15th basic algebra

2022年10月30日 星期日　　下午12:04

Determinants.

k- linear functional

alternating.

Group. vector space

definition of determinant: is a certain scalar. if attached initially to any square matrix

ultimately to any linear map from a finite-dimensional vector space into itself.

example:

$$\det \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{x} = \underbrace{ad - bc}_{f(x)}.$$

$$\det \underbrace{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}}_{x} = aei + bfg + cdh - afh - bdi - ceg.$$

For n-by-n square matrices the determinant function will have the following important properties:

(i) $\det(AB) = \det(A)\det(B)$

(ii): $\det(I) = 1$

$$\det(A)\cdot\det(A^{-1}) \overset{(i)}{=} \det(A\cdot A^{-1}) = \det(I) = 1$$

(ii): $\det(I) = 1$

(iii) $\det(A) = 0$ iff A has no inverse.

$= \det(I)$

$= 1$

↓ extend the function to be defined on all linear maps.

$L: V \to V$. V. finite-dimensional. Let Γ be any ordered basis of V.

$$\det(L) = \det\left(\underset{\Gamma\Gamma}{L}\right)$$

Δ is another ordered basis.

$$\det\left(\underset{\Delta\Delta}{L}\right) = \det\left(\underset{\Delta\Gamma}{I}\right)\det\left(\underset{\Gamma\Gamma}{L}\right)\det\left(\underset{\Gamma\Delta}{I}\right) = \det\left(\underset{\Gamma\Gamma}{L}\right)$$

$\left(\underset{\Delta\Gamma}{I}\right)\left(\underset{\Gamma\Delta}{I}\right)$ inverse.

$\det\left(\underset{\Delta\Gamma}{I}\right) = \dfrac{1}{\det\left(\underset{\Gamma\Delta}{I}\right)}$ reciprocals.

$\det\left(\underset{\Delta\Delta}{L}\right) = \det\left(\underset{\Gamma\Gamma}{L}\right) \Rightarrow \det(L)$ unique.

Hence the definition of $\det L$ is independent of the choice of ordered basis, and determinant is well-defined on the linear map. $L: V \to V$.

$x_1 = x_2$

$\det x_1 = \det x_2$.

$\det(L_1)\cdot\det(L_2) = \det(L_1 \cdot L_2)$ satisfies (i) (ii) (iii)

$A_1 \cdot A_2 = A_1 \cdot A_2$.

It is enough to establish the determinant function on n-by-n matrices.

signs of permutation.

det

We view det on n-by-n matrices over $\mathbb{F}$ as a function of n rows of the matrix.

V. for the vector space $M_{1n}(\mathbb{F})$ of all n-dimensional row vectors.

$f: V \to \mathbb{F}$.

A function: $f: V \times V \cdots \times V \to \mathbb{F}$ defined on ordered k-tuples of members of V is called k-multilinear functional or k-linear functional if it depends linearly on each of the k vector variables members of V is called k-multilinear functional or k-linear functional if it depends linearly on each of the k vector variables when the other $k-1$ vector variables are held fixed. △

$k=2$ $\quad f((a\ b),(c\ d)) = ac + b(c+d) + \frac{1}{2}ad.$ 2-linear functional on $M_{12}(\mathbb{F}) \times M_{12}(\mathbb{F})$

$g((a\ b),(c\ d)) = l_1(a\ b)l_2(c\ d) + l_3(a\ b)\,l_4(c\ d)$

g is 2-linear functional on $M_{12}(\mathbb{F}) \times M_{12}(\mathbb{F})$

l_1, l_2, l_3, l_4 are linear functionals on $M_{12}(\mathbb{F})$

$\boxed{g(\underbrace{(a\ b\ c)}_{M_{13}(\mathbb{F})}\underbrace{(d\ e\ f)}_{})}$ 2-linear $M_{13}(\mathbb{F})$

v_1, v_2, v_3 2-tuples of basis vectors.

Let $\{v_1, \cdots v_n\}$ be a basis of V.

△ Then a k-linear functional as above is determined by its value on all k-tuples of basis vectors $(V_{i_1}, \cdots V_{i_k})$ ($i_1, \cdots i_k$ are integers between 1 and n).

(V_1, V_2) (V_2, V_1)

The reason is that: we fix all but first variables and expand out the expression by linearity so that only a basis vector remains in each term for the first variable. For the resulting terms, we fix all but second ....

A k-linear functional f on k-tuples from $M_{1n}(\mathbb{F})$ is said to be alternating if f is 0 whenever two of the variables are equal.

example: $k=2$. $n=2$ let $\{V_1 = (1\ 0).\ V_2 = (0\ 1)\}$. as basis.

2-linear functional f is determined by $f(V_1, V_1)$ ✓

$f(V_1, V_2)$

$f(V_2, V_1)$ ✓

$f(V_2, V_2)$ ✓

If f is alternating: $f(V_1, V_1)=0$ $f(V_2, V_2)=0$ $\underset{\text{def.}}{\Rightarrow}$ $f(V_1+V_2, V_1+V_2)=0$

$f(V_1+V_2, V_1+V_2) \overset{\text{linearity.}}{=} f(V_1+V_2, V_1) + f(V_1+V_2, V_2)$

$= f(V_1, V_1) + f(V_2, V_1) + f(V_1, V_2) + f(V_2, V_2) = 0$

$f(V_2, V_1) = -f(V_1, V_2)$ ✓

$= f(v_1, v_1) + f(v_2, v_1) + f(v_1, v_2) + f(v_2, v_2) = 0$

$f(v_2, v_1) = -f(v_1, v_2)$ ✓

f is completely determined by $f(v_1, v_2)$.

