Heuristic Search

- Reading note: Chapter 4 covers heuristic search.

Heuristic functions

We can encode each notion of the “ merit” of a state into a heuristic function, h(n).

A heuristic function maps a state onto an estimate of the cost to the goal from that state.

Can you think of examples of heuristics?

Heuristics are sensitive to the problem domain.

Heuristic for planning a path through a maze?

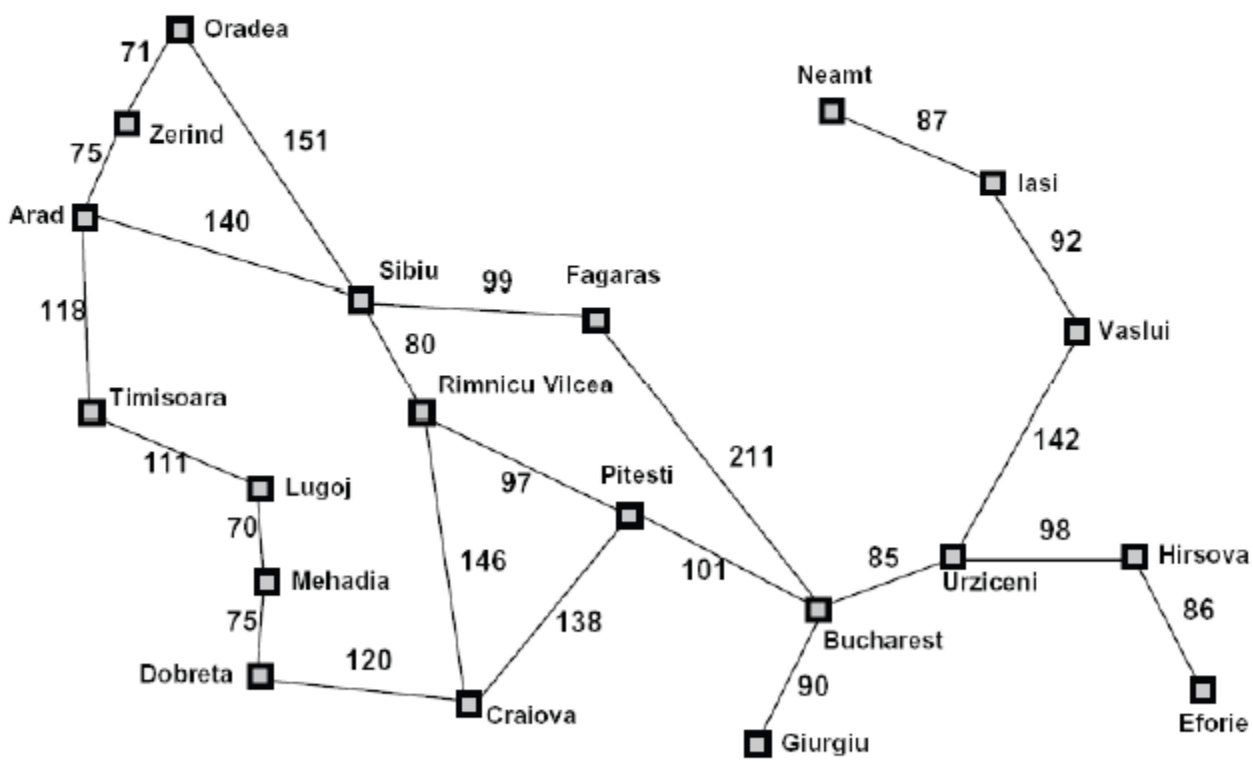
For solving a Rubick’s cube?

Heuristic Search

- If $h(n_1) < h(n_2)$ this means that we guess that it is cheaper to get to the goal from n_1 than from n_2.

- We require that
  - $h(n) = 0$ for every node n whose terminal state satisfies the goal.
  - Zero cost of achieving the goal from node that already satisfies the goal.

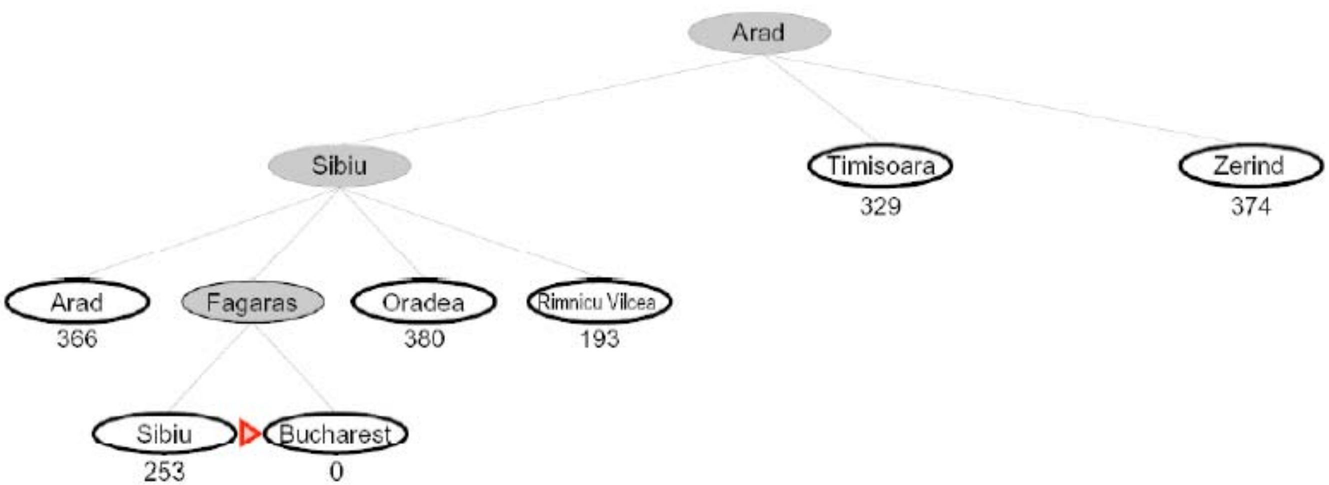
Euclidean distance as h(s)



Straight−line distance to Bucharest	
Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	178
Giurgiu	77
Hirsova	151
Iasi	226
Lugoj	244
Mehadia	241
Neamt	234
Oradea	380
Pitesti	98
Rimnicu Vilcea	193
Sibiu	253
Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

Say we want to plan a path from Arad to Bucharest, and we know the straight line distance from each city to our goal. This lets us plan our trip by picking cities at each time point that minimize the distance to our goal (or maximize our heuristic).

Best first (greedy) search



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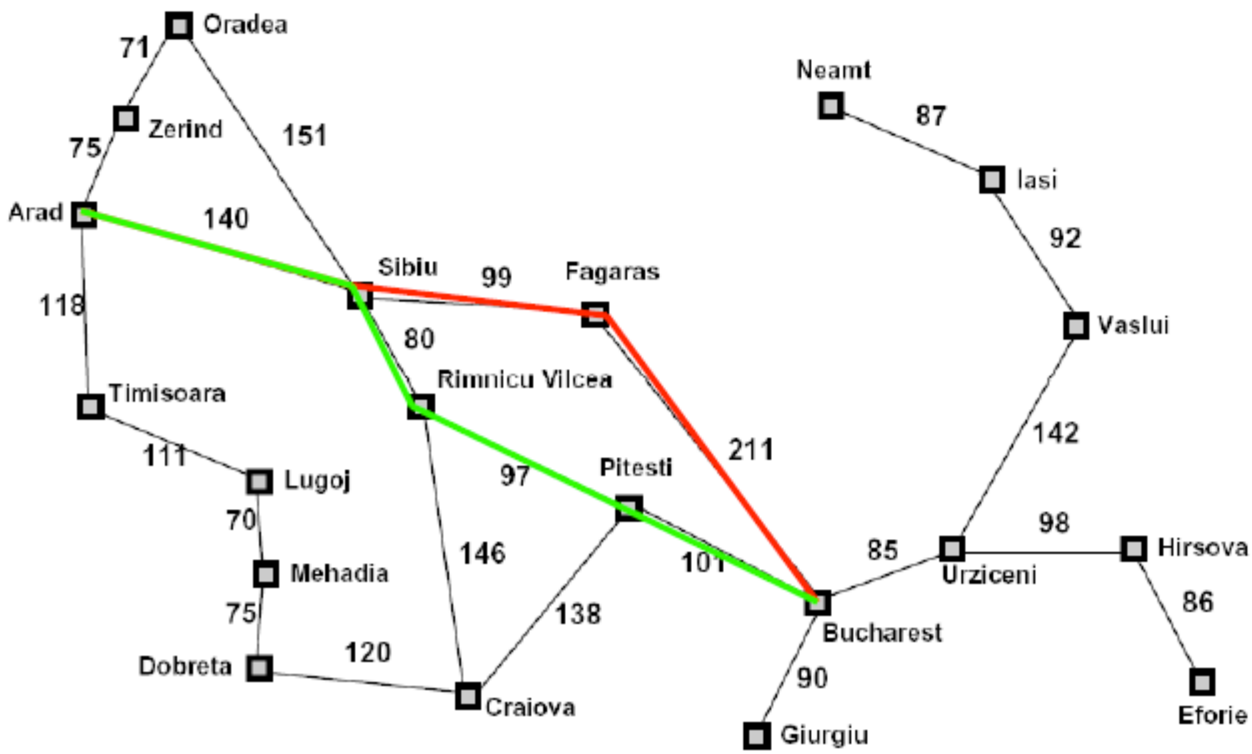
If we only use h(n) to guide our search, the search strategy is called Greedy or Best First Search

How can it possibly go wrong??

