ECE608, Homework #2 Solution

(1) CLR 3.1-2

To show that $(n+a)^b = \Theta(n^b)$, we want to find constants c_1, c_2, and $n_0 > 0$ such that $0 \le c_1 n^b \le (n+a)^b \le c_2 n^b$ for all $n \ge n_0$. Note that $n + a \le n + |a| \le 2n$, when $|a| \le n$, and $n + a \ge n - |a| \ge \dfrac{n}{2}$, when $|a| \le \dfrac{n}{2}$. Thus, $0 \le \dfrac{n}{2} \le n + a \le 2n$, when $n \ge 2|a|$. Since $b > 0$, the inequality continues to hold when all parts are raised to the power of b: $0 \le (\dfrac{n}{2})^b \le (n+a)^b \le (2n)^b$ and $0 \le (\dfrac{1}{2})^b n^b \le (n+a)^b \le 2^b n^b$. Thus, $c_1 = (\dfrac{1}{2})^b$, $c_2 = 2^b$, and $n_0 = 2|a|$ satisfy the definition.

Another way to look at this is as follows:

$$0 \le c_1 n^b \le (n+a)^b \le c_2 n^b$$
$$\downarrow$$
$$0 \le c_1 n^b \le (n(1 + \frac{a}{n}))^b \le c_2 n^b$$
$$\downarrow$$
$$0 \le c_1 n^b \le n^b\,(1 + \frac{a}{n})^b \le c_2 n^b$$
$$\downarrow$$
$$0 \le c_1 \le (1 + \frac{a}{n})^b \le c_2$$

Let $n_0 = 2|a|$, then $c_1 = (\frac{1}{2})^b$ and $c_2 = (\frac{3}{2})^b$.

(2) CLR 3.1-6

Prove that the running time of an algorithm $T(n) = \Theta(g(n))$ if and only if the worst-case running time is $O(g(n))$ and the best-case running time is $\Omega(g(n))$.

Proof

($\Leftarrow$)

Let $T(n)$ be the running time of the algorithm. Then, if the worst-case running time of the algorithm is $O(g(n))$, it follows that $T(n) = O(g(n))$, because the algorithm cannot operate more slowly than the worst case. If the best-case running time of the algorithm is $\Omega(g(n))$, it also follows that $T(n) = \Omega(g(n))$, because it is impossible for the algorithm to operate faster than the best case.

Hence, by the Theorem 3.1 in CLR, $T(n) = \Theta(g(n))$

($\Rightarrow$)

If the running time $T(n) = \Theta(g(n))$, there exist constants $c_1 > 0$, $c_2 > 0$, and $n_0 > 0$

such that $0 \le c_1 g(n) \le T(n) \le c_2 g(n)$ for all $n \ge n_0$. Thus, by the definition of $O(\cdot), \Omega(\cdot)$, $T(n) = O(g(n))$ and $T(n) = \Omega(g(n))$.

(3) CLR 3.1-7

Prove that $\omega(g(n)) \cap o(g(n))$ is the empty set.

Proof

If $\omega(g(n)) \cap o(g(n))$ is non-empty then $\exists f(n)$ such that $f(n) \in o(g(n))$ and $f(n) \in \omega(g(n))$. For every $f(n) \in o(g(n))$ we know that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. But such an $f(n) \notin \omega(g(n))$ because that requires $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$. Thus by contradiction we know that $\omega(g(n)) \cap o(g(n))$ must be empty.

(4) CLR 3.2-2

Proof

$a^{log_b c} = (c^{log_c a})^{log_b c} = c^{(log_c a \cdot log_b c)} = c^{\frac{log_c a}{log_c b}} = c^{log_b a}$

(5) CLR 3.2-3

Stirling's approximation: $n! = \sqrt{2\pi n}(\frac{n}{e})^n(1 + \Theta(\frac{1}{n}))$

(1) $\lg(n!) = \Theta(n \lg n)$

Proof

By Stirling's approximation,

$$\begin{aligned} \lg(n!) &= \lg\{\sqrt{2\pi n}(\frac{n}{e})^n(1 + \Theta(\frac{1}{n}))\} \\ &= \frac{\lg 2\pi}{2} + \frac{\lg n}{2} + n \lg n - n \lg e + \lg\{1 + \Theta(\frac{1}{n})\} \end{aligned}$$

Because $n \lg n$ is the dominant term in the above equation, $\lg(n!) = \Theta(n \lg n)$.

