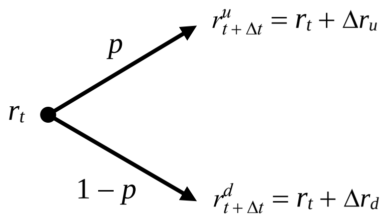
Random normal short rate

Consider the risk-neutral process of short rate given by,



$$\Delta r_t = \mu(r_t, t)\, \Delta t + \sigma(t)\, \Delta z_t$$

From the stochastic model, $E(\Delta r_t \mid r_t) = \mu(r_t, t)\, \Delta t$

$$S^2(\Delta r_t \mid r_t) = \sigma^2(t)\, \Delta t$$

From the binomial tree, $E(\Delta r_t \mid r_t) = p\, \Delta r_u + (1 - p)\, \Delta r_d$

$$S^2(\Delta r_t \mid r_t) = p\, (\Delta r_u)^2 + (1 - p)\, (\Delta r_d)^2 - [\, p\, \Delta r_u + (1 - p)\, \Delta r_d\,]^2$$

Suppose, $E(\Delta r_t \mid r_t) = p\, \Delta r_u + (1 - p)\, \Delta r_d = \mu(r_t, t)\, \Delta t$ (matching the mean)

$$\Rightarrow p = \frac{\mu(r_t, t)\Delta t - \Delta r_d}{\Delta r_u - \Delta r_d} \qquad \text{and} \qquad (1 - p) = \frac{\Delta r_u - \mu(r_t, t)\Delta t}{\Delta r_u - \Delta r_d}$$

This gives the variance according to the binomial tree as

$$S^2(\Delta r_t \mid r_t) = p\, (\Delta r_u)^2 + (1 - p)\, (\Delta r_d)^2 - [\, p\, \Delta r_u + (1 - p)\, \Delta r_d\,]^2$$

$$= \frac{\mu(r_t, t)\Delta t - \Delta r_d}{\Delta r_u - \Delta r_d}\, (\Delta r_u)^2 + \frac{\Delta r_u - \mu(r_t, t)\Delta t}{\Delta r_u - \Delta r_d}\, (\Delta r_d)^2 - \mu^2(r_t, t)\, (\Delta t)^2$$

$$= \mu(r_t, t)\, \Delta t\, (\, \Delta r_u + \Delta r_d\,) - \Delta r_u\, \Delta r_d - \mu^2(r_t, t)\, (\Delta t)^2$$

Consider $\Delta r_u = (J + 1)\, \sigma(t)\sqrt{\Delta t}$ and $\Delta r_d = (J - 1)\, \sigma(t)\sqrt{\Delta t}$ (it is possible to match variance with this parameterization using one factor)

$$S^2(\Delta r_t \mid r_t) = \mu(r_t, t)\Delta t\, (\, 2J\, \sigma(t)\sqrt{\Delta t}\,) - (J^2 - 1)\sigma^2(t)\Delta t - \mu^2(r_t, t)\, (\Delta t)^2$$

$$= \sigma^2(t)\, \Delta t \quad \text{(matching the variance)}$$

$$\Rightarrow \sigma^2(t)J^2 - 2\mu(r_t, t)\sigma(t)\sqrt{\Delta t}\, J + \mu^2(r_t, t)\, \Delta t = 0$$

$$(\, \sigma(t)J - \mu(r_t, t)\sqrt{\Delta t}\,)^2 = 0 \quad \text{or,} \quad \sigma(t)J = \mu(r_t, t)\sqrt{\Delta t}$$

$$\Rightarrow \Delta r_u = (\, J + 1)\, \sigma(t)\, \sqrt{\Delta t} = \mu(r_t, t)\Delta t + \sigma(t)\sqrt{\Delta t}$$

$$\Delta r_d = (\, J - 1)\, \sigma(t)\, \sqrt{\Delta t} = \mu(r_t, t)\Delta t - \sigma(t)\sqrt{\Delta t}$$

These give $p = \dfrac{\mu(r_t, t)\Delta t - \Delta r_d}{\Delta r_u - \Delta r_d} = \dfrac{\mu(r_t, t)\Delta t - \mu(r_t, t)\Delta t + \sigma(t)\sqrt{\Delta t}}{2\sigma(t)\sqrt{\Delta t}} = \dfrac{1}{2}$

Starting from r_t , goes up and then goes down

$$r_t \to \underbrace{r_t + \mu(r_t, t)\Delta t + \sigma(t)\sqrt{\Delta t}}_{r^u_{t+\Delta t}} \to r_t + \mu(r_t, t)\Delta t + \sigma(t)\sqrt{\Delta t} + \mu(r^u_{t+\Delta t}, t+\Delta t)\Delta t - \sigma(t+\Delta t)\sqrt{\Delta t}$$

Starting from r_t , goes down and then goes up

$$r_t \to \underbrace{r_t + \mu(r_t, t)\Delta t - \sigma(t)\sqrt{\Delta t}}_{r^d_{t+\Delta t}} \to r_t + \mu(r_t, t)\Delta t - \sigma(t)\sqrt{\Delta t} + \mu(r^d_{t+\Delta t}, t+\Delta t)\Delta t + \sigma(t+\Delta t)\sqrt{\Delta t}$$

Binomial tree nodes are recombining if $\mu(r_t, t) = \mu(t)$ and $\sigma(t) = \sigma$.

