ECE608 Homework #4 Solution

(1) CLR 5.1-3

Perform two calls to BIASED-RANDOM obtaining two bits. The following outcomes are possible with the shown probability:

1) 1 and 0 with probability $= p(1-p)$

2) 0 and 1 with probability $= p(1-p)$

3) 1 and 1 with probability $= p^2$

4) 0 and 0 with probability $= 1 - p^2$

The first two cases occur with equal probability, so we output a 1 if the first case occurs and a 0 if the second case occurs. If the third or fourth case occur then we must repeat the two-call procedure again. This procedure has to be repated until the output is either case 1 or case 2. We can estimate the expected running time of the algorithm by viewing the algorithm as a repeated bernoulli trial and computing the expected number of trials before a success.

The probability of success in one trial is the sum of probabilities for case 1 and case 2. Which is $2p(1-p)$. The expected number of trials for a success to occur is the simply $\frac{1}{2p(1-p)}$. So the expected running time of the algorithm is $T(p) = \frac{1}{2p(1-p)}$.

(2) CLR 5.2-1

Exactly on hire takes place only in the case when the best candidate appears for the very first interview. We can estimate the probability of this event by counting the total permutations that have the best candidate for the first interview. We choose the best candidate and assign him/her to the first interview slot. The remaining $n-1$ candidates can be permuted to the remaining $n-1$ interview slots in $(n-1)!$ ways.

So the probability of exactly one hire is:

$$P(\text{Exactly one hire}) = P(\text{Best candidate fixed to first interview}) =$$
$$\frac{\text{Number of permutations that place best candidate first}}{\text{Total permutations possible}} = \frac{(n-1)!}{n!} = \frac{1}{n}$$

Exactly n hires will take place if each candidate is better than the previous ones. This means that the candidates must come for interview, sorted in an ascending order of rank. Because all ranks are unique there is only one permutation of the candidates that causes this ordering. Hence

$$P(\text{Exactly n hires}) = \frac{1}{n!}$$

(3) CLR 5.2-3

Let the following indicator variables represent the possible outcomes for one die.

$$I_1 = 1\text{if the die rolls a } 1, 0 \text{ otherwise}$$

$$I_2 = 1\text{if the die rolls a } 2, 0 \text{ otherwise}$$

$$I_3 = 1\text{if the die rolls a } 3, 0 \text{ otherwise}$$

$$I_4 = 1\text{if the die rolls a } 4, 0 \text{ otherwise}$$

$$I_5 = 1\text{if the die rolls a } 5, 0 \text{ otherwise}$$

$$I_6 = 1\text{if the die rolls a } 6, 0 \text{ otherwise}$$

Each of these indicator variables have a $\frac{1}{6}$ probability of being true.

The values rolled by n dice are represented by n random variables $X_1, X_2 ... X_n$. The sum X of all the random variables represents the sum obtained by rolling these n dice. We wish to compute the expected value for X.

$$E[X] = E[\sum_{i=1}^{n} X_i]$$

$$E[X] = \sum_{i=1}^{n} E[X_i]$$

We can compute $E[X_i]$ using the indicator variables as follows:

$$E[X_i] = 1.Pr(X_i = 1) + 2.Pr(X_i = 2) + 3.Pr(X_i = 3) + 4.Pr(X_i = 4) + 5.Pr(X_i = 5) + 6.Pr(X_i = 6)$$

$$E[X_i] = 1.Pr(I_1 = 1) + 2.Pr(I_2 = 1) + 3.Pr(I_3 = 1) + 4.Pr(I_4 = 1) + 5.Pr(I_5 = 1) + 6.Pr(I_6 = 1)$$

$$E[X_i] = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}$$

$$E[X_i] = \frac{21}{6} = 3\frac{1}{2}$$

Now using $E[X_i]$ we can compute $E[X]$ as:

$$E[X] = \sum_{i=1}^{n} E[X_i]$$

$$E[X] = \sum_{i=1}^{n} 3\frac{1}{2}$$

$$E[X] = 3\frac{1}{2}n$$

(4) CLR 5.2-4

This problem is similar to taking an input sequence of n numbers and permuting them to some random order. If the permuting is uniformly random then each number is equally likely to permute to any given position in the new order. Thus given any one particular position in the array, each of the n numbers is equally likely to end up at that position.

Assuming that the return of the hats is completely randomized, each customer gets back some random hat with uniform probability. The probability of each hat being given to that customer is $\frac{1}{n}$. Hence the probability that a customer gets back his/her own hat is also $\frac{1}{n}$. Let the indicator variables $I_1, I_2, ...I_n$ indicate the event that customers $1, 2, ...n$ get their hats back, respectively . The expected number of customers who get their hat back can be computed as the expectation of the sum of these indicator variables.

$$E[X] = E[\sum_{i=1}^{n} I_i]$$

$$E[X] = \sum_{i=1}^{n} E[I_i]$$

$$E[X] = \sum_{i=1}^{n}(1.Pr(I_i = 1) + 0.Pr(I_i = 0))$$

$$E[X] = \sum_{i=1}^{n}(1.\frac{1}{n})$$

$$E[X] = 1$$

So the expected number of customers that get their hats back is one.

