STOCHASTIC SIDE INFORMATION IN ONLINE CONVEX OPTIMIZATION

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ABSTRACT

***Index Terms*—** Online learning, Expert advice, Minimax regret, Side information

1. INTRODUCTION

The online convex optimization(OCO) investigate the standard optimization problems with the time-varying convex loss functions, and it is becoming a popular topic in recent years due to its practical meaning. Hazan et al. [1] studied the OCO problems systematically including the detailed definition, algorithms, applications, and analysis.

Cesa-Bianchi and Lugosi [2] studied the effectiveness of the deterministic side information in online learning problems. Our previous work also evaluated the stochastic side information under online learning setup[3].

This work focus on exploring the effect of evolving the stochastic side information in online convex optimization problems, and find the improved upper and lower bound comparing with the problems with no side information.

2. PROBLEM FORMULATION

2.1. Online Convex Optimization(OCO) with Side Information

We investigate an online convex optimization problem with side information. The standard formulation of convex optimization problem is well defined by []. Differ from the standard optimization problem, the loss functions for each iteration is drawn from a function set in OCO problems, i.e. $f_1(x), f_2(x), ..., f_T(x) \in \mathcal{F}$, and these functions are defined by: $\forall f \in \mathcal{F}, f(x) : \mathcal{X} \mapsto \mathbb{R}$, where the input x takes value from the input space $\mathcal{X} \subseteq \mathbb{R}$.

Throughout this paper, the input space $\mathcal{X}$ is a bounded, convex, and nonempty subset of $\mathbb{R}^d$, it has a diameter D: $\forall x, y \in \mathcal{X}$, $\|x - y\| \leq D$. The functions $f_t(x)$ are nonnegative, convex and having bounded gradient. More precisely, $\forall x, y \in \mathcal{X}$ and $\forall f \in \mathcal{F}$, $f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y)$, and $\forall x \in \mathcal{X}, \forall f \in \mathcal{F}$, $\|\nabla f(x)\| \leq G$.

An algorithm A is designed to produce x_t as the estimation of the minimizer of the sum of the loss functions, denoted by x^*, i.e.,

$$x^* = \min_{x \in \mathcal{X}} \sum_{t=1}^{T} f_t(x) \tag{1}$$

To evaluate the performance, we will introduce the static regret in section**??** our work is then concentrate on deriving the upper and lower bound for it given the side information introduced in section**??**

2.2. Side information

We first introduce a general form of side information for stochastic cases, and the later sections will discuss the details. We denoted by $s_t \in \mathbb{R}$ the side instance appearing in the t_{th} iteration, are drawn from a known conditional distribution $P(s_t | x_t^*)$. Under most circumstance, we regard the side information as a noisy version of the the current round minimizer x_t^*. The side information is provided to the algorithm but not known to the environment, in which case the performance of our proposed algorithm will be tightly related to the side information.

2.3. Expected Static Regret

The algorithm makes its prediction by using the information from the previous functions up to round $t-1$ and side information $\hat{x}_t^*$ up to round t. In round t, the output of the algorithm is denoted by x_t, i.e.,

$$x_t = A(f_1, f_2, ..., f_{t-1}, s_1, s_2, ..., s_t) \tag{2}$$

Then the environment select the loss function $f_t(x)$ stochastically or adversarially. We then evaluate the performance of the forecaster by the minimax expected static regret:

$$R_T = \inf_{x_1, ..., x_T} \sup_{f_1, f_2, ..., f_T \in \mathcal{F}} \mathbb{E}_{s_1, ..., s_T} [\sum_{t=1}^{T} (f_t(x_t) - f_t(x^*))]$$

3. MAIN RESULTS

3.1. The MMADU Estimator

To encounter the side information for the learning bound analysis, we first introduce an Minimum Mean Absolute Deviation Unbiased(MMADU) estimator. Here, we give a definition of the MMADU estimator.

