

# CSC373

## Weeks 7 & 8: Complexity

# Recap

- Linear Programming
  - Standard formulation
  - Slack formulation
  - Simplex
  - Duality

# And Now...

- Applications of linear programming
  - Shortest path
  - Network flow
- A note about integer programming
- Complexity
  - Turing machines, computability, efficient computation
  - P, NP, and NP-completeness
  - Reductions
  - Idea behind NP-completeness of SAT and 3SAT
  - NP vs co-NP
  - Other complexity classes

# Network Flow via LP

- **Problem**

- **Input:** directed graph  $G = (V, E)$ , edge capacities  $c: E \rightarrow \mathbb{R}_{\geq 0}$
- **Output:** Value  $v(f^*)$  of a maximum flow  $f^*$

- Flow  $f$  is valid if:

- **Capacity constraints:**  $\forall (u, v) \in E: 0 \leq f(u, v) \leq c(u, v)$
- **Flow conservation:**  $\forall u: \sum_{(u,v) \in E} f(u, v) = \sum_{(v,u) \in E} f(v, u)$

- Maximize  $v(f) = \sum_{(s,v) \in E} f(s, v)$

Linear constraints

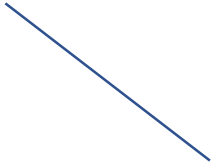
Linear objective!

# Network Flow via LP

$$\text{maximize} \quad \sum_{(s,v) \in E} f_{sv}$$

$$0 \leq f_{uv} \leq c(u, v) \quad \text{for all } (u, v) \in E$$

$$\sum_{(u,v) \in E} f_{uv} = \sum_{(v,w) \in E} f_{v,w} \quad \text{for all } v \in V \setminus \{s, t\}$$



Exercise: Write the dual of this LP.  
What is the dual trying to find?

# Shortest Path via LP

- **Problem**

- **Input:** directed graph  $G = (V, E)$ , edge weights  $w: E \rightarrow \mathbb{R}_{\geq 0}$ , source vertex  $s$ , target vertex  $t$
- **Output:** weight of the shortest-weight path from  $s$  to  $t$

- **Variables:** for each vertex  $v$ , we have variable  $d_v$

Why max?

maximize  $d_t$   
subject to

$$\begin{aligned} d_v &\leq d_u + w(u, v) \quad \text{for each edge } (u, v) \in E, \\ d_s &= 0. \end{aligned}$$

Exercise: prove formally  
that this works!

If objective was min., then we  
could set all variables  $d_v$  to 0.

# But...but...

- For these problems, we have different combinatorial algorithms that are much faster and run in strongly polynomial time
- Why would we use LP?
- For some problems, we don't have faster algorithms than solving them via LP

# Multicommodity-Flow

- **Problem:**

- **Input:** directed graph  $G = (V, E)$ , edge capacities  $c: E \rightarrow \mathbb{R}_{\geq 0}$ ,  $k$  commodities  $(s_i, t_i, d_i)$ , where  $s_i$  is source of commodity  $i$ ,  $t_i$  is sink, and  $d_i$  is demand.
- **Output:** valid multicommodity flow  $(f_1, f_2, \dots, f_k)$ , where  $f_i$  has value  $d_i$  and all  $f_i$  jointly satisfy the constraints

The only known polynomial time algorithm for this problem is based on solving LP!

$$\begin{aligned}
 \sum_{i=1}^k f_{iuv} &\leq c(u, v) && \text{for each } u, v \in V, \\
 \sum_{v \in V} f_{iuv} - \sum_{v \in V} f_{ivu} &= 0 && \text{for each } i = 1, 2, \dots, k \text{ and} \\
 &&& \text{for each } u \in V - \{s_i, t_i\}, \\
 \sum_{v \in V} f_{i, s_i, v} - \sum_{v \in V} f_{i, v, s_i} &= d_i && \text{for each } i = 1, 2, \dots, k, \\
 f_{iuv} &\geq 0 && \text{for each } u, v \in V \text{ and} \\
 &&& \text{for each } i = 1, 2, \dots, k.
 \end{aligned}$$



# Integer Linear Programming

- Variable values are restricted to be integers

- **Example:**

- **Input:**  $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$

- **Goal:**

Maximize  $c^T x$

Subject to  $Ax \leq b$

$x \in \{0, 1\}^n$

- **Does this make the problem easier or harder?**
  - Harder. We'll prove that this is "NP-complete".

# LPs are everywhere...

- Microeconomics
- Manufacturing
- VLSI (very large scale integration) design
- Logistics/transportation
- Portfolio optimization
- Bioengineering (flux balance analysis)
- Operations research more broadly: maximize profits or minimize costs, use linear models for simplicity
- Design of approximation algorithms
- Proving theorems, as a proof technique
- ...

# Introduction to Complexity

- You have a problem at hand
- You try every technique known to humankind for finding a polynomial time algorithm, but fail.
- You try every technique known to humankind for proving that there cannot exist a polynomial time algorithm for your problem, but fail.
- **What do you do?**
  - Prove that it is NP-complete, of course!

# Turing Machines

- “Which problems can a computer (not) solve in a certain amount of time?”
  - How do we mathematically define what a computer is?
- Alan Turing (“Father of Computer Science”), 1936
  - Introduced a mathematical model
  - “**Turing machine**”
  - All present-day computers can be simulated by a Turing machine
  - **Fun fact:** So can all the quantum computers
    - But TM might take longer to solve the same problem

# Turing Machines

- We won't formally introduce...but at a high level...
- Turing machine
  - Tape
    - Input is given on tape
    - Intermediate computations can be written there
    - Output must to be written there
  - Head pointer
    - Initially pointing at start of input on tape
  - Maintains an internal “state”
  - A transition function describes how to change state, move head pointer, and read/write symbols on tape

# Computability

- Church-Turing Hypothesis

- “Everything that is computable can be computed by a Turing machine”
- Widely accepted, cannot be “proven”
- There are problems which a Turing machine cannot solve, regardless of the amount of time available
  - E.g., the halting problem

- What about the problems we *can* solve? How do we define the time required?

- Need to define an encoding of the input and output

# Encoding

- What can we write on the tape?
  - $S$  = a set of finite symbols
  - $S^* = \bigcup_{n \geq 0} S^n$  = set of all finite strings using symbols from  $S$
- Input:  $w \in S^*$ 
  - Length of input =  $|w|$  = length of  $w$  on tape
- Output:  $f(w) \in S^*$ 
  - Length of output =  $|f(w)|$
  - Decision problems: output = “YES” or “NO”
    - E.g. “does there exist a flow of value at least 7 in this network?”

# Encoding

- Example:

- “Given  $a_1, a_2, \dots, a_n$ , compute  $\sum_{i=1}^n a_i$ ”

- Suppose we are told that  $a_i \leq C$  for all  $i$

- What  $|S|$  should we use?

- $S = \{0,1\}$  ( $|S| = 2$ , binary representation)

- Length of input =  $O(\log_2 a_1 + \dots + \log_2 a_n) = O(n \log_2 C)$

- What about 3-ary ( $|S| = 3$ ) or 18-ary ( $|S| = 18$ )?

- Only changes the length by a constant factor, still  $O(n \log C)$

- What about unary (conceptually,  $|S| = 1$ )?

- Length blows up exponentially to  $O(nC)$

- Binary is already good enough, but unary isn't



# Efficient Computability

- Polynomial-time computability

- A TM solves a problem in polynomial time if there is a polynomial  $p$  such that on every instance of  $n$ -bit input and  $m$ -bit output, the TM halts in at most  $p(n, m)$  steps
- Polynomial:  $n, n^2, 5n^{100} + 1000n^3, n \log^{100} n, o(n^{1.001})$
- Non-polynomial:  $2^n, 2^{\sqrt{n}}, 2^{\log^2 n}$

- Extended Church-Turing Thesis

- “Everything that a TM can compute in polynomial time”

If you ask the Turing machine to write a  $2^n$ -bit output, it's only reasonable to let it take  $2^n$  time...but usually, we'll look at problems where output is  $O(\text{length of input})$ , so we can ignore this  $m$

- Much less widely accepted, especially today
- But in this course, **efficient = polynomial-time**

# P

- P (polynomial time)

- The class of all decision problems computable by a TM in polynomial time

- Examples

- Addition, multiplication, square root
- Shortest paths
- Network flow
- Fast Fourier transform
- Checking if a given number is a prime  
[Agrawal-Kayal-Saxena 2002]
- ...

# NP

- NP (nondeterministic polynomial time) intuition
  - Subset sum problem:

Given an array  $\{-7, -3, -2, 5, 8\}$ , is there a zero-sum subset?
  - Enumerating all subsets is exponential
  - BUT given a subset  $\{-7, -2\}$  or  $\{-3, -2, 5\}$ , we can check in polynomial time whether it has zero sum
  - A nondeterministic Turing machine could “guess” the subset and then test if it has zero sum in polynomial time

# NP

- NP (nondeterministic polynomial time)
  - The class of all decision problems for which a YES answer can be verified by a TM in polynomial time given polynomial length “advice” or “witness”.
  - There is a polynomial-time verifier TM  $V$  and another polynomial  $p$  such that
    - For all YES inputs  $x$ , there exists  $y$  with  $|y| = p(|x|)$  on which  $V(x, y)$  returns YES
    - For all NO inputs  $x$ ,  $V(x, y)$  returns NO for every  $y$
  - Informally: “Whenever the answer is YES, there’s a short proof of it.”