(2) $n! = \omega(2^n)$

Proof

By Stirling's approxmation,

$$\begin{aligned} \lim_{n\to\infty} \frac{n!}{2^n} &= \lim_{n\to\infty} \sqrt{2\pi n}(\frac{n}{2e})^n(1 + \Theta(\frac{1}{n})) \\ &= \lim_{n\to\infty} \frac{(\sqrt{2\pi})n^{(n+\frac{1}{2})}}{(2e)^n} \\ &= \infty \end{aligned}$$

Hence, $n! = \omega(2^n)$.

(3) $n! = o(n^n)$

Proof

By Stirling's approxmation,

$$\begin{aligned}
\lim_{n\to\infty} \frac{n!}{n^n} &= \lim_{n\to\infty} \sqrt{2\pi n}(\frac{1}{e})^n(1+\Theta(\frac{1}{n})) \\
&= \lim_{n\to\infty} \frac{\sqrt{2\pi n}}{e^n} \\
&= \lim_{n\to\infty} \frac{\sqrt{2\pi}}{2\sqrt{n}e^n} \qquad \text{by} \quad \text{L'Hospital's} \quad \text{rule} \\
&= 0
\end{aligned}$$

Hence, $n! = o(n^n)$.

(6) CLR 3.2-5

$\lg^*(\lg n)$ is asymptotically larger than $\lg(\lg^* n)$.

Proof

Let $m = \lg^* n$, and assume that $n \geq 4$. Hence $\lg^*(\lg n) = m - 1$. We are now comparing between $\lg(\lg^* n) = \lg m$ and $m - 1$. Clearly $m - 1$ is asymptotically larger than $\lg m$ when m is sufficiently large. Thus we can conclude that $\lg^*(\lg n)$ is asymptotically larger than $\lg(\lg^* n)$.

(7) CLR 3-2 (refer to Figure31 for table)

(a) If $f(n) = \lg^k n$, then $f'(n) = \frac{k \lg^{k-1} n \lg e}{n}$; hence, by using L'Hôpital's rule as follows:
$\lim_{n\to\infty} \frac{\lg^k n}{n^\epsilon} = \lim_{n\to\infty} \frac{k\lg^{k-1} n \lg e}{\epsilon n^\epsilon} = \lim_{n\to\infty} \frac{[k(k-1)\lg^{k-2} n \lg^2 e]}{\epsilon^2 n^\epsilon} = \cdots = \lim_{n\to\infty} \frac{k! \lg^k e}{\epsilon^k n^\epsilon} =$ 0, we conclude that $\lg^k n = \mathrm{o}(n^\epsilon) \Rightarrow$ hence $\mathrm{O}(n^\epsilon)$.

(b) If $f(n) = c^n$, then $f'(n) = c^n \ln c$; hence, by using L'Hôpital's rule as follows:
$\lim_{n\to\infty} \frac{n^k}{c^n} = \lim_{n\to\infty} \frac{kn^{k-1}}{c^n \ln c} = \cdots = \lim_{n\to\infty} \frac{k!}{c^n \ln^k c} = 0,$
we conclude that $n^k = \mathrm{o}(c^n) \Rightarrow$ hence $\mathrm{O}(c^n)$.

(c) $\lim_{n\to\infty} \frac{\sqrt{n}}{n^{\sin(n)}} = \lim_{n\to\infty} n^{\frac{1}{2}-\sin(n)}$
Since sin(n) oscillates between $+1$ and -1, $n^{\frac{1}{2}-\sin(n)}$ takes a value between $n^{-\frac{1}{2}}$ and $n^{+\frac{3}{2}}$. Thus, an asymptotic comparison cannot be made.