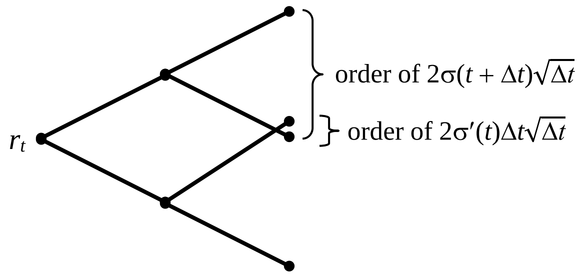
Suppose, $\mu(r_t, t) = \mu(t)$ and $\sigma(t)$ remains to be time dependent.

Starting from r_t , goes up and then goes down

$$r^{ud}_{t+2\Delta t} = r_t + \mu(t)\Delta t + \mu(t+\Delta t)\Delta t + \sigma(t)\sqrt{\Delta t} - \sigma(t+\Delta t)\sqrt{\Delta t}$$

Starting from r_t , goes down and then goes up

$$r^{du}_{t+2\Delta t} = = r_t + \mu(t)\Delta t + \mu(t+\Delta t)\Delta t - \sigma(t)\sqrt{\Delta t} + \sigma(t+\Delta t)\sqrt{\Delta t}$$



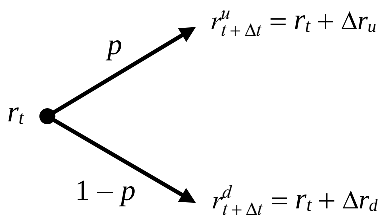
The discrepancy for a recombining binomial lattice is in higher order relative to the tree gap.

$$r^{du} - r^{ud} = 2[\sigma(t+\Delta t) - \sigma(t)\ ]\ \sqrt{\Delta t} \cong 2\sigma'(t)\ \Delta t\ \sqrt{\Delta t}$$

$$r^{uu} - r^{ud} = 2\sigma(t+\Delta t)\sqrt{\Delta t}$$

The Ho-Lee Model

$\Delta r_t = \mu(t)\Delta t + \sigma\Delta z_t$, where $\mu(t) = \frac{\partial}{\partial t}F_0(t) + \sigma^2 t$



The binomial tree nodes are recombining.

$\Delta r_u = (J + 1)\,\sigma\sqrt{\Delta t} = \mu(t)\Delta t + \sigma\sqrt{\Delta t}$

$\Delta r_d = (J - 1)\,\sigma\sqrt{\Delta t} = \mu(t)\Delta t - \sigma\sqrt{\Delta t}$

$p = ½$

Define discrete time as $\{\Delta t, 2\Delta t, \ldots\}$

Short rate at time $t_i = i\Delta t$,

$r_{i,j} = r_i\,(i - j \text{ up}, j \text{ down}) = r_0 + [\,\mu(t_0) + \mu(t_1) + \ldots + \mu(t_{i-1})\,]\Delta t + (i - j)\sigma\sqrt{\Delta t} - j\sigma\sqrt{\Delta t}$

with $j = 0, 1, \ldots, i$ running from top to bottom nodes.

It is easy to see that

$r_{i,j+1} = r_0 + [\,\mu(t_0) + \mu(t_1) + \ldots + \mu(t_{i-1})\,]\Delta t + (i - j - 1)\sigma\sqrt{\Delta t} - (j + 1)\sigma\sqrt{\Delta t} = r_{i,j} - 2\sigma\sqrt{\Delta t}$

This gives $r_{0,0} = r_0 \equiv \gamma_0$

$r_{1,0} = r_0 + \mu(t_0)\Delta t + \sigma\sqrt{\Delta t} \equiv \gamma_1$
$r_{1,1} = \gamma_1 - 2\sigma\sqrt{\Delta t}$

$r_{2,0} = r_0 + [\,\mu(t_0) + \mu(t_1)\,]\Delta t + 2\sigma\sqrt{\Delta t} \equiv \gamma_2$
$r_{2,1} = \gamma_2 - 2\sigma\sqrt{\Delta t}$
$r_{2,2} = \gamma_2 - 4\sigma\sqrt{\Delta t}$

$r_{3,0} = r_0 + [\,\mu(t_0) + \mu(t_1) + \mu(t_2)\,]\Delta t + 3\sigma\sqrt{\Delta t} \equiv \gamma_3$
$r_{3,1} = \gamma_3 - 2\sigma\sqrt{\Delta t}$
$r_{3,2} = \gamma_3 - 4\sigma\sqrt{\Delta t}$
$r_{3,3} = \gamma_3 - 6\sigma\sqrt{\Delta t}$

In general, $r_{i,0} = r_0 + [\,\mu(t_0) + \mu(t_1) + \mu(t_2) + \ldots + \mu(t_{i-1})\,]\Delta t + i\sigma\sqrt{\Delta t} \equiv \gamma_i$
$r_{i,j} = \gamma_i + j\phi$, $j = 1, 2, \ldots, i$

