Numerical Analysis
Homework #1

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1. Find the roots of the following equations by using Bisection, Newton's and Secant methods. Comapare the convergence(verify with real numbers).

(1) $e^x - 3x^2 = 0$

(Solution.)
We can find roots using Rootfinding Method. I choose error boundary less than ϵ
First, We find intersection point of e^x and $3x^2$ or using $f(x)f(y) < 0$

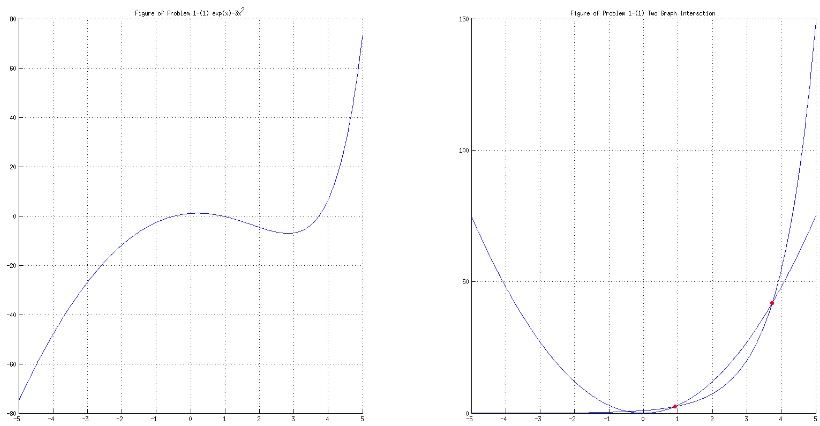

Figure 1: Problem 1-(1), Graph.

Redpoint of Right Graph of Figure 1. is Roots. Problem 1-(1) have two roots.
Therefore, Using rootfinding method,

Bisection				Newton				Secant			
n	x	y	Err	n	x	y	Err	n	x	y	Err
1	0.50000	0.89872	1.00000	1	1.00000	-0.28172	0.00000	1	1.00000	-0.28172	1.00000
2	0.75000	0.42950	0.50000	2	2.00000	-4.61094	1.00000	2	0.78020	0.35577	0.28172
3	0.87500	0.10200	0.25000	3	1.00000	-0.28172	0.09391	3	0.90287	0.02116	0.13586
4	0.93750	-0.08313	0.12500	4	0.91416	-0.01237	0.00455	4	0.91062	-0.00183	0.00852
5	0.90625	0.01116	0.06250	5	0.91002	-0.00003	0.00001	5	0.91000	0.00001	0.00068
6	0.92188	-0.03556	0.03125	6	0.91001	-0.00000	0.00000	6	0.91001	0.00000	0.00000
7	0.91406	-0.01210	0.01562	7	0.91001	-0.00000	0.00000	7	0.91001	-0.00000	0.00000
8	0.91016	-0.00044	0.00781	...	...	...	...	8	0.91001	0.00000	0.00000
9	0.90820	0.00536	0.00391	...	...	...	...	9	0.91001	0.00000	0.00000
...	...	...	...	...	...	...	...	...	...	...	...
50	0.91001	0.00000	0.00000	...	...	...	...	...	...	...	...
51	0.91001	0.00000	0.00000	...	...	...	...	...	...	...	...
52	0.91001	0.00000	0.00000	...	...	...	...	...	...	...	...

Figure 2: Problem 1-(1), Result of 1 near root

We choose initial condition,

Bisection: $x_0 = 0, x_1 = 1$**, Newton:** $x_0 = 1$**, Secant:** $x_0 = 0, x_1 = 1$

We aleady know this result. Because we learned convergence speed. We have result as expected.

Therefore, We know convergence speed which is Bisection << Secant < Newton.
Next, We once more find root which is near 3. Similary, using rootfinding method,

RESULT

Bisection				Newton				Secant			
n	x	y	Err	n	x	y	Err	n	x	y	Err
1	3.50000	-3.63455	1.00000	1	4.00000	6.59815	0.00000	1	4.00000	6.59815	0.25000
2	3.75000	0.33358	0.50000	2	5.00000	73.41316	0.14155	2	3.51170	-3.49088	0.13905
3	3.62500	-1.89715	0.25000	3	4.38003	22.28619	0.10497	3	3.68066	-0.96924	0.04590
4	3.68750	-0.84811	0.12500	4	3.96393	5.52558	0.05072	4	3.74560	0.24581	0.01734
5	3.71875	-0.27446	0.06250	5	3.77260	0.79545	0.01021	5	3.73246	-0.01200	0.00352
6	3.73438	0.02518	0.03125	6	3.73446	0.02687	0.00037	6	3.73307	-0.00014	0.00016
7	3.72656	-0.12572	0.01562	7	3.73308	0.00003	0.00000	7	3.73308	0.00000	0.00000
8	3.73047	-0.05054	0.00781	8	3.73308	0.00000	0.00000	8	3.73308	-0.00000	0.00000
9	3.73242	-0.01275	0.00391	9	3.73308	0.00000	0.00000	9	3.73308	-0.00000	0.00000
10	3.73340	0.00620	0.00195	...	...	...	...	10	3.73308	0.00000	0.00000
11	3.73291	-0.00328	0.00098	...	...	...	...	11	3.73308	0.00000	0.00000
...	...	...	...	...	...	...	...	...	...	...	...
50	3.73308	0.00000	0.00000	...	...	...	...	...	...	...	...
51	3.73308	-0.00000	0.00000	...	...	...	...	...	...	...	...
52	3.73308	0.00000	0.00000	...	...	...	...	...	...	...	...

