17th basic algebra

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Vandermonde matrix

Cramer's rule.

eigenvector.

Corollary: If $r_1, \cdots r_n$ are scalars, then:

$$\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \vdots & & & \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{pmatrix} = \prod_{j>i} (r_j - r_i)$$

proof: we show that the determinant is $= \prod_{j>1} (r_j - r_1) \det\begin{pmatrix} 1 & \cdots & 1 \\ r_2 & \cdots & r_n \\ \vdots & \ddots & \vdots \\ r_2^{n-2} & \cdots & r_n^{n-2} \end{pmatrix}$

and then the result follows by induction.

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \vdots & & & \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{pmatrix} \begin{matrix} \\ -r_1 \\ -r_1 \\ \\ -r_1 \end{matrix} \Rightarrow \det\begin{pmatrix} \boxed{1} & 1 & \cdots & 1 \\ 0 & r_2 - r_1 & \cdots & r_n - r_1 \\ \vdots & & & \\ 0 & r_2^{n-2} - r_1 r_2^{n-3} & \cdots & r_n^{n-2} - r_1 r_n^{n-3} \\ 0 & r_2^{n-1} - r_1 r_2^{n-2} & \cdots & r_n^{n-1} - r_1 r_n^{n-2} \end{pmatrix} =$$

$$\det\begin{pmatrix} r_2 - r_1 & \cdots & r_n - r_1 \\ \vdots & & \vdots \\ r_2^{n-2} - r_1 r_2^{n-3} & \cdots & r_n^{n-2} - r_1 r_n^{n-3} \\ \underbrace{r_2^{n-1} - r_1 r_2^{n-2}}_{r_2 - r_1} & \cdots & \underbrace{r_n^{n-1} - r_1 r_n^{n-2}}_{r_n - r_1} \end{pmatrix} = (r_2 - r_1)\cdots(r_n - r_1) \begin{pmatrix} 1 & \cdots & 1 \\ r_2 & \cdots & r_n \\ \vdots & & \vdots \\ r_2^{n-2} & \cdots & r_n^{n-2} \end{pmatrix} \quad \square.$$

classical adjoint: of a square matrix A. denoted by A^{adj}

is a matrix with entries: $A^{adj}_{ij} = (-1)^{i+j} \det \hat{A}_{ji}$ $\quad \hat{A}_{kl}$: the matrix A with k^{th} row, l^{th} column deleted.

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Example: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{adj} = \begin{pmatrix} \boxed{d} & -b \\ \boxed{-c} & a \end{pmatrix}$ $A \cdot A^{adj} = A^{adj} A = (\det A) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & -ab+ab \\ cd-cd & -bc+ad \end{pmatrix} = (\det A) \cdot I$$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & db-db \\ -ac+ac & -bc+ad \end{pmatrix}$$

$n \times n$ case

Proposition. If A is an n-by-n matrix, then $A \cdot A^{adj} = A^{adj} \cdot A = (\det A) \cdot I$.

Cramer's rule. thus $\det A \neq 0$ implies $A^{-1} = (\det A)^{-1} \cdot A^{adj}$.

Consequently: If $\det A \neq 0$, then the unique solution of the simultaneous system: $A \cdot x = b$ of n equations in n unknowns. in which $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ has. $x_j = \dfrac{\det B_j}{\det A}$, n, 1 $\Leftarrow$

with B_j equals to the n-by-n matrix obtained from A by replacing the j^{th} column of A by b.

Remark. n-by-n. determinant: n^3 steps.

cramer's rule: $n^3(n+1)$ steps

theoretical application.

row reduction: more efficient.

observation: each entry of inverse of a matrix is the quotient of a polynomial function of the entries divided by the determinant.

proof: $(i,j)^{th}$ entry of $A^{adj}A$ is

$$(A^{adj}A)_{ij} = \sum_{k=1}^{n} A^{adj}_{ik} A_{kj} = \sum_{k=1}^{n} (-1)^{i+k} (\det \hat{A}_{ki}) A_{kj}.$$

If. $i=j$. expansion in cofactors. about the j^{th} column. $= \det A$.

If $i \neq j$. Consider matrix B obtained from A by replacing the i^{th} column of A by the j^{th} column.

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$\det B = 0$

Expanding $\det B$ in cofactors about the i^{th} column.

$$0 = \det B = \sum_{k=1}^{n} (-1)^{i+k} (\det \hat{B}_{ki}) B_{ki} = \sum_{k=1}^{n} (-1)^{i+k} (\det \hat{A}_{ki}) A_{kj}.$$

$(A^{adj} A) = \det A \cdot I$ similiarly: $(A \cdot A^{adj}) = (\det A) \cdot I$

$Ax = b$. $\overset{A^{adj}}{\Rightarrow}$ $A^{adj} \cdot Ax = A^{adj} \cdot b$ $\Rightarrow$ $\det A \cdot x = A^{adj} \cdot b$. Hence:

$$\det A \cdot x_j = \sum_{i=1}^{n} (A^{adj})_{ji} b_i = \sum_{i=1}^{n} (-1)^{i+j} b_i \det \hat{A}_{ij} = \det B_j.$$

$$x_j = \frac{\det B_j}{\det A}.$$ □

Eigenvectors and Characteristic Polynomials.