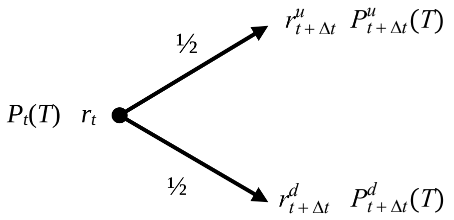
where $\phi = (-2\sigma\sqrt{\Delta t}\,)$

Model parameters $\{\gamma_0, \gamma_1, \ldots, \gamma_{N_{tree}}, \phi\}$

Model Calibration :

Risk-neutral pricing of interest rate derivatives, $f_t = \hat{E}(e^{-\int_t^T r_s\, ds} f_T \mid r_t)$

For one time step Δt, $f_t = e^{-r_t \Delta t}\, \hat{E}(f_{t+\Delta t} \mid r_t)$



Risk-neutral valuation of bond price, $P_t(T) = e^{-r_t \Delta t} \left(\frac{1}{2} P^u_{t+\Delta t}(T) + \frac{1}{2} P^d_{t+\Delta t}(T) \right)$

Suppose, bond price $P_t(T)$ has a volatility structure $\nu_t(T)$ defined in its stochastic process as,

$$\Delta P_t(T)/P_t(T) = (...)\Delta t + \nu_t(T)\Delta z_t$$

So, $\Delta log P_t(T) = (...)\Delta t + \nu_t(T)\Delta z_t$ with variance $S^2(\Delta log P_t(T) \mid r_t) = \nu_t^2(T)\Delta t$

From the binomial tree,

$$S^2(\Delta log P_t(T) \mid r_t) = \frac{1}{2}\left(log P^u_{t+\Delta t}(T)\right)^2 + \frac{1}{2}\left(log P^d_{t+\Delta t}(T)\right)^2 - \left(\frac{1}{2} log P^u_{t+\Delta t}(T) + \frac{1}{2} log P^d_{t+\Delta t}(T) \right)^2$$

$$= \frac{1}{4}\left(log\left(\frac{P^u_{t+\Delta t}(T)}{P^d_{t+\Delta t}(T)}\right)\right)^2$$

Bond price volatility term structure, $\nu_t(T)\sqrt{\Delta t} = \frac{1}{2}\left| log\left(\frac{P^u_{t+\Delta t}(T)}{P^d_{t+\Delta t}(T)}\right)\right| = \frac{1}{2} log\left(\frac{P^d_{t+\Delta t}(T)}{P^u_{t+\Delta t}(T)}\right)$

Calibrating model parameters using current yield curve $\{P_0(\Delta t), P_0(2\Delta t), ... , P_0((N_{tree} + 1)\Delta t)\}$ and

current bond price volatility $\{\nu_0(2\Delta t), \nu_0(3\Delta t), ... , \nu_0((N_{tree} + 1)\Delta t)\}$

- Matching the current yield curve, $P_0(m\Delta t) = P_{00}(m\Delta t)$, $m \geq 1$

  by backward iterating forward bond price as

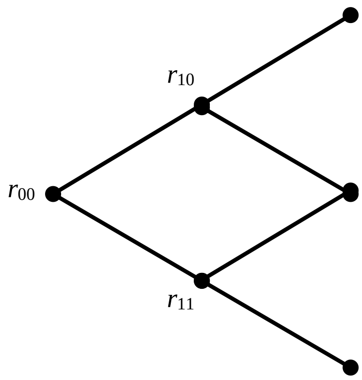
$$P_{ij}(m\Delta t) = e^{-(\gamma_i + j\phi)\Delta t} \left(\frac{1}{2} P_{i+1\, j}(m\Delta t) + \frac{1}{2} P_{i+1\, j+1}(m\Delta t) \right) \quad \text{, starting with } P_{m\, j}(m\Delta t) = 1$$

- Matching the current bond price volatilities,

$$\nu_0(m\Delta t)\sqrt{\Delta t} = \frac{1}{2} log\left(\frac{P_{11}(m\Delta t)}{P_{10}(m\Delta t)}\right) \text{ , } m = 2, 3, ... , N_{tree} + 1$$

Calibrating the model parameters $\{\gamma_0, \gamma_1, \dots, \gamma_{N_{tree}}, \phi\}$ in the Ho-Lee tree.

For $m = 1$,



$$P_{0,0}(\Delta t) = e^{-r_{0,0}\,\Delta t}\left(\frac{1}{2}(\$1) + \frac{1}{2}(\$1)\right) = e^{-\gamma_0\,\Delta t}$$

$$\Rightarrow\; e^{-\gamma_0\,\Delta t} = P_0(\Delta t)$$

For $m = 2$,

$$P_{1,0}(2\Delta t) = e^{-r_{1,0}\,\Delta t}\left(\frac{1}{2}(\$1) + \frac{1}{2}(\$1)\right) = e^{-\gamma_1\,\Delta t}$$

$$P_{1,1}(2\Delta t) = e^{-r_{1,1}\,\Delta t}\left(\frac{1}{2}(\$1) + \frac{1}{2}(\$1)\right) = e^{-r_{1,1}\,\Delta t} = e^{-(\gamma_1 + \phi)\Delta t} \quad , \; \phi = (-\,2\sigma\sqrt{\Delta t}\,)$$

