

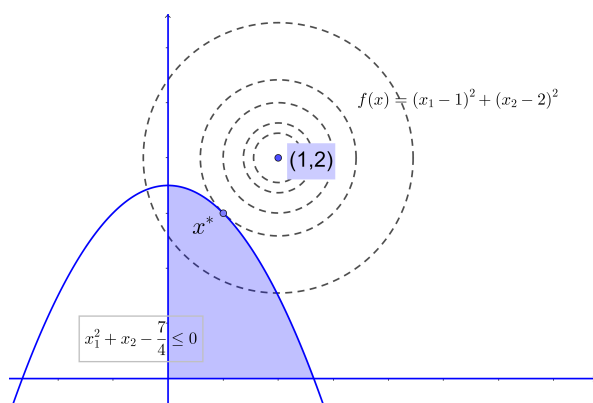
# 实用优化算法习题

## 0. 考虑问题

$$\begin{aligned} \min & (x_1 - 1)^2 + (x_2 - 2)^2 \\ \text{s.t. } & x_1^2 + x_2 - \frac{7}{4} \leq 0, \\ & x_1, x_2 \geq 0. \end{aligned}$$

画出问题的可行域和等高线图, 据此求出问题的解.

【解:】见第一章课件例2.6.



1. 用斐波那契法求函数  $f(x) = x^2 - x + 2$  在区间  $[-1, 3]$  上的极小点, 要求精度为0.1, 至少迭代3次.

【解:】先确定迭代次数:

$$\frac{3 - (-1)}{0.1} = 40.$$

写出斐波那契数列

$$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, F_9 = 34, F_{10} = 55.$$

故需迭代  $n = 10 - 1 = 9$  次.

第一次迭代  $a = -1, b = 3,$

$$\begin{aligned} x_1 &= a + \frac{F_8}{F_{10}}(b - a) = -1 + \frac{21}{55} \times 4 = 0.5273, \quad f_1 = f(x_1) = 1.7507; \\ x_2 &= a + \frac{F_9}{F_{10}}(b - a) = -1 + \frac{34}{55} \times 4 = 1.4727, \quad f_2 = f(x_2) = 2.6962. \end{aligned}$$

因为  $f_1 < f_2$ , 故而  $b = x_2 = 1.4727, x_2 = x_1 = 0.5273, f_2 = f_1 = 1.7507.$

第二次迭代  $a = -1, b = 1.4727$ .

$$x_1 = a + \frac{F_7}{F_9}(b - a) = -1 + \frac{13}{34}(1.4727 - (-1)) = -0.0545, f_1 = f(x_1) = 2.0575;$$

$$x_2 = 0.5273, f_2 = 1.7507.$$

因为  $f_1 > f_2$ , 故而  $a = x_1 = -0.0545, x_1 = x_2 = 0.5273, f_1 = f_2 = 1.7507$ .

第三次迭代  $a = -0.0545, b = 1.4727$ .

$$x_1 = 0.5273, f_1 = 1.7507;$$

$$x_2 = a + \frac{F_7}{F_8}(b - a) = 0.8909, f_2 = f(x_2) = 1.9028.$$

因  $f_1 < f_2$ , 故而  $b = x_2 = 0.8909, x_2 = x_1 = 0.5273, f_2 = f_1 = 1.7507$ .

## 2. 用黄金分割法求

$$f(x) = -2x^3 + 21x^2 - 60x + 50$$

在区间  $[-1, 4]$  内的最小值, 迭代三次.

【解:】第一次迭代:

$$a = -1, b = 4; x_1 = a + 0.382(b - a) = 0.91, f_1 = f(x_1) = 11.2830;$$

$$x_2 = 2.09, f_2 = f(x_2) = -1.9286.$$

因为  $f_1 > f_2$ , 故而  $a = x_1 = 0.91, x_1 = x_2 = 2.09, f_1 = f_2 = -1.9286$ .

第二次迭代:

$$a = 0.91, b = 4; x_1 = 2.09, f_1 = -1.9286;$$

$$x_2 = a + 0.618(b - a) = 2.8196, f_2 = f(x_2) = 2.9448.$$

因  $f_1 < f_2$ , 故而  $b = x_2 = 2.8196, x_2 = x_1 = 2.09, f_2 = f_1 = -1.9286$ .

第三次迭代:

$$a = 0.91, b = 2.8196; x_2 = 2.09, f_2 = -1.9286;$$

$$x_1 = a + 0.618(b - a) = 1.6395, f_1 = f(x_1) = -0.7364.$$

因  $f_1 > f_2$ , 故而  $a = x_1 = 1.6395, x_1 = x_2 = 2.09, f_1 = f_2 = -1.9286$ .

