Abstract

The report evaluates results of Heston stochastic volatility model (1993) and its extensions, assuming that the model follows a random process to measure volatility of the asset. Heston model is an augmentation of renowned Black & Scholes methodology that incorporates an important case including leverage effect, mean reversion property of volatility and non-lognormal distributions of stocks. The stochastic variance has been modelled as a mean reverting process, further Euler discretization scheme has been carried out such that negative variances issue is removed.Four Schemes have been used to investigate the minimum positive bias while pricing the European options. The full truncation scheme has produced best results providing lowest bias and root-mean-squared error volatility model.

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Introduction

Heston Stochastic Volatility model works on the assumption that variance of the stock is neither constant nor deterministic, but a random walk process. Volatility clustering and asset return distribution that are fat tailed and highly peaked phenomena are taken into account by the model; volatility follows mean reverting process indicates extremely high or low stochastic variance periods do not sustain for a long time.

The Model

The Heston Volatility model is discussed and computation of stock prices method is also specified. The parameters of the model are following:

S(t): Asset Spot Price
V(t): Variance
C: European call option price.
K: Strike price.
$W_{1,2}$: Standard Brownian movements.
r: Interest rate.
κ: Mean reversion rate.
θ: Long run variance.
V_0: Initial variance.
σ Volatility of variance.
ρ Correlation parameter.
t_0 Current date.
T Maturity date.

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t} S_t dW_t^1 \\ dV_t &= -\kappa(V_t - \theta)dt + \omega\sqrt{V_t} dW_t^2 \\ dW_t^1 dW_t^2 &= \rho dt \end{aligned} \tag{1}$$

The Stochastic process dW_1, dW_2 are correlated with each other, therefore leverage effect can be investigated. The variance of stochastic model dV_t is linked with the square-root process of Feller (1951). The square-root process dV_t, it variance is always positive and if $2\kappa\theta > \sigma^2$, then it cannot approach zero. Notice the deterministic part of dV_t is stable asymptotically if κ >0, further the equilibrium is obtained at point $V_t = \theta$.

The call price is of the form:

$$c_T(K) = e^{-rT}E[(S_T - K)^+] \tag{2}$$

Euler schemes for Heston model

There are four Euler schemes for Monte Carlo simulation under constant elasticity of variance (CEV) stochastic volatility models; absorption, reflection, partial truncation and full truncation schemes. The four Euler discretizations for the CEV-SV are unified under one framework such that:

$$\hat{V}(t + \Delta t) = f_1(\hat{V}(t)) - \kappa\Delta t(f_2(\hat{V}(t) - \bar{V}) + \omega f_3(\hat{V}(t))^{1/2}\Delta W_v(t) \tag{3}$$

$$V(t + \Delta t) = f_3(\hat{V}(t + \Delta t)) \tag{4}$$

where $\hat{V}(0) = V(0)$ and the functions f_i, 1 through 3 have to satisfy:

- $f_i(x) = x$ for x $\geq$ ‘ 0, and i=1,2,3.
- $f_i(x) \geq x$ for x in R, and i=1,3.

For the asset price we switch to logarithms, as in Andersen and Brotherton-Ratcliffe (2005). This guarantees nonnegativity:

$$lnS(t + \Delta t) = lnS(t) + [\mu - \frac{1}{2}S(t^{2(\beta-1)})V(t)]\Delta t + S(t)^{\beta-1}\sqrt{V(t)}\Delta W_s(t) \tag{5}$$

and automatically ensures that the first moment of the asset is matched exactly. In an implementation of $lnS(t + \Delta t)$ one would use the Cholesky decomposition to arrive at $\Delta W_S(t) = \rho\Delta W_v(t) + \sqrt{1-\rho^2}\Delta Z(t)$, with Z(t) independent of $W_V(t)$.

Table 1: Description of four Euler schemes

Scheme	Paper	$f_1(x)$	$f_2(x)$	$f_3(x)$
Absorption	Unknown	x^+	x^+	x^+
Reflection	Diop (2003), Bossy and Diop (2004),	$\mid x \mid$	$\mid x \mid$	$\mid x \mid$
Partial truncation	Deelstra and Delbaen (1998)	x	x	x^+
Full truncation	Lord, Koekkoek and Dijk (2010)	x	x^+	x^+

Numerical Results

Estimation Accuracy

The table 2 draws a comparison among absorption, reflection, partial truncation and full truncation schemes to price the European call option based on given parameters. The table clearly indicates the biases of all schemes for European Call, the accuracy of the estimates is evaluated by conducting Monte Carlo Simulation. The computation involves three different simulation

involving simulation paths; 10,000, 40,000 and 160,000 repeating 100 times. The lower the bias, the more accurate is the scheme therefore if the bias is not significantly different from zero at 95% confidence interval then it significant hence denoted by bold text in the table.
After exterminating the four schemes, there has been a significant difference in the magnitude of the bias validating their accuracies in estimating the option price by incorporating a random volatility variable. The full truncation scheme only produced desirable results, since its bias in all the simulations is close to zero hence providing most accurate estimations. Whereas, Reflection scheme has the highest bias as such it is least preferred model for estimation purposes. An important similarity among all schemes is that as the number of simulations increased, the bias reduced which is relevant to Monte Carlo theory. Nevertheless, the full truncation method presented Roger Lord , Remmert Koekkoek & Dick Van Dijk (2010) is the most accurate and stable method.