$$P_{0,0}(2\Delta t) = e^{-r_{0,0}\,\Delta t}\left(\frac{1}{2}P_{1,0}(2\Delta t) + \frac{1}{2}P_{1,1}(2\Delta t)\right)$$

$$= e^{-\gamma_0\,\Delta t}\left(\frac{1}{2}e^{-\gamma_1\,\Delta t} + \frac{1}{2}e^{-(\gamma_1 + \phi)\Delta t}\right)$$

$$\Rightarrow e^{-\gamma_0\,\Delta t}\left(\frac{1}{2}e^{-\gamma_1\,\Delta t} + \frac{1}{2}e^{-(\gamma_1 + \phi)\Delta t}\right) = P_0(2\Delta t)$$

and continue on for $m = 2, 3, \dots, (N_{tree} + 1)$

To calibrate a constant ϕ, we might consider, for example, calibrating with the current bond price volatility for the shortest maturity term as,

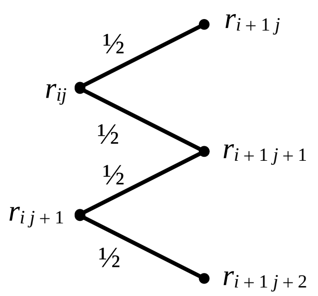
$$\nu_0(2\Delta t)\sqrt{\Delta t} = \frac{1}{2}\,log\left(\frac{P_{11}(2\Delta t)}{P_{10}(2\Delta t)}\right) = \frac{1}{2}\,(-\,\phi\Delta t) = \sigma\sqrt{\Delta t}\;\Delta t$$

However, the calibrated value of ϕ will be inconsistent using other maturity terms.

For constant σ, the entire bond volatility term structure cannot be included in the calibration. This means that the binomal tree does not necessary consistent with the market values of bond price volatilities $\{\nu_0(2\Delta t)$, $\nu_0(3\Delta t)$, ... $\}$.

Risk-neutral process of short rate, $\Delta r_t = \mu(t)\Delta t + \sigma(t)\Delta z_t$

The binomial lattice are not recombining, but the error is suppressed by $\Delta t\sqrt{\Delta t}$ relative to $\Delta r \sim \sqrt{\Delta t}$.
Thus, we can consider a recombining lattice as in the previous Model, and the binomial tree should converge to the above short rate process as $\Delta t \to 0$.



Since $r_{i+1\,j} = r_{i\,j} + \mu(t_i)\Delta\mathrm{t} + \sigma(t_i)\sqrt{\Delta t}$

$$r_{i+1\,j+1} = r_{i\,j} + \mu(t_i)\Delta\mathrm{t} - \sigma(t_i)\sqrt{\Delta t} = r_{i+1\,j} - 2\sigma(t_i)\sqrt{\Delta t}$$

Thus, the branching rule for r_{ij} satisfies the condition that $r_{i+1\,j+1} = r_{i+1\,j} + \phi_{i+1}$, where $\phi_{i+1} = -\,2\sigma(t_i)\sqrt{\Delta t}$

Similarly, the branching rule for $r_{i\,j+1}$ is given by $r_{i+1\,j+2} = r_{i+1\,j+1} + \phi_{i+1}$

The recombining requirement then suggests that $r_{i+1\,j+2} = r_{i+1\,j+1} + \phi_{i+1} = r_{i+1\,j} + 2\phi_{i+1}$

Running down from top to bottom nodes, we should have $r_{i+1\,j} = r_{i+1\,0} + j\phi_{i+1}$

For a recombining binomial lattice, we have $r_{ij} = \gamma_i + j\phi_i$, $j = 0, 1, \ldots , i$

Model paremeters $\{\gamma_0, (\gamma_1, \phi_1) ,\ldots , (\gamma_{N_{tree}}, \phi_{N_{tree}})\}$.

Calibration :

For $m = 1$, $P_{0,0}(\Delta t) = e^{-r_{0,0}\,\Delta t}\,(\frac{1}{2}(\$1) + \frac{1}{2}(\$1)\,) = e^{-\gamma_0\,\Delta t} \;\Rightarrow\; e^{-\gamma_0\,\Delta t} = P_0(\Delta t)$

For $m = 2$, $P_{1,0}(2\Delta t) = e^{-r_{1,0}\,\Delta t}\,(\frac{1}{2}(\$1) + \frac{1}{2}(\$1)\,) = e^{-\gamma_1\,\Delta t}$

$$P_{1,1}(2\Delta t) = e^{-r_{1,1}\,\Delta t}\,(\frac{1}{2}(\$1) + \frac{1}{2}(\$1)\,) = e^{-r_{1,1}\,\Delta t} = e^{-\left(\gamma_1 + \phi_1\right)\Delta t}$$

$$P_{0,0}(2\Delta t) = e^{-r_{0,0}\,\Delta t}\,(\frac{1}{2}P_{1,0}(2\Delta t) + \frac{1}{2}P_{1,1}(2\Delta t)\,)$$

$$= e^{-\gamma_0\,\Delta t}\,(\frac{1}{2}e^{-\gamma_1\,\Delta t} + \frac{1}{2}e^{-\left(\gamma_1 + \phi_1\right)\Delta t}\,)$$

$$\Rightarrow e^{-\gamma_0\,\Delta t}\,(\frac{1}{2}e^{-\gamma_1\,\Delta t} + \frac{1}{2}e^{-\left(\gamma_1 + \phi_1\right)\Delta t}\,) = P_0(2\Delta t)$$

Together with $\nu_0(2\Delta t)\sqrt{\Delta t} = \frac{1}{2}\,log\left(\frac{P_{11}(2\Delta t)}{P_{10}(2\Delta t)}\right) = \frac{1}{2}(-\,\phi_1\Delta t)$, we can solve for γ_1 and ϕ_1.

