Analysis and Prediction of the Implied Volatility

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1 Introduction and Motivation

1.1 In deriving the option pricing formula, one of the most famous formula is the Black-Scholes formula. Under the condition that the stock follows geometric Brownian motion, Black-Scholes formula provides a theoretical estimate of the price of European-style options, and also underpins the risk-management practices of modern financial institutions. While discussing the uncertainty in the prices of option, volatility in Black-Scholes formula is widely characterized as a practical measurement of risk. The forecasting of volatility has significant implications for all investors focused on risk-adjusted returns, especially those that employ asset allocation, risk parity, and volatility targeting strategies. The value of a call option for a non-dividends-paying underlying stock in terms of the Black-Scholes parameters is:

$$
\begin{aligned}
&C(S_t, t) = N(d_1)S_t - N(d_2)PV(K) \\
&d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \\
&d_2 = d_1 - \sigma\sqrt{T-t} \\
&PV(K) = Ke^{-r(T-t)}
\end{aligned}
$$

where N(·) is the cumulative distribution function of the standard normal distribution, T-t is the time to maturity (expressed in years), S_t is the spot price of the underlying asset, K is the strike price, r is the risk free rate (annual rate, expressed in terms of continuous compounding), σ is the volatility of returns of the underlying asset.

2 Problem Statement

2.1 An understanding of distinct numerical analysis methods and interpolation methods from Math104A used to forecast implied volatility of their assumptions and dependencies provides a robust framework for the process of risk budgeting. However, it is hard to find the implied volatility as a function of other parameters in the Black-Scholes formula, so in this project, we are interested in using distinct numerical analysis methods to approximate the value of implied volatility in each data. Hence, our goal is to approximate implied volatility of the option of AUPH by applying zero finding method and use interpolation methods to approximate the path of implied volatility in a given certain period.

3 Methods and Results Analysis

3.1 In order to find the implied volatility in each data, we select five data as our sample to build the simulation model:

Expiration Time	Name	Stock Price	Strike Price	Interest Rate	Price of Option
162/365 (11/07)	AUPH	5.13	4	1.56%	2.7
164/365 (11/05)	AUPH	5.13	4	1.56%	2.49
165/365 (11/04)	AUPH	5.13	4	1.56%	2.63
168/365 (11/01)	AUPH	5.13	4	1.56%	2.5
169/365 (10/31)	AUPH	5.13	4	1.56%	2.36

3.2 Then, we need to prove that Newton's Method and Secant Method can be effective approaches to calculate the implied volatility in the Black-Scholes model.Here is our proof:
First, we establish a zero-finding function:
$f(\sigma) = S_0N(d_1) - xe^{-rT}N(d_2) - C(S_0, 0)$.
In order to prove the availability of Newton's method, we need to prove that $f'(\sigma)$ and $f^{(2)}(\sigma)$ both exist and are continuous first.
According to Black-Scholes formula, $f^{(2)}(\sigma) = -\frac{S_0d_1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}(\frac{\sqrt{T}}{2} - \frac{\ln\frac{S_0}{x}+rT}{\sigma^2\sqrt{T}})^2 + \frac{S_0}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\frac{2(\ln\frac{S_0}{x}+rT)}{\sigma^3\sqrt{T}} + \frac{xe^{-rT}d_2}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}}(-\frac{\sqrt{T}}{2} - \frac{\ln\frac{S_0}{x}+rT}{\sigma^2\sqrt{T}})^2 + \frac{xe^{-rT}}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}}\frac{2(\ln\frac{S_0}{x}+rT)}{\sigma^3\sqrt{T}}$.
Given fixed S_0, r, T, x, C_0, let $g(\sigma) = S_0N(d_1) - xe^{-rT}N(d_2)$ and $g(\sigma_0) = C_0$, then by the definition

of Vega we have $\frac{dC_0}{d\sigma} > 0$ for all σ. This means $g(\sigma)$ is monotonic increasing.
Thus, for $\sigma < \sigma_0$, $f(\sigma) = g(\sigma) - g(\sigma_0) < 0$, and the smaller the difference between σ and σ_0, the smaller $f(\sigma)$ will be. Therefore, $f(\sigma)$ is monotonic increasing. This gives $f'(\sigma) > 0$ for all σ. Thus, when $f(\sigma_0) = 0, f'(\sigma_0) \neq 0$.
Since the interval we choose is always positive, $\sigma \neq 0$. Then $f \in C^2[a, b]$ at here.
Then according to Theorem 2.6, we can use Newton's Method Iteration to find the zero of $f(\sigma)$. Also, since $f'(\sigma)$ is more difficult and need more arithmetic operations than $f(\sigma)$, so Secant method is also an effective method to find the implied volatility we need in our project. The prerequisite for Newton's Method also satisfies the prerequisite for Secant method.
For False Position Method, it always converges in our project, since the root is always bracketed between successive iterations. The prerequisite for Newton's method also validates False Position Method.

