10th basic algebra

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linear maps. ①
composite. ②
linear extension. ③
kernel. ④

Linear maps. homomorphism.

① $\mathbb{F}^n \to \mathbb{F}^k$ functions.

② $\mathbb{F}$ any two vector spaces

$\underbrace{(a_1, \cdots a_n)}_{n}$ $\underbrace{(a_1 \cdots a_k)}_{k}$

e_1 $(1, 0, 0, \cdots 0)$
e_2 $(0, 1, 0, \cdots 0)$
$\vdots$
e_n $(0, 0 \cdots 1)$

k-by-n matrix define a function $L: \mathbb{F}^n \to \mathbb{F}^k$:

$$k\left[\quad\quad\right]_{n}$$

$L(v) = Av$.

$v \in \mathbb{F}^n$ $Av \in \mathbb{F}^k$

Linear.

$$\begin{cases} \text{i).} \ L(u+v) = L(u) + L(v) \\ \text{ii)} \ L(cv) = c \cdot L(v) \\ u, v \in \mathbb{F}^n, \ c \text{ all scalars.} \end{cases}$$

Linear function.
Linear maps.
Linear mappings.
Linear transformations.
Linear operators.

proposition: If. $L: \mathbb{F}^n \to \mathbb{F}^k$ is a linear map, then there exists a unique k-by-n matrix A such that $L(v) = A \cdot v$ for all v in $\mathbb{F}^n$.

proof: How to obtain the matrix A.

$\mathbb{F}^n$: e_j (j^{th} standard basis) $1 \leq j \leq n$.

$\mathbb{F}^n$: $\underline{e_j}$ (j^{th} standard basis) $1 \le j \le n$.

A: $k \left[\quad \right] \cdots$ j^{th} column of A: k-dimensional column vector $L(e_j)$.

$\uparrow L(e_1)$ $\uparrow L(e_2)$ n

v: $(\underline{c_1}, c_2, \cdots c_n)$

scalar.

$$L(v) = L\left(\sum_{j=1}^{n} c_j e_j\right) \overset{i)}{=} \sum_{j=1}^{n} L(c_j e_j) \overset{ii)}{=} \sum_{j=1}^{n} c_j L(e_j)$$

$$= \sum_{j=1}^{n} c_j \,(j^{th} \text{ column of } A)$$

$L(v)_i$ denotes the i^{th} entry of the column vector $L(v)$ $\mathbb{F}^k$.

$$L(v)_i = \sum_{j=1}^{n} c_j A_{ij}.$$

$A \cdot v$: i^{th} entry.

uniqueness: since $L(e_j) = A \cdot e_j$. j^{th} column of A has to be $L(e_j)$. $\square$ existence.

Example: $L: \underline{\mathbb{R}^2} \to \underline{\mathbb{R}^2}$, counterclockwise through the angle θ.

$L(0,1) = (-\sin\theta, \cos\theta)$ $\qquad e_1 \begin{pmatrix}1\\0\end{pmatrix}$ $\quad L(e_1) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$

$L(1,0) = (\cos\theta, \sin\theta)$ $\qquad e_2 \begin{pmatrix}0\\1\end{pmatrix}$ $\quad L(e_2) = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$

$$L(v) = A \cdot v = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \cdot v.$$

v: $\begin{pmatrix}0\\1\end{pmatrix}$ $L(v) = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$ $\qquad v$: $\begin{pmatrix}1\\0\end{pmatrix}$ $L(v) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$

linear maps:

$L: \mathbb{F}^n \to \mathbb{F}^k$.

$M: \mathbb{F}^n \to \mathbb{F}^k$.

$(L + M)\cdot(v) = L(v) + M(v)$

$(L \cdot M)(v) = L(v) \cdot M(v)$

$\Rightarrow L + M$. linear

N.

$(L \cdot M)(v) = L(v) \cdot M(v)$

$\Rightarrow$ $L+M$, $c \cdot L$ linear.

N.

$\underline{(L+M)}(u+v) = L(u+v) + M(u+v)$

$\overset{i)}{=} L(u)+L(v)+M(u)+M(v)$

$= \underbrace{(L+M)}_{N}(u) + \underbrace{(L+M)}_{N}(v)$ linear.

the set of linear maps from $\mathbb{F}^n$ to $\mathbb{F}^k$. is a vector subspace. of the vector space of all functions from $\mathbb{F}^n$ to $\mathbb{F}^k$. $\{L, M, N \cdots\}$

$\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}^n, \mathbb{F}^k)$: Hom: $L(u+v) = L(u) + L(v)$

$\mathbb{F}$: $L(cv) = c \cdot L(v) \quad \forall c \in \mathbb{F}$.