Whenever a multilinear functional f is alternating and two of its arguments are interchanged, then the value of f is multiplied by -1.

let's suppress all variables except for the i^{th} and j^{th}:

$$0 = f(v+w, w+v) = f(v+w, w) + f(v+w, v) = f(v, w) + f(w, w) + f(v, v) + f(w, v)$$
$$= f(w, v) + f(v, w).$$

△

Theorem: For $M_{n}(\mathbb{F})$, the vector space of alternating n-multilinear functional has dimensional 1. and a nonzero such functional has nonzero value on $(e_1^t, \cdots e_n^t)$. where $\{e_1, \cdots e_n\}$ is the standard. basis of $\mathbb{F}^n$. Let f_0 be the unique such alternating n-multilinear functional taking the value 1 on $(e_1^t, \cdots e_n^t)$. If a function

$$\det: M_{nn}(\mathbb{F}) \to \mathbb{F} \text{ is defined by } \det A = f_0(A_1, \cdots A_n)$$

where A has rows $A_1 \dots A_n$, then det has the properties that:

(a) $\det(AB) = \det A \cdot \det B$

(b) $\det I = 1$

(c) $\det A = 0$ if and only if A has no inverse.

(d) $\det A = \sum_\sigma (\operatorname{sgn} \sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}$, the sum. being taken over all permutations σ of $\{1, \cdots n\}$ △

proof: $\boxed{\text{uniqueness.}}$ Let f be an alternating n-multilinear functional. and let $\{u_1, \cdots u_n\}$ be the basis of the space of row vectors defined by $u_i = e_i^t$.

since f is multilinear. f is determined by its value on n-tuples $(u_{k_1}, \cdots u_{k_n})$.

Since f is alternating $f(u_{k_1}, \cdots u_{k_n}) = 0$ unless u_{k_i} are distinct. i.e. unless

since f is multilinear. f is determined by its value on n-tuples $(u_{k_1}, \cdots u_{k_n})$.
since f is alternating. $f(\underline{u_{k_1}, \cdots u_{k_n}}) = 0$ unless u_{k_i} are distinct. i.e., unless.
unless
$(u_{k_1}, \cdots u_{k_n})$ is the form of $(u_{\sigma(1)} \cdots u_{\sigma(n)})$ for some permutation σ.

We have seen the value of f on n-tuple of rows is multiplied by -1 if two of the rows are interchanged.

from 5th basic algebra class, the value of f on an n-tuple is multiplied by $\mathrm{sgn}(\sigma)$ if the members of the n-tuple are permuted by σ. Therefore.

$$f(u_{\sigma(1)} \cdots u_{\sigma(n)}) = \underline{\mathrm{sgn}(\sigma)}\, f(u_1, \cdots u_n).$$

f is determined completely by its value on $(u_1, \cdots u_n)$. we conclude that the vector space of alternating. n-linear functionals has dimension at most 1.

$\boxed{\text{Existence.}}$ Define det A. and therefore also f_0. by (d) Each term. in this definition is the product of n-linear functionals. the k^{th} linear functional being applied to the k^{th} argument of f_0. f_0 is consequently n-multilinear.

To see f_0 is alternating. suppose i^{th} row and j^{th} row are equal. $i \neq j$.

If τ is the transposition of i and j. then

(d) $f_0(A_1 \cdots A_n) = \sum_\sigma \mathrm{sgn}\,\sigma\, A_{1\sigma(1)} \cdots A_{n\sigma(n)}$ $\triangle$

$$A_{1\sigma\tau(1)} A_{2\sigma\tau(2)} \cdots A_{n\sigma\tau(n)} = A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}$$

$\sigma, \sigma\tau$.

Then from lemma: $(\mathrm{sgn}(\sigma\tau) = -\mathrm{sgn}(\sigma))$?

$\sum_{\sigma, \sigma\tau} = 0$.

$$(\mathrm{sgn}\,\sigma\tau) A_{1\sigma\tau(1)} \cdots A_{n\sigma\tau(n)} + \mathrm{sgn}\,\sigma (A_{1\sigma(1)} \cdots A_{n\sigma(n)}) = 0. \triangle$$

Thus if we compute the terms (d) by grouping pairs of terms. the one for $\sigma\tau$ and. the one for σ if $\mathrm{sgn}\,\sigma = +1$. we see the whole sum is 0. Thus f_0 is alternating.

Finally when A is the identity matrix I. we see that $A_{1\sigma(1)} \cdots A_{n\sigma(n)} = 0$ unless σ is the identity permutation. and the product is 1 $\mathrm{sgn}(\sigma) = +1$.

det $I = +1$.

unless σ is the identity permutation. and the product is 1 $\operatorname{sgn}(\sigma) = +1$.

$\det I = +1$.

We conclude that the vector space of alternating n-linear functionals has dimension exactly 1.

$\square$

$\boxed{\text{properties of det}}$

13. Nov. 2022

4pm.