19th basic algebra

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polarization.

Cauchy-Schwarz inequality.

orthogonal and unit vector

distributive law: $(a+bi)(c+di)$

Complex numbers. $\mathbb{C}$ is 2-dim vector space over $\mathbb{R}$.

$\{1, i\}$. $1\cdot 1 = 1$, $1\cdot i = i$, $i\cdot i = -1$ $\quad a+bi$ or $a+ib$, $a, b \in \mathbb{R}$

multiplication: associative commutative.

$(a+bi)(c+di) = (c+di)(a+bi)$

identity: 1. $1+0i$ $\in \mathbb{C}$.

inverse: $(a+bi)\cdot\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i\right) = \frac{a^2+abi - abi - b^2i^2}{a^2+b^2}$ ($+b^2$) $= \frac{a^2+b^2}{a^2+b^2} = 1$

complex conjugation: $\overline{a+bi} = a-bi$

linear.

$\overline{z+w} = \overline{a+c+(b+d)i} = \overline{z} + \overline{w}$. $\quad z = a+bi$, $w = c+di$

$C(z+w) = C(z) + C(w)$

$r\cdot C(z) = C(rz)$.

scalar $\overline{rz} = \overline{ac+adi} = ac - adi = a(c-di) = r\cdot\overline{z}$ $\quad r = a$, $z = c+di$

$\overline{z}\cdot\overline{w} = \overline{(a+bi)}\,\overline{(c+di)} = (a-bi)(c-di) = ac - adi - bci + bdi^2$

$z\cdot w = ac + adi + bci - bd$ $\qquad = ac - bd - (ad+bc)i$.

$= (ac-bd) + (ad+bc)i$

$\overline{z}\cdot\overline{w} = \overline{z\cdot w}$.

real and imaginary parts. $\quad z = a+bi$. $\quad \operatorname{Re} z = a = \frac{1}{2}(z+\overline{z})$

$\operatorname{Im} z = b = \frac{1}{2i}(z - \overline{z})$

$a+bi - (a-bi) = 2bi$.

absolute value: $z = a+bi$ $\quad |z| = \sqrt{a^2+b^2}$ $\quad |z|^2 = (a+bi)(a-bi) = z\cdot\overline{z}$.

$a+bi-(a-bi)=2bi.$

absolute value:

$z=a+bi \qquad |z|=\sqrt{a^2+b^2} \qquad |z|^2=(a+bi)(a-bi)=z\cdot\bar z.$

$\bar z=a-bi \qquad |\bar z|=\sqrt{a^2+b^2}=|z| \qquad |\mathrm{Re}\, z|\le|z|.$ "=" with $b=0$

$|\mathrm{Im}\, z|\le|z|$ "=" with $a=0$

$\underset{a+bi}{z}\,\underset{c+di}{w}$

$$|zw|=|z|\cdot|w| \Rightarrow |zw|^2=zw\cdot\overline{zw}=zw\cdot\bar z\cdot\bar w$$
$$=z\cdot\bar z\cdot w\cdot\bar w$$
$$=|z|^2|w|^2.$$

$$|z+w|\le|z|+|w| \Rightarrow |z+w|^2=(z+w)(\overline{z+w})$$
$$=(z+w)(\bar z+\bar w)=z\cdot\bar z+\underbrace{z\cdot\bar w+w\bar z}+w\bar w.$$
$$=|z|^2+2\mathrm{Re}(z\bar w)+|w|^2$$
$$\le|z|^2+2|z\bar w|+|w|^2$$
$$=|z|^2+2|z||w|+|w|^2$$
$$=(|z|+|w|)^2$$

$(a+bi)(c-di)+(a-bi)(c+di)$

$ac+bci-adi+bd+ac-bci+adi+bd$

$2ac+2bd. \quad 2(ac+bd)$

$2\mathrm{Re}(z\bar w)$

$$z\cdot\bar w+w\bar z=2\mathrm{Re}(z\bar w)=2\mathrm{Re}(w\bar z)$$

Euler formula:

$$e^{i\theta}=\cos\theta+i\sin\theta. \qquad \theta=\pi.\ e^{i\pi}=-1$$

$e^{i\pi}+1=0$

$\in\mathbb{R}$

$$z=a+bi=|z|\cdot(\cos\theta+i\sin\theta)=|z|\cdot e^{i\theta}.$$
$$\bar z=a-bi=|z|(\cos\theta-i\sin\theta)=|z|e^{-i\theta}$$