Definition 1. An unbiased estimator δ for θ, i.e., $\mathbb{E}[\delta]=\theta$, is minimum mean absolute deviation unbiased(MMADU), if

$$\mathbb{E}\|\delta-\theta\| \leq \mathbb{E}\|\delta^*-\theta\|, \quad \forall \theta \in \Omega \tag{3}$$

for any competing unbiased estimator δ^*, where Ω is the space formed from all possible θ.

We now propose a simple but effective algorithm, which is a time-varying linear combination of the MMADU estimator $\hat{x}_t$ and gradient descent predictor denoted by $\bar{x}_t$.

$$x_t=(1-\beta_t)\hat{x}_t+\beta_t\bar{x}_t \tag{4}$$

We select the MMADU estimator as part of the algorithm as it can help to minimize the derived upper bound in section 3.2.

3.2. Upper Bound for OCO with Side Information

Theorem 1. Under the online convex optimization with side information setup, denoting by Δ^* the mean average deviation produced from the MMADU estimator of x_t^*, i.e. $\Delta^*=\mathbb{E}\|\hat{x}_t-x_t^*\|$, the static regret is upper bounded by:

$$R_T \leq \frac{D^2\beta_T}{2\eta_T}+\frac{G^2}{2}\sum_{t=1}^{T}\beta_t\eta_t+G\sum_{t=1}^{T}(1-\beta_t)(\Delta^*+D_t^*) \tag{5}$$

$$\tag{6}$$

where $\beta_1,...,\beta_t$ and $\eta_1,...,\eta_t$ are the parameters from the algorithm above, and D^* is the deviation of the online functions, i.e., $D_t^*=f_t(x_t^*)-f_t(x^*)$

Proof.

$$R_T \leq \mathbb{E}\left(\sum_{t=1}^{T}f_t((1-\beta_t)\hat{x}_t+\beta_t\bar{x}_t)-f_t(x^*)\right) \tag{7}$$

$$\leq \sum_{t=1}^{T}\beta_t(f_t(\bar{x}_t)-f_t(x^*))+(1-\beta_t)\mathbb{E}((f_t(\hat{x}_t)-f_t(x^*))) \tag{8}$$

$$\leq \frac{D^2\beta_T}{2\eta_T}+\frac{G^2}{2}\sum_{t=1}^{T}\beta_t\eta_t \tag{9}$$

$$+\sum_{t=1}^{T}(1-\beta_t)\mathbb{E}((f_t(\hat{x}_t)-f_t(x_t^*))+(f_t(x_t^*)-f_t(x^*))) \tag{10}$$

$$\leq \frac{D^2\beta_T}{2\eta_T}+\frac{G^2}{2}\sum_{t=1}^{T}\beta_t\eta_t+G\sum_{t=1}^{T}(1-\beta_t)(\Delta^*+D_t^*) \tag{11}$$

$$\tag{12}$$

□

We determine both η_t and β_t by the design of the algorithm A. With the same optimal set up for η_t in [] that $\eta_t=\frac{D}{G\sqrt{t}}$, we can then optimize over $\beta_1,...,\beta_t$ by the observed side information $\hat{x}_1^*,...,\hat{x}_t^*$. One should note that the learning rate of the upper bound given in theorem **??** is not fully derived, as the rate of Δ^* varies through different cases. In section **??**, we present two examples that Δ^* can be calculated, which leads to a concrete learning rate for the minimax regret. To investigate the tightness of the obtained upper bound, we now provide the lower bound for OCO with side information.

3.3. Gaussian Channel

We now use the Gaussian channel to provide a specialized result for the upper bound. We consider the side information to be the current round minimizer with additive zero mean Gaussian noise. More precisely, we have $s_t \sim N(x_t^*,\sigma^2)$, i.e., $P(s_t|x_t^*)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{s_t-x_t^*}{\sigma}\right)^2}$.

