9th basic algebra

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row, column, null space ①
dim ( ) ②
methods of calculation. ③

Corollary. If V is a finite-dimensional vector space and U is a vector subspace of V, then U is finite-dimensional, and $\dim U \leq \dim V$.

proof: $\{v_1, \cdots v_m\}$ be the basis of V.

U is a vector subspace of V.

according to proposition, any linear independent set in U has $\leq m$ elements. being linear independent in V.

Choose a maximal linear independent subset of U $\leq m$ elements. according to proposition, the result is the basis of U. □

Vector spaces Defined by Matrices.

$M_{kn}(\mathbb{F}) \ni A$ k-by-n matrix

$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

row space of A: linear span of rows of A.

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ subset.

vector subspace of vector space of all n-dimensional row vectors.

column space of A. columns of A.

k-dimensional

column space of A. columns of A.

.... . . , k- dimensional.

null space of A. is a vector subspace of $\underline{n\text{-dimensional}}$ column vectors v for which. $A \cdot v = 0$ $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

matrix product.

$$\boxed{\begin{array}{l} A: \text{homomorphism.} \\ V \in \text{elements.} \qquad V: \text{kernel.} \end{array}}$$

a_{11} a_{12} a_{21} $\vdots$ $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

vector subspace: of $\mathbb{F}^n$. $V_1, V_2 \in V$. $\quad V_1 + V_2 \in V \quad A \cdot (V_1 + V_2) = 0 = A \cdot V_1 + A \cdot V_2$

$C \cdot V_1 \in V. \quad A \cdot (cV_1) = 0 = c \cdot \underset{0}{\underline{A(V_1)}}$

$\underset{\mathbb{F}^n}{V} \longrightarrow \underset{\mathbb{F}^k}{A \cdot V}$: $A(V_1 + V_2) = AV_1 + AV_2$
$A(cV) = c \cdot A(V)$

we can view A as a function.

null space of A: a set in the domain mapped $\mathbb{F}^n$ to 0.

column space of A $\quad A \cdot e_j$. $\quad j^{th}$ column of A.

image of the function $v \to Av$ as a subset of the range $\mathbb{F}^k$.

Theorem: If A is in $M_{kn}(\mathbb{F})$. then

$$\dim(\text{column space}(A)) + \dim(\text{null space}(A)) = \#(\text{columns of } A). = n$$

proof: null space is finite-dimensional. $\triangle$.

null space has a basis (according to corollary)

$\{V_1, \cdots V_r\}$.

adjoin $\{V_{r+1} \cdots V_n\}$. so that $\underline{\{V_1, \cdots V_n\} \text{ is a basis}}$ of $\mathbb{F}^n$.

$v \in \mathbb{F}^n$: $\quad V = C_1 V_1 + \cdots + C_n V_n$.

$$A \cdot \underline{v} = A(C_1 V_1 + \cdots C_n V_n) = \underbrace{C_1 \cdot A \cdot V_1 + \cdots \quad C_r A V_r}_{} + C_{r+1} A V_{r+1} + \cdots \ C_n A V_n$$

$$= C_{r+1} \cdot A \cdot V_{r+1} + \cdots + C_n \cdot A \cdot V_n. \Leftarrow$$

image.

$AV_{r+1} \cdots AV_n$. span the column space. $\Leftarrow$

form a basis: suppose $c_{r+1}A\cdot v_{r+1}+\cdots c_nA\cdot v_n=0 \Leftarrow$ only $c_j=0$

$A(c_{r+1}v_{r+1}+\cdots c_n\cdot v_n)=0$

$c_{r+1}\cdot v_{r+1}+\cdots c_n\cdot v_n$ is in the null space. $\underset{\text{basis}}{\{v_1\cdots v_r\}}$

$c_{r+1}v_{r+1}+\cdots+c_n\cdot v_n = a_1\cdot v_1+\cdots a_rv_r.$

$(-a_1)\underline{v_1}+\cdots+(-a_r)\underline{v_r}+c_{r+1}\underline{v_{r+1}}+\cdots c_n\underline{v_n}=0.$

linearly independent.

c_j are 0.

$A\cdot v_{r+1},\cdots A v_n$ is linearly independent. $\Leftarrow$ form a basis of the column space.

