Option Pricing with Heston's Stochastic Volatility Model

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1 Introduction and Motivation

1.1 When deriving the option price formula, one of the most famous formula is Black-Scholes formula. In Math104A, we used Black-Scholes formula as a coherent framework for pricing European options. Due to its simplicity and accuracy, Black-Scholes model is widely used in asset and option pricing industry by treating the volatility of an option as a constant. During the last decades several alternatives have been proposed to improve volatility modelling in the context of derivatives pricing. One of such approaches is to model volatility as a stochastic quantity.

Stochastic volatility models, on the other hand, allow for variation in both the asset's price and its price volatility, or standard deviation. One of the most widely used stochastic volatility model was proposed by Steven Heston in 1993.The Heston model extends Black-Scholes model by adding a stochastic process for the stock volatility. The basic Heston model assumes the S_t the price of the asset is given by:

$$dS_t = \mu S_t \, dt + \sqrt{\nu_t} S_t \, dW_t^S$$
$$d\nu_t = \kappa(\theta - \nu_t) \, dt + \xi \sqrt{\nu_t} \, dW_t^\nu$$

where S_t is the asset price, μ is the rate of return of the asset, $\sqrt{\nu_t}$ is the volatility (standard deviation) of the asset price, ξ is the volatility of volatility, θ is the long-term price variance, κ is the rate of reversion to the long-term price variance, dW_t^S is the Brownian motion of the asset price, dW_t^ν is the Brownian motion of the asset's price variance, ρ is the correlation coefficient for dW_t^S and dW_t^ν.

1.2 For each stochastic volatility model, it has a unique characteristic function that describes the probability density function of that model. The expression for a European Call option, derived from the Heston PDE, is shown below,

$$C_0 = S_0 \Pi_1 - \mathrm{e}^{-rT} K \Pi_2$$

where

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left[\frac{\mathrm{e}^{-iwln(K)} \Psi_{lnS_T}(w-i)}{iw \Psi_{lnS_T}(-i)} \right] \mathrm{d}w$$
$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left[\frac{\mathrm{e}^{-iwln(K)} \Psi_{lnS_T}(w)}{iw} \right] \mathrm{d}w$$

where the characteristic function is

$$\Psi_{lnS_T}(w) = \mathrm{e}^{[C(t,w)\tilde{V} + D(t,w)V_0 + iwln(S_0 \mathrm{e}^{rt})]}$$

where

$$C(t,w) = a \left[r_- t - \frac{2}{\eta^2} .ln \left(\frac{1 - g\mathrm{e}^{-ht}}{1-g} \right) \right]$$
$$D(t,w) = r_- \frac{1 - \mathrm{e}^{-ht}}{1 - g\mathrm{e}^{-ht}}$$
$$r_\pm = \frac{\beta \pm h}{\eta^2}; h = \sqrt{\beta^2 - 4\alpha\gamma}$$
$$g = \frac{r_-}{r_+}$$
$$\alpha = -\frac{w^2}{2} - \frac{iw}{2}; \beta = \alpha - \rho\eta iw; \gamma = \frac{\eta^2}{2}$$

In the above characteristic function for the Heston model, the variable V_0,$\tilde{V}$,α,η,ρ need to be calibrated in order to model the marketplace.

2 Problem Statement

2.1 After getting the value of different variable in the above characteristic function, the understanding of numerical analysis methods from Math104B can be really effective ways to compute the value of Π_1 and Π_2 after Fourier Transformation. Hence, in our project, after obtaining the value of Π_1 and Π_2, by applying the expression of European Call option of Heston PDE, we can calculate the price of the option and asset we want in Heston model. After we getting the result, we compare different numerical integration methods we used and make a conclusion. Later, we compare the price of option by Heston model to actual value and Black-Scholes model and make an analysis.

3 Methods and Results Analysis

3.1 In order to find the price of option, we select select fifteen dataset as our sample to build our simulation model, which includes the value of different variable after being calibrated, as our sample to build the simulation model:

A	B	C	D
Expiration	Stock Price	Strike Price	Actual Call
24/365	425.73	395	36.75
24/365	425.73	400	27.88
24/365	425.73	405	25
24/365	425.73	410	17.5
24/365	425.73	415	15.88
87/265	425.73	395	33
87/265	425.73	400	28.5
87/265	425.73	405	24.13
87/265	425.73	410	20.38
87/265	425.73	415	16.13
115/365	425.73	380	47.25
115/365	425.73	390	38.13
115/365	425.73	400	29.38
115/365	425.73	410	21.19
115/365	425.73	420	13.88

