18th basic algebra

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basis of eigenvectors

eigenvector (Linear map)

inner products

orthonormal sets.

Example. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ characteristic polynomial.

$\det(\lambda I - A) \overset{\det}{=} \begin{pmatrix} \lambda & -1 \\ +1 & \lambda \end{pmatrix} = \underline{\lambda^2 + 1}$ $\quad \underline{\lambda^2+1=0}$ $\lambda = i$ $\quad \boxed{\mathbb{F} = \mathbb{R}.}$

does not factor into first-degree factors.

$\mathbb{F} = \mathbb{C}$. has at least one root. matrix or linear map always has at least one eigenvalue hence an eigenvector.

Even when we do have a factorization into first-degree factors, we still can fail to has a basis of eigenvectors.

Example. $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ characteristic polynomial.

$\det(\lambda I - A) = \begin{pmatrix} \lambda-1 & 1 \\ 0 & \lambda-1 \end{pmatrix} = (\lambda-1)^2$ $\quad \lambda = 1.$

$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{matrix} x_2 = 0 \\ x_1 \in \mathbb{F}. \end{matrix}$ $\quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{x_1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we do not have a basis of eigenvectors.

$\begin{cases} 0 \cdot x_1 + 1 \cdot x_2 = 0 \\ 0 \cdot x_1 + 0 \cdot x_2 = 0 \end{cases}$

$(\lambda - \underline{c})^k$ ensures r-parameter family of eigenvector for eigenvalue $\underline{c}$.
$1 \le r \le k.$

In example: r can be strictly less than k.

What will happen?

Proposition: If A is an n-by-n matrix, then eigenvectors for distinct proposition: If A is an n-by-n matrix, then eigenvectors for distinct eigenvalues are linealy independent.

it follows that:

If the characteristic polynomial of A has n distinct eigenvalues. then it has a basis of eigenvectors

$(n-1)$ distinct. $(n-1)$ vector space basis of eigenvectors.

proof: Let
$$A v_1 = \lambda_1 v_1$$
$$A v_2 = \lambda_2 v_2 \qquad \underline{\lambda_1, \lambda_2 \cdots \lambda_k} \text{ distinct.}$$
$$\vdots$$
$$A v_k = \lambda_k v_k.$$

$$\begin{bmatrix} \quad & \quad \end{bmatrix}\begin{bmatrix} | \end{bmatrix} = \lambda_1 \begin{bmatrix} | \end{bmatrix}$$

suppose. $c_1 \cdot v_1 + \cdots + c_k v_k = 0$ multiplied by A.

$(A - \lambda_1 I) v_1 = 0$

$$\begin{cases} c_1 \lambda_1 v_1 + \cdots + c_k \lambda_k v_k = 0. \\ c_1 \lambda_1^2 v_1 + \cdots + c_k \lambda_k^2 v_k = 0 \\ \vdots \\ c_1 \lambda_1^{k-1} v_1 + \cdots + c_k \lambda_k^{k-1} v_k = 0 \end{cases}$$

$c_1 \cdot \underbrace{A v_1}_{\lambda_1 v_1} + \cdots c_k \cdot \underbrace{A v_k}_{\lambda_k v_k} = 0 \cdot A = 0$

multiplied by A

$c_1 \cdot \lambda_1 \cdot \underbrace{A v_1}_{\lambda_1 v_1} + \cdots + c_k \lambda_k \cdot \underbrace{A v_k}_{\lambda_k v_k} = 0$

$$\underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \vdots & \vdots & & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{pmatrix}}\begin{pmatrix} c_1 \underline{v_1^{(j)}} \\ \vdots \\ c_k \underline{v_k^{(j)}} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

j^{th} entry of v_i is denoted by $v_i^{(j)}$

$1 \le j \le n$

Vandermonde matrix. invertible. since $\lambda_1, \cdots \lambda_k$ are distinct.

therefore $\underline{c_i} \cdot \underline{v_i^{(j)}} = 0$ for all i and j.