Continue on with $m = 2, 3, \ldots , (N_{tree} + 1)$

Random lognormal short rate

Consider the risk-neutral process of short rate given by

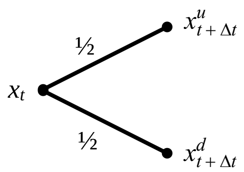
$$\Delta log(x_t) = \theta(x_t, t)\Delta t + \sigma(t)\Delta z_t$$

Since, $\Delta log(x_t) = log\left(\frac{x_{t+\Delta t}}{x_t}\right) \quad \Rightarrow \quad log(x_{t+\Delta t}) = log(x_t) + \theta(x_t, t)\Delta t + \sigma(t)\Delta z_t$

Mean of $log(x_{t+\Delta t})$ conditional to x_t , $E(log(x_{t+\Delta t}) \mid x_t) = log(x_t) + \theta(x_t, t)\Delta t$

Variance of $log(x_{t+\Delta t})$ conditional to x_t , $S^2(log(x_{t+\Delta t}) \mid x_t) = \sigma^2(t)\Delta t$

Use Jarrow and Rudd parameterization,



Under a binomial representation, the same mean and variance are calculated to be

$$E(log(x_{t+\Delta t}) \mid x_t) = \frac{1}{2}\, log(x^u_{t+\Delta t}) + \frac{1}{2}\, log(x^d_{t+\Delta t}) = log(x_t) + \theta(x_t, t)\Delta t \quad \text{(matching the mean)}$$

$$\Rightarrow log(x^u_{t+\Delta t}\; x^d_{t+\Delta t}) = log(x_t^2) + 2\theta(x_t, t)\Delta t$$

$$\Rightarrow x^u_{t+\Delta t}\; x^d_{t+\Delta t} = x_t^2\; e^{2\theta(x_t,\, t)\Delta t}$$

$$S^2(log(x_{t+\Delta t}) \mid x_t) = \frac{1}{2}\left(log(x^u_{t+\Delta t})\right)^2 + \frac{1}{2}\left(log(x^d_{t+\Delta t})\right)^2 - \left(\frac{1}{2}\, log(x^u_{t+\Delta t}) + \frac{1}{2}\, log(x^d_{t+\Delta t})\right)^2$$

$$= \frac{1}{4}\left(log(x^u_{t+\Delta t}) - log(x^d_{t+\Delta t})\right)^2$$

$$= \frac{1}{4}\left(log\left(\frac{x^u_{t+\Delta t}}{x^d_{t+\Delta t}}\right)\right)^2$$

$$= \sigma^2(t)\Delta t \quad , \text{ matching the variance} \quad \Rightarrow \left|log\left(\frac{x^u_{t+\Delta t}}{x^d_{t+\Delta t}}\right)\right| = 2\sigma(t)\sqrt{\Delta t} \quad , \text{ take } x^u_{t+\Delta t} > x^d_{t+\Delta t}$$

$$\Rightarrow \left(\frac{x^u_{t+\Delta t}}{x^d_{t+\Delta t}}\right) = e^{2\sigma(t)\sqrt{\Delta t}}$$

Together with the previous result, $\quad \Rightarrow x^d_{t+\Delta t} = x_t\; e^{\theta(x_t,\, t)\Delta t - \sigma(t)\sqrt{\Delta t}}$

$$x^u_{t+\Delta t} = x_t\; e^{\theta(x_t,\, t)\Delta t + \sigma(t)\sqrt{\Delta t}}$$

Starting from x_t, goes up and then goes down

$$x_t \rightarrow \underbrace{x_t\; e^{\theta(x_t,\, t)\Delta t + \sigma(t)\sqrt{\Delta t}}}_{x^u_{t+\Delta t}} \rightarrow x_t\; e^{\theta(x_t,\, t)\Delta t + \sigma(t)\sqrt{\Delta t}}\;\; e^{\theta(x^u_{t+\Delta t},\, t+\Delta t)\Delta t - \sigma(t+\Delta t)\sqrt{\Delta t}}$$

Starting from x_t, goes down and then goes up

$$x_t \rightarrow \underbrace{x_t\; e^{\theta(x_t,\, t)\Delta t - \sigma(t)\sqrt{\Delta t}}}_{x^d_{t+\Delta t}} \rightarrow x_t\; e^{\theta(x_t,\, t)\Delta t - \sigma(t)\sqrt{\Delta t}}\;\; e^{\theta(x^d_{t+\Delta t},\, t+\Delta t)\Delta t + \sigma(t+\Delta t)\sqrt{\Delta t}}$$

Binomial lattice are recombining when $\theta(x_t, t) = \theta(t)$ and $\sigma(t) = \sigma$.

