8th basic algebra

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Sparning ①
Linear independence ②
Bases. ③

① $\mathbb{Q}^n$, $\mathbb{R}^n$, $\mathbb{C}^n$. column.

$(3, 5, 7.5)$ $(3+5i, 8+7i, 6-3i)$

Let $\mathbb{F}$ denote any of $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$. members of $\mathbb{F}$ are called scalars.

Group. $(a+b)+c = a+(b+c)$

A vector space over $\mathbb{F}$ is a set V. with two operations.

addition carrying $V \times V$ into V ←
scalar multiplication, $\mathbb{F} \times V$ into V. ← with following properties:
carrying $V \div V =$ $V \to V$. $v \to v^{-1}$

i) + satisfies. $v_1, v_2, v_3 \in V$

a). $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$. $\boxed{\text{associative law}}$ ←

b). $\exists$ 0 in V. $v + 0 = 0 + v = v$, $\forall v \in V$. → vectors.

c) to each $v \in V$, corresponds to an element $-v \in V$. s.t. $v + (-v) = 0. = (-v) + v$.

d). $v_1 + v_2 = v_2 + v_1$, $\forall v_1, v_2 \in V$. $\boxed{\text{commutative law}}$ ←

$v_1 + \cdots v_n = v_5 + v_3 + \cdots v_1$
σ: permutation of $\{1, \cdots n\}$.
$= v_{\sigma(1)} + \cdots v_{\sigma(n)}$.

ii). written without a sign, satisfies.

a) $a(bv) = (ab)v$, $\forall v \in V$, $a, b \in \mathbb{F}$.

b). $1 \cdot v = v$, $\forall v \in V$, scalar 1

iii). distributive law.

a) $a(v_1 + v_2) = av_1 + av_2$

b) $(a+b)v = av + bv$

1). 0 is unique

2). $-v$ is unique $\qquad \exists\ \underline{(-v)'} = (-v)' + 0 = \underline{(-v)' + (v + (-v))} = \underline{(-v)' + v} + (-v)$

$= 0 + (-v) = \underline{-v}.$

3). $0 \cdot v = 0$

4) $(-1)v = -v$

5) $a0 = 0 \qquad a0 = a(0+0) = a0 + a0.$

1). $V = M_{kn}(\mathbb{F})$ the space of all $\underline{k\text{-by-}n}$ matrices.

$V = \mathbb{F}^k \quad n = 1.$ $\qquad$ 1-by-2. V_1

3-by-5. V_2

$V = \mathbb{F}. \quad k = n = 1.$ $\quad \mathbb{C}$. is a vector space.

2). S: nonempty set. $\xrightarrow{\text{functions}}$ $\mathbb{F}$. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.

V. $\quad (f+g)(s) = \underline{f(s) + g(s)}$

$(cf)(s) = c \cdot (f(s))$ $\quad$ vector space.

i)

a) $(f + \underline{(g+h)})(s) = f(s) + \underline{(g+h)(s)} = f(s) + g(s) + h(s)$

$((f+g)+h)(s) = (f+g)(s) + h(s) = f(s) + g(s) + h(s)$

b) $g: S \to 0 \quad (\underline{f+g})(s) = f(s) + 0 = \underline{f}(s)$

$(g+f)(s) \cdots = f(s)$

c). given f. $f(s)$, $g: S \to -f(s)$

$(f+g)(s) = f(s) + \underline{g(s)} = f(s) - f(s) = 0$

d) $(f+g)(s) = f(s) + g(s) = g(s) + f(s) = (g+f)(s)$

ii): a) $(a(\underline{bf}))(s) = a \cdot (bf)(s) = a \cdot b \cdot f(s).$

$((a.b)f)(s) = \underbrace{(a \cdot b)}_{\in \mathbb{F}} \cdot \underbrace{f(s)}_{\in \mathbb{F}} = a\,b\,f(s)$

b). $(1 \cdot f)(s) = 1 \cdot f(s) = f(s)$

3). S: nonempty set. $\xrightarrow{\text{functions}}$ Let $\underline{V}$ be a $\underline{\text{vector space}}$ over $\underline{\mathbb{F}}$.