$\underline{(n-r)}+\underline{r}=n.$ $\square$

proposition: If A is in $M_{k n}(\mathbb{F})$. then each elementary row operation on A perserves the row space of A.

proof: let rows of A be $\{r_1,\cdots r_k\}$.

interchange. ✓

multiply one of them by a nonzero scalar ✓

replace $\underline{r_i}$ by $\underline{r_i+cr_j}$ $a_1r_1+\cdots+a_nr_n.$

$\underline{a_i\underline{r_i}+a_j\underline{r_j}} = a_i(\underline{r_i+c\cdot r_j})+(a_j-a_ic)\cdot\underline{r_j}.$

$b_i(\underline{r_i+cr_j})+b_j\underline{r_j}=b_i\underline{r_i}+(b_ic+b_j)\cdot\underline{r_j}.$ $\square$.

Theorem: If A in $M_{kn}(\mathbb{F})$ has a reduced row-echelon form R. then.

proof.

$\dim(\text{row space}(A)) = \dim(\text{row space}(R)) = \#(\text{nonzero rows of } R)$

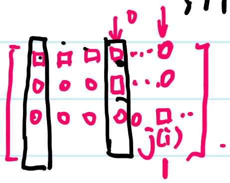
proposition. $= \#(\text{corner variables of } R)$ definition.

$\dim(\text{null space}(A)) = \dim(\text{null space}(R)) = \#(\text{independent variables of } R)$

dim (null space (A)) = dim (null space (R)) = # (independent variables of R)

proof: 1). nonzero rows of R are linearly independent.

$\{r_1 \cdots r_t\}$. suppose $C_1\underline{r_1}+\cdots+C_t\underline{r_t}=0$. $C_i=0$.



$\{r_1 \cdots r_t\}$ linearly independent.

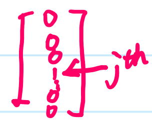
$A \cdot v = 0$. $\{ \; - \; | = 0$ $k \le n$.

2). solution procedure for homogeneous systems of equations.

solution is a finite linear combination of specific vectors.

coefficient: independent variables. x_j. form a basis.

prove they are linear independent. v_j.



v_i j^{th} is 0.

if we want finite linear combination of vectors is 0. $a_1v_1+\cdots+a_nv_n=0$

then j^{th} coefficient is 0. $\underline{a_j v_j}$ $\square$.

Corollary: If A is in $M_{kn}(\mathbb{F})$, then.

dim (row space (A)) + dim (null space (A)) = # (columns of A) = n.

proof: # (corner variables$_R$) + # (independent variables$_R$) = n. Δ. $\square$.

Corollary: If A is in $M_{kn}(\mathbb{F})$, then.

dim (row space (A)) = dim (column space (A)) rank

row rank

column rank

proof. Δ $\square$.

If A is in $M_{kn}(\mathbb{F})$. A^t $\boxed{\text{transpose}}$ of A: is the member of.

$M_{nk}(\mathbb{F})$. $(A^t)_{ij}=A_{ji}$.

Methods of calculation.

(1) Basis of the row space. (of A).

Row reduce A. use the nonzero rows of the reduced row-echelon form.

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Row reduce A. use the nonzero rows of the reduced row-echelon form.

(2). Basis of the column space.

Transpose A. use (1). calculate the row space of A^t.

transpose the resulting row vectors into column vectors.

(3). Basis of the null space. coefficient: ind. var..

solute $A \cdot v = 0$. set of solutions are all finite linear combinations of certain column vectors. form a basis of the null space.

(4). Basis of the linearly span of column vectors $v_1 \cdots v_n$.

Arrange the columns into a matrix A, column space of A. basis. by (2)

(5). Extension of a linearly independent set $\{v_1 \cdots v_r\}$. of column vectors in $\mathbb{F}^n$ to a basis $\mathbb{F}^k$.

Arrange. the columns into a matrix. transpose. row reduce.

adjoin. additional row vectors. one for each independent variables.

if x_j is an independent variables. row vector corresponding to x_j is to be 1 in j^{th} entry and 0 elsewhere.

Transpose all additional row vectors. these are vectors that may be adjoined to obtain a basis.

(6). shrinking of a set $\{v_1 \cdots v_r\}$ of column vectors to a subset that is a basis for the linear span of $\{v_1 \cdots v_r\}$..

$\forall i.\ 0 \leq i \leq r.$ $d_i = \dim(\text{span}\{v_1, \cdots v_i\})$.

Retain v_i for $i \geq 0$ if $d_{i-1} < d_i$. and discard v_i otherwise.

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