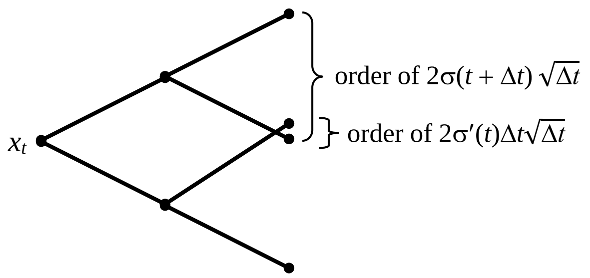
Suppose, $\theta(x_t, t) = \theta(t)$

Starting from x_t , goes up and then goes down

$$x^{ud}_{t+2\Delta t} = x_t\, e^{\theta(t)\Delta t + \theta(t+\Delta t)\Delta t + \sigma(t)\sqrt{\Delta t} - \sigma(t+\Delta t)\sqrt{\Delta t}}$$

Starting from x_t , goes down and then goes up

$$x^{du}_{t+2\Delta t} = x_t\, e^{\theta(t)\Delta t + \theta(t+\Delta t)\Delta t - \sigma(t)\sqrt{\Delta t} + \sigma(t+\Delta t)\sqrt{\Delta t}}$$



The discrepancy for a recombining binomial lattice is in higher order relative to the tree gap.

$$x^{du} - x^{ud} = x_t\, e^{\theta(t)\Delta t + \theta(t+\Delta t)\Delta t - \sigma(t)\sqrt{\Delta t} + \sigma(t+\Delta t)\sqrt{\Delta t}} - x_t\, e^{\theta(t)\Delta t + \theta(t+\Delta t)\Delta t + \sigma(t)\sqrt{\Delta t} - \sigma(t+\Delta t)\sqrt{\Delta t}}$$

$$= x_t\, e^{\theta(t)\Delta t + \theta(t+\Delta t)\Delta t} \left(e^{[-\sigma(t) + \sigma(t+\Delta t)]\sqrt{\Delta t}} - e^{[\sigma(t) - \sigma(t+\Delta t)]\sqrt{\Delta t}} \right)$$

$$\cong x_t\, e^{\theta(t)\Delta t + \theta(t+\Delta t)\Delta t} \left(2[\sigma(t+\Delta t) - \sigma(t)]\sqrt{\Delta t} \right)$$

$$\cong x_t\, e^{\theta(t)\Delta t + \theta(t+\Delta t)\Delta t + \sigma(t)\sqrt{\Delta t} - \sigma(t+\Delta t)\sqrt{\Delta t}} \left(2\sigma'(t)\,\Delta t\,\sqrt{\Delta t} \right)$$

$$x^{uu} - x^{ud} = x_t\, e^{\theta(t)\Delta t + \theta(t+\Delta t)\Delta t + \sigma(t)\sqrt{\Delta t} + \sigma(t+\Delta t)\sqrt{\Delta t}} - x_t\, e^{\theta(t)\Delta t + \theta(t+\Delta t)\Delta t + \sigma(t)\sqrt{\Delta t} - \sigma(t+\Delta t)\sqrt{\Delta t}}$$

$$= x_t\, e^{\theta(t)\Delta t + \theta(t+\Delta t)\Delta t} \left(e^{[\sigma(t) + \sigma(t+\Delta t)]\sqrt{\Delta t}} - e^{[\sigma(t) - \sigma(t+\Delta t)]\sqrt{\Delta t}} \right)$$

$$\cong x_t\, e^{\theta(t)\Delta t + \theta(t+\Delta t)\Delta t} \left(2\sigma(t+\Delta t)\sqrt{\Delta t} \right)$$

Black Model

Risk-neutral process of lognormal short rate, $\Delta log(r_t) = \theta(t)\Delta t + \sigma\Delta z_t$

The binomial lattice are recombining for constant σ.

Define discrete time as $\{\Delta t, 2\Delta t, ... \}$

Short rate at time $t_i = i\Delta t$,

$r_{i,j} = r_i$ ($i - j$ up, j down) $= r_0\, exp\big([\theta(t_0) + \theta(t_1) + ... + \theta(t_{i-1})]\Delta t + (i - j)\sigma\sqrt{\Delta t} - j\sigma\sqrt{\Delta t}\,\big)$

with $j = 0, 1, ... , i$ running from top to bottom nodes.

$r_{i,j+1} = r_0\, exp\big([\theta(t_0) + \theta(t_1) + ... + \theta(t_{i-1})]\Delta t + (i - j - 1)\sigma\sqrt{\Delta t} - (j + 1)\sigma\sqrt{\Delta t}\,\big)$

$= r_{i,j}\,\beta$, where $\beta = e^{-2\sigma\sqrt{\Delta t}}$

At time $t_i = i\Delta t$ and running from top to bottom nodes with $j = 0, 1, ... , i$,

$r_{i,0} = \alpha_i$

$r_{i,1} = r_{i,0}\,\beta = \alpha_i\,\beta$

$r_{i,2} = r_{i,1}\,\beta = \alpha_i\,\beta^2$

....