# co-NP

- co-NP

- Same as NP, except whenever the answer is NO, we want there to be a short proof of it

- Open questions

- $NP = co-NP$ ?
- $P = NP \cap co-NP$ ?
- And...drum roll please...

$$P = NP?$$

# P versus NP

- Lance Fortnow in his article on P and NP in Communications of the ACM, Sept 2009

*“The P versus NP problem has gone from an interesting problem related to logic to perhaps the most fundamental and important mathematical question of our time, whose importance only grows as computers become more powerful and widespread.”*

# Millenium Problems

- Award of \$1,000,000 for each problem by Clay Math institute

1. Birch and Swinnerton-Dyer Conjecture

2. Hodge Conjecture

3. Navier-Stokes Equations

4.  $P = NP?$

Claim: Worth  $\gg$  \$1M

5. Poincare Conjecture (Solved)<sup>1</sup>

6. Riemann Hypothesis

7. Yang-Mills Theory

<sup>1</sup>Solved by Grigori Perelman (2003): Prize unclaimed

# Cook's Conjecture

- Cook's conjecture

- (And every sane person's belief...)
- $P$  is likely not equal to  $NP$

- Why do we believe this?

- There is a large class of problems (NP-complete)
- By now, contains thousands and thousands of problems
- Each problem is the “hardest problem in NP”
- If you can efficiently solve *any one of them*, you can efficiently solve *every problem in NP*
  - Despite decades of effort, no polynomial time solution has been found for *any of them*



# Reductions

- Problem  $A$  is **p-reducible** to problem  $B$  if an “oracle” (subrouting) for  $B$  can be used to efficiently solve  $A$ 
  - You can solve  $A$  by making polynomially many calls to the oracle and doing additional polynomial computation
- **Question:** If  $A$  is p-reducible to  $B$ , then which of the following is true?
  - a) If  $A$  cannot be solved efficiently, then neither can  $B$ .
  - b) If  $B$  cannot be solved efficiently, then neither can  $A$ .
  - c) Both.
  - d) None.

# Reductions

- Problem  $A$  is **p-reducible** to problem  $B$  (denoted  $A \leq_p B$ ) if an “oracle” (subrouting) for  $B$  can be used to efficiently solve  $A$ 
  - You can solve  $A$  by making polynomially many calls to the oracle and doing additional polynomial computation
- **Question:** If I want to prove that my problem  $X$  is “hard”, I should:
  - a) Reduce my problem to a known hard problem.
  - b) Reduce a known hard problem to my problem.
  - c) Both.
  - d) None.

# NP-completeness

- NP-completeness

- A problem  $B$  is NP-complete if it is in NP and **every** problem  $A$  in NP is p-reducible to  $B$
- Hardest problems in NP
- If one of them can be solved efficiently, every problem in NP can be solved efficiently, implying  $P=NP$

- Observation:

- If  $A$  is in NP, and some NP-complete problem  $B$  is p-reducible to  $A$ , then  $A$  is NP-complete too
  - “If I could solve  $A$ , then I could solve  $B$ , then I could solve any problem in NP”

# NP-completeness

- But this uses an already known NP-complete problem to prove another problem is NP-complete
- How do we find the *first* NP-complete problem?
  - How do we know there are *any* NP-complete problems at all?
  - Key result by Cook
  - First NP-complete problem: SAT
    - By a reduction from an arbitrary problem in NP to SAT
    - “From first principles”

# CNF Formulas

- **Conjunctive normal form (CNF)**
  - Boolean **variables**  $x_1, x_2, \dots, x_n$
  - Their **negations**  $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$
  - **Literal**  $\ell$ : a variable or its negation
  - **Clause**  $C = \ell_1 \vee \ell_2 \vee \dots \vee \ell_r$  is a disjunction of literals
  - **CNF formula**  $\varphi = C_1 \wedge C_2 \wedge \dots \wedge C_m$  is a conjunction of clauses
    - **kCNF**: Each clause has at most  $k$  literals
      - We'll abuse notation a little and assume there are *exactly*  $k$
      - Example of 3CNF

$$\varphi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4) \wedge (\bar{x}_3 \vee \bar{x}_4 \vee x_1)$$

# SAT and 3SAT

- Example of 3CNF

$$\varphi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4) \wedge (\bar{x}_3 \vee \bar{x}_4 \vee x_1)$$

- SAT

- A CNF formula  $\varphi$  is **satisfiable** if there is an assignment of truth values (TRUE/FALSE) to variables under which the formula evaluates to TRUE
  - That means, in each clause, at least one literal is TRUE
- SAT: “Given a CNF formula  $\varphi$ , is it satisfiable?”
- 3SAT: “Given a 3CNF formula  $\varphi$ , is it satisfiable?”

# SAT and 3SAT

- Cook-Levin Theorem

- SAT (and even 3SAT) is NP-complete

- Doesn't use any known NP-complete problem

- Directly reduces any NP problem to SAT

- Reduction is a bit complex, so we'll defer it until a bit later, after we've seen some other reductions and are more comfortable with the reduction framework

- But for now, let's assume SAT is NP-complete, and reduce it to a bunch of other problems (and then those problems to other problems...)

# NP-Complete Examples

- NP-complete problems

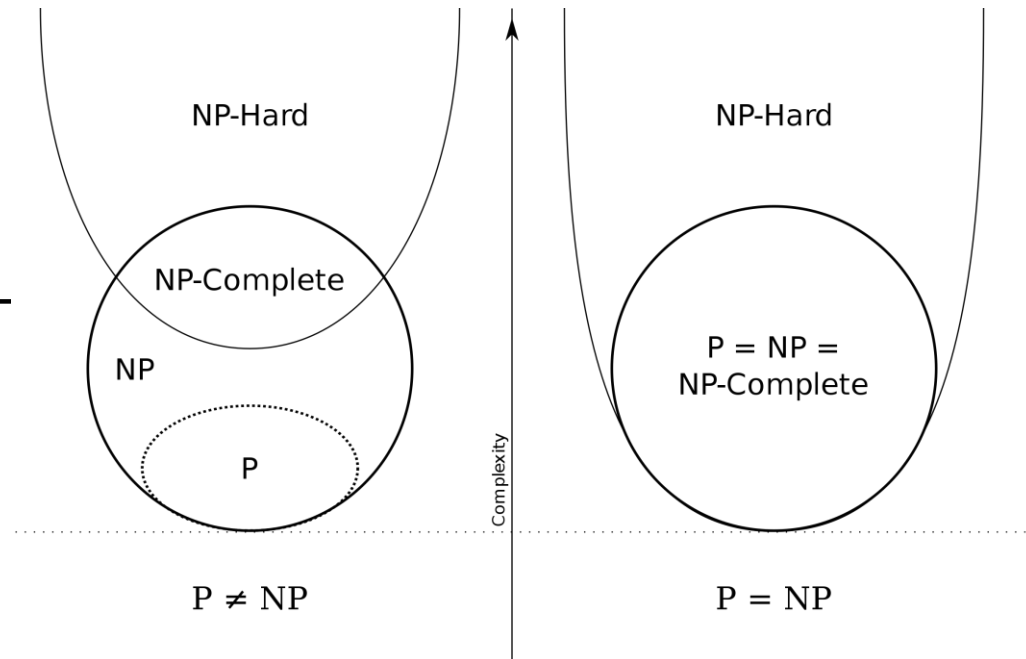
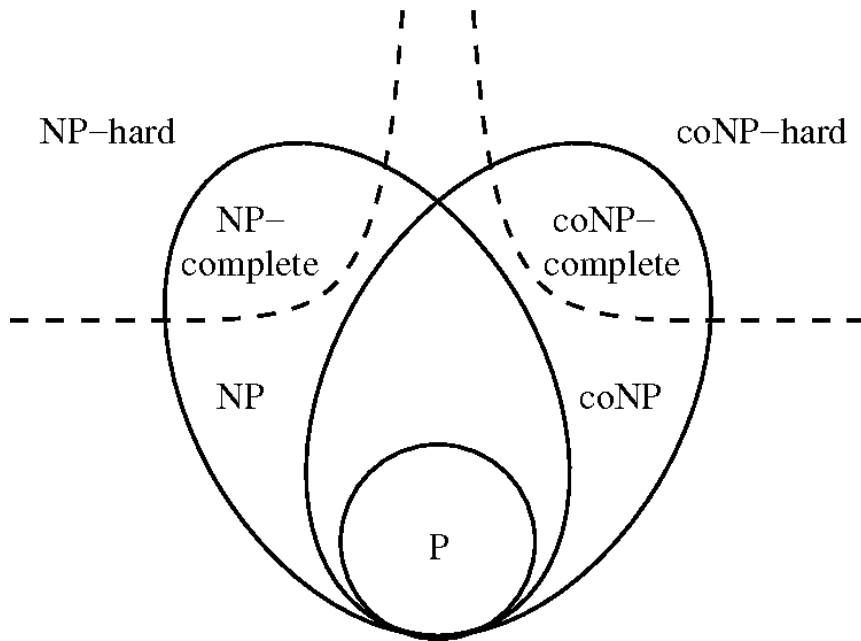
- SAT = first NP complete problem
- Decision TSP: Is there a route visiting all  $n$  cities with total distance at most  $k$ ?
- 3-Colorability: Can the vertices of a graph be colored with at most 3 colors such that no two adjacent vertices have the same color?
- Karp's 21 NP-complete problems

- co-NP-complete

- Tautology problem (“negation” of SAT)



