5th basic algebra

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permutations ①
k-cycle ②
transposition ③
Sign (Sgn) ④
Contents

S be a finite nonempty set of n elements.

A permutation of S is one-one function from S onto S.

$\{a_1, a_2, \cdots a_n\}$
$\{1, 2, \cdots n\}$

permutation σ at $j \Rightarrow \sigma(j)$

composition of τ followed by σ

$\sigma \circ \tau$
$\sigma\tau \rightarrow \sigma\tau(j) = \sigma(\tau(j))$

$f \cdot g$.

composition is associative. $\rho(\sigma\tau) = (\rho\sigma)\tau$ product.

$\rho(\sigma(\tau(j)))$

identity permutation: 1　　$\sigma\sigma^{-1} = \sigma^{-1}\sigma = 1$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} \Rightarrow \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 1 & 3 \end{pmatrix}.$$

$2 \le k \le n$. cycles.

A k-~~circle~~ is a permutation σ that fixes each element in some subset of n-k elements and moves the remaining elements $C_1, \cdots C_k$. according to $\sigma(C_1) = C_2$, $\sigma(C_2) = C_3$, $\cdots$ $\sigma(C_{k-1}) = C_k$. $\sigma(C_k) = C_1$.

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix}$　3-circle.　$\sigma = (C_1\ C_2\ \cdots C_k) \Rightarrow \sigma = (2\ 3\ 5)$.
$(5\ 2\ 3)$
$(3\ 5\ 2)$

A system of ~~circles~~ cycles. is said to be disjoint if the sets that each of them moves are disjoint in pairs.

$(2\ 3\ 5)$ $(1\ 4)$ are disjoint.　σ, τ.

$(2\ 3\ 5)$ $(1\ 3)$ are not　Any two disjoint ~~circles~~ cycles. σ and τ. commute: $\sigma\tau = \tau\sigma$.

proposition: Any permutation σ of $\{1, 2, \cdots n\}$ is a product of disjoint cycles the individual cycles are unique.

Proposition: Any permutation σ of $\{1,2,\cdots n\}$ is a product of disjoint cycles

the individual cycles are unique.

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix} = \underbrace{(2\ 3\ 5)}_{\sigma\ \uparrow}\underbrace{(1\ 4)}_{\Delta} = (2\ 5)(2\ 3)(1\ 4)$

$\begin{pmatrix} 2 & 3 & 5 \\ 3 & 5 & 2 \end{pmatrix}$ $\begin{matrix} 2 & 3 \\ 3 & 2 \end{matrix} \Rightarrow \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{matrix} \Leftarrow ①$

$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 4 & 2 \end{matrix} \Leftarrow ②$

$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 4 & 2 \end{matrix}$ $(2\ 3\ 5)$.

proof: existence. $\{1,2,\cdots n\}$. we show that any σ is the disjoint product of cycles in such a way that no cycle moves an element j unless σ moves j.

induction: suppose σ fixes the elements in the subset T of r elements of $\{1,2,\cdots n\}$ with $r<n$.

Let j be an element not in T. so that $\sigma(j) \neq j$.

Choose k as small as possible so that some element is repeated among $j, \sigma(j), \sigma^2(j), \cdots \sigma^k(j)$

$\quad 2 \quad 3 \quad 5 \quad 2$

This means $\sigma^l(j) = \sigma^k(j)$ for some l $0 \le l \le k$.

$\sigma^{k-l}(j) = j$

we obtain a contradiction to the minimality of k unless $k = k-l$. $l=0$.

$\sigma^k(j) = j$.

we may thus form the k-cycle: $\gamma = (j, \sigma(j)\ \sigma^2(j)\ \sigma^{k-1}(j))$.

The permutation $\gamma^{-1}\sigma$ fixes $r+k$ elements of $T \cup U$. where U is the set of element $j, \sigma(j), \sigma^2(j) \cdots \sigma^{k-1}(j)$.

By the inductive hypothesis. $\gamma^{-1}\sigma$ is the product $\tau_1 \cdots \tau_p$ of disjoint cycles that moves only elements not in $T \cup U$.

since γ moves only the element in U, γ is disjoint from each of $\tau_1 \cdots \tau_p$.

$\sigma = \gamma \tau_1 \cdots \tau_p$ provides the required decomposition of σ.

uniqueness:

each element j generates a k-cycle C_j for some $k \ge 1$ depending on j.