3.3 Next, by plugging T,S_t,r, K into the function, we use Newton's Methods, Secant Method and False Position Method (zero-finding methods) to find the volatility we need. At the same time, in order to find the best approximation methods of finding the implied volatility in our project. We decide to find the speed of convergences according to the iterations in each method.
For example, in our one of examples from Nov07, S_t=5.13, t=162/365, K=4, r=0.0156, C(S_t,t)=2.7. Let error within 10^{-10}, we run the code and get the result:

```
11/07
Test chosen boundary: -9.585221505403752e-12 0.19173114968001226
Newton's method: (1.8272918429898295, 1)
Secant method: (1.8272918429898293, 2)
False position method: (1.8272918429898293, 2)
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The result (all five data) shows that Newton's Method has fewer iterations than Secant Method and False Position Method, so we decide to use Newton's Method as the primary method to approximate implied volatility in each data, and the implied volatility for each data is:

Expiration Time	Implied Volatility
162/365 (11/07)	1.827291843
164/365 (11/05)	1.605725715
165/365 (11/04)	1.739736242
168/365 (11/01)	1.595981554
169/365 (10/31)	1.456429213

From the table above, it shows the implied volatility on Oct31, Nov01, Nov04, Nov05, Nov07.

3.4 For the purpose of estimating the value of implied volatility on Nov06, we decide to select the best approximation interpolation method from Lagrange Polynomial Approximation (theorem 3.2), Natural Cubic Spline Method (theorem 3.11) and Divided Difference Interpolation Method (theorem 3.6). The way of seeking out the best interpolation methods is to select four data (Oct31, Nov01, Nov04, Nov07), and use different interpolation methods to estimate the value of implied volatility on Nov05. Then, we compare each volatility from different methods and find the interpolation method with minimum error.
The approximated implied volatility by using different interpolation methods are:

Methods	Implied Volatility	Absolute Error
Natural Cubic Spline	1.7641307100070585	0.14682709086585932
Lagrange Polynomial	1.7525528060399094	0.1468270908658591
Newton Divided Difference Method	1.7525528060399094	0.15840499483300818

After we comparing the implied volatility by using Newton's Method on Nov05, the absolute error by using Lagrange Polynomial Approximation is lower than Divided Difference interpolation Method and Natural Cubic Spline Method, so we decide to use Divided Difference Interpolation Method to approximate the volatility on Nov06, and the result shows that

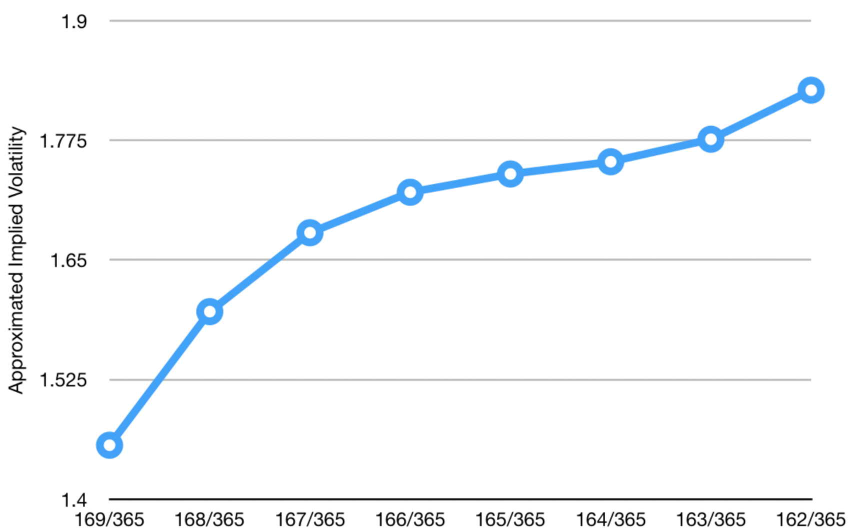
Expiration Time	Implied Volatility
162/365 (11/07)	1.827291843
163/365 (11/06)	1.776084762
164/365 (11/05)	1.752552806
165/365 (11/04)	1.739736242
168/365 (11/01)	1.595981554
169/365 (10/31)	1.456429213

3.5 Also, by using Black-Scholes formula, we put approximation implied volatility on Nov06, S_t=5.13, t=162/365, K=4, r=0.0156 into the formula, we can calculate the price of option on Nov06 C(S_t,t)=C(5.13,162/365)=2.655.

```
print("The expected price of option is:",test(163/365,5.13,4,1.7760847617821316))
```

```
The expected price of option is: 2.655282245868393
```

3.6 In order to find the trend of implied volatility, we use Lagrange Polynomial Methods to approximate the implied volatility on Nov02 and Nov06 as well. After getting the value of implied volatility for eight days, we make a plot of volatility from 10/31 and 11/07.



Based on the graph, we find that the systematic risk of AUPH's option is slightly fluctuating, which means the systematic risk is also getting a little bit larger, but the whole trend of implied volatility of AUPH's option in our simulation market is increasing. Hence, according to the increasing trend of price of option, we can also predict the price of option on Nov08 is higher than Nov07. This prediction may not be true, but for investors who want to predict future price of option to make a benefit, this prediction builds a theoretical foundation for them to make an investment.

4 Conclusion

4.1 Mathematics Perspective
Theoretically, under the condition that Newton's Method converges in Black-Scholes formula, False Position Method and Bisection Method (we didn't discuss it here) always converges in our porject, but they converge slowly than Newton's method. In our project, we compare different numerical methods and use data to show that the speed of convergence in Newton's Method is faster than Secant Method and False Position Method.
Also, we find that by using different interpolation method to calculate the implied volatility on Nov05,

there exists an error of 15 percents between approximated volatility by using Divided Difference Method and Newton's Method. We interpret this error as the lack of enough data to build the relationship between time and implied volatility or those interpolation methods are not perfect choices to find the implied volatility we need.

4.2 Financial Perspective

From our analysis, the fluctuation of volatility from Nov04 to Nov06 represents the risk of buying a call option. However, as we can see, since price of option has a positive relationship with implied volatility we approximated, the whole trend of implied volatility and price of option is increasing. Hence, when people decide to enter our simulation market, based on the graph and information I provided, they prefer buying options and sell it in next day to make a benefit.

5 Reference

[1] R.L.Burden, J.D.Faires and A.M.Burden, *Numerical Analysis*, Cengage Learning, 2016.