Deterministic Reallocation Algorithm for Evenly Weighted Particle Filter

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This paper deals with the particle degeneracy problem by reallocating particles in a deterministic fashion. The purpose of this study is to replace the resampling process as a whole or in part. By considering the evenness of weights and spatial occupancy of particles over the state space simultaneously, the resultant ensemble becomes a better representation of the posterior distribution in terms of the weight variance, the heaviness of the tail, and thus the robustness against the unmodelled system. The proposed algorithm is tested using a 2D tracking problem.

Key Words: Particle Filter, Degeneracy, Reallocation

Nomenclature

x	:	each particle
w	:	weight of a particle
y	:	measurement
q	:	proposal distribution
N	:	the number of particles
Subscripts		
k	:	time step
Superscripts		
$[i]$	:	index of a particle

1. Introduction

From the navigation of aerial vehicles to the general tracking problems in the field of aerospace engineering, the particle filter (PF) which performs Bayesian statistical inference in a Monte Carlo way is widely adopted. Especially for the nonlinear and non-Gaussian problems, PF shows its true value. Moreover, It is indeed the advantage of the PF that its applicable problems have very little, or almost no requirements to be satisfied. It is also worth mentioning that convergence of the PF has a good scalability with the dimension of the problem.[1)] Nevertheless, required computation cost is traded of for the performance and the universality.

The state space is occupied with a finite number of samples and their corresponding weights in representing the probability distribution. Then, an expectation of a certain function $g(x)$ over the specific density, i.e., posterior, can be approximated as

$$\int_{X_k} g(x_k)dx_k \approx \sum_{i=1}^{N} w_k^{[i]} g(x_k^{[i]}) \tag{1}$$

where $x_k^{[i]}$ denotes i^{th} particle of k^{th} time step. Meanwhile, stochastic sampling and weighting of the particles according to the system dynamics and measurement model complete the filtering recursion as

$$\begin{aligned} p(x_{k+1}|y_{1:k}) &= \int_{X_k} p(x_{k+1}|x_k)p(x_k|y_{1:k})dx_k, \\ p(x_k|y_{1:k}) &= \frac{p(y_k|x_k)p(x_k|y_{1:k-1})}{\int_{X_k} p(y_k|x_k)p(x_k|y_{1:k-1})dx_k} \end{aligned} \tag{2}$$

where the latter equation is represented in terms of weight recursion as

$$w_k^{[i]} \propto w_{k-1}^{[i]} \frac{p(y_k|x_k^{[i]})p(x_k^{[i]}|x_{k-1}^{[i]})}{q(x_k^{[i]}|x_{k-1}^{[i]}, y_k)} \tag{3}$$

in association with the proposal density q of our choosing.

One of the critical pitfalls in the naïve application of the PF, however, is the loss of particle diversity. It should be the natural consequence of keep multiplying likelihoods to the discretized samples as in Eq. (3). Within a few update steps, weights of all but a couple of the particles are nearly zeroed out and associated particles become ineffective. The filter immediately diverges at the case. This phenomenon is easily observed in the PF of sequential importance sampling (SIS) setup.[4)] It becomes more dramatic when the nonlinearity of the system gets more severe, or when the system has a peaky likelihood compared to the process noise.[1)] The measurements corrupted with the extreme outliers also contribute the imbalance. The improper choice of proposal density also contributes to the situation.[2,3)] All of the mentioned cases are present in the real-world applications and raise practical issues.

Such the problem, which is commonly referred to as a degeneration, is remedied first by a sensible choice of proposal distribution and further refined by the resampling process. The choice of proposal density, as shown in Eq. (3), however, should depends upon the characteristics of the problem which involves a detailed analysis of the system. Acquiring a good proposal density in general is not an easy task and has no universally well-behaving solution. Therefore, prior sampling is a typical choice. Although there is a notion of the optimal proposal distribution,[5,6)] realization of the optimal proposal distribution is either infeasible or impractical unless the system is both linear and Gaussian.

Meanwhile, the core of the resampling process is to duplicate particles of high weights while removing the low-weighted ones in a stochastic fashion. The principle of the inverse transform sampling enables the PF to resampling equally weighted particles. However, the resultant particles are often arbitrary. More importantly, spatial importance of particles is ignored as illustrated in Ref. 7). Despite the fact that it is important to take significant care for maintaining heavy tails of the distribution when the target system is highly nonlinear and exhibits non-Gaussian characteristics, the standard resampling process tends to eliminate particles located at the tail of the distribution that may potentially be influential and important in subsequent steps. Finally, resampling of an already degenerated ensemble is nothing but a replica of few dominant particles and only accelerates the lack of diversity. Therefore, a process that can restore degenerated condition and/or a condition that is highly likely to degenerate back to the healthy condition, is required other than the resampling or at least the one that works in conjunction with the resampling process. In this study we call evenly weighted situation as healthy, which can be easily identified by a variance of particle weights and indicates how much each particle weight is deviated from its the most even value.

