12th basic algebra

2022年10月9日 星期日 下午12:00

Isomorphism of vector spaces. ①
dual spaces ②
dual basis ③
Annihilator. ④

$V = \{000, 111\}$ dim $V = ?$ 1

basis: 111

finite-dimensional example:

Let U: n-dimensional ordered basis Γ
let V: k-dimensional ordered basis Δ

$\mathrm{Hom}_{\mathbb{F}}(U, V) \cong M_{nk}(\mathbb{F})$ carries a member L of $\mathrm{Hom}_{\mathbb{F}}(U,V)$ to k-by-n matrix $\left({}_{\Delta}L_{\Gamma}\right)$

"Is isomorphic to" is an equivalence relation.

mod 5
[0] [1] [2] [3] [4]

1. $x \sim x$ reflexive: V is isomorphic to V

2. $x \sim y \Rightarrow y \sim x$. symmetry. $L: U \to V$
$L^{-1}: V \to U$ function.
$\Downarrow$ L^{-1} linear map.

$$L(L^{-1}(v_1) + L^{-1}(v_2)) = L(L^{-1}(v_1)) + L(L^{-1}(v_2))$$
$$= v_1 + v_2 = I(v_1+v_2)$$
$$= L(L^{-1}(v_1+v_2)).$$

$L(L^{-1}(cv)) = cv = c \cdot L(L^{-1}(v)) = L(c \cdot L^{-1}(v))$

$$\mathcal{L}(\mathcal{L}^{-1}(cv)) = cv = c\cdot \mathcal{L}\cdot(\mathcal{L}^{-1}(v)) = \mathcal{L}(c\cdot \mathcal{L}^{-1}(v))$$

3. transitive. $x \sim y,\ y\sim z \Rightarrow x \sim z$. Equivalence classes.
$\mathcal{L}: U \to V$ $\ M: V \to W$. $\ M\mathcal{L}: U \to W$.

proposition: Two finite-dimensional vector spaces over $\mathbb{F}$ are isomorphic iff they have the same dimension.

proof. $\Rightarrow$ if $U \cong V$ then isomorphism carries any basis of U to a basis of V.

U, V have the same dimension.

$\Leftarrow$

$U: (u_1, \cdots u_n)$
$V: (v_1, \cdots v_n)$

Define $l(u_j) = v_j$ for $1 \le j \le n$ According to proposition.

$\exists\ \mathcal{L}: U \to V$ is the linear extension of l.
linear. one-to-one. onto. $U \cong V$ $\square$

isomorphism

isomorphism via a canonically constructed linear map.

Dual spaces

Let V be a vector space over $\mathbb{F}$.

A linear functional on V is a linear map from V to $\mathbb{F}$.

The space of all such linear maps is a vector space.

Denote it by V' and call it the dual space of V.

$\mathbb{F}^n$: standard ordered basis: Σ_1

dim($\mathbb{C}$ over $\mathbb{R}$) = 2

$\mathbb{F}$: basis 1.

$\{a+bi,\ a,b\in\mathbb{R}\}$

(1, 0)
(0, 1)

linear functional: v' on $\mathbb{F}^n$.

$$\begin{pmatrix} v' \\ 1\Sigma_1 \end{pmatrix} = (\underline{v'(e_1)} \quad v'(e_2) \ \cdots \ v'(e_n))_{\Delta}$$ n-dimensional row vector

$$\boxed{\begin{pmatrix} L(u) \\ \Delta \end{pmatrix} = \begin{pmatrix} L \\ \Delta P \end{pmatrix}\begin{pmatrix} u \\ P \end{pmatrix}}$$

column vector $v = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ $\Rightarrow v'$ $v'(v)$ is a multiple of the scalar 1.

$$\begin{pmatrix} v'(v) \\ 1 \end{pmatrix} = \begin{pmatrix} v' \\ 1\Sigma \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (v'(e_1) \quad v'(e_2) \ \cdots \ v'(e_n))\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

The space of all linear functionals on $\mathbb{F}^n$ may be identified with the space of all n-dimensional row vectors. and the effect of the row vector on a column vector is given by matrix multiplication.