Best first (greedy) search



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In red is the path we selected. In green is the shortest path between Arad and Bucharest. What happened?

Modifying the search

How to avoid the mistake?

Modifying the search

How to avoid the mistake?

Take into account the cost of getting to the node as well as our estimate of the cost of getting to the goal from the node.

Define an evaluation function f(n):

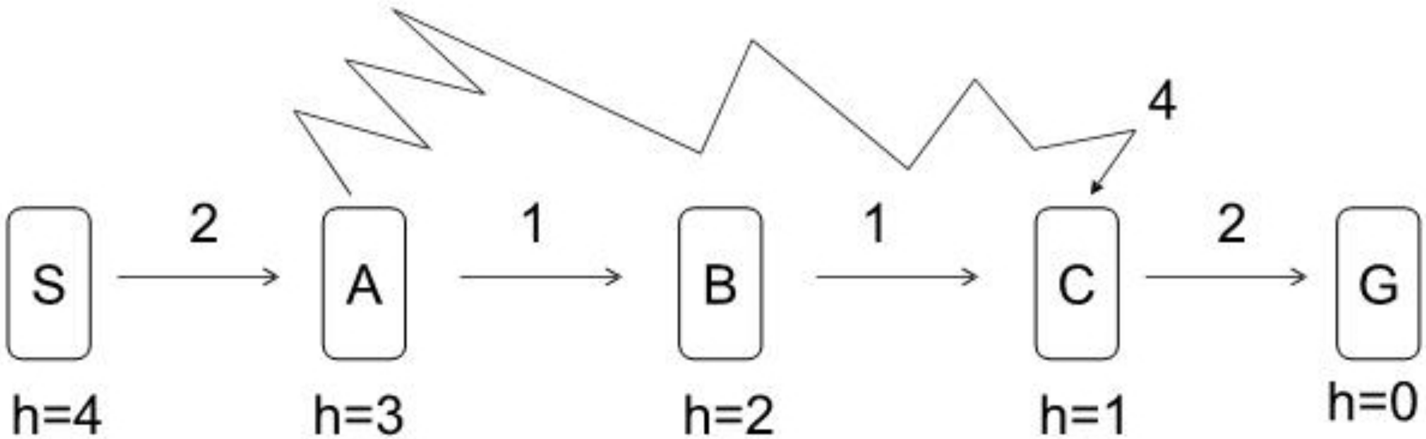
$$f(n) = g(n) + h(n)$$

g(n) is the cost of the path to node n
h(n) is the heuristic estimate of the cost of achieving the goal from n.

Always expand the node with lowest f-value on Frontier.

The f-value, f(n) is an estimate of the cost of getting to the goal via the node (path) n. I.e., we first follow the path n then we try to get to the goal. f(n) estimates the total cost of such a solution.

A* Looking Non-Stupid

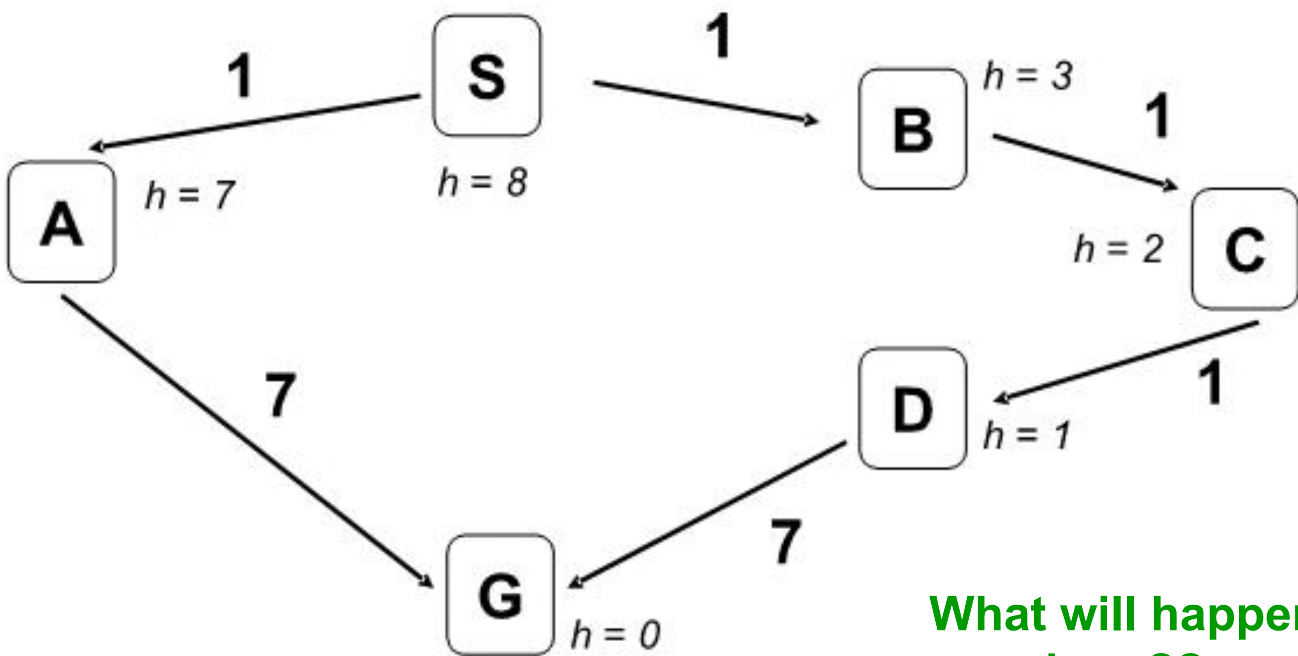


What will A* do here?

When should A* terminate?

Idea: As soon as it generates a goal state?

Look at this example:



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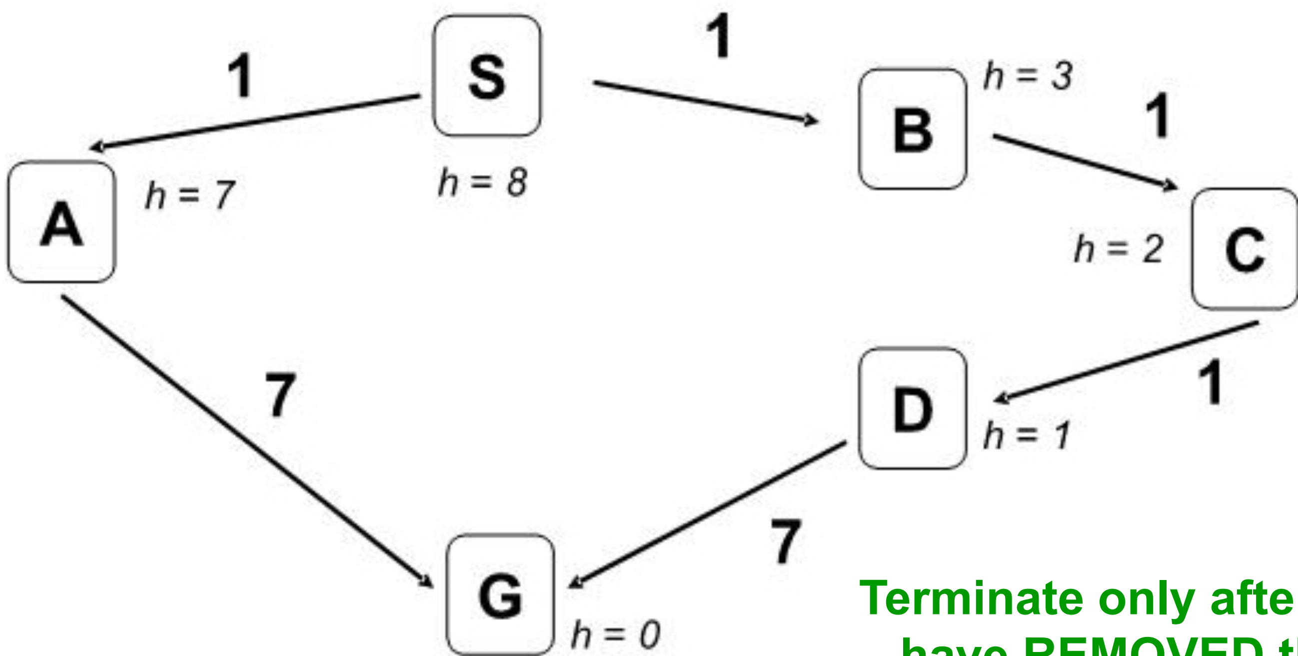
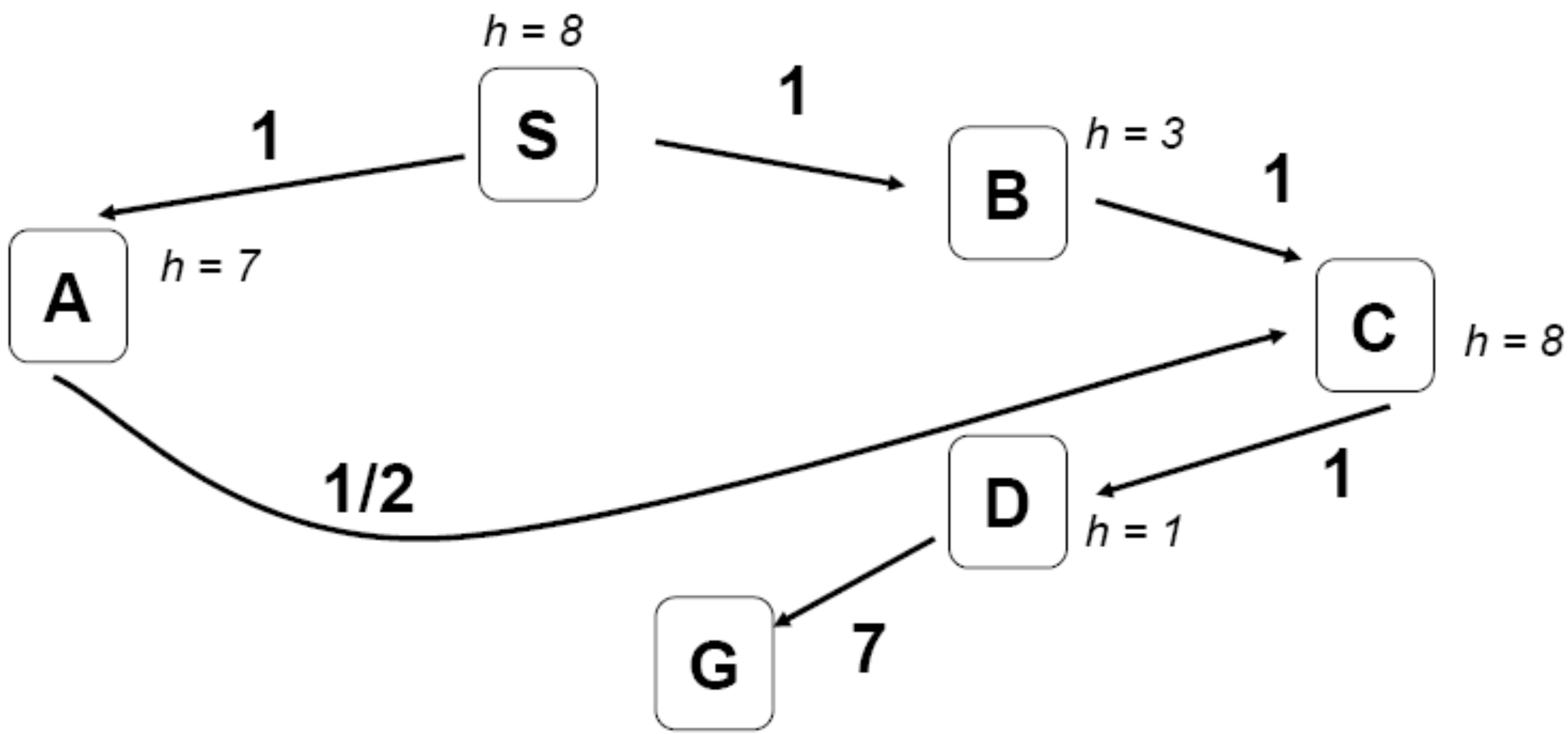


