Group action 2

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Cayley table.

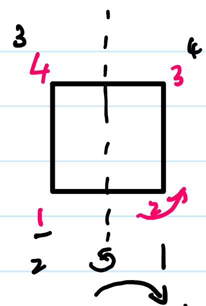
subgroup.

group with order 4.

isomorphism.

proof: 1). $\underbrace{e, r \cdots\cdots}_{2n}$

2). $\underbrace{e, r \cdots r^{n-1}, s, rs, \cdots r^{n-1}s}_{\text{distinct}}$ ✓ *



r: $e. r, \cdots r^{n-1}$ (rotation)

anticlockwise → clockwise.

$e, r, \cdots r^{n-1}$. 1 → different place anticlockwise

$s, rs, \cdots r^{n-1}s$. 1 → different place. clockwise.

Assume 1 to k $(1 \le k \le n)$

$\underline{r^{1-k}}\,\underline{t} = e$. $t = r^{k-1}$.

rotation

$\underline{t}s = r^{k-1}$. $t = r^{k-1} \cdot \underline{s^{-1}} = r^{k-1} \cdot s$. *

changes the vertice to a clockwise

Dihedral groups □

product group: Let $(G, *_G)$ ✓ and $(H, *_H)$ ✓ be groups.

Then the operation $*$ defined on $G \times H$ by

$$(g_1, h_1) * (g_2, h_2) = (g_1 *_G g_2, h_1 *_H h_2)$$ ✓

is a group operation $(G \times H, *)$ is called the product group. the product of G and H.

proof: $*$. is associative ⇐ $*_G$ is associative
$*_H$ is associative.

$$[(g_1, h_1) * (g_2, h_2)] * (g_3, h_3) = (g_1, h_1) * [(g_2, h_2) * (g_3, h_3)]$$

$e_{G\times H} = (e_G, e_H)$

$$(g,h) * (e_G, e_H) = (g,h) = (e_G, e_H) * (g,h)$$

$(g,h)^{-1} = (g^{-1}, h^{-1})$

$$(g,h) * (g^{-1}, h^{-1}) = (g * g^{-1}, h * h^{-1}) = (e_G, e_H) \qquad \square$$

Definition: The cardinality $|G|$ of a group G is called the order of G.

We say a group is finite if $|G|$ is finite.

Definition: let $G = \{e, g_2, g_3, \dots g_n\}$ be a finite group.

Arthur Cayley (1821–1895).

The Cayley table of G is a square grid which contains all the possible products of two elements from G

$g_i g_j \Rightarrow$ ith row, jth column of the Cayley table.

Remark: A group is abelian if and only if (iff) Cayley table is symmetric about the main (top-left to bottom-right) diagonal.

D_6.

$*$	e	r	r^2	s	rs	r^2s.
e	e	r	r^2	s	rs	r^2s
r	r	r^2	e	rs	r^2s	s
r^2	r^2	e	r	r^2s	s	rs
s	s	$s\cdot r = r^2s$	rs	e	r^2	r
rs	rs	s	r^2s	r	e	r^2
r^2s	r^2s	rs	s	r^2	r	e

$r^3 = e$

$s^2 = e$

$sr = r^2s$.

not abelian.

$$(r^2s)(rs) = r^2 s \cdot r \cdot s = r^2 \cdot r^2 \cdot s \cdot s = r^4 s^2 = r$$

proposition: A Cayley table is a Latin square.

Every group element appears precisely once in each row and in each column.

proof: Given g_k: $G \to G$ bijection.

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$g \to g_k g$. $\quad g_k g \to g_k^{-1} \cdot g_k g = g$.

k^{th} row contains each element of G percisely once.

Given $g_k : G \to G$ bijection.
$g \to g \cdot g_k$

k^{th} column contains each element of G percisely once. $\square$.

D_6.

$*$	e	r	r^2
e	e	r	r^2
r	r	r^2	e
r^2	r^2	e	r

$\{e, r, r^2\}$. makes a group.
subgroup. of D_6

D_8.

