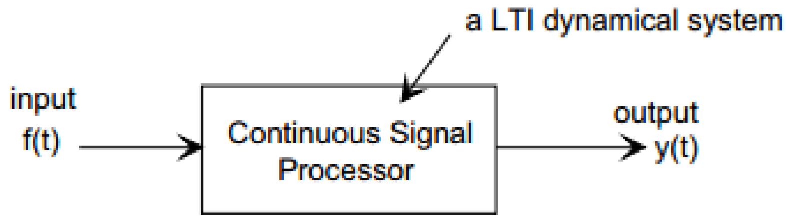
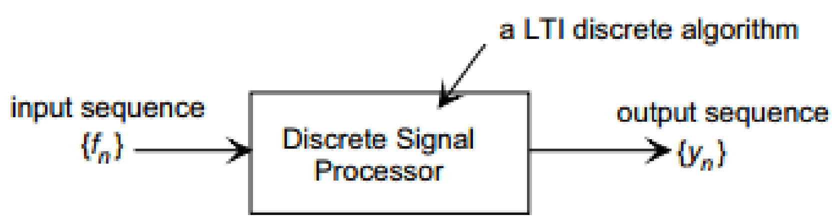
Fundamentals of Signal Processing

Introduction to Signal Processing

- Signal processing of continuous waveforms $f(t)$, using continuous LTI systems.

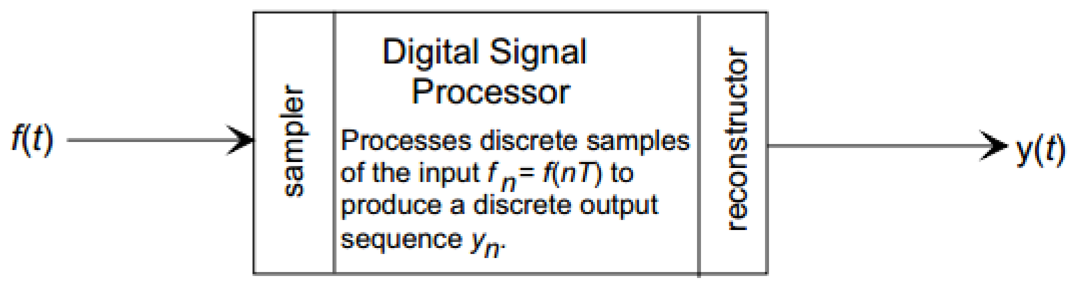


- Discrete-time signal processing of data sequences $\{f_n\}$ that might be samples of real continuous experimental data.



http://ocw.mit.edu/courses/mechanical-engineering/2-161-signal-processing-continuous-and-discrete-fall-2008/lecture-notes/

Digital Signal Processors



Properties of LTI Continuous Filter

(a) **The Principle of Superposition** This is the fundamental property of linear systems. For a system at rest at time $t = 0$, if the response to input $f_1(t)$ is $y_1(t)$, and the response to $f_2(t)$ is $y_2(t)$, then the response to a linear combination of $f_1(t)$ and $f_2(t)$, that is $f_3(t) = af_1(t) + bf_2(t)$ (a and b constants) is

$$y_3(t) = ay_1(t) + by_2(t).$$

(b) **The Differentiation Property** If the response to input $f(t)$ is $y(t)$, then the response to the derivative of $f(t)$, that is $f_1(t) = df/dt$ is

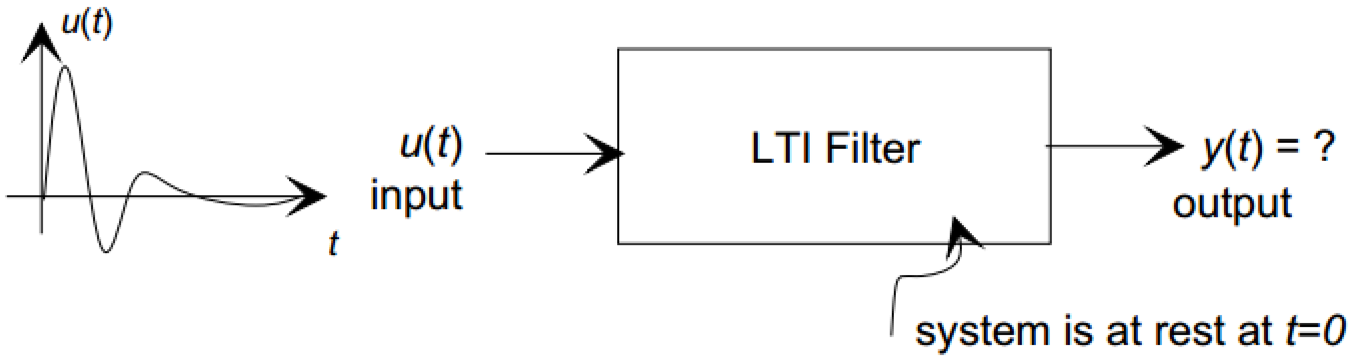
$$y_1(t) = \frac{dy}{dt}.$$

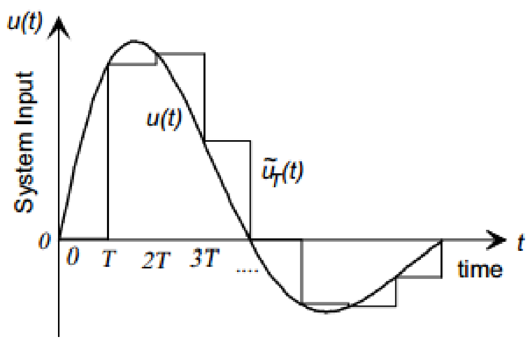
(c) **The Integration Property** If the response to input $f(t)$ is $y(t)$, then the response to the integral of $f(t)$, that is $f_1(t) = \int_{-\infty}^{t} f(t)dt$ is