Terminate only after we have REMOVED the goal from the Frontier.

What happens when we visit nodes twice?



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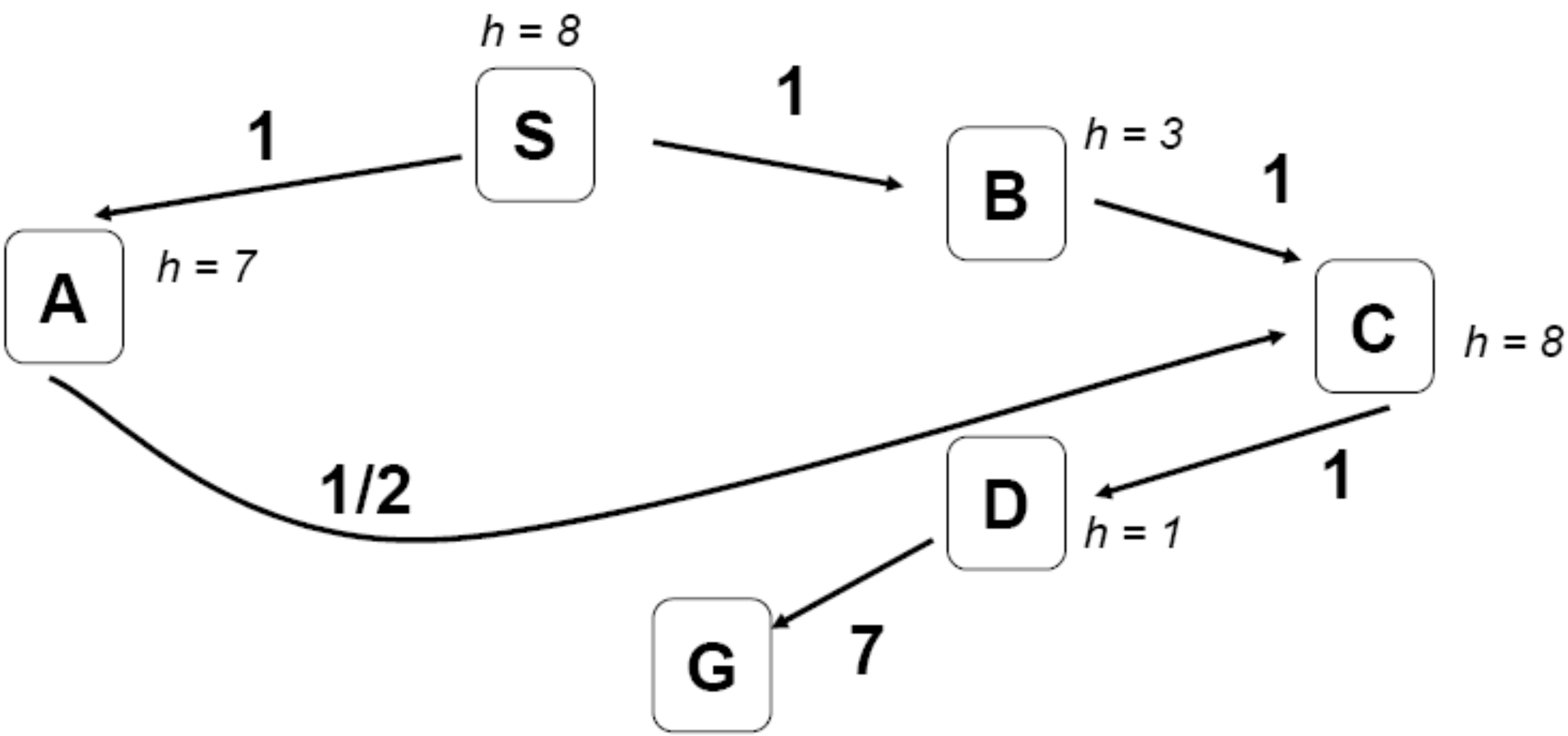
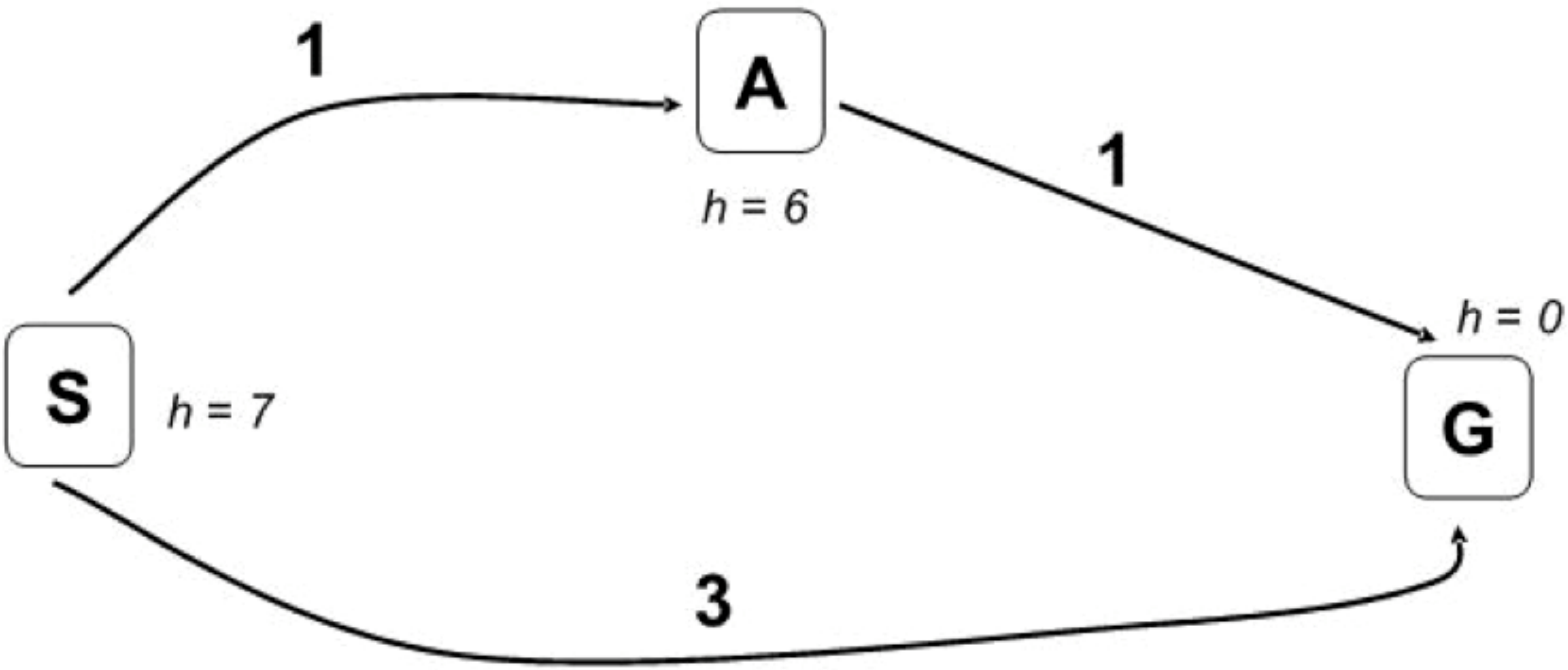


If A* discovers a lower cost path through a state as it is searching, it should update the order of that state on the Frontier, based on the lower path cost.