## 3. 考虑函数

$$f(\mathbf{x}) = \frac{1}{3}x_1^3 - 16x_1 + \frac{1}{3}x_2^3 - x_2$$

写出  $\min f(x)$  的一阶必要条件并利用该条件求  $f(x)$  的极小点.

【解:】一阶必要条件

$$\nabla f(\mathbf{x}) = \begin{pmatrix} x_1^2 - 16 \\ x_2^2 - 1 \end{pmatrix} = 0.$$

求解上述方程组得

$$x_1 = \pm 4, x_2 = \pm 1.$$

由于 Hesse 矩阵

$$\nabla^2 f(\mathbf{x}) = \begin{pmatrix} 2x_1 & \\ & 2x_2 \end{pmatrix}$$

当  $x_1 = 4, x_2 = 1$  时正定, 故而  $\mathbf{x} = (4, 1)^T$  为问题的极小点.

## 4. 考虑下列函数

$$(1) f(\mathbf{x}) = 2x_1^2 + x_2^2 - 2x_1x_2 + 2x_1^3 + x_1^4$$

$$(2) f(\mathbf{x}) = 2x_1^3 - 3x_1^2 - 6x_1x_2(x_1 - x_2 - 1).$$

的所有驻点.哪些是极小点, 是否是整体极小点.

【解:】(1) 函数都梯度和Hesse 矩阵分别为

$$\nabla f(\mathbf{x}) = \begin{pmatrix} 4x_1 - 2x_2 + 6x_1^2 + 4x_1^3 \\ 2x_2 - 2x_1 \end{pmatrix}, \quad G(\mathbf{x}) = \begin{pmatrix} 4 + 12x_1 + 12x_1^2 & -2 \\ -2 & 2 \end{pmatrix}.$$

令 $\nabla f(\mathbf{x}) = 0$ , 得

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \quad \text{或} \quad \begin{cases} x_1 = -1 \\ x_2 = -1 \end{cases} \quad \text{或} \quad \begin{cases} x_1 = -\frac{1}{2} \\ x_2 = -\frac{1}{2} \end{cases}.$$

在这三个点处, Hesse 矩阵分别为

$$\begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & -2 \\ -2 & 2 \end{pmatrix}.$$

其中前两个矩阵是正定矩阵, 第三个矩阵不是正定的, 也不是半正定的. 故而 $(0, 0)^T$ ,  $(-1, -1)^T$  为局部极小点.

又因为 $f(0, 0) = 0$ ,  $f(-1, -1) = 0$ , 故而这两个点均为整体极小点.

(2) 函数的梯度和Hesse矩阵分别为

$$\nabla f(\mathbf{x}) = \begin{pmatrix} 6x_1^2 - 12x_1x_2 - 6x_1 + 6x_2^2 + 6x_2 \\ 12x_1x_2 - 6x_1^2 + 6x_1 \end{pmatrix},$$

$$G(\mathbf{x}) = \begin{pmatrix} 12x_1 - 12x_2 - 6 & 12x_2 - 12x_1 + 6 \\ 12x_2 - 12x_1 + 6 & 12x_1 \end{pmatrix}.$$

令 $\nabla f(\mathbf{x}) = 0$ , 得

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \text{或} \quad \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

在这四个点处的Hesse矩阵分别是

$$G = \begin{pmatrix} -6 & 6 \\ 6 & 0 \end{pmatrix}, \begin{pmatrix} 6 & -6 \\ -6 & 12 \end{pmatrix}, \begin{pmatrix} 6 & -6 \\ -6 & 0 \end{pmatrix}, \quad \text{和} \quad \begin{pmatrix} -6 & 6 \\ 6 & -12 \end{pmatrix}.$$

这四个矩阵分别是鞍点矩阵, 正定矩阵, 鞍点矩阵和负定矩阵. 所以上述四个点分别为鞍点、极小点、鞍点、极大点. 其中,  $(1, 0)$  也是整体极小点.

5. 对正定二次函数 $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T G \mathbf{x} + c^T \mathbf{x}$ , 在点 $\mathbf{x}_k$  处, 求出沿方向 $d_k$  做精确搜索的步长 $\alpha_k$ .