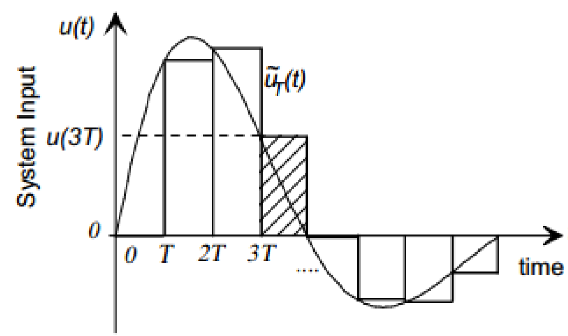
$$y_1(t) = \int_{-\infty}^{t} y(t)dt.$$

LTI: Linear Time-Invariant

Continuous LTI System Time-Domain Response







$$\tilde{u}_T(t) = u(nT) \qquad \text{for } nT \le t < (n+1)T.$$

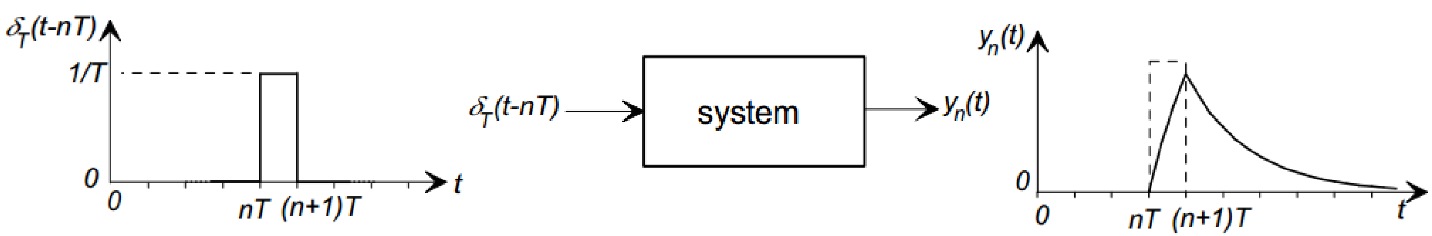
The approximation $\tilde{u}_T(t)$ is written as a superposition of non-overlapping pulses

$$\tilde{u}_T(t) = \sum_{n=-\infty}^{\infty} p_n(t)$$

where

$$p_n(t) = \begin{cases} u(nT) & nT \le t < (n+1)T \\ 0 & \text{otherwise} \end{cases}$$

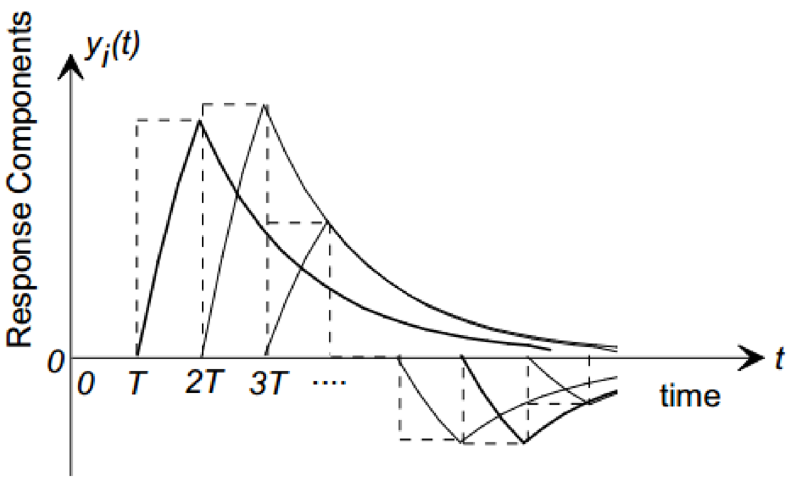
$$y_n(t) = h_T(t - nT)$$

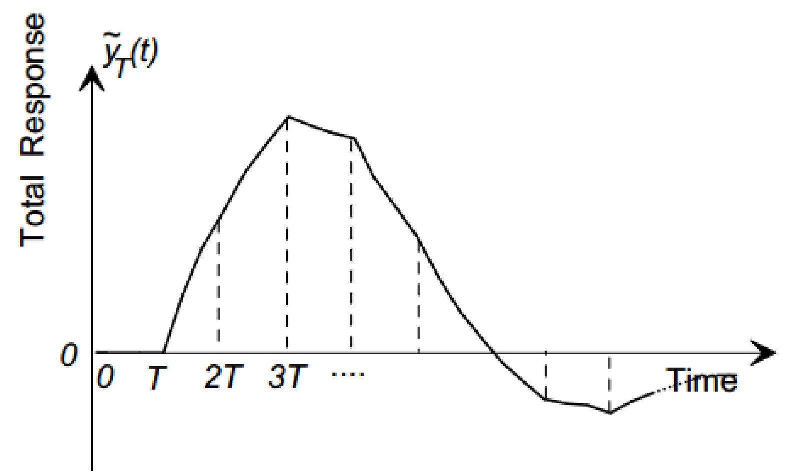


$$\delta_T(t) = \begin{cases} 0 & \text{for } t \le 0 \\ 1/T & 0 < t \le T \\ 0 & \text{for } t > T. \end{cases}$$

System Response

$$\tilde{y}_T(t) = \sum_{n=-\infty}^{\infty} u(nT)h_T(t - nT)T$$



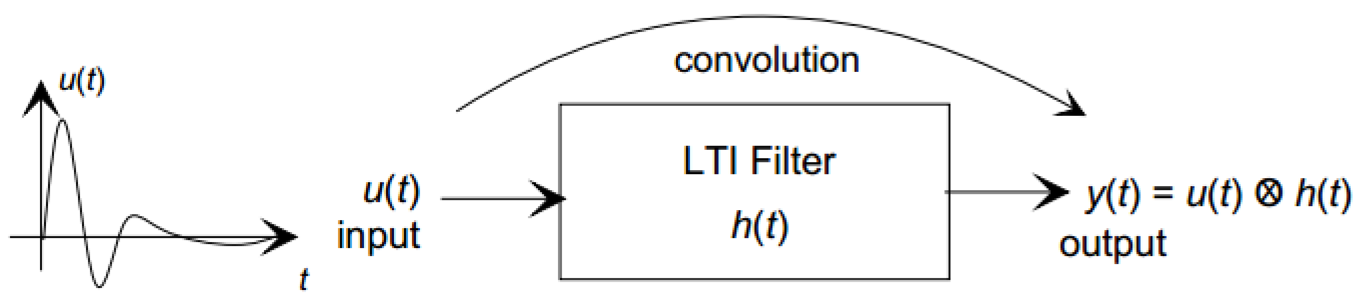


- We now let the pulse width T become very small, and write $nT = \tau$, $T = d\tau$, and note that $\lim_{T\to 0} \delta_T(t) = \delta(t)$. As $T \to 0$ the summation becomes an integral and