Is A* Guaranteed Optimal?



Properties of A* depend on conditions on h(n)

- To achieve completeness, optimality, and desirably time and space complexity with A* search, we must put some conditions on the heuristic function h(n) and the search space.

Condition on h(n): Admissible

- Assume each transition due to an action a has cost $\geq \varepsilon > 0$.
- Let h*(n) be the **cost of an optimal path** from n to a goal node (∞ if there is no path). Then an admissible heuristic satisfies the condition:

$$h(n) \leq h^*(n)$$

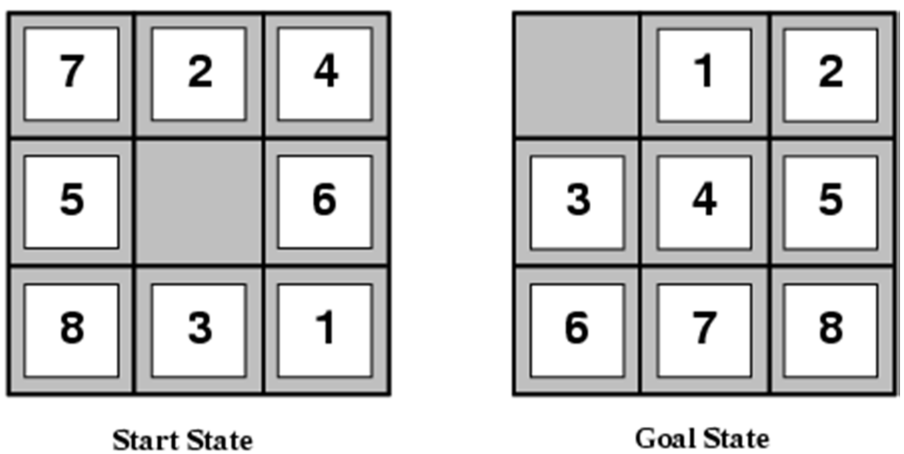
an admissible heuristic never over-estimates the cost to reach the goal, i.e., it is optimistic

- Hence $h(g) = 0$, for any goal node g
- Also $h^*(n) = \infty$ if there is no path from n to a goal node

Admissible heuristics

Which heuristics are admissible for the 8 puzzle?

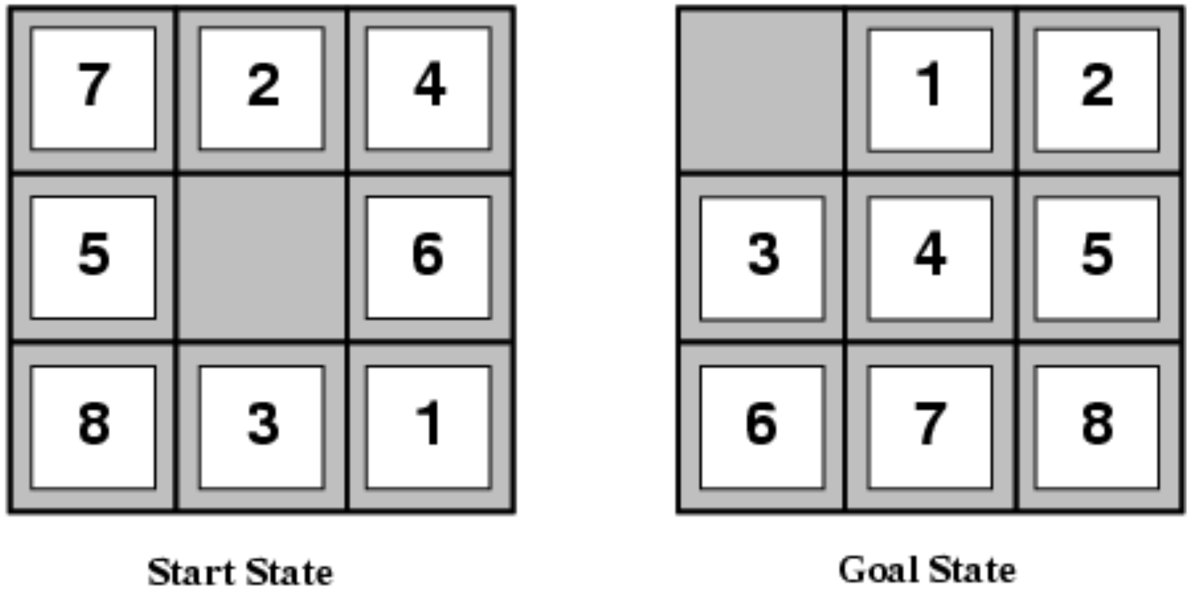
- *h(n)* = number of misplaced tiles
- *h(n)* = total Manhattan distance between tile locations in S and goal locations in G
- $h(n)$ = min (2, $h^*[n]$)
- $h(n) = h^*(n)$
- $h(n)$ = max (2, $h^*[n]$)
- $h(n) = 0$



Admissible heuristics

Say for the 8-puzzle:

$h_1(n)$ = number of misplaced tiles
$h_2(n)$ = total Manhattan distance
(i.e., no. of squares from desired location of each tile)



- $h_1(S) = ?$ 8
- $h_2(S) = ?$ 3+1+2+2+2+3+3+2 = 18

Admissible heuristics: Dominance

- If $h_2(n) \geq h_1(n)$ for all n and both are admissible, then h_2 dominates h_1

  Which would you prefer to use and why?

- Typical search costs (average number of nodes expanded):

- *d=12* IDS = 3,644,035 nodes
  $A^*(h_1)$ = 227 nodes
  $A^*(h_2)$ = 73 nodes

- *d=24* IDS = too many nodes
  $A^*(h_1)$ = 39,135 nodes
  $A^*(h_2)$ = 1,641 nodes

Admissibility implies optimality

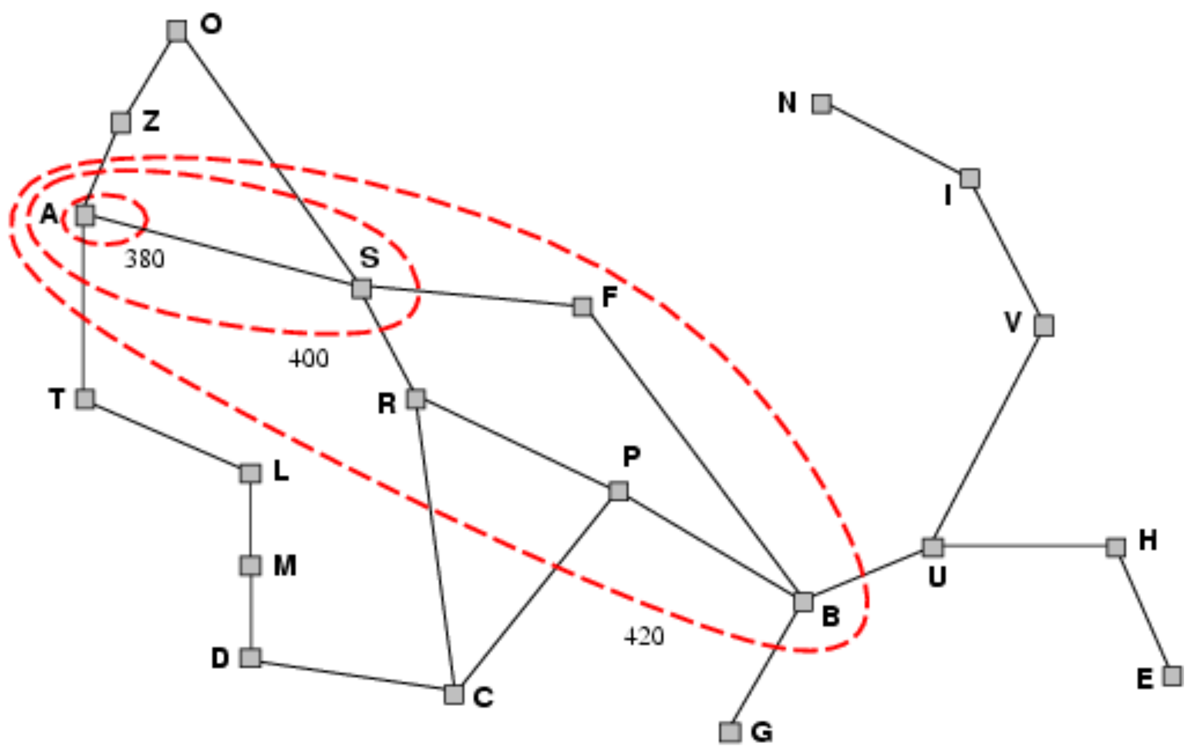
The intuition is this: $h(n) \leq h^*(n)$ means that the search won't miss any promising paths.

Why?

Because if it really is cheap to get to a goal via node n (i.e., both $g(n)$ and $h^*(n)$ are low), then $f(n) = g(n) + h(n)$ will also be low, and the search won't ignore n in favor of more expensive options.