$\mathbb{C}\to\mathbb{R}\cdot e^{i\theta}$

law of cosines:

$$\|u+v\|^2=\|u\|^2+2\,\mathrm{Re}(u,v)+\|v\|^2$$

$=x_1\bar y_1+\cdots x_n\bar y_n.$

dot product: $u=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} \quad v=\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}$ $\quad (u,v)=x_1y_1+\cdots+x_ny_n$

$\|v\|=\sqrt{(v,v)}$

$\begin{pmatrix}x_1+y_1\\ \vdots\\ x_n+y_n\end{pmatrix} \qquad (x_1+y_1)^2+\cdots(x_n+y_n)^2=\|u+v\|^2$

$x_1^2+\cdots x_n^2=\|u\|^2$

$y_1^2+\cdots y_n^2=\|v\|^2$

$2\mathrm{Re}(u,v)=2x_1y_1+\cdots+2x_ny_n \Leftarrow$

$(-i)^0=1 \quad (-i)^2=-1$

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$(-i)^2=(-1)^2\cdot i^2=-1$

polarization: $$(u,v)=\frac14\sum_k i^k\|u+i^kv\|^2$$

$k\in\{0,2\}$ if scalars are real

$k\in\{0,1,2,3\}$ if scalars are complex.

$$\|u+i^kv\|^2=\|u\|^2+2\mathrm{Re}(u,i^kv)+\|v\|^2$$
$$=\|u\|^2+2\mathrm{Re}((-i)^k(u,v))+\|v\|^2.$$

real. $\|u\|^2+2i^0\mathrm{Re}((-i)^k(u,v))+\|v\|^2$

$-\|u\|^2+2i^2\mathrm{Re}((-i)^k(u,v))-\|v\|^2$

$$\sum_k i^k\|u+i^kv\|^2=2\sum_k i^k\,\mathrm{Re}((-i)^k(u,v)).$$

$-1.\ \mathrm{Re}(z).$

k is even: $i^k\cdot\mathrm{Re}((-i)^k\cdot \underset{(a+bi)}{z})=\mathrm{Re}\, z.$

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$(a+bi)$

$k=0,\ i^k = 1.\ Re(1\cdot z) = Re\, z.$

$k=2,\ i^k=-1.\ Re(-1\cdot z) = -Re\, z$

$\underbrace{Re\, z}_{a} + i \underbrace{Im\, z}_{b} = \underset{a+bi}{z}.$

k is odd: $i^k \cdot Re((-i)^k \cdot z) = i \cdot Im\, z.$

$$2 \times \sum_k i^k Re((-i)^k \cdot z)) = 2z. \times 2$$

$$\triangle\ \sum_k i^k \|u + i^k v\|^2 = 4(u,v) \Rightarrow (u,v) = \frac{1}{4}\sum_k i^k \|u+i^k v\|^2. \ \divideontimes$$

Schwarz inequality: In any inner product space V. $\|(u,v)\| \le \|u\|\|v\|$. $\forall u,v \in V$.

definition. (i). integrals.

proof. $\boxed{\text{homework}}$ $u \to e^{i\theta} \cdot u.$ $(u,v) \to e^{i\theta}(u,v)$

law of cosines. $\underbrace{\|u - \|v\|^{-2}(u,v)\, v\|^2}_{\ge 0} \ge 0.$ $\|v\| \ne 0.$

$\|u + \underset{\ge 0}{c}v\|^2 \Rightarrow \underset{\text{must}}{(u,v) = 0.}$ $\|v\| = 0.$

Proposition: In any inner-product space V, the norm satisfies:

(a) $\|v\| \ge 0$, $\forall v \in V$. $\|v\| = 0$ iff $v = 0$

(b). $\|cv\| = |c|\, \|v\|$ $\forall v \in V$, $\forall c$ scalars

(c). $\|u+v\| \le \|u\| + \|v\|$ $\forall u,v \in V$.