Each v_i is nonzero in some entry $v_i^{(j)}$ with j. perhaps depending on i. hence. $c_i = 0$

$v_1, \cdots v_k$ are linearly independent. $\square$

$\underline{A} \cdot v = \lambda v$

$L(v) = \lambda v$.

Linear maps: $L: V \to V$ $\quad V$: n-dimensional vector space. over $\mathbb{F}$.

If L is such a function, a vector $v \neq 0$ in V is an eigenvector of L if $L(v) = \lambda v$ for some λ. we call λ the eigenvalue.

If L is such a function, a vector $v \neq 0$ in V is an eigenvector of L iff $L(v) = \lambda v$ for some λ. we call λ the eigenvalue.

When λ is an eigenvalue. the vector space of all v with $L(v) = \lambda v$ is called the eigenspace for λ under L.

We can compute the eigenvalues and eigenvectors of L by working on any ordered basis Γ of V.

$$L(v) = \lambda(v) \Rightarrow \begin{pmatrix} L \\ \Gamma\Gamma \end{pmatrix}\begin{pmatrix} v \\ \Gamma \end{pmatrix} = \lambda \begin{pmatrix} v \\ \Gamma \end{pmatrix} \text{ is satisfies iff}$$

the column vector $\begin{pmatrix} v \\ \Gamma \end{pmatrix}$ is an eigenvector of the matrix $A = \begin{pmatrix} L \\ \Gamma\Gamma \end{pmatrix}$ with eigenvalue λ.

L has eigenvectors and eigenvalues λ iff $\det(\lambda I - L) = 0$.

eigenspace: kernel of $\lambda I - L$.

Different ordered basis Δ $\quad A = \begin{pmatrix} L \\ \Gamma\Gamma \end{pmatrix} \quad B = \begin{pmatrix} L \\ \Delta\Delta \end{pmatrix}$ are similiar.

eigenvectors

eigenvalues. same.

$B = C^{-1}AC. \quad C = \begin{pmatrix} I \\ \Gamma\Delta \end{pmatrix}$

$A \Rightarrow u = \begin{pmatrix} v \\ \Gamma \end{pmatrix}$ as eigenvector for the eigenvalue λ.

$B \Rightarrow B \cdot (C^{-1}u) = C^{-1}AC \cdot (C^{-1}u) = C^{-1} \cdot A \cdot u = C^{-1} \lambda \cdot u = \lambda \cdot C^{-1} \cdot u.$

Thus $C^{-1}u = \begin{pmatrix} I \\ \Delta\Gamma \end{pmatrix}\begin{pmatrix} v \\ \Gamma \end{pmatrix} = \begin{pmatrix} v \\ \Delta \end{pmatrix}$ is an eigenvector of B with eigenvalue λ.

$\Gamma = \begin{matrix} (1,0) \\ (0,1) \end{matrix} \quad \Delta = \begin{matrix} (1,0) \\ (1,1). \end{matrix} \quad v = (5,5) \qquad \begin{pmatrix} v \\ \Gamma \end{pmatrix} = 5 \cdot (1,0) + 5(0,1)$

$\begin{pmatrix} v \\ \Delta \end{pmatrix} = 0 \cdot (1,0) + 5(1,1)$

Similiar matrices.

$B = C^{-1} \cdot A C.$

$\det A \cdot B = \det A \cdot \det B$

$$\det(\lambda I - B) = \det(\lambda I - C^{-1}AC) = \det(C^{-1}(\lambda I - A) \cdot C)$$
$$= \det(C^{-1}) \cdot \det(\lambda I - A) \cdot \det(C)$$
$$= \det(C^{-1}) \cdot \det(C) \det(\lambda I - A) = \det(\lambda I - A).$$

trace are same.

eigenvector of such a linear map.

Inner products and Orthonormal Sets

$\mathbb{F}$: $\mathbb{R}$. $\mathbb{C}$. scalars. finite-dimensional.

addition, subtraction, multiplication. division.

Definition: Let V be a vector space over $\mathbb{F}$. An inner product on V is a function from $V \times V$ into $\mathbb{F}$, which we here denoted by $(\cdot,\cdot)$ with the following properties:

(i) the function $u \to (u, v)$ of V into $\mathbb{F}$ is linear.

