Assignment 7-Lie Theory

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Identities

$$[a] \times b = -[b] \times a \tag{1}$$
$$[Ab]\times = A[b] \times A^T \tag{2}$$

Q1: Left and Right derivatives on Lie Groups

For the following function

$$f : \mathrm{SO}(3) \to \mathbb{R}^3; f(R,p) = Rp, \tag{3}$$

calculate the left and right derivatives of f with respect to R, by applying the definitions of the left and right derivatives:

$$\frac{{}^R D f(R,p)}{DR} \tag{4}$$
$$\frac{{}^L D f(R,p)}{DR} \tag{5}$$

1. **Right Derivative:**

   The right derivative of a function f at a point R in the direction θ is defined as the limit:

   $$\frac{{}^R D f(R,p)}{DR} = \lim_{\theta \to 0} \frac{f(R\exp(\theta), p) - f(R,p)}{\theta}$$

   Substitute $f(R,p) = Rp$ from Equation (3):

   $$\frac{{}^R D f(R,p)}{DR} = \lim_{\theta \to 0} \frac{R\exp(\theta)p - Rp}{\theta}$$

   The first-order approximation of the Taylor series is used for the small θ, $\exp(\theta) \approx I + \theta$, where I is the identity matrix.

Substitute this approximation into the expression:

$$\frac{{}^{R}Df(R,p)}{DR} = \lim_{\theta \to 0} \frac{R(I + [\theta]_x)p - Rp}{\theta}$$

Simplify the expression inside the limit:

$$\frac{{}^{R}Df(R,p)}{DR} = \lim_{\theta \to 0} \frac{R[\theta]_x p}{\theta}$$

Using the identities from equation (1):

$$\frac{{}^{R}Df(R,p)}{DR} = \lim_{\theta \to 0} \frac{-R[p]_x \theta}{\theta}$$

Finally, the θ terms cancel out:

$$\frac{{}^{R}Df(R,p)}{DR} = -R[p]_x$$

2. **Left Derivative:**

   The left derivative of a function f at a point R in the direction θ is defined as the limit:

$$\frac{{}^{L}Df(R,p)}{DR} = \lim_{\theta \to 0} \frac{f(\exp(\theta)R, p) - f(R,p)}{\theta}$$

   Substitute $f(R,p) = Rp$ from Equation (3):

$$\frac{{}^{L}Df(R,p)}{DR} = \lim_{\theta \to 0} \frac{\exp(\theta)Rp - Rp}{\theta}$$

   The first-order approximation of the Taylor series is used for the small θ, $\exp(\theta) \approx I + \theta$, where I is the identity matrix.

   Substitute this approximation into the expression:

$$\frac{{}^{L}Df(R,p)}{DR} = \lim_{\theta \to 0} \frac{(I + [\theta]_x)Rp - Rp}{\theta}$$

   Simplify the expression inside the limit:

$$\frac{{}^{L}Df(R,p)}{DR} = \lim_{\theta \to 0} \frac{[\theta]_x Rp}{\theta}$$

   Using the identities in equation (1):

$$\frac{{}^{L}Df(R,p)}{DR} = \lim_{\theta \to 0} \frac{-[p]_x R\theta}{\theta}$$

Finally, the θ terms cancel out:

$$\frac{{}^{L}Df(R,p)}{DR} = -R[p]_x$$

As Lie groups are not always commutative, the order of the multiplication can matter. For a Lie group G and an element g, the right derivative is defined by $R_g(h) = hg$ for all points h in G, and the left derivative is defined by $L_g(h) = gh$ for all points h in G.

Q2: Jacobian

Given a Lie group $\mathcal{M}$ with a composition operation $\circ$, and elements $\mathcal{X}, \mathcal{Y} \in \mathcal{M}$, calculate the derivative of $\mathcal{X} \circ \mathcal{Y}$ with respect to $\mathcal{Y}$, i.e.

$$\frac{{}^{\mathcal{Y}}D(\mathcal{X} \circ \mathcal{Y})}{D\mathcal{Y}} \tag{6}$$