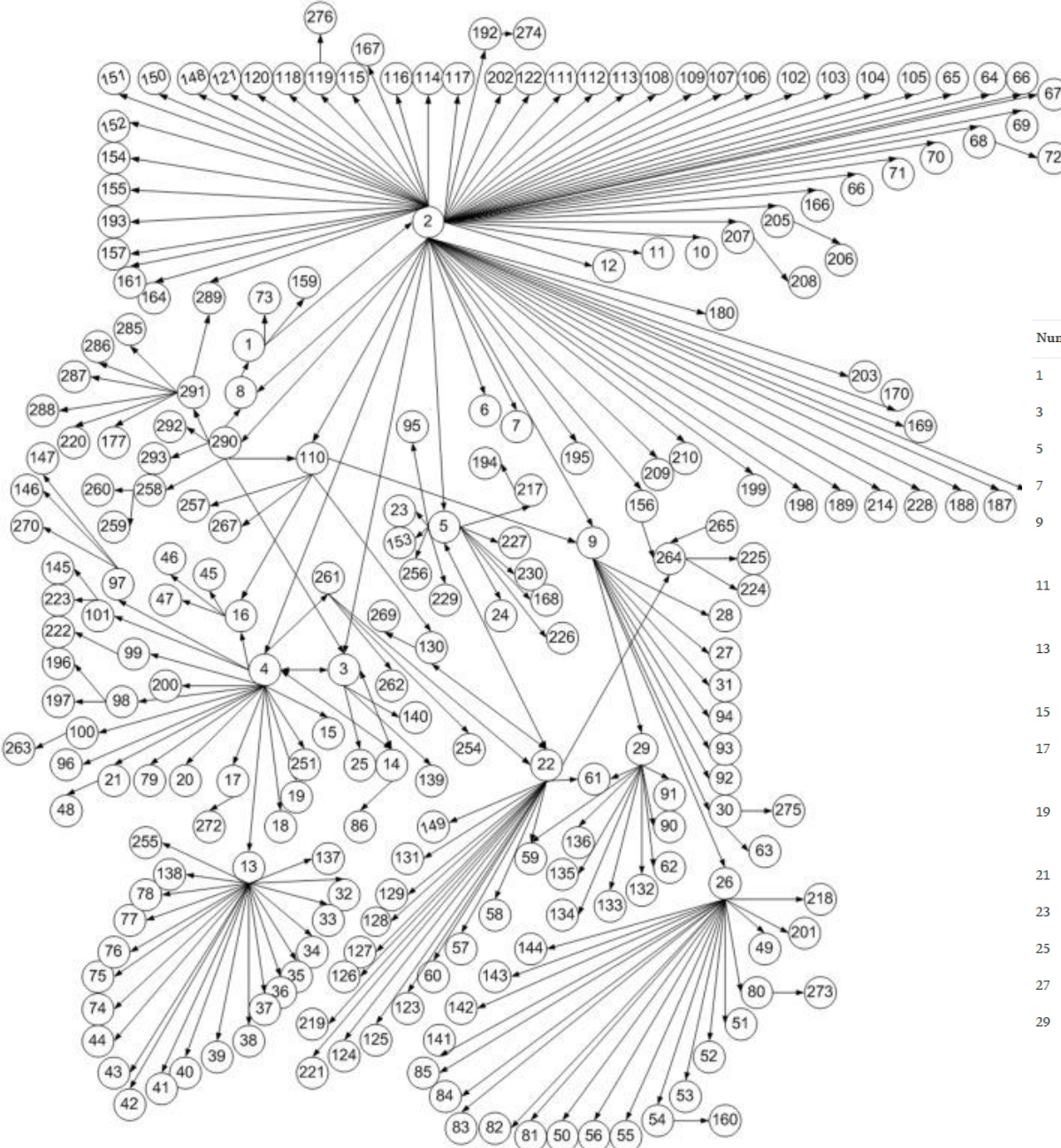
# Complexity



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# Survey of polynomial transformations between NP-complete problems

Jorge A. Ruiz-Vanoye <sup>a</sup>✉, Joaquín Pérez-Ortega <sup>b</sup>✉, Rodolfo A. Pazos R. <sup>c</sup>✉, Ocotlán Díaz-Parra <sup>d</sup>✉, Juan Frausto-Solís <sup>a</sup>✉, Hector J. Fraire Huacuja <sup>c</sup>✉, Laura Cruz-Reyes <sup>c</sup>✉, José A. Martínez F. <sup>c</sup>✉

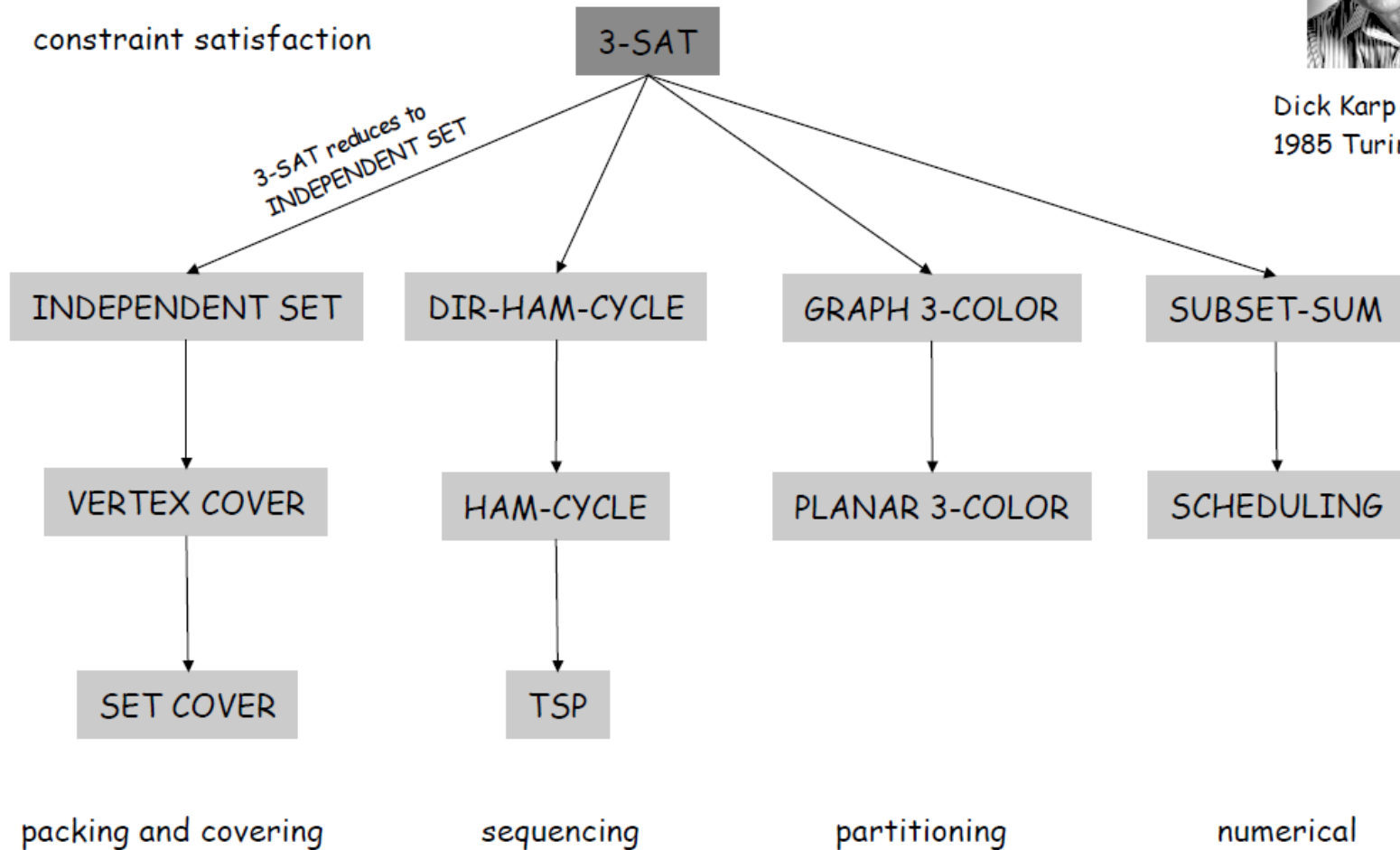


Number	Name of problem	Number	Name of problem
1	Satisfiability (SAT)	2	3-Satisfiability (3SAT)
3	Clique (clique cover)	4	Vertex cover
5	Subset sum	6	Hitting string
7	Chinese postman for mixed graphs	8	Graph colorability
9	Three-Dimensional matching (3DM)	10	Rectilinear picture compression
11	Tableau equivalence	12	Consistency of databases frequency tables
13	Hamiltonian Circuit (Directed Hamiltonian path, Undirected Hamiltonian path)	14	Independent set
15	Setbasis	16	Hitting set
17	Comparative containment	18	Multiple copy file allocation
19	Shortest common supersequence	20	Longest common subsequence
21	Minimum cardinality key	22	Partition
23	$K$ 'th largest subset	24	Capacity assignment
25	Conjunctive Boolean query	26	Exact cover by 3-sets (X3C)
27	Minimum test set	28	3-Matroid intersection
29	3-Partition	30	Numerical three-dimensional matching

# Polynomial-Time Reductions



Dick Karp (1972)  
1985 Turing Award



# Just A Tad Bit of History

- [Cook 1971]
  - Proved 3SAT is NP-complete in seminal paper
- [Karp 1972]
  - Showed that 20 other problems are also NP-complete
  - “Karp's 21 NP-complete problems”
  - Renewed interest in this idea
- 1982: Cook won the Turing award

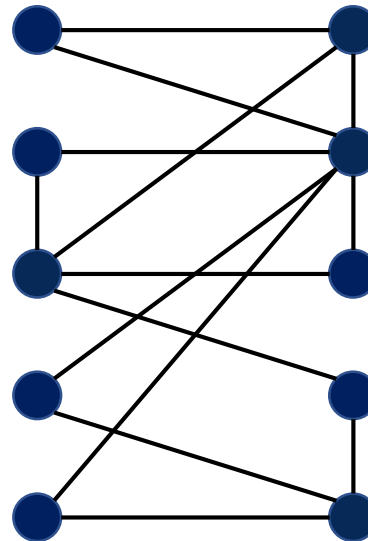
# Independent Set

- **Problem**

- **Input:** Undirected graph  $G = (V, E)$ , integer  $k$
- **Question:** Does there exist a subset of vertices  $S \subseteq V$  with  $|S| = k$  such that for each edge, **at most one** of its endpoints is in  $S$ ?