Figure 3: Problem 1-(1), Result of 3 near root

We can know convergence speed, as known above. Therefore, we can check that i learned in theory.

(2) $x^3 = x^2 + x + 1$

(Solution.)
Similary, we can find roots using Rootfinding Method. I choose error boundary less than ϵ
First, We find intersection point of x^3 and $x^2 + x + 1$ or using $f(x)f(y) < 0$

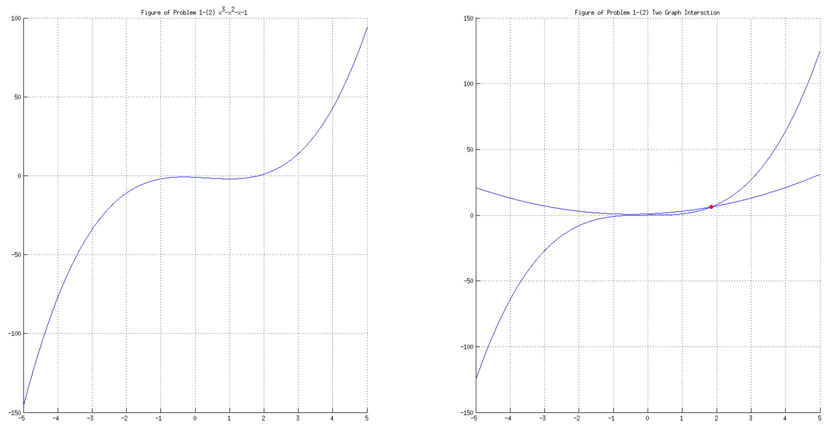

Figure 4: Problem 1-(2), Graph.

Problem 1-(2) have one root near 2.Therefore, Using rootfinding method,
We choose initial condition,

Bisection: $x_0 = 1, x_1 = 2$**, Newton:** $x_0 = 1$**, Secant:** $x_0 = 1, x_1 = 2$

Like Problem (1), We have similar result. We can know centain comparison of convergence speed.

Bisection << Secant < Newton.

(3) $e^x = \frac{1}{0.1+x^2}$

(Solution.)
Similary, we can find roots using Rootfinding Method. I choose error boundary less than ϵ
First, We find intersection point of e^x and $\frac{1}{0.1+x^2}$ or using $f(x)f(y) < 0$
Problem 1-(3) have one root near 1.Therefore, Using rootfinding method,
We choose initial condition,

Bisection: $x_0 = 0, x_1 = 1$**, Newton:** $x_0 = 1$**, Secant:** $x_0 = 0, x_1 = 1$

We can get different result from above. Newton Method don't have convergence point.
Since there is not differential value of $f(x) = e^x - \frac{1}{0.1+x^2}$ at x=0 and $f' = 0$ where $x < -1$.Therefore,

Bisection				Newton				Secant			
n	x	y	Err	n	x	y	Err	n	x	y	Err
1	1.50000	-1.37500	1.00000	1	1.00000	-2.00000	0.00000	1	2.00000	1.00000	0.50000
2	1.75000	-0.45312	0.50000	2	2.00000	1.00000	0.07692	2	1.66667	-0.81481	0.20000
3	1.87500	0.20117	0.25000	3	1.85714	0.09913	0.00957	3	1.81633	-0.12323	0.08240
4	1.81250	-0.14331	0.12500	4	1.83954	0.00141	0.00014	4	1.84299	0.02034	0.01447
5	1.84375	0.02451	0.06250	5	1.83929	0.00000	0.00000	5	1.83922	-0.00039	0.00205
6	1.82812	-0.06050	0.03125	...	...	...	...	6	1.83929	-0.00000	0.00004
7	1.83594	-0.01827	0.01562	...	...	...	...	7	1.83929	0.00000	0.00000
8	1.83984	0.00305	0.00781	...	...	...	...	8	1.83929	-0.00000	0.00000
...	...	...	...	...	...	...	...	...	...	...	...
50	1.83929	0.00000	0.00000	...	...	...	...	...	...	...	...
51	1.83929	0.00000	0.00000	...	...	...	...	...	...	...	...
52	1.83929	0.00000	0.00000	...	...	...	...	...	...	...	...

