CSC373

Week 2: Greedy Algorithms

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Recap

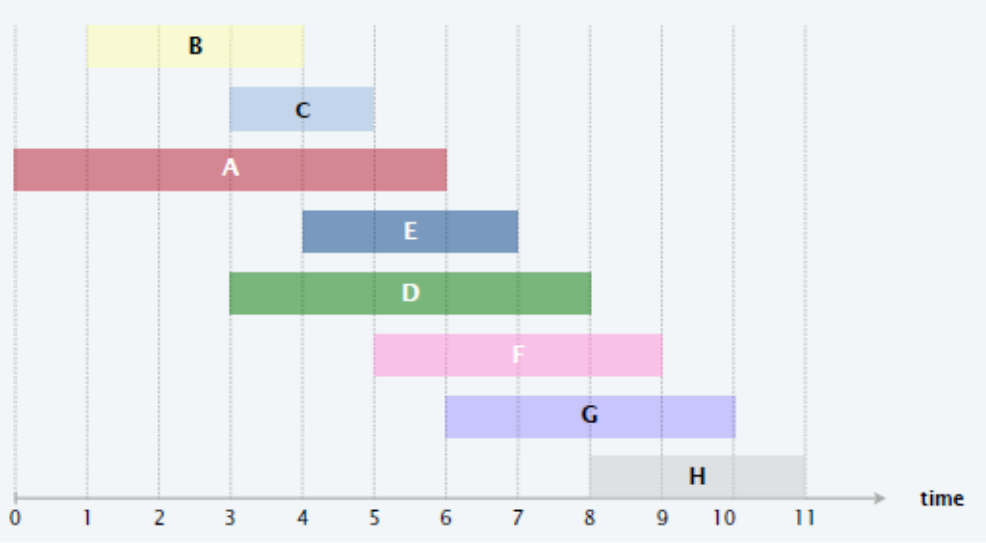
- Divide & Conquer
  - Master theorem
  - Counting inversions in $O(n \log n)$
  - Finding closest pair of points in $\mathbb{R}^2$ in $O(n \log n)$
  - Fast integer multiplication in $O\left(n^{\log_2 3}\right)$
  - Fast matrix multiplication in $O\left(n^{\log_2 7}\right)$
  - Finding k^{th} smallest element (in particular, median) in $O(n)$

Greedy Algorithms

- Greedy (also known as myopic) algorithm outline
  - We want to find a solution x that maximizes some objective function f
  - But the space of possible solutions x is too large
  - The solution x is typically composed of several parts (e.g. x may be a set, composed of its elements)
  - Instead of directly computing x…
    - Compute it one part at a time
    - Select the next part "greedily" to get maximum immediate benefit (this needs to be defined carefully for each problem)
    - May not be optimal because there is no foresight
    - But sometimes this can be optimal too!

Interval Scheduling

- Problem
  - Job j starts at time s_j and finishes at time f_j
  - Two jobs are compatible if they don't overlap
  - Goal: find maximum-size subset of mutually compatible jobs

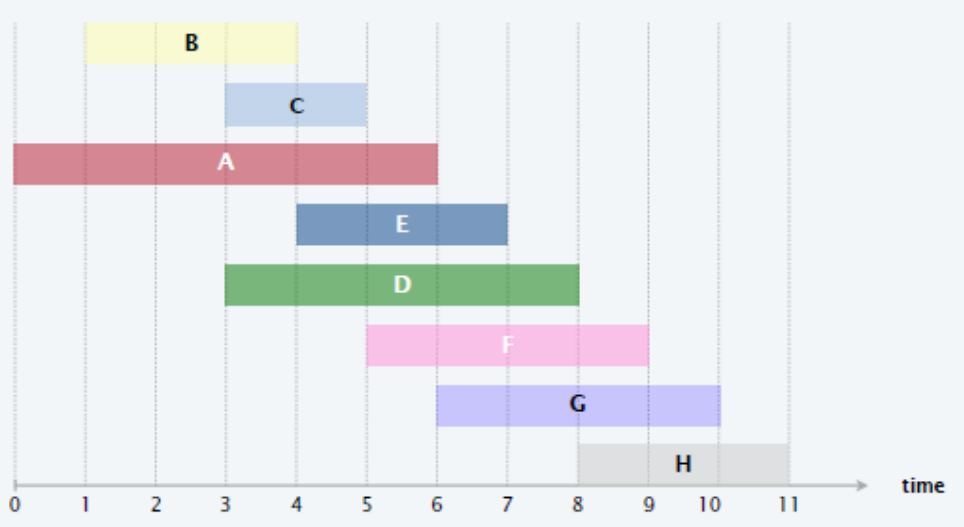


Interval Scheduling

- Greedy template
  - Consider jobs in some "natural" order
  - Take each job if it's compatible with the ones already chosen
- What order?
  - Earliest start time: ascending order of s_j
  - Earliest finish time: ascending order of f_j
  - Shortest interval: ascending order of $f_j - s_j$
  - Fewest conflicts: ascending order of c_j, where c_j is the number of remaining jobs that conflict with j

Example

- Earliest start time: ascending order of s_j
- Earliest finish time: ascending order of f_j
- Shortest interval: ascending order of $f_j - s_j$
- Fewest conflicts: ascending order of c_j, where c_j is the number of remaining jobs that conflict with j



Interval Scheduling

- Does it work?

Counterexamples for

earliest start time

one job takes all time, other three can only be scheduled in another room

shortest interval

although shortest, the job incompatible with two more jobs

fewest conflicts

may not conflict with jobs that conflict with each other

Interval Scheduling

- Implementing greedy with earliest finish time (EFT)
  - Sort jobs by finish time. Say $f_1 \leq f_2 \leq \cdots \leq f_n$
  - When deciding whether job j should be included, we need to check whether it's compatible with all previously added jobs
    - We only need to check if $s_j \geq f_{i^*}$, where i^* is the *last added job*
    - This is because for any jobs i added before i^*, $f_i \leq f_{i^*}$
    - So we can simply store and maintain the finish time of the last added job
  - Running time: $O(n \log n)$

Interval Scheduling

- Optimality of greedy with EFT
  - Suppose for contradiction that greedy is not optimal
  - Say greedy selects jobs $i_1, i_2, \dots, i_k$ sorted by finish time
  - Consider the optimal solution $j_1, j_2, \dots, j_m$ (also sorted by finish time) which matches greedy for as long as possible
    - That is, we want $j_1 = i_1, \dots, j_r = i_r$ for greatest possible r

Interval Scheduling

Another standard method is induction

- Optimality of greedy with EFT
  - Both i_{r+1} and j_{r+1} were compatible with the previous selection ($i_1 = j_1, \dots, i_r = j_r$)
  - Consider the solution $i_1, i_2, \dots, i_r, i_{r+1}, j_{r+2}, \dots, j_m$
    - It should still be feasible (since $f_{i_{r+1}} \leq f_{j_{r+1}}$)
    - It is still optimal
    - And it matches with greedy for one more step (contradiction!)