Corollary 1. With the Gaussian channel setup for side information, when $D^*=\sum_{t=1}^{T}f_t(x_t^*)-f_t(x^*)$ satisfies $D^*<-\sigma\sqrt{\frac{2}{\pi}}T$, the static regret is upper bounded by $O(-T)$:

$$R_T \leq G(\sigma\sqrt{\frac{2}{\pi}}+D^*)T \tag{13}$$

and for other cases of L^*, the upper bound remains $O(\sqrt{T})$:

$$R_T \leq \frac{3}{2}GD\sqrt{T} \tag{14}$$

Proof. We first calculate

$$\Delta^* = \mathbb{E}\|\hat{x}_t - x_t^*\| \overset{(a)}{=} \mathbb{E}\|s_t - x_t^*\| = \sigma\sqrt{\frac{2}{\pi}} \tag{15}$$

Where (a) is done by substituting the estimator $\hat{x}_t$ by the MMADU estimator which is the side information itself. Then if $D^* < -\sigma\sqrt{\frac{2}{\pi}}T$, by setting $\beta_t = 0$, we obtain $R_T \leq G(\sigma\sqrt{\frac{2}{\pi}} + D^*)T$. If $D^* \geq -\sigma\sqrt{\frac{2}{\pi}}T$, by setting $\beta_t = 1, \eta_t = \frac{D}{G\sqrt{t}}$, we have $R_T \leq \frac{3}{2}DG\sqrt{T}$. □

Note that the right hand side of equation (14) is negative, in which case we produced a negative regret upper bound in scale with $O(-T)$.

3.4. Uniform Channel

Suppose the side information s_t is now the minimizer with additive uniform noise: $s_t \sim U(x_t^* - \theta, x_t^* + \theta)$, i.e., $P(s_t|x_t^*) = \frac{1}{\theta}$ when $x_t^* - \theta \leq s_t \leq x_t^* + \theta$ and $P(s_t|x_t^*) = 0$ elsewhere.

Corollary 2. With the uniform channel setup for side information, when $D^* = \sum_{t=1}^T f_t(x_t^*) - f_t(x^*)$ satisfies $D^* < -\frac{\theta}{2}T$, the static regret is upper bounded by $O(-T)$:

$$R_T \leq G(\frac{\theta}{2} + D^*)T \tag{16}$$

and when $D^* \geq -\frac{\theta}{2}T$, the upper bound is in scale with $O(\sqrt{T})$:

$$R_T \leq \frac{3}{2}GD\sqrt{T} \tag{17}$$

Proof. Similar to the Gaussian channel proof, we first calculate $\Delta^* = \frac{\theta}{2}$ Where (a) is done by substituting the estimator $\hat{x}_t$ by the MMADU estimator which is the side information itself. Then by setting $\beta_t = 0$ when $D^* < -\frac{\theta}{2}T$, and $\beta_t = 1$ when $D^* \geq -\frac{\theta}{2}T$, we complete the proof. □

3.5. Lower Bound

Theorem 2. Under the setup of OCO with side information, if the input space $\mathcal{X}$ is an n-dimensional cube, and the side information on each dimension is independent of other dimensions, the lower bound of the regret is $O(-T)$.

Proof. To derive the lower bound for OCO problems with side information, we reduce the problem by substituting the loss function $f_t(x)$ to any convex function to derive the lower bound of regret and restrict the input space $\mathcal{X}$ to a n-dimensional cube, i.e., $\mathcal{X} = [-1, +1]^n$. More precisely, we consider the linear function $l_t(x) = c_t^\top x$, where we denote by $c_t = (v_t^1, v_t^2, ..., v_t^n)$ the parameter vector of the linear function f_t. To lower bound the regret, we also consider a stochastic c_t, , where $v_t^1, ..., v_t^n$ follow the Rademacher distribution, i.e. $P(v_t^1 = 1) = P(v_t^2 = 1) = ... = P(v_t^n = 1) = P(v_t^1 = -1) = P(v_t^2 = -1) = ... = P(v_t^n = -1) = \frac{1}{2}$, and then take expectation with respect to it.

