11th abstract algebra

Saturday, 18. December 2021 10:07

Definition: Let G be a group and let x and y be two elements of G $x^{-1}y^{-1}xy$ is called the commutator of x and y.

The commutator subgroup of G is the group generated by all of the commutators.

Lemma: Let G be a group and let H be the commutator subgroup. Then H is characteristically normal in G and quotient group G/H is abelian.

Moreover this quotient is universal amongst all abelian quotients in the following sense:

Suppose that $\phi: G \to G'$ is any homomorphism of groups, where G' is abelian. Then there is a unique homomorphism. $f: G/H \to G'$ such that $f \circ u = \phi$.

proof: Suppose. ϕ is an automorphism of G.

Let x, y be two elements of G $\quad G \to G$

$$\phi(x^{-1}y^{-1}xy) = \phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y)$$

commutator of x and y $\quad$ commutator of $\phi(x)$ and $\phi(y)$

$\phi(H)$ is generated by the commutators. $\phi(H) = H$.

H is characteristically normal in G.

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G/H : aH. bH.

$$(bH)(aH) = baH = ba \cdot a^{-1} \cdot b^{-1} \cdot a \cdot b \cdot H = abH$$
$$a^{-1}b^{-1}abH \qquad = (aH)(bH)$$

G/H is abelian.

$\phi: G \to G'$ be a homomorphism, G' is abelian.

kernel ϕ contains H.

H is generated by the commutators $x.y \in G$. $\phi(x) \in G'$ $\phi(y) \in G'$

$$\phi(x)\phi(y) = \phi(y)\phi(x)$$

$\phi(x)^{-1} \cdot \phi(y)^{-1} \cdot \phi(x) \cdot \phi(y)$ is the identity in G'.

$x^{-1}y^{-1}xy$ is sent to identity.

$x^{-1}y^{-1}xy \in \ker \phi$.

$H \in \ker \phi$. $\square$.

Definition Let G and H be any two groups.

Lemma. The product of G and H. denoted $G \times H$ is the group whose elements are the ordinary elements of the cartesian product of G and H as sets: with multiplication defined. as:

$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$$

proof. $(g.h)$. ① $[(g_1, h_1)(g_2h_2)](g_3, h_3) = (g_1g_2g_3, h_1h_2h_3)$

$[\quad]$ ....

② (e, f) $\quad (g, h) \cdot (e, f) = (\underset{g}{g \cdot e}, \underset{h}{hf})$

G H.

(e, f) in G, H. $\quad (g,h)\cdot(e,f) = (\underbrace{g\cdot e}_{g}, \underbrace{hf}_{h})$

③ $(g_1,h_1)(g_2,h_2) = (e, f)$.

$g_1 g_2 = e$ under G

$\underline{h_1 h_2 = f}$. under H. $\square$

Definition: Let C be a category and let X and Y be two objects of C. The categorical product of X and Y, denoted by $X \times Y$, is an object together with two morphisms.

$$p: X \times Y \to X$$
$$q: X \times Y \to Y.$$

that are universal amongst all such morphisms, in the following sense.

Suppose that there are morphisms $f: Z \to X$
$g: Z \to Y$

Then there is a unique morphisms $Z \to X \times Y$, which makes the following diagram commute.

$$\begin{tikzcd} & & X \\ Z \arrow[urr, "f"] \arrow[rr, dashed, "u"] \arrow[drr, "g"'] & & X \times Y \arrow[u, "p"'] \arrow[d, "q"] \\ & & Y \end{tikzcd}$$

$X \times Y$ is unique.

Lemma: The product of groups is a categorical product.

proof: $p: G \times H \to G \quad (g, h) \to g \quad$ homomorphisms.

proof: $p: G \times H \to G$ $\quad (g, h) \to g$
$q: G \times H \to H.$ $\quad (g, h) \to h.$

homomorphisms.
$\phi(xy) = \phi(x) \cdot \phi(y)$

$$\begin{matrix}(g_1, h_1) \\ (g_2, h_2).\end{matrix} \Rightarrow (g_1 g_2, h_1 h_2) \Rightarrow g_1 g_2 \quad \phi(xy)$$
$$g_1 \cdot g_2 \Rightarrow g_1 g_2. \quad \phi(x)\phi(y).$$

Given a group K. $\quad f: K \to G \quad k \to f(k) \in G$
$g\ K \to H. \quad k \to g(k). \in H.$

$u: K \to G \times H$ $\quad$ homomorphism.
$k \to (f(k), g(k))$

$$\phi(k_1 k_2) = (f(k_1 k_2), g(k_1 k_2)) = (f(k_1) \cdot f(k_2), g(k_1) g(k_2))$$
$$\phi(k_1)\phi(k_2) = (f(k_1), g(k_1))(f(k_2), g(k_2)) = (f(k_1) \cdot f(k_2), g(k_1) g(k_2))$$

The diagram commutes. $\square$

The Alternating group.

$S_3.\ \{(1,2,3),\ \{\},\ (1,2),\ (2,3)\}.$

$\tau\sigma$

S_3 contains a normal subgroup. H generated by a 3-cycle.

$(1,2)$ $\quad (2,3)$
$123 \quad 123$
$213 \quad 132$
$\sigma(1)=2 \quad \tau(2)=3$
$123\ \tau(\sigma(1))=3$
312
$(1,3,2)$

S_3: identity. product of zero transpositions. $\quad$ 3-cycle.
transpositions $\quad (1,2)(2,3)$
one transpositions.