**Example:**

- Does this graph have an independent set of size 6?
  - Yes!
- Does this graph have an independent set of size 7?
  - No!



● = independent set

# Independent Set

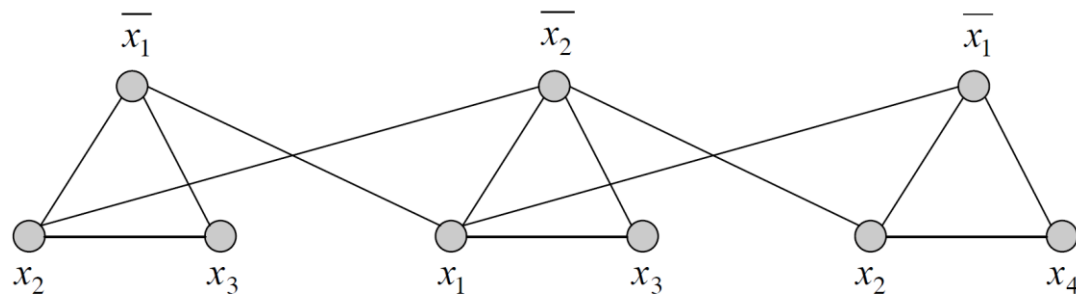
- Claim: Independent Set is in NP

- Recall: We need to show that there is a polynomial-time algorithm which
  - Can accept every YES instance with the right polynomial-size advice
  - Will not accept a NO instance with any advice
- Advice: the actual independent set  $S$ 
  - Algorithm: Simply check if  $S$  is an independent set and  $|S| = k$
  - Simple!

# Independent Set

- Claim:  $3SAT \leq_p$  Independent Set

- Given a formula  $\varphi$  of 3SAT with  $k$  clauses, construct an instance  $(G, k)$  of Independent Set as follows
  - Create 3 vertices for each clause (one for each literal)
  - Connect them in a triangle
  - Connect the vertex of each literal to each of its negations

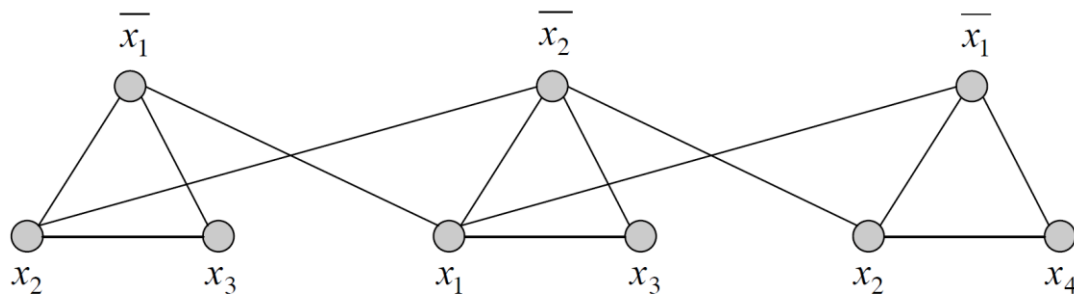


$$\Phi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4)$$

# Independent Set

## ➤ Why does this work?

- 3SAT = YES ( $\varphi$  has satisfying assignment)  $\Rightarrow$  Independent Set = YES
  - From each clause, take any literal that is TRUE in the assignment
- Independent Set = YES  $\Rightarrow$  3SAT = YES
  - Independent set  $S$  must contain one vertex from each triangle
  - No literal and its negation are both in  $S$
  - Set literals in  $S$  to TRUE, their negations to FALSE, and the rest to arbitrary values



$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$



# Different Types of Reductions

- $A \leq B$

- Karp reductions

- Take an arbitrary instance of  $A$ , and in polynomial time, **construct a single instance of  $B$  with the same answer**
    - Very restricted type of reduction
    - The reduction we just constructed was a Karp reduction

- Turing/Cook reductions

- Take an arbitrary instance of  $A$ , and solve it by **making polynomially many calls to an oracle for solving  $B$  and some polynomial-time extra computation**
    - Very general reduction
    - In this course, we'll allow Turing/Cook reductions, but whenever possible, see if you can construct a Karp reduction

# Subset Sum

- Problem

- Input: Set of integers  $S = \{w_1, \dots, w_n\}$ , integer  $W$
- Question: Is there  $S' \subseteq S$  that adds up to exactly  $W$ ?

- Example

- $S = \{1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344\}$   
and  $W = 3754$ ?
- Yes!
  - $1 + 16 + 64 + 256 + 1040 + 1093 + 1284 = 3754$

# Subset Sum

- Claim: Subset Sum is in NP

- Recall: We need to show that there is a polynomial-time algorithm which
  - Can accept every YES instance with the right polynomial-size advice
  - Will not accept a NO instance with any advice
- Advice: the actual subset  $S'$ 
  - Algorithm: Simply check that elements of  $S'$  sum to  $W$
  - Simple!

# Subset Sum

• Claim:  $3SAT \leq_p \text{Subset Sum}$

- Given a formula  $\varphi$  of 3SAT, we want to construct  $(S, W)$  of Subset Sum with the same answer
- In the table in the following slide:
  - Columns are for variables and clauses
  - Each row is a number in  $S$ , represented in decimal
  - Number for literal  $\ell$  : has 1 in its variable column and in the column of every clause where that literal appears
    - Number selected = literal set to TRUE
  - “Dummy” rows: can help make the sum in a clause column 4 if and only if at least one literal is set to TRUE

# Subset Sum

- Claim:  $3SAT \leq_p \text{Subset Sum}$

$$C_1 = \bar{x} \vee y \vee z$$

$$C_2 = x \vee \bar{y} \vee z$$

$$C_3 = \bar{x} \vee \bar{y} \vee \bar{z}$$

dummies to get  
clause columns  
to sum to 4

	x	y	z	$C_1$	$C_2$	$C_3$
x	1	0	0	0	1	0
$\neg x$	1	0	0	1	0	1
y	0	1	0	1	0	0
$\neg y$	0	1	0	0	1	1
z	0	0	1	1	1	0
$\neg z$	0	0	1	0	0	1
<div> </div>	0	0	0	1	0	0
	0	0	0	2	0	0
	0	0	0	0	1	0
	0	0	0	0	2	0
	0	0	0	0	0	1
	0	0	0	0	0	2
	0	0	0	0	0	2
W	1	1	1	4	4	4

# Subset Sum

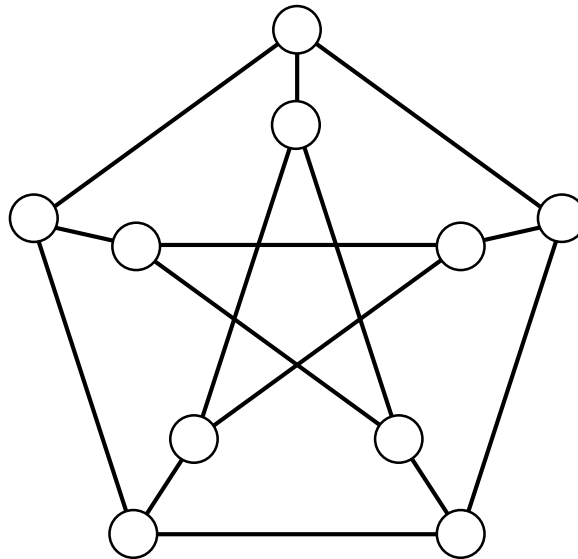
- Note

- The Subset Sum instance we constructed has large numbers
  - Something like  $10^{\#variables + \#clauses}$
  - Numbers are exponential in size of the original 3CNF instance, but it only takes polynomially many bits to write these numbers
- Can we hope to construct Subset Sum instance with numbers that are only  $\text{poly}(\#variables, \#clauses)$  large?
  - Unlikely.
  - Like Knapsack, Subset Sum can be solved in pseudo-polynomial time (i.e. we can solve Subset Sum in polytime if the numbers are only polynomially large in value).

# 3-Coloring

- **Problem**

- **Input:** Undirected graph  $G = (V, E)$
- **Question:** Can we color each vertex of  $G$  using at most three colors such that no two adjacent vertices have the same color?



# 3-Coloring

- Claim: 3-coloring is in NP

- Recall: We need to show that there is a polynomial-time algorithm which
  - Can accept every YES instance with the right polynomial-size advice
  - Will not accept a NO instance with any advice
- Advice: colors of the nodes in a valid 3-coloring
  - Algorithm: Simply check that this is a valid 3-coloring
  - Simple!