Figure 5: Problem 1-(2), Result of 2 near root

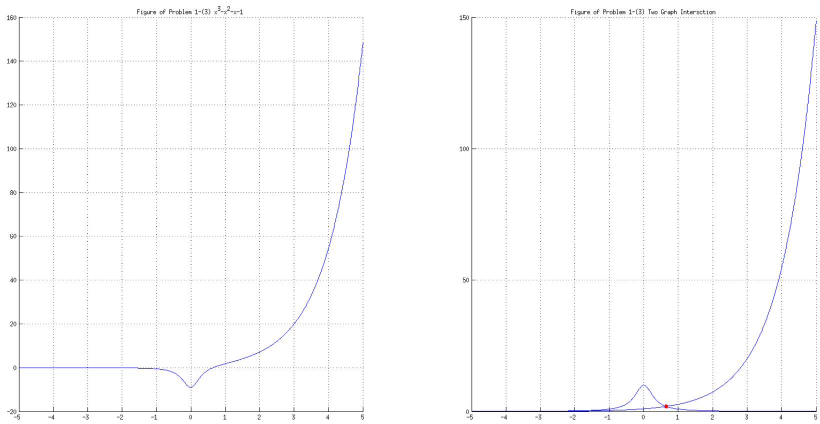

Figure 6: Problem 1-(3), Graph.

Bisection				Newton				Secant			
n	x	y	Err	n	x	y	Err	n	x	y	Err
1	0.50000	-1.20842	1.00000	1	1.00000	1.80919	0.00000	1	1.00000	1.80919	1.00000
2	0.75000	0.60757	0.50000	2	2.00000	7.14515	1.04235	2	0.83262	1.03873	0.20102
3	0.62500	-0.16997	0.25000	3	0.97926	1.71817	0.95553	3	0.60697	-0.30001	0.37177
4	0.68750	0.24249	0.12500	...	...	...	...	4	0.65754	0.05160	0.07691
5	0.65625	0.04312	0.06250	...	...	...	...	5	0.65012	0.00245	0.01141
6	0.64062	-0.06158	0.03125	...	...	...	...	6	0.64975	-0.00002	0.00057
7	0.64844	-0.00879	0.01562	...	...	...	...	7	0.64975	0.00000	0.00000
8	0.65234	0.01728	0.00781	...	...	...	...	8	0.64975	0.00000	0.00000
9	0.65039	0.00427	0.00391	...	...	...	...	9	0.64975	-0.00000	0.00000
10	0.64941	-0.00225	0.00195	...	...	...	...	10	0.64975	0.00000	0.00000
11	0.64990	0.00101	0.00098	...	...	...	...	11	0.64975	-0.00000	0.00000
...	...	...	...	...	...	...	...	...	...	...	...
50	0.64975	-0.00000	0.00000	...	...	...	...	...	...	...	...
51	0.64975	0.00000	0.00000	...	...	...	...	...	...	...	...
52	0.64975	0.00000	0.00000	...	...	...	...	...	...	...	...

Figure 7: Problem 1-(3), Result of 1 near root

no longer newton method don't work. We can know disadvantage of Newton method above result.

(4) $x = 1 + 0.3\cos(x)$

(Solution.)
Similary, we can find roots using Rootfinding Method. I choose error boundary less than ϵ
First, We find intersection point of x and $1 + 0.3\cos(x)$ or using $f(x)f(y) < 0$

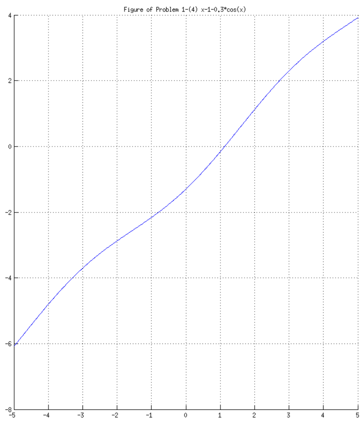
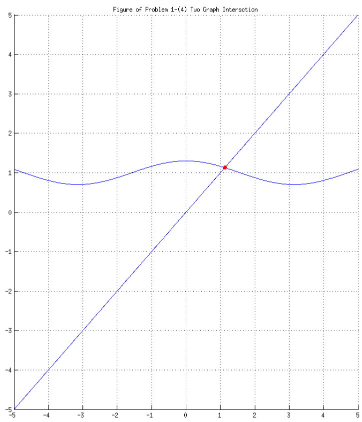

Figure 8: Problem 1-(4), Graph.

