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MODELING OF PACKING PROBLEMS

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In this paper the modeling of packing problems (cutting and allocation problems) is investigated. Especially, linear mixed-integer models are proposed for two-dimensional packing problems of convex and non-convex polygons. The models are based on the so-called (outer) hodograph and the inner hodograph. Rotations of the polygons are not allowed. The principles of modeling in the two-dimensional case can be applied also for three- and higher dimensional packing problems.

KEY WORDS Cutting and packing problems, packing of polygons, modeling, linear mixed-integer modeling, 0/1-problems, hodograph.

Mathematics Subject Classification 1991:
Primary: 90C11; Secondary: 90C09.

1. INTRODUCTION AND PROBLEM FORMULATION

In this paper packing problems of the following type are investigated:
Let be given a closed and bounded region B *with* $B \subset R^n, (n = 1, 2, 3)$ *whose interior* int B *is connected, and* m *(closed and bounded) objects (pieces, figures)* T_i *with* $T_i \subset R^n$, int T_i *connected,* $i = 1, \ldots, m$. *To each object* T_i *a non-negative valuation* c_i *is assigned. The objects may be translated but not rotated.*

An allocation (or packing) of a subset of the objects within B *is called feasible if the packed objects do not overlap each other.*

Find a feasible allocation such that the total valuation is a maximum.

In the literature, problems of this kind are called packing problems but also allocation or cutting problems. A typology of such problems is given in [4]. Because of their practical relevance a lot of work is done with respect to modeling, exact and heuristic solutions. A wide field of such investigations is contained in [8]. Bibliographies are given in [5] and [7].

Packing problems belong in general to the NP-hard problems but in many cases problems of medium size can be solved exactly with acceptable computational amount, especially in the case of rectangular regions and objects.

In [2] a linear optimization model with 0/1-variable for the two-dimensional non-guillotine cutting problem of rectangular objects is proposed. The branch and bound algorithm given there uses a Lagrangean relaxation and a subgradient iteration to compute upper bounds.

A mixed-integer nonlinear programming model is used in [3] to describe the two-dimensional assortment problem with rectangular objects. The model contains a nonlinear objective function but only linear constraints.

The problem of deciding whether a polygon can be translated to fit inside another polygon is investigated in [1] from an algorithmical point of view.

In this paper we develop linear optimization models with continuous and 0/1-variables for packing problems with convex and non-convex polygons. The number of 0/1-variables is essentially reduced in comparision to [2] and [3]. This reduction is based on the use of very helpful tools – the so-called (outer) hodograph and the inner hodograph of two polygons. The (outer) hodograph can be used to handle the non-overlapping condition of two packed objects. The containment of a polygon in another polygon can be characterized by using the inner hodograph.

2. SET-THEORETICAL MODEL

Within the paper an orthogonal x, y, z-coordinate system is used for the description of the figures T_i and the region B. It is called reference system. Depending on the considered dimension the investigations are made with respect to the x-axis, the x, y-plane or the x, y, z-space. (The x-axis corresponds to the length, the y-axis to the width and the z-axis to the height.) The normalized position of an object T_i with respect to the reference system is characterized by the following conditions:

$$\min\{x : (x, y, z) \in T_i\} = 0, \ \min\{y : (x, y, z) \in T_i\} = 0, \ \min\{z : (x, y, z) \in T_i\} = 0.$$

The normalized position of the region B is defined analogously. The translation of T_i to $v^i \in R^n$ yields the set of points

$$T_i(v^i) := T_i + v^i := \{v \in R^n : v = v^i + w, w \in T_i\}.$$

The point v_i is called the allocation point of $T_i(v^i)$ or for short, the allocation point of T_i.