If we have two decompositions as in the proposition.

then the cycle within each decomposition that contains j must be C_j.

then the cycle within each decomposition that contains j must be C_j.
Hence the cycles in the two decompositions must match. $\square$.

A 2-cycle is often called a transposition.

Corollary: Any k-cycle σ permuting $\{1,2,\cdots n\}$ is a product of $k-1$ transpositions if $k>1$.

Therefore any permutation σ of $\{1,2,\cdots n\}$ is a product of transpositions.

proof: $(c_1\ c_2 \cdots c_{k-1}\ c_k) = (c_1\ c_k)(c_1\ c_{k-1})\cdots \underline{(c_1\ c_2)}$ $\square$.

(c_1, c_3)

$$\prod_{1\le j\le k\le n} |\sigma(k)-\sigma(j)|. \qquad \prod_{1\le j\le k\le n} (\sigma(k)-\sigma(j)).$$

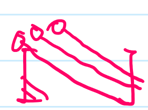
If (r, s) is any pair of integers with $1\le r<s\le n$. $s-r$. appears once and only once as a factor of first product.

Therefore the first product is independent of σ, $\prod_{1\le j<k\le n}(k-j)$.

each factor of the second product is ± 1 times the corresponding factor of the first product.

$$\prod_{1\le j<k\le n} (\sigma(k)-\sigma(j)) = \underbrace{(\operatorname{sgn}\sigma)}_{\pm 1} \prod_{1\le j<k\le n}(k-j).$$

sign of permutation σ.



Lemma. Let σ be a permutation of $\{1,\cdots n\}$. let $(a\ b)$ be a transposition. and form the product $\sigma(a\ b)$, $\operatorname{sgn}(\sigma(a\ b)) = -\operatorname{sgn}\sigma$.

proof: For the pairs (j,k) with $j<k$, we are to compare $\sigma(k)-\sigma(j)$ with $\sigma(a\ b)(k)-\sigma(a\ b)(j)$ we assume $a<b$. 5 cases.

case 1. neither j nor k equals a and b.

$$\sigma(a\ b)k - \sigma(a\ b)j = \sigma(k) - \sigma(j).$$

Thus (j,k) can be ignored.

case 2. one of j,k equals a and b.

$(a,t)\ (t,b)$ with $a<t<b$.

contribute $(\sigma(t)-\sigma(a))$, $(\sigma(b)-\sigma(t))$ to σ.

contribute $(\sigma(t)-\sigma(b))$, $(\sigma(a)-\sigma(t))$ to $\sigma(a\ b)$

$$(\sigma(t)-\sigma(a))(\sigma(b)-\sigma(t)) = (\sigma(t)-\sigma(b))(\sigma(a)-\sigma(t))$$

Thus can be ignored

case 3 continue.

$(a,t)\ (b,t)$ with $a<b<t$.

contribute $(\sigma(t)-\sigma(a))$, $(\sigma(t)-\sigma(b))$ to σ

contribute $(\sigma(t)-\sigma(b))$, $(\sigma(t)-\sigma(a))$ to $\sigma(a\ b)$. similiarly

Thus can be ignored.

case 4. continue.

$(t,a)\ (t,b)$ with $t<a<b$.

$$\underbrace{(\sigma(a)-\sigma(t))(\sigma(b)-\sigma(t))}_{\sigma} = \underbrace{(\sigma(b)-\sigma(t))(\sigma(a)-\sigma(t))}_{\sigma(a\ b)}$$

Thus can be ignored.

case 5. $(a\ b)$ itself.

$\sigma(b)-\sigma(a)$ to σ

$\sigma(a)-\sigma(b)$ to $\sigma(a\ b)$.

$$\sigma(b)-\sigma(a) = -(\sigma(a)-\sigma(b)). \qquad \square$$

Proposition: The signs of permutations of $\{1, 2, \cdots n\}$ have the following:

(a) $\operatorname{sgn} 1 = +1$

(b) $\operatorname{sgn} \sigma = (-1)^k$ product of k transpositions.

(c) $\operatorname{sgn}(\sigma\tau) = (\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)$

(d). $\operatorname{sgn}(\sigma^{-1}) = \operatorname{sgn}(\sigma)$.

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4 pm China.