Problem 1-(4) have one root near 1.Therefore, Using rootfinding method,

Bisection				Newton				Secant			
n	x	y	Err	n	x	y	Err	n	x	y	Err
1	1.50000	0.47878	1.00000	1	2.00000	1.12484	0.00000	1	2.00000	1.12484	0.50000
2	1.25000	0.15540	0.50000	2	3.00000	2.29700	0.03981	2	1.12595	-0.00314	0.77628
3	1.12500	-0.00435	0.25000	3	2.88515	2.17534	0.04321	3	1.12839	-0.00005	0.00216
4	1.18750	0.07531	0.12500	4	2.76564	2.04469	0.04716	4	1.12843	0.00000	0.00003
5	1.15625	0.03542	0.06250	5	2.64108	1.90428	0.05179	5	1.12843	-0.00000	0.00000
6	1.14062	0.01552	0.03125	6	2.51104	1.75335	0.05725	6	1.12843	-0.00000	0.00000
...	...	...	...	...	...	...	...	...	...	...	...
40	1.12843	-0.00000	0.00000	40	1.18554	0.07280	0.07832	...	...	...	...
41	1.12843	0.00000	0.00000	41	1.09943	-0.03680	0.07263	...	...	...	...
42	1.12843	0.00000	0.00000	42	1.18554	0.07280	0.07832	...	...	...	...
43	1.12843	-0.00000	0.00000	43	1.09943	-0.03680	0.07263	...	...	...	...
44	1.12843	0.00000	0.00000	44	1.18554	0.07280	0.07832	...	...	...	...
45	1.12843	-0.00000	0.00000	45	1.09943	-0.03680	0.07263	...	...	...	...
46	1.12843	0.00000	0.00000	46	1.18554	0.07280	0.07832	...	...	...	...
47	1.12843	-0.00000	0.00000	47	1.09943	-0.03680	0.07263	...	...	...	...
48	1.12843	0.00000	0.00000	48	1.18554	0.07280	0.07832	...	...	...	...
49	1.12843	0.00000	0.00000	49	1.09943	-0.03680	0.07263	...	...	...	...
50	1.12843	-0.00000	0.00000	50	1.18554	0.07280	0.07832	...	...	...	...
51	1.12843	0.00000	0.00000	51	1.09943	-0.03680	0.07263	...	...	...	...
52	1.12843	0.00000	0.00000	52	1.18554	0.07280	0.07832	...	...	...	...
...	...	...	...	53	1.09943	-0.03680	0.07263	...	...	...	...
...	...	...	...	54	1.18554	0.07280	0.07832	...	...	...	...
...	...	...	...	55	1.09943	-0.03680	0.07263	...	...	...	...

Figure 9: Problem 1-(4), Result of 1 near root

We choose initial condition,

Bisection: $x_0 = 1, x_1 = 2$, **Newton:** $x_0 = 2$, **Secant:** $x_0 = 1, x_1 = 2$

We can know disadvantage of Newton Method. This situation is Cycling of Newton Method.
Newton Method don't convergence a point.

- Comment

Newton Method have the fastest convergence speed. But we should careful use it.
Since (3) and (4) situation are able to occur. Because We can know Newton Method always don't haveguarantee to find root. Maybe, I think Newton Method use diffential value. On the other hand, Bisection and Secant Method converge well.
(Bisection and Secant Method are covergence speed less than Newton method.)
I think most important choose initial condition and looking to characteristics of the function.

2. Witch of the following iterations will converge to the indicated fixed point α (provided x_0 is sufficiently close to α)? If it does converge, give the order of convergence; for linear convergence, give the rate of linear convergence.

(a) $x_{n+1} = -16 + 6x_n + 12/x_n, \quad \alpha = 2$

(Solution.)
Before using Fixed Point Method, Looking at Graph of $f(x) = -16 + 6x + 12/x$

