Finite Difference Methods

Solve Black-Scholes PDE

$$\frac{\partial}{\partial t}F(S_t, t) + rS_t\,\frac{\partial}{\partial S}F(S_t, t) + \tfrac{1}{2}\,\sigma^2(S_t, t)S_t^2\,\frac{\partial^2}{\partial S^2}F(S_t, t) = rF(S_t, t) \quad \text{with} \;\; F(S_T, T) = \psi(S_T)$$

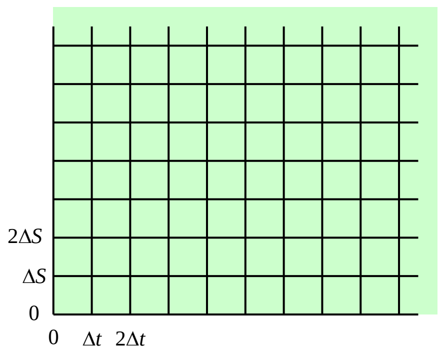
with intermediate boundary conditions for $F(S_t, t)$.

Discretize t and S_t in grids as

$$0 \,,\, \Delta t \,,\, 2\Delta t \,,\, \ldots \,,\, i\Delta t \,,\, \ldots \,,\, i_{max}\Delta t = T$$

$$0 \,,\, \Delta S \,,\, 2\Delta S \,,\, \ldots \,,\, j\Delta S \,,\, \ldots \,,\, j_{max}\Delta S$$

such that $t_i = i\Delta t$ and $S_j = j\Delta S$



Consider the Taylor expansions as,

$$F(S_{j+1}, t_i) = F(S_j + \Delta S, t_i) \cong F(S_j\,, t_i) + \frac{\partial}{\partial S}F(S_j\,, t_i)\;\Delta S + \frac{1}{2}\,\frac{\partial^2}{\partial S^2}F(S_j\,, t_i)\;(\Delta S)^2 + \frac{1}{6}\,\frac{\partial^3}{\partial S^3}F(S_j\,, t_i)\;(\Delta S)^3 + \mathrm{O}[(\Delta S)^4]$$

$$F(S_{j-1}, t_i) = F(S_j - \Delta S, t_i) \cong F(S_j\,, t_i) - \frac{\partial}{\partial S}F(S_j\,, t_i)\;\Delta S + \frac{1}{2}\,\frac{\partial^2}{\partial S^2}F(S_j\,, t_i)\;(\Delta S)^2 - \frac{1}{6}\,\frac{\partial^3}{\partial S^3}F(S_j\,, t_i)\;(\Delta S)^3 + \mathrm{O}[(\Delta S)^4]$$

$$F(S_j\,, t_{i+1}) = F(S_j, t_i + \Delta t) \cong F(S_j\,, t_i) + \frac{\partial}{\partial t}F(S_j\,, t_i)\;\Delta t + \frac{1}{2}\,\frac{\partial^2}{\partial t^2}F(S_j\,, t_i)\;(\Delta t)^2 + \frac{1}{6}\,\frac{\partial^3}{\partial t^3}F(S_j\,, t_i)\;(\Delta t)^3 + \mathrm{O}[(\Delta t)^4]$$

$$F(S_j\,, t_{i-1}) = F(S_j, t_i - \Delta t) \cong F(S_j\,, t_i) - \frac{\partial}{\partial t}F(S_j\,, t_i)\;\Delta t + \frac{1}{2}\,\frac{\partial^2}{\partial t^2}F(S_j\,, t_i)\;(\Delta t)^2 - \frac{1}{6}\,\frac{\partial^3}{\partial t^3}F(S_j\,, t_i)\;(\Delta t)^3 + \mathrm{O}[(\Delta t)^4]$$

Take $\displaystyle \frac{\partial}{\partial S}F(S_j\,, t_i) = \frac{F(S_{j+1}, t_i) - F(S_{j-1}, t_i)}{2\Delta S} + \mathrm{O}[(\Delta S)^2]$

$$\frac{\partial^2}{\partial S^2}F(S_j\,, t_i) = \frac{F(S_{j+1}, t_i) - 2F(S_j, t_i) + F(S_{j-1}, t_i)}{(\Delta S)^2} + \mathrm{O}[(\Delta S)^2]$$

$$\frac{\partial}{\partial t}F(S_j\,, t_i) = \frac{F(S_j, t_{i+1}) - F(S_j, t_i)}{\Delta t} + \mathrm{O}[(\Delta t)] \quad \text{(Forward in time)}$$

$$\frac{\partial}{\partial t}F(S_j\,, t_i) = \frac{F(S_j, t_i) - F(S_j, t_{i-1})}{\Delta t} + \mathrm{O}[(\Delta t)] \quad \text{(Backward in time)}$$

Note that we cannot use the following definition to improve the error

$$\frac{\partial}{\partial t}F(S_j\,, t_i) = \frac{F(S_j, t_{i+1}) - F(S_j, t_{i-1})}{2\Delta t} + \mathrm{O}[(\Delta t)^2]$$

as the difference equation will involve t_{i+1} , t_i , and t_{i-1}. The resulting iteration equation will not be able to start in our case.