proof: (a) $\|v\| \ge 0 \Leftarrow \sqrt{(v,v)} = \|v\| \Leftarrow (v,v) \ge 0,$ $(v,v) = 0$ iff $v = 0$

(b). $\|cv\|^2 = (cv, cv). = c(v, cv) = c \cdot \bar{c}(v,v) = |c|^2 \|v\|^2.$

(c). $\|u+v\|^2 = \|u\|^2 + 2Re(u,v) + \|v\|^2 \le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$
$= (\|u\| + \|v\|)^2.$ $\square$

$u \in V$, $v \in V$,

u, v are orthogonal if $(u,v) = 0$. $u \perp v$. perpendicular

An orthogonal set in V is a set of vectors such that each pair is orthogonal.

The nonzero members of an orthogonal set are linearly independent.

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if $\{v_1, \cdots v_k\}$ is an orthogonal set of nonzero vectors.

$\exists\ c_1v_1 + \cdots + c_kv_k = 0. \Rightarrow c_1 \cdots c_k = 0.$ linealy independent.

$$0 = (c_1v_1 + \cdots + c_kv_k,\ v_j) \underset{\text{orthogonal}}{=} c_j \|v_j\|^2 \qquad c_j = 0.\ \text{for every } j.$$

A unit vector in V is a vector u with $\|u\| = 1$.

if v is any nonzero vector. $\frac{v}{\|v\|}$ is a unit vector.

An orthonormal set. in V is an orthogonal set of unit vectors.

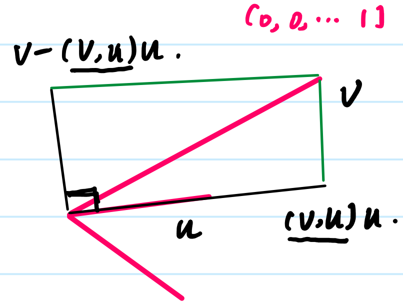
V is finite. an orthonormal basis of V is an orthonormal set that is a vector-space basis.

Example: $\mathbb{R}^n$. $\mathbb{C}^n$: $\{e_1, \cdots e_n\}$. standard basis $(1, 0, \cdots 0)$ $(0, 1, \cdots 0)$ $\vdots$ $(0, 0, \cdots 1)$

u is a unit vector.

$$v = (v,u)u + (v - (v,u)u).$$

decomposition is unique.



$$(u, v - (v,u)u) = (u,v) - \overline{(v,u)}(u,u)$$
$$= (u,v) - (u,v) \quad 1 \quad \text{orthogonal}$$
$$= 0.$$

unique: assume.

$v = v_1 + v_2.$ $\quad v_1 = cu.$ $\quad (v_2, u) = 0.$ $\Rightarrow (v, u) = (v_1 + v_2, u) = \underbrace{(cu,u)}_{c} + \underbrace{(v_2,u)}_{0}$

$= c.$ must be (v,u).

$v_1 = cu = (v,u)u.$

$v_2 = v - (v,u)u.$

proposition: Let V be an inner-product space.

$\{u_1, \cdots u_k\}$ is an orthonormal set in V. given $v \in V$.

then there exists a unique decomposition:

$$v = c_1u_1 + \cdots + c_ku_k + v^{\perp}.$$

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with $v^{\perp}$ orthogonal to u_j for $1 \le j \le k$, $c_j = (v, u_j)$.

proof: uniqueness: Taking the inner product of both sides with u_j

$$(v, u_j) = (c_1 u_1 + \cdots + c_k u_k + v^{\perp}, u_j) = \underbrace{c_j}_{\text{is unique}}. \text{ for each } j.$$

$v^{\perp}$ can only be $v - (v, u_1)u_1 - \cdots - (v, u_k)u_k$.

existense: put $c_j = (v, u_j)$

we need. $v - (v, u_1)u_1 - \cdots (v, u_k)u_k$. is orthongonal to each u_j. $1 \le j \le k$.

$$(v - \sum_i (v, u_i)u_i, u_j) = (v, u_j) - \boxed{\sum_i ((v, u_i)u_i, u_j)}. \quad (u_i, u_j) = 0.$$

$$= (v, u_j) - \boxed{(v, u_j)} = 0 \qquad \square$$

11. Dec. 2022.

4pm. (UTC+8).