Table 2: Bias, Standard Error, RMSE and CPU Time Estimation Estimation:
Pricing an European call in Heston Model : S(0)=100, r=0.05, Maturity 5 yrs. True option price: 34.9998.

Method	Paths	10,000	40,000	160,000
	Steps/year	20	40	80
Absorption scheme	Bias	1.9298	1.6171	1.1147
	Standard error	0.6127	0.1267	0.1374
	RMSE	2.0247	1.6221	1.1231
	Run time	7.093	53.130	567.15
Reflection scheme	Bias	4.3482	3.0645	2.3882
	Standard error	0.6888	0.3979	0.1338
	RMSE	4.4024	3.0902	2.392
	Run time	6.970	53.024	576.101
Partial truncation scheme	Bias	0.4407	0.2366	0.1190
	Standard error	0.3872	0.1838	0.10192
	RMSE	0.5866	0.2996	0.15668
	Run time	6.837	52.533	619.11
Full truncation scheme	Bias	0.0546	**0.0375**	**0.0308**
	Standard error	0.5011	0.2771	0.1616
	RMSE	0.5041	0.2796	0.1645
	Run time	6.888	53.543	579.376

Greeks Assessment

The Greeks in finance enable in evaluation of sensitivity in derivative prices, such as the intensity of impact on portfolio of financial options or assets is dependent on the change in parameters of the option. These Greeks are utilized to assess risk measures, hedging parameters and sensitivities. The Greeks have been calculated as:

$$\Delta : \frac{\partial C(S(t), t)}{\partial S(t)} \Rightarrow \ at\ time\ 0 : \Delta : \frac{\partial C(S(0), 0)}{\partial S(0)} \tag{6}$$

$$F.D.S.\ \Delta_{num} : \frac{C(S(0)+h,0) - C(S(0)-h,0)}{2h} \tag{7}$$

$$\Gamma : \frac{\partial^2 C(S(t),t)}{\partial S(t)^2} \Rightarrow \ at\ time\ 0 : \Gamma : \frac{\partial^2 C(S(0),0)}{\partial S(0)^2} \tag{8}$$

$$F.D.S.\ \Gamma_{num} : \frac{C(S(0)+h,0) - 2C(S(0),0) + C(S(0)-h,0)}{h^2} \tag{9}$$

Following are the delta and gamma of the above European call option under Heston model with full truncation scheme (assuming increments $\Delta S = 0.01S_0$, 100,000 simulations and 50 time steps/year):

$$\Delta_{num}^{Heston} = 0.77165$$

$$\Gamma_{num}^{Heston} = 0.0195$$

The Greeks under Black-Scholes model are as follow:

$$\Delta_{num}^{Black-Scholes} = 0.7606$$

$$\Gamma_{num}^{Black-Scholes} = 0.0046$$

The Δ is important measure in examining value of the option that how it is impacted due fluctuations in the price underlying financial instrument, keeping all other parameters constant. The Δ values under both models is quite close, and near to 1 hence indicating in the money call option.
The Γ measures sensitivity of Δ to a change in an asset or stock price, hence it analyzes Δ. It is usually small, and in fact both models have a small Γ that signifies option is further away from the date of expiration thus less sensitive to Δ.

Further Numerical Results

The bias of all four schemes are further evaluated in table 3 with $\omega = 0.3$, due to a lower ω overall bias has declined in all three simulations for each model. But there is no CEV-scheme that provides an accurate and stable results.

Conclusion

During the discretization process of Heston Model, it is carried out in such a way that the result is non-negative irrespective of the size of time steps. A unified framework is derived such that all four scheme can be described taking into account the issue negative stochastic volatility term. The Full Truncation scheme provided the most optimum results being highly stable and convergent, with minimum bias that it is close to zero hence providing the best estimation for valuing European Option price.

Table 3: Bias, Standard Error, RMSE and CPU Time Estimation Estimation:
Pricing an European call in Heston Model : S(0)=100, r=0.05, $\omega = 0.3$, T=5 yrs. True option price: 34.9998.

Method	Paths	10,000	40,000	160,000
Absorption scheme	Steps/year	20	40	80
	Bias	0.8127	0.7918	0.8597
	Standard error	0.6135	0.3183	0.1526
	RMSE	1.0183	0.8534	0.8691
	Run time	7.762	53.29	576.623
Reflection scheme	Bias	0.6456	0.7505	0.8122
	Standard error	0.6073	0.2884	0.1419
	RMSE	0.8864	0.8040	0.8257
	Run time	7.516	52.318	589.755
Partial truncation scheme	Bias	0.7930	0.7834	0.7822
	Standard error	0.5585	0.2798	0.1341
	RMSE	1.0791	0.8318	0.7936.
	Run time	7.356	52.211	600.481
Full truncation scheme	Bias	0.6949	0.7796	0.7161
	Standard error	0.7279	0.2744	0.127075
	RMSE	1.0063	0.8265	0.7272
	Run time	7.032	52.254	576.165

Codes