This gives $r_{ij} = \alpha_i\,\beta^{\,j}$, $j = 0, 1, ... , i$

Model parameters $\{\alpha_0, \alpha_1, ... , \alpha_{N_{tree}}, \beta\}$.

Calibrating the parameters $\{\alpha_0, \alpha_1, \ldots, \alpha_{N_{tree}}, \beta\}$ in Black's model.

For $m = 1$,

$$P_{00}(\Delta t) = e^{-\alpha_0 \Delta t} \left(\frac{1}{2} (\$1) + \frac{1}{2} (\$1) \right)$$

$$\Rightarrow \alpha_0 = -\left(\frac{1}{\Delta t}\right) log(P_0(\Delta t)/\$1)$$

For $m = 2$,

$$P_{10}(2\Delta t) = e^{-\alpha_1 \Delta t} \left(\frac{1}{2} P_{20}(2\Delta t) + \frac{1}{2} P_{21}(2\Delta t) \right) = e^{-\alpha_1 \Delta t} \left(\frac{1}{2} (\$1) + \frac{1}{2} (\$1) \right) = e^{-\alpha_1 \Delta t} (\$1)$$

$$P_{11}(2\Delta t) = e^{-\alpha_1 \beta \Delta t} \left(\frac{1}{2} P_{21}(2\Delta t) + \frac{1}{2} P_{22}(2\Delta t) \right) = e^{-\alpha_1 \beta \Delta t} \left(\frac{1}{2} (\$1) + \frac{1}{2} (\$1) \right) = e^{-\alpha_1 \beta \Delta t} (\$1)$$

$$P_{00}(2\Delta t) = e^{-\alpha_0 \Delta t} \left(\frac{1}{2} P_{10}(2\Delta t) + \frac{1}{2} P_{11}(2\Delta t) \right) = e^{-\alpha_0 \Delta t} \left(\frac{1}{2} e^{-\alpha_1 \Delta t} (\$1) + \frac{1}{2} e^{-\alpha_1 \beta \Delta t} (\$1) \right)$$

$$\nu_0(2\Delta t)\sqrt{\Delta t} = \frac{1}{2} log\left(\frac{P_{11}(2\Delta t)}{P_{10}(2\Delta t)}\right) = \frac{1}{2} log\left(\frac{e^{-\alpha_1 \beta \Delta t}}{e^{-\alpha_1 \Delta t}}\right)$$

Solve the following two equations for α_1 and β.

$$P_0(2\Delta t) = e^{-\alpha_0 \Delta t} \left(\frac{1}{2} e^{-\alpha_1 \Delta t} (\$1) + \frac{1}{2} e^{-\alpha_1 \beta \Delta t} (\$1) \right)$$

$$\nu_0(2\Delta t)\sqrt{\Delta t} = \frac{1}{2} log\left(\frac{e^{-\alpha_1 \beta \Delta t}}{e^{-\alpha_1 \Delta t}}\right)$$

For $m = 3$,

We can backward iterate $P_{ij}(3\Delta t) = e^{-\alpha_i \beta^j \Delta t} \left(\frac{1}{2} P_{i+1\, j}(3\Delta t) + \frac{1}{2} P_{i+1\, j+1}(3\Delta t) \right)$

to $P_{11}(3\Delta t)$, and $P_{10}(3\Delta t)$, and then $P_{00}(3\Delta t)$

Solve α_2 that satisfies $P_0(3\Delta t) = P_{00}(3\Delta t)$

When $m = k + 1$, determine α_k based on $\{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \beta\}$

We can backward iterate $P_{ij}((k+1)\Delta t) = e^{-\alpha_i \beta^j \Delta t} \left(\frac{1}{2} P_{i+1\, j}((k+1)\Delta t) + \frac{1}{2} P_{i+1\, j+1}((k+1)\Delta t) \right)$

to $P_{00}((k+1)\Delta t)$

Solve α_k that satisfies $P_0((k+1)\Delta t) = P_{00}((k+1)\Delta t)$

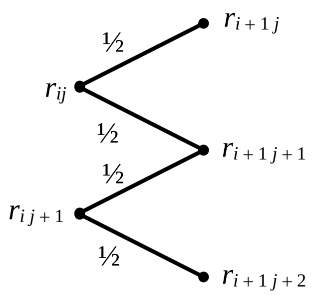
For constant σ, the entire bond volatility term structure cannot be included in the calibration. This means that the binomal tree does not necessary consistent with the market values of bond price volatilities $\{\nu_0(2\Delta t), \nu_0(3\Delta t), \ldots\}$.

$$\nu_0(m\Delta t)\sqrt{\Delta t} = \frac{1}{2} log\left(\frac{P_{11}(m\Delta t)}{P_{10}(m\Delta t)}\right) \text{ is only true for } m = 2.$$

Black-Derman-Toy Model

Risk-neutral process of lognormal short rate, $\Delta log(r_t) = \theta(t)\Delta t + \sigma(t)\Delta z_t$

The binomial lattice are not recombining, but the error is suppressed by $\Delta t\sqrt{\Delta t}$ relative to $\Delta r \sim \sqrt{\Delta t}$.
Thus, we can consider a recombining lattice as in the previous Model, and the binomial tree should converge to the above short rate process as $\Delta t \to 0$.