【解:】该二次函数的梯度为

$$\nabla f(\mathbf{x}) = G\mathbf{x} + c.$$

定义  $\varphi(\alpha) = f(\mathbf{x}_k + \alpha d_k)$ , 则  $\alpha_k$  满足

$$\varphi'(\alpha) = 0,$$

也就是

$$\nabla f(\mathbf{x}_k + \alpha_k d_k)^T d_k = 0.$$

又

$$\begin{aligned} \nabla f(\mathbf{x}_k + \alpha_k d_k)^T d_k &= (G(\mathbf{x}_k + \alpha_k d_k) + c)^T d_k \\ &= (G\mathbf{x}_k + c + \alpha_k Gd_k)^T d_k \\ &= (\nabla f_k + \alpha_k Gd_k)^T d_k \\ &= \nabla f_k^T d_k + \alpha_k d_k^T Gd_k = 0. \end{aligned}$$

从而有

$$\alpha_k = -\frac{\nabla f_k^T d_k}{d_k^T Gd_k}.$$

6. 用最速下降法求解

$$\min f(\mathbf{x}) = \frac{1}{2}x_1^2 + \frac{9}{2}x_2^2,$$

设初始点为  $(9, 1)^T$ .

【见教材、课件例题】

7. 试用最速下降法求解

$$\min f(\mathbf{x}) = 2x_1^2 + x_2^2,$$

设初始点取为  $\mathbf{x}_0 = (4, 4)^T$ , 迭代2次, 并验证相邻两次迭代的搜索方向是正交的.

【解:】记

$$g(\mathbf{x}) = \nabla f(\mathbf{x}) = \begin{pmatrix} 2x_1 \\ 4x_2 \end{pmatrix}.$$

在初始点处  $f(\mathbf{x}_0) = 48$ ,  $g(\mathbf{x}_0) = (8, 16)^T$ , 搜索方向为  $d_0 = -g(\mathbf{x}_0) = (-8, -16)$ . 考虑

$$\min_{\alpha \geq 0} f(\mathbf{x}_0 + \alpha d_0) = 16(3 - 20\alpha + 36\alpha^2).$$

其极小点为  $\alpha_0 = 5/18$ . (或者  $\alpha_0 = -\frac{g_0^T d_0}{d_0^T G d_0} = \frac{5}{18}$ )

则

$$\mathbf{x}_1 = \mathbf{x}_0 + \alpha_0 d_0 = \begin{pmatrix} 16/9 \\ -4/9 \end{pmatrix}.$$

此时

$$g_1 = g(\mathbf{x}_1) = \begin{pmatrix} 32/9 \\ -16/9 \end{pmatrix}, d_1 = -g_1 = \begin{pmatrix} -32/9 \\ 16/9 \end{pmatrix}.$$

考虑

$$\min_{\alpha \geq 0} f(\mathbf{x}_1 + \alpha d_1) = \frac{32}{81}(9 - 40\alpha + 48\alpha^2).$$

其极小点为 $\alpha_1 = 5/12$ . (或者 $\alpha_1 = -\frac{g_1^T d_1}{d_1^T G d_1} = \frac{5}{12}$ ) 则

$$\mathbf{x}_2 = \mathbf{x}_1 + \alpha_1 d_1 = \begin{pmatrix} 8/27 \\ 8/27 \end{pmatrix}.$$

此时

$$g_2 = g(\mathbf{x}_2) = \begin{pmatrix} 16/27 \\ 32/27 \end{pmatrix}, d_2 = -g_2 = \begin{pmatrix} -16/27 \\ -32/27 \end{pmatrix}.$$

考虑

$$\min_{\alpha \geq 0} f(\mathbf{x}_2 + \alpha d_2) = \left(\frac{8}{27}\right) (3 - 20\alpha + 36\alpha_2).$$

其极小点为 $\alpha_2 = 5/18$ . (或者 $\alpha_2 = -\frac{g_2^T d_2}{d_2^T G d_2} = \frac{5}{18}$ ) 则

$$\mathbf{x}_3 = \mathbf{x}_2 + \alpha_2 d_2 = \begin{pmatrix} 32/243 \\ -8/243 \end{pmatrix}.$$

8. 用Newton 法求解

$$\min f(x) = \frac{1}{2}x_1^2 + \frac{9}{2}x_2^2.$$

【解：】Newton 法用于求解严格凸二次函数时可任选初始点，这里选 $\mathbf{x}_0 = (1, 1)^T$ . 函数都梯度和Hesse矩阵分别是

$$\nabla f(\mathbf{x}) = \begin{pmatrix} x_1 \\ 9x_2 \end{pmatrix}, G = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}.$$

令

$$d_0 = -G^{-1}\nabla f(\mathbf{x}_0) = -\begin{pmatrix} 1 & \\ & -\frac{1}{9} \end{pmatrix} \begin{pmatrix} 1 \\ 9 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

则

$$\mathbf{x}_1 = \mathbf{x}_0 + d_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

算法终止.

9. 用FR 共轭梯度法求解

$$\min f(\mathbf{x}) = \frac{3}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1x_2 - 2x_1,$$

取初始点 $\mathbf{x}_0 = (0, 0)^T$ .