Interval Partitioning

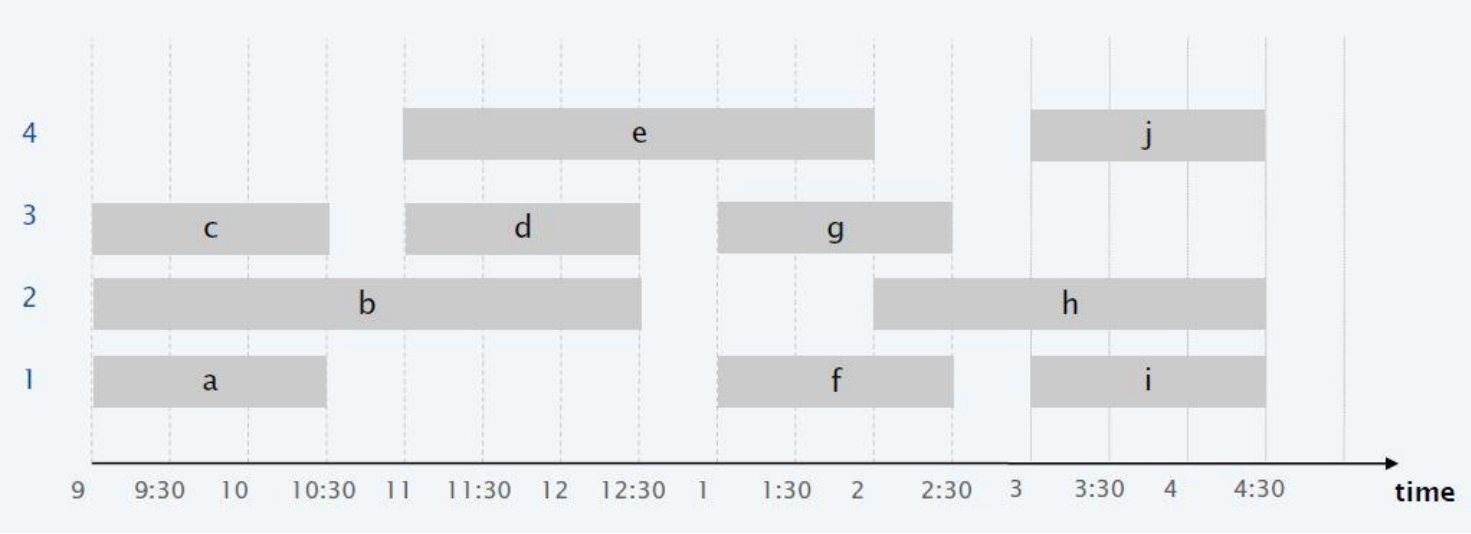
- Problem
  - Job j starts at time s_j and finishes at time f_j
  - Two jobs are compatible if they don't overlap
  - Goal: group jobs into fewest partitions such that jobs in the same partition are compatible

- One idea
  - Find the maximum compatible set using the previous greedy EFT algorithm, call it one partition, recurse on the remaining jobs.
  - Doesn't work (check by yourselves)

Interval Partitioning

- Think of scheduling lectures for various courses into as few classrooms as possible

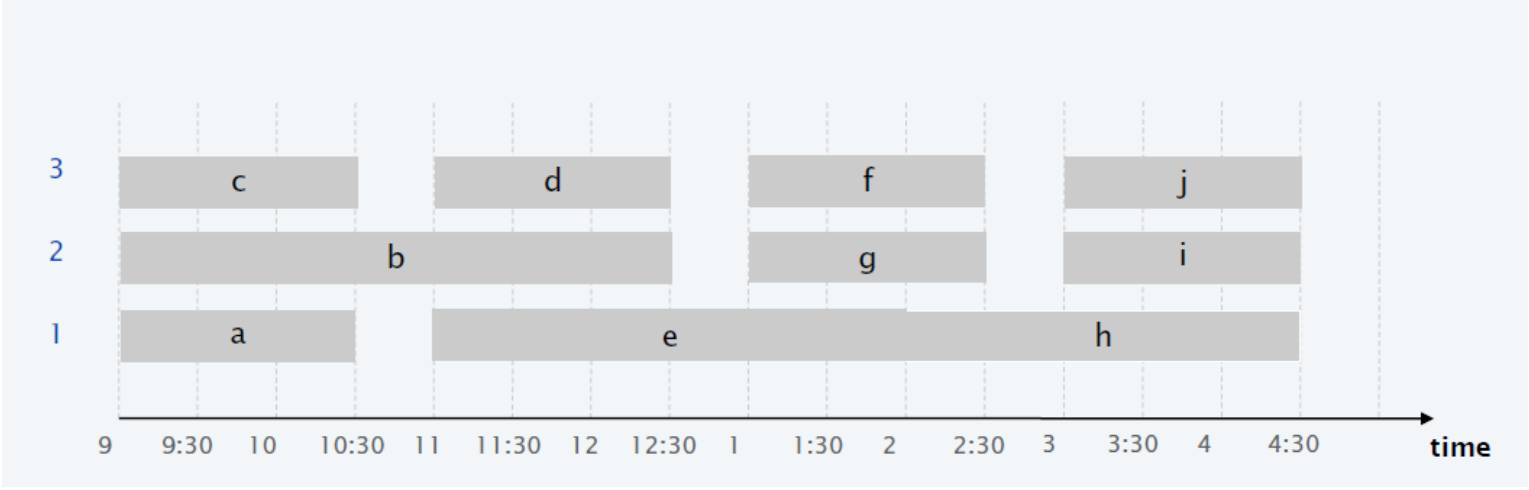
- This schedule uses 4 classrooms for scheduling 10 lectures



Interval Partitioning

- Think of scheduling lectures for various courses into as few classrooms as possible

- This schedule uses 3 classrooms for scheduling 10 lectures

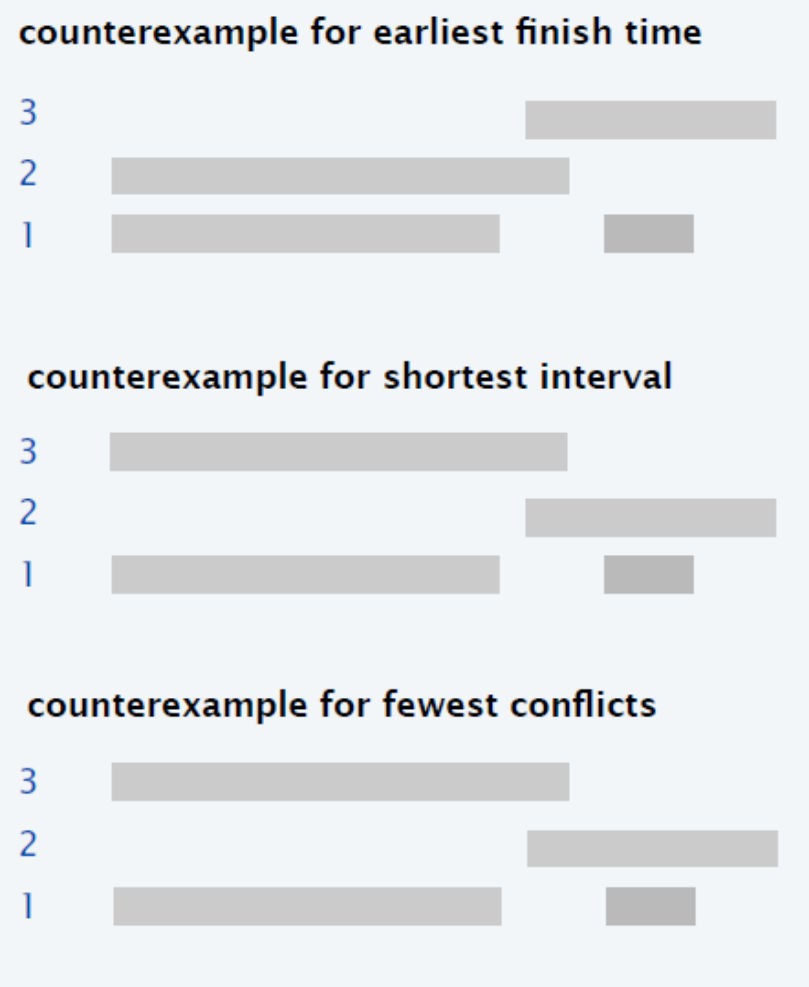


Interval Partitioning

- Let's go back to the greedy template!
  - Go through lectures in some "natural" order
  - Assign each lecture to a compatible classroom (which?), and create a new classroom if the lecture conflicts with every existing classroom

