Exam

ECE 608

March 1, 2016, 3:00pm-4:15pm

Name: Solutions.

Read all of the following information before starting the timed period of the exam:

1. **NOTE:** Unanswered questions are worth 25% credit, rounded down. Writing any answer loses this "free" credit, which must be earned back by the quality of the answer. If you wish a question to be treated as unanswered, but you have written on it, clearly write "DO NOT GRADE" in the answer area. In a multi-part question, unanswered *parts* are worth 25% (to label "DO NOT GRADE" for a part, you must include the part name, e.g. "DO NOT GRADE PART A.") This is an option only for parts that are numbered or lettered on the exam: you may not create your own "parts" for this purpose.
2. Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
3. No calculators, or materials other than pen/pencil and blank paper are allowed except those we distribute during the exam. This is a closed book closed notes exam.
4. Please keep your written answers brief; be clear and to the point. Points may be deducted for rambling and for incorrect or irrelevant statements.
5. There are 7 problems. Each of the problems is approximately equal in value, except where indicated. Multi-part problems divide the problem score approximately equally among the parts.
6. Good luck!

1. Suppose an array $A[0..(2n-1)]$ of $2n$ integers is initialized to all zeros and then modified by the following process. Repeatedly, a uniformly random index i into the array is generated and the element $A[i]$ is set to 1. This repeating halts when one of two conditions holds: either (a) $A[2j]$ has been set to 1 for **any one** even index $2j$, or (b) $A[2j+1]$ has been set to 1 for **all** odd indices $2j+1$.

i. What is the asymptotic class, in terms of n, of the expected number of random indices generated, given an assumption that the termination condition reached is (b)? 7.

$$\sum_{i=1}^{n} \frac{n}{n-(i-1)} = \sum_{i=1}^{n} \frac{n}{i} = n \sum_{i=1}^{n} \frac{1}{i}.$$

$$= \Theta(n \log n)$$

ii. (Harder) Show that the probability that termination condition (b) is reached is 8.

$$\frac{(n!)^2}{(2n)!}$$

Hint: You can ignore generation of previously generated indices, as nothing changes. So, analyze the probability that the next index selected is odd given that it is not previously selected. Combining over the process gives the above bound.

$$\prod_{i=0}^{n-1} \frac{n-i}{2n-i} = \frac{n!}{(2n)!/n!} = \frac{(n!)^2}{2n!}.$$

2. Prove carefully an asymptotic relationship between $f(x) = 3x+4$ and $g(x) = x^2 - 5$. For full credit, prove the strongest relationship you can (among those defined in class). 14.

$$f(x) = o\, g(x)$$

small 'O'

$$\lim_{n \to \infty} \frac{n^2 - 5}{3n+4} = \infty.$$

$$\therefore f(x) = o\, g(x).$$

OR

Show for any arbitrary c, $\exists\, n_0$ s.t.

$$\forall n \geq n_0, \quad c f(n) < g(n).$$

$$c(3n+4) < n^2 - 5 \Rightarrow c < \frac{n^2-5}{3n+4}$$

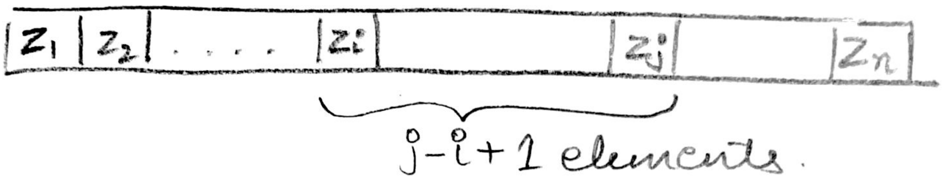
$$3cn + 4c < n^2 - 5.$$

We can choose n to make the above statement true for any c.

$$\therefore f(x) = o\, g(x)$$

3. Consider the problem of sorting n integers $x_1, \ldots, x_n$. Let z_i denote the i^{th} smallest of the integers, for each i in $[1..n]$. For particular arbitrary i and j in $[1..n]$ with $i < j$, what is the probability in terms of i and j that a single run of QUICKSORT using RANDOMIZEDPARTITION (which selects the pivot uniformly randomly among the elements to be partitioned) will compare z_i and z_j. Justify your answer carefully.

14.