Now, the considered packing problem can be formulated as follows: Find a subset I^* of $I := \{1, \ldots, m\}$ (a subset of the objects) and the corresponding allocation points $v^i, i \in I^*$, such that the packing conditions are fulfilled and the total value of the packed objects is a maximum. Hence:

$$\sum_{i \in I^*} c_i \to \max$$

subject to

$$I^* \subset I, \tag{1}$$
$$T_i(v^i) \subset B \quad \text{for } i \in I^*,$$
$$\text{int}\,(T_i(v^i)) \;\cap\; \text{int}\,(T_j(v^j)) = \emptyset \quad \text{for } i \neq j,\, i, j \in I^*.$$

This model is contained in [6]. More detailed theoretical and practical investigations require the knowledge of the structure of the region B and the objects T_i

3. MODELS OF ONE-DIMENSIONAL PACKING PROBLEMS

In the one-dimensional case the region B and the objects T_i can be considered as intervals $[0, L]$ and $[0, l_i]$, $i = 1, \ldots, m$, respectively. The task is to find allocation points $v^i = x_i$ of the objects T_i.

For any feasible packing which contains the objects T_i and T_j with $x_i > x_j$ there exists a packing with the same valuaton for which $x_i < x_j$ holds. For that reason, without loss of generality, it can be assumed that $x_i + l_i \leq x_j$ is valid for any two packed objects T_i and T_j with $i < j$.

To describe the set I^* of the packed objects T_i 0/1-variables a_i are defined as follows:

$$a_i = \begin{cases} 1 & , T_i \text{ is contained in } B, \\ 0 & , \text{otherwise.} \end{cases} \tag{2}$$

The set-theoretical model (1) has now the following form:

$$z = \sum_{i=1}^{m} c_i a_i \;\rightarrow\; \max$$

subject to

$$x_1 \geq 0,\; x_i + a_i l_i \leq x_{i+1},\; i = 1, \ldots, m-1,\; x_m + a_m l_m \leq L, \tag{3}$$

$$a_i \in \{0, 1\}, \;\; i = 1, \ldots, m.$$

Here, m continuous, m 0/1-variables and $m+1$ inequalities are used in model (3). Fixing of x_i according to

$$x_1 = 0,\; x_i = \sum_{j=1}^{i-1} l_j a_j, \quad i = 2, \ldots, m,$$

corresponds to a dense packing of the objects, or also to the consideration of so-called "normalized patterns" ([2], "left-justified"). Doing so, the well-known 0/1 knapsack problem arises from the model (3):

$$z = \sum_{i=1}^{m} c_i a_i \;\rightarrow\; \max$$

subject to

$$\sum_{i=1}^{m} l_i a_i \leq L \quad a_i \in \{0,1\}, i = 1, \ldots, m.$$

Only m 0/1-variables and 1 inequality are contained in the knapsack problem.

4. MODELS FOR TWO-DIMENSIONAL PACKING PROBLEMS

4.1 Packing of Rectangles within a Rectangle

Within this section it is assumed that the region B is rectangular as well as all objects T_i which are edge-parallel to the reference system. Then B can be characterized by its length L and width W. T_i has length l_i and width $w_i (i = 1, \ldots, m)$. Only edge-parallel allocations are allowed.

The aim of the following investigations is to develop a linear mixed-integer optimization model. To describe the index set I^* 0/1-variables a_i are introduced in analogy to section 3 according to (2).

The modeling is based on the following idea:

Let be given a sufficiently large rectangle B' of length $L', L' \geq L$, and width $W', W' \geq W$, in which all objects can be packed. But only the total containment of the piece T_i within B gets a positive valuation. This can be realized using the inequalities (5) where x_i, y_i are the coordinates of the allocation point v^i of $T_i (i = 1, \ldots, m)$.

To characterize the mutual position of two objects $T_i(v^i)$ and $T_j(v^j)$ the notions "right-left" and "above-below" are used.

Thereby

" T_j right (left) to T_i " if $p = (x, y) \in T_j(v^j) \Rightarrow x \geq x_i + l_i (\text{resp. } x \leq x_i)$,
" T_j above (below) to T_i " if $p = (x, y) \in T_j(v^j) \Rightarrow y \geq y_i + w_i (\text{resp. } y \leq y_i)$.