$f : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ defined by $f(\mathcal{X}, \mathcal{Y}) = \mathcal{X} \circ \mathcal{Y}$, where $\mathcal{M}$ is the Lie group, § and $\mathcal{Y}$ are elements of $\mathcal{M}$, and $\circ$ is the composition operation.
The Jacobian $J_{\mathcal{Y}}^{\mathcal{X} \circ \mathcal{Y}}$ is the derivative of f with respect to the †.
Considering a small change in $\mathcal{Y} \approx \mathcal{Y} + \tau$:

$$f(\mathcal{X}, \mathcal{Y} + \tau) - f(\mathcal{X}, \mathcal{Y})$$

The change in $f(\mathcal{X}, \mathcal{Y} + \tau) - f(\mathcal{X}, \mathcal{Y})$ is given by $\mathcal{X} \circ (\mathcal{Y} + \tau) - \mathcal{X} \circ \mathcal{Y}$.
This means that $\mathcal{X} \circ (\mathcal{Y} + \tau) - \mathcal{X} \circ \mathcal{Y}$ is approximately equal to $\mathcal{X} \circ \tau$. This can be written as:

$$\lim_{\tau \to 0} \frac{\mathcal{X} \circ (\mathcal{Y} + \tau) - \mathcal{X} \circ \mathcal{Y}}{\tau} \approx \lim_{\tau \to 0} \frac{\mathcal{X} \circ \tau}{\tau}$$

Divide this by τ and take the limit as τ approaches 0:

$$\lim_{\tau \to 0} \frac{\mathcal{X} \circ \tau}{\tau} = I$$

Therefore, $\frac{{}^{\mathcal{Y}}D(\mathcal{X} \circ \mathcal{Y})}{D\mathcal{Y}}$= Identity Matrix ($I$)

Q3-Adjoint Matrix Properties

Given the Lie group of $M = SE(3)$ with the composition operation $\circ$, and elements $\mathcal{X}, \mathcal{Y} \in \mathcal{M}$, show that

$$Ad_{\mathcal{X}} Ad_{\mathcal{Y}} = Ad_{\mathcal{X} \circ \mathcal{Y}} \tag{7}$$

$$Ad_{\mathcal{X}^{-1}} = Ad_{\mathcal{X}}^{-1} \tag{8}$$

Adjoint Matrix Approach

- Group element $M = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$
- Lie algebra element $\tau^\wedge = \begin{bmatrix} [\theta]_\times & \rho \\ 0 & 0 \end{bmatrix}$
- Vector $\tau = \begin{bmatrix} \rho \\ \theta \end{bmatrix}$

The adjoint matrix is identified:

$$Ad_M \tau = (M\tau^\wedge M^{-1})^\vee = \begin{bmatrix} R & [t]_\times R \\ 0 & R \end{bmatrix} \begin{bmatrix} \rho \\ \theta \end{bmatrix} = \begin{bmatrix} R\rho + [t]_\times R\theta \\ R\theta \end{bmatrix}$$

So the adjoint matrix is $Ad_M = \begin{bmatrix} R & [t]_\times R \\ 0 & R \end{bmatrix} \in R^{6x6}$

1. **Adjoint Property 1:** $Ad_X Ad_Y = Ad_{X \circ Y}$
   $Ad_{X \circ Y} = \begin{bmatrix} R_X R_Y & [t_X + R_X t_Y]_\times R_X R_Y \\ 0 & R_X R_Y \end{bmatrix}$
   $Ad_X Ad_Y = \begin{bmatrix} R_X & [t_X]_\times R_X \\ 0 & R_X \end{bmatrix} \begin{bmatrix} R_Y & [t_Y]_\times R_Y \\ 0 & R_Y \end{bmatrix} = \begin{bmatrix} R_X R_Y & [t_X + R_X t_Y]_\times R_X R_Y \\ 0 & R_X R_Y \end{bmatrix}$
   Therefore, $Ad_{X \circ Y} = Ad_X Ad_Y$
2. **Adjoint Property 2:** $Ad_{X^{-1}} = Ad_X^{-1}$
   $Ad_{X^{-1}} = \begin{bmatrix} R_X^T & -[t_X]_\times R_X^T \\ 0 & R_X^T \end{bmatrix}$
   $Ad_X^{-1} = \begin{bmatrix} R_X & [t_X]_\times R_X \\ 0 & R_X \end{bmatrix}^{-1} = \begin{bmatrix} R_X^T & -[t_X]_\times R_X^T \\ 0 & R_X^T \end{bmatrix}$
   Therefore, $Ad_{X^{-1}} = Ad_X^{-1}$