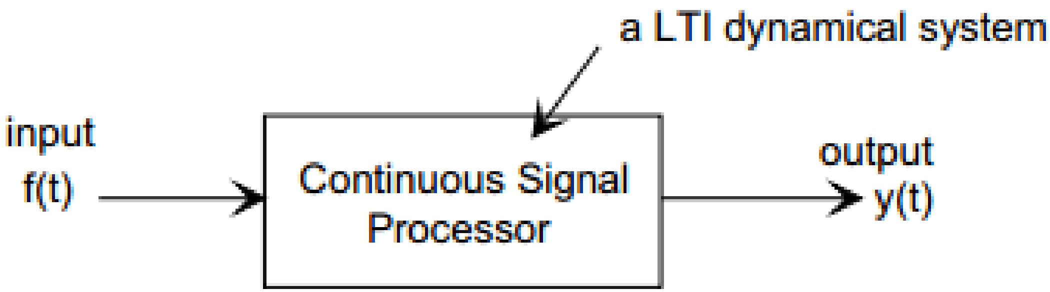
$$\begin{aligned} y(t) &= \lim_{T\to 0} \sum_{n=-\infty}^{N} u(nT) h_T(t - nT) T \\ &= \int_{-\infty}^{t} u(\tau) h(t-\tau) d\tau \end{aligned} \tag{1}$$

where $h(t)$ is defined to be the system *impulse response*,

$$h(t) = \lim_{T\to 0} h_T(t).$$



Convolution



$y(t) = f(t) * h(t)$

h(t) : impulse response of a system

Fourier Transform

- Taylor series represents any function using polynomials.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots.$$

- Polynomials are not the best.

http://www.cs.cmu.edu/afs/cs/academic/class/15385-s12/www/lec_slides/lec-4.ppt

Joseph Fourier

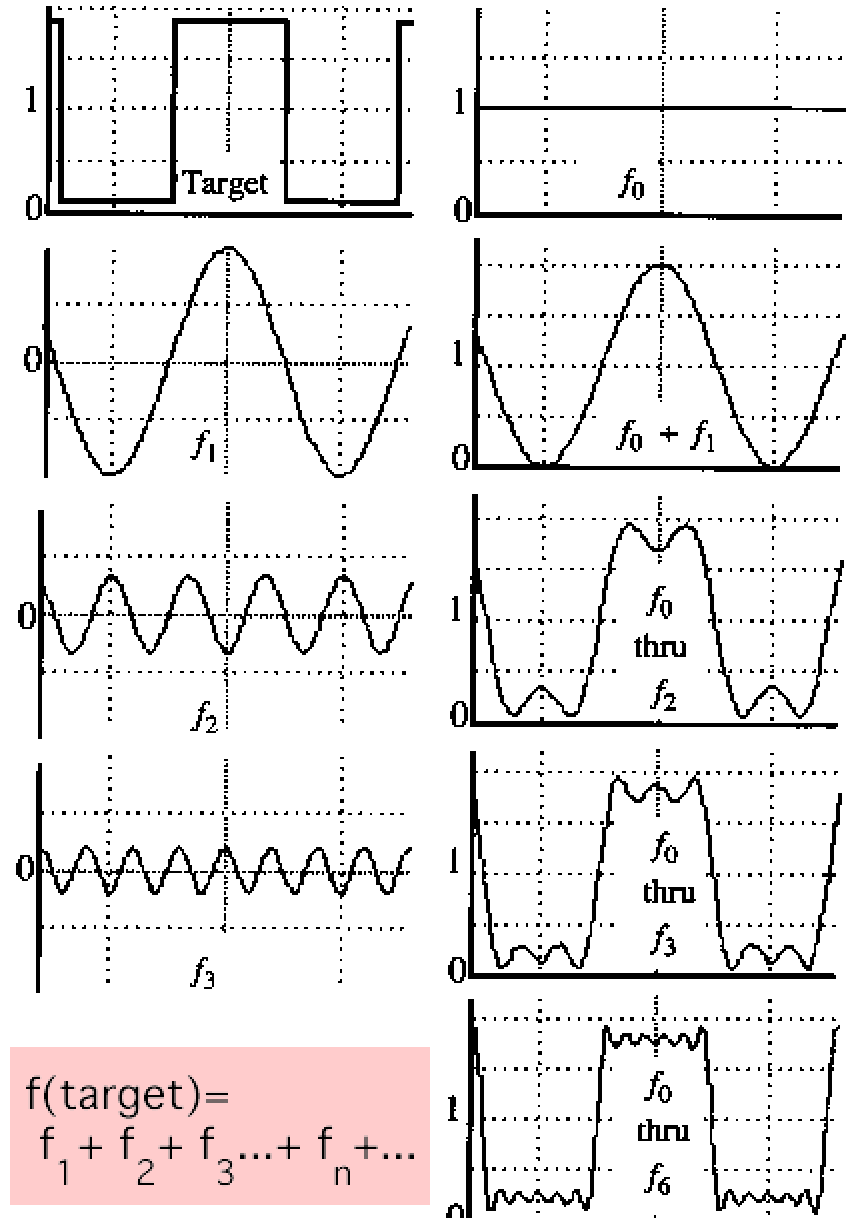
Joseph's father was a tailor in Auxerre
Joseph was the ninth of twelve children
His mother died when he was nine and his father died the following year

Fourier demonstrated talent on math at the age of 14.
In 1787 Fourier decided to train for the priesthood - a religious life or a mathematical life?
In 1793, Fourier joined the local Revolutionary Committee