- Order of lectures?
  - Earliest start time: ascending order of s_j
  - Earliest finish time: ascending order of f_j
  - Shortest interval: ascending order of $f_j - s_j$
  - Fewest conflicts: ascending order of c_j, where c_j is the number of remaining jobs that conflict with j

Interval Partitioning



- At least when you assign each lecture to an arbitrary feasible classroom, three of these heuristics do not work.
- The fourth one works! (next slide)

Interval Partitioning

EARLIESTSTARTTIMEFIRST(n, s_1, s_2, …, s_n, f_1, f_2, …, f_n)

SORT lectures by start time so that $s_1 \leq s_2 \leq \ldots \leq s_n$.

$d \leftarrow 0$ ← number of allocated classrooms

FOR $j = 1$ TO n

  IF lecture j is compatible with some classroom

    Schedule lecture j in any such classroom k.

  ELSE

    Allocate a new classroom $d + 1$.

    Schedule lecture j in classroom $d + 1$.

    $d \leftarrow d + 1$

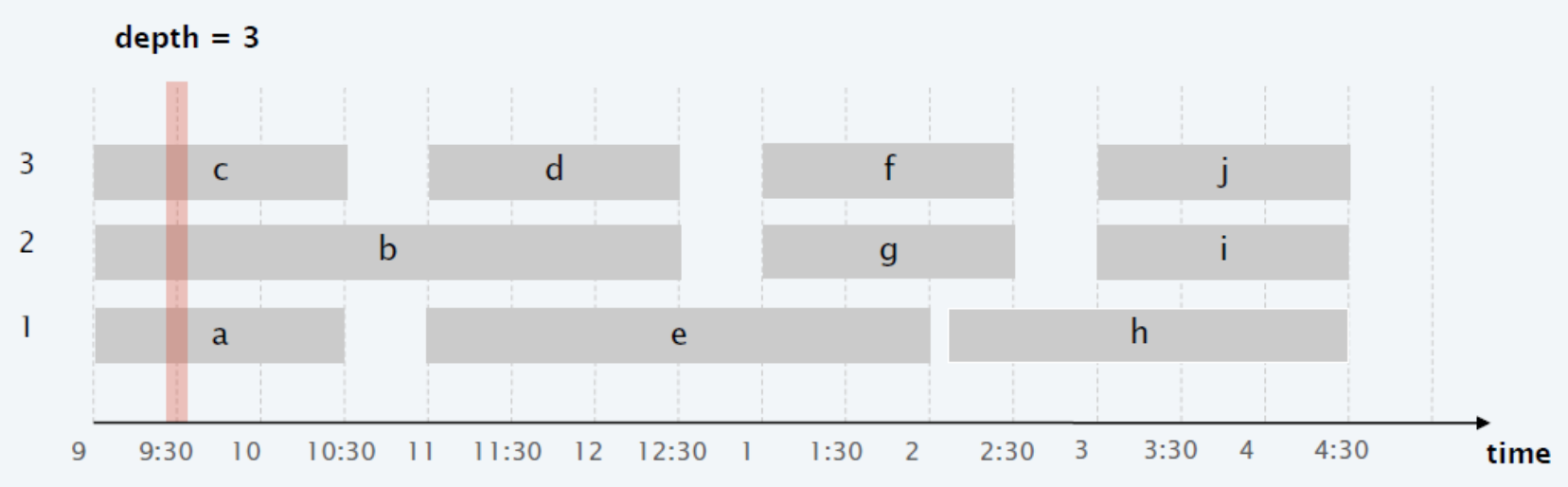
RETURN schedule.

Interval Partitioning

- Running time
  - Key step: check if the next lecture can be scheduled at some classroom
  - Store classrooms in a priority queue
    - key = finish time of its last lecture
  - Is lecture j compatible with some classroom?
    - Same as "Is s_j at least as large as the minimum key?"
    - If yes: add lecture j to classroom k with minimum key, and increase its key to f_j
    - Otherwise: create a new classroom, add lecture j, set key to f_j
  - $O(n)$ priority queue operations, $O(n \log n)$ time

Interval Partitioning

- Proof of optimality (lower bound)
  - # classrooms needed $\geq$ maximum "depth" at any point
    - depth = number of lectures running at that time
  - We now show that our greedy algorithm uses only these many classrooms!



Interval Partitioning

- Proof of optimality (upper bound)
  - Let d = # classrooms used by greedy
  - Classroom d was opened because there was a schedule j which was incompatible with some lectures already scheduled in each of $d - 1$ other classrooms
  - All these d lectures end after s_j
  - *Since we sorted by start time*, they all start at/before s_j
  - So at time s_j, we have d overlapping lectures
  - Hence, depth $\geq d$
  - So all schedules use $\geq d$ classrooms.
  - QED!

Interval Graphs

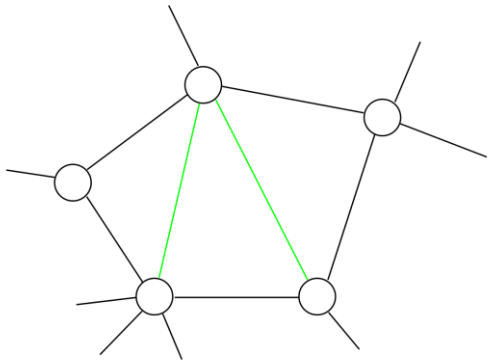
- Interval scheduling and interval partitioning can be seen as graph problems

- Input
  - Graph $G = (V, E)$
  - Vertices V = jobs/lectures
  - Edge $(i, j) \in E$ if jobs i and j are incompatible

- Interval scheduling = maximum independent set (MIS)
- Interval partitioning = graph colouring

Interval Graphs

- MIS and graph colouring are NP-hard for general graphs
- But they're efficiently solvable for interval graphs
  - Interval graphs = graphs which can be obtained from incompatibility of intervals
  - In fact, this holds even when we are not given an interval representation of the graph
- Can we extend this result further?
  - Yes! Chordal graphs
    - Every cycle with 4 or more vertices has a chord



Minimizing Lateness

- Problem
  - We have a single machine
  - Each job j requires t_j units of time and is due by time d_j
  - If it's scheduled to start at s_j, it will finish at $f_j = s_j + t_j$
  - Lateness: $\ell_j = \max\{0, f_j - d_j\}$
  - Goal: minimize the maximum lateness, $L = \max_j \ell_j$