Since $\quad r_{i+1\,j} = r_{i\,j}\, e^{\theta(t_i)\Delta t + \sigma(t_i)\sqrt{\Delta t}}$

$$r_{i+1\,j+1} = r_{i\,j}\, e^{\theta(t_i)\Delta t - \sigma(t_i)\sqrt{\Delta t}} = r_{i+1\,j}\, e^{-2\sigma(t_i)\sqrt{\Delta t}}$$

Thus, the branching rule for r_{ij} satisfies the condition that $\dfrac{r_{i+1\,j+1}}{r_{i+1\,j}} = e^{-2\sigma(t_i)\sqrt{\Delta t}} = \beta_{i+1}$

Similarly, the branching rule for r_{ij+1} is given by $\dfrac{r_{i+1\,j+2}}{r_{i+1\,j+1}} = \beta_{i+1}$

The recombining requirement then suggests that $\dfrac{r_{i+1\,j+2}}{r_{i+1\,j}} = \beta^2_{i+1}$

Running down from top to bottom nodes, we should have $r_{i+1\,j} = r_{i+1\,0}\, \beta^j_{i+1}$

For a recombining binomial lattice, we have $r_{ij} = \alpha_i\, \beta^j_i$, $j = 0, 1, \ldots , i$

Model paremeters $\{\alpha_0, (\alpha_1, \beta_1) ,\ldots , (\alpha_{N_{tree}}, \beta_{N_{tree}})\}$

Calibrating the paremeters $\{\alpha_0, (\alpha_1, \beta_1), \dots, (\alpha_{N_{tree}}, \beta_{N_{tree}})\}$ in Black-Derman-Toy Model.

For $m = 1$,

$P_{00}(\Delta t) = e^{-\alpha_0 \Delta t} \left(\frac{1}{2} (\$1) + \frac{1}{2} (\$1) \right)$

$\Rightarrow \alpha_0 = -\left(\frac{1}{\Delta t}\right) log(P_0(\Delta t)/\$1)$

When $m = 2$,

$P_{10}(2\Delta t) = e^{-\alpha_1 \Delta t} \left(\frac{1}{2} P_{20}(2\Delta t) + \frac{1}{2} P_{21}(2\Delta t) \right) = e^{-\alpha_1 \Delta t} \left(\frac{1}{2} (\$1) + \frac{1}{2} (\$1) \right) = e^{-\alpha_1 \Delta t} (\$1)$

$P_{11}(2\Delta t) = e^{-\alpha_1\beta_1 \Delta t} \left(\frac{1}{2} P_{21}(2\Delta t) + \frac{1}{2} P_{22}(2\Delta t) \right) = e^{-\alpha_1\beta \Delta t} \left(\frac{1}{2} (\$1) + \frac{1}{2} (\$1) \right) = e^{-\alpha_1\beta_1 \Delta t} (\$1)$

$P_{00}(2\Delta t) = e^{-\alpha_0 \Delta t} \left(\frac{1}{2} P_{10}(2\Delta t) + \frac{1}{2} P_{11}(2\Delta t) \right) = e^{-\alpha_0 \Delta t} \left[\frac{1}{2} e^{-\alpha_1 \Delta t} (\$1) + \frac{1}{2} e^{-\alpha_1\beta_1 \Delta t} (\$1) \right]$

$v_0(2\Delta t)\sqrt{\Delta t} = \frac{1}{2} log\left(\frac{P_{11}(2\Delta t)}{P_{10}(2\Delta t)}\right) = \frac{1}{2} log\left(\frac{e^{-\alpha_1\beta_1 \Delta t}}{e^{-\alpha_1 \Delta t}}\right)$

Solve the following two equations for α_1 and β_1.

$P_0(2\Delta t) = e^{-\alpha_0 \Delta t} \left(\frac{1}{2} e^{-\alpha_1 \Delta t} (\$1) + \frac{1}{2} e^{-\alpha_1\beta_1 \Delta t} (\$1) \right)$

$v_0(2\Delta t)\sqrt{\Delta t} = \frac{1}{2} log\left(\frac{e^{-\alpha_1\beta_1 \Delta t}}{e^{-\alpha_1 \Delta t}}\right)$

When $m = k + 1$, determine α_k and β_k based on $\{\alpha_0, \alpha_1, \dots, \alpha_{k-1}\}$ and $\{\beta_1, \dots, \beta_{k-1}\}$

We can backward iterate $P_{ij}((k + 1)\Delta t) = e^{-\alpha_i\beta_i^j \Delta t} \left(\frac{1}{2} P_{i+1\,j}((k + 1)\Delta t) + \frac{1}{2} P_{i+1\,j+1}((k + 1)\Delta t) \right)$

to $P_{10}((k + 1)\Delta t)$, $P_{11}((k + 1)\Delta t)$, and $P_{00}((k + 1)\Delta t)$.

Solve α_k and β_k that satisfies $P_0((k + 1)\Delta t) = P_{00}((k + 1)\Delta t)$

and $v_0((k + 1)\Delta t)\sqrt{\Delta t} = \frac{1}{2} log\left(\frac{P_{11}((k + 1)\Delta t)}{P_{10}((k + 1)\Delta t)}\right)$