L corresponds to matrix A

M. corresponds to matrix B.

$$\boxed{\begin{array}{l} L(v) = Av \\ M(v) = B \cdot v \\ (L+M)(v) = (A+B) \cdot v \end{array}}$$

Then:

$L+M$ corresponds to matrix $A+B$.

$c \cdot L$ corresponds to matrix $c \cdot A$.

Proposition: Let $L: \mathbb{F}^n \to \mathbb{F}^m$ be the linear map corresponding to an m-by-n matrix A, and let $M: \mathbb{F}^m \to \mathbb{F}^k$ be the linear map corresponding to an k-by-m matrix B. Then the composite function: $M \circ L: \mathbb{F}^n \to \mathbb{F}^k$ is linear and it corresponds to the k-by-n matrix $B \cdot A$. $M \circ L(v) = B \cdot A \cdot v$.

proof: $(M \circ L)(u+v) = M(L(u+v)) \overset{i) \text{ of } L}{=} M(L(u)+L(v)) \overset{i) \text{ of } M}{=}$

$M(L(u)) + M(L(v)) = (M \circ L)(u) + (M \circ L)v$.

$(M \circ L) \cdot (cv) = c(M \circ L) \cdot v$

Therefore it is linear.

$(M \circ L) \cdot (v) = M(L(v)) = B \cdot (L(v)) = B \cdot (Av) = (B \cdot A) \cdot v$. $\square$.

Treating arbitrary linear maps:

$\mathcal{L}: U \to V$ between two vector spaces over $\mathbb{F}$.

$U = \mathbb{F}^n$ $\quad V = \mathbb{F}^k$. infinite dimentional (Section 9)

We say $\mathcal{L}: U \to V$ is linear. or $\mathbb{F}$ linear. if:

$$\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v) \quad \forall\, u, v \in U$$

$$\mathcal{L}(cv) = c \cdot \mathcal{L}(v) \quad c \in \mathbb{F}.$$

$\mathrm{Hom}_{\mathbb{F}}(U, V)$

Proposition: Let U and V be vector space over $\mathbb{F}$, let Γ be the basis of U. Then to each function $\ell: \Gamma \to V$ corresponds one and only one linear map $\mathcal{L}: U \to V$ whose restriction to Γ has $\Gamma \subset U$

$$\mathcal{L}|_{\Gamma} = \ell.$$

$\mathcal{L}$ linear extension of ℓ

proof: Suppose $\ell: \Gamma \to V$ is given.

since Γ is a basis of U $\quad \forall\, u \in U$

$$u = c_{\alpha_1} u_{\alpha_1} + \cdots + c_{\alpha_r} u_{\alpha_r}. \quad \text{members of } \Gamma.$$

$$\mathcal{L}(u) = \mathcal{L}(c_{\alpha_1} u_{\alpha_1} + \cdots + c_{\alpha_r} u_{\alpha_r}) = c_{\alpha_1} \mathcal{L}(u_{\alpha_1}) + \cdots c_{\alpha_r} \mathcal{L}(u_{\alpha_r})$$

is unique dertermined.

Existence: define $\mathcal{L}$ by this formula. Expanding u. and v.

$$\mathcal{L}(u+v) = \mathcal{L}(c_{\alpha_1} u_{\alpha_1} + \cdots c_{\alpha_r} u_{\alpha_r} + c_{\beta_1} u_{\alpha_1} + \cdots c_{\beta_r} u_{\alpha_r})$$

$$= c_{\alpha_1} \mathcal{L}(u_{\alpha_1}) + \cdots c_{\alpha_r} \mathcal{L}(u_{\alpha_r}) + c_{\beta_1} \mathcal{L}(u_{\alpha_1}) + \cdots c_{\beta_r} \mathcal{L}(u_{\alpha_r})$$

$$= \mathcal{L}(u) + \mathcal{L}(v)$$

$L(cu) = c \cdot L(u)$ Therefore ℓ has a linear extension. $\square$

Since U, V arbitrary finite-dimentional vector spaces.

Fix/arbitrary bases?

