3.8 Computational Formulas for All Markov Chains

Name	Notation	Formula	Matrix Form
Walk Probability	$p_{j_0 j_1 j_2, \ldots, j_n}$	$p_{j_0 j_1} p_{j_1 j_2} p_{j_2 j_3}, \ldots, p_{j_{n-1} j_n}$	none
Transition Probability	$p_{ij}^{(n)}$	$p_{ij}^{(n)} = \sum_k p_{ik}^{(n-1)} p_{kj}$	$\mathbf{P}^n = \mathbf{P}^{n-1}\mathbf{P}$
State Probability	$p_j^{(n)}$	$p_j^{(n)} = \sum_i p_i^{(0)} p_{ij}^{(n)}$	$\mathbf{p}^{(n)} = \mathbf{p}^{(0)}\mathbf{P}^{(n)}$
Expected Number of Visits in n Epochs	$e_{ij}^{(n)}$	$e_{ij}^{(n)} = \sum_{k=1}^{n} p_{ij}^{(k)} \quad \text{for } i \neq 0$ $e_{ii}^{(n)} = 1 + \sum_{k=1}^{n} p_{ii}^{(k)}$	$\mathbf{E}^{(n)} = \sum_{k=0}^{n} \mathbf{P}^{(n)}$
First Passage (or Return) Probability	$f_{ij}^{(n)}$	$f_{ij}^{(n)} = p_{ij}^{(n)} - \sum_{k=1}^{n-1} f_{ij}^{(k)} p_{jj}^{(n-k)}$	none

3.13 Computational Formulas for Ergodic Markov Chains

Name	Notation	Formula	Matrix Form
Steady-State Probability	π_j	$\pi_j = \sum_k \pi_k p_{kj}$ $\sum_j \pi_j = 1$	$\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$ $\boldsymbol{\pi}\mathbf{1} = 1$
Mean First Passage Time	m_{ij}	$m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj}$	$\mathbf{m}_j = (\mathbf{I} - \mathbf{P}_j^*)\mathbf{1}$
Mean Recurrence Time	m_{ii}	$m_{ii} = \frac{1}{\pi_i}$	none

3.20 Computational Formulas for Terminating Processes

Name	Notation	Formula	Matrix Form
Expected Number of Visits (to transient state j over the full duration of the process)	e_{ij}	$e_{ii} = \sum_{k=1}^{K} p_{ik} e_{kj}$ $e_{ij} = 1 + \sum_{k=1}^{K} p_{ik} e_{ki}$	$\mathbf{E} = (\mathbf{I} - \mathbf{Q})^{-1}$
Expected Duration	d_i	$d_i = \sum_{k=1}^{K} e_{ik}$	Sum across *i*th row of **E**
Absorption Probability	a_{ij}	$a_{ij} = p_{ij} + \sum_{k=1}^{K} p_{ik} a_{kj}$	$\mathbf{A} = \mathbf{E}\mathbf{R}$
Hit Probability	f_{ij}	$f_{ij} = p_{ij} + \sum_{\text{transient } k \neq j} p_{ik} f_{kj}$	$\mathbf{f}_j = (\mathbf{I} - \mathbf{Q}_j^*)^{-1} \mathbf{q}_j$
Conditional Mean First Passage Time to an Absorbing State j	$m_{ij}^{(c)}$	$a_{ij} m_{ij}^{(c)} = a_{ij} + \sum_{k=1}^{K} p_{ik} a_{kj} m_{kj}^{(c)}$	none

■ **TABLE 5.1**

Continuous Time Markov Results

Name	Notation	Type	Formula	Matrix Form
Transition probability function	$p_{ij}(t)$	Any	$\frac{d}{dt}p_{ij}(t) = \sum_k p_{ik}(t)\lambda_{kj}$	$\frac{d}{dt}\mathbf{P}(t) = \mathbf{P}(t)\mathbf{\Lambda}$
Absolute probability	$p_j(t)$	Any	$p_j(t) = \sum_i p_i(0)p_{ij}(t)$	$\mathbf{p}(t) = \mathbf{p}(0)\mathbf{P}(t)$
Mean sojourn time	h_i	Any	$h_i = 1/\left(\sum_{k \neq i} \lambda_{ik}\right)$	none
Steady-state probability	π_j	Ergodic	$\sum_k \pi_k\lambda_{kj} = 0, \text{ for } j = 1, 2, \ldots$ $\sum_j \pi_j = 1$	$0 = \boldsymbol{\pi}\mathbf{\Lambda}$ $\boldsymbol{\pi}\mathbf{1} = 1$
Mean first passage time $(i \neq j)$	m_{ij}	Ergodic	$0 = 1 + \sum_{k \neq j} \lambda_{ik}m_{kj}$	$\mathbf{m}_j = (-\mathbf{P}^*_{jj})^{-1}\mathbf{1}$
Mean recurrence time	m_{ii}	Ergodic	$m_{ii} = h_i \sum_{j \neq i} \lambda_{ij}(1 + m_{ji})$	none
Mean time accumulated in a transient state j	e_{ij}	Terminating	$0 = \sum_{\text{Trans } k} \lambda_{ik}e_{kj} \text{ for } i \neq j$ $0 = 1 + \sum_{\text{Trans } k} \lambda_{ik}e_{ki}$	$\mathbf{E} = (-\mathbf{Q})^{-1}$
Expected duration	d_i	Terminating	$d_i = \sum_{\text{Trans } j} e_{ij}$	$\mathbf{d} = \mathbf{E}\mathbf{1}$
Absorption probability	a_{ij}	Terminating	$0 = \lambda_{ij} + \sum_{\text{Trans } k} \lambda_{ik}a_{kj}$	$\mathbf{A} = (-\mathbf{Q})^{-1}\mathbf{R}$
Hit probability	f_{ij}	Terminating	$0 = \lambda_{ij} + \sum_{\text{Trans } k \neq j} \lambda_{ik}f_{kj}$ $0 = \sum_{\text{Trans } k \neq j} \lambda_{ik}f_{ki}$	none
Conditional mean first passage time	$m^{(c)}_{ij}$	Terminating	$0 = a_{ij} + \sum_{\text{Trans } k} \lambda_{ik}a_{kj}m^{(c)}_{kj}$	none