【解：】记

$$G = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}, b = \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

则 $f(\mathbf{x})$ ,  $\nabla f(\mathbf{x})$  可表示成

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T G \mathbf{x} + b^T \mathbf{x}, \nabla f(\mathbf{x}) = G \mathbf{x} + b.$$

第一次迭代.  $\mathbf{x}_0 = (0, 0)^T$ .

$$d_0 = -\nabla f(\mathbf{x}_0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \alpha_0 = -\frac{\nabla f(\mathbf{x}_0)^T d_0}{d_0^T G d_0} = \frac{1}{3}.$$

故

$$\mathbf{x}_1 = \mathbf{x}_0 + \alpha_0 d_0 = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}.$$

因

$$\nabla f(\mathbf{x}_1) = \begin{pmatrix} 0 \\ -\frac{2}{3} \end{pmatrix} \neq 0,$$

故算法不终止.

第二次迭代.

$$\beta_0 = \frac{\|\nabla f(\mathbf{x}_1)\|^2}{\|\nabla f(\mathbf{x}_0)\|^2} = \frac{1}{9}, \quad d_1 = -\nabla f(\mathbf{x}_1) + \beta_0 d_0 = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}.$$

计算步长和新迭代点

$$\alpha_1 = -\frac{\nabla f(\mathbf{x}_1)^T d_1}{d_1^T G d_1} = \frac{3}{2}, \quad \mathbf{x}_2 = \mathbf{x}_1 + \alpha_1 d_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

因  $\nabla f(\mathbf{x}_1) = (0, 0)^T$ , 故而算法终止.

函数  $f(\mathbf{x})$  的极小点为  $\mathbf{x}^* = (1, 1)^T$ .

10. 考虑函数

$$f(\mathbf{x}) = 1/2x_1^2 + 1/2x_2^2$$

设初始点为  $\mathbf{x}_0 = (1, 1)^T$ , 取  $d_0 = (-1, 0)^T$ ,

- (a) 沿方向  $d_0$  进行精确一维搜索得  $\alpha_0$ . 令  $\mathbf{x}_1 = \mathbf{x}_0 + \alpha_0 d_0$ , 用FR 公式求  $d_1$ .  
 (b) 设  $G$  为目标函数的Hesse 矩阵, 证明  $d_0$  与  $d_1$  不是关于  $G$  共轭的. 试说明原因.

**【解:】** 在初始点处

$$g_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

沿  $d_0$  方向进行精确搜索得最优步长为

$$\alpha_0 = -\frac{(g_0)^T d_0}{(d_0)^T G d_0} = 1.$$

则  $\mathbf{x}_1 = \mathbf{x}_0 + \alpha_0 d_0 = (0, 1)^T$ . 计算

$$g_1 = \nabla f(\mathbf{x}_1) = (0, 1)^T, \quad \beta_0 = \frac{\|g_1\|^2}{\|g_0\|^2} = \frac{1}{2}.$$

则

$$d_1 = -g_1 + \beta_0 d_0 = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}.$$

由于

$$(d_0)^T G d_1 = (d_0) d_1 = \frac{1}{2} \neq 0$$

所以  $d_0$  与  $d_1$  不是关于

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

共轭的.

#### 11. 用DFP 算法求解

$$\min f(x) = x_1^2 + 2x_2^2 - 2x_1x_2 - 4x_1,$$

$$\text{取 } x_0 = (1, 1)^T, H_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

【解:】记  $c = (-4, 0)^T$ . 则目标函数可记为  $f = \frac{1}{2} \mathbf{x}^T G \mathbf{x} + c^T \mathbf{x}$ .

1)  $g_0 = \nabla f(\mathbf{x}_0) = G \mathbf{x}_0 + c = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ . 搜索方向  $d_0 = -H_0 g_0 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ . 用精确搜索, 步长为

$$\alpha_0 = -\frac{g_0^T d_0}{d_0^T G d_0} = \frac{1}{4}.$$

$$2) \mathbf{x}_1 = \mathbf{x}_0 + \alpha_0 d_0 = \begin{pmatrix} 2 \\ \frac{1}{2} \end{pmatrix}, g_1 = G \mathbf{x}_1 + c = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

$$s_0 = \alpha_0 d_0, y_0 = g_1 - g_0 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, H_1 = H_0 - \frac{H_0 y_0 y_0^T H_0}{y_0^T H_0 y_0} + \frac{s_0 s_0^T}{y_0^T s_0} = \begin{pmatrix} \frac{21}{50} & \frac{19}{100} \\ \frac{19}{50} & \frac{41}{100} \end{pmatrix}.$$

$$d_1 = -H_1 g_1 = \begin{pmatrix} \frac{8}{5} \\ \frac{6}{5} \end{pmatrix}, \text{用精确搜索 } \alpha_1 = \frac{5}{4}.$$

$$3) \mathbf{x}_2 = \mathbf{x}_1 + \alpha_1 d_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, g_2 = G \mathbf{x}_2 + c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

由于  $d_0^T G d_1 = 0$ , 故而  $d_0, d_1$  关于  $G$  共轭.