functions

$V.$ $(f+g)(s) = \underline{f(s)+g(s)}$
$(cf)(s) = cf(s).$

a). $(f+(g+h))(s) = f(s)+(g+h)(s) = \underbrace{f(s)+(g(s)+h(s))}_{\in U} = (f(s)+g(s))+h(s)$
$= (f+g)(s)+h(s)$
$= ((f+g)+h)(s)$

4). V is a vector space over $\mathbb{C}$.
scalar multiplication. $\underline{\mathbb{R}} \times V \to V$. V becomes a vector space over $\mathbb{R}$.
$\mathbb{C}$ is a vector space over $\mathbb{R}$.

5) $V = \mathbb{F}[X]$ all polynomials in one indeterminate with coefficients in $\mathbb{F}$.
addition, scalar multiplication. (3rd class).
V is a vector space.

6). V is a vector space over $\mathbb{F}$. U nonempty <u>subset</u>. <u>closed under. addition and scalar multiplication</u>.
U is a vector space over $\mathbb{F}$.
U. vector $\boxed{\text{subspace}}$ of V.

7). Let V be any vector space over $\mathbb{F}$. $U = \{v_\alpha\}$ be any subset of V.
a $\boxed{\text{finite linear combination}}$ of the members of U is any vector of the form.
FLC $c_{\alpha_1} v_{\alpha_1} + \cdots c_{\alpha_n} + v_{\alpha_n}$.
$c_{\alpha_j} \in \mathbb{F}$. $v_{\alpha_j} \in U$. $n \geq 0$

The $\boxed{\text{linear span}}$ of U is the set of all finite linear combination of members of U
vector subspace of V. span$\underline{\{v_\alpha\}}$.
convention
span $\emptyset = 0$.

8). V. vector space / of functions $\mathbb{R}^3 \to \mathbb{R}$.

8). V. vector space / of functions. $\mathbb{R}^3 \to \mathbb{R}$.

U is the continuous member of V. U: vector subspace.

addition: $\mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

scalar multiplication. $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$.

twice continuously differentiable members f of V. satisfying PDE:

$$c\frac{\partial^2 f}{\partial x_1^2} + c\frac{\partial^2 f}{\partial x_2^2} + c\frac{\partial^2 f}{\partial x_3^2} + cf = 0 \quad c=0 \text{ on } \mathbb{R}^3.$$

Let V be a vector space over $\mathbb{F}$. A subset $\{V_\alpha\}$ of V spans V or is a spanning set for V if.

the linear span of $\{V_\alpha\}$ is all of V. $\Leftarrow$

A subset $\{V_\alpha\}$ is $\boxed{\text{linearly independent}}$ if whenever. a FLC.

$C_{\alpha_1}V_{\alpha_1} + \cdots C_{\alpha_n}V_{\alpha_n} = \underline{0} \in V$. then all of C_{α_j} must be 0. $\Leftarrow$

$C_{\alpha_1}\underline{V_{\alpha_1}} + \cdots C_{\alpha_n}V_{\alpha_n} = d_{\alpha_1}\underline{V_{\alpha_1}} + \cdots d_{\alpha_n}V_{\alpha_n}$. implies.

$C_{\alpha_j} = d_{\alpha_j}$, $1 \le j \le n$ (subtraction)

A subset $\{V_\alpha\}$ is a $\boxed{\text{basis}}$ if it spans V and is linear independent.

In this case. each member of V has one and only one expansion as a FLC of the members of $\{V_\alpha\}$.

Example: $\mathbb{F}^n$.

$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ $\cdots$ $e_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$ form a basis of $\mathbb{F}^n$.

standard basis of $\mathbb{F}^n$.

