10th abstract algebra

Monday, 6. December 2021 14:11

proof:

$$\begin{tikzcd} g \quad G \arrow[r, "\phi"] \arrow[d, "u"'] & G' \quad \phi(g) \\ G/H \arrow[ur, "f"'] & \end{tikzcd}$$

diagram commutes. $H = \ker(\phi)$ *

ϕ surjective $g \to \phi(g)$

f surjective. $g \to u(g) \to f(u(g)) = \phi(g)$

$g \xrightarrow{u} gH$ $G' \cong G/H$

bijection. f. injective. $x \in \ker(f)$. $\Rightarrow$ $h \in H$ $\phi(h) = e$

$g \in H$

identity in G'

$\phi: G \to G'$ homomorphism

$H \in G$ u: homomorphism.

$\phi(g \in H) = e \in G'$

gH

$g \to gH \to \phi(g)$ $=H$ x

$f(x) = \phi(g)$ $g \to e$

$f(x) = e = \phi(g)$ $\to gH$

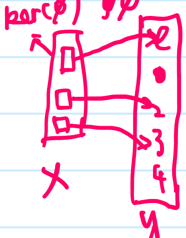
$f(gH)$ $x = H$ gH

$gH = H$

gH

$\{H\} \in \ker f$

ker(ϕ)



$\ker f = \{e\}$. then f is injective

G/H

$x = eH \Rightarrow$ identity.

$f(eH) = \phi(h)$ from lemma.

kernel $\square$

Definition: Let G be a group. Let H and K be two subgroups of G. $H \vee K$ denotes the subgroup generated by the union of H and K. $H \vee K$ is hard to identify.

$H \vee K$ set subgroup. $h \cdot k \Rightarrow l$.

Second isomorphism Theorem: Let G be a group. Let H be a subgroup and let N be a normal subgroup. Then

$$H \vee N = HN = \{hn \mid h \in H, n \in N\}. \checkmark$$

Furthermore, $H \cap N$ is a normal subgroup of H.

$$H / H \cap N \cong HN / N.$$

G' G/H

$H \to HN/N$

$H \cap N$

Proof. $h n \in H \vee N$. Show $\{h \mid h \in H, n \in N\}$ is closed under multiplication

G' $\quad G/H$

$H \cap N$

Proof: $hn \in H \cup N$. $\{hn \mid h \in H, n \in N\}$ is closed under multiplication and taking inverses.

$x = h_1 n_1$, $\quad h_i \in H$ $\qquad n_3 = h_2^{-1} \cdot n_1 h_2 \in N$ (normal)

$y = h_2 n_2$. $\quad n_i \in N$ normal subgroup $\qquad n_1 h_2 = h_2 n_3$.

$$xy = (h_1 n_1)(h_2 n_2) = h_1 (n_1 h_2) n_2 = \underbrace{(h_1 h_2)}_{\in H}\underbrace{(n_3 n_2)}_{\in N} \in \{hn \mid h \in H, n \in N\}$$

closed under multiplication

$x = hn$. $\quad m = hnh^{-1} \in N$ (normal) invariant.

$$m^{-1} = hn^{-1}h^{-1} \Rightarrow h^{-1}m^{-1} = n^{-1}h^{-1}$$

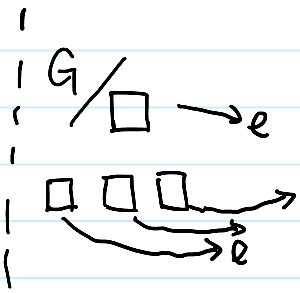
$$x^{-1} = n^{-1} \cdot h^{-1} = \underbrace{h^{-1}}_{\in H}\underbrace{m^{-1}}_{\in N} \in \{hn \mid h \in H, n \in N\}$$

Closed under taking inverses

$H \longrightarrow HN$ be the natural inclusion.

N is normal. $\qquad$ HN is normal

$g \cdot N \quad N$ $\qquad$ $\underbrace{gh}_{g' \in G} \cdot N \quad g'N \quad N$.



$\phi: H \longrightarrow HN/N$. homomorphism.

$\phi(h) \longrightarrow hN$. $\quad eN$

$x \in HN/N \qquad x = hn \cdot N = hN$. $\quad h \in H$. $\qquad$ surjective.

$h \in H$ is $\ker(\phi)$ $\quad hN = N$ the identity (coset) $\quad h \in N$.

$h \in H \cap N$

following First isomorphism theorem. $\square$

Third isomorphism theorem: Let $K \subset H$ be two normal subgroups of group G.

$$G/H \cong (G/K)/(H/K).$$

$$G/H \cong (G/K)/\underbrace{(H/K)}_{\text{kernel}}.$$

proof: $G \to G/H$ Kerner H contains K

$G/K \to G/H$ homomorphism.

$gK \to gH$. surjective

gK is the kernel gH is the identity. $gH=H$. $g\in H$.

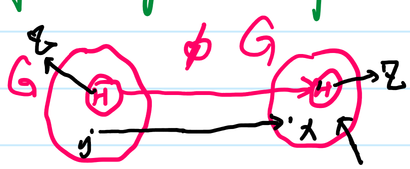
$g\in H$ $H/K=\{gK \mid g\in H\}$.

first isomorphism theorem.

$$G/H \cong (G/K)/(H/K).$$ □

Definition: Let G be a group and let H be a subgroup

We say that H is a characteristic subgroup of G, if for every automorphism ϕ of G. $\phi(H)=H$.



commutes with every element of G.

Lemma: Let G be a group. Let $Z=Z(G)$ be the centre. Then Z is characteristic normal. $yZ=Zy$.

proof. Let ϕ be an automorphism of G. $\phi(Z)=Z$. $xZ=Zx$.

bijection

$z\in Z$. $x\in G$. $x=\phi(y)$ $y\in G$.

$$x\cdot\phi(z)=\phi(y)\cdot\phi(z)$$
$$=\phi(yz)=\phi(zy)=\phi(z)\phi(y)=\phi(z)x.$$

$\phi(z)$ commutes with every element of G $\phi(z)\in Z$.

$\phi^{-1}(z)$ commutes $\phi(Z)\subset Z$.

$\phi^{-1}(Z)$ commutes

$\phi(Z) \subset Z$.

$Z \subset \phi(Z)$

$\phi(Z) = Z$

$\square$.

18. December: 4:00 (China)