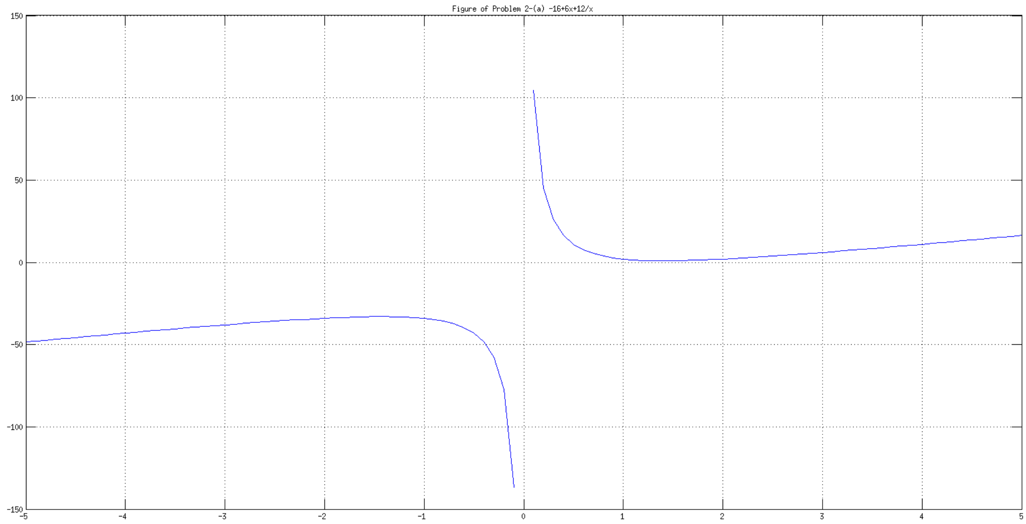

Figure 10: Problem 2-(a), $-16 + 6x + 12/x$

We can know $f(x)$ don't have root. Using Fixed Point Method,

R E S U L T

F i x e d P o i n t

Iteration	x	y	Error
1	2.00000000000000000000	2.00000000000000000000	0.33333333333333331483
2	3.00000000000000000000	6.00000000000000000000	0.50000000000000000000
3	6.00000000000000000000	22.00000000000000000000	0.72727272727272729291
4	22.00000000000000000000	116.54545454545454674644	0.81123244929797189151
5	116.54545454545454674644	683.37569139129197992588	0.82945624783904980948
6	683.37569139129197992588	4084.27170823486949302605	0.83268113871723004049
7	4084.27170823486949302605	24489.63318750953112612478	0.83322446371642788776
8	24489.63318750953112612478	146921.79961506044492125511	0.83331518364413503797
9	146921.79961506044492125511	881514.79777203872799873352	0.83333030825302756295
10	881514.79777203872799873352	5289072.78664584551006555557	0.83333282914961237520
...	...	...	...
Inf	Inf	Inf	...

Figure 11: Problem 2-(a), Result

We choose initial condition. $x_0 = 2 + \epsilon$ and relative error boundary less than ϵ.
Therefore, We can know this function is divergence using fixed point method.

(b) $x_{n+1} = -2x_n/3 + 1/x_n^2, \quad \alpha = 3^{\frac{1}{3}}$

(Solution.)
Similary, Before using Fixed Point Method, Looking at Graph of $f(x) = -2x/3 + 1/x^2$

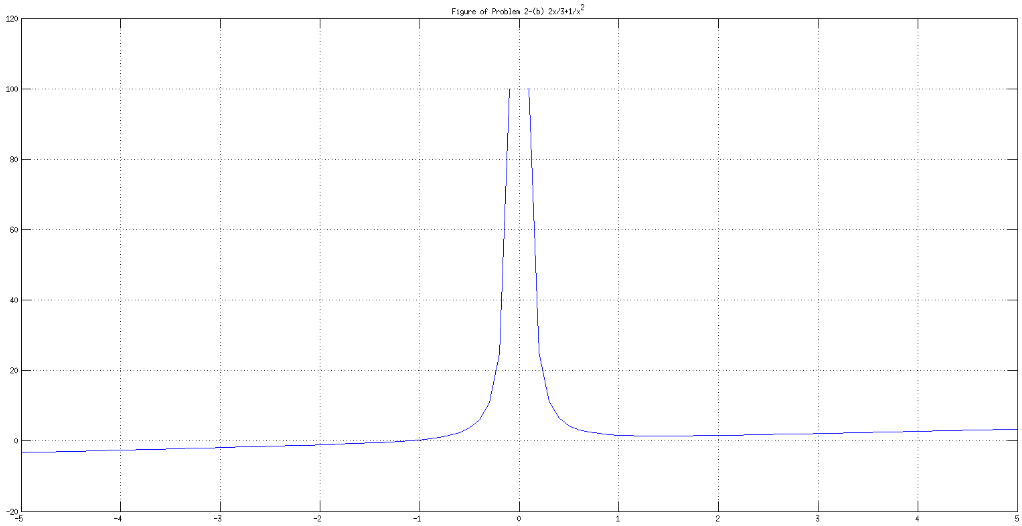

Figure 12: Problem 2-(b), $-2x/3 + 1/x^2$

We can know $f(x)$ have root. Using Fixed Point Method,

RESULT

Fixed Point

Iteration	x	y	Error
1	1.44224957030740852382	1.44224957030740830177	0.40945856318612394720
2	2.44224957030740874586	1.79582269517138404247	0.35996141315851509779
3	1.79582269517138404247	1.50729465395461681432	0.19142112689099641254
4	1.50729465395461681432	1.44501611897609216761	0.04309885139734905002
5	1.44501611897609216761	1.44225486360962551480	0.00191454051301021694
6	1.44225486360962551480	1.44224957032683565039	0.00000367015730062923
7	1.44224957032683565039	1.44224957030740830177	0.00000000001347017120

Figure 13: Problem 2-(b), Result

We choose initial condition. $x_0 = 3^{\frac{1}{3}} + \epsilon$ and relative error boundary less than ϵ.
We can know this function is convergence using fixed point method.Therefore, convergence rate is

$$\begin{aligned} f(x_n) - (\alpha + \epsilon) = r(x_n - (\alpha + \epsilon)) &\rightarrow x_{n+1} - (\alpha + \epsilon) = r(x_n - (\alpha + \epsilon)) \\ &\rightarrow r = \tfrac{x_{n+1} - (\alpha+\epsilon)}{x_n - (\alpha+\epsilon)} \\ &\rightarrow r = g(\alpha) \approx \tfrac{x_{n+1} - \alpha + \epsilon}{x_n - \alpha + \epsilon} \text{ (Using I.V.T)} \\ &\rightarrow r = -\tfrac{1}{3}. \end{aligned}$$

Since $f'(x) = \frac{2}{3} - 3x^{-3}$. Therefore, convergence rate is $-\frac{1}{3}$.