#### 12. 用DFP 算法求解

$$\min f(\mathbf{x}) = \frac{1}{2} \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 7.$$

初始点为  $x_0 = (-1, 0)^T$ , 初始矩阵为单位矩阵.

【解:】记

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, b = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, c = 7.$$

第一次迭代.

$$\mathbf{x}_0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \nabla f(\mathbf{x}_0) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, d_0 = -\nabla f(\mathbf{x}_0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

$$\alpha_0 = -\frac{\nabla f_0^T d_0}{d_0^T G d_0} = \frac{5}{6}, \mathbf{x}_1 = \mathbf{x}_0 + \alpha_0 d_0 = \begin{pmatrix} \frac{2}{3} \\ -\frac{5}{6} \end{pmatrix}.$$

因为

$$\nabla f(\mathbf{x}_1) = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \neq 0,$$

故算法不终止.

第二次迭代.

$$s_0 = \alpha_0 d_0 = \begin{pmatrix} \frac{5}{3} \\ -\frac{5}{6} \end{pmatrix}, y_0 = \begin{pmatrix} \frac{5}{3} \\ -\frac{5}{6} \end{pmatrix}.$$

DFP 修正:

$$H_1 = H_0 - \frac{H_0 y_0 y_0^T H_0}{y_0^T H_0 y_0} + \frac{s_0 s_0^T}{y_0^T s_0} = \begin{pmatrix} \frac{7}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} \end{pmatrix}.$$

计算搜索方向和步长

$$d_1 = -H_1 \nabla f_1 = \left(\frac{1}{2}, \frac{1}{2}\right), \alpha_1 = -\frac{\nabla f_1^T d_1}{d_1^T G d_1} = \frac{2}{3}.$$

故

$$\mathbf{x}_2 = \mathbf{x}_1 + \alpha_1 d_1 = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}.$$

因

$$\nabla f(\mathbf{x}_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

故而算法终止.

13. 用DFP 算法求解

$$\min x_1^2 - x_1 x_2 + x_2^2 + 2x_1 - 4x_2$$

初始点取为 $\mathbf{x}_0 = (2, 2)^T$ , 初始矩阵取为单位矩阵, 并验证算法所产生的两个方向关于 $G = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ 共轭的.

【解:】记 $c = (2, -4)^T$ . 则目标函数可记为 $f = \frac{1}{2} \mathbf{x}^T G \mathbf{x} + c^T \mathbf{x}$ .

1)  $g_0 = \nabla f(\mathbf{x}_0) = G \mathbf{x}_0 + c = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ . 搜索方向 $d_0 = -H_0 g_0 = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ . 用精确搜索, 步长为

$$\alpha_0 = -\frac{g_0^T d_0}{d_0^T G d_0} = \frac{5}{14}.$$



$$2) \mathbf{x}_1 = \mathbf{x}_0 + \alpha_0 d_0 = \begin{pmatrix} \frac{4}{7} \\ \frac{19}{7} \end{pmatrix}, g_1 = G\mathbf{x}_1 + c = \begin{pmatrix} \frac{3}{7} \\ \frac{6}{7} \end{pmatrix}.$$

$$s_0 = \alpha_0 d_0, y_0 = g_1 - g_0 = \begin{pmatrix} -\frac{25}{7} \\ \frac{20}{7} \end{pmatrix}, H_1 = H_0 - \frac{H_0 y_0 y_0^T H_0}{y_0^T H_0 y_0} + \frac{s_0 s_0^T}{y_0^T s_0} = \begin{pmatrix} \frac{194}{287} & \frac{99}{574} \\ \frac{287}{99} & \frac{391}{574} \end{pmatrix}.$$

$$d_1 = -H_1 g_1 = \begin{pmatrix} -\frac{24}{41} \\ -\frac{30}{41} \end{pmatrix}, \text{用精确搜索 } \alpha_1 = \frac{41}{42}.$$

$$3) \mathbf{x}_2 = \mathbf{x}_1 + \alpha_1 d_1 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, g_2 = G\mathbf{x}_2 + c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

由于  $d_0^T G d_1 = 0$ , 故而  $d_0, d_1$  关于  $G$  共轭.

14. 考虑问题

$$\begin{aligned} \min f(x) &= x_1^2 + x_2^2 \\ \text{s.t. } c(x) &= x_1^2 + 2x_2^2 - 1 = 0. \end{aligned}$$

的最优性条件和极值点. (相当于用Lagrange 乘子法求极小点)

【解:】见课件第四章例题2.1.

