<u>Interest Rate Derivatives</u>

Bond prices :

In the Cox-Ingersoll-Ross model, the risk-neutral process of short rate is given by

$$\Delta r_t = a(\, b - r_t \,)\, \Delta t + \sigma\sqrt{r_t}\,\sqrt{\Delta t}\,\varepsilon(0, 1)$$

where a, b, σ, and r_0 are constant parameters. Interest rate derivatives are considered to be function over this stochastic variable for which current value of derivatives can be determined through risk-neutral expectation of discounted payoff at maturity. Use Monte-Carlo simulation to estimate the risk-neutral pricing of pure discount bond with maturity at T given by

$$P_0(T) = (\$1)\, \widehat{E}\left(e^{-\int_0^T r_t\, dt} \mid r_0\right)$$

where $\{a, b, \sigma, r_0\}$ are taken to be input parameters.

(1) As control variate in the simulation, consider the analytic solution from the Vasicek model written as

$$P_{0,\,Vasicek}(T) = (\$1)\; exp\left(\frac{(\,B(T) - T\,)(\,a^2 b - ½\sigma^2\,)}{a^2} - \frac{\sigma^2 B(T)^2}{4a}\right) e^{-B(T)\, r_0} \;\;,\;\; B(T) = \frac{(1 - e^{-aT})}{a}$$

In this model, short rate follows instead the risk-neutral process given by

$$\Delta r_t^{vasicek} = a(\, b - r_t^{vasicek} \,)\, \Delta t + \sigma\sqrt{\Delta t}\,\varepsilon(0, 1)$$

with the same parameters. The Monte-Carlo pricng of the discount bond then becomes

$$P_0(T) = P_{0,Vasicek}(T) + \widehat{E}\left(e^{-\int_0^T r_t\, dt}\,(\$1) - e^{-\int_0^T r_t^{vasicek}\, dt}(\$1) \mid r_0\right)$$

(2) For the antithetic variate method, the Monte-Carlo pricng of the discount bond is given by

$$P_0(T) = \widehat{E}\left(\tfrac{1}{2}\left(e^{-\int_0^T r_t\, dt}\,(\$1) + e^{-\int_0^T r_t^a\, dt}\,(\$1)\right) \mid r_0\right)$$

Note : We can estimate the discount path as

$$\int_0^T r_t\, dt \cong (r_{t_0} + r_{t_1} + ... + r_{t_{N-1}})\Delta t \;\;,\;\text{where } t_i = i\,\Delta t\;\;,\;\;\Delta t = \tfrac{T}{N}$$

or $\int_0^T r_t\, dt \cong \frac{1}{3}(r_{t_0} + 4r_{t_1} + 2r_{t_2} + 4r_{t_3} + ... + 4r_{t_{N-1}} + r_{t_N})\Delta t$ (Simpson's rule with even N)

$CMC\_CIRZeroCouponBondPrice(\ a\ ,\ b\ ,\ \sigma\ ,\ r_0\ ,\ T\ ,\ par,\ N,\ n\ ,\ f_0\ ,\ error\ )$

\# define the size of time interval
$\Delta t = T\ /\ N$

\# zeroize the sample sum and squared sum
$sum = 0\ ,\ sum2 = 0$

For($L_s = 1$ to n){

\# initialize the short rates and discount paths
$ReSampling : r = r_0\ ,\ y = r\Delta t$
$r_c = r_0\ ,\ y_c = r_c\,\Delta t$

\# generate the short rates at each intermediate time and accumulate the discount paths
For($i = 1$ to $N - 1$){ ε = StdNormNum( )
$\Delta r = a(\ b - r\ )\ \Delta t + \sigma\sqrt{r}\,\sqrt{\Delta t}\ \varepsilon$
$r = r + \Delta r$
If($r \leq 0$) goto *ReSampling*
$y = y + r\Delta t$

$\Delta r = a(\ b - r_c\ )\ \Delta t + \sigma\sqrt{\Delta t}\ \varepsilon$
$r_c = r_c + \Delta r$
If($r_c \leq 0$) goto *ReSampling*
$y_c = y_c + r_c\Delta t$ }

\# evaluate the payoff function
$PV = e^{-y}\ par - e^{-y_c}\ par$

\# accumulate the sample sum and squared sum
$sum = sum + PV$
$sum2 = sum2 + PV^2$ }

\# evaluate the estimates of mean and variance
$m = sum\ /\ n$

$$s = \sqrt{\frac{1}{n-1}\ sum2 - \frac{n}{n-1} m^2}$$

\# return the estimation of option price and standard error
$f_0 = m + VasicekZeroCouponBondPrice(\ r_0\ ,\ a\ ,\ b\ ,\ \sigma\ ,\ T\ )$
$error = s\ /\sqrt{n}$