(c) $x_{n+1} = 12/(1+x_n), \quad \alpha = 3$

(Solution.)
Similary, Before using Fixed Point Method, Looking at Graph of $f(x) = 12/(1+x_n)$

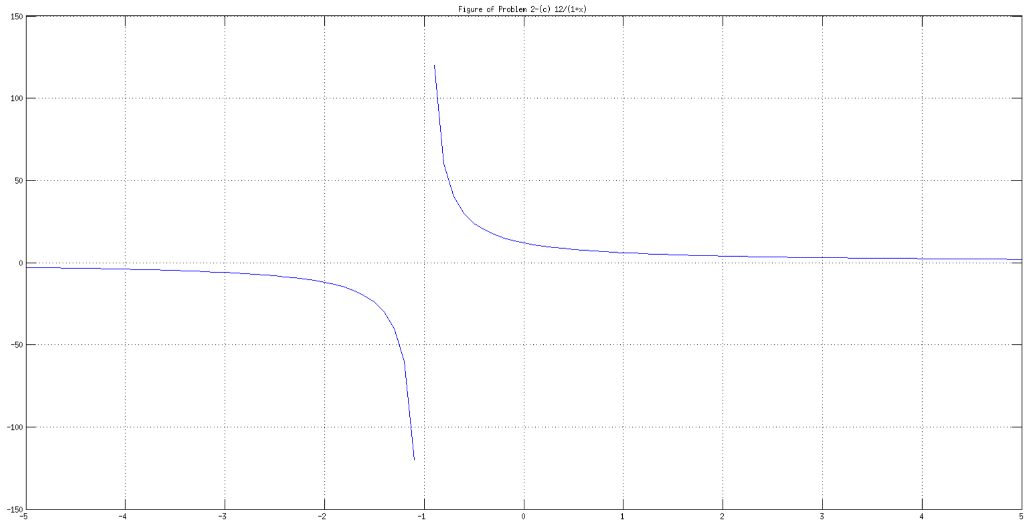

Figure 14: Problem 2-(c), $12/(1+x_n)$

We can know $f(x)$don't have root. Using Fixed Point Method,

RESULT

Fixed Point

Iteration	x	y	Error
1	3.00000000000000000000	3.00000000000000000000	0.25000000000000000000
2	4.00000000000000000000	2.39999999999999991118	0.66666666666666674068
3	2.39999999999999991118	3.52941176470588224845	0.32000000000000000666
4	3.52941176470588224845	2.64935064935064934488	0.33217993079584773319
5	2.64935064935064934488	3.28825622775800718856	0.19429920728622029902
6	3.28825622775800718856	2.79834024896265587401	0.17507377059561041177
7	2.79834024896265587401	3.15927463403976371836	0.11424596683941375119
8	3.15927463403976371836	2.88511845353784757151	0.09502423727723723068
9	2.88511845353784757151	3.08870891415746129383	0.06591442129312764797
10	3.08870891415746129383	2.93491179047965466964	0.05240263921276879650
...	...	...	...
121	2.99999999999999866773	3.00000000000000088818	0.00000000000000074015
122	3.00000000000000088818	2.99999999999999955591	0.00000000000000044409
123	2.99999999999999955591	3.00000000000000044409	0.00000000000000029606

Figure 15: Problem 2-(c), Result

We choose initial condition. $x_0 = 3 + \epsilon$ and relative error boundary less than ϵ.
We can know this function is convergence using fixed point method.Therefore, convergence rate is

$$f(x_n) - (\alpha + \epsilon) = r(x_n - (\alpha + \epsilon)) \to x_{n+1} - (\alpha + \epsilon) = r(x_n - (\alpha + \epsilon)) \\ \to r = -\frac{3}{4}.$$

Since $f'(x) = -\frac{12}{(1+x)^2}$. Therefore, convergence rate is $-\frac{3}{4}$.

- Comment

We investigated the convergence. We calculated convergence rate of (b) and (c), When the other problem (a) compute convergence rate $r = 3$.
We can know $|r| < 1$ is convergence, $|r| > 1$ is divergence. And we can know slow convergence and divergence when r is closer 1. Above all the reason is

$$x_{n+1} - \alpha = r(x_n - \alpha) \to E_{n+1} = rE_n.$$

Therefore, If r increase, Error is more and more increased by r.

3. Consider the system

$$x = \frac{0.5}{1+(x+y)^2}, \quad y = \frac{0.5}{1+(x-y)^2}$$

Find a bouded region D for which the hypotheses of Theorem in page 10 of NA lecture note #1 are satisfied.

