This was originally written in Fall 2014, later comments are in blue and red.

STAT 350 Notes on Continuous Distributions

Fall 2014

1 Learning Objectives:

- Modeling continuous random variables by density functions
- Sketching functions and determining if a function is a density function
- Calculate probabilities for continuous random variables
- Calculate means and medians of continuous random variables
- Calculate the variance and standard deviation of continuous random variables

2 Density Curves

This is called just the density function in the book.

Probability density function (PDF) $f(x)$ is a mathematical model for continuous distributions. It must satisfies the following two conditions:

This is why I don't like using f(x) as a generic function. f in probability means the pdf.

$$(1) f(x) \geq 0, \text{ and } (2) \int_{-\infty}^{+\infty} f(x)dx = 1.$$

The probability that a random variable X falls in between a and b is

$$P(a \leq X \leq b) = \int_a^b f(x)dx = \text{ area under } f(x) \text{ from } a \text{ to } b$$

Remark 2.1. *It is important to notice that all density functions are defined on* $(-\infty, +\infty)$*. Sometimes we consider a density function of the form* $f(x), a < x < b$*. Here what we mean is that*

$$f(x) = \begin{cases} f(x), & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

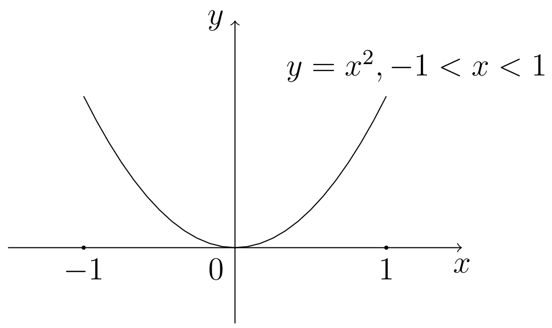
f(x) identically equal to
0 in the last two integrals

In this sense, we have

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{a}^{b} f(x)dx + \int_{\infty}^{a} f(x)dx + \int_{b}^{-\infty} f(x)dx$$

Example 2.1. *Determine if* $f(x) = x^2$ *for* $-1 < x < 1$ *is a density function.*

Solution. *The graph for the given function is shown as below:*



Although

$$f(x) = x^2 \geq 0 \text{ for } -1 < x < 1$$

we have

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^2 dx = 1/3x^3|_{-1}^{1} = 2/3 \neq 1$$

Therefore, $f(x) = x^2$ *for* $-1 < x < 1$ *is not a density function.*

Remark 2.2. *As mentioned in Remark 2.1, what we really mean is*

$$f(x) = \begin{cases} x^2, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

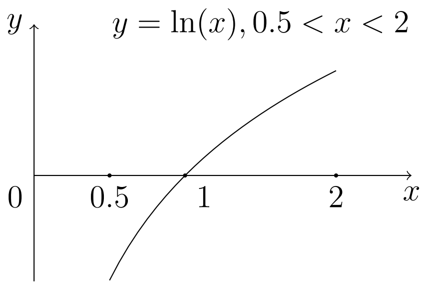
Example 2.2. *Determine if*

$$f(x) = ln(x) \text{ for } 0.5 < x < 2$$

We will not be integrating log functions
for this class. We will just be integra
exponentials and polynomials.

is a density function.

Solution. *The graph for the given function is shown as below:*



Since

$$f(x) = ln(x) < 0 \text{ for } 0.5 < x < 1$$

we know that

$$f(x) = ln(x) \text{ for } 0.5 < x < 2$$

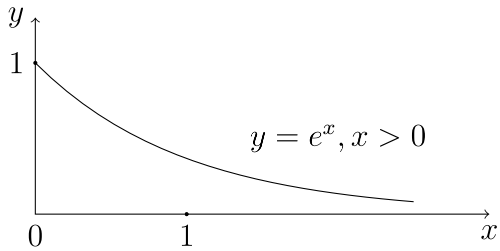
is not a density function.