$AMC\_CIRZeroCouponBondPrice(a , b , \sigma , r_0 , T , par, N, n , f_0 , error)$

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# define the size of time interval
Δt = T / N

# zeroize the sample sum and squared sum
sum = 0 , sum2 = 0

For( L_s = 1 to n ){

# initialize the short rates and discount paths
  ReSampling : r = r_0 ,  y = rΔt
               r_a = r_0 ,  y_a = r_a Δt

# generate the short rates at each intermediate time and accumulate the discount paths
        For( i = 1 to N − 1){   ε = StdNormNum( )
                                Δr = a( b − r ) Δt + σ√r √Δt ε
                                r = r + Δr
                                If( r ≤ 0 ) goto ReSampling
                                y = y + rΔt

                                Δr = a( b − r_a ) Δt + σ√r_a √Δt (−ε)
                                r_a = r_a + Δr
                                If( r_a ≤ 0 ) goto ReSampling
                                y_a = y_a + r_a Δt                }

# evaluate the payoff function
        PV = ½ ( e^(−y) par + e^(−y_a) par )

# accumulate the sample sum and squared sum
        sum = sum + PV
        sum2 = sum2 + PV^2      }

# evaluate the estimates of mean and variance
m = sum / n
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$$s = \sqrt{\frac{1}{n-1}\, sum2 - \frac{n}{n-1} m^2}$$

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# return the estimation of option price and standard error
f_0 = m
error = s / √n
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Bond option prices :

Use Monte-Carlo simulation to estimate the risk-neutral pricing of a call option, with maturity at T and strike K, written on a zero coupon bond with maturity at T^* and face value L.

$$c_0(T) = \widehat{E}\left(e^{-\int_0^T r_s\,ds}\, max\{\, B_T\left(T^*, r_T\right) - K, 0\} \mid r_0\right)$$

It is convenient to use the antithetic variate method and consider the pricing given by

$$c_0(T) = \widehat{E}\left(\frac{1}{2}\left(e^{-\int_0^T r_s\,ds}\, max\{\, B_T\left(T^*, r_T\right) - K, 0\} + e^{-\int_0^T r_s^a\,ds}\, max\{\, B_T\left(T^*, r_T^a\right)\right) \mid r_0\right)$$

$AMC\_CIRBondCallPrice(a , b , \sigma , r_0 , T , K, T^*, par, N, n , f_0 , error)$

define the size of time interval and the number steps from T to T^*

$\Delta t = T / N$, $M = \text{CINT}\left(N \left(\frac{T^* - T}{T}\right)\right)$

zeroize the sample sum and squared sum

$sum = 0$, $sum2 = 0$

For($L_s = 1$ to n){

initialize the short rates and discount paths

$ReSampling$: $r = r_0$, $y = r\Delta t$

$r_a = r_0$, $y_a = r_a \Delta t$

generate the short rates at each intermediate time and accumulate the discount paths

For($i = 1$ to N){ ε = StdNormNum( )

$\Delta r = a(b - r) \Delta t + \sigma\sqrt{r}\ \sqrt{\Delta t}\ \varepsilon$

$r = r + \Delta r$

If($r \leq 0$) goto $ReSampling$

$y = y + r\Delta t$

$\Delta r = a(b - r_a) \Delta t + \sigma\sqrt{r_a}\ \sqrt{\Delta t}\ (-\varepsilon)$

$r_a = r_a + \Delta r$

If($r_a \leq 0$) goto $ReSampling$

$y_a = y_a + r_a \Delta t$ }

evaluate the payoff function

Call $AMC\_CIRZeroCouponBondPrice(a , b , \sigma , r , T^* - T, par, M, n , B , PFerror)$

Call $AMC\_CIRZeroCouponBondPrice(a , b , \sigma , r_a , T^* - T, par, M, n , B_a , PFerror)$

$PV = ½ (e^{-y}\ max\{ B - K , 0 \} + e^{-y_a}\ max\{ B_a - K , 0 \})$

accumulate the sample sum and squared sum

$sum = sum + PV$

$sum2 = sum2 + PV^2$ }

evaluate the estimates of mean and variance

$m = sum / n$

$$s = \sqrt{\frac{1}{n-1}\ sum2 - \frac{n}{n-1} m^2}$$

return the estimation of option price and standard error

$f_0 = m$

$error = s / \sqrt{n}$