~~The normal subgroup can be represented by a product of an even number of transpositions.~~

Definition: Let $\sigma \in S_n$ be a permutation.
σ is even if it can be represented as a product of an

σ is even if it can be represented as a product of an even number of transpositions.

σ is odd ...... odd number of transpositions.

Lemma: $\sigma \in S_n$ is not both an even and an odd transposition

Definition: Let $x_1, x_2, \cdots x_n$ be indeterminate and set.

$$f(x_1, x_2 \cdots x_n) = \prod_{i<j} (x_i - x_j)$$

$$f(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = f.$$

~~(123)~~ ~~123~~ 231

Definition: $\sigma \in S_n$. Let. $g = \sigma^*(f) = \prod_{i<j} (x_{\sigma(i)} - x_{\sigma(j)})$.

S_4

1 2 3 4

2 1 3 4

$x_{\sigma(1)}, x_{\sigma(2)} = x_2, x_1.$

$x_{\sigma(3)} - x_{\sigma(4)} = x_3 - x_4.$

$\sigma = (1,2) \in S_3.$

$$g = \sigma^*(f) = \underbrace{(x_2 - x_1)}_{x_{\sigma(1)} - x_{\sigma(2)}} \underbrace{(x_2 - x_3)}_{x_{\sigma(1)} - x_{\sigma(3)}} (x_1 - x_3) = -f$$

Lemma: σ and τ be two permutations. $\rho = \sigma\tau$. Then

1) $\sigma^*(f) = \pm f.$

2) $\rho^*(f) = \sigma^*(\tau^*(f))$

3) $\sigma^*(f) = -f$ whenever σ is a transposition.

$\sigma(\{3\}) = f.$

$\sigma(\tau(f)) = -f$

Proof: 1) g is a product of $(x_i - x_j)$ or $(x_j - x_i)$ hence 1).

2). $$\sigma^*(\tau^*(f)) = \sigma^*\left(\prod_{i<j} x_{\tau(i)} - x_{\tau(j)}\right)$$

$$= \prod_{i<j} (x_{\sigma(\tau(i))} - x_{\sigma(\tau(j))})$$

$$= \prod_{i<j} (x_{\rho(i)} - x_{\rho(j)}).$$

$\sigma = (70, 100)$

$= \prod_{i<j} (x_{\rho(i)} - x_{\rho(j)}).$

$= \rho^*(f).$ hence 2).

$\sigma = (70, 100)$

$(x_i - x_j)$

$i \neq 70 \quad j \neq 100$

3). suppose $\sigma = (a,b)$ $a<b$.

f is affected only by the terms of x_a, and x_b.

① $i \neq a$ and $j = b \Leftarrow$

② $i = a$ and $j \neq b \Leftarrow$

③ $i = a$ and $j = b \Leftarrow$

$1 \cdots 100$

$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \cdots (x_{99} - x_{100}) = f.$

$x_{\sigma(1)} - x_{\sigma(2)} = x_1 - x_2$

$(x_{\sigma(70)} - x_{81})$ ②

$(x_{\sigma(70)} - x_{92})$

$(x_{56} - x_{\sigma(100)})$ ①

x_{70}

$(x_{81} - x_{70})$

$(x_{81} - x_{\sigma(100)})$

$(x_{81} - x_{100})$

$a-b-1$

① $i<a$ $x_i - x_b \Rightarrow x_i - x_{\sigma(b)} \Rightarrow x_i - x_a.$

$a<i<b.$ $x_a - x_i \Rightarrow x_{\sigma(a)} - x_i \Rightarrow x_b - x_i$ changed.

$i>b>a$ $\underset{-}{x_a - x_i} \Rightarrow \underset{-}{x_b - x_i}$

$x_{70} - x_{100}$

$(x_{\sigma(70)} - x_{\sigma(100)})$

$\underset{-}{(x_{100} - x_{70})}$

② $i = a$ and $j \neq b$ $j<a<b$ $x_{\sigma(a)} - x_j$

$\underset{-}{x_a - x_j} \Rightarrow \underset{-}{x_b - x_j}.$

$j>b>a$ $\underset{+}{x_a - x_j} \Rightarrow \underset{+}{x_b - x_j}$

$5 \; 6 \; 7 \; 8 \; 9 \; 10$

$10 - 5 - 1 = 4.$

$a<j<b.$ $\underset{+}{x_a - x_j} \Rightarrow \underset{-}{x_b - x_j}$ changed. $a-b-1$

$a \to b$ $a-b-1$ sign chages

$+ \underset{\uparrow}{} - \underset{\uparrow}{} +$

2

in total: $2(a-b-1)$ sign changes. (even).

③. $\underset{+}{x_a - x_b} \Rightarrow \underset{-}{x_b - x_a}$ changed.

$\sigma^*(f) = -f$ $\square$.

proof: σ is even transpositions (suppose).

$\sigma^*(f) = f$

proof: σ is even transpositions (suppose).

$$\sigma^*(f) = f.$$

σ is odd transpositions.

$$\sigma^*(f) = -f.$$

thus σ cannot be both even and odd. $\square$

8.1.2022. China (4:00) afternoon.