To describe the mutual position of T_i and T_j the following 0/1-variables are defined:

$$u_{ij}^r = \begin{cases} 1 & , \quad \text{if } T_j \text{ is right to } T_i, \\ 0 & , \quad \text{otherwise}, \end{cases} \qquad u_{ij}^o = \begin{cases} 1 & , \quad \text{if } T_j \text{ is above to } T_i, \\ 0 & , \quad \text{otherwise}. \end{cases}$$

The condition $\text{int}\,(T_i(v^i)) \cap \text{int}\,(T_j(v^j)) = \emptyset$ is fulfilled if and only if T_j is right or left or above or below to T_i.

Hence, the condition int $(T_i(v^i)) \cap$ int $(T_j(v^j)) = \emptyset$ can be described by the system of inequalities (6). Summarizing, the following linear mixed-integer model is obtained:

$$z = \sum_{i=1}^{m} c_i a_i \rightarrow \max \tag{4}$$

subject to

$$\begin{gathered} 0 \le x_i, \quad x_i \le (L - l_i)a_i + L'(1 - a_i), \quad i = 1, \dots, m, \\ 0 \le y_i, \quad y_i \le (W - w_i)a_i + W'(1 - a_i), \quad i = 1, \dots, m, \\ a_i \in \{0, 1\}, \quad i = 1, \dots, m; \end{gathered} \tag{5}$$

and

$$\begin{gathered} x_i + l_i \le x_j + L'(1 - u^r_{ij}), \quad x_j + l_j \le x_i + L'(1 - u^r_{ji}), \\ y_i + w_i \le y_j + W'(1 - u^o_{ij}), \quad y_j + w_j \le y_i + W'(1 - u^o_{ji}), \\ u^r_{ij} + u^r_{ji} + u^o_{ij} + u^o_{ji} \ge 1, \\ u^r_{ij}, u^r_{ji}, u^o_{ij}, u^o_{ji} \in \{0, 1\}, \quad i = 1, \dots, m - 1, \quad i = 1 + 1, \dots, m. \end{gathered} \tag{6}$$

The model (4) – (6) contains $2m$ continuous, $m + 2m(m - 1)$ 0/1-variables and $4m + 5m(m - 1)/2$ inequalities.

Instead of the 4 variables $u^r_{ij}, u^r_{ji}, u^o_{ij}, u^o_{ji}$ in (6) only 2 0/1-variables u_{ij}, v_{ij} are sufficient to describe the four possibly mutual positions of two objects. (6) is equivalent to the system of inequalities (7):

$$\begin{gathered} x_i + l_i \le x_j + L'(2 - u_{ij} - v_{ij}) \\ x_j + l_j \le x_i + L'(1 - u_{ij} + v_{ij}) \\ y_i + w_i \le y_j + W'(1 + u_{ij} - v_{ij}) \\ y_j + w_j \le y_i + W'(u_{ij} + v_{ij}) \\ u_{ij}, v_{ij} \in \{0, 1\}, \quad i = 1, \dots, m - 1, \; j = i + 1, \dots, m. \end{gathered} \tag{7}$$

Since $u_{ij}, v_{ij} \in \{0, 1\}$ exactly one of the four conditions is non-trivial. Hence, $2m$ continuous, but only $m + m(m - 1)$ 0/1-variables and $4m + 2m(m - 1)$ inequalities are used in model (4,5,7).

4.2. PACKING OF CONVEX POLYGONS

Within this section it is assumed that the region B as well as the objects $T_i (i = 1, \dots, m)$ are convex polygons.

