CSC373   Fall'19
Assignment 3
Due Date: Nov. 17, 2019, by 11:59pm

Instructions

1. Be sure to include your name and student number with your assignment. Typed assignments are preferred (e.g., PDFs created using LaTeX or Word), especially if your handwriting is possibly illegible or if you do not have access to a good quality scanner. Please submit a single PDF on MarkUS at https://markus.teach.cs.toronto.edu/csc373-2019-09.

2. You will receive 20% of the points for any (sub)problem for which you write "I do not know how to approach this problem." (you will receive 10% if you leave the question blank and do not write this or a similar statement). Not applicable to BONUS questions.

3. You may receive partial credit for the work that is clearly on the right track. But if your answer is largely irrelevant, you will receive 0 points.

4. This assignment has 4 questions (worth 20, 10, 20, 20 marks).

Q1 [20 Points] LP and IP

Consider the following primal LP and IP in standard form:

$$
\begin{aligned}
\text{Maximize} \quad & x_2 \\
\text{Subject to} \quad & -3x_1 + 5x_2 \leq 8 \\
& 7x_1 + 3x_2 \leq 12 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

For the IP, add the constraints that x_1 and x_2 are integers.

Plot the feasible region of this program. *Note: You do not need to submit this with the assignment, but it will be helpful to plot the feasible region.* You can use any online graphing programs such as desmos, fooplot, etc.

(a) [5 Points] What are the vertices of the feasible region of the primal LP? (No explanation is needed.)

(b) [5 Points] What are the optimal solutions of the primal LP and IP? What are the corresponding optimal objective values? (No explanation is needed.)

(c) [5 Points] Provide the dual LP of the primal LP above. Clearly indicate which dual variable in your formulation corresponds to which primal constraint.

(d) [5 Points] What are the optimal solutions of the dual LP and its IP version? What are the corresponding optimal objective values? Does strong duality hold for this particular pair of primal and dual IP?

Q2 [10 Points] Team Building

You are putting together a team of m players. The m positions in your team are ranked (so there is a position k for every $k \in \{1, 2, \ldots, m\}$. You can select your players from a pool of n people, denoted $N = \{1, \ldots, n\}$. Assume $n \geqslant m$.

Each person $i \in N$ has a celebrity rating c_i and suitability $s_{ik} \in [0, 1]$ that measures how well person i can play in position k on the team, where $k \in \{1, 2, \ldots, m\}$. You are also given a relation $I \subseteq N \times N$, where $(i, j) \in I$ indicates that person i and person j are *incompatible*, and should never be put on the team together. You can assume that this relation is symmetric, so $(i, j) \in I$ if and only if $(j, i) \in I$.

Your goal is to pick a team that maximizes the total celebrity rating of all selected players, subject to making sure that the total suitability of players for their assigned positions is at least 1 and no pair of players on the team is incompatible.

Give an linear or integer programming formulation for choosing the desired optimal team. Please include a high-level verbal description of your program and justify the correctness of your solution.

Q3 [20 Points] P, NP, and $coNP$

For each decision problem below, state whether it belongs to P, NP, or $coNP$. Make the strongest claim that you can. E.g. if you can show that a problem is in P, then you should claim so, as this implies membership in NP and $coNP$. Similarly, e.g., if you think it is not in P but is in both NP and $coNP$, then you should claim so, instead of claiming membership in just one of them. Note that if you claim a problem is in NP or $coNP$, you *do not* have to show NP- or $coNP$-completeness.

In all the problems below, you may assume that a cycle in a graph means a *simple* cycle with no repeated vertices. Assume all graphs are undirected. $\mathbb{Z}^+$ is the set of *positive* integers.

Justify your answers. If you claim a problem is in P, give a polynomial-time algorithm and argue its correctness and running time. If you claim a problem is in NP and/or $coNP$, then prove this membership.

(a) [5 Points] AllSmallCycles ("ASC" for short)
Input: Graph $G = (V, E)$, edge weights $w : E \to \mathbb{Z}^+$, vertex $s \in V$, bound $B \in \mathbb{Z}^+$.
Question: Does EVERY cycle in G that includes vertex s have total weight at most B?

(b) [5 Points] AllLargeCycles ("ALC" for short)
Input: Graph $G = (V, E)$, edge weights $w : E \to \mathbb{Z}^+$, vertex $s \in V$, bound $B \in \mathbb{Z}^+$.
Question: Does EVERY cycle in G that includes vertex s have total weight at least B?

(c) [5 Points] SomeLargeCycles ("SLC" for short)
Input: Graph $G = (V, E)$, edge weights $w : E \to \mathbb{Z}^+$, vertex $s \in V$, bound $B \in \mathbb{Z}^+$.
Question: Does SOME cycle in G include vertex s and have total weight at least B?

(d) [5 Points] SomeSmallCycles ("SSC" for short)
Input: Graph $G = (V, E)$, edge weights $w : E \to \mathbb{Z}^+$, vertex $s \in V$, bound $B \in \mathbb{Z}^+$.
Question: Does SOME cycle in G include vertex s and have total weight at most B?

Q4 [20 Points] Friendly Representatives

There is a set of n people, denoted $N = \{1, \ldots, n\}$. Some of them are friends with some other people. This is captured by a friendship relation $F \subseteq N \times N$, where $(i, j) \in F$ indicates that person i and person j are friends. You can assume that friendship is symmetric, so $(i, j) \in F$ if and only if $(j, i) \in F$.

Here is a decision problem, termed **FriendlyRepresentatives**.

Input: Set of people N, friendship relation F, integer m.
Question: Does there exist $S \subseteq N$ with $|S| = m$ such that every person who is not in S is friends with someone who is in S?

(a) [5 Points] Show that this problem is in NP.

(b) [15 Points] Show that this problem is NP-complete. For this part, you can use the fact that **ConnectedVertexCover** problem, which takes a connected graph $G = (V, E)$ as input and decides whether it admits a vertex cover of size exactly k, is NP-complete.

[Hint: The following gadget might be useful!]