- Contrast with interval scheduling
  - We can decide the start time
  - All jobs must be scheduled on a single machine

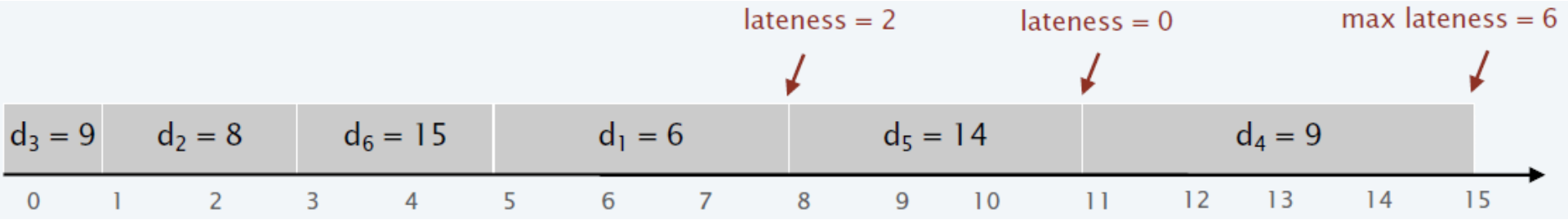
Minimizing Lateness

- Example

Input

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15

An example schedule



Minimizing Lateness

- Let's go back to greedy template
  - Consider jobs one-by-one in some "natural" order
  - Schedule jobs in this order (nothing special to do here, since we have to schedule all jobs and there is only one machine available)

- Natural orders?
  - Shortest processing time first: ascending order of processing time t_j
  - Earliest deadline first: ascending order of due time d_j
  - Smallest slack first: ascending order of $d_j - t_j$

Minimizing Lateness

- Counterexamples
  - Shortest processing time first
    - Ascending order of processing time t_j

first job 1, then job 2, finish at day 11, miss the deadline for job 2 (day 10)

	1	2
t_j	1	10
d_j	100	10

  - Smallest slack first
    - Ascending order of $d_j - t_j$

first job 2, then job 1, finish at day 11, miss the deadline for job 1 (day 2)

	1	2
t_j	1	10
d_j	2	10

Minimizing Lateness

- By now, you should know what's coming…
- We'll prove that earliest deadline first works!

EARLIESTDEADLINEFIRST(n, t_1, t_2, …, t_n, d_1, d_2, …, d_n)

SORT n jobs so that $d_1 \leq d_2 \leq \ldots \leq d_n$.

$t \leftarrow 0$

FOR $j = 1$ TO n

  Assign job j to interval $[t, t + t_j]$.

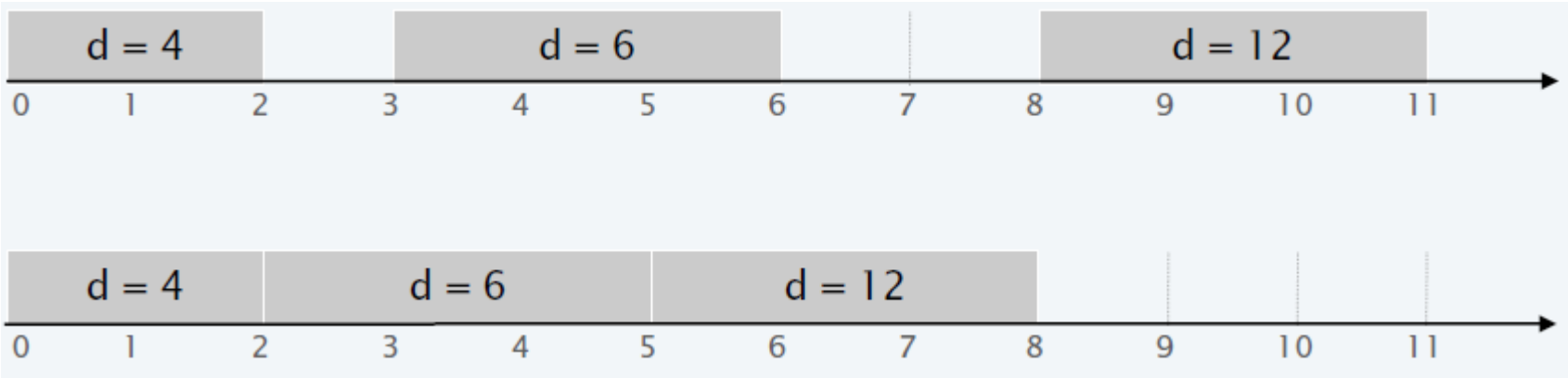
  $s_j \leftarrow t$; $f_j \leftarrow t + t_j$

  $t \leftarrow t + t_j$

RETURN intervals $[s_1, f_1]$, $[s_2, f_2]$, …, $[s_n, f_n]$.

Minimizing Lateness

- Observation 1
  - There is an optimal schedule with no idle time

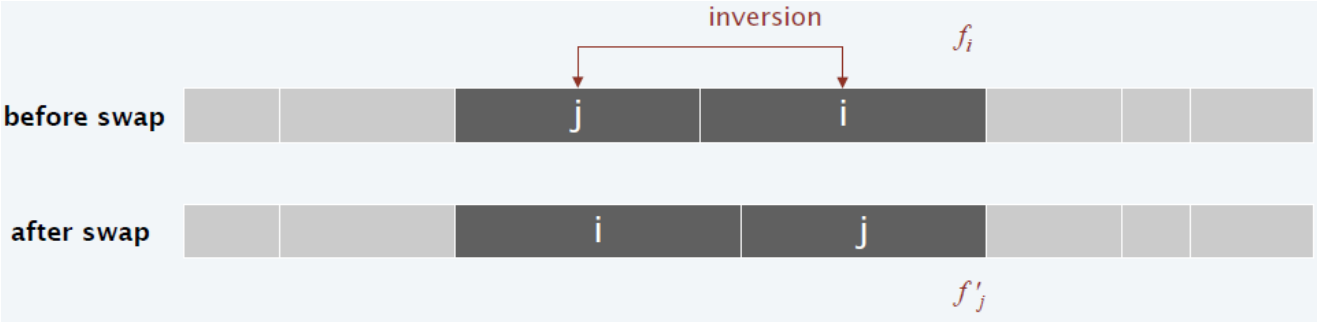


Minimizing Lateness

- Observation 2
  - Earliest deadline first has no idle time
- Let us define an "inversion"
  - (i, j) such that $d_i < d_j$ but j is scheduled before i
- Observation 3
  - By definition, earliest deadline first has no inversions
- Observation 4
  - If a schedule with no idle time has an inversion, it has a pair of inverted jobs scheduled consecutively

Minimizing Lateness

- Claim
  - Swapping adjacently scheduled inverted jobs doesn't increase lateness but reduces #inversions by one
- Proof
  - Let ℓ and ℓ' denote lateness before/after swap
  - Clearly, $\ell_k = \ell'_k$ for all $k \neq i, j$
  - Also, clearly, $\ell'_i \leq \ell_i$



Minimizing Lateness

- Claim
  - Swapping adjacently scheduled inverted jobs doesn't increase lateness but reduces #inversions by one
- Proof
  - $\ell'_j = f'_j - d_j = f_i - d_j \leq f_i - d_i = \ell_i$
  - $L' = \max\left\{\ell'_i, \ell'_j, \max_{k \neq i,j} \ell'_k\right\} \leq \max\left\{\ell_i, \ell_i, \max_{k \neq i,j} \ell_k\right\} \leq L$

Minimizing Lateness

- Proof of optimality of earliest deadline first
  - Suppose for contradiction that it's not optimal
  - Consider an optimal schedule S^* which has fewest inversions among all optimal schedules
    - We can assume it has no idle time
    - If S^* has zero inversions, it's exactly earliest deadline first
    - So assume S^* has at least one inversion
    - So it must have an adjacent inversion (i, j)
    - But swapping these jobs doesn't increase lateness (so new schedule stays optimal) and reduces the number of inversions by 1
    - Contradiction given that S^* has fewest inversions among all optimal schedules.
    - QED!

