GroupAction9

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First isomorphism theorem.

Group action.

$(a+b)+c = a+(b+c)$

$\mathbb{Z} = \{\cdots -2, -1, 0, 1, 2 \cdots\}$ $+$ identity. 0

inverse. $a+(-a)=0$

abelian. $a+b=b+a$.

$\{-2,-1\}$ ✗ identity.

$\{-1,0\}$ ✗ identity

$\{-1,0,1\}$ subgroup (normal)

$n\mathbb{Z}$ $n\in\mathbb{Z}$. (normal)

$8 = \{\cdots -2n, -n, 0, n, 2n, \cdots\} \in \mathbb{Z}$. subgroup

$\{\cdots -16, -8, 0, 8, 16 \cdots\}$

normal subgroup ① subgroup H $n\mathbb{Z}$ $\mathbb{Z}$ $h=-8$

② $g^{-1}hg \in H$ $h\in H$. $g\in G$ 3 $g\in\mathbb{Z}$ $g^{-1}=-g\in\mathbb{Z}$

$g^{-1}\cdot g\cdot h = h \in H$.

$g^{-1}hg = g+h+(-g)$

$3-8+(-3)=-8 = h \in H$

-8

$\in H$.

Quotient group $gH = \{gh, h\in H\}$.

$\{-16, -8, 0, 8, 16\} = 0+H = 8+H$

$\{\cdots -15, -7, 1, 9, 17 \cdots\} = 1+H$

8 cosets

$\mathbb{Z} + n\mathbb{Z}$.

$\mathbb{Z}/n\mathbb{Z} = \bmod 8$.

$19 \in 3+H$ $19 \bmod 8 = 3$.

proposition: Let G be a group and H a subset of G. Then H is a normal subgroup of G iff. it is the kernel of some homomorphism from G.

$G \xrightarrow{\phi} K \Leftrightarrow \ker\phi \lhd G$.

H.

$0+H$. identity.

proof: $\Rightarrow G \xrightarrow{\phi} K$. $\ker\phi \lhd G$

H. $\qquad\qquad$ 0+H. identity.

proof: $\Rightarrow G \xrightarrow{\phi} K$. $\ker\phi \trianglelefteq G$ $\qquad \mathbb{Z} \qquad \mathbb{Z}/8\mathbb{Z}$

$\Leftarrow H \trianglelefteq G$. normal $\qquad \pi: G \to G/H \qquad$ homomorphism ? ✓

$\qquad\qquad\qquad\qquad g \to gH. \qquad \ker(\pi) = H$? ✓

$\pi(g_1 g_2) = g_1 g_2 H = g_1H * g_2H = \pi(g_1)\pi(g_2)$

$\pi(g) = H. \Rightarrow g \in \ker(\pi) \qquad H = \ker(\pi)$.

def: $gH = H. \Rightarrow g \in H$ $\qquad \square$.

proposition: Let $\phi: G \to H$ be a group map. $\qquad f(ab) = f(a)f(b)$

ϕ is injective iff. $\ker\{\phi\} = 1$. $f(1) = f(1\cdot 1) = f(1)\cdot f(1)$

proof: $\Rightarrow \phi$ is injective. $\phi(1) = 1$. $\{1\} \subset \ker\phi$.

$\boxed{\text{if } \phi(a) = \phi(b) \text{ then } a = b}$ $\triangle$

$g \in \ker\phi$. $\phi(g) = 1. = \phi(1)$

$g = 1$

$\ker\phi = \{1\}$

$\Leftarrow$. $\ker\phi = \{1\}$, $\phi(a) = \phi(b)$.

$\phi(a)\cdot\phi(b)^{-1} = 1$

$\phi(a)\,\phi(b^{-1}) = 1 \qquad ab^{-1} \in \ker\phi$.

$\phi(ab^{-1}) = 1 \qquad ab^{-1} = 1. \Rightarrow a = b$. $\square$

First isomorphism theorem: Let $\phi: G \to H$ be a homomorphism between two groups. Then.

i) $\ker\phi \trianglelefteq G$.

ii) $\mathrm{Im}\,\phi \leq H$

iii) $g\ker\phi \mapsto \phi(g)$ gives an isomorphism

$$\begin{tikzcd} g \arrow[r, bend left, mapsto] & \phi(g) \\ G \arrow[r, "\phi"] \arrow[d, "\pi"'] & \mathrm{Im}\,\phi \\ G/\ker\phi \arrow[ur, "\bar{\phi}"'] & \\ g\ker\phi & \end{tikzcd}$$

iii) $g\ker\phi \mapsto \phi(g)$ gives an isomorphism $G/\ker\phi$. T

$$\frac{G}{\ker\phi} \cong \operatorname{Im}\phi.$$

$g\ker\phi$

$\pi: G \to G/\ker\phi.$ H.