Admissibility implies optimality

- A^* expands nodes in order of increasing *f* value
- Gradually adds "*f*-contours" of states
- Contour *i* contains all states with $f=f_i$, where $f_i < f_{i+1}$



Search animations: Pac Man

https://www.youtube.com/watch?v=2XjzjAfGWzY

Condition on h(n): Monotonic (or consistent)

- Stronger condition than global admissibility

- A monotone heuristic satisfies the condition

  $h(n1) \leq c(n1, a, n2) + h(n2)$

- Note that there might more than one transition (action) that joins n1 and n2, and the inequality must hold for all of them.

- If h(n) is admissible and monotonic, search will be both optimal *and* not "locally" mislead.

Consistency implies Admissibility

Assume consistency: $h(n1) \le c(n1,a,n2) + h(n2)$

Prove admissible: $h(n) \le h^*(n)$

Proof:

If no path exists from n to a goal then $h^*(n) = \infty$ and $h(n) \le h^*(n)$

Else let n → n1 → … → n* be an OPTIMAL path from n to a goal (with actions a1, a2). Note the cost of this path is h*(n), and each sub-path (ni → … → n*) has cost equal to h*(ni).

Prove $h(n) \le h^*(n)$ by induction on the length of this optimal path.

Proof by Induction

Assume consistency: $h(n1) \leq c(n1,a,n2) + h(n2)$

Prove admissible: $h(n) \leq h^*(n)$

Base Case:

$h(n_{goal}) = 0 \leq h(n_{goal})^* = 0$

$h(n_1) \leq c(n_1,a_1,n_{goal}) + h(n_{goal}) \leq c(n_1,a_1,n_{goal}) + h^*(n_{goal}) = h(n_1)^*$

Induction:

Assume $h(n_i) \leq h(n_i)^*$

$h(n_{i-1}) \leq c(n_{i-1},a_{i-1},n_i) + h(n_i) \leq c(n_{i-1},a_{i-1},n_i) + h^*(n_i) = h^*(n_{i-1})$

Some consequences of Monotonicity

1. f-values of states in a path are non-decreasing.

   i.e. if n1 and n2 are nodes along a path, then $f(n1) \leq f(n2)$

   Proof: $f(n1) = g(n1) + h(n1) = \text{cost(path to n1)} + h(n1)$

   $\leq g(n1) + c(n1, a, n2) + h(n2)$

   But $g(n1) + c(n1, a, n2) + h(n2) = g(n2) + h(n2) = f(n2)$

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But $g(n1) + c(n1, a, n2) + h(n2) = g(n2) + h(n2) = f(n2)$

So $f(n1) \leq f(n2)$

Some consequences of Monotonicity

2. If n2 is expanded after n1, then f(n1) ≤ f(n2).

   i.e. f-values of nodes that are expanded cannot decrease during the search.

   Why? When n1 was selected for expansion, n2 was either:

   1. Already on the frontier, meaning f(n1) ≤ f(n2). Otherwise we would have expanded n2 before n1.
   2. Added to the frontier as a result of n1's expansion, meaning n2 and n1 lie along the same path. If this is the case, as we demonstrated on the prior slide, f(n1) ≤ f(n2).

Some consequences of Monotonicity

3. If node n has been expanded, every path with a lower f-value than n has already been expanded.
   - Say we just expanded node ni on a path to node nk, and that f(nk) < f(n).
   - This means ni+1 is on the frontier and f(ni+1) ≤ f(nk), because they are both on the same path.
   - BUT if ni+1 were on the frontier at the same time as node n, it would have been expanded before n because f(ni+1) ≤ f(nk) < f(n).
   - Thus, n *can't* have been expanded before every path with a lower f-value has been expanded.

Some consequences of Monotonicity

4. The first time A* expands a node, it has found the minimum cost path to that node.

f(of the first discovered path to n) = cost(of the first discovered path to n) + h(n).

Likewise,

f(of any other path to n) = cost(of any other path to n) + h(n).

From the prior slide we know:

f(of the first discovered path to n) ≤ f(of any other path to n).

This means, by substitution:

cost(of 1st discovered path to n) ≤ cost(of any other path to n)

Hence, the first discovered path is the optimal one!

Monotonic, Admissible A*

Complete?

YES. Consider a least cost path to a goal node

–SolutionPath = <Start→ n1→ …→ G> with cost c(SolutionPath).

–Since each action has a cost ≥ ε > 0, there are only a finite number of paths that have f-value < c(SolutionPath). None of these paths lead to a goal node since SolutionPath is a least cost path to the goal.

–So eventually SolutionPath, or some equal cost path to a goal must be expanded.

Time and Space complexity?

–When h(n) = 0 for all n, h is monotone (A* becomes uniform-cost search)!

–When h(n) > 0 for some n and still admissible, the number of nodes expanded will be no larger than uniform-cost.

–Hence the same bounds as uniform-cost apply. (These are worst case bounds). Still exponential complexity unless we have a very good *h*!

–In real world problems, we sometimes run out of time and memory. IDA* can sometimes be used to address memory issues, but IDA* isn't very good when many cycles are present.

Monotonic, Admissible A*

Optimal?

YES. As we saw, the first path to a goal node must be optimal.

Cycle Checking?

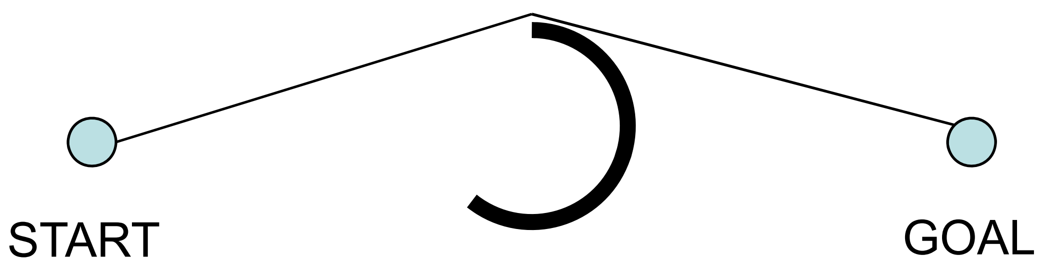
We can use a simple implementation of cycle checking (multiple path checking) - just reject all search nodes that visit a state already visited by a previously expanded node. We need keep only the first path to a state, rejecting all subsequent paths.

Admissibility without monotonicity

- Optimal?
- Complete?
- Time and Space Complexity?
- Cycle Checking?

Effect of Heuristic Functions

- What portion of the state space will be explored by UCS? A*?



Effect of Heuristic Functions

- Observation: While A* may expand less of the state space, it is still constrained by speed or memory (many states are explored, on Frontier).
- Tools to address these problems:
  - IDA* (Iterative Deepening A*) - similar to Iterative Deepening Search.
  - Weighted A* - A* with an added weight, to bias exploration toward goal.

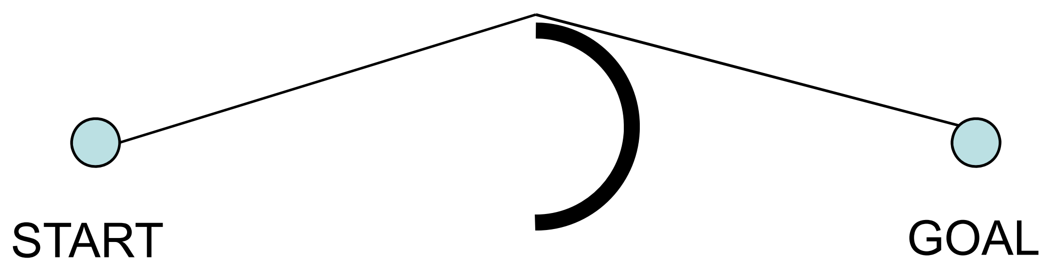
IDA* - Iterative Deepening A*

Objective: reduce memory requirements for A*

- Like iterative deepening, but now the "cutoff" is the f-value (g+h) rather than the depth
- At each iteration, the cutoff value is the smallest f-value of any node that exceeded the cutoff on the previous iteration
- Avoids overhead associated with keeping a sorted queue of nodes, and the open list occupies only linear space.
- Two new parameters:
  - curBound (any node with a bigger f-value is discarded)
  - smallestNotExplored (the smallest f-value for discarded nodes in a round); when Frontier becomes empty, the search starts a new round with this bound.
  - To compute "smallestNotExplored" most readily, expand all nodes with f-value EQUAL to the f-limit.

IDA* - Iterative Deepening A*

- Optimal?
- Complete?
- Time and Space Complexity?
- Cycle Checking?



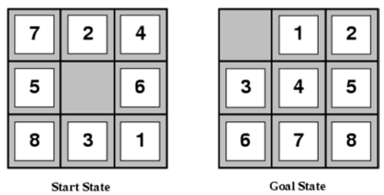
Building Heuristics by Relaxing Problems

A useful technique is to simplify a problem when building heuristics, and to let h(n) be the cost of reaching the goal in the easier problem.

For example, in the 8-Puzzle you can only move a tile from square A to B if A is adjacent (left, right, above, below) to B and B is blank

We can relax some of these conditions and:

1. allow a move from A to B if A is adjacent to B (i.e. we can ignore whether or not position is blank),
2. allow a move from A to B if B is blank (i.e. we can ignore adjacency),
3. allow all moves from A to B (ignore both conditions).