Lossless Compression

- Problem
  - We have a document that is written using n distinct labels
  - Naïve encoding: represent each label using $k = \log n$ bits
  - If the document has length m, this uses $m \log n$ bits

  - Say for English documents with no punctuations etc, we have $n = 26$, so we can use 5 bits.
    - $a = 00000$
    - $b = 00001$
    - $c = 00010$
    - $d = 00011$
    - …

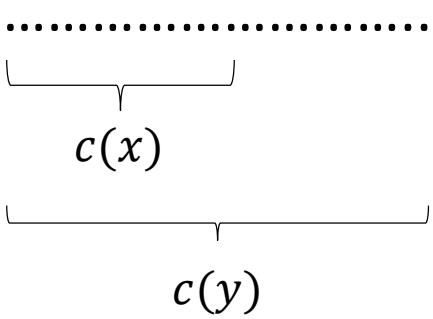
Lossless Compression

- Is this optimal?
  - What if a, e, r, s are much more frequent in the document than x, q, z?
  - Can we assign shorter codes to more frequent letters?

- Say we assign…
  - $a = 0, b = 1, c = 01, \ldots$
  - See a problem?
    - What if we observe the encoding ‘01’?
    - Is it ‘ab’? Or is it ‘c’?

Lossless Compression

- To avoid conflicts, we need *prefix-free encoding*
  - Map each label x to a bit-string $c(x)$ such that for all distinct labels x and y, $c(x)$ is not a prefix of $c(y)$
  - Then it's impossible to have a scenario like this



  - So we can read left to right, find the first point where it becomes a valid encoding, decode the label, and continue

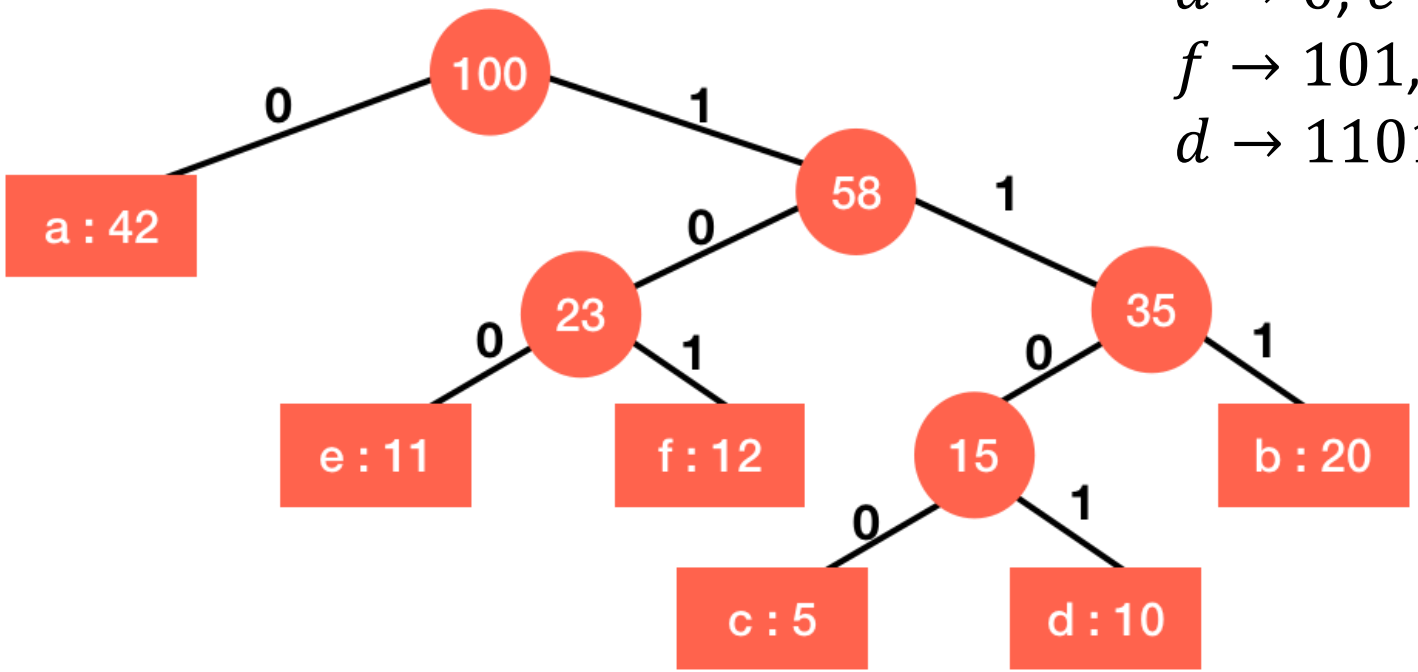
Lossless Compression

- Formal problem
  - Given n symbols and their frequencies $(w_1, \dots, w_n)$, find a prefix-free encoding with lengths $(\ell_1, \dots, \ell_n)$ assigned to the symbols which minimizes $\sum_{i=1}^{n} w_i \cdot \ell_i$
    - Note that $\sum_{i=1}^{n} w_i \cdot \ell_i$ is the length of the compressed document

- Example
  - $(w_a, w_b, w_c, w_d, w_e, w_f) = (42,20,5,10,11,12)$
  - No need to remember the numbers ☺