(d) $\lim_{n\to\infty} \frac{2^n}{2^{n/2}} = \lim_{n\to\infty} 2^{n/2} = \infty$. Thus, $2^n = \omega(2^{n/2}) \Rightarrow \Omega(2^{n/2})$.

(e) $\lim_{n\to\infty} \frac{n^{\lg(m)}}{m^{\lg(n)}} = \lim_{n\to\infty} 1 = 1$, because $n^{\lg(m)} = m^{\lg(n)}$. Thus, $n^{\lg(m)} = \Theta(m^{\lg(n)}) \Rightarrow$ $n^{\lg(m)} = \mathrm{O}(m^{\lg(n)})$ and $n^{\lg(m)} = \Omega(m^{\lg(n)})$.

(f) $\lg n! = \Theta(n \lg n)$ and $\lg(n^n) = n \lg n$. Thus, $\lg n! = \Theta(n \lg n) = \Theta(\lg(n^n))$ and we have $\lg n! = \mathrm{O}(\lg(n^n))$ and $\lg n! = \Omega(\lg(n^n))$.

item	O	o	Ω	ω	Θ
a	yes	yes	no	no	no
b	yes	yes	no	no	no
c	no	no	no	no	no
d	no	no	yes	yes	no
e	yes	no	yes	no	yes
f	yes	no	yes	no	yes

Figure 1: Table for Problem CLR 3-2.

(8) CLR 3-3

(a) Ranking by asymptotic growth rate, equivalent classes are enclosed by '[ ]'.

$[1, n^{1/\lg n}]$, $\lg(\lg^* n)$, $[\lg^*(\lg n), \lg^*(n)]$, $2^{\lg^* n}$, $\ln \ln n$, $\sqrt{\lg n}$, $\ln n$, $\lg^2 n$, $2^{\sqrt{2 \lg n}}$, $(\sqrt{2})^{\lg n}$, $2^{\lg n}$, $[n \lg n, \lg(n!)]$, $[4^{\lg n}, n^2]$, n^3, $(\lg n)!$, $[n^{\lg \lg n}, (\lg n)^{\lg n}]$, $(3/2)^n$, 2^n, $n2^n$, e^n, $n!$, $(n+1)!$, 2^{2^n}, $2^{2^{n+1}}$

(b) $2^{2^{n+5}}(\sin(n) + 1)$

(9) CLR 3-4

(a) False. Let $g(n) = n^2$ and $f(n) = n$, so that $f(n) = \mathrm{O}(g(n))$, i.e., $n = \mathrm{O}(n^2)$. But this does not imply that $g(n) = \mathrm{O}(f(n))$ as $n^2 \neq \mathrm{O}(n)$.

(b) False. Let $f(n) = n^2$, $g(n) = n$, then $f(n) + g(n) = n^2 + n = \Theta(n^2)$.
$\Theta(\min(f(n), g(n))) = \Theta(n)$ and $\Theta(n^2) \neq \Theta(n)$.
Thus, $f(n) + g(n) \neq \Theta(\min(f(n), g(n)))$.

(c) If we assume that $f(n)$ and $g(n)$ represent the time complexities for an algorithm, then they are monotonically increasing functions. Given these assumptions, the claim is true. Given $f(n) = \mathrm{O}(g(n))$ and $f(n) \geq 1$, we know $1 \leq f(n) \leq c_1 g(n)$ for all $n \geq n_0$ and $c_1 > 0$. Since $f(n) \geq 1$, $\lg(f(n)) \geq 0$, $\lg(g(n))$ is positive, and $\lg(f(n))$ is positive, $\lg 1 \leq \lg(f(n)) \leq \lg(c_1 g(n))$.
$\Rightarrow 0 \leq \lg(f(n)) \leq \lg c_1 + \lg(g(n))$
$\Rightarrow$ for $c_1 \geq 1$ and $0 \leq \lg(f(n)) \leq c_2 \lg(g(n))$, for $c_2 \geq 1$.