Proposition: Let V be a vector space over $\mathbb{F}$.

a) If $\{V_\alpha\}$ is a linearly independent subset of V that is $\underline{\text{maximal}}$ with respect to the propert of being linearly independent. _ is a basis of $\mathbb{F}$.

a) If $\{V_\alpha\}$ is a linearly independent subset of V that is maximal with respect to the propert of being linearly independent _ is a basis of $\mathbb{F}$.

b). If $\{V_\alpha\}$ is a spanning set for V that is minimal with respect to the property of spanning _ is a basis of V.

proof: a). Let v be given. show v is the span of $\{V_\alpha\}$. (want)

W.L.O.G. v is not in $\{V_\alpha\}$. $\{V_\alpha\} \cup \{v\}$. is not linear independent. / maximal.

$c \cdot v + c_{\alpha_1} v_{\alpha_1} + \cdots c_{\alpha_n} v_{\alpha_n} = 0$. for some $c.\ c_{\alpha_1} \cdots c_{\alpha_n}$ not all 0

$c \neq 0$ since $\{V_\alpha\}$ is linear independent.

$$v = -c^{-1} c_{\alpha_1} v_{\alpha_1} - \cdots - c^{-1} c_{\alpha_n} v_{\alpha_n}.$$

v is exhibited as in the linear span of $\{V_\alpha\}$.

b). $c_{\alpha_1} v_{\alpha_1} + \cdots c_{\alpha_n} v_{\alpha_n} = 0$ $\quad c_{\alpha_1} \cdots c_{\alpha_n}$ not all 0. say $c_{\alpha_1} \neq 0$.

v_{α_1} is a FLC of $v_{\alpha_2} \cdots v_{\alpha_n}$.

substitution shows that any FLC of v_α's is a FLC of v_α's other than v_{α_1}.

We obtain a contradiction to the assumed minimality. □.

Proposition: Let V be a vector space over $\mathbb{F}$. If V has a finite spanning set. $\{v_1, \cdots v_m\}$. then any linearly independent set in V has $\leq m$ elements.

proof. no subset of $m+1$ vectors can be linearly independent.

suppose $\{u_1, \cdots u_n\}$ is a linearly independent set with $n = m+1$.

$$u_1 = c_{11} v_1 + \cdots c_{m1} v_m$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$u_n = c_{1n} v_1 + \cdots c_{mn} v_m$$

system of linear equations

$$c_{11} x_1 + \cdots + c_{1n} x_n = 0$$
$$\vdots \qquad \vdots$$
$$c_{m1} x_1 + \cdots + c_{mn} x_n = 0.$$

is homogenous system

more unknows than equations.

it has a nonzero solution $(x_1, \cdots x_n)$. $\Leftarrow$ proposition (d).

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$$x_1 u_1 + \cdots x_n u_n \overset{\triangle}{=} \begin{array}{cccc} c_{11} x_1 v_1 & + c_{12} x_1 v_2 & + \cdots & c_{m1} x_1 v_m \\ + & + & & + \\ \vdots & \vdots & & \vdots \\ + & + & & + \\ c_{1n} x_n v_1 & + c_{2n} x_n v_2 & + \cdots & c_{mn} x_n v_m . \end{array}$$

$$= 0 + 0 + \cdots + 0 = 0$$

in contradiction to the linear independence of $\{u_1 \cdots u_n\}$. $\square$.

Corollary: if the vector space V has a finite spanning set $\{v_1, \cdots v_m\}$. then

a). $\{v_1, \cdots v_m\}$. has a subset that is basis.

b). any linearly independent set in V can be extended to a basis.

c). V has a basis

d). Any two bases have the same finite number of elements. necessarily $\leq m$.

in this case. V: finite - dimensional

the number of element in a basis: dimension of V. $\underline{\dim V}$. $\dim(V)$.

V has no finite spanning set. V. infinite-dimensional.

analog of corollary is valid. (Section 9.).

proof. a). from proposition (minimal)

b) from proposition: linearly independent set has $\leq m$ elements.

adjoin one at a time.

from proposition (maximal)

c). from a)

d) from two propositions. $\square$

Example. $M_{kn}(\mathbb{F})$. k-by-n. dimension: kn.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \cdots$$

The vector space of all polynomials in one indeterminate is infinite-dimensional.