Building Heuristics by Relaxing Problems

- #3 leads to the misplaced tiles heuristic.
  - To solve the puzzle, we need to move each tile into its final position.
  - Number of moves = number of misplaced tiles.
  - Clearly $h(n)$ = number of misplaced tiles ≤ the $h^*(n)$ the cost of an optimal sequence of moves from n.
- #1 leads to the Manhattan distance heuristic.
  - To solve the puzzle we need to slide each tile into its final position.
  - We can move vertically or horizontally.
  - Number of moves = sum over all of the tiles of the number of vertical and horizontal slides we need to move that tile into place.
  - Again $h(n)$ = sum of the Manhattan distances ≤ $h^*(n)$
    - in a real solution we need to move each tile at least that that far and we can only move one tile at a time.

Comparing Iterative Deepening with A*

From Russell and Norvig

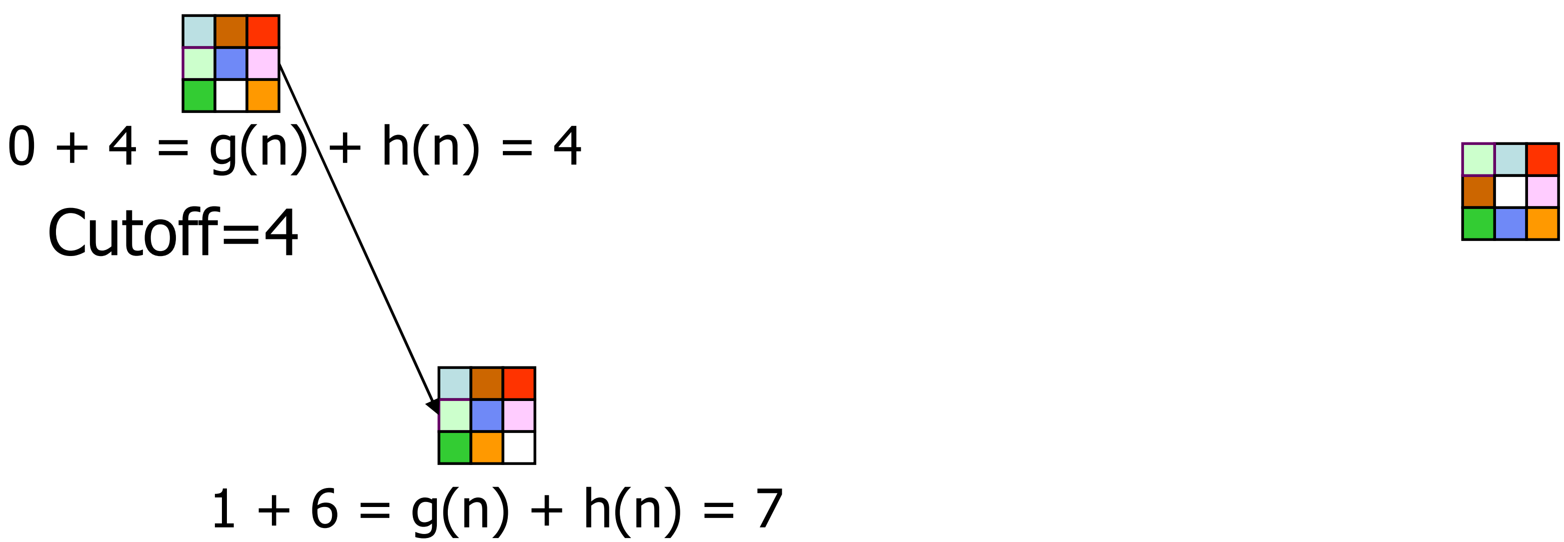
	For 8-puzzle, average number of states expanded over 100 randomly chosen problems in which optimal path is length…		
	…4 steps	…8 steps	…12 steps
Iterative Deepening (see previous slides)	112	6,300	3.6×10^6
A* search using "number of misplaced tiles" as the heuristic	13	39	227
A* using "Sum of Manhattan distances" as the heuristic	12	25	73

IDA* Example: 8-Puzzle

$f(n) = g(n) + h(n)$

h(n) = number of misplaced tiles

blank tile is white



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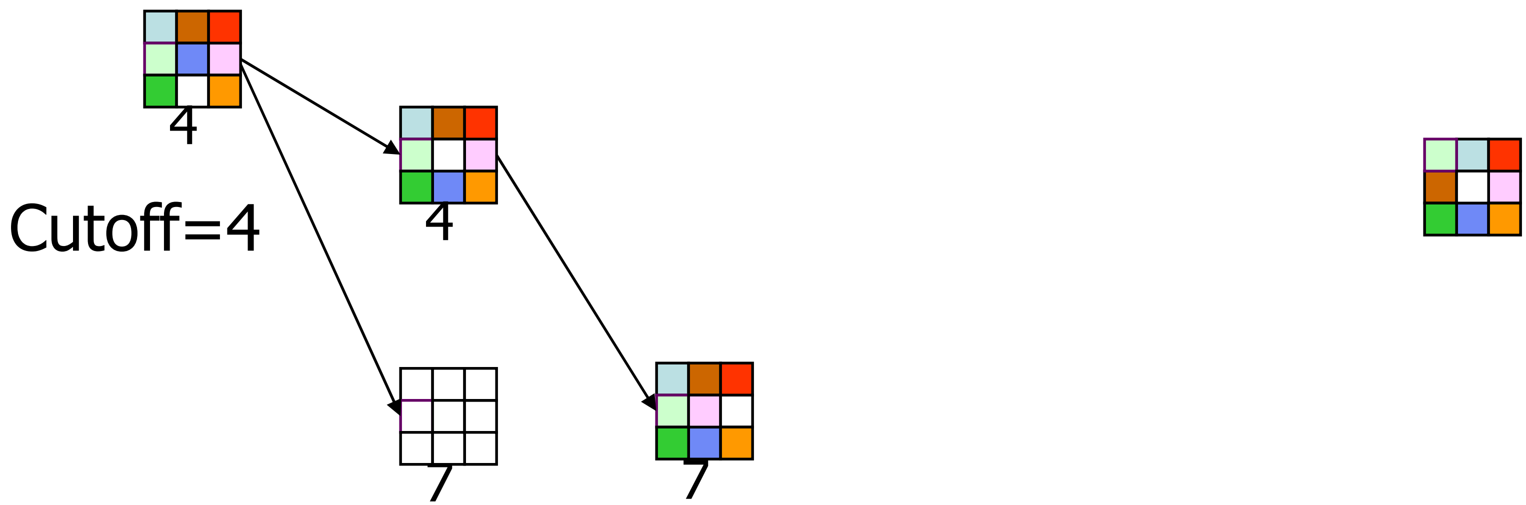

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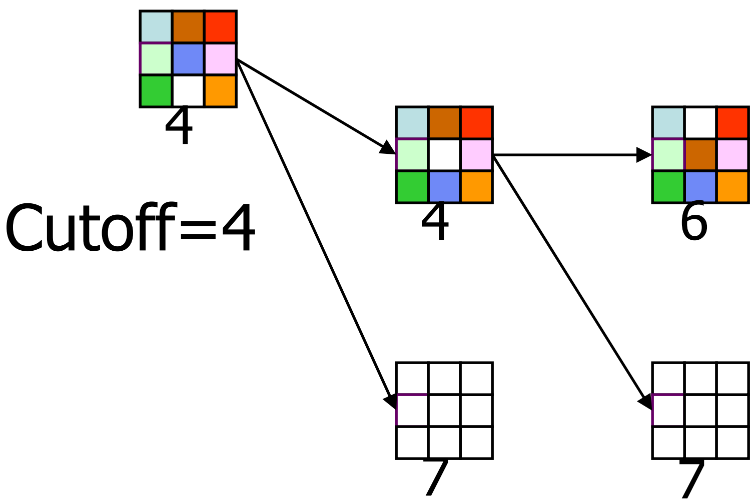