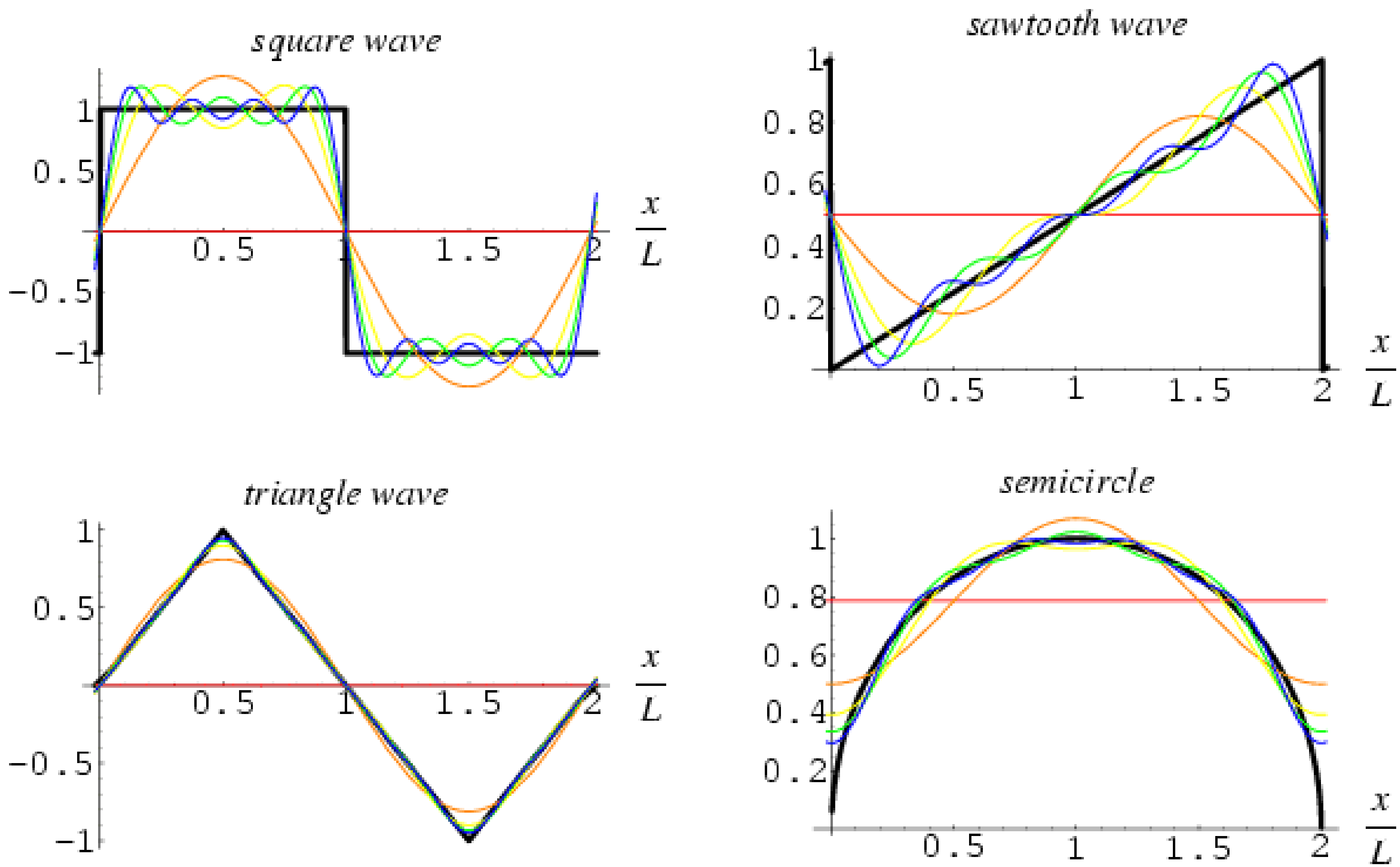
Born: 21 March 1768 in Auxerre, Bourgogne, France
Died: 16 May 1830 in Paris, France

Had crazy idea (1807):
Any periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies.

Examples



f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

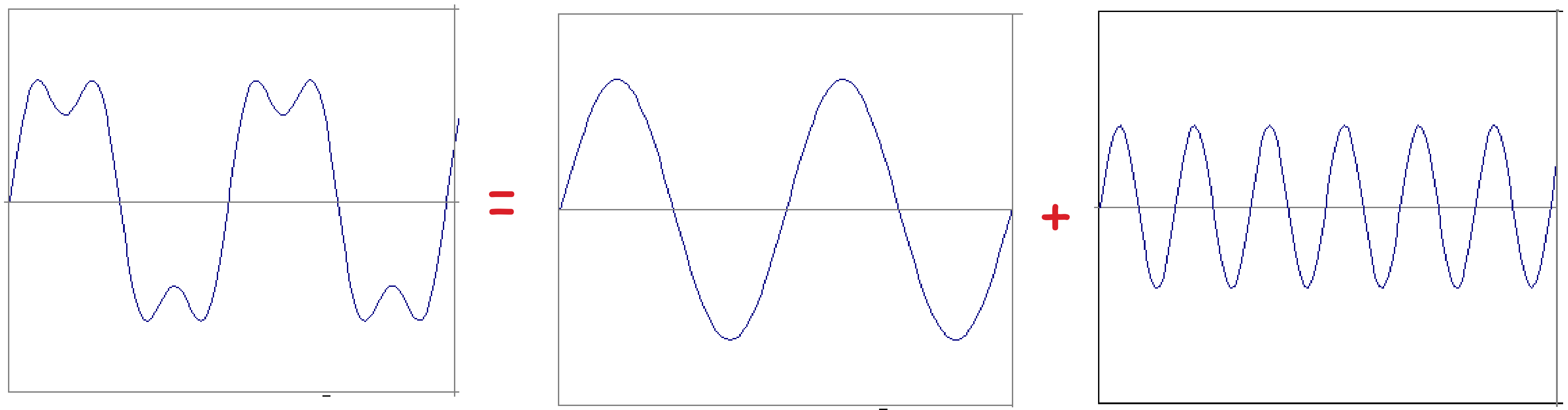

square wave
sawtooth wave
triangle wave
semicircle
x
L
0.5
1
1.5
2
-0.5
-1
0.2
0.4
0.6
0.8