Lossless Compression

- Observation: prefix-free encoding = tree



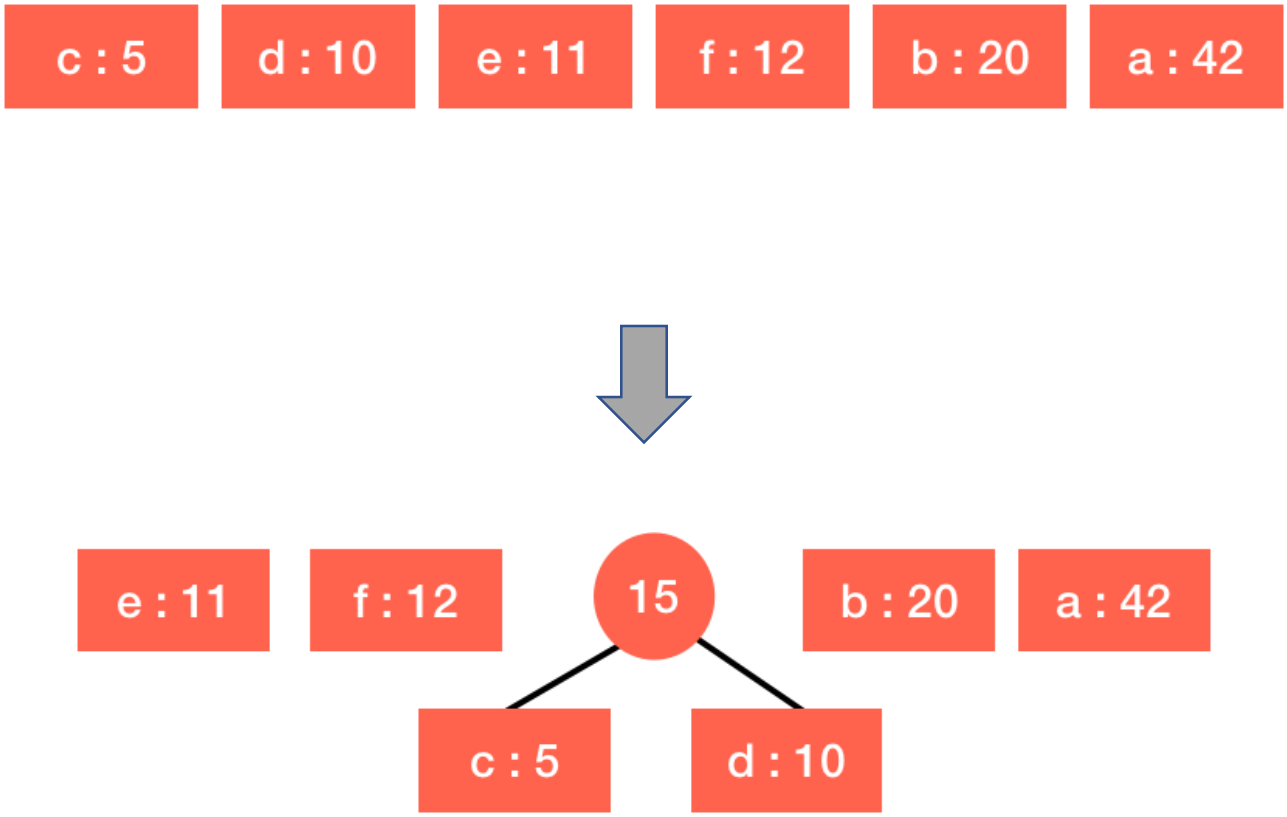
$a \to 0, e \to 100,$
$f \to 101, c \to 1100,$
$d \to 1101, b \to 111$

Lossless Compression

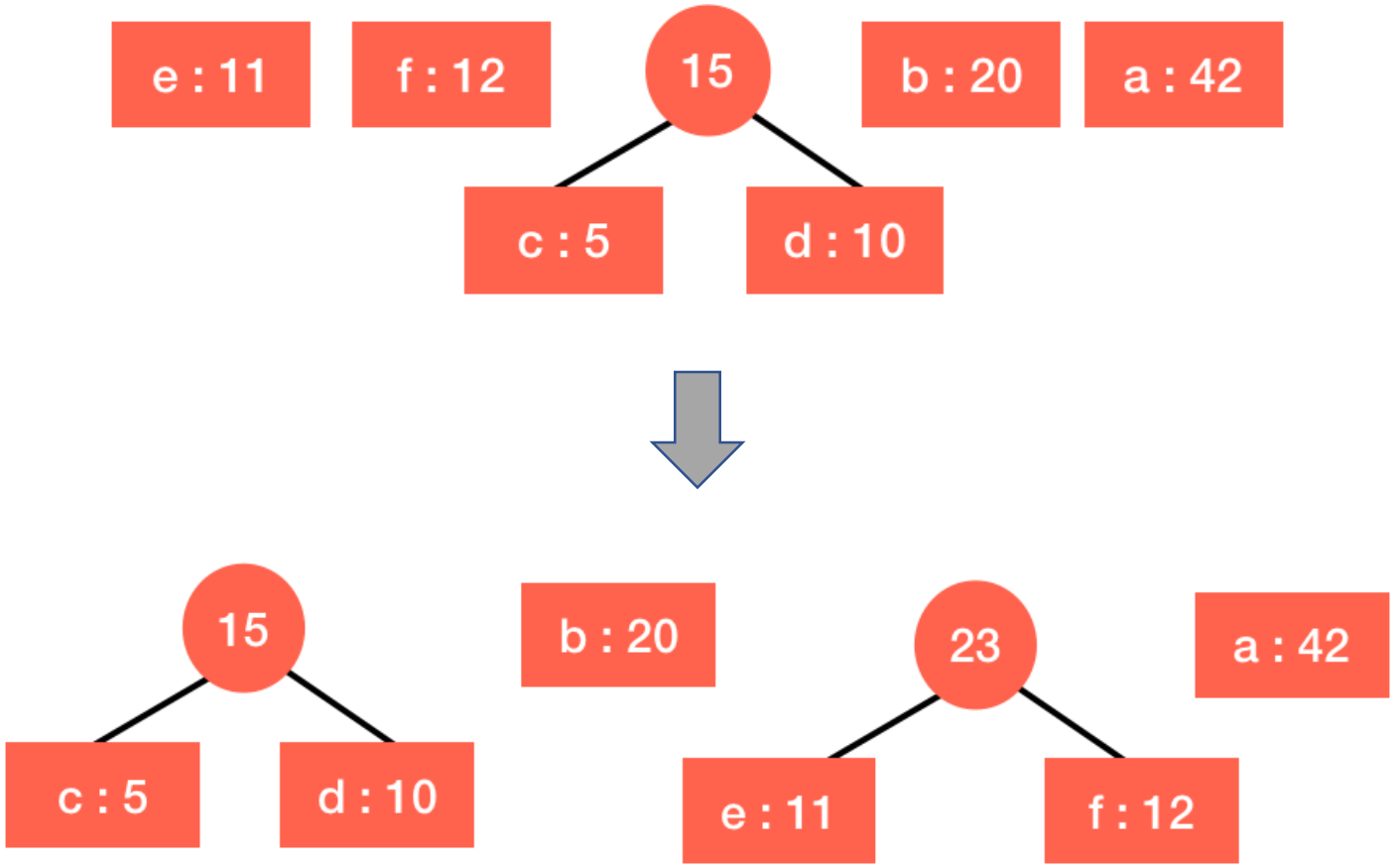
- Huffman Coding
  - Build a priority queue by adding (x, w_x) for each symbol x
  - While $|\text{queue}| \geq 2$
    - Take the two symbols with the lowest weight (x, w_x) and (y, w_y)
    - Merge them into one symbol with weight $w_x + w_y$