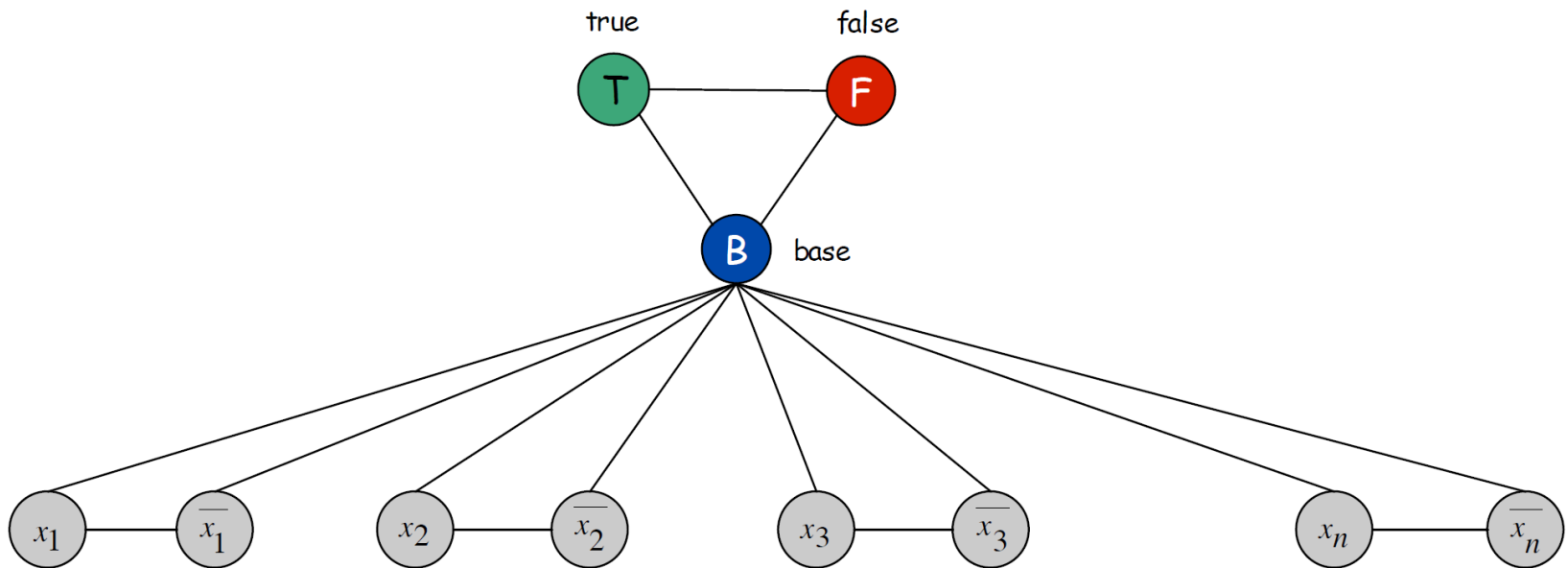
# 3-Coloring

• Claim:  $3SAT \leq_p 3\text{-Coloring}$

- Given a 3SAT formula  $\varphi$ , we want to construct a graph  $G$  such that  $G$  is 3-colorable if and only if  $\varphi$  has a satisfying assignment
- We want a satisfying assignment of  $\varphi$  to correspond to a valid 3-coloring of  $G$ 
  - Each true literal should have color T
  - Each false literal should have color F
  - We need to make sure they don't get the third color

# 3-Coloring

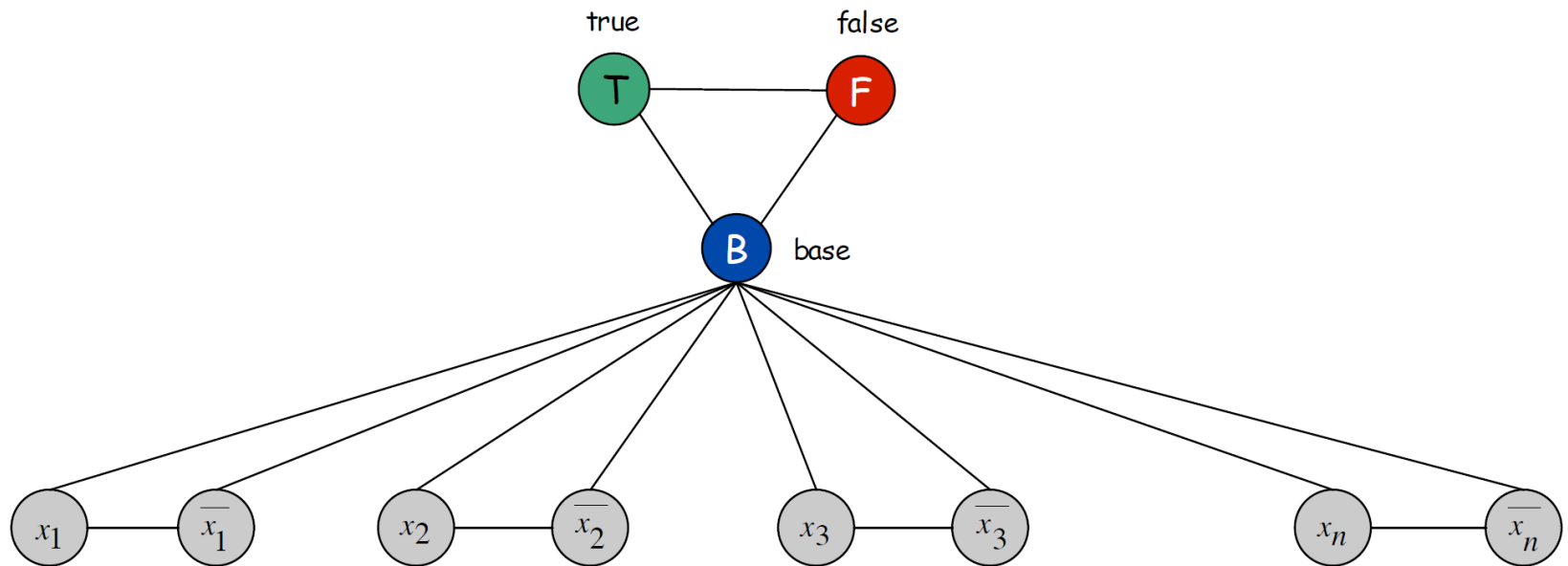
- Given a 3SAT formula  $\varphi$ , we construct a graph  $G$  as follows
  - Create 3 new nodes T, F, and B; connect them in a triangle
  - Create a node for each literal, connect it to its negation and to B
  - So T-F-B have different colors and  $B-x_i-\bar{x}_i$  have different colors
    - Each literal has the color of T/F and its negation has the other color



# 3-Coloring

➤ **Claim:** valid 3-coloring  $\Rightarrow$  valid truth assignment

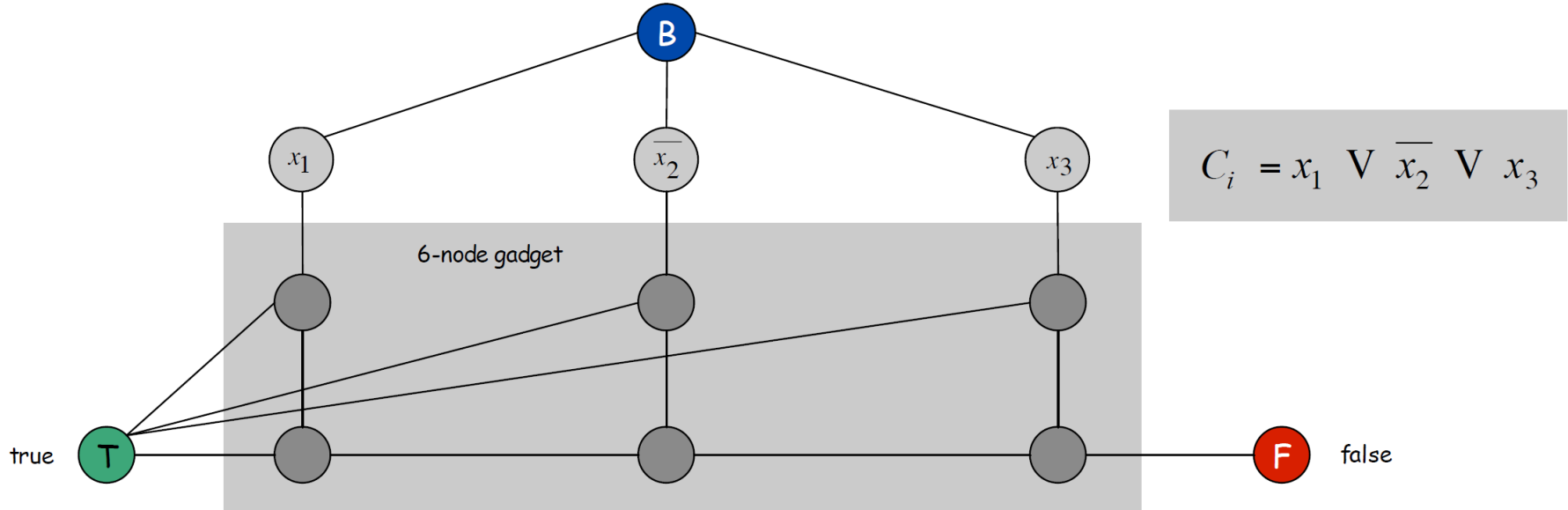
- Each literal node must be colored T or F
- If a literal is T, its negation must be F
- We can set all literals with color T to be TRUE
  - Valid truth assignment