Fourier Transform

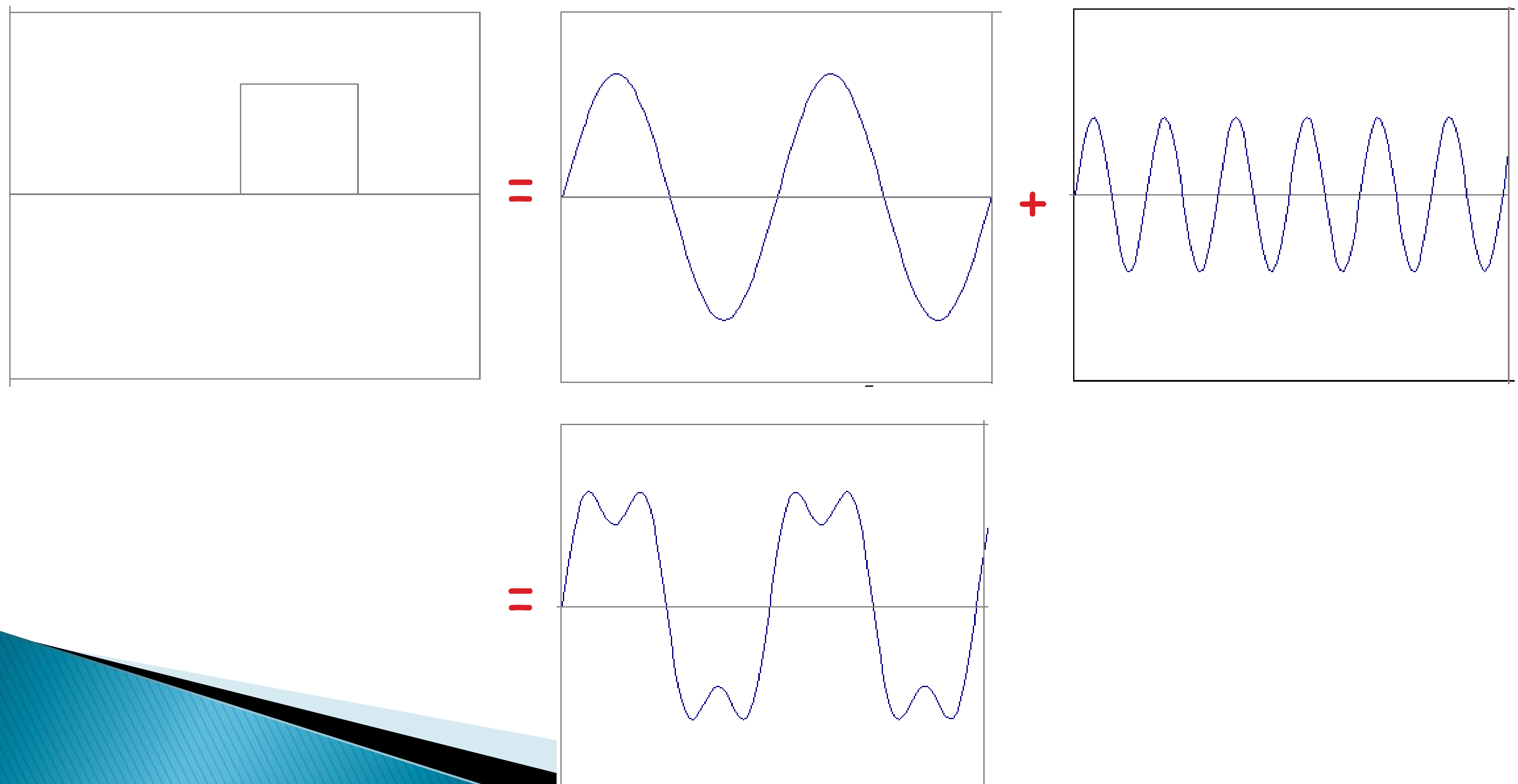
- We want to understand the frequency ω of our signal. So, let's reparametrize the signal by ω instead of x:

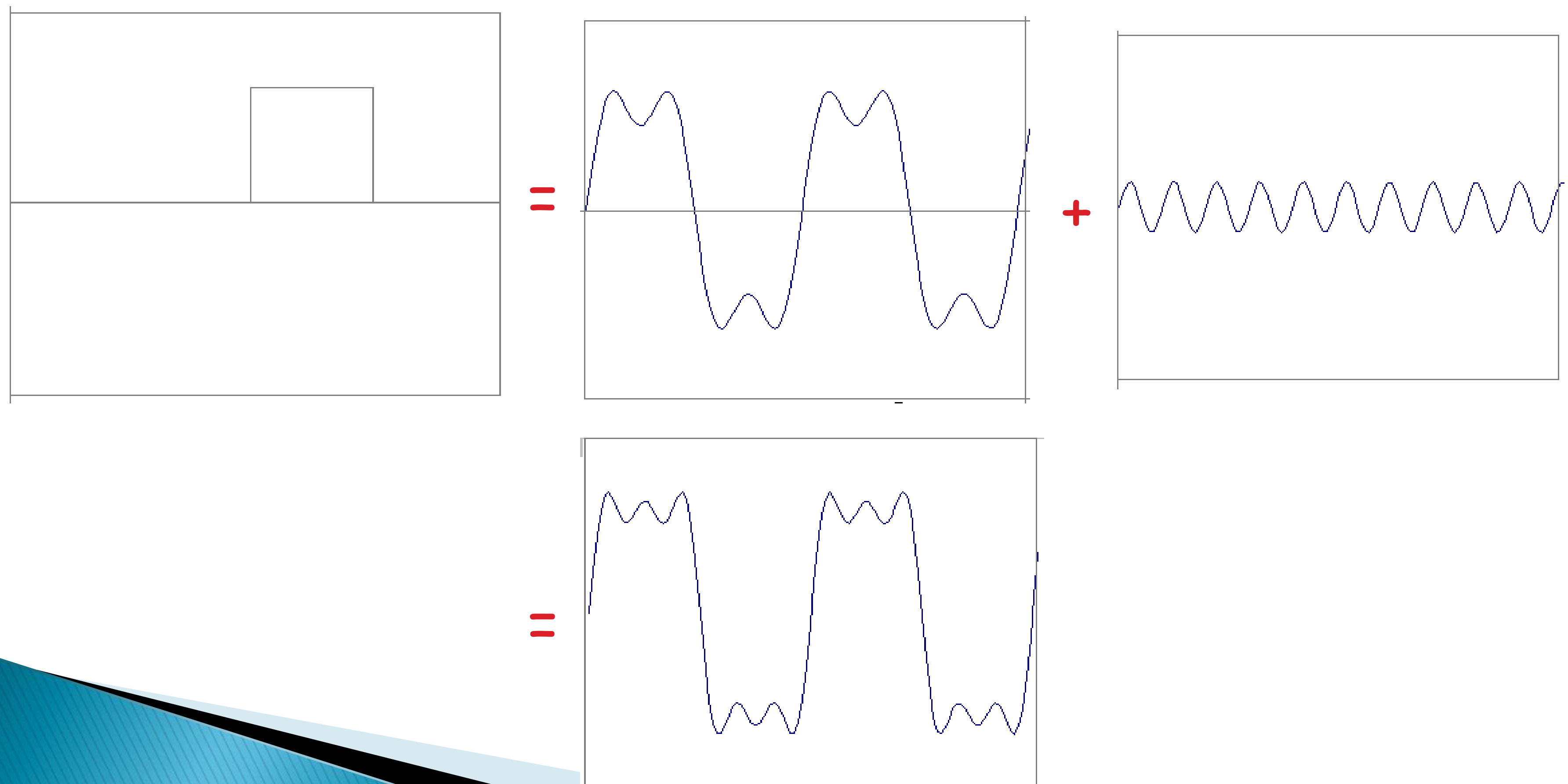
$$f(x) \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow F(\omega)$$