Example 2.3. *Determine if*

$$f(x) = e^{-x} \text{ for } x > 0$$

is a density function.

Solution. *The graph of the given function is shown as below:*



Since

$$f(x) = e^{-x} > 0 \text{ for } x > 0$$

and

$$\int_0^\infty f(x)dx = \int_0^\infty e^{-x}dx = -e^{-x}|_0^\infty = 1$$

we know that

$$f(x) = e^{-x} > 0 \text{ for } x > 0$$

is a density function.

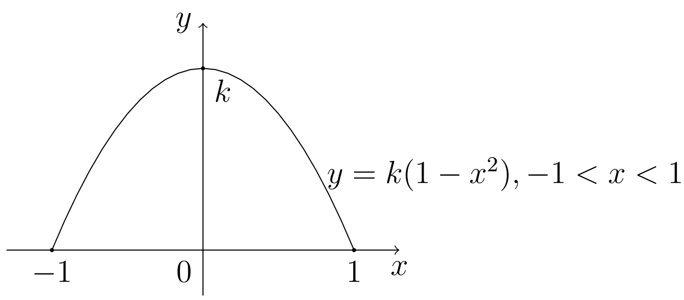
Example 2.4. *The following function is a density function, where k is a constant.*

$$f(x) = k(1 - x^2), \text{ for } -1 \leq x \leq 1$$

1) *Draw the graph of the density curve $f(x)$.*
2) *Determine the value of k.*

I find it easier to do this problem the other direct
First determine k, then graph the density function.

Solution. *The graph for the given function is shown as below:*



First, $f(x) = k(1 - x^2) \geq 0$ *for positive* k *and* $-1 \leq x \leq 1$
In order for $f(x)$ *to be a probability density function, we need*

$$\int_{-1}^{1} f(x)dx = 1$$

Thus, we need to have

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} k(1 - x^2)dx = k(x - \frac{1}{3}x^3)]_{-1}^{1} = \frac{4}{3}k = 1$$

Thus, $k = 3/4$.

Problem 2.1. *For each of the functions below, draw the graph of the function and determine if it is a density curve.*

These problems will be part of your homework assignment on LaunchPad.

- $f(x) = \cancel{2}e^{-2x}$ *for* $x > 0$
- $f(x) = 3(8x - x^2)/256$ *for* $0 < x < 8$
- ~~$f(x) = \cos x$ *for* $-\pi/2 < x < \pi/2$~~
- $f(x) = 2/x^3$ *for* $x > 1$
- $f(x) = 1/x$ *for* $x > 1$

Problem 2.2. *The following function is a density function, where* k *is a constant.*

$$f(x) = kx, \quad 0 \leq x \leq 4$$

1) Draw the graph of the density curve $f(x)$.
2) Determine the value of k.

3 Finding Probability, Mean, Median, Variance and Standard Deviation

Suppose X is a continuous random variable with p.d.f. $f(x)$. We have the following:

- The probability that X is between a and b for $-\infty \leq a \leq b \leq +\infty$ is defined as:

$$P(a < X < b) = \int_a^b f(x)dx$$

.

Note that for continuous random variable, we have

This is in the book.

$$P(a < X < b) = P(a \leq X < b) = P(a < x \leq b) = P(a \leq X \leq b)$$

This is because

Also note that if $a = -\infty$, we have

$$P(X < b) = \int_{-\infty}^{b} f(x)dx$$

It is similar for the case of $b = +\infty$.