The resampling is triggered by monitoring the effective sample size (ESS) given as,[1,8]

$$\begin{aligned} \mathrm{ESS} &= \frac{N}{1 + N^2 \mathrm{Var}(w_k)} \\ &\approx \frac{1}{\sum_{i=1}^{N} \left(w_k^{[i]}\right)^2}. \end{aligned} \tag{4}$$

When the value of Eq. (4) is less than a certain value, e.g., 3/4, the resampling is triggered. The readers are referred to Ref. 10) for the details of resampling process.

A deterministic resampling scheme as shown in Ref. 7) which considers the spatial density of weights provides a fair remedy, yet its computation time grows exponentially with respect to the system dimension and it involves determination of more than three user parameters based upon which the performance varies a lot. This stduy, as a variant of particle distribution optimization (PDO) described in Ref. 9) and deterministic resampling, seeks an effective technical alternative to the conventional resampling process that can enhance the diversity of particles and the longevity of the filter by carefully managing the location and thus the weight of particles. Specifically, a deterministic reallocation algorithm which considers both the spatial density of weights and the evenness of weight distribution is suggested. The internal interaction model acting between each pair of particles that mutually pushes or pull each other is proposed. The effectiveness of the proposed method is demonstrated using a 2D radar tracking example.

2. Particle Reallocation Algorithm

In the proposed method each particle is classified as, for instance, either positive or negative charge and force-like interaction among particles automatically regulate their locations based on both their state values and weights. What immediately follows is the re-weighting of particles based on the kernel regression approach so that the resultant ensemble is equivalent to the posterior distribution, which should be the result of Eq. (2). At the same time, the authors also intended to make the algorithm less dependent on the handcrafted tuning of parameters.

The suggesting deterministic algorithmic is twofold: relocation and re-weighting. As mentioned, the former process is realized by mutual repulsion or attraction considering each particle as an individual charge, or a mass capable of possessing negative value. Then the authors model the interaction between pairs of the particles that can automatically and deterministically relocate the particle distributions. Note that when the particle shifts, its associated weight must be adjusted too.

The critical question of the realization should be how to determine the magnitude of each repulsion or attraction.

2.1. Relocating particles

This subsection describes the proposed model of repulsion (or attraction) between pairs of particles. In replacement of resampling $\{x_k^{[i]}, w_k^{[i]}\}_{i=1}^N$ into equal-weight ensemble $\{x'^{[i]}_k, \frac{1}{N}\}_{i=1}^N$, each particle is shifted as much as $\delta x_k^{[i]}$ so that

$$x'^{[i]}_k = x_k^{[i]} + \delta x_k^{[i]} \tag{5}$$

where the deviation term is modelled as

$$\delta x_k^{[i]} = \alpha_k \sum_{\substack{j=1 \\ j\neq i}}^{N} g\left(x_k^{[i]}, x_k^{[j]}\right) f\left(x_k^{[i]}, x_k^{[j]}\right). \tag{6}$$

Here, α_k denotes a gain, $g(\cdot,\cdot)$ denotes a gating function given as

$$g\left(x_k^{[i]}, x_k^{[j]}\right) = \begin{cases} 1, & \text{if } \left|h(x_k^{[i]}) - h(x_k^{[j]})\right| > \epsilon \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

and $f(\cdot,\cdot)$ denotes a force-like interaction between each pair of particles given as

$$f\left(x_k^{[i]}, x_k^{[j]}\right) = -C(w_k^{[j]}) \frac{x_k^{[j]} - x_k^{[i]}}{\left\|x_k^{[j]} - x_k^{[i]}\right\|^3}. \tag{8}$$

Note that the function is in a form somewhat analogous to the Couloumb's law or the Newton's law of universal gravitation except that the self-weight is eliminated. Here, $C(\cdot)$ denotes a function that determines whether each particle should be classified as negative or positive. The function is given as

$$C(w_k^{[j]}) = \begin{cases} w_k^{[j]}, & \text{if } w_k^{[j]} > \mathrm{median}[w_k] \\ -w_k^{[j]}, & \text{otherwise.} \end{cases} \tag{9}$$

This kind of mutual forcing between particles can deterministically relocate the entire particles every time step. The gating function prevents overly reallocating the ensemble by approving $f(\cdot,\cdot)$ only for sufficiently distinguishable pairs.