Also, in order to calculate the value of Π_1 and Π_2 first, after calibrating (MLE), we obtain the parameters we want:

Parameter Estimates	Value
V_0	$6.47*10^{-5}$
$\tilde{V}$	$6.47*10^{-5}$
α	$6.57*10^{-3}$
η	$5.09*10^{-4}$
ρ	$-1.98*10^{-3}$

3.2 Since $\frac{\mathrm{e}^{-iwln(K)}\Psi_{lnS_T}(w-i)}{iw\Psi_{lnS_T}(-i)} \in C^4[0,\infty)$ and $\frac{\mathrm{e}^{-iwln(K)}\Psi_{lnS_T}(w-i)}{iw} \in C^4[0,\infty)$, by Theorem 4.4 [1], we can apply Composite Simpson's Rule to integrate Π_1 and Π_2.

Since $\frac{\mathrm{e}^{-iwln(K)}\Psi_{lnS_T}(w-i)}{iw\Psi_{lnS_T}(-i)} \in C^2[0,\infty)$ and and $\frac{\mathrm{e}^{-iwln(K)}\Psi_{lnS_T}(w-i)}{iw} \in C^2[0,\infty)$, by Theorem 4.5 and Theorem 4.6 [1], we can apply Composite Trapezoidal Rule and Composite Midpoint Rule to integrate $\Pi_1$1 and Π_2, where the number of subintervals must be even.