# 3-Coloring

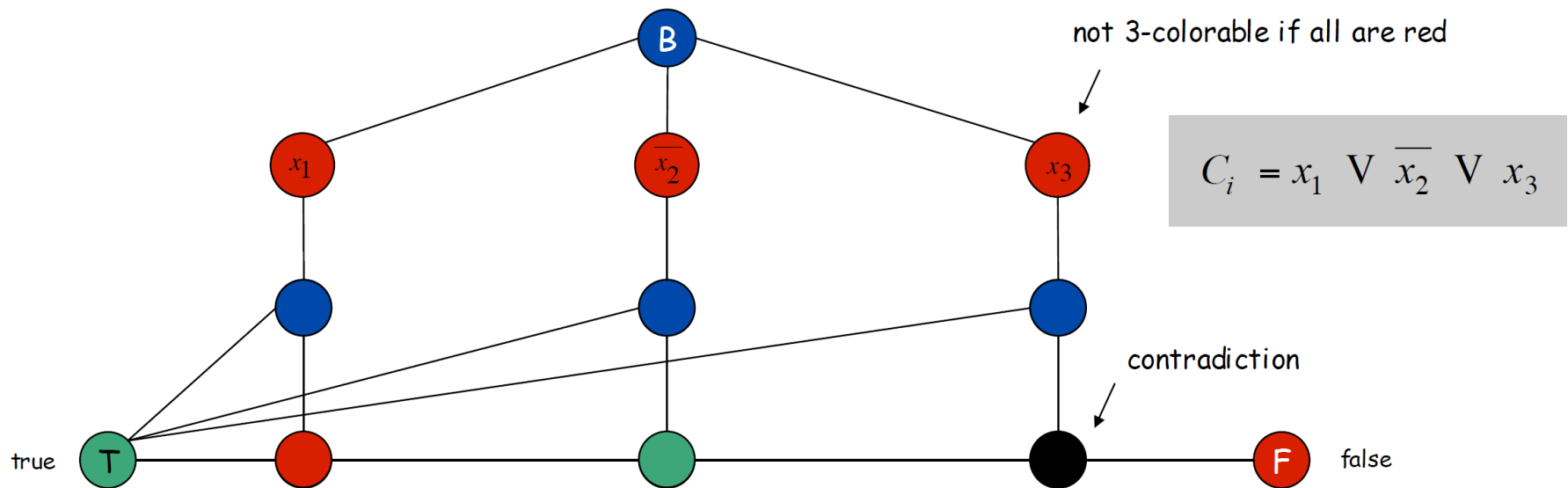
## ➤ What about clauses?

- For each clause, add the following gadget with 6 nodes and 13 edges
- **Claim:** Clause gadget is 3-colorable if and only if at least one of the nodes corresponding to the literals in the clause is assigned color of T



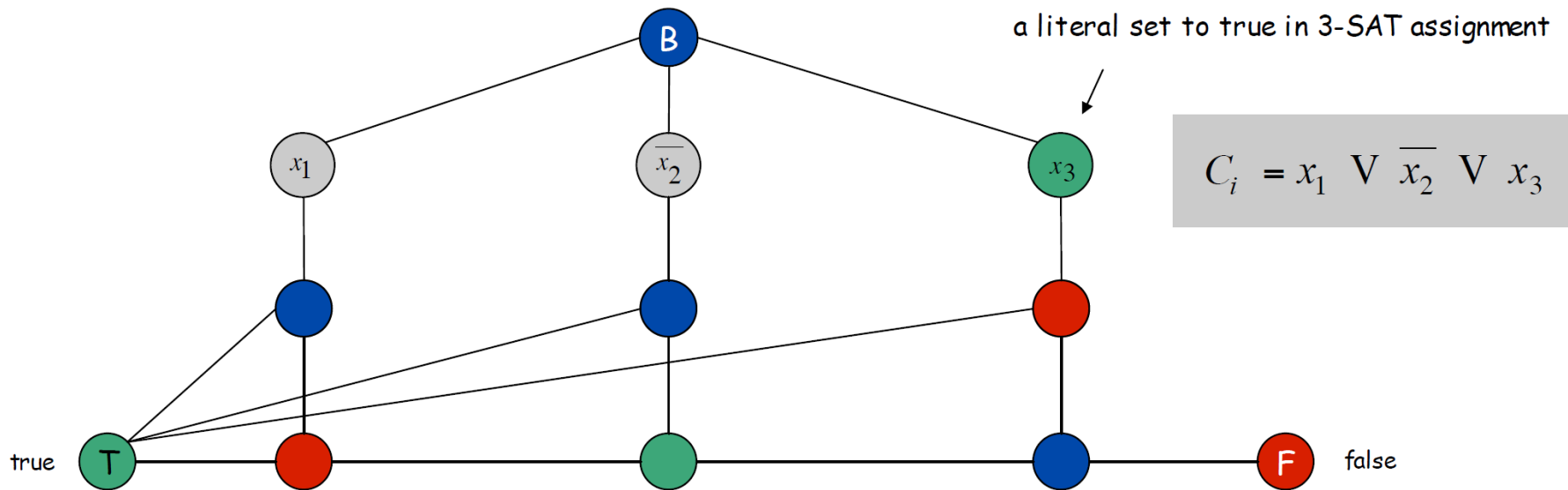
# 3-Coloring

- **Claim:** Valid 3-coloring  $\Rightarrow$  truth assignment satisfies  $\varphi$
- Suppose a clause  $C_i$  is not satisfied, so all its three literals must be F
  - Then the 3 nodes in top layer must be B
  - Then the first two nodes in bottom layer must be F and T, resp.
  - Then no color left for the remaining node  $\Rightarrow$  contradiction!



# 3-Coloring

- We just proved: valid 3-coloring  $\Rightarrow$  satisfying assignment
- **Claim:** satisfying assignment  $\Rightarrow$  valid 3-coloring
  - Color all true literals as T and their negations as F
    - Valid 3-coloring for the literal widget
  - Each clause widget with at least one T literal can be 3-colored



# Review of Reductions

- If you want to show that problem B is NP-complete
- **Step 1: Show that B is in NP**
  - Some polynomial-size advice should be sufficient to verify a YES instance in polynomial time
  - No advice should work for a NO instance
- Usually, the solution of the “search version” of the problem works
  - But sometimes, the advice can be non-trivial
  - For example, to **check LP optimality**, one possible advice is the **values of both primal and dual variables**, as we saw in the last lecture

# Review of Reductions

- If you want to show that problem B is NP-complete
- Step 2: Find a known NP-complete problem A and reduce it to B (i.e. show  $A \leq_p B$ )
  - This means taking an arbitrary instance of A, and solving it in polynomial time using an oracle for B
    - Caution 1: Remember the direction. You are “reducing known NP-complete problem to your current problem”.
    - Caution 2: The size of the B-instance you construct should be polynomial in the size of the original A-instance
  - This would show that if B can be solved in polynomial time, then A can be as well
  - Some reductions are trivial, some are notoriously tricky...



# Integer Linear Programming (ILP)

- **Problem**

- **Input:**  $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}, k \in \mathbb{R}$
- **Question:** Does there exist  $x \in \{0,1\}^n$  such that  $c^T x \geq k$  and  $Ax \leq b$ ?
- Decision variant of “maximize  $c^T x$  subject to  $Ax \leq b$ ” but instead of any  $x \in \mathbb{R}^n$  with  $x \geq 0$ , we are restricting  $x$  to binary.
- Does restricting search space make the problem easier or harder?
  - This is actually NP-complete!

# IP Feasibility

- An even simpler problem
  - Special case where  $c = k = 0$ , so  $c^T x \geq k$  is always true

- **Problem**

- **Input:**  $b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$
  - **Question:** Does there exist  $x \in \{0,1\}^n$  such that  $Ax \leq b$ ?
- 
- Just need to find a feasible solution
  - This is still NP-complete!

# IP Feasibility

- Claim: IP Feasibility is in NP

- Recall: We need to show that there is a polynomial-time algorithm which
  - Can accept every YES instance with the right polynomial-size advice
  - Will not accept a NO instance with any advice
- Advice: simply a vector  $x$  satisfying  $Ax \leq b$ 
  - Algorithm: Check if  $Ax \leq b$
  - Simple!

# IP Feasibility

- Claim:  $3SAT \leq_p \text{IP Feasibility}$

- Take any formula  $\varphi$  of 3SAT
- Create a binary variable  $x_i$  for each variable  $x_i$  in  $\varphi$ 
  - We'll represent its negation  $\bar{x}_i$  with  $1 - x_i$
- For each clause  $C$ , we want at least one of its three literals to be TRUE
  - Just make sure their sum is at least 1
  - E.g.  $C = x_1 \vee \bar{x}_2 \vee \bar{x}_3 \Rightarrow x_1 + (1 - x_2) + (1 - x_3) \geq 1$
- Easy to check that
  - this is a polynomial reduction
  - Resulting system has a feasible solution iff  $\varphi$  is satisfiable

# So far...

- To establish NP-completeness of problem B, we always reduced 3SAT to B
  - But we can reduce any other problem A that we have already established to be NP-complete
  - Sometimes this might lead to a simpler reduction because A might already be “similar” to B
- Let’s see an example!

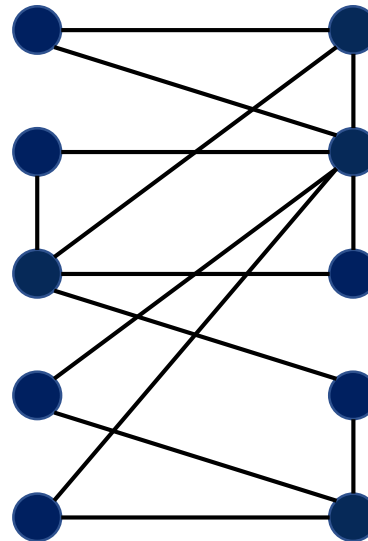
# Vertex Cover

- **Problem**

- **Input:** Undirected graph  $G = (V, E)$ , integer  $k$
- **Question:** Does there exist a vertex cover of size  $k$ ?
  - That is, does there exist  $S \subseteq V$  with  $|S| = k$  such that every edge is incident to at least one vertex in  $S$ ?

**Example:**

- Does this graph have a vertex cover of size 4?
  - Yes!
- Does this graph have a vertex cover of size 3?
  - No!