$\Uparrow$: canonical.

For any V. $\not\Rightarrow V'$.

We first fix an ordered basis $\underline{\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}}$ of $\underline{V}$. if $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ are permuted

V': space of n-dimensional row vectors.

The members of V' that corresponds to the standard basis of row vectors.

$\left.\begin{matrix}(1, 0, 0 \cdots) \\ (0, 1, 0 \cdots) \\ \vdots\end{matrix}\right\}$ the space of row vectors.

linear functionals: v_i'

$\underline{v_i'(v_j) = \delta_{ij}}$ δ_{ij}: Kronecker delta $\delta_{ij} = \begin{cases} 0 & i\neq j \\ 1 & i=j. \end{cases}$

$(v_1', \cdots v_n')$ form a basis of V'

$\left\{ \begin{array}{l} v_1'(v_1)=1 \\ \quad v_2 = 0 \\ \quad v_3 = 0 \end{array} \right.$ $\quad v_2'(v_1)=0 \quad v_2'(v_2)=1 \quad v_j'(v_1)=0 \; \vdots \; v_j'(v_j)=1$

$\begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix}$ are. permuted in the same way.

$\{v_1', \cdots v_n'\}$ is called the dual basis of V related to $\{v_1, v_2, \cdots v_n\}$

Proposition. If V is a finite-dimensional vector space with dual V', then V' is finite-dimensional with $\dim V = \dim V'$.

proof: homework.

Every vector subspace U of a finite dimensional V can be described as the intersection of the kernels of a finite set of linear functionals.

Take a basis of a vector subspace U. $\{u_1, \cdots u_r\}$. $\triangle$ $(u_1', \cdots u_r')$

extend it to a basis of V

$\{u_1, \cdots u_r, u_{r+1}, \cdots u_n\}$. $\triangle$

$\{u_1', \cdots u_r', u_{r+1}', \cdots u_n'\}$.

basis of V'.

U is described as the set of all vectors in V.

such that $\underbrace{v_j'(v)=0}_{\text{kernels}}$ for $\boxed{r+1 \leq j \leq n.}$

annihilator of U. $\quad \mathrm{Ann}(U)$

for the vector subspace of all members v' of V' with $v'(u)=0$ for all u in U.

proposition. Let V be a finite-dimensional vector space, let U be a vector subspace of V. Then.

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1). $\dim U + \dim \mathrm{Ann}(U) = \dim V$.

2) every linear functional on U extends to a linear functional on V.

3). whenever v_0 is a member of V that is not in U. there exists a linear functional on V that is 0 on U and is 1 on v_0.

proof: $\{v_1, \cdots v_r\}$ a basis of U

$\{v_1, \cdots v_r, v_{r+1} \cdots v_n\}$ a basis of V

$\{v_1', \cdots v_n'\}$. a basis of V'.

1). $\{v_{r+1}', \cdots v_n'\}$. span $\mathrm{Ann}(U)$

These linear functionals are 0 on every members of the basis.

$\{v_1, \cdots v_r\}$. of U. hence are in $\mathrm{Ann}(U)$

Assume v' is a member of $\mathrm{Ann}(U)$

$v' = c_1 v_1' + \cdots c_n v_n'$. for some scalars $c_1, \cdots c_n$.

v' is 0 in U. $\Rightarrow$. $v'(v_i) = 0$. $i \leq r$.

$v'(v_i) = c_i \quad c_i = 0$ for $i \leq r$.

Therefore v' is a linear combination of $v_{r+1}', \cdots v_n'$.

2). $v_1'|_U \cdots v_r'|_U$ form a dual basis of U'.

if u' in U'. $u' = c_1 v_1'|_U + \cdots c_r v_r'|_U$

$v' = c_1 v_1' + \cdots c_r v_r'$ is the required extension of u' to all of V.

3). $\{v_1, \cdots v_r\}$. basis of V. $\quad v_{r+1} = \underline{v_0}$.

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we adjoin $v_{r+2}, \cdots v_n$. to obtain a basis $\{v_1, \cdots v_n\}$ of V.

v'_{r+1} has the required property. $\square$.

23. October. 2022.
3pm (+8).