A vector $v \neq 0$ in $\mathbb{F}^n$ is an $\boxed{\text{eigenvector}}$ of the n-by-n matrix A if $A \cdot v = \lambda \cdot v$ for some scalar λ.

We call λ the $\boxed{\text{eigenvalue}}$ associated with v.

When λ is an eigenvalue. the vector space of all v with $A \cdot v = \lambda \cdot v$. is called the $\boxed{\text{eigenspace}}$ for λ.

0 vector + eigenvectors. $v_1, v_2 \in V$

$v_1 + v_2 \in V$.

$A \cdot (v_1 + v_2) = Av_1 + Av_2 = \lambda v_1 + \lambda v_2 = \lambda(v_1 + v_2)$

$A(v) \to v$. A giving a linear map L from $\mathbb{F}^n$ to itself.

$\lambda(v) \to (\lambda v)$. eigenvector. mapped. a vector to multiple of itself. by λ.

Geometric view: the eigenvector yields a 1-dimensional subspace $U = \mathbb{F} \cdot v$ that is invariant. or stable under L in the sense of satisfing $L(U) \subseteq U$.

Proposition: An n-by-n matrix A has an eigenvector with eigenvalue iff. $\det(\lambda I - A) = 0$. In this case, the eigenspace for λ is the kernel of $\lambda I - A$.

proof: We have $A \cdot v = \lambda v$ iff. $\lambda I v - Av = 0$

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iff $(\lambda I - A) v = 0$

iff. v is in kernel $\lambda I - A$.

This kernel is nonzero iff $\det(\lambda I - A) = 0$. $\square$.

Fix A, the expression $\det(\lambda I - A)$ is a polynomial. in λ of degree n. and is called the characteristic polynomial. of A.

$\det(A - \lambda I)$ → same n is even
→ negative n is odd.

Explaination:

$$\det \begin{pmatrix} \lambda - A_{11} & -A_{12} & \cdots & -A_{1n} \\ -A_{21} & \lambda - A_{22} & & -A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{n1} & -A_{n2} & & \lambda - A_{nn} \end{pmatrix} = \sum_{\sigma} (\operatorname{sgn} \sigma) \cdot T_{1,\sigma(1)} \cdots T_{n,\sigma(n)}.$$

T for permutation $\sigma = 1$. $\sigma(k) = k$ $\forall k$. gives $\prod_{j=1}^{n} (\lambda - A_{jj})$

other permutations. have $\sigma(k) = k$ for at most $n-2$ values of k.

λ therefore occurs at most $n-2$ times.

$1, \cdots n$
$1, \cdots n$
$1, \underset{\curvearrowleft}{i\,j}\, n$
$1, \cdots j, i \cdots n.$

$$= \prod_{j=1}^{n} (\lambda - A_{jj}) + \left\{ \begin{matrix} \text{other } T \text{ with powers} \\ \text{of } \lambda \text{ at most } n-2 \end{matrix} \right\}$$

$$\boxed{= \lambda^n - \underbrace{\Big(\sum_{j=1}^{n} A_{jj}\Big)}_{\operatorname{Tr} A} \lambda^{n-1} + \left\{ \begin{matrix} \text{other } T \text{ with powers} \\ \text{of } \lambda \text{ from } n-2 \text{ to } 1 \end{matrix} \right\}. + \underline{(-1)^n \det(A)}.}$$

$f(\lambda)$. $\lambda = 0$. $\det(-A)$.

$\det(\lambda I - A)$ characteristic polynomials are polynomial functions.

λ: indeterminate. x.

The negative of the coefficient of λ^{n-1} is the trace of A.
denoted by $\operatorname{Tr} A$. $\operatorname{Tr} A = \sum_{j=1}^{n} A_{jj}$.

Trace is a linear functional on the vector space $M_{nn}(\mathbb{F})$ of n-by-n matrices.

Example: $A = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}$. characteristic polynomial.

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$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda-4 & -1 \\ 2 & \lambda-1 \end{pmatrix} = (\lambda-4)(\lambda-1)+2$$
$$= \lambda^2 - 5\lambda + 6$$
$$= (\lambda-2)(\lambda-3) \qquad = 0.$$

The roots and hence the eigenvalues are $\lambda=2$, $\lambda=3$.

The eigenvectors for $\lambda=2 \Rightarrow (2I-A)\cdot V = 0$

$$\left(\begin{array}{cc|c} 2-4 & -1 & 0 \\ 2 & 2-1 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -2 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} -2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ $x_1 + \frac{1}{2}x_2 = 0$ eigenvectors for $\lambda=2$ are the nonzero vectors

$x_1 = -\frac{1}{2}x_2$ of the form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$

Similiarly, eigenvectors for $\lambda=3$ are the nonzero vectors of the form. $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

In this example, there is a basis of eigenvectors.

Corollary: An n-by-n matrix A has at most n eigenvalues.

proof: $\det(\lambda I - A)$ is a polynomial of degree n. from 4th class. it has at most n eigenvalues. $\square$.

27. Nov.
4pm (+8).