$*$	e	r	r^2	r^3
e	e	r	r^2	r^3
r	r	r^2	r^3	e
r^2	r^2	r^3	e	r
r^3	r^3	e	r	r^2

$\{e, r, r^2, r^3\}$. makes a group.
subgroup of D_8.

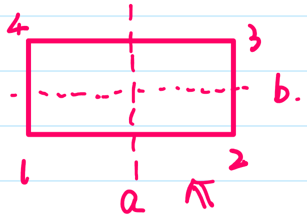
Definition. Let G be a group. We say that $H \subseteq G$ is a subgroup of G. if the group operation $*$ restricts to make a group of H.

H is a subgroup of G if:

i). $e \in H$. ✓
ii). $g_1, g_2 \in H$, then $g_1 g_2 \in H$. ✓.
iii) $g \in H$. $g^{-1} \in H$. ✓

associativity.
$(h_1 * h_2) * h_3 = h_1 * (h_2 * h_3)$.
$h_1 \in H$. $h_1 \in G$.
$(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$

Example.



$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ $\quad b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ $\quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = ab$

permutation. Klein four-group. V. V_4.

Felix Klein. (1849-1925).

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b

vier-four V V_4

e	e	a	b	c
a	a	e	c	b
b	b	c=ab	e	a
c	c	b	a	e

Felix Klein. (1849–1925).

vier. – four. V V_4

order of G.

*	e
e	e

2.

*	e	a
e	e	a
a	a	e

$a^2=a$, $a=e$

$\{0,1\}$.

3.

*	e	a	b
e	e	$\boxed{a}$	b
a	a	~~b~~ e	e
b	$\boxed{b}$	e	a

latin square

$ab = e$.

$ba = e$.

*	e	a	b	c
e	e	a	b	c
a	a	$c = a^2$	e	$b = c\cdot a = a^2 \cdot a = a^3$
b	b	e	~~e~~ a	
c	c	b	a	e

$\to a^3$ $\to a^2$

case i). $ba = e$. $b = a^{-1}$. $ab = e$.

$ba = e, a, b, c$.

$b = e$.

*	e	a	a^2	a^3
e	e	a	a^2	a^3
a	e	a^2	a^3	e
a^2	a^2	a^3	e	a
a^3	a^3	e	a	a^2

rotation of D_8.

Case ii). $ba = c$.

$ab \neq e$. ab

$ba \neq e$.

*	e	a	b	c
e	e	a	b	c
a	a	b/e	c	e/b
b	b	c	e	a
c	c	e/b	a	b

$b = a^2$

$c = b\cdot a = a^3$.

Case ii) a). $a^2 = b$.

Case ii) b'). $a^2 = e$

1). $b^2 = a$.

a	e
e	a

$a = b^2$

$a\cdot b = b^3 = c$

2). $b^2 = e$

e	a
a	e

*	e	a	a^2	a^3
e	e	a	a^2	a^3
a	a	a^2	a^2	e
a^2	a^2	a^3	e	a
a^3	a^3	e	a	a^2

*	e	a	b	c
e	e	a	b	c
a	b	e	c	b
b	b	c	e	a
c	c	b	a	e

V_4.

latin square represented a group

G

Definition Let G be a group. $g \in G$. The order of g. $o(g)$.

is the least integer s.t. $g^r = e$. no such r exists. g has infinite order.

is the least integer s.t. $g^r = e$. no such r exists. g has infinite order.

Definition. An isomorphism $\phi: G \to H$ is a bijection. $\forall g_1, g_2 \in G$.

$(G, *_G)$ $(H, *_H)$

$$\phi(g_1 *_G g_2) = \phi(g_1) *_H \phi(g_2)$$

G, H are isomorphic.

$a \in G_1$.

$f: G_1 \to G_2$ $b \in G_2$.

$f(ab) = f(a) f(b)$.

$e_{G_1} \to e_{G_2}$.

25. Feb. 2023.

4pm.