习题——常微分方程

2020 年 10 月 4 日

Homework 01 (Sep 7, 2020)

问题 0.1. Try to solve the following ODEs

(1) $\dot{y} + y^2 \sin t = 0, \qquad y(0) = 1$

(2) $\dot{y} + y^2 \sin t = 0, \qquad y(0) = 0$

(3) $\dot{y}(t + 2y) = 1$

(4) $\dot{y} = \cos(t - y)$

(5) $t^2\dot{y} + ty + 1 = 0$

(6) $2xy\mathrm{d}x + (x^2 - y^2)\mathrm{d}y = 0$

(7) $e^{-y}\mathrm{d}x - (xe^{-y} + 2y)dy = 0$

问题 0.2. Assume $f \in C[0, +\infty[$, $\int_0^\infty |f| < \infty$. Prove that the solution of

$$\dot{x}(t) = -x(t) + f(t)x(t), \qquad x(0) = a$$

satisfies

$$x(t) \sim (c + \delta(t)) \cdot e^{-t} \qquad \text{as } t \to +\infty$$

where $c = \text{const}$, and $\delta(+\infty) = 0$.

问题 0.3. If f is bounded, then

$$\dot{x} + x = f(t), \qquad t \in \mathbb{R}$$

has a bounded solution x. Furthermore, if f is periodic, then so is x.

问题 0.4. Let $(x(t), y(t))$ be the unique solution of the ODE

$$\begin{cases} \dot{x} = y, & x(0) = 0 \\ \dot{y} = -\sin x & y(0) = 2 \end{cases}$$

(1) Show that $\dot{x} = 2\cos\frac{x(t)}{2}$.

(2) Show that

$$\frac{x(t)}{2} = \arctan e^t - \arctan e^{-t}, \qquad y(t) = \frac{2}{\cosh t}$$

问题 0.5. Let

$$u(t) = \int_0^\infty e^{-x}\sin tx \frac{1}{\sqrt{x}}\mathrm{d}x, \qquad t \in \mathbb{R}$$

$$v(t) = \int_0^\infty e^{-x}\cos tx \frac{1}{\sqrt{x}}\mathrm{d}x, \qquad t \in \mathbb{R}$$

Try to compute u, v.

Homework 02 (Sep 14, 2020)

问题 0.6. Prove the Lemma:

If $V(r) = -Ar^s(-2 < s < 0, A > 0)$, then

$$\Theta(0^-) := \lim_{E\to 0^-} \Theta(E) = \frac{\pi}{2+s}$$

Homework 03 (Sep 21, 2020)

问题 0.7. (For self-thinking)

(1) Let $A \in \mathbb{C}^{p\times p}$, $B \in \mathbb{C}^{q\times q}$, $C \in \mathbb{C}^{p\times q}$. If $\|A\|\,\|B\| < 1$, then the matrix equation

$$X - AXB = C$$

has a unique solution $X \in \mathbb{C}^{p\times q}$.

(2) Let $A \in \mathbb{C}^{p\times p}$, $B \in \mathbb{C}^{q\times q}$, $C \in \mathbb{C}^{p\times q}$. If $\|A^n\|\,\|B^n\| < 1$, then the matrix equation

$$X - AXB = C$$

has a unique solution $X \in \mathbb{C}^{p\times q}$.

问题 0.8. Let $A \in C(\mathbb{R}, \mathbb{R}^{n\times n})$, $B \in C(\mathbb{R}, \mathbb{R}^n)$. Prove that there exists a unique solution of the IVP

$$\dot{x}(t) = A(t)x + B(t), \qquad x(0) = x_0$$

问题 0.9. Prove that $0 \le f \in C[0,T]$ and $f(t) \le \int_0^t f$ on $[0,T]$ imply $f = 0$.

问题 0.10. Let $F \in C^2(\mathbb{R}^2, \mathbb{R})$ and $\partial_x F + F\partial_y F = 0$ on $\mathbb{R}^2$. Prove that $F = \text{const}$.

问题 0.11. Let $F \in C(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $\rho > 0$. Given $v_1, v_2 \in \{v \in \mathbb{R}^n : |v| < \rho\}$, $v_1 \ne v_2$. Let x_j be the solution of

$$\ddot{x_j} = F(x_j, \dot{x_j}), \quad x_j(0) = 0, \quad \dot{x_j}(0) = y, \quad j = 1, 2$$

Prove that $\exists \epsilon > 0$, s.t. $x_1 \ne x_2$ on $]0, \epsilon]$.

Homework 04 (Sep 28, 2020)

For self-thinking

问题 0.12. Let $A \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ and $\int_0^\infty \|A(s)\|_{op}\, \mathrm{d}s < \infty$. Prove that the solution of

$$\dot{x}(t) = A(t)x(t)$$

exists on $[0, +\infty[$. Moreover, $\lim_{t\to+\infty} x(t)$ exists.

问题 0.13. * Let $F \in C^1(\mathbb{R}^{n+1}, \mathbb{R}^n)$. If the maximal solution ϕ of the IVP

$$\dot{x} = F(t, x), \qquad x(0) = x_0$$

is bounded, then it is defined on $]-\infty, +\infty[$.

As Homework

问题 0.14. Let $A \in C(\mathbb{R}, \mathbb{R}_{>0})$ and

$$|F(t, x)| \leq A(t)\,|x|\,, \qquad (t, x) \in \mathbb{R}_{>0} \times \mathbb{R}^n$$

Prove that the solution of $\dot{x} = F(t, x), \quad x(0) = x_0$ can be extended on $t \in [0, +\infty[$.

问题 0.15. Let $F \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be bounded. Then the solution of the IVP

$$\dot{x} = F(t, x), \qquad x(0) = x_0$$

exists on $]-\infty, +\infty[$.

证明. Hint: Consider a compact set $B_m(0)$, $|(t, x)| \leq m$. Then let $m \to \infty$. $\square$