$$R_T = \inf_{x_1,...,x_T} \sup_{f_1,f_2,...,f_T \in \mathcal{F}} \mathbb{E}_{s_1,...,s_T}\left[\sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(x^*)\right] \tag{18}$$

$$\geq \inf_{x_1,...,x_T} \mathbb{E}_{c_1,...,c_T,s_1,...,s_T}\left[\sum_{t=1}^T l_t(x_t) - \sum_{t=1}^T l_t(x^*)\right] \tag{19}$$

$$\overset{(a)}{=} \mathbb{E}_{c_1,...,c_T,s_1,...,s_T}\left[\sum_{t=1}^T c_t^\top \tilde{x}_t(s_t) - c_t^\top x^*\right] \tag{20}$$

Where we denote by $\tilde{x}_t(s_t)$ the optimal estimator which minimizes the expectation $\mathbb{E}[(l_t(x)]$ which only related to the current round side information s_t. Equality (a) is done by substituting x_t with $\tilde{x}_t(s_t)$, since x_t^* is the minimizer of R_T and it is conditionally independent with all the previous round minimizer $x_1^*, x_2^*, ..., x_{t-1}^*$ and $s_1, s_2, ..., s_{t-1}$ given s_t. Nevertheless, the regret is now lower bounded by an expectation with respect to the side information, instead of finding the minimax regret, in which case all the other information will not contribute on reducing the result even with a stochastic algorithm. We denote by s_t^i the stochastic side information variable on the i-th dimension, and $\tilde{x}_t^i$ the i-th entry of the estimator $\tilde{x}_t$. We then give the derivation of the $\tilde{x}_t(s_t)$ and the produced cumulative loss. note: relationship between sigma and T. plot upper and lower bound given same setup.

$$\mathbb{E}[c_t^\top \tilde{x}_t(s_t)] = \sum_{i=1}^n \mathbb{E}\left[v_t^i \tilde{x}_t^i(s_t^i)\right] \tag{21}$$

$$= \sum_{i=1}^n \int_{s_t^i}\int_{v_t^i} v_t^{i\top} \tilde{x}_t^i(s_t^i) P(v_t^i, s_t^i) d(v_t^i) d(s_t^i) \tag{22}$$

$$= \sum_{i=1}^n \int_{s_t^i}\int_{v_t^i} v_t^{i\top} \tilde{x}_t^i(s_t^i) P(v_t^i|s_t^i) d(v_t^i) P(s_t^i) d(s_t^i) \tag{23}$$

Take derivative of the integral part of equation 23 with respect to $\tilde{x_t^i}_t$, we have

$$\frac{d}{d\tilde{x_t^i}} \int_{s_t^i}\int_{v_t^i} v_t^{i\top} \tilde{x}_t^i(s_t^i) P(v_t^i|s_t^i) d(v_t^i) P(s_t^i) d(s_t^i) \tag{24}$$

$$= \int_{s_t^i}\int_{v_t^i} v_t^{i\top} \frac{d}{d\tilde{x_t^i}} \tilde{x}_t^i(s_t^i) P(v_t^i|s_t^i) d(v_t^i) P(s_t^i) d(s_t^i) \tag{25}$$

$$= \int_{s_t^i}\int_{v_t^i} v_t^{i\top} P(v_t^i|s_t^i) d(v_t^i) P(s_t^i) d(s_t^i) \tag{26}$$

$$= \int_{s_t^i} \mathbb{E}[v_t^i|s_t^i] P(s_t^i) d(s_t^i) \tag{27}$$

As the expectation in equation 27 is a constant, in which case we can get the minimizer $\tilde{x}_t^i = 1$ when $\mathbb{E}[v_t^i|s_t^i] \leq 0$, and $\tilde{x}_t^i = -1$ when $\mathbb{E}[v_t^i|s_t^i] > 0$. We now denote by $\phi(x)$ the cumulative probability function of the conditional distribution $P(v_t^i|s_t^i)$, and then calculate the expected loss induced by $\tilde{x}$,

$$\mathbb{E}[\sum_{t=1}^{T} c_t^\top \tilde{x}_t(s_t)] = nT(\phi(0)\times 1 + (1-\phi(0)\times(-1))) \tag{28}$$
$$= nT(2\phi(0)-1) \tag{29}$$