(Solution.)
The hypotheses of Theorem in page 10 of NA lecture note

$$g(D) \subset D, \quad \lambda \equiv max_{x \in D} \|G(x)\|_\infty \rightarrow \exists!\text{solution and convergenc in } D \text{ to } \alpha.$$

We can calculate Boundary condition, substituting $(x+y)^2$ to T.
Boundary of$x = \frac{0.5}{1+T^2}$ is

$$\rightarrow T^2 \geq 0$$
$$\rightarrow 1 + T^2 \geq 1$$
$$\rightarrow \frac{1}{1+T^2} \leq 1$$
$$\rightarrow \frac{0.5}{1+T^2} \leq 0.5$$

Therefore, $x \leq 0.5$, we can compute y using simlar manner. $y \leq 0.5$.
For recheck, Using Matlab, (Initical condition $x_0 = 0.5$, $y_0 = 0.5$)

RESULT

Fixed Point

Iteration	x	y	Error
1	0.50000000000000000000	0.50000000000000000000	0.00000000000000000000
2	1.50000000000000000000	1.50000000000000000000	0.96666666666666667407
3	0.05000000000000000278	0.50000000000000000000	6.67754318618042219668
4	0.38387715930902110983	0.41580041580041576799	0.20554282814836677407
5	0.30497396232308376929	0.49949097159544042501	0.00466232054893498532
6	0.30355207595165473178	0.48177128405092389807	0.01882256023979731255
7	0.30926570318717028218	0.48460784391947586691	0.00828250362714415614
8	0.30670420887877125615	0.48508609292110482780	0.00203040262954292837
9	0.30732694191097059688	0.48458059856522084363	0.00011411015658246712
10	0.30729187278550712525	0.48476910785113086488	0.00014934881947028532
...	...	...	...
32	0.30726479114590404818	0.48473329168512946374	0.00000000000000234861
33	0.30726479114590476982	0.48473329168512946374	0.00000000000000072265
34	0.30726479114590454778	0.48473329168512951926	0.00000000000000018066

Figure 16: Problem 3, Result

Therefore, we can know that calculated boudary $D = \{(x, y) | x \in (0, 0.5],\ y \in (0, 0.5]\}$.
Once again, we are running iteration 1000 using random function to select initial condition.

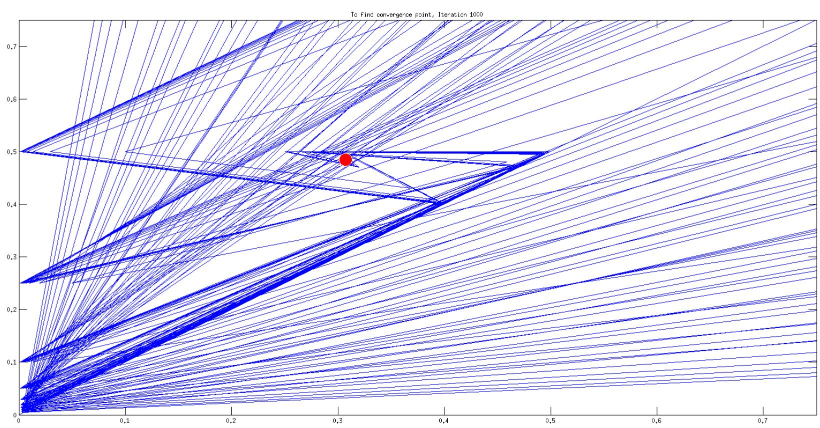

Figure 17: Problem 3, Random initial condition, iteration 1000

Therefore, we know this function is started at some initial point. Convergence point is about $(0.30, 0.48)$.

4. Consider

$$f(x) = \frac{1}{1+x^2}, \quad -5 \leq x \leq 5$$

Use Lagrange's formula and Newton's divided difference formula to construct p_6, p_8, p_{10} and p_{16} interpolation polynomials in uniformly divided interval.

(Solution.)
To solve this problem, We can know disadvantage of Lagrange and Newton interpolation.
We already knew that Newton is computing better than Langrange. Priority, comparing Original with Numerical function,

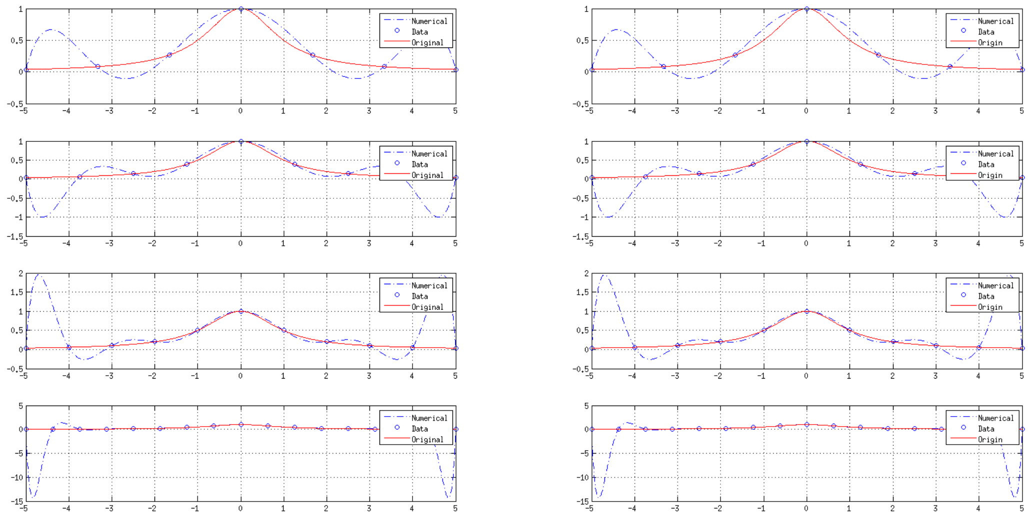

Figure 18: Problem 4, Left figure is f(x) vs Lagrange.Right figure is f(x) vs Newton

Two method is similar result. Results show that each degree.