4.2.1 First Model

The region $T_0 := B$ and the objects T_i can be described as the intersection of k_i closed half-spaces (half-plains). Without loss of generality, it is assumed that the k_i corner points $e_{ik}, k = 1, \ldots, k_i$, of $T_i (i = 0, 1, \ldots, m)$ are numbered clockwise and let e_{i1} be that corner point having x-coordinate 0 and smallest y-coordinate (the normalized position of T_i with respect to the reference system is assumed). Hence T_i can be described as follows:

$$T_i = \{(x, y) \in R^2 \, : \, g_{ik}(x, y) := a_{ik}x + b_{ik}y + c_{ik} \leq 0, \; k = 1, \ldots, k_i\},$$

where the corner point $e_{ik} = (x_{ik}, y_{ik})$ is the intersection of g_{ik} and $g_{i,k+1}$, $k = 1, \ldots, k_i$, $(k_i + 1 \equiv 1)$, $i = 1, \ldots, m$.

The translation of T_i from the normalized position along the vector $v^i = (x_i, y_i)$ yields

$$T_i(v^i) = T_i(x_i, y_i) = \{(x, y) \in R^2 \, : \, g_{ik}(x - x_i, y - y_i) \leq 0, \; k = 1, \ldots, k_i\}.$$

(x_i, y_i) is the allocation point of T_i.

The translated objects $T_i(v^i)$ and $T_j(v^j)$ do not overlap each other if they can be separated by a hyperplain resp. $T_i(v^i) \cap T_j(v^j)$ is part of a hyperplain. In both cases there exists a bounding hyperplain $g_{ik}(x - x_i, y - y_i)$ of $T_i(v^i)$ such that for $(x, y) \in T_j(v^j)$ $g_{ik}(x - x_i, y - y_i) \geq 0$ is valid $(k \in \{1, \ldots, k_i\})$, or a corresponding hyperplain of $T_j(v^j)$.

To test whether two fixed objects T_i and T_j do not overlap each other, it is sufficient, because of the convexity of the objects, to consider the bounding hyperplains of $T_i(v^i)$ and the corner points of $T_j(v^j)$ resp. the hyperplains of $T_j(v^j)$ and the corner points of $T_i(v^i)$. It holds:

Statement 1: *If there exists an index* k, $k \in \{1, \ldots, k_i\}$, *such that for all corner points* (x_{jp}, y_{jp}), $p = 1, \ldots, k_j$,

$$g_{ik}(x_{jp} + x_j - x_i, y_{jp} + y_j - y_i) = a_{ik}x_{jp} + b_{ik}y_{jp} + g_{ik}(x_j - x_i, y_j - y_i) \geq 0,$$

then $T_i(v^i)$ *and* $T_j(v^j)$ *do not overlap each other.*

Let

$$d_{ik}^j := \min_{p=1,\ldots,k_j} \{a_{ik}x_{jp} + b_{ik}y_{jp}\}, \; i = 1, \ldots, m, \; k = 1, \ldots, k_i, \; j = 1, \ldots, m, \; j \neq i.$$

The statement $\text{int}\,(T_i(v^i)) \cap \text{int}\,(T_j(v^j)) = \emptyset$ is true if at least one of the following $k_i + k_j$ conditions is fulfilled:

$$-g_{ik}(x_j - x_i, y_j - y_i) \le d_{ik}^j, \quad k = 1, \dots, k_i \tag{8}$$

(Separating of $T_j(v^j)$ using a hyperplain of $T_i(v^i)$),

$$-g_{jk}(x_i - x_j, y_i - y_j) \le d_{jk}^i, \quad k = 1, \dots, k_j. \tag{9}$$

(Separating of $T_i(v^i)$ using a hyperplain of $T_j(v^j)$).

