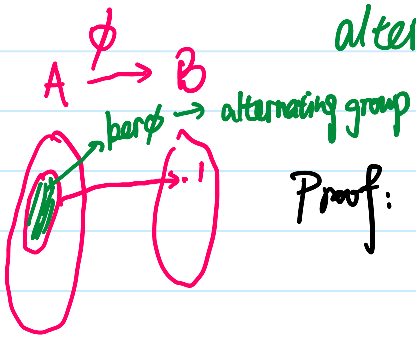
12th abstract algebra

2022年2月12日 星期六 上午9:56

Lemma / Definition There is a surjective homomorphism

$\phi: S_n \to \mathbb{Z}_2$ {1,2} or {-1,1} or {1,0}

Ker ϕ consists of the even transpositions, is called alternating group A_n.



Proof: ϕ: sends even transposition to 1 (identity), odd transposition to -1

$f(a)\cdot f(b) = f(ab)$ $\left(\phi(\tau)\cdot\phi(\sigma) = \phi(\tau\sigma)\right)$ checked

$\phi(\tau(\sigma(f)))$

τ	σ		$\tau\sigma$	
even		1	even	1
1	-1	-1	odd	-1
-1	1	-1	odd	-1
-1	-1	1	even	1

$\square$.

S_n: permutation group contains $n!$ elements

1 2 3 4 .... n
(n) (n-1) (n-2) . . . .

$Ord(A_n) = \frac{n!}{2}$

Theorem: Suppose $n \neq 4$.

The only normal subgroup of S_n is A_n. Moreover A_n is simple. (A_n has no proper / normal subgroups).

$A_n \trianglelefteq A_n$

$n=4$. $\{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \trianglelefteq S_4$.

$4\times3\times2\times1 = 24$. $A_4 = \frac{24}{2} = 12$ ↑ $\trianglelefteq A_4$.

$4\times3\times2\times1=24$. $A_4=\frac{24}{2}=12$ $\quad V_4 \trianglelefteq A_4$.

A_4 is not the only normal subgroup of S_4

A_4 is not simple.

Presentations and Groups of small order

Definition: Let A be a set. $\{a\}$.

A word in A is a string of elements in A and their inverses. (a. a^{-1}) aa aaa $aa^{-1}a$ aaa^{-1} ...

the word w' is obtained from w by a reduction if we can get from w to w' by repeatedly applying the following rule:

$w = aa^{-1}aaaa^{-1}a$ $\quad w_1 = aaaa^{-1}a$

- replace aa^{-1} (or $a^{-1}a$) by the empty string. $w' = aaa$

Given any word w, the reduced word w' associated to w is any word obtained from w by induction, such that w' cannot be reduced any further.

$w_1 = aa^{-1}a^{-1}$ $\quad w_2 = aaa$ $\quad w_1w_2 = aa^{-1}a^{-1}aaa$

Given two words w_1 and w_2 of A, the concatenation of w_1 and w_2 is the word $w = w_1w_2$. The empty word is denoted by e. The set of all reduced words is denoted F_A.

With product defined as the reduced concatenation, this set becomes a group. called the free group with generators A.

$A=\{a\}$. $\quad F_A = F_a$ using only a and a^{-1}.

word. $w = aaaa^{-1}a^{-1}aaa$ $\quad =$ is reduction

$w' = aaaa = a^4$.

free group on one generator is isomorphic to $\mathbb{Z}$.

$a^4 \cdot a^6$ $\quad a^{\mathbb{Z}}$ $\quad \phi.\ a^8 \to \mathbb{Z}$. $\quad a^8\cdot a^5 = a^{8+5} = a^{13}$

aaaa aaaaaa

$a^4 \cdot a^6$ $\quad a^{\square}$

$\underline{aaaa}\ \underline{aaaaaa}$

a^{10}. abelian.

$\phi: \underline{a^8} \to \underline{\mathbb{Z}}$.

$a^8 \cdot a^5 = a^{8+5} = a^{13}$
$\Downarrow$ $= a^5 \cdot a^8$
13

$A = \{a, b\}$. is not abelian. $a^3 b^{-2} a^5 b^{13}$ $\quad F_{a,b}$ has quite a few elements

$ab \quad b^{-1}a$

$abb^{-1}a$
$\neq$
$b^{-1}aab$

Lemma: Let $F = F_S$ be a free group with generators S. Let G be any group. Suppose that we are given a function $f: S \to G$.

Then there is a unique homomorphism $\phi: F \to G$ that extends f.

In other words, the following diagram commutes.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & G \\ F \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

proof. Given a reduced word w in F. sending this w to the element given by replacing every letter by its image in G.

