ b \in I$.

$y = b + I = I = 0$

$a+I,\ b+I,\ c+I,\ I$

$(a+I)\cdot I = \underset{\in I}{aI} + \underset{\in I}{I^2} = I$ 0.

R/I is a domain.

$\Rightarrow$. R/I is a domain. $a \in R.\ b \in R$ suppose $a \notin I$ want $b \in I$.

I is an ideal s.t. $ab \in I$.

$x \in R/I$ $\begin{cases} x = a + I \\ y = b + I \end{cases}$ $\quad xy = \underset{ab \in I}{ab + I} = I = 0.$ $\quad x = a + I,\ a \notin I$
$y \in R/I$ $\quad R/I.\ I = 0.$ $\quad \neq I \quad x \neq 0.$

domain. $\underset{\neq 0}{x} \cdot \underset{=0}{y} = 0 \Rightarrow b + I = I \Rightarrow b \in I$, I is prime.

$R = \mathbb{Z}$ every ideal in R has the form $\langle n \rangle = n\mathbb{Z}$. I is prime iff n is prime

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$n=4$. $I = 4\mathbb{Z} = \{4, 8, 12, \cdots\}$.

$a \in \mathbb{Z}$. $aI = \{$ ideal

$a=1$ $\{4, 8, 12 \cdots\} \in I$

$a=2$ $\{8, 12, 16 \cdots\} \in I$

$ab \in I$ either $a \in I$ or $b \in I$

$12 = 3 \times 2 \times 2$ $\notin I$ $\notin I$

$n=5$ $I = 5\mathbb{Z} = \{5, 10, 15 \cdots\}$.

$10 = 2 \times 5$ $\notin I$ $\in I$ $25 = 5 \times 5$ $\in I$ $\in I$.

$I = \{n, 2n, 3n \cdots\}$.

$15n = 3 \cdot 5n \in I$.

$an = a \cdot n \Rightarrow n$ is prime

$\neq a \cdot \frac{n}{k} \cdot k = ak \cdot \frac{n}{k} \notin I$.

Definition: Let R be an integral domain, a be a non-zero element of R.

We say a is prime, if $\langle a \rangle$ is a prime ideal, not equal to the whole of R. 1 is not a prime.

$ar \in I$, $a \in I$ or $r \in I$.

$\langle a \rangle = R$.

$a = 1 \Rightarrow r_1 \cdot 1 = r_1$, $r_2 \cdot 1 = r_2$, $\vdots$ $\langle a \rangle = R$

$I \neq R$.

Definition: I be an ideal we say that I is maximal if for every ideal J, such that $I \subset J$, either $J = I$ or $J = R$.

Proposition: Let R be a commutative ring

R is a field iff the only ideals are $\{0\}$ and R.

proof. $\Rightarrow$ R is a field, R contain no non-trivial ideals.

$\Leftarrow$ R contains no non-trivial ideals.

Let $a \in R$.

$a = 0$ $I = \langle a \rangle$. $I = \{0\}$.

$a \neq 0$ $I = \langle a \rangle = R$. $1 \in R \Rightarrow 1 = a \cdot b$

$a \in R$. $\exists b$. s.t. $ab = 1 \Rightarrow$ division ring $\Rightarrow$ commutative. arbitrary field $\square$.

Proposition: Let R be a commutative ring

R/M is a field. iff M is a maximal ideal.

the ideals of R/M = the ideal of R contains M

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From

$\square$

Corollary: R be a commutative ring.

Every maximal ideal is prime.

field is an integral domain. $\longrightarrow$ R/I is a domain iff I is prime.

R/M is a field iff M is a maximal ideal.

$\Downarrow$ M is prime. ideal

Every maximal ideal is prime.

$R = \mathbb{Z}$. p is a prime. $I = \langle p \rangle$ is prime and maximal.

$R/M = \mathbb{Z}/\langle p \rangle = \mathbb{Z}_p$. is a field. $\Rightarrow$ ~~$\mathbb{Z}/\langle p \rangle$ is~~

$p=5$. $\langle 5 \rangle = \{5, 10, 15 \cdots\} = I$

$$\begin{aligned}
\mathbb{Z}/\langle p \rangle = 1+I &= \{6, 11, 16 \cdots\} \\
2+I &= \{7, 12, 17 \cdots\} \\
&\vdots \\
5+I &= \{10, 15 \cdots\} \\
6+I &= \{11, 16 \cdots\} \\
&\vdots
\end{aligned}$$

$= \{1, 2, 3, 4, 5\}. = \mathbb{Z}_p$. is a field.

$\langle p \rangle$ is maximal.

Let R be the ring of Gaussian integers. Let I be the ideal of all Gaussian integers $a+bi$ where both a and b are divisible by 3.

I is maximal.

Suppose we have $I \subset J \subset R$. $J \neq I$.

$a=5$. $b=6$.

$a+bi \in J$ 3 is not divide one of a and b. $\Rightarrow$ 3 doesn't divide a^2+b^2.

$25+36=61$

$c = (a+bi)(a-bi) \in J$.

$= a^2+abi-abi+b^2 = a^2+b^2$. $\in J$. 3 doesn't divide $c = a^2+b^2$.

$3 \in I \subset J$.

we can find r, s. s.t. $3 \cdot r + c \cdot s = 1$ $1 \in J$. $\Rightarrow J = R$

$\in J$. $\in J$. $\to$ $cs \in J$ ideal.

$I \subset J \subset R$.

$1 \in J$ $j=1$

I is maximal.

ideal: $\forall a \in R$. $\forall j \in J$.

$aj \in J$.

$aj \in J.$

$a_1 \cdot 1 \in J$
$a_2 \cdot 1 \in J$ $\Rightarrow J = R.$