① For z_i and z_j to be compared, one of them has to be the pivot element.

② If a ~~...~~ z_k is chosen such that $k < i$ or $k > j$, z_i and z_j fall into the same partition and doesnot affect the probability of being compared.

③ If an element ~~to~~ z_k is chosen as pivot such that $i < k < j$, z_i and z_j fall into different partition and never get compared.

④ Therefore, since its a uniformly random selection of pivot, the probability that z_i and z_j are compared is that either z_i or z_j are chosen as pivots out of $j-i+1$ elements.

$$\therefore \quad \underline{\frac{2}{j-i+1}}.$$

4.

1. Give and carefully prove a $\Theta()$ bound for the asymptotic growth of the following summation as a function of n: 7.

$$\sum_{k=1}^{n} k^3$$

$\sum_{k=1}^{n} k^3 \leqslant \sum_{k=1}^{n} n^3 = n^4 \quad \therefore \quad \sum_{k=1}^{n} k^3 = O(n^4)$ — 1

$\sum_{k=1}^{n} k^3 \geqslant \sum_{k=\lceil \frac{n}{2} \rceil}^{n} k^3 \geqslant \sum_{k=\lceil \frac{n}{2} \rceil}^{n} \left(\frac{n}{2}\right)^3 \approx \frac{n}{2} \frac{n^3}{8}$.

$= \Theta(n^4)$.

$\sum_{k=1}^{n} k^3 = \Omega(n^4)$. — 2

From, 1 and 2. $S = \Theta(n^4)$.

2. Write two recurrences that the Master Theorem assigns $\Theta(n^3)$ bounds to. The two recurrences should go through different cases of the Master Theorem. Explain. 8.

1 $T(n) = 8T\left(\frac{n}{2}\right) + n^2$ $\quad \log_b a = 3$.

n^2 is $O(n^{3-\varepsilon})$ for $\varepsilon = 0.5$

2 $T(n) = 4T\left(\frac{n}{2}\right) + n^3$ $\quad \log_b a = 2$.

n^3 is $\Omega(n^{2+\varepsilon})$ for $\varepsilon = 0.5$

5. Group the following functions into equivalence classes under the $\Theta()$ asymptotic growth relationship and order the classes from smallest to largest under the $O()$ relationship. Partial credit only for answers that are close to correct.

$n^{1/\lg n}$	$\lg^* n$	$\lg(\lg^* n)$	2^{2^n}
$(n+1)!$	3^n	$n!$	2^n
$\lg^*(\lg n)$	1	$\lg^2 n$	

Homework problem.

6. Use mathematical induction to demonstrate that the following recurrence is $O(2^n)$. 14.

$$T(n) = 2T(n-1) + 5$$
$$T(1) = 1$$

To show

$\forall n \; T(n) \leq c \cdot 2^n + d.$

Base case

$T(1) = 1 \leq 2c + d.$

Holds.

Hypothesis:

$T(k) \leq c2^k + d.$

To show

$T(k+1) \leq c2^{k+1} + d.$

$T(k+1) = 2T(k) + 5 \leq 2(c2^k + d) + 5.$

(From hypothesis).

$T(k+1) \leq c2^{k+1} + \underbrace{2d + 5}.$

$2d + 5 \leq d.$

Therefore for $d \leq -5$, $2d + 5 < d.$

Choose $d = -6$. and $c = 4$,

$T(k+1) \leq c2^{k+1} + d.$

Hence proved

7. Given the following pseudo-code for HEAPSORT, state a fully comprehensive loop invariant for the main loop that implies the correctness of the sort. You do **not** need to argue for the invariant's initialization, preservation, and termination properties, but these properties must hold. 14.

HEAPSORT(A)
1. BUILDMAXHEAP(A)
2. **for** $i = A.length$ downto 2
3. Exchange $A[1]$ with $A[i]$
4. $A.heap\text{-}size = A.heap\text{-}size - 1$
5. MAXHEAPIFY($A, 1$)

Just after i is updated,

(1) Elements $A[1 \ldots n]$ are a permutation of the original $A[1 \ldots n]$.

(2) Elements $A[i+1 \ldots n]$ are all larger than the largest element in $A[1 \ldots i]$.

(3) Elements $A[i+1 \ldots n]$ are sorted in non decreasing order.