ϕ is homomophism since there are no relations between the elements of F.

$\underline{f(a)}\,\underline{f(b)} = f(ab)$ $\square$.

$S = \{a, b\}$. $\quad G = \{g, h, \ldots\}$.

$w = a^2 b^{-3} a$

$\underline{g^2 h^{-3} g}$

$\underline{f(a)} f(b)$ $\quad \underline{ba^{-2}a} \to hg^{-2}g$

$f(\underline{ab})$ $\quad a^{-1}b \to g^{-1}h$.

$hg^{-2}\underline{g \cdot g^{-1}}h$

$ba^{-2}\underline{aa^{-1}}b$. $\quad = hg^{-2}h$.

$ba^{-2}b \to hg^{-2}h$.

This gives us a convenient way to present a group G.

Pick generators S of G. $\quad \phi: \underline{F_S} \to G$.

S generates G. ϕ is surjective. Let the ker ϕ be H.

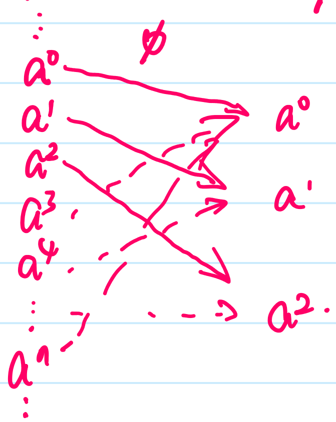
S generates G. ϕ is surjective. Let the ker ϕ be H.

$$\boxed{G \cong F_S / H}$$

To describe H. we need generators R of H.

R is called relations. $\phi: F_a \to G$. $G = \{a^0, a^1, a^2\}$.

$n=3$



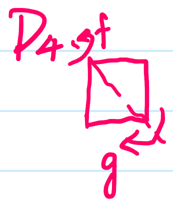
ker $\phi = \langle a^n \rangle$ R.

Definition A representation of a group G is a choice of generators S of G and a description of relations R amongst those generators.

$G = \{a^0, a^1, a^2\} \Rightarrow \underbrace{a}_{\text{generator}},\ \underbrace{a^3 = e}_{\text{relation}}$.

cyclic group. $a,\ a^n = e$.

D_4



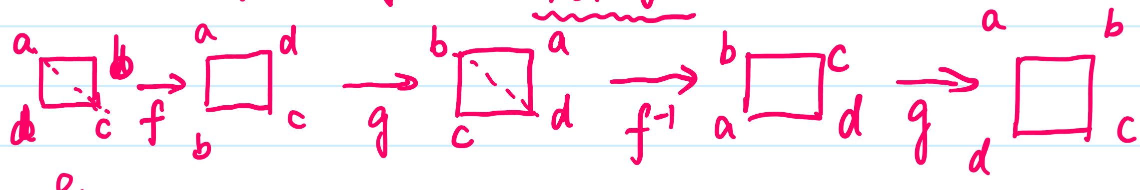
g. f. reflection about a diagonal.

rotation through $\frac{\pi}{2}$

$F_{a,b} \to D_4$

$a \to g \quad b \to f$

$f^2 = e \quad g^4 = e. \quad fgf^{-1}g = e.$



$\{w : f^i g^j \quad i \in \{0,1\}.\ j \in \{0,1,2,3\}\}$

$\begin{cases} W: f^i g^j \quad i \in \{0,1\}. \quad j \in \{0,1,2,3\} \\ R: f^2 = e,\ g^4 = e,\ fgf^{-1} = g^{-1}. \end{cases}$ $\curvearrowright$ D_4.

e

Definition: Let S be a set. The free abelian group A_S generated by S is the quotient of F_S. the free group generated by S, and the relations R given by commutators of the elements of S.

$x^{-1}y^{-1}xy = e.$ $\quad \ker\phi = R = \{\text{commutators}\}$

$x^{-1}y^{-1}yx = e.$

e

Let $S = \{a, b\}$. $A_{a,b} \cong \mathbb{Z}\times\mathbb{Z}$. $\quad S = \{a,b,c\}$ $A_S \cong \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ ✓.

$a^{\#} b^{\#} \cong \mathbb{Z}\times\mathbb{Z}$

Lemma. Let S be a ~~set~~ any set and G be any abelian group. Given any map $f: S \to G$. there is a unique homomorphism $A_S \to G$

proof: Let F_S be the free group. $\phi: F_S \to G$. is a unique homomorphism. G is abelian. $\ker\phi$ contains the commutator subgroup. $A_S = F_S / \ker\phi$. there is a unique homomorphism $A_S \to G$. $\square$

20. Feb. 2022.
16:00. pm (China).