Crank-Nicholson Scheme :

We can improve the error to be in the order of $(\Delta t)^2$ and $(\Delta S)^2$.

Consider the Black-Scholes PDE at t_i and t_{i+1}.

$$\frac{\partial}{\partial t}F(S_j, t_i) + rS_j \frac{\partial}{\partial S}F(S_j, t_i) + \frac{1}{2}\sigma^2(S_j, t_i)S_j^2 \frac{\partial^2}{\partial S^2}F(S_j, t_i) = rF(S_j, t_i)$$

$$\frac{\partial}{\partial t}F(S_j, t_{i+1}) + rS_j \frac{\partial}{\partial S}F(S_j, t_{i+1}) + \frac{1}{2}\sigma^2(S_j, t_{i+1})S_j^2 \frac{\partial^2}{\partial S^2}F(S_j, t_{i+1}) = rF(S_j, t_{i+1})$$

Taking the average to get

$$\frac{1}{2}\left[\frac{\partial}{\partial t}F(S_j, t_i) + \frac{\partial}{\partial t}F(S_j, t_{i+1}) \right] + \frac{1}{2}\left[rS_j \frac{\partial}{\partial S}F(S_j, t_i) + rS_j \frac{\partial}{\partial S}F(S_j, t_{i+1}) \right]$$

$$+ \frac{1}{2}\left[½\sigma^2(S_j, t_i)S_j^2 \frac{\partial^2}{\partial S^2}F(S_j, t_i) + \frac{1}{2}\sigma^2(S_j, t_{i+1})S_j^2 \frac{\partial^2}{\partial S^2}F(S_j, t_{i+1}) \right] = \frac{1}{2}\left[rF(S_j, t_i) + rF(S_j, t_{i+1}) \right]$$

Recall

$$F(S_j , t_{i+1}) = F(S_j, t_i + \Delta t) \cong F(S_j , t_i) + \frac{\partial}{\partial t}F(S_j , t_i)\, \Delta t + \frac{1}{2}\frac{\partial^2}{\partial t^2}F(S_j , t_i)\, (\Delta t)^2 + O[(\Delta t)^3]$$

$$F(S_j , t_i) = F(S_j, t_{i+1} - \Delta t) \cong F(S_j , t_{i+1}) - \frac{\partial}{\partial t}F(S_j , t_{i+1})\, \Delta t + \frac{1}{2}\frac{\partial^2}{\partial t^2}F(S_j , t_{i+1})\, (\Delta t)^2 + O[(\Delta t)^3]$$

$$\cong F(S_j , t_{i+1}) - \frac{\partial}{\partial t}F(S_j , t_{i+1})\, \Delta t + \frac{1}{2}\frac{\partial^2}{\partial t^2}F(S_j , t_i)\, (\Delta t)^2 + O[(\Delta t)^3]$$

Subtracting these two equations to give

$$\frac{1}{2}\left(\frac{\partial}{\partial t}F(S_j , t_i) + \frac{\partial}{\partial t}F(S_j , t_{i+1}) \right) = \frac{F(S_j, t_{i+1}) - F(S_j, t_i)}{\Delta t} + O[(\Delta t)^2]$$

Use also $$\frac{\partial}{\partial S}F(S_j , t_i) = \frac{F(S_{j+1}, t_i) - F(S_{j-1}, t_i)}{2\Delta S} + O[(\Delta S)^2]$$

$$\frac{\partial^2}{\partial S^2}F(S_j , t_i) = \frac{F(S_{j+1}, t_i) - 2F(S_j, t_i) + F(S_{j-1}, t_i)}{(\Delta S)^2} + O[(\Delta S)^2]$$

Substitute in the above Black-Scholes PDE gives the difference equation.