15. 考虑优化问题

$$\begin{aligned} \min f(x) &= -(x_1 + x_2) \\ \text{s.t. } c_1(x) &= 1 - x_1^2 - 4x_2^2 \geq 0, \\ x_1 &\geq 0, \\ x_2 &\geq 0. \end{aligned}$$

初始点为  $x_0 = \left[\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right]^T$ . 试分别判断方向向量  $d_1 = [1, 0]^T$ ,  $d_2 = [1, -0.5]^T$ ,  $d_3 = [0, -1]^T$  是否是初始点处的下降方向? 是否是可行方向? 是否是可行下降方向?

【解:】见课件第四章例题2.4.

16. 问题

$$\begin{aligned} \min x_1^2 + x_2^2, \\ \text{s.t. } (x_1 - 1)^2 + x_2^2 \leq 1, \\ x_2^2 - x_1 + 1 \leq 0 \end{aligned} \quad (1)$$

的最优解是  $x^* = (1, 0)^T$ . 求点  $x^*$  处的有效集  $\mathcal{I}^*$ .

【解:】 $\mathcal{I}^* = \{2\}$ .

17. 求问题

$$\begin{aligned} \min f(x) &= (x_1 - 1)^2 + (x_2 - 1)^2, \\ \text{s.t. } c_1(x) &= 1 - x_1 - x_2 \geq 0, \\ c_2(x) &= x_1 \geq 0, \\ c_3(x) &= x_2 \geq 0. \end{aligned}$$

的KKT 点.

【解:】见课件第四章例题2.6.

18. 求问题

$$\begin{aligned} \min f(x) &= (x_1 + x_2)^2 + 2x_1 + x_2^2, \\ \text{s.t. } x_1 + 3x_2 &\leq 4, \\ 2x_1 + x_2 &\leq 3, \\ x_1, x_2 &\geq 0 \end{aligned}$$

的KKT 点.

【解:】问题的KKT 条件为

$$\begin{cases} 2x_1 + 2x_2 + 2 + \lambda_1 + 2\lambda_2 - \lambda_3 = 0, \\ 2x_1 + 4x_2 + 3\lambda_1 + \lambda_2 - \lambda_4 = 0, \\ 4 - x_1 - 3x_2 \geq 0, \lambda_1 \geq 0, \lambda_1(4 - x_1 - 3x_2) = 0, \\ 3 - 2x_1 - x_2 \geq 0, \lambda_2 \geq 0, \lambda_2(3 - 2x_1 - x_2) = 0, \\ x_1 \geq 0, \lambda_3 \geq 0, x_1\lambda_3 = 0, \\ x_2 \geq 0, \lambda_4 \geq 0, x_2\lambda_4 = 0. \end{cases}$$

由于

$$\lambda_3 = 2x_1 + 2x_2 + 2 + \lambda_1 + 2\lambda_2 \geq 2 > 0,$$

故而 $x_1 = 0$ .

若 $\lambda_4 > 0$ , 则 $x_2 = 0$ . 则

$$4 - x_1 - 3x_2 = 4 > 0, \quad 3 - 2x_1 - x_2 = 3 > 0,$$

从而 $\lambda_1 = 0, \lambda_2 = 0$ . 那么

$$\lambda_4 = \lambda_1 + 2\lambda_2 = 0,$$

矛盾.

从而 $\lambda_4 = 0$ . 那么

$$4x_2 + 3\lambda_1 + \lambda_2 = 0.$$

这意味着

$$x_2 = 0, \lambda_1 = 0, \lambda_2 = 0.$$

所以原问题的解为

$$x_1 = 0, x_2 = 0,$$

相应的Lagrange 乘子为

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 2, \lambda_4 = 0.$$

## 19. 已知约束问题

$$\begin{aligned}
 \min f(x) &= -3x_1^2 - x_2^2 - 2x_3^2, \\
 \text{s.t. } c_1(x) &= x_1^2 + x_2^2 + x_3^2 - 3 = 0, \\
 c_2(x) &= -x_1 + x_2 \geq 0, \\
 c_3(x) &= x_1 \geq 0, \\
 c_4(x) &= x_2 \geq 0, \\
 c_5(x) &= x_3 \geq 0.
 \end{aligned}$$

试验证最优解  $x^* = (1, 1, 1)^T$  为KKT 点.

【解：】见课件第四章例题2.10.

## 20. 用外罚函数法求解约束优化问题

$$\begin{aligned}
 \min f(x) &= x_1^2 + x_2^2, \\
 \text{s.t. } x_1 + x_2 - 2 &= 0.
 \end{aligned}$$

【解：】见课件第四章例题3.1.