R E S U L T

Interpolation Method

degree	Lagrange Time	Newton Time	norm(f(x)-lagrange,inf)	norm(f(x)-newton,inf)	norm(lagrange-newton,inf)
6	0.2671290	0.1681380	0.61691948634373006133	0.61691948634372717475	0.00000000000001387779
8	0.0550630	0.0446817	1.04517391178372132110	1.04517391178377327954	0.00000000000006594725
10	0.0537940	0.0453469	1.91564305021916880811	1.91564305021923408923	0.00000000000036393111
16	0.0549450	0.0480365	14.38626843693654855372	14.38626843683218226033	0.00000000018136962765

Figure 19: Problem 4, Result

You can see Newton time is shorter than Lagrange. And Lagrange is similar to Newton.
But two interpolation have ocillation at boundary. Therefore, we need other method to minimize ocillation.
Other method mean piecewise interporation using some a condition. Next problem (5) is Spline which is piecewise interporation.

5. Consider the same function $f(x)$ given in "problem4" and consider

$$x_k = -5 + k(10/16), \quad y_k = f(x_k), \text{ for } k = 0, 1, ..., 16$$

Construct cubic sspline interpolation and compare the result with the results when p_{16} polynomial was used in "problem4"-interpolation.

(Solution.)
Cubic Spline is piecewise interpolation. Therefore, we can expect more and more exact solution.
Prior to the description, Cubic Spline end-conditions are Natural, Complete and Not-A-Knot. Each condition is simila to boundary condition.
In some cases, We can select end-condition. This problem compare Problem (4) with 3 case of Cubic Spline. First, 3 condition of Cubic Spline are

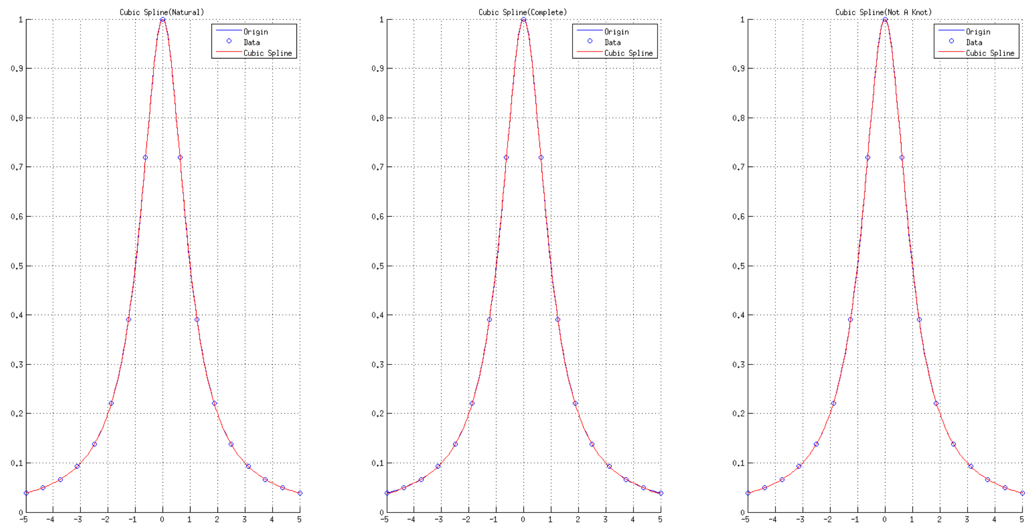

Figure 20: Problem 5, Natural Condition vs. Complete Condition vs. Not-A-Knot Condition.

We can know that this problem is not good using Complete Condition. Because boudary have a few ocillation. But Cubic Spline have result better than problem (4). Examined error is

R E S U L T

Interpolation Method	norm(f(x)-'method',2)	norm(f(x)-'method',inf)
Lagrange	41.376270277118948115457897074520587921 1426	14.245994254998173289550322806462645530 7007
Newton	41.37627027675149093965956126339733600616 46	14.24599425489270920763829053612425923347 47
Cubic(Natural)	0.0138760852338742184108877353310163016431	0.0037367980487872376471614188631065189838
Cubic(Complete)	0.0147586100400862001158497349706522072665	0.0037362126028780417996699725335929542780
Cubic(N-A-K)	0.0138674421184685064956099154187540989369	0.0037367653486699570208884324529208242893

Figure 21: Problem 5, Result

Therefore, We can exactly know that Cubic Spline of piecewise interporation is better than Lagrange and Newton interpolation.