The difference equation now reads

$$[-\tfrac{1}{2}a_j(t_{i+1})]F(S_{j-1}, t_{i+1}) + [1 - \tfrac{1}{2}d_j(t_{i+1})]F(S_j, t_{i+1}) + [-\tfrac{1}{2}c_j(t_{i+1})]F(S_{j+1}, t_{i+1})$$

$$= [\tfrac{1}{2}a_j(t_i)]F(S_{j-1}, t_i) + [1 + \tfrac{1}{2}d_j(t_i)]F(S_j, t_i) + [\tfrac{1}{2}c_j(t_i)]F(S_{j+1}, t_i) \quad , \quad \text{for } j = 1, \ldots , j_{max} - 1$$

$$i = 0, \ldots , i_{max} - 1$$

where $a_j(t_i) = ½\, rj\Delta t - ½\, \sigma^2(S_j, t_i)\, j^2\, \Delta t$

$d_j(t_i) = r\Delta t + \sigma^2(S_j, t_i)\, j^2\, \Delta t$

$c_j(t_i) = -\, ½\, rj\Delta t - ½\, \sigma^2(S_j, t_i)\, j^2\, \Delta t$

For completeness, we include also the transformations

$$F(S_0, t_{i+1}) = b_0\, F(S_0, t_i)$$

$$F(S_{j_{max}}, t_{i+1}) = b_{j_{max}}\, F(S_{j_{max}}, t_i)$$

where $b_{j_{max}} = 1$ and $b_0 = e^{r\Delta t}$ or 1 for European or American-style options.

Assume $S_{j_{max}}$ to be sufficiently large such that the change in time premium between t_i and t_{i+1} is insignificant. This gives

$$b_{j_{max}} = F(S_{j_{max}}, t_{i+1})/F(S_{j_{max}}, t_i) \cong 1$$

For European and American call options, we have $F(S_0, t_i) = 0$ and thus b_0 is arbitrary. For European put options, $F(S_0, t_i) = Ke^{-r(T-t_i)}$ from put-call parity. This gives $b_0 = e^{r\Delta t}$. For American put options, $F(S_0, t_i) = K$ due to early exercise and $b_0 = 1$.

The difference equation can be written in matrix representation as

$$\boldsymbol{Q}(t_{i+1})\begin{bmatrix} F(S_0, t_{i+1}) \\ F(S_1, t_{i+1}) \\ \vdots \\ F(S_{j_{max}-1}, t_{i+1}) \\ F(S_{j_{max}}, t_{i+1}) \end{bmatrix} = \boldsymbol{P}(t_i)\begin{bmatrix} F(S_0, t_i) \\ F(S_1, t_i) \\ \vdots \\ F(S_{j_{max}-1}, t_i) \\ F(S_{j_{max}}, t_i) \end{bmatrix}$$

where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are $(j_{max}+1)\times(j_{max}+1)$ tridiagonal matrices given by

$$\boldsymbol{P}(t_i) = \begin{bmatrix} b_0 & 0 & 0 & \ldots & & & & \\ \tfrac{1}{2}a_1 & 1+\tfrac{1}{2}d_1 & \tfrac{1}{2}c_1 & 0 & \ldots & & & \\ 0 & \tfrac{1}{2}a_2 & 1+\tfrac{1}{2}d_2 & \tfrac{1}{2}c_2 & 0 & \ldots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & \ldots & \tfrac{1}{2}a_{j_{max}-1} & 1+\tfrac{1}{2}d_{j_{max}-1} & \tfrac{1}{2}c_{j_{max}-1} \\ & & & & \ldots & 0 & 0 & b_{j_{max}} \end{bmatrix} \text{at } t_i$$

$$\boldsymbol{Q}(t_{i+1}) = \begin{bmatrix} 1 & 0 & 0 & \ldots & & & & \\ -\tfrac{1}{2}a_1 & 1-\tfrac{1}{2}d_1 & -\tfrac{1}{2}c_1 & 0 & \ldots & & & \\ 0 & -\tfrac{1}{2}a_2 & 1-\tfrac{1}{2}d_2 & -\tfrac{1}{2}c_2 & 0 & \ldots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & \ldots & -\tfrac{1}{2}a_{j_{max}-1} & 1-\tfrac{1}{2}d_{j_{max}-1} & -\tfrac{1}{2}c_{j_{max}-1} \\ & & & & \ldots & 0 & 0 & 1 \end{bmatrix} \text{at } t_{i+1}$$

The difference equation can be iterated forward or backward in time by inverting $\boldsymbol{Q}$ or $\boldsymbol{P}$, respectively. It is unconditionally stable and the errors are proportional to $(\Delta t)^2$ and $(\Delta S)^2$.