Associate a matrix to a linear map $L: U \to V$.

$\Gamma = (u_1, \cdots u_n)$ $\Delta = (v_1, \cdots v_n)$. order bases of U and V.

vector basis

If a member u of U can be expanded in terms of Γ as

$u = c_1 u_1 + \cdots + c_n u_n$. we write.

$\left(\begin{matrix} u \\ \Gamma \end{matrix}\right) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ the column vector expressing u in the ordered basis Γ

$L: U \to V$.

Define k-by-n matrix $\boxed{\left(\begin{matrix} L \\ \Delta\Gamma \end{matrix}\right)}$: j^{th} column of $\left(\begin{matrix} L \\ \Delta\Gamma \end{matrix}\right)$ is $\boxed{\left(\begin{matrix} L(u_j) \\ \Delta \end{matrix}\right)}$.

$\left(\begin{matrix} L(u_1) \\ \Delta \end{matrix}\right) = \begin{pmatrix} d_{11} \\ \vdots \\ d_{n1} \end{pmatrix}$. $d_{11} \cdot v_1 + \cdots d_{n1} \cdot v_n$.

Example: let V be the space of all complex-valued solution on $\mathbb{R}$ of the differential equation $y''(t) = y(t)$ $c_1 \cdot e^t + c_2 \cdot e^{-t}$.

if $y(t)$ is a solution. $y'(t)$ is a solution V

$$V \xrightarrow[d/dt]{} V.$$

$\Gamma = (e^t, e^{-t})$ $\Delta = (\frac{1}{2}(e^t + e^{-t}), \frac{1}{2}(e^t - e^{-t})) = (\cosh t, \sinh t)$.

$\left(\begin{matrix} d/dt \\ \Delta\Gamma \end{matrix}\right)$ $(d/dt)e^t$ $(d/dt)e^{-t}$ in terms of $\cosh t$, $\sinh t$.

$\begin{pmatrix} d/dt \\ \Delta\Gamma \end{pmatrix}$ $\quad \dots (d/dt)e^{t}$, $(d/dt)e^{-t}$ in terms of $\cosh t$, $\sinh t$.

$$\begin{pmatrix} (d/dt)(e^{t}) \\ \Delta \end{pmatrix} = \begin{pmatrix} e^{t} \\ \Delta \end{pmatrix} = \begin{pmatrix} \cosh t + \sinh t \\ \Delta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} (d/dt)(e^{-t}) \\ \Delta \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ \Delta \end{pmatrix} = \begin{pmatrix} -\cosh t + \sinh t \\ \Delta \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} d/dt \\ \Delta\Gamma \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Theorem: If $\mathcal{L}: U \to V$ is a linear map between finite-dimensional vector space over $\mathbb{F}$ and if Γ, Δ are ordered bases of U and V, respectively. then $\underline{\begin{pmatrix} \mathcal{L}(u) \\ \Delta \end{pmatrix}} = \underline{\begin{pmatrix} \mathcal{L} \\ \Delta\Gamma \end{pmatrix}}\underline{\begin{pmatrix} u \\ \Gamma \end{pmatrix}}$ $\quad \forall u \in U$.

Proof: It is enough to prove the identity. for the members u of some ordered basis of U.

Γ be the basis.

For the basis vector u equals to the j^{th} member u_j of Γ.

$\underline{\begin{pmatrix} u_j \\ \Gamma \end{pmatrix}}$ is the column vector e_j. $(0\,0 \cdots \underset{j}{1}, 0\,0 \cdots 0)$

$(c_{\alpha_1}u_{\alpha_1} + \cdots c_{\alpha_r}u_{\alpha_r})$

$\underline{u_{\alpha_k}}$

$c_{\alpha_k} = 1.\ c_{\alpha_1} \cdots c_{\alpha_r} = 0$

$\underline{\begin{pmatrix} \mathcal{L} \\ \Delta\Gamma \end{pmatrix}}\underline{e_j}$ $\quad j^{th}$ column of $\begin{pmatrix} \mathcal{L} \\ \Delta\Gamma \end{pmatrix}$

was defined to be $\begin{pmatrix} \mathcal{L}(u_j) \\ \Delta \end{pmatrix}$.

Thus the identity is valid for u_j. $\square$

9. October. 2022

Sunday 3pm (+8)

7. October. 2022.

Sunday 3pm (+8)