● = vertex cover

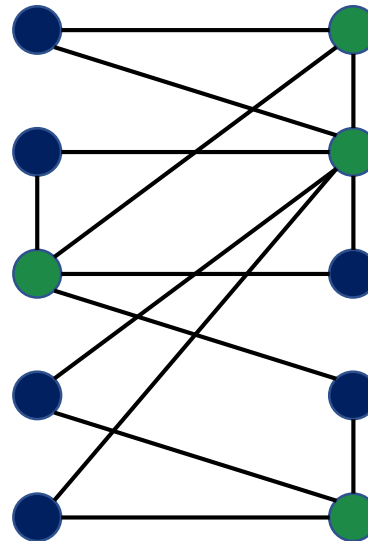
# Vertex Cover

- **Problem**

- **Input:** Undirected graph  $G = (V, E)$ , integer  $k$
- **Question:** Does there exist a vertex cover of size  $k$ ?
  - That is, does there exist  $S \subseteq V$  with  $|S| = k$  such that every edge is incident to at least one vertex in  $S$ ?

**Question:**

- Did we see this graph in the last lecture?
  - Yes!
  - For independent set of size 6



- = vertex cover
- = independent set

# Vertex Cover

- Vertex cover and independent set are intimately connected!
- **Claim:**  $G$  has a vertex cover of size  $k$  if and only if  $G$  has an independent set of size  $n - k$
- **Stronger claim:**  $S$  is a vertex cover if and only if  $V \setminus S$  is an independent set



# Vertex Cover

- **Claim:**  $S$  is a vertex cover if and only if  $V \setminus S$  is an independent set
- **Proof:**
  - $S$  is a vertex cover
  - IFF: For every  $(u, v) \in E$ , at least one of  $\{u, v\}$  is in  $S$
  - IFF: For every  $(u, v) \in E$ , at most one of  $\{u, v\}$  is in  $V \setminus S$
  - IFF: No two vertices of  $V \setminus S$  are connected by an edge
  - IFF:  $V \setminus S$  is an independent set ■

# Vertex Cover

- Claim: Independent Set  $\leq_p$  Vertex Cover
  - Take an arbitrary instance  $(G, k)$  of Independent Set
  - We want to check if there is an independent set of size  $k$
  - Just convert it to the instance  $(G, n - k)$  of Vertex Cover
  - Simple!
    - A reduction from 3SAT would have basically repeated the reduction we already did for  $3\text{SAT} \leq_p \text{Independent Set}$
  - **Note:** I didn't argue that Vertex Cover is in NP
    - This is simple as usual. Just give the actual vertex cover as the advice.

# Set Cover

- Problem

- **Input:** A universe of elements  $U$ , a family of subsets  $S$ , and an integer  $k$
- **Question:** Do there exist  $k$  sets from  $S$  whose union is  $U$ ?

- Example

- $U = \{1,2,3,4,5,6,7\}$
- $S = \{\{1,3,7\}, \{2,4,6\}, \{4,5\}, \{1\}, \{1,2,6\}\}$
- $k = 3$ ? Yes!  $\{\{1,3,7\}, \{4,5\}, \{1,2,6\}\}$
- $k = 2$ ? No!

# Set Cover

- Claim: Set Cover is in NP

- Easy. Let the advice be the actual  $k$  sets whose union is  $U$ .

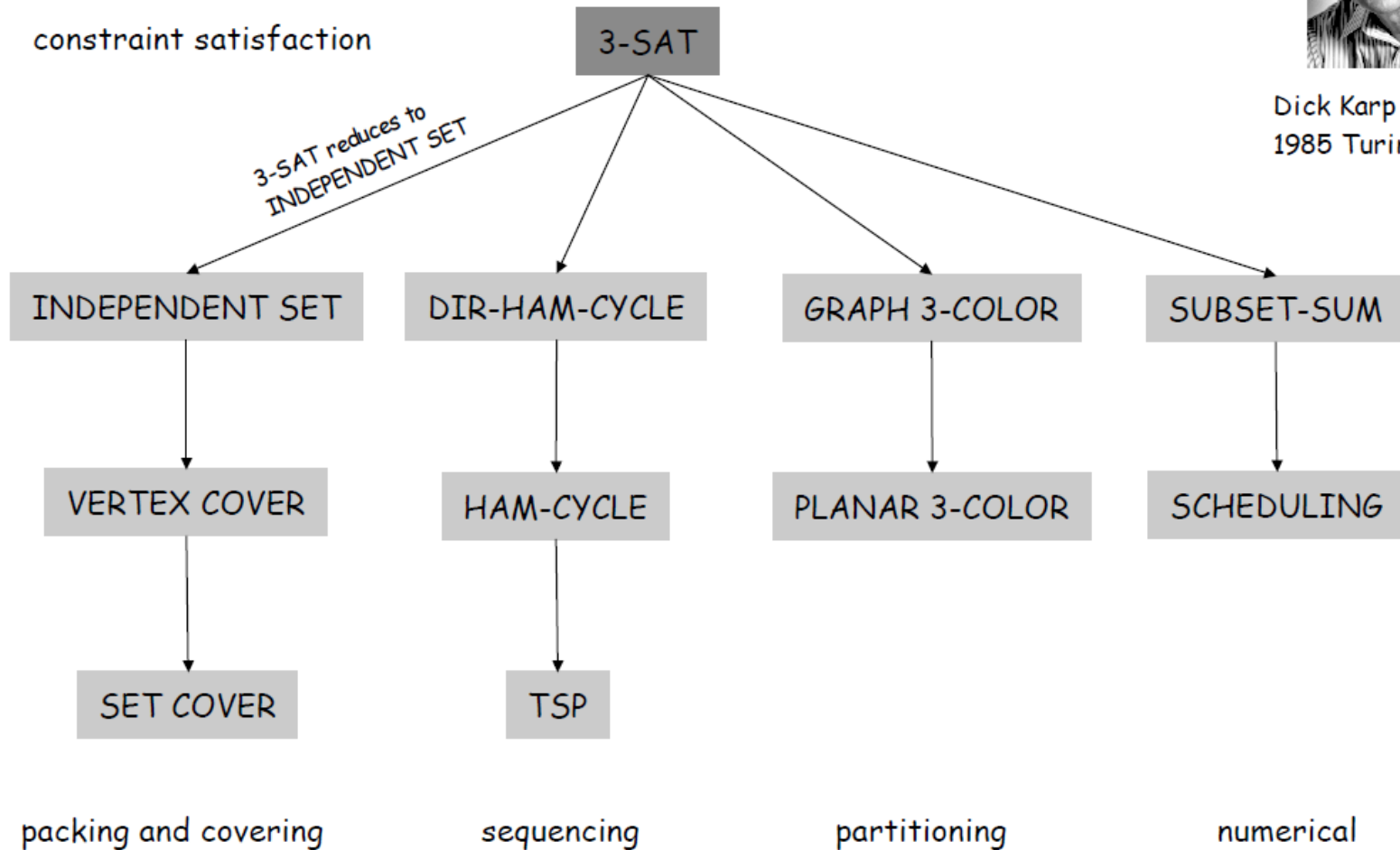
- Claim: Vertex Cover  $\leq_p$  Set Cover

- Given an instance of vertex cover with graph  $G = (V, E)$  and integer  $k$ , create the following set cover instance
  - Set  $U = E$
  - For each  $v \in V$ ,  $S$  contains a set  $S_v$  of all edges incident on  $v$
  - Selecting  $k$  set whose union is  $U$  = selecting  $k$  vertices such that union of their incident edges covers all edges
  - Hence, the two problems obviously have the same answer

# Polynomial-Time Reductions



Dick Karp (1972)  
1985 Turing Award



# Cook-Levin Theorem

- We did not prove “the first NP-completeness” result
- **Theorem: 3SAT is NP-complete**
  - We need to prove this without using any other “known NP-complete” problem
  - We want to directly show that *every problem in NP* can be reduced to 3SAT

# Cook-Levin Theorem

- We're not going to prove it in this class, but the key idea is as follows
  - If a problem is in NP, then  $\exists$  Turing machine  $T(x, y)$  which
  - ...takes a problem instance  $x$ , an advice  $y$  of  $p(n)$  size, and verifies in  $p(n)$  time whether  $x$  is a YES instance...
  - ...where  $p$  is some polynomial and  $n = |x|$
  - $x$  is a YES instance iff  $\exists y T(x, y) = ACCEPT$

# Cook-Levin Theorem

- We're not going to prove it in this class, but the key idea is as follows
  - $x$  is a YES instance iff  $\exists y T(x, y) = ACCEPT$
  - We can introduce a bunch of variables...
    - $T_{i,j,k}$  = True if machine's tape cell  $i$  contains symbol  $j$  at step  $k$  of the computation
    - $H_{i,k}$  = True if the machine's read/write head is at tape cell  $i$  at step  $k$  of the computation
    - $Q_{q,k}$  = True if machine is in state  $q$  at step  $k$  of the computation
  - Then express how these variables must be related using a bunch of constraints (clauses)
    - This shows SAT is NP-complete. Then we can show  $SAT \leq_p 3SAT$ .