$g \to g\ker\phi.$

proof: surjective. $\phi(g) \in \operatorname{Im}\phi$. $\forall g$.

$\exists$ $\bar\phi(\pi(g)) = \phi(g)$. $\bar\phi$ surjective.

injective: $\ker\bar\phi = \{1\}$. $\Rightarrow \ker\bar\phi = \{\ker\phi\}$.

$g\ker\phi \in \ker\bar\phi$ iff. $\bar\phi(g\ker\phi) = 1 \Rightarrow \bar\phi(\pi(g)) = 1$.

$\Rightarrow \phi(g) = 1$

$\Rightarrow g \in \ker\phi$. $\square$

Group action.

$\{1, 2, \cdots n\} \leftarrow S_n$.

S_3. $\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}\{1,3\} = \{1,2\}$.

$S = \{\phi, \{1\}, \{2\}, \{3\}, \_\ \_\ \_\ \_\}$.

Definition: A left action of a group G on a set S is a map

$$\rho: G \times S \to S.$$

s.t.

(i) $\rho(e, s) = s$ for all $s \in S$.

(ii) $\rho(g, \rho(h, s)) = \rho(gh, s)$ for all $s \in S$ and $g, h \in G$.

S_3.

$\rho(e, s) = e \cdot s$. (i) $e \cdot s = s$ for all $s \in S$

(ii) $g(h \cdot s) = (gh) \cdot s$ for all $s \in S$ and $g, h \in G$.

$g \cdot s \in S$ gives a point that s is moved by g.

$\{s, gs, g^2 s, \cdots g^{n-1} s\}$. $g^n \cdot s = e \cdot s = s$.

$GL(n,\mathbb{R}) = G$. $\mathbb{R}^n = S$ $A \cdot v = Av$.

$A \cdot A^{-1} = I$ [ ]

i). $I_n \cdot v = v$ $\forall\ v \in \mathbb{R}^n$

ii) $(AB) \cdot v = A \cdot (Bv)$ $A.B \in GL(n.\mathbb{R})$.

$GL(n,\mathbb{R})$ $M_{nn}(\mathbb{R})$ $A \cdot M = AMA^{-1}$.

[ ]

i) $I_n \cdot M = I_n M I_n^{-1} = M$

ii) $(AB) \cdot M = (AB)M(AB)^{-1}$
$= AB\,M\,B^{-1} \cdot A^{-1}$.
$= A(BMB^{-1})A^{-1}$
$= A \cdot (B \cdot M)$

group

G left action of G on itself:

$g \cdot h = gh$. $\forall, g.h \in G$.

conjugation. $g \cdot h = ghg^{-1}$. $\forall g. h \in G$.

i). $e \cdot h = h$.

ii) $g \cdot (k \cdot h) = g \cdot (khk^{-1})$
$= g \cdot khk^{-1} g^{-1}$
$= gk\,h\,(gk)^{-1} = (gk)h$.

G G/H. subgroup.

$g_1 \cdot (g_2 H) = (g_1 g_2) H$.

left action.

Definition: If a group G acts on a set S and $s \in S$ then:

i) the orbit of s. is defined as

$Orb(s) = \{g \cdot s : g \in G\} \subseteq S$.

If there is only one orbit we say that the action is transitive.

ii) the stabilizer of s is defined as

$Stab(s) = \{g \in G,\ g \cdot s = s\} \subseteq G$.

$$\mathrm{Stab}(s) = \{g \in G,\ g\cdot s = s\} \subseteq G.$$

Example: S_n (right) acts on $\{1,2,\cdots n\}$ by $\rho(k,\sigma) = k\sigma$ then there is only one orbit.

$$S_n * \{1,2,3\cdots n\} = \{1,2,3,\cdots n\}.$$

$$\mathrm{Stab}(n) = \{\sigma \in S_n,\ n\sigma = n\} = \mathrm{Sym}\{1,2,\cdots n-1\} \cong S_{n-1}.$$

proposition: The orbits of an action partition the set.

proof: G acts on S.