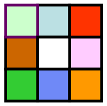

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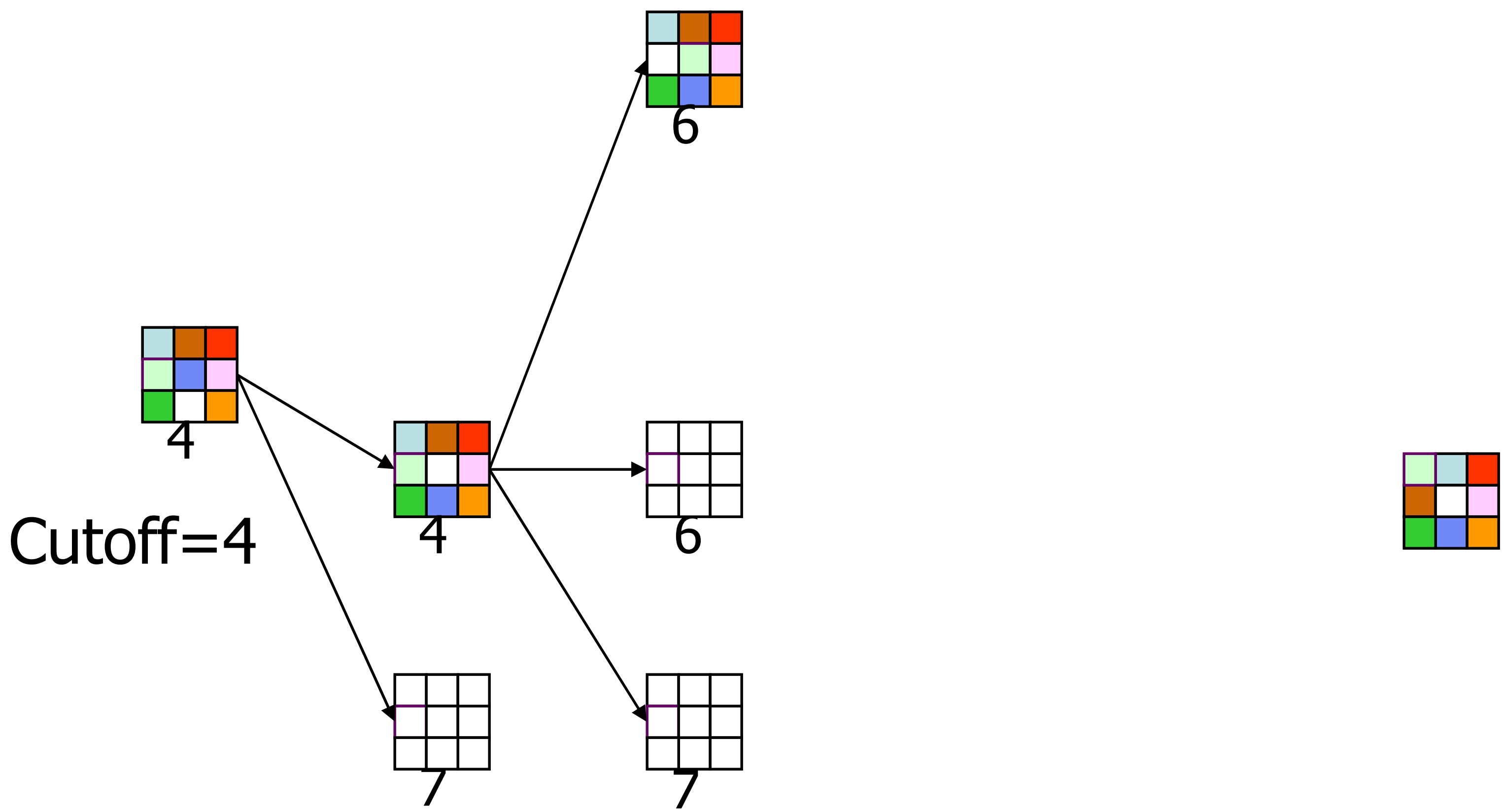

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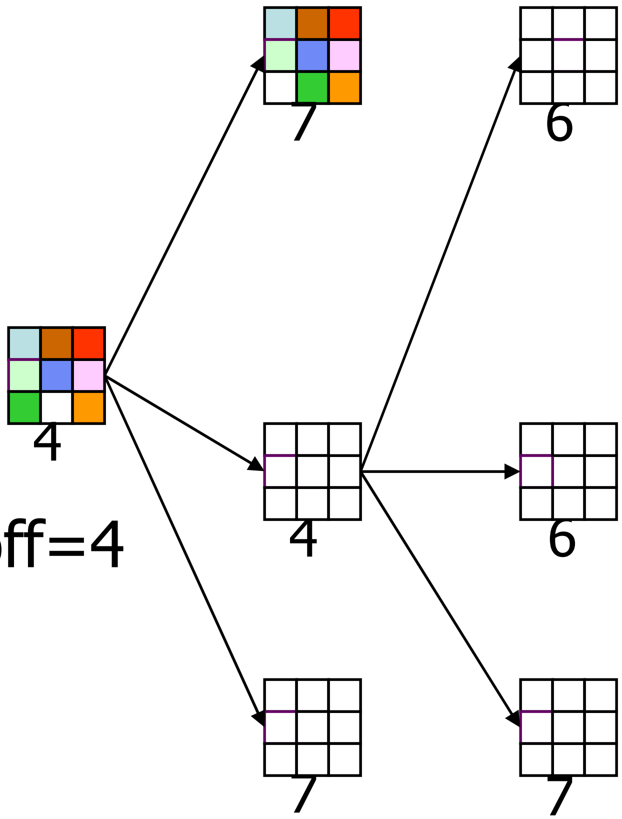
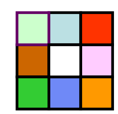


Cutoff=4



IDA* Example: 8-Puzzle

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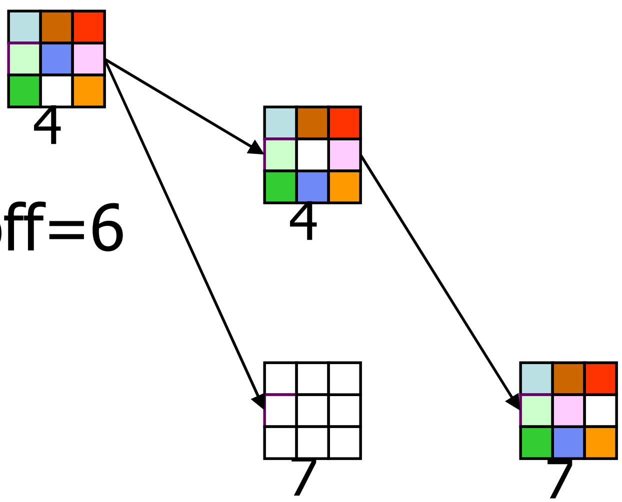
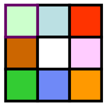
IDA* Example: 8-Puzzle

f(n) = g(n) + h(n)

h(n) = number of misplaced tiles

Cutoff=6

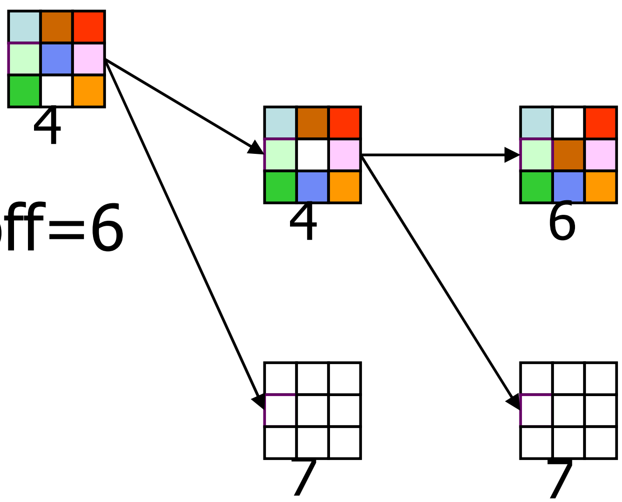




8-Puzzle

$f(n) = g(n) + h(n)$

$h(n)$ = number of misplaced tiles



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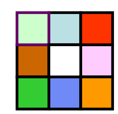

8-Puzzle

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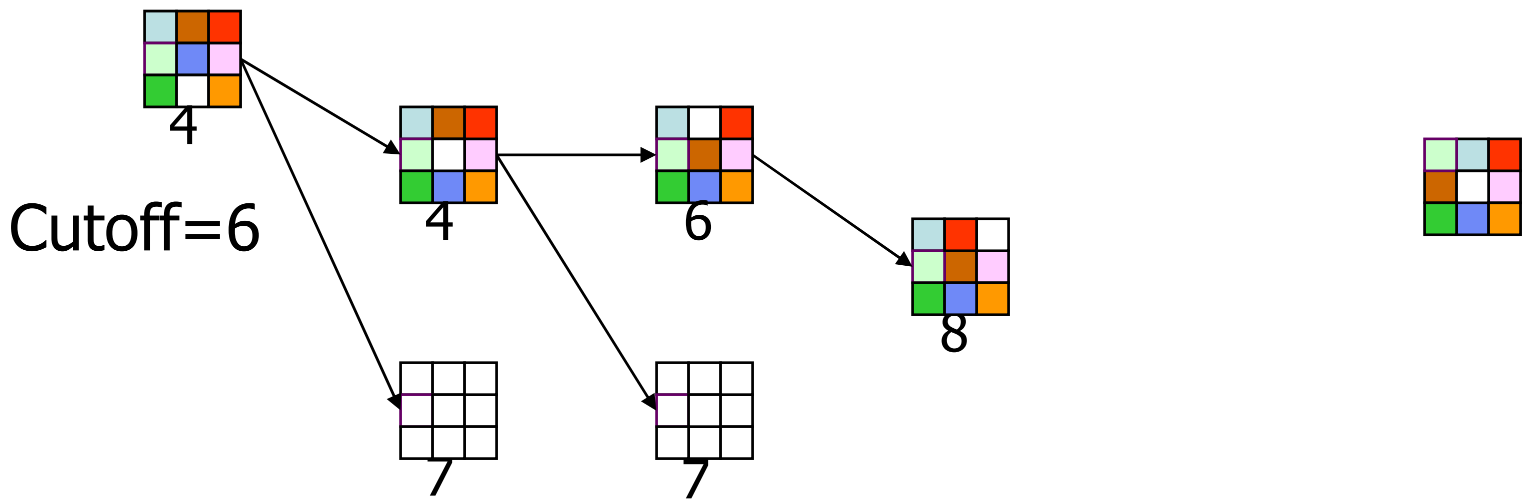