(ii) the function $v \to (u, v)$ of V into $\mathbb{F}$ is conjugate linear.

$$\left.\begin{array}{l} (u, v_1 + v_2) = (u, v_1) + (u, v_2) \text{ for } v_1 \text{ and } v_2 \text{ in } V. \\ (u, cv) = \bar{c}(u, v) \text{ for } v \text{ in } V \text{ and } c \text{ in } \mathbb{F}. \end{array}\right\}$$

(iii) $(u, v) = \overline{(v, u)}$ for u and v in V.

complex conjugation

$a + bi \to a - bi$

(iv) $(v, v) \geq 0 \ \forall v \in V$.

(v). $(v, v) = 0$ iff $v = 0$ in V.

(ii): $\mathbb{F} = \mathbb{R}$. $v \to (u, v)$ linear. (i)(ii) $\to$ bilinear. (iii) $\to$ $(\cdot,\cdot)$ is symmetric.

$\mathbb{F} = \mathbb{C}$. $\to$ sesquilinear (iii) $\to$ Hermitian symmetric.

An inner-product space. is a vector space over $\mathbb{R}$ and $\underline{\mathbb{C}}$ with an inner product in the above sense.

Hermitian inner product.

Example: (1). $V=\mathbb{R}^n$ with $(\cdot,\cdot)$ as the dot product.

$(x,y)=y^t\cdot x=x_1y_1+\cdots+x_ny_n$ if $x=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$ and $y=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}$ (iii) $xy=yx$.

traditional notation: $x\cdot y$. linear (i)(ii)

$(x,y+y')=x_1(y_1+y_1')+\cdots x_n(y_n+y_n')$

$=x_1\cdot y_1+x_1y_1'+\cdots+x_ny_n+x_ny_n'$

$=(x,y)+(x,y')$

(iv): $(x,x)\geq 0$ (v): $(x,x)=0$ $x=0$.

(2). $V=\mathbb{C}^n$ with $(\cdot,\cdot)$ defined by $(x,y)=\bar{y}^t\cdot x=x_1\cdot\bar{y}_1+\cdots+x_n\bar{y}_n$ if $x=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$ and $y=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}$

entry-by-entry complex conjugate of y.

The sesquilinear expression $(\cdot,\cdot)$ is different from the complex bilinear dot product. $x\cdot y=x_1y_1+\cdots+x_ny_n$.

(3). V equal to the vector space of all complex-valued polynomials with $(f,g)=\int_0^1 f(x)\overline{g(x)}\,dx$.

Let V be an inner-product space If $v\in V$. define $\|v\|=\sqrt{(v,v)}$. calling $\|\cdot\|$ the norm associated with the inner product.

$\mathbb{R}^n$: $\|x\|$ is the Euclidean distance: $\sqrt{x_1^2+\cdots x_n^2}$ from: $x=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$.

dot product: $x\cdot y=\|x\|\|y\|\cdot\cos\theta$. θ is the angle between the vectors x and y.

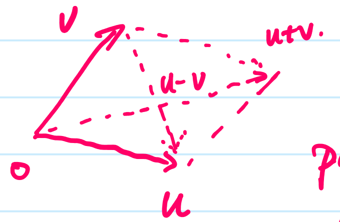
$\|u+v\|^2=\|u\|^2+2\cdot\mathrm{Re}(u,v)+\|v\|^2$. generalizes from $\mathbb{R}^2$ a version of the law of cosines in trigonometry relating the lengths of the three sides of a triangle when one of the angles is known.

when one of the angles is known.

hypothesis: $(u,v)=0$. $\|u+v\|^2=\|u\|^2+\|v\|^2$.

parallelogram: $\underline{\|u+v\|^2}+\underline{\|u-v\|^2}=2\|u\|^2+2\|v\|^2$. $\forall$ u, v in V.

by law of cosines



the sum of the squared length of the four sides of a parallelogram = the sum of the squared lengths of diagonals.

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4 pm.