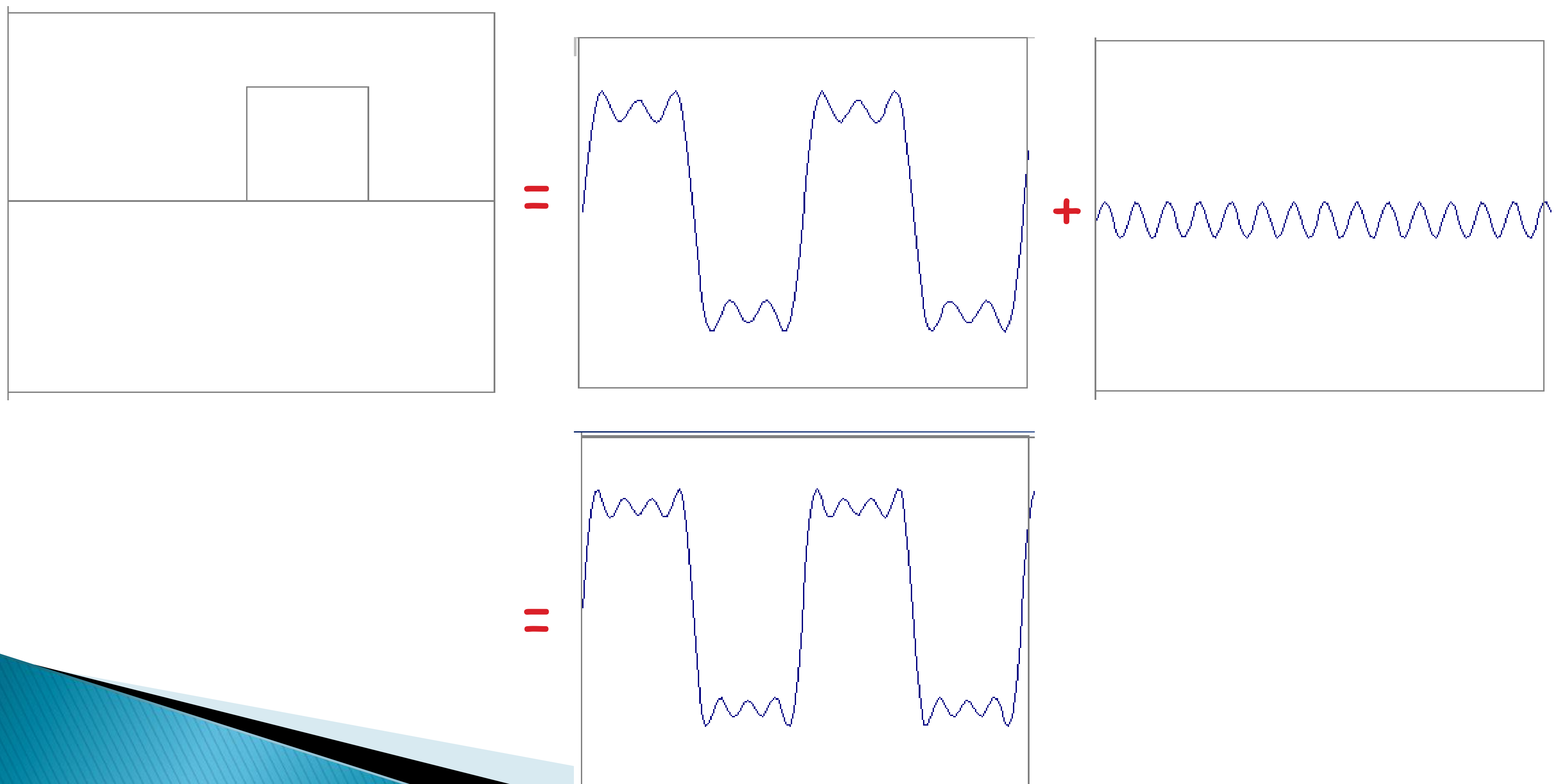
Example #1

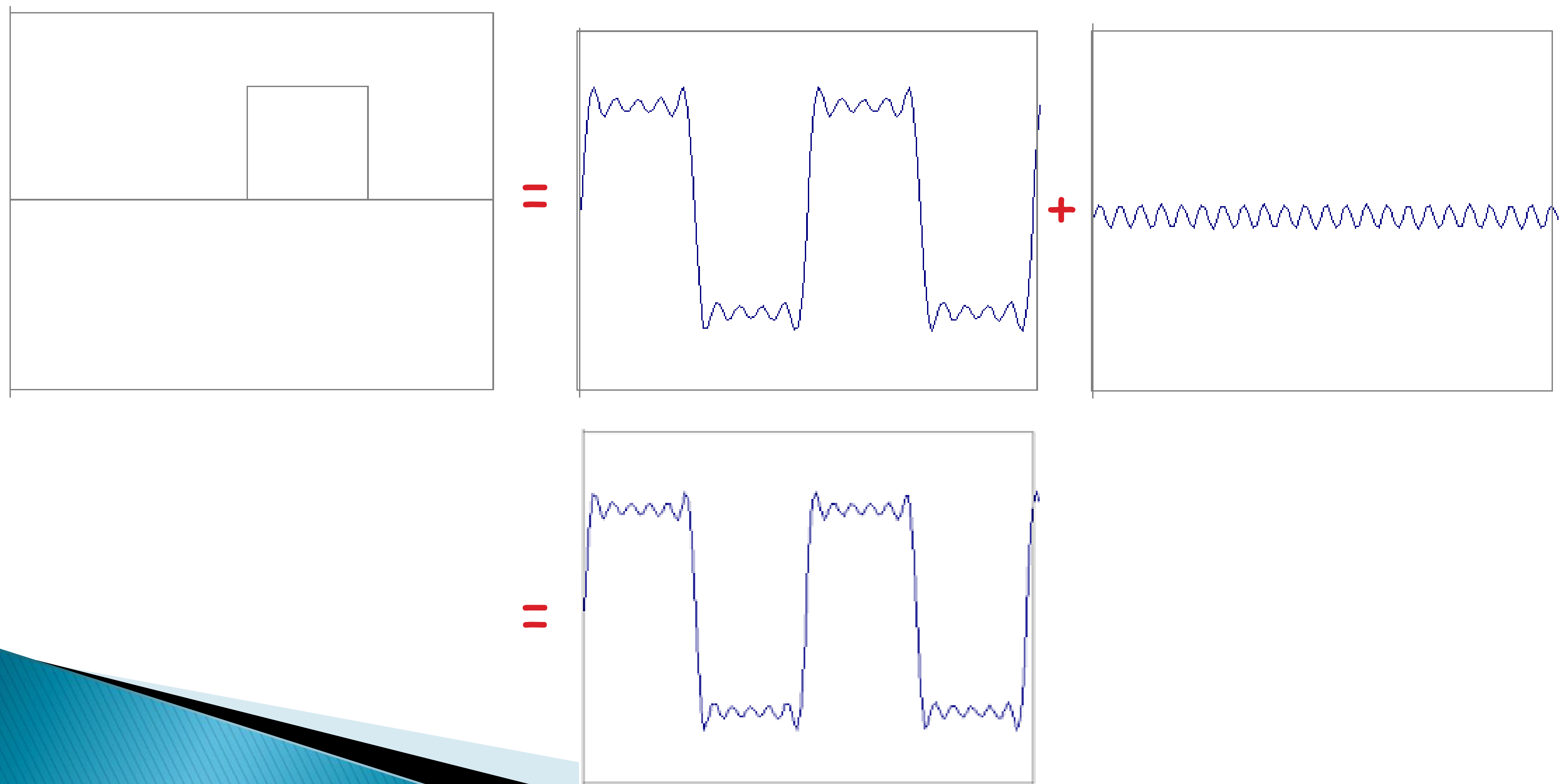


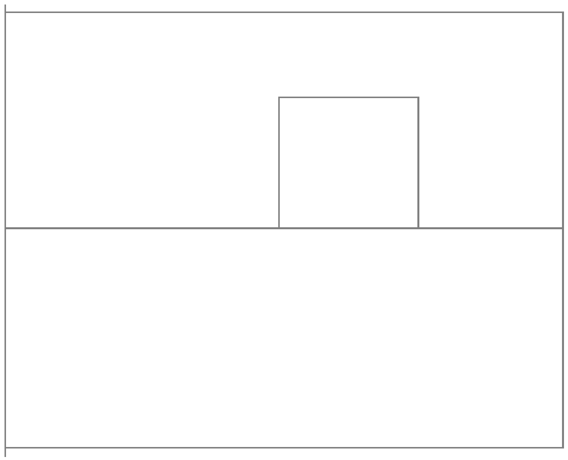
Example #2



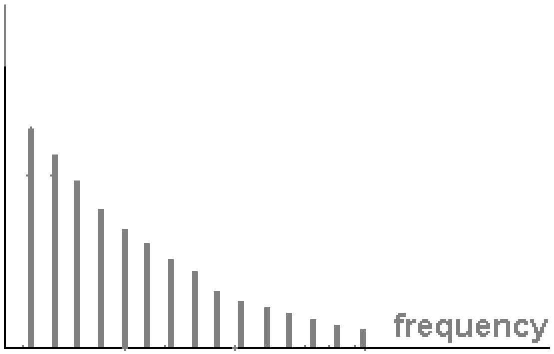








$$= \boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$



Fourier Transform

- Fourier transform

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx$$

$$e^{ik} = \cos k + i \sin k \qquad i = \sqrt{-1}$$

- Inverse Fourier Transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{iux} dx$$

Fourier Transform of Convolution

$$g = f * h$$

$$G(u) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi ux} dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau) h(x-\tau) e^{-i2\pi ux} d\tau dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \left[h(x-\tau) e^{-i2\pi u(x-\tau)} dx \right]$$

$$= \int_{-\infty}^{\infty} \left[f(\tau) e^{-i2\pi u\tau} d\tau \right] \int_{-\infty}^{\infty} \left[h(x') e^{-i2\pi ux'} dx' \right]$$

$$= F(u) H(u)$$

Fourier Transform of Convolution

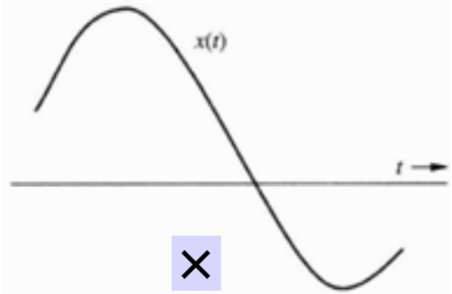
Spatial Domain *(x)*		Frequency Domain *(u)*
$g = f * h$	$\longleftrightarrow$	$G = FH$
$g = fh$	$\longleftrightarrow$	$G = F * H$