(d) False. Given $\mathrm{f}(n) = \mathrm{O}(\mathrm{g}(n))$, we have $0 \le \mathrm{f}(n) \le c\mathrm{g}(n)$ for positive c, n_0, and $n > n_0$. Then if it is true that $0 \le 2^{\mathrm{f}(n)} \le c2^{\mathrm{g}(n)}$ for some c, n_0, and $n > n_0$, then $0 \le \dfrac{2^{\mathrm{f}(n)}}{2^{\mathrm{g}(n)}} \le c$ and $0 \le 2^{\mathrm{f}(n)-\mathrm{g}(n)} \le c$.

However, if $\mathrm{f}(n) = 5n$ and $\mathrm{g}(n) = n$, then $0 \le 2^{4n} \le c$ is impossible.

(e) If $0 \le \mathrm{f}(n) \le c(\mathrm{f}(n))^2$ for some positive c, n_0 and $n \ge n_0$, then $0 \le \dfrac{\mathrm{f}(n)}{(\mathrm{f}(n))^2} \le c$ and $0 \le \dfrac{1}{\mathrm{f}(n)} \le c$.

With additional assumptions as stated in (c), this claim is true. But without those additional assumptions about $\mathrm{f}(n)$, then if $\mathrm{f}(n) = \dfrac{1}{n}$, this claim is false.

(f) True. $\mathrm{f}(n) = \mathrm{O}(\mathrm{g}(n))$ implies that for some positive c_1 and n_0, $0 \le \mathrm{f}(n) \le c_1\mathrm{g}(n)$, for all $n \ge n_0$. $\mathrm{g}(n) = \Omega(\mathrm{f}(n))$ implies that for some positive c_2 and n_0, $0 \le c_2\mathrm{f}(n) \le \mathrm{g}(n)$, for all $n \ge n_0$.

$\lim_{n\to\infty} \dfrac{\mathrm{f}(n)}{\mathrm{g}(n)} = c$, $0 \le c < \infty$, given that $\mathrm{f}(n) = \mathrm{O}(\mathrm{g}(n))$.

Case 1: If $c = 0$, $\lim_{n\to\infty} \dfrac{\mathrm{f}(n)}{\mathrm{g}(n)} = 0$. Here $\lim_{n\to\infty} \dfrac{\mathrm{g}(n)}{\mathrm{f}(n)} = \infty$, so $\mathrm{f}(n)$ is a lower bound of $\mathrm{g}(n)$.

Case 2: If $c > 0$, $\lim_{n\to\infty} \dfrac{\mathrm{f}(n)}{\mathrm{g}(n)} = c$. Here $\lim_{n\to\infty} \dfrac{\mathrm{g}(n)}{\mathrm{f}(n)} = \dfrac{1}{c} = c'$, where $c' > 0$.

Based on those two cases, $\mathrm{g}(n) = \Omega(\mathrm{f}(n))$.

(g) False. Consider $\mathrm{f}(n) = 2^n$ and $\mathrm{f}(\frac{n}{2}) = 2^{\frac{n}{2}}$, then if $\mathrm{f}(n) = \Theta(\mathrm{f}(\frac{n}{2}))$, we must have $2^n \le c_2 2^{\frac{n}{2}} \Rightarrow 2^{\frac{n}{2}} \le c_2$, which is impossible as there is no c_2 for fixed n_0.

(h) True. $0 \le c_1\mathrm{f}(n) \le \mathrm{f}(n) + \mathrm{o}(\mathrm{f}(n)) \le c_2\mathrm{f}(n)$

$\Rightarrow 0 \le c_1 \le 1 + \dfrac{\mathrm{o}(\mathrm{f}(n))}{\mathrm{f}(n)} \le c_2$, but $\lim_{n\to\infty} \dfrac{\mathrm{o}(\mathrm{f}(n))}{\mathrm{f}(n)} = 0$ by the definition of $o(f(n))$.

Hence, there is a c_1, say 1 and a c_2 for sufficiently large n and $n \ge n_0$.