- Let's see this on the previous example

Lossless Compression



Lossless Compression



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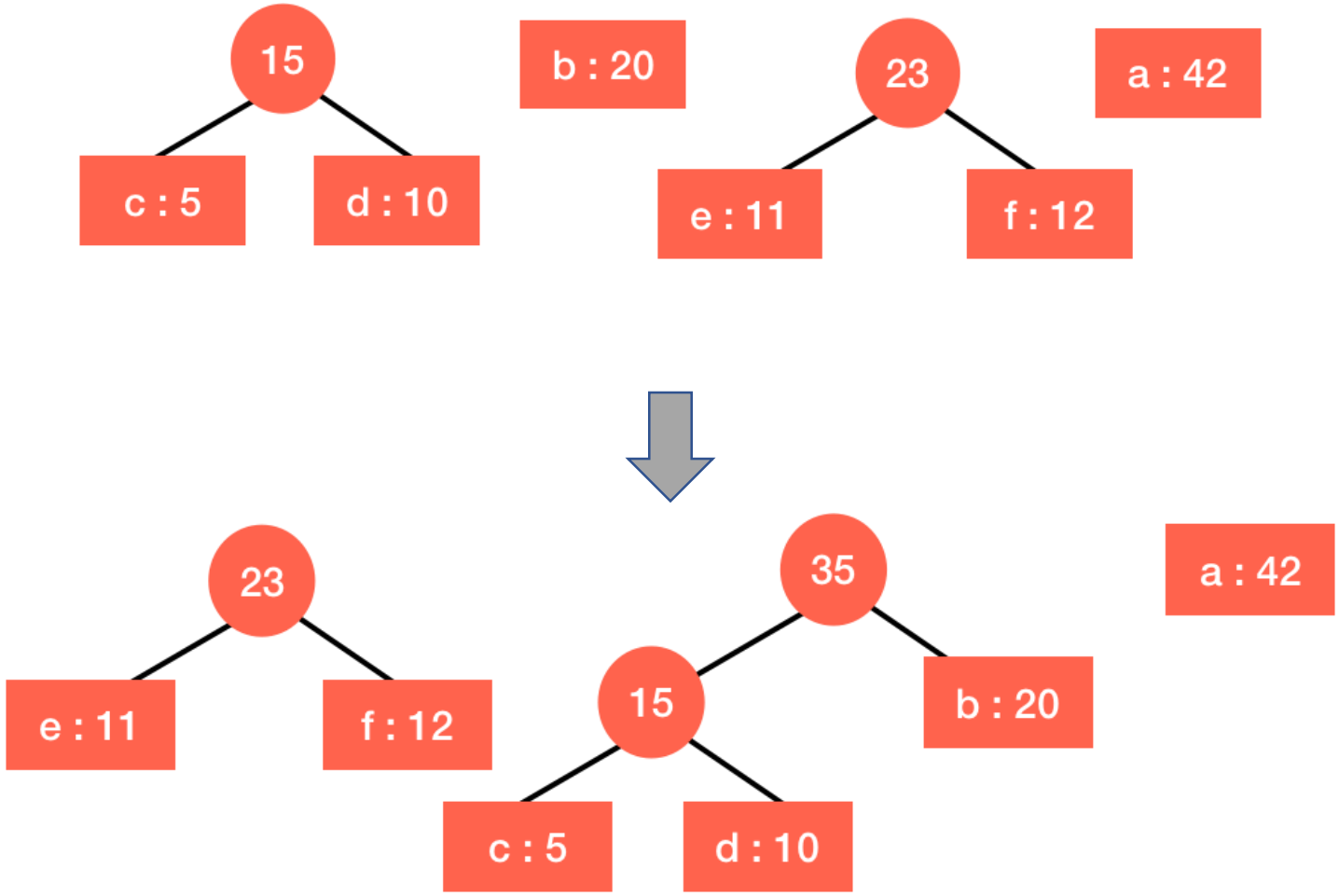

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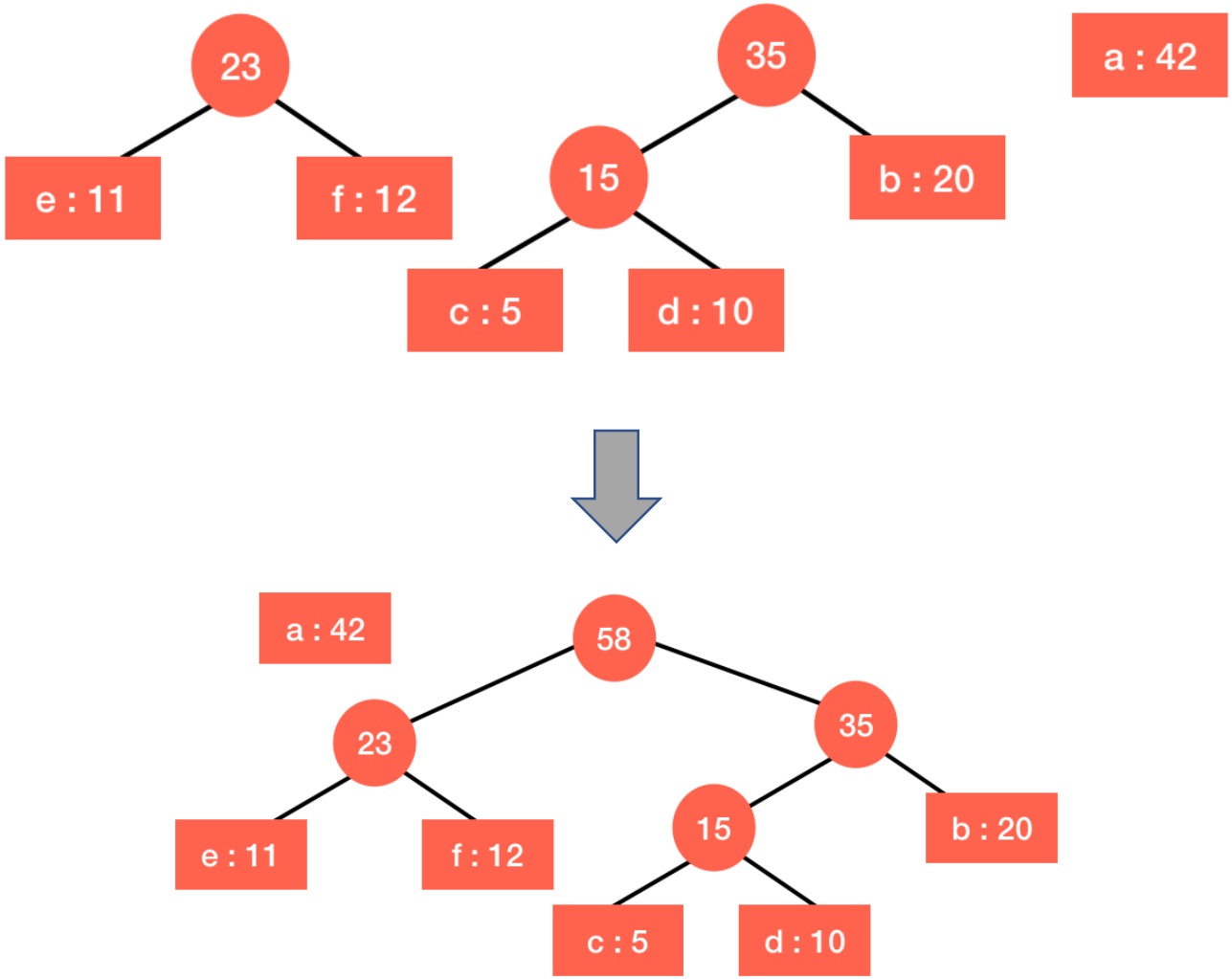
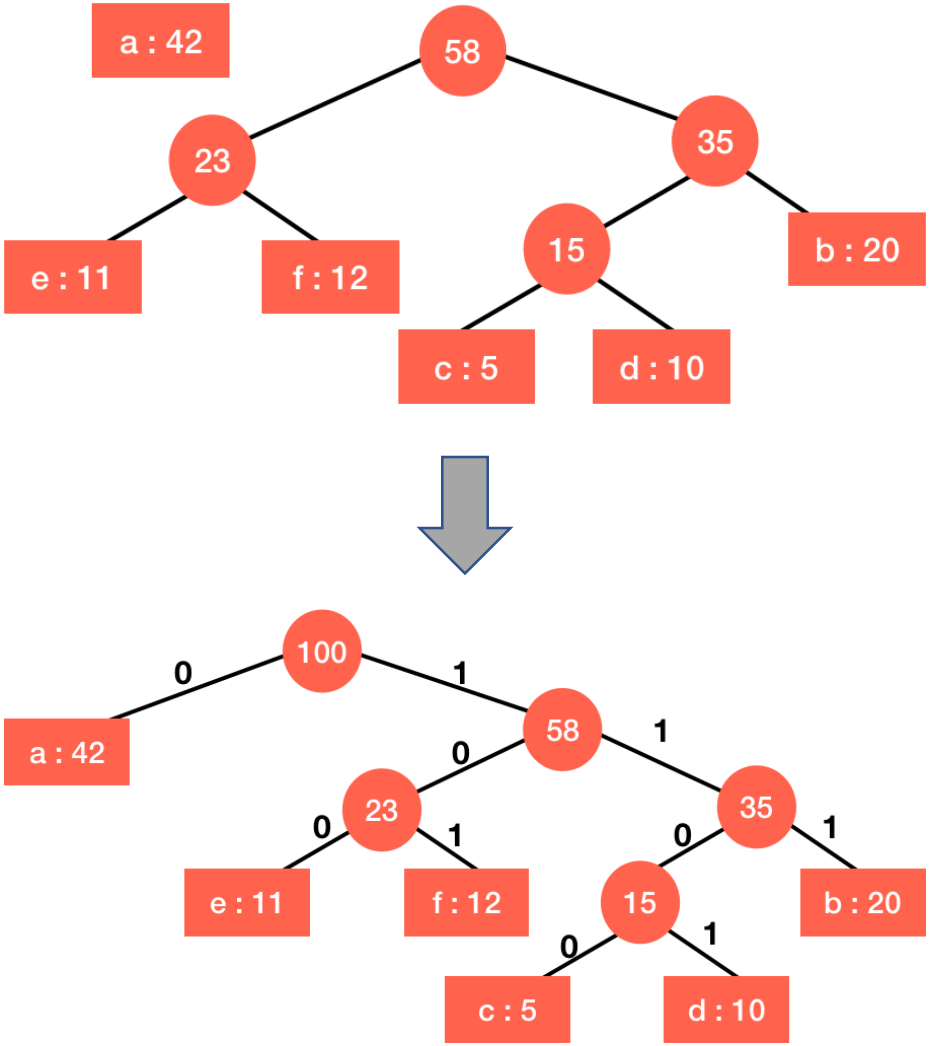