(5) CLR 5.3-3

The algorithm starts with a given permutation of n numbers as the input. Consider what happens during the first iteration of the loop. One element is randomly chosen from n possiblities, and is swapped with the first element of the array. So for a given input array, at the end of the first iteration, there are n permutations possible each with equal probability of $\frac{1}{n}$. For each of these n permutations, the second iteration randomly chooses an element from n possibilities and swaps it with the second element of the array. So for a given input, after the second iteration, there are n^2 permuations possible each with an equal probability of $\frac{1}{n^2}$. Note that these permutations may not necessarily be unique i.e., the count n^2 may contain repeats of the same permutation. The algorithm iterates for n loops, thus at the end of the n'th iteration there should be n^n possible permutations each with equal probability of $\frac{1}{n^n}$. Because there are a total of $n!$ ($\leq n^n$) distinct permutations possible, the n^n count must contain repeats of the same permutations.

For a uniformly random permutation, we require that each permuation be equally likely with a probability of $\frac{1}{n!}$. This means that each permuation must have equal repeats in the n^n count. Say that each permutation is repeated x times. Then we require that

$$\frac{x}{n^n} = \frac{1}{n!}$$
$$x = \frac{n^n}{n!}$$
$$x = \frac{n^{n-1}}{(n-1)!}$$

Because $(n-1)!$ does not divide into n^{n-1} for any $n \geq 3$, we cannot have all $n!$ permutations be equally likely for the given algorithm. Thus it is not uniformly random.

(6) CLR 5.3-5

PERMUTE-BY-SORTING generates random numbers between 0 and n^3 and uses these as ranks to sort the array over. We need to show that in choosing n random numbers from a range of 0 to n^3, the probability that all n numbers are unique is at least $1 - \frac{1}{n}$.

When the first number is chosen, it will be unique with a probability of 1. When the second number is chosen, it will be unique with a probability of $\frac{n^3-1}{n^3}$. Similarly the third choice will be unique with probability $\frac{n^3-2}{n^3}$, and so on. We can write the expression for the probability that all numbers are unique as:

$$P = 1.(\frac{n^3-1}{n^3})(\frac{n^3-2}{n^3})(\frac{n^3-3}{n^3})...(\frac{n^3-n+1}{n^3})$$
$$P = 1(1-\frac{1}{n^3})(1-\frac{2}{n^3})(1-\frac{3}{n^3})...(1-\frac{n-1}{n^3})$$
$$P = 1 - \frac{1}{n^3} - \frac{2}{n^3} - \frac{3}{n^3}... - \frac{n-1}{n^3} + \frac{2}{n^6} + \frac{3}{n^6}... + \frac{n-1}{n^6} - \frac{6}{n^9} - \frac{8}{n^9}....$$

Ignoring $\frac{1}{n^6}$ and smaller terms

$$P \geq 1 - \frac{1}{n^3}\sum_{k=1}^{n-1} k$$

$$P \geq 1 - \frac{1}{n^3}\frac{n(n-1)}{2}$$

$$P \geq 1 - \frac{n^2 - n}{2n^3}$$
$$P \geq 1 - \frac{1}{2n} + \frac{1}{2n^2}$$
$$P \geq 1 - \frac{1}{n} + (\frac{1}{2n} + \frac{1}{2n^2})$$
$$P \geq 1 - \frac{1}{n}$$

(7) CLR 5.4-4

The textbook presents two approaches to analyzing the birthday paradox: first, by computing the probability for all unique birthdays, and, second, by computing the expected number of pairs with the same birthday. We present a solution below based on the second approach.

Regarding the first approach, while we have formulated a summation expression for the probability needed using the first technique, we were unable to obtain a closed-form expression for the summation, i.e. for the probability of three persons having the same birthday. If you have adopted this approach and you believe you have a solution we would like you to share it with us. But, note that this approach involves complex counting/combinatorics formulations, and it is very easy to overcount. We believe there is no easy solution along this path, so, if what you have is not too elaborate, take a close look to make sure you are not counting the same birthday arrangement twice (or not at all).

Now for our solution based on the second approach.