So, we can find *g(x)* by Fourier transform

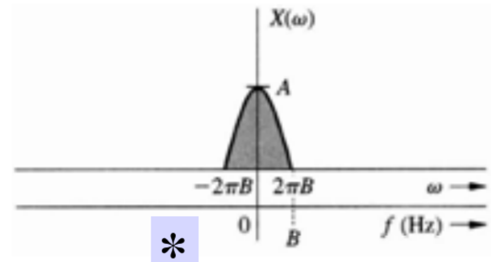
$$
\begin{tikzcd}
g & = & f \arrow[d, "\textbf{FT}"] & * & h \arrow[d, "\textbf{FT}"] \\
G \arrow[u, "\textbf{IFT}"] & = & F & \times & H
\end{tikzcd}
$$

Sampling Theorem (Intuitive)

- Consider a bandlimited signal x(t) and is spectrum X(ω):



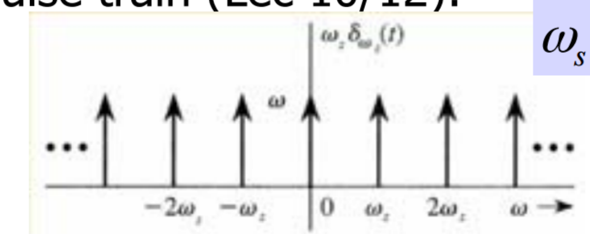
$\Leftrightarrow$



$\times$ $*$

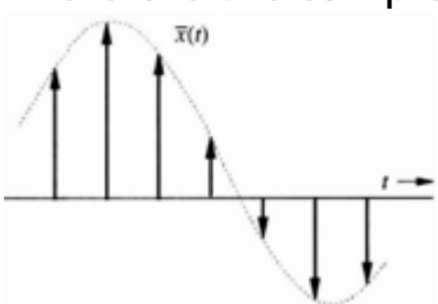
- Ideal sampling = multiply x(t) with impulse train (Lec 10/12):

$\Leftrightarrow$

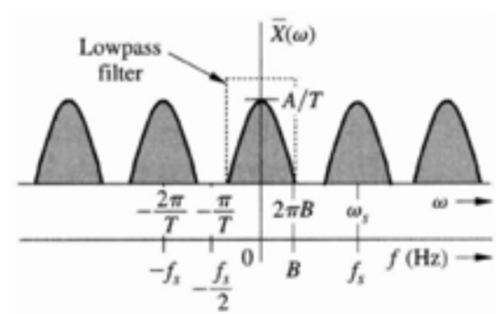


$$\omega_s = 2\pi / T_s$$

- Therefore the sampled signal has a spectrum:



$\Leftrightarrow$

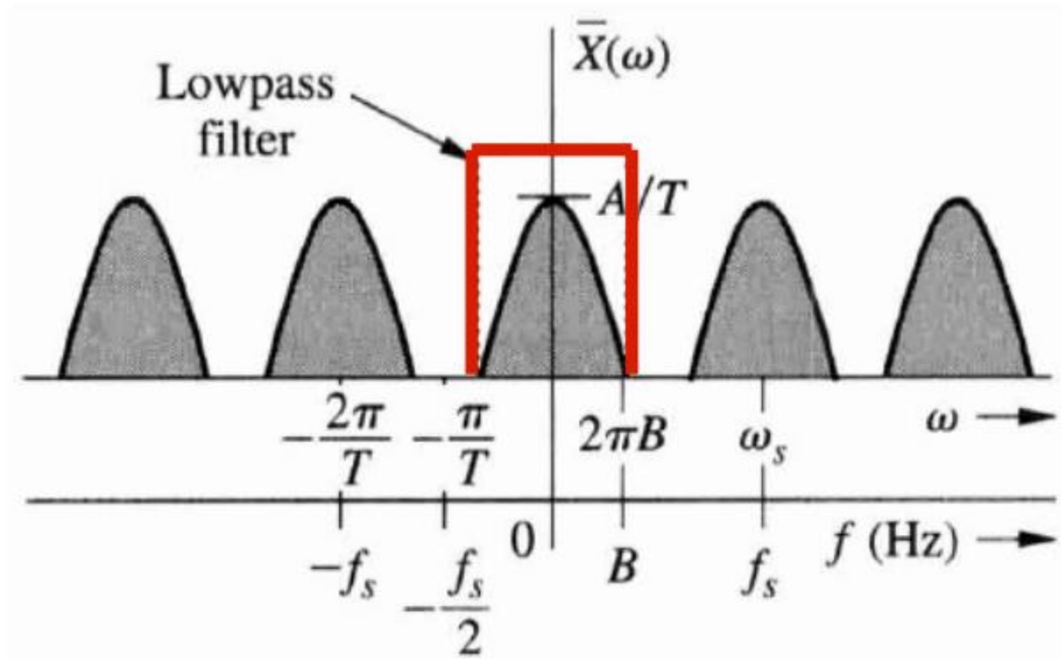


L8.1 p771

http://www.ee.ic.ac.uk/pcheung/teaching/ee2_signals/Lecture%2013%20-%20Sampling%20&%20discrete%20signals.pdf

Sampling Theorem (Intuitive)

- Therefore, to reconstruct the original signal x(t), we can use an ideal lowpass filter on the sampled spectrum:



Sampling Theorem

- The sampled version can be expressed as:

$$\bar{x}(t) = x(t)\delta_{Ts}(t) = \sum_n x(nT_s)\delta(t - nTs)$$

- We can express the impulse train as a Fourier series:

$$\delta_{Ts}(t) = \frac{1}{Ts}[1 + 2\cos\omega_s t + 2\cos 2\omega_s t + ...] \quad \text{where} \quad \omega_s = 2\pi / Ts$$

- Therefore:

$$\bar{x}(t) = \frac{1}{Ts}[x(t) + 2x(t)\cos\omega_s t + 2x(t)\cos 2\omega_s t + ...]$$

- Since $2x(t)\cos\omega_s t \quad \Leftrightarrow \quad X(\omega-\omega_s) + X(\omega+\omega_s)$

$$\bar{X}(\omega) = \frac{1}{Ts}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

- Which is essentially the spectrum shown in the previous slide.