3.3 Then, by plugging V_0,$\tilde{V}$,α,η,ρ into the function, we apply Composite Midpoint Method, Composite Trapezoidal Method and Composite Simpson's method to calculate the value of each value in Π_1 and Π_2 to the function $C_0 = S_0$ Π_1 - $\mathrm{e}^{-rT}K$ Π_2 we can get the results (One example of our dataset [Value & Error]):

```
K1=395
Actual value: 33.0
Composite Mid Point:  [32.66995011254102, 0.33004988745898345]
Composite Trapezoidal:  [32.63316406116252, 0.36683593883748244]
Composite Simpsons:  [32.669958806198395, 0.33004119380160546]
Black-Schole model:  [30.790251842591942, 2.2097481574080575]
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After comparing all the value that calculated by different integration methods, we find that, in some cases, Composite Trapezoidal Method is closer to the actual value than Composite Simpson's Method and Composite Midpoint Method.

However, from the general term of Composite Trapezoidal Method and Composite Simpson's Method, the error term for Composite Simpson's Method is $\frac{b-a}{180}h^4 f^{(4)}(\xi)$ for some $\xi \in (a,b)$, and the error term of Composite Trapezoidal Rule is $\frac{b-a}{12}h^2 f^{(2)}(\xi)$ for some $\xi \in (a,b)$. Hence, Composite Simpson's Method should be more accurate than Composite Trapezoidal Method due to its $O(h^4)$ error term involving $f^{(4)}$ to $O(h^2)$ error term involving $f^{(2)}$ of Composite Trapezoidal Method.

Then at here why in some cases Composite Trapezoidal Method will tend to be more accurate than Composite Simpson's Method is because, from our perspectives, the interval and value we choose sometimes make $f^{(2)}$ makes more dominate than $f^{(4)}$ when strike price are changed. If we choose interval with a comparatively smaller value of h, $f^{(2)}$ and $f^{(4)}$ may have different degrees of influences on the results we get, which explains why Composite Trapezoidal Method in some situation can get more accurate value than Composite Simpson's Method.

3.4 Then we decide to use Composite Trapezoidal Method as the integration method to calculate the price of option. At the same time, in order to compare the accuracy of the Heston model and the Black-Scholes model, according to fifteen call options on the SP 100 exchange-traded fund from June 1997,by plugging K, S_t, r, σ, T into Black-Scholes function, we can obtain the results of call option as well.

Formula of Black-Scholes Model

$$C(S_t,t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$$

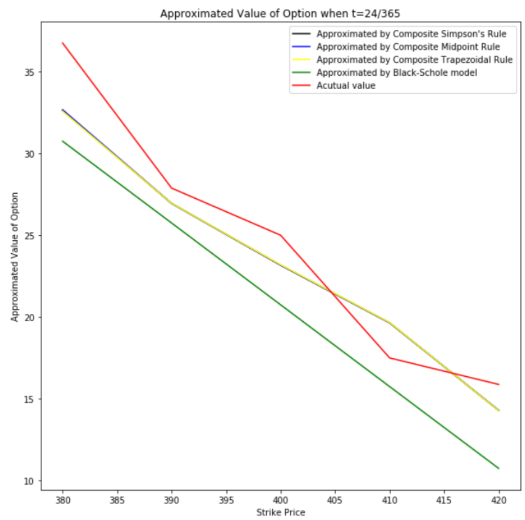
$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$

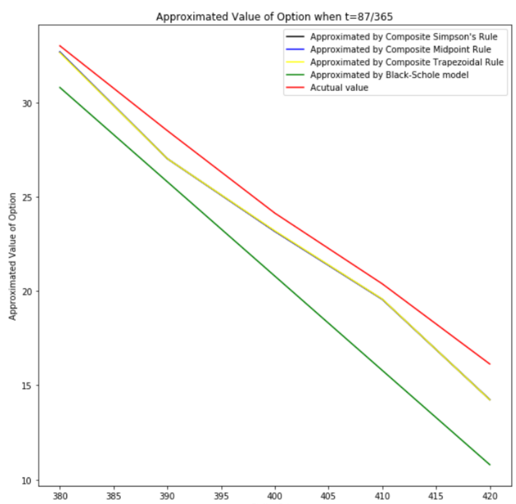
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

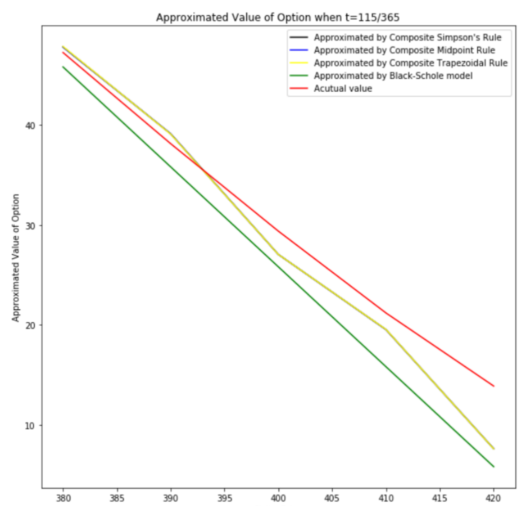
where N($\cdot$) is the cumulative distribution function of the standard normal distribution, T-t is the time to maturity (expressed in years), S_t is the spot price of the underlying asset, K is the strike price, r is the risk free rate (annual rate, expressed in terms of continuous compounding), σ is the volatility of returns of the underlying asset.
The result we get:

A	E	F	G	H
Expiration	Heston Model	Black-Scholes Model	Heston Error	Black-Scholes Error
24/365	32.58	30.75	4.17	6
24/365	26.94	25.75	0.94	2.13
24/365	23.2	20.75	1.8	4.25
24/365	19.66	15.75	2.16	1.75
24/365	14.28	10.75	1.6	5.13
87/265	32.63	30.79	0.37	2.21
87/265	27.02	25.79	1.48	2.71
87/265	23.2	20.79	0.93	3.34
87/265	19.57	15.79	0.81	4.59
87/265	14.23	10.79	1.9	5.34
115/365	47.85	45.81	0.6	1.44
115/365	39.1	35.81	0.97	2.32
115/365	27.05	25.81	2.33	3.57
115/365	19.54	15.81	1.65	5.38
115/365	7.6	5.81	6.28	8.07

According to our results, we can find that in most cases, Heston model has less error than Black-Scholes model. After plotting, as seen in the below graph for option call price comparison, the Heston model's call price estimates are closer than the Black-Scholes estimates to the observed call prices.







4 Conclusion

4.1 Mathematics Perspective

Theoretically, with the $O(h^4)$ error term involving $f^{(4)}$, Composite Simpson's Method is supposed to be more accurate than the method we used in this project. In our project, in some situations, Composite Trapezoidal Method become more accurate than Composite Simpson's Method, even though it is possible that we need more data to make an analysis. However,after calculating the value of call option, it really intrigues me to find the discrepancies from the basic numerical knowledge I learned in Math104B class. In most cases, those different numerical methods build a solid foundation to apply them to use in other field, like in finance, physics and so on, and their exists also make a huge progress to the development of the whole society, no matter in mathematics field or other fields.

4.2 Financial Perspective

From our analysis, in most situations, the value of option pricing by Heston's model seems closer than the value calculated by Black-Scholes Method. Indeed, stochastic volatility models tackle one of the most restrictive hypotheses of the Black-Scholes model, which assumes that volatility remains constant. Nevertheless, by observing financial markets,it is possible that volatility may change dramatically in short-time periods and its behavior is clearly not deterministic. In our real market, the randomness and stochasticity of volatility make it harder for us to predict the price of call/put option. There are still lots of works we need to analyze and study in the future.

5 Reference

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