## 21. 对问题

$$\begin{aligned}
 \min x_2^2 - 3x_1, \\
 \text{s.t. } x_1 + x_2 &= 1 \\
 x_1 - x_2 &= 0
 \end{aligned}$$

考虑外罚函数法, 求出问题的局部最优解和相应的Lagrange 乘子.

【解：】构造外罚函数

$$P(x; \sigma) = x_2^2 - 3x_1 + \sigma(x_1 + x_2 - 1)^2 + \sigma(x_1 - x_2)^2.$$

$$\begin{cases} \frac{\partial P}{\partial x_1} = -3 + 2\sigma(x_1 + x_2 - 1) + 2\sigma(x_1 - x_2) = 0, \\ \frac{\partial P}{\partial x_2} = 2x_2 + 2\sigma(x_1 + x_2 - 1) - 2\sigma(x_1 - x_2) = 0. \end{cases}$$

解之得

$$\begin{cases} x_1 = \frac{3+2\sigma}{4\sigma}, \\ x_2 = \frac{2\sigma}{4\sigma+2}. \end{cases}$$

令  $\sigma \rightarrow +\infty$ , 得

$$\begin{cases} x_1 = \frac{1}{2}, \\ x_2 = \frac{1}{2}. \end{cases}$$

所以, 问题的解为  $x^* = (1/2, 1/2)^T$ .

$\nabla f - \nabla c \lambda = 0$  得

$\lambda_i = \lim_{\sigma \rightarrow +\infty} -2\sigma(c_i(x(\sigma)))^m$

$\lambda_1 = \lim_{\sigma \rightarrow +\infty} -2\sigma(x_1 + x_2 - 1)$   
 $= -1$

$\lambda_2 = \lim_{\sigma \rightarrow +\infty} -2\sigma(x_1 - x_2)$   
 $= -2$

$\begin{pmatrix} -3 \\ 2x_2 \end{pmatrix} - \lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0 \Rightarrow \begin{matrix} \lambda_1 = -1 \\ \lambda_2 = -2 \end{matrix}$

22. 对问题

$$\begin{aligned} \min \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & 1 - 2x_1^2 - x_2^2 \geq 0 \end{aligned}$$

考虑对数障碍函数, 求出问题的解.

【解:】构造障碍函数

$$B(\mathbf{x}; r) = 2x_1 + 3x_2 - r \ln(1 - 2x_1^2 - x_2^2).$$

$$\begin{cases} \frac{\partial B}{\partial x_1} = 2 + \frac{4rx_1}{1-2x_1^2-x_2^2} = 0, \\ \frac{\partial B}{\partial x_2} = 3 + \frac{2rx_2}{1-2x_1^2-x_2^2} = 0. \end{cases} \Rightarrow x_2 = 3x_1$$

解之得,

$$\begin{cases} x_1 = \frac{r+\sqrt{r^2+11}}{11}, \\ x_2 = \frac{3(r+\sqrt{r^2+11})}{11} \end{cases} \quad \begin{cases} x_1 = \frac{r-\sqrt{r^2+11}}{11}, \\ x_2 = \frac{3(r-\sqrt{r^2+11})}{11} \end{cases}$$

由于第一组解不在可行域内, 故而取

$$\begin{cases} x_1 = \frac{r-\sqrt{r^2+11}}{11}, \\ x_2 = \frac{3(r-\sqrt{r^2+11})}{11}. \end{cases}$$

令  $r \rightarrow 0$ , 得解  $\mathbf{x}^* = (\frac{1}{11}, \frac{3}{11})^T$

23. 写出如下不等式约束最优化问题

$$\begin{aligned} \min \quad & x_1^2 + x_2^2, \\ \text{s.t.} \quad & x_1 + x_2 - 2 \geq 0. \end{aligned}$$

的外罚函数.

【按定义写,略】

24. 写出如下不等式约束最优化问题

$$\begin{aligned} \min \quad & -x_1x_2 \\ \text{s.t.} \quad & c_1 = -x_1 - x_2^2 + 1 \geq 0, \\ & c_2 = x_1 + x_2 \geq 0. \end{aligned}$$

的外罚函数.

【按定义写,略】

25. 用障碍函数法

$$\begin{aligned} \min \quad & f(x_1 + x_2) = \frac{1}{3}(x_1 + 1)^3 + x_2 \\ \text{s.t.} \quad & 1 - x_1 \leq 0, \\ & x_2 \geq 0. \end{aligned}$$

【解:】见课件第四章例题3.5.