As each persons birthday is an independent random variable, the event that 3 persons have the same birthday can be computed using product of the independent probabilities of each person's birthday. Consider three persons i, j and l. Each person's birthday is equally likely to fall on any day of the year. Given n days in a year, the probability for a person's birthday to fall on a given day is $\frac{1}{n}$. The probability that all three birthdays fall on a given day r is

$$P(b_i = r \text{ and } b_j = r \text{ and } b_l = r) = P(b_i = r)P(b_j = r)P(b_l = r)$$

$$P(b_i = r \text{ and } b_j = r \text{ and } b_l = r) = \frac{1}{n^3}$$

Thus, the probability that all three persons have the same birthday is simply

$$P(b_i = b_j = b_l) = \sum_{r=1}^{n} P(b_i = r \text{ and } b_j = r \text{ and } b_l = r)$$

$$P(b_i = b_j = b_l) = \sum_{r=1}^{n} \frac{1}{n^3}$$

$$P(b_i = b_j = b_l) = \frac{1}{n^2}$$

X_{ijl} is the indicator variable that indicates persons i,j and l having the same birthday. We have from above that $E[X_{ijl}] = 1/n^2$. Let X be the random variable that counts the number of triples of individuals with the same birthday. If there are k persons in the room, we have

$$X = \sum_{i=1}^{k} \sum_{j=i+1}^{k} \sum_{l=j+1}^{k} X_{ijl}$$

$$E[X] = E[\sum_{i=1}^{k} \sum_{j=i+1}^{k} \sum_{l=j+1}^{k} X_{ijl}]$$

$$E[X] = \sum_{i=1}^{k} \sum_{j=i+1}^{k} \sum_{l=j+1}^{k} E[X_{ijl}]$$

$$E[X] = \sum_{i=1}^{k} \sum_{j=i+1}^{k} \sum_{l=j+1}^{k} \frac{1}{n^2}$$

$$E[X] = \binom{k}{3} \frac{1}{n^2}$$

$$E[X] = \frac{k(k-1)(k-2)}{6n^2}$$

To obtain an $E[X]$ equal to 1 for $n = 365$, we can solve

$$k^3 - 3k^2 + 2k = 6n$$

and get a solution of $k = 94$ persons.

(8) CLR 5.4-6

Let $I_{X1}, I_{X2}, I_{X3}, ...I_{Xn}$ be the indicator variables indicating that bins $1, 2, 3...n$ are empty, respectively. Where

$$I_{Xi} = 1 \text{ if bin i is empty}, 0 \text{ otherwise}$$

We can compute $E[I_{Xi}]$ as follows:

$$E[I_{Xi}] = 1.P(\text{ bin i receives no ball}) + 0.P(\text{ bin i receives at least 1 ball})$$

When a ball is tossed it is equally likely to go into any bin with a probability equal to $\frac{1}{n}$. The probability that it will go into bin i is $\frac{1}{n}$, and the probability that it will not go into bin i is $1 - \frac{1}{n} = \frac{n-1}{n}$. For n balls being tossed, we can view this as n trials for the same bernoulli experiment. Bin i receives no ball only when all trails fail. So

$$E[I_{Xi}] = 1.(\frac{n-1}{n})^n + 0.P(\text{ bin i receives at least 1 ball})$$

$$E[I_{Xi}] = (\frac{n-1}{n})^n$$

Let the random variable X represent the number of empty bins at the end of n tosses. X is simply the sum of the indicator variables for all bins.

$$X = I_{X1} + I_{X2} + I_{X3} + ...I_{Xn}$$

$$E[X] = E[\sum_{i=1}^{n} I_{Xi}]$$

$$E[X] = \sum_{i=1}^{n} E[I_{Xi}]$$

$$E[X] = \sum_{i=1}^{n} (\frac{n-1}{n})^n$$

$$E[X] = n(\frac{n-1}{n})^n$$

To compute the expected number of bins with exactly 1 ball, we use another set of indicator variables $I_{Y1}, I_{Y2}, I_{Y3}, ...I_{Yn}$ that indicate the event that bins $1, 2, 3, ...n$ have exactly 1 ball. So

$$I_{Yi} = 1 \text{ if bin i receives exactly 1 ball}, 0 \text{ otherwise}$$

$E[I_{Yi}] = 1.P($ bin i receives exactly 1 ball$)+0.P($ bin i receives 0 or more than 1 balls$)$

$$E[I_{Yi}] = 1.n\frac{1}{n}(\frac{n-1}{n})^{n-1} + 0.P(\text{ bin i receives 0 or at least 2 balls})$$

$$E[I_Y i] = (\frac{n-1}{n})^{n-1}$$

Let Y be the random variable representing the number of bins that have exactly 1 ball

$$Y = I_{Y1} + I_{Y2} + I_{Y3} + ...I_{Yn}$$

$$E[Y] = E[\sum_{i=1}^{n} I_{Yi}]$$

$$E[Y] = \sum_{i=1}^{n} E[I_{Yi}]$$

$$E[Y] = \sum_{i=1}^{n} (\frac{n-1}{n})^{n-1}$$

$$E[Y] = n(\frac{n-1}{n})^{n-1}$$