2.2. Re-weighting of relocated particles

This subsection describes re-weighting scheme of the proposed method. This step is to reassign the importance of each particle based on the modified state value of the particle. The kernel density estimation technique based on the regularized particle filter[11] is utilized. One can approximate the continuous probability density function of the posterior out of a set of particles and weights as

$$p(x_k) \approx \sum_{i=1}^{N} w_k^{[i]} K\left(||x_k - x_k^{[i]}||\right) \tag{10}$$

where $K(\cdot)$ denotes a statistic kernel function. Typical examples of the kernel function includes the Gaussian kernel, and Epanechnikov (quadratic) kernel. Also, a bandwidthed kernel of the form

$$K_h(x) = \frac{1}{h^d} K\left(\frac{x}{h}\right) \tag{11}$$

is a valid kernel as long as the given K is a valid one for positive h. Here, d is the dimension of the problem of interest.

After each particle is relocated using Eq. (5), reassigned weight is calculated as

$$w'^{[i]}_k = p(x'^{[i]}_k) \tag{12}$$

using Eq. (10) so that the resultant particle set is equivalent to the original posterior. Then, the normalization follows immediately. Putting all together, the resampling process is replaced with the proposed algorithm that yields a new ensemble $\{x'^{[i]}_k, w'^{[i]}_k\}_{i=1}^N$ where each element is calculated as Eq. (5) and Eq. (12).

3. Experiments

The proposed method is tested using a toy problem of tracking a maneuvering target where the maneuver is not informed to the filter. Therefore, when the prediction model of the filter cannot capture the boundaries of the target maneuver it may diverge. The target is simulated to have an accelerating turn rate, where the filters do not aware of the fact. The constant velocity (CV) model given as

$$\begin{aligned} x_k &= \begin{bmatrix} \mathrm{x}_k \\ \mathrm{y}_k \\ V_k \\ \psi_k \end{bmatrix} \\ x_{k+1} &= x_k + \begin{bmatrix} V_k \cos(\psi_k) \\ V_k \cos(\psi_k) \\ 0 \\ 0 \end{bmatrix} + v_k \end{aligned} \tag{13}$$

is utilized in this study. Here, x, y denote 2D position of the target, and V, ψ denote the velocity and heading angle of the target. Note that the maneuvering target of nonzero turn rate The radar measurement is assumed so that range and look angle to the target given as

$$\begin{aligned} y_k &= h(x_k) + w_k \\ &= \begin{bmatrix} \sqrt{\mathrm{x}_k^2 + \mathrm{y}_k^2} \\ \tan^{-1} \frac{\mathrm{y}_k}{\mathrm{x}_k} \end{bmatrix} + w_k \end{aligned} \tag{14}$$

is measured.

Figure 1 illustrates the numerical results of tracking a maneuvering target where $h^{-1}(y)$ denotes the inverse relationship of Eq. (14). One can clearly see that when the maneuver exceeds the predetermined uncertainty of CV model, the standard PF struggles correctly tracking the target. Meanwhile, the PF driven by the proposed algorithm can converge to the true target location even at the same situation. Note that the proposed method does not undergo the resampling process.

This is due to the wide occupancy of the state space thanks to Eq. (5) and thus the maintenance of heavy tail. Moreover, correctly weighting them according to Eq. (12) also contributes to the result.

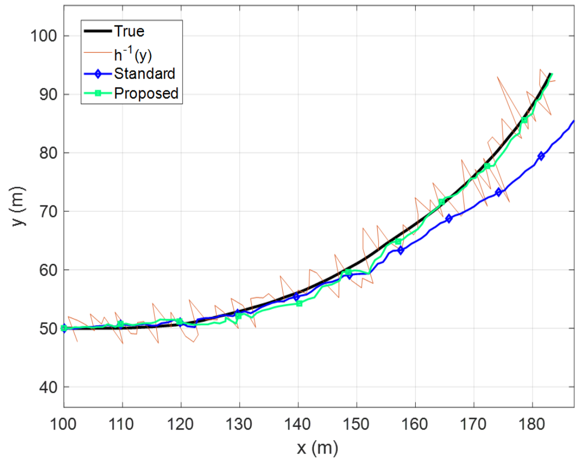

Fig. 1. Simulation result of tracking a maneuvering target.

4. Conclusion

This paper proposed a deterministic reallocation algorithm for evenly weighted particle filter. The algorithm is composed of two steps which are relocation and re-weighting. Numerical results of 2D tracking problem shows that this approach can effectively replace the resampling process. Moreover, maintaining heavy tails of the distribution enables the filter to be robust to the unmodelled information.

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