Following the same procedures from [], we have the rest of equation (20) calculated,

$$\mathbb{E}[c_t^\top x^*] = G\sqrt{nT} \tag{30}$$

Then we have,

$$R_T = n(2\phi(0)-1)T + G\sqrt{nT} \tag{31}$$

Then we have, when $n(2\phi(0)-1)T < G\sqrt{nT}$, i.e. $T > \frac{G^2}{n(1-2\phi(0))^2}$, the lower bound of the regret is in scale with $-T$. □

Corollary 3. Under Gaussian channel setup in section **??**, we have,

$$R_T \leq G\sqrt{nT} - erf(\frac{1}{\sigma\sqrt{2}})nT \tag{32}$$

Proof. We first find $\phi(0)$ for Gauusian channel. As in Gauusian channel, s_t is the x_t^* with an additive zero-mean Gaussian noise, $P(v_t^i|s_t^i)$ is symmetric and centering on s_t^i. We can then show that when s_t^i is larger than 0, $\mathbb{E}[v_t^i||s_t^i]$ is also larger than 0, and vice versa. Thus we have,

$$\phi(0) = P(N_t \geq 1) = \frac{1}{2}(1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{\sigma\sqrt{2}}} e^{-t^2} dt) \tag{33}$$
$$= \frac{1}{2}(1 - erf(\frac{1}{\sigma\sqrt{2}})) \tag{34}$$

Then we have,

$$R_T \geq n(1 - erf(\frac{1}{\sigma\sqrt{2}}) - 1)T + G\sqrt{nT} \tag{35}$$
$$= G\sqrt{nT} - erf(\frac{1}{\sigma\sqrt{2}})nT \tag{36}$$

□

note: Bound erf(x) instead of directly use it. We can now check that the lower bound is related to the σ, and when σ is satisfying $erf(\frac{1}{\sigma\sqrt{2}}) \leq \frac{G}{\sqrt{n}}$, the lower bound of R_T is in scale with $O(-T)$, which coincide with the derived upper bound in corollary **??**. Meanwhile, when $erf(\frac{1}{\sigma\sqrt{2}}) > \frac{G}{\sqrt{n}}$, which refers to a weak side information, the lower bound remains same with the case without any side information. One will also note that when σ approaches ∞, $erf(\frac{1}{\sigma\sqrt{2}})$ will approach 0, which vanished the second term in corollary 3.

4. EXPECTED DYNAMIC REGRET

In this section, we will discuss the setup for dynamic regret, where we define:

Where x_t^* is the minimizer to the current round:

$$x_t^* = \min_{x\in\mathcal{X}} f_t(x) \tag{37}$$

4.1. Upper Bound

For dynamic regret set up, we first propose an similar algorithm to the static setup,

$$x_t = \beta\hat{x}_t + (1-\beta)\bar{x}_t \tag{38}$$

Then we can get the following theorem,

Theorem 3.

$$R_T^D \leq \beta(\frac{7D^2}{4\eta} + \frac{LD}{\eta} + \frac{T\eta G^2}{2}) + (1-\beta)G\Delta^* T \tag{39}$$

Proof.