# Cook-Levin Theorem

- Claim:  $SAT \leq_p 3SAT$

- Given an instance  $\varphi = C_1 \vee C_2 \vee \dots$  of SAT, we can take each clause, and replace it with a bunch of clauses with exactly 3 literals each

- For a clause with one literal  $C = \ell_1$ :

- Add two variables  $z_1, z_2$ , and replace it with four clauses  
 $(\ell_1 \vee z_1 \vee z_2) \wedge (\ell_1 \vee \bar{z}_1 \vee z_2) \wedge (\ell_1 \vee z_1 \vee \bar{z}_2) \wedge (\ell_1 \vee \bar{z}_1 \vee \bar{z}_2)$
- Verify that this is indeed always equal to  $\ell_1$

- For a clause with two literals  $C = (\ell_1 \vee \ell_2)$ :

- Add one variable  $z_1$  and replace it with the following:  
 $(\ell_1 \vee \ell_2 \vee z_1) \wedge (\ell_1 \vee \ell_2 \vee \bar{z}_1)$
- Verify that this is indeed logically equal to  $(\ell_1 \vee \ell_2)$

# Cook-Levin Theorem

- Claim:  $\text{SAT} \leq_p \text{3SAT}$

- For a clause with three literals  $C = \ell_1 \vee \ell_2 \vee \ell_3$ :
  - Perfect. No need to do anything!
- For a clause with 4 or more literals  $C = (\ell_1 \vee \ell_2 \vee \dots \vee \ell_k)$ :
  - Add variables  $z_1, z_2, \dots, z_{k-3}$  and replace it with:  
 $(\ell_1 \vee \ell_2 \vee z_1) \wedge (\ell_3 \vee \bar{z}_1 \vee z_2) \wedge (\ell_4 \vee \bar{z}_2 \vee z_3) \wedge \dots$   
 $\wedge (\ell_{k-2} \vee \bar{z}_{k-4} \vee z_{k-3}) \wedge (\ell_{k-1} \vee \ell_k \vee \bar{z}_{k-3})$
  - If one of the  $\ell$ 's is TRUE, you can make this TRUE by setting appropriate values for  $z$  variables (check!)
  - If all  $\ell$  are FALSE, then there is no way to make the above conjunction TRUE (check!)

# NP vs co-NP

- Complements of each other

- NP = short proof for YES, co-NP = short proof for NO
- If a problem “Does there exist...” is in NP, then its complement “Does there not exist...” is in co-NP, and vice-versa
- The same goes for NP-complete and co-NP-complete

- Example

- SAT is NP-complete (“Does there exist  $x$  satisfying  $\varphi$ ?”)
- Tautology is coNP-complete (“Does there exist no  $x$  satisfying  $\varphi$ ?” = “Is  $\varphi$  always FALSE?”)

# $NP \cap co-NP$

- Clearly,  $P \subseteq NP \cap co-NP$ 
  - No advice needed; can just solve the problem in polytime
  - Major open question: Is  $P = NP \cap co-NP$ ?
- How about a short proof of both YES and NO?
  - Hunt for problems not known in  $P$  but still in  $NP \cap co-NP$

# NP $\cap$ co-NP

- Linear programming
  - [Gale–Kuhn–Tucker 1948]: LP is in NP  $\cap$  co-NP

## CHAPTER XIX

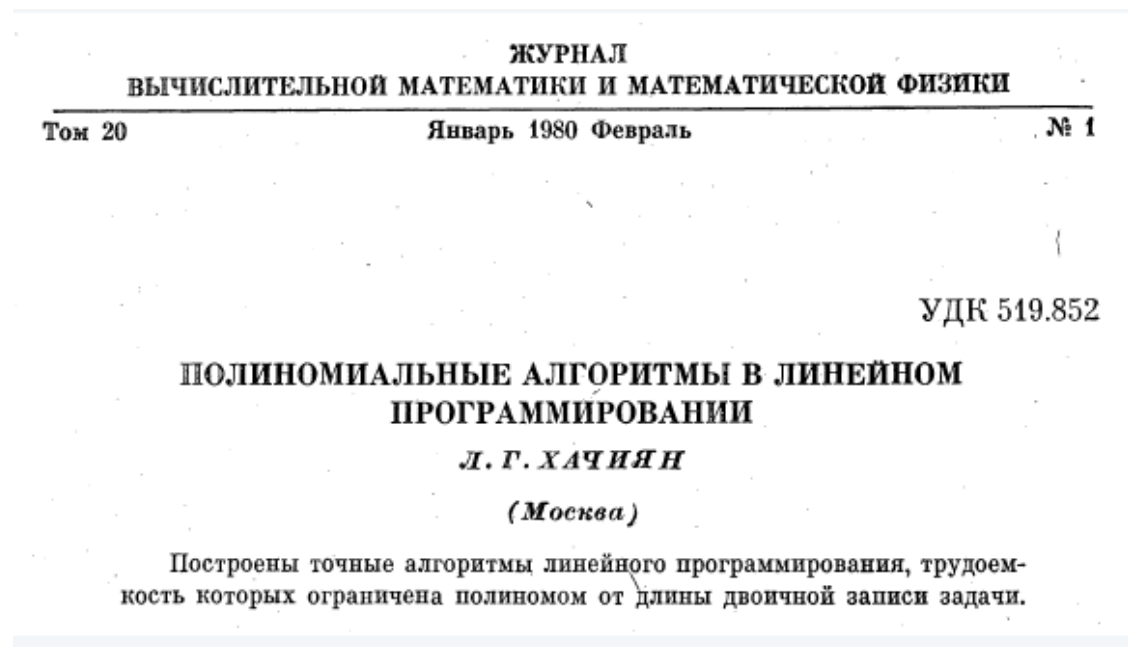
### LINEAR PROGRAMMING AND THE THEORY OF GAMES <sup>1</sup>

BY DAVID GALE, HAROLD W. KUHN, AND ALBERT W. TUCKER <sup>2</sup>

The basic “scalar” problem of *linear programming* is to maximize (or minimize) a linear function of several variables constrained by a system of linear inequalities [Dantzig, II]. A more general “vector” problem calls for maximizing (in a sense of partial order) a system of linear functions of several variables subject to a system of linear inequalities and, perhaps, linear equations [Koopmans, III]. The purpose of this chapter is to establish theorems of duality and existence for general “matrix” problems of linear programming which contain the “scalar” and “vector” problems as special cases, and to relate these general problems to the theory of zero-sum two-person games.

# $NP \cap co-NP$

- Linear programming
  - But later, we found out:
  - [Khachiyan 1979]: LP is in P



# NP $\cap$ co-NP

- Primality testing (“Is  $n$  a prime?”)
  - [Pratt 1975]: PRIMES is in NP  $\cap$  co-NP
    - A short NO proof is easy, but a short YES proof relies on some interesting math

SIAM J. COMPUT.  
Vol. 4, No. 3, September 1975

## EVERY PRIME HAS A SUCCINCT CERTIFICATE\*

VAUGHAN R. PRATT†

**Abstract.** To prove that a number  $n$  is composite, it suffices to exhibit the working for the multiplication of a pair of factors. This working, represented as a string, is of length bounded by a polynomial in  $\log_2 n$ . We show that the same property holds for the primes. It is noteworthy that almost no other set is known to have the property that short proofs for membership or nonmembership exist for all candidates without being known to have the property that such proofs are easy to come by. It remains an open problem whether a prime  $n$  can be recognized in only  $\log_2^\alpha n$  operations of a Turing machine for any fixed  $\alpha$ .

The proof system used for certifying primes is as follows.

AXIOM.  $(x, y, 1)$ .

INFERENCE RULES.

$R_1$ :  $(p, x, a), q \vdash (p, x, qa)$  provided  $x^{(p-1)/q} \not\equiv 1 \pmod{p}$  and  $q|(p-1)$ .

$R_2$ :  $(p, x, p-1) \vdash p$  provided  $x^{p-1} \equiv 1 \pmod{p}$ .

THEOREM 1.  $p$  is a theorem  $\equiv p$  is a prime.

THEOREM 2.  $p$  is a theorem  $\supset p$  has a proof of  $[4 \log_2 p]$  lines.

# $NP \cap co-NP$

- Primality testing (“Is  $n$  a prime?”)
  - But later we found out:
  - [Agrawal–Kayal–Saxena 2004]: PRIMES is in P
    - Milestone result!

Annals of Mathematics, 160 (2004), 781–793

## PRIMES is in P

By MANINDRA AGRAWAL, NEERAJ KAYAL, and NITIN SAXENA\*

### Abstract

We present an unconditional deterministic polynomial-time algorithm that determines whether an input number is prime or composite.



# NP $\cap$ co-NP

- Factoring (“Does  $n$  have a factor  $\leq k$ ?”)
  - FACTOR is in NP  $\cap$  co-NP
    - Short YES proof: Just present the factor
    - Short NO proof:
      - Present the entire prime factorization of  $n$ , along with a short proof that each presented factor is a prime
      - A TM can check that each factor is a prime
        - Actually, proofs of primality are not required now that we know the TM can just run AKS algorithm to check primality
      - The TM can also verify that none of the factors is  $\leq k$

# NP $\cap$ co-NP

- Factoring (“Does  $n$  have a factor  $\leq k$ ?”)
  - Major open question: Is FACTOR in P?
    - Basis of several cryptographic procedures
  - Challenge: Factor the following number.

74037563479561712828046796097429573142593188889231289  
08493623263897276503402826627689199641962511784399589  
43305021275853701189680982867331732731089309005525051  
16877063299072396380786710086096962537934650563796359

RSA-704

(\$30,000 prize if you can factor it)

# NP $\cap$ co-NP

- Factoring (“Does  $n$  have a factor  $\leq k$ ?”)
  - [Shor 1994]: We can factor an  $n$ -bit integer in  $O(n^3)$  steps on a quantum computer.
  - \*Scalable\* quantum computers can help
    - 2001: Factored  $15 = 3 \times 5$  (with high probability)
    - 2012: Factored  $21 = 3 \times 7$

# Other Complexity Classes

- Based on the exact time complexity
  - $\text{DTIME}(n)$ ,  $\text{NTIME}(n^2)$ , ...
    - Deterministic / nondeterministic time complexity
- Based on space complexity
  - $\text{DSPACE}(n)$ ,  $\text{NSPACE}(\log n)$
- Using randomization
  - ZPP (expected polytime, no errors)
- Allowing probabilistic errors
  - RP (polytime, one-sided error)
  - BPP (polytime, two-sided errors)