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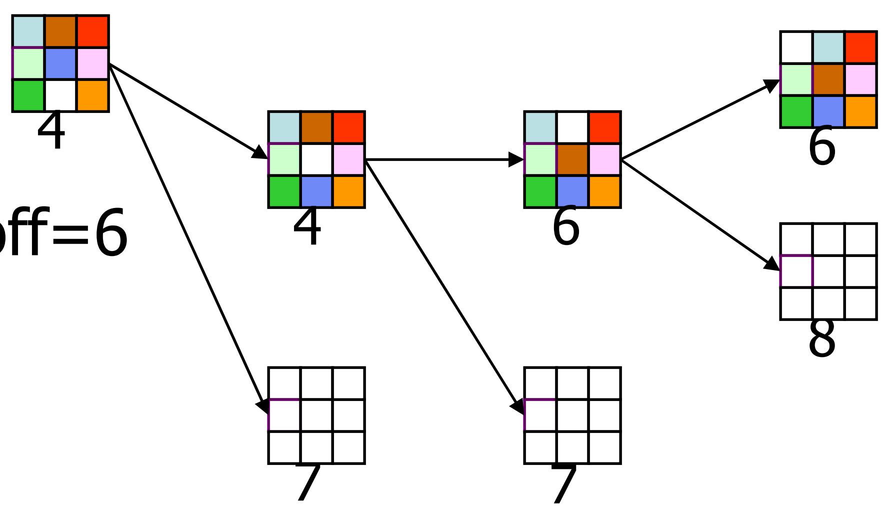

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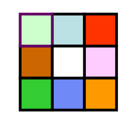

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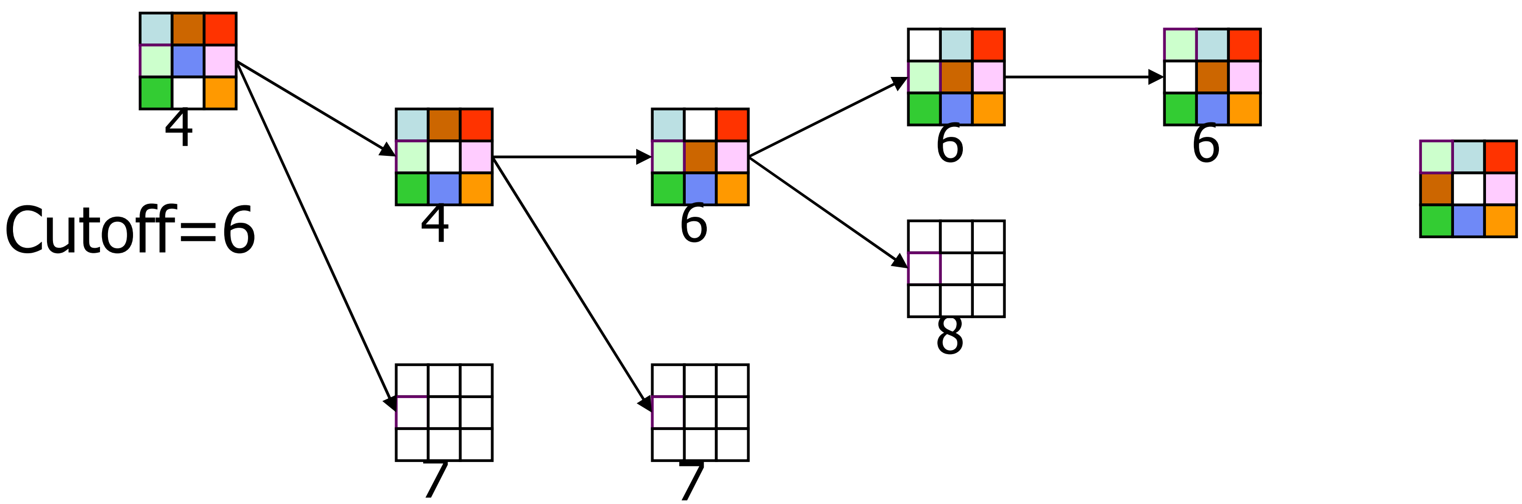
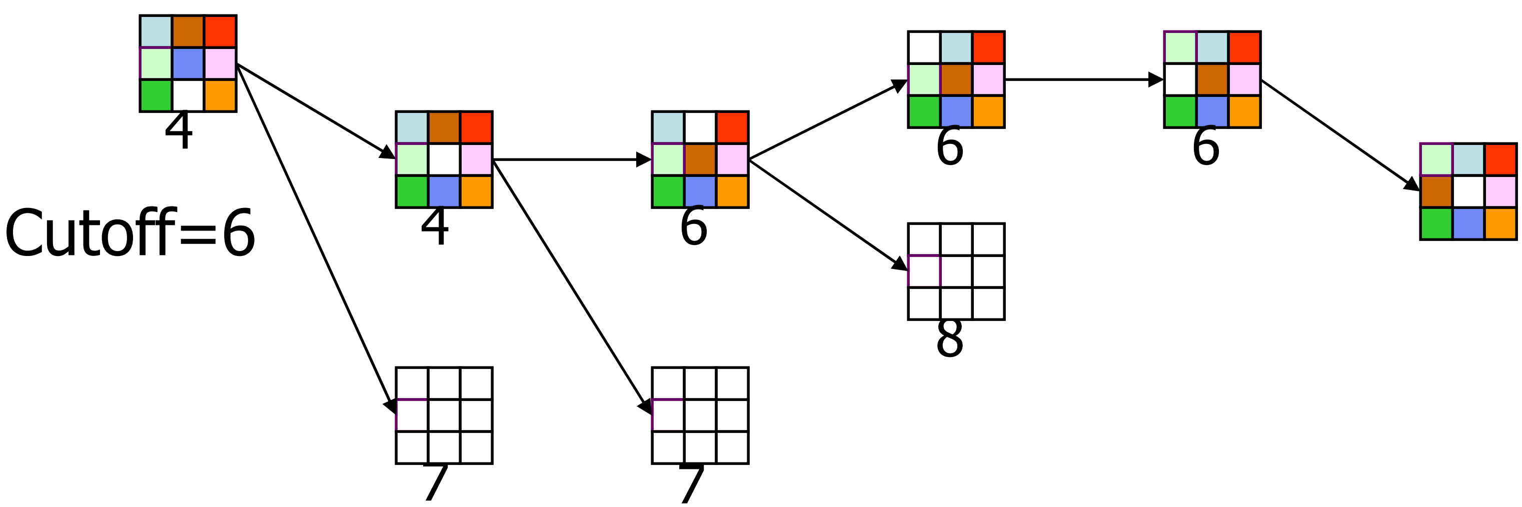
h(n) = number of misplaced tiles



8-Puzzle

f(n) = g(n) + h(n)

h(n) = number of misplaced tiles

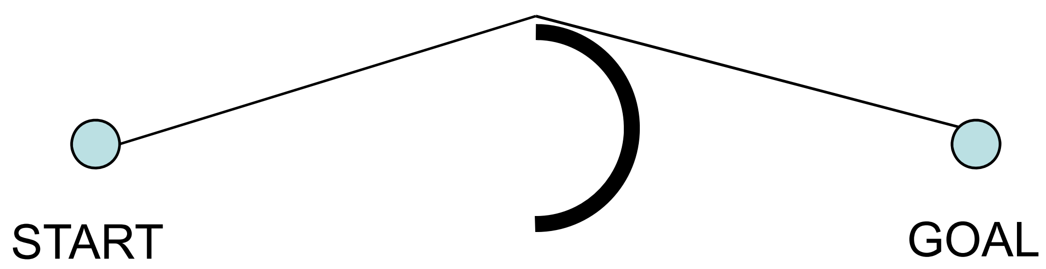


Weighted A*

- Weighted A* defines an evaluation function f(n):

$$f(n) = g(n) + \varepsilon * h(n)$$

  - **ε > 1** introduces a bias towards states that are closer to the goal.
  - **ε == 1** generates a provably optimal solution (assuming admissible heuristic).



Weighted A*

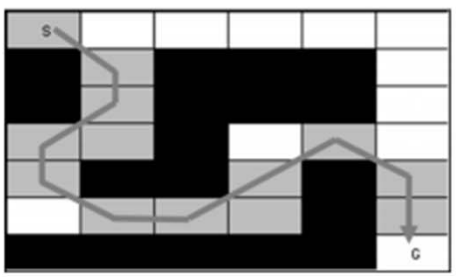
- Weighted A* defines an evaluation function f(n):

$$f(n) = g(n) + \varepsilon^* h(n)$$

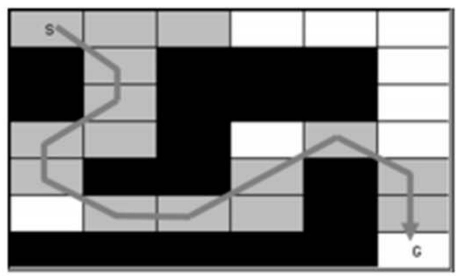
  - **ε > 1** introduces a bias towards states that are closer to the goal.
  - **ε == 1** generates a provably optimal solution (assuming admissible heuristic).
- Search is typically orders of magnitude faster
- Path that is discovered may be sub-optimal (by factor that depends on ε)

Anytime A*

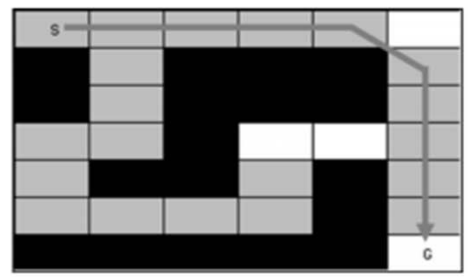
- Weighted A* can be used to construct an anytime algorithm:
  - Find the best path for a given ε
  - Reduce ε and re-plan



ε = 2
13 node expansions
Solution length: 12



ε = 1.5
15 node expansions
Solution length: 12



ε = 1
20 node expansions
Solution length: 10