$$R_T^D = \mathbb{E}\left[\sum_{t=1}^{T}(f_t(x_t) - f_t(x_t^*))\right] \tag{40}$$
$$= \mathbb{E}\left[\sum_{t=1}^{T}(f_t(x_t) - f_t(x_t^*))\right] \tag{41}$$
$$\leq \sum_{t=1}^{T}(\beta(f_t(\bar{x}_t) - f_t(x_t^*)) + (1-\beta)\mathbb{E}(f_t(\hat{x}_t) - f_t(x_t^*))) \tag{42}$$
$$= \beta\sum_{t=1}^{T}(f_t(\bar{x}_t) - f_t(x_t^*)) + (1-\beta)\sum_{t=1}^{T}\mathbb{E}(f_t(\hat{x}_t) - f_t(x_t^*)) \tag{43}$$
$$\leq \beta(\frac{7D^2}{4\eta} + \frac{LD}{\eta} + \frac{T\eta G^2}{2}) + (1-\beta)G\Delta^* T \tag{44}$$

□

Given different L, we could reach various rate of the regret. We first discuss the case when $\mathcal{L} = O(T)$, that we can obtain,

Corollary 4. When the functions in all rounds satisfying $L = c_L T$, we have the upper bound of the regret is $O(T)$.

Proof. We can substitute the L in theorem 3, then we have,

$$R_T^D \leq \beta(\frac{7D^2}{4\eta} + \frac{c_L DT}{\eta} + \frac{T\eta G^2}{2}) + (1-\beta)G\Delta^* T \tag{45}$$
$$= \beta\frac{7D^2}{4\eta} + G\Delta^* T + (\frac{c_L D}{\eta} + \frac{\eta G^2}{2} - G\Delta^*)\beta T \tag{46}$$

As the expression above is linear with β, we can further get the optimal β: when $\Delta^* < \left(\frac{c_L D}{G\eta} + \frac{\eta G}{2}\right)$, set $\beta = 0$ and when $\Delta^* \geq \left(\frac{c_L D}{G\eta} + \frac{\eta G}{2}\right)$, set $\beta = 1$. For all β, the upper bound of the regret will be positive and linear with T. □

4.2. Lower Bound

Theorem 4. When the side information provides an unbiased estimator of x_t^*, if L is in scale with O(T), the lower bound of the expected dynamic regret is O(T).

Proof. Similar to the proof of theorem 2, we have, as $L = c_L T$, the regret can be lower bounded by substituting the worst case with stochastic functions,

$$R_T^D = \inf_{x_1,\ldots,x_T} \sup_{f_1,f_2,\ldots,f_T \in \mathcal{F}} \mathbb{E}_{s_1,\ldots,s_T} \left[\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_t^*) \right] \tag{47}$$

$$\geq \inf_{x_1,\ldots,x_T} \mathbb{E}_{c_1,\ldots,c_T,s_1,\ldots,s_T} \left[\sum_{t=1}^{T} l_t(x_t) - \sum_{t=1}^{T} l_t(x_t^*) \right] \tag{48}$$

$$= \mathbb{E}_{c_1,\ldots,c_T,s_1,\ldots,s_T} \left[\sum_{t=1}^{T} c_t^\top \tilde{x}_t(s_t) - c_t^\top x_t^* \right] \tag{49}$$

$$= \mathbb{E}_{c_1,\ldots,c_T,s_1,\ldots,s_T} \left[\sum_{t=1}^{T} c_t^\top \tilde{x}_t(s_t) \right] - \mathbb{E}_{c_1,\ldots,c_T} \left[\sum_{t=1}^{T} c_t^\top x_t^* \right] \tag{50}$$

$$= (2\phi(0) - 1 + \frac{D^2}{4})T \tag{51}$$

noteAdd proof that the lower bound is averaging all linear with T case is larger than this results. explain why we only use st for round t, why s1..st-1 does not influence even with a randomized algorithm(not minimax problem, is a expectation).

James stein paradox Upper write out all cases discussion clearly

Write out the commments immiediately

□

5. CONCLUSION AND FUTURE WORKS

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