若算乘子, 利用  
 $\lambda_i = \lim_{r \rightarrow 0} \frac{r}{c_i(x)}$   
 ② 约束为有效约束  
 $\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \lambda \begin{pmatrix} -4x_1 \\ -2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$   
 将  $\mathbf{x}^*$  代入解出  $\lambda$

26. 用内罚函数法求如下问题的最优点。

$$\begin{aligned} \min f(x) &= x_1^2 + x_2^2 \\ s.t. & x_1 - x_2 + 1 \leq 0 \end{aligned}$$

【解: 注意不等号的方向】

构造内罚函数(障碍函数)

$$B(x, \sigma) = x_1^2 + x_2^2 - r \ln(-x_1 + x_2 - 1).$$

令

$$\begin{cases} \frac{\partial B}{\partial x_1} = -2x_1 + \frac{r}{-x_1+x_2-1} = 0, \\ \frac{\partial B}{\partial x_2} = 2x_2 - \frac{r}{-x_1+x_2-1} = 0. \end{cases}$$

两式相加, 得  $x_1 + x_2 = 0$ . 将其回代入第一式并求解有

$$x_1 = -\frac{2 \pm \sqrt{4 + 16r^2}}{8}.$$

那么

$$x_2 = -x_1 = \frac{2 \pm \sqrt{4 + 16r^2}}{8}.$$

如果取

$$x_1 = -\frac{2 - \sqrt{4 + 16r^2}}{8},$$

则

$$x_2 = -x_1 = \frac{2 - \sqrt{4 + 16r^2}}{8}.$$

那么约束条件

$$x_1 - x_2 + 1 = \frac{\sqrt{4 + 16r^2}}{4} + 1 > 0$$

这说明该解不在可行域内部, 不符合内点法的要求. 从而舍弃这个解.

因此取

$$x_1 = -\frac{2 + \sqrt{4 + 16r^2}}{8},$$

求乘子?

则

$$x_2 = -x_1 = \frac{2 + \sqrt{4 + 16r^2}}{8}.$$

令  $r \rightarrow 0$ , 得解

$$x = (-1/2, 1/2).$$

27. 用增广Lagrange 函数法求解等式约束最优化问题

$$\begin{aligned} \min & x_1^2 + x_2^2, \\ s.t. & x_1 + x_2 - 2 = 0. \end{aligned}$$

【解:】见课件第四章例题4.2.

28. 用增广Lagrange 函数法求解等式约束最优化问题

$$\begin{aligned} \min f(\mathbf{x}) &= 2x_1^2 - x_2^2 + x_1 - x_2 \\ \text{s.t. } x_1 - x_2 &= 0 \end{aligned}$$

【解:】构造增广Lagrange 函数

$$M(\mathbf{x}, \lambda; \sigma) = 2x_1^2 - x_2^2 + \lambda(x_1 - x_2)$$

$$\begin{cases} \frac{\partial M}{\partial x_1} = 4x_1 + 1 - \lambda + \sigma(x_1 - x_2) = 0, \\ \frac{\partial M}{\partial x_2} = -2x_2 - 1 + \lambda - \sigma(x_1 - x_2) = 0. \end{cases}$$

解之得

$$\begin{cases} x_1 = \frac{\lambda-1}{4-\sigma}, \\ x_2 = \frac{2\lambda-2}{4-\sigma}. \end{cases}$$

将解代入约束条件, 得 $\lambda = 1$ . 从而得问题的解为 $\mathbf{x}^* = (0, 0)^T$ .

29. 写出不等式约束最优化问题

$$\begin{aligned} \min x_1^2 + x_2^2, \\ \text{s.t. } x_1 + x_2 - 2 \geq 0. \end{aligned}$$

的增广Lagrange 函数.

【按定义写,略】

30. 用增广Lagrange 函数法求解等式约束最优化问题

$$\begin{aligned} \min f(\mathbf{x}) &= 2x_1^2 - x_2^2 + x_1 - x_2 \\ \text{s.t. } x_1 - x_2 &\leq 0 \end{aligned}$$

(写出增广Lagrange 函数, 不需求解).

【按定义写,略】

31. 用有效集法求解

$$\begin{aligned} \min q(\mathbf{x}) &= (x_1 - 1)^2 + (x_2 - 2.5)^2 \\ \text{s.t. } x_1 - 2x_2 + 2 &\geq 0 \\ -x_1 - 2x_2 + 6 &\geq 0 \\ -x_1 + 2x_2 + 2 &\geq 0 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

初始点为 $\mathbf{x}_0 = (2, 0)^T$ . (至少能迭代一次)

【见课件例6.2】

$$M(\mathbf{x}, \lambda; \sigma) = f(\mathbf{x}) - \lambda(x_1 - x_2) + \frac{\sigma}{2}(x_1 - x_2)^2$$