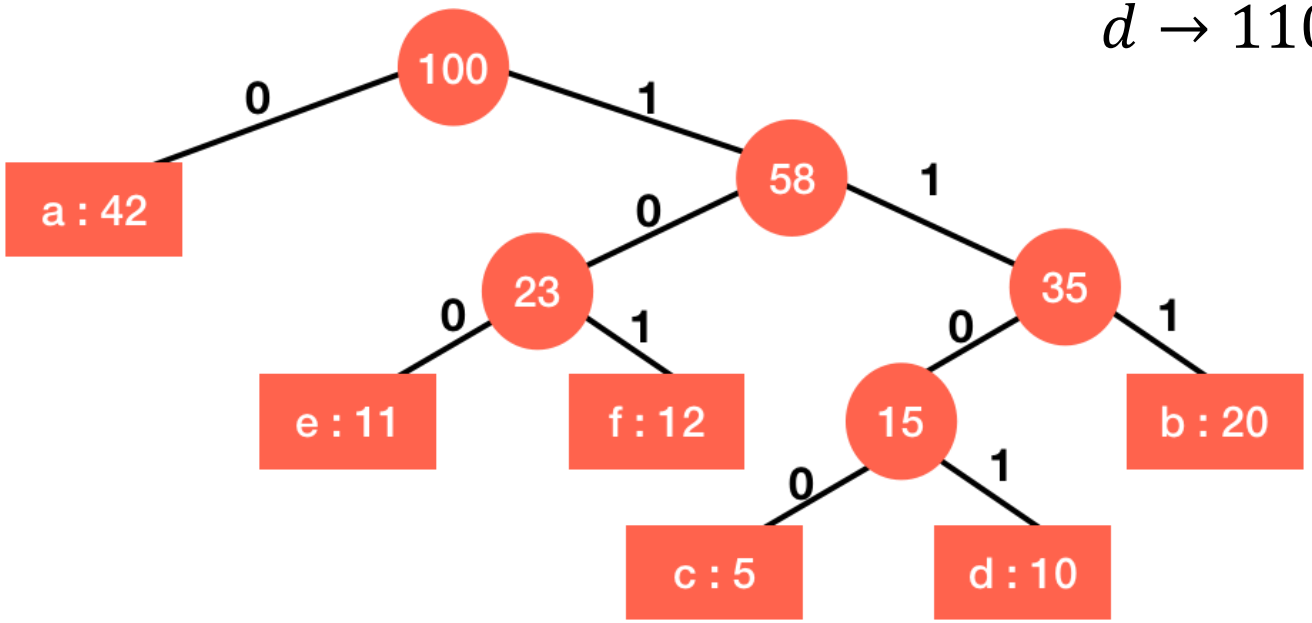
Lossless Compression



Lossless Compression

- Final Outcome

$a \to 0, e \to 100,$
$f \to 101, c \to 1100,$
$d \to 1101, b \to 111$



Lossless Compression

- Running time
  - $O(n \log n)$
  - Can be made $O(n)$ if the labels are given to you sorted by their frequencies

- Proof of optimality
  - Induction on the number of symbols n
  - Base case: For $n = 2$, there are only two possible encodings, both are optimal, assign 1 bit to each symbol
  - Hypothesis: Assume it returns an optimal encoding with $n - 1$ symbols

Lossless Compression

- Proof of optimality
  - Consider the case of n symbols
  - Lemma 1: If $w_x < w_y$, then $\ell_x \geq \ell_y$ in any optimal tree.
    - Proof sketch: Otherwise, swapping x and y would strictly reduce the overall length (exercise!).
  - Lemma 2: There is an optimal tree T in which the two least frequent symbols are siblings.
    - Proof sketch: First prove that they must have the same longest length assigned to them. Then, if they're not siblings, chop and rearrange the tree to make them siblings (exercise!).
  - Now, we can compare the tree H produced by Huffman vs such an optimal tree T

Lossless Compression

- Proof of optimality
  - Let x and y be the two least frequency symbols
  - In Huffman, we combine them in the first step into "xy"
  - Let H' and T' be trees obtained from H and T by treating xy as one symbol with frequency $w_x + w_y$
  - Use induction hypothesis: $Length(H') \leq Length(T')$
  - $Length(H) = Length(H') + (w_x + w_y) \cdot 1$
  - $Length(T) = Length(T') + (w_x + w_y) \cdot 1$
  - QED!

one node expands to two children, each new node has depth increases by 1, thus plus (w_x + w_y) * 1

Other Greedy Algorithms

- If you aren't familiar with the following algorithms, spend some time checking them out!
  - Dijkstra's shortest path algorithm
  - Kruskal and Prim's minimum spanning tree algorithms