- The mean of X is defined as:

Normally $\mathbb{E}X$ is written E(X)

The mean is also called the Expected Value

$$\mathbb{E}X = \int_{-\infty}^{+\infty} xf(x)dx$$

- The median of X is defined as a real number M such that:

This can be generalized for any percentile where y is the answer and p is the percentile.

$$\int_{-\infty}^{M} f(x)dx = \frac{1}{2}$$

- The variance of X, denoted by $\sigma^2 = Var(X)$, is defined as:

The only difference between this and what is in the book for discrete ran variables is that we are using integ here.

$$\sigma^2 = Var(X) = \mathbb{E}(X - \mathbb{E}X)^2$$

Some deductions show that:

$$\begin{aligned}
\sigma^2 = Var(X) = \mathbb{E}(X - \mathbb{E}X)^2 &= \mathbb{E}(X^2) - (\mathbb{E}X)^2 \\
&= \int_{-\infty}^{+\infty} (x - \mu)^2 f(x)dx \\
&= \int_{-\infty}^{+\infty} x^2 f(x)dx - \mu^2
\end{aligned}$$

Here $\mu = \mathbb{E}X$ is the mean.

- The standard deviation of X, denoted by σ, is defined to be the squared root of $Var(X)$. Thus, we have

$$\sigma = \sqrt{Var(X)}$$

Example 3.1. *The duration of phone calls between Ravi and his girlfriend (hours) can be modeled by a continuous random variable X with a probability density function:*

$$f(x) = \begin{cases} 0.5x, & 0 < x < 2 \\ 0, & otherwise \end{cases}$$

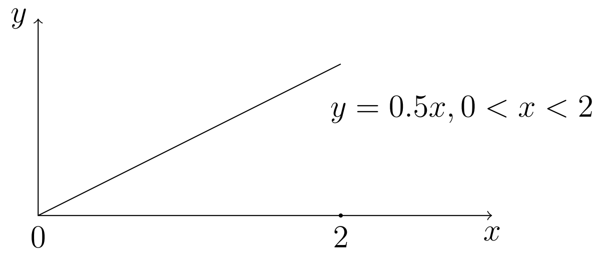
1) Verify that $f(x)$ is indeed a density function.

2) What proportion of the calls last longer than 1 hour?

3) What is the mean and median of the duration of the phone calls?

4) What is the standard deviation of the duration of the phone calls?

Solution. *The graph of $f(x)$ is shown as below:*



1) Note that we have

$$f(x) \begin{cases} \geq 0, & for\ 0 < x < 2 \\ = 0, & otherwise \end{cases}$$

and

$$\int_0^2 f(x)dx = \int_0^2 0.5xdx = 0.25x^2|_0^2 = 1$$

Thus, $f(x)$ is indeed a p.d.f. `or density function`

2)

$$P(X > 1) = \int_1^2 0.5xdx = 0.25x^2|_1^2$$
$$= (1 - 0.25) = 0.75 = 75\%$$

Thus, 75% of the phone calls are longer than one hour.

3) The mean is calculated as follow:

$$\mu = \mathbb{E}X = \int_0^2 xf(x)dx = \int_0^2 0.5x^2 dx$$
$$= \frac{1}{6}x^3|_0^2 = \frac{4}{3} = 1.33$$

Suppose the median is M. Then we have

$$\frac{1}{2} = \int_0^M f(x)dx = \frac{M^2}{4}$$

Thus, we have $M = \sqrt{2} = 1.41$.

When you solve a quadratic, you will always have two answers.
Choose the one that occurs in the region where the function is pos

Therefore, the mean is 1.33 hours and the median is 1.41 hours.

4) Recall that we have the following formula for variance:

$$Var(X) = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{+\infty} x^2 f(x)dx - \mu^2$$

The variance of X can be calculated as follow:

Note the use of the computing formu

$$Var(X) = \int_{-\infty}^{+\infty} x^2 f(x)dx - (\frac{4}{3})^2 = \int_0^2 0.5x^3 dx - (\frac{4}{3})^2$$
$$= \frac{1}{8}x^4|_0^2 - (\frac{4}{3})^2 = 2 - (\frac{4}{3})^2$$
$$= \frac{2}{9}$$

Thus, the standard deviation σ is

$$\sigma = \sqrt{Var(X)} = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$$