$\sim$ on S: $s \sim t \Leftrightarrow \exists g \in G$ s.t. $g\cdot s = t$. equivalence relation.

i). reflexive: $s \sim s$ $\quad s = e\cdot s$.

ii) symmetry: $s \sim t \Rightarrow g\cdot s = t$ $\exists g \in G \Rightarrow s = g^{-1}\cdot t. \Rightarrow t \sim s.$
$\Rightarrow t \sim s$

iii) transitive. $s \sim t, t \sim u \Rightarrow g\cdot s = t$ $\exists g.h \in G$ $\Rightarrow (hg)\cdot s = u.$ $hg \in G$
$h\cdot t = u.$
$\Rightarrow s \sim u.$

equivalence classes partition S.

$s \in S.$ $\quad \bar{s} = \{g\cdot s : g \in G\} = \mathrm{Orb}(s).$ $\square$.

proposition: The stabilizer of an action are subgroups.

proof: G acts on S. $\quad s \in S$.

$e \in \mathrm{Stab}(s).$ $\quad e\cdot s = s$

If $g.h \in \mathrm{Stab}(s)$ then

$$(gh)\cdot s = g\cdot(h\cdot s) = g\cdot s = s.$$

$$gh \in \mathrm{Stab}(s)$$

$$g^{-1}\cdot s = g^{-1}\cdot (g\cdot s) = g^{-1}\cdot g\cdot s = s.$$

$g^{-1} \in Stab(s)$ $\square$.

proposition: If two elements lie in the same orbit then their stabilizers are conjugate.

proof: $Stab(s)$ $Stab(t)$

s, t lie in the same orbit, $g\cdot s = t$ $\exists g \in G$.

$h \in Stab(s) \Rightarrow h\cdot s = s.$

$\Rightarrow h\cdot(g^{-1}\cdot t) = g^{-1}\cdot t$

$\Rightarrow g\cdot h(g^{-1}\cdot t) = t.$

$\Rightarrow ghg^{-1}\cdot t = t \Rightarrow ghg^{-1} \in Stab(t).$

$\Rightarrow h \in g^{-1} Stab(t) g.$

Hence $Stab(s) = g^{-1} Stab(t) g.$ $\square$.

Orbit-Stablizer Theorem: Let G be finite group acting on a set S and let $s \in S$.

Then: $|G| = |Stab(s)| \times |Orb(s)|.$

the size of an orbit must divide the order of the group

proof: $\dfrac{|G|}{|Stab(s)|}$ = the number of cosets of $Stab(s)$

we want $|Orb(s)|$.

bijection: the cosets of $Stab(s) \leftrightarrow Orb(s)$

$\phi: G/Stab(s) \rightarrow Orb(s)$

$\phi(g\,Stab(s)) = g\cdot s.$

$x_1 = x_2$

$\Rightarrow f(x_1) = f(x_2)$

$\phi(g\,\mathrm{Stab}(s)) = g\cdot s.$ $\Rightarrow f(x_1)=f(x_2)$

well-defined: $g\,\mathrm{Stab}(s) = h\,\mathrm{Stab}(s).$

$\Rightarrow h^{-1}g \in \mathrm{Stab}(s).$

$\xRightarrow{\text{def}} h^{-1}g\cdot s = s \xRightarrow{\cdot h\times} gs = hs.$

injective. surjective. $\phi^{-1}(g\cdot s) = g\,\mathrm{Stab}(s)$ $\square$.

$\phi: \underset{\text{set}}{S} \to \underset{\text{set}}{J}.$ isomorphism.

$\phi(g\cdot s) = g\cdot\phi(s).$

Corollary: Let G be a group and $H \le G$. Then G acts on G/H by

$$g\cdot(kH) = (gk)H.$$

$\mathrm{Stab}(H) = H.$ $gH = H.$

$\mathrm{Orb}(H) = G/H.$ $g\cdot s$

$|G/H| \times |H| = |G|.$ langrange's theorem.

Corollary: Let G be a group $g \in G$. and.

$C_G(g) = \{h \in G;\ gh = hg\}$ centralizer of g.

$C(g) = \{h^{-1}gh : h \in G\}.$ conjugacy class of g.

Then $|C_G(g)| \times |C(g)| = |G|.$

proof: $g\cdot h = ghg^{-1}.$ $g.h \in G.$

$g \in G.$ $\mathrm{Stab}(g) = \{h \in G.\ hgh^{-1} = g\}. = C_G(g).$

$\cancel{h\cdot g}$ $hg = gh.$

$\mathrm{Orb}(g) = \{h^{-1}gh.\ h \in G\} = C(g).$ $\square$

$\cancel{h\cdot g}$ $\quad hg = gh.$

$$\mathrm{Orb}(g) = \{h^{-1}gh \,.\, h \in G\}. = C(g)$$

$\square$

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