The containment of $T_i(v^i)$, $i = 1, \dots, m$, within $B(= T_0)$ is equivalent to

$$g_{0k}(x_{ip} + x_i, y_{ip} + y_i) \le 0, \quad p = 1, \dots, k_i, \quad k = 1, \dots, k_0.$$

Let $d_{0k}^i := \max_{p=1,\dots,k_i}\{a_{0ik}x_{ip} + b_{0k}y_{ip}\}$. Then $T_i(v^i) \subset B$ if and only if the following inequalities are fulfilled:

$$g_{0k}(x_i, y_i) \le -d_{0k}^i, \quad k = 1, \dots, k_0 \tag{10}$$

Again, 0/1-variables $a_i, i = 1, \dots, m$, are used according to (2) to characterize the containment of T_i within B. Additionally, 0/1-variables u_{ik}^j are defined as follows to describe the mutual position of two objects $T_i(v^i)$ and $T_j(v^j), i \ne j$:

$$u_{ik}^j = \begin{cases} 1 & , \quad \text{if } T_j(v^j) \text{ lies in the half-plain } g_{ik}(x - x_i, y - y_i) \ge 0, \\ 0 & , \quad \text{otherwise ,} \end{cases}$$
$$i = 1, \dots, m, \; k = 1, \dots, k_i, \; j = 1, \dots, m, \; i \ne j.$$

By this a natural generalization is given of the 0/1-variables in section 4.1. Using the same linear objective function (4) and the restrictions (8)–(10) the following linear mixed-integer model is obtained where M is a sufficiently large number.

$$z = \sum_{i=1}^{m} c_i a_i \rightarrow \max \tag{11}$$

subject to

$$g_{0k}(x_i, y_i) + d_{0k}^i \le M(1 - a_i), \quad a_i \in \{0, 1\}, \; i = 1, \dots, m, \; k = 1, \dots, k_0 \tag{12}$$

and

$$-g_{ik}(x_j - x_i, y_j - y_i) - d^j_{ik} \le M(1 - u^j_{ik}),$$
$$k = 1, \dots, k_i, \ i = 1, \dots, m, \ j = 1, \dots, m, \ i \ne j. \tag{13}$$

$$\sum_{k=1}^{k_i} u^j_{ik} + \sum_{k=1}^{k_j} u^i_{jk} \ge 1, \quad i = 1, \dots, m-1, \ j = i+1, \dots, m, \tag{14}$$

$$u^j_{ik} \in \{1, 0\}, \ i = 1, \dots, m, \ k = 1, \dots, k_i, \ j = 1, \dots, m, \ i \ne j.$$

If $a_i = 1$ then (12) secures $T_i(v^i) \subset B$. (13,14) ensures that $T_i(v^i)$ and $T_j(v^j)$ do not overlap each other.

The model (11)–(14) uses $2m$ continuous, $m + (m-1)\sum_{i=1}^m k_i$ 0/1-variables and $mk_0 + m(m-1)/2 + (m-1)\sum_{i=1}^m k_i$ inequalities. With respect to the fact that for fixed $i, j, i \ne j$, only one of the inequalities

$$-g_{ik}(x_j - x_i, y_j - y_i) - d^j_{ik} \le M(1 - u^j_{ik}), \quad k = 1, \dots, k_i$$
$$-g_{jk}(x_i - x_j, y_i - y_j) - d^i_{jk} \le M(1 - u^i_{jk}), \quad k = 1, \dots, k_j$$

has to be fulfilled, an equivalent model with[1] $\lceil log_2(k_i + k_j)\rceil$ 0/1-variables can be obtained in analogy to (7). Then the inequalities (14) are unnecessary.

Hence, the model has $2m$ continuous, $m + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\lceil log_2(k_i + k_j)\rceil$ 0/1-variables and $mk_0 + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(k_i + k_j) = mk_0 + (m-1)\sum_{i=1}^m k_i$ inequalities.

In the case of rectangular objects ($k_i = 4, \ i = 1, \dots, m$) we have the same number of continuous and 0/1-variables but $4m + 4m(m-1)$ inequalities. In section 4.1. additionally the fact could be exploited that the rectangular objects have to be parallel to the edges of the region.

4.2.2 Example

Let be given a region B and two convex polygons T_1 and T_2 as defined below and shown in Fig. 1.

$B = \{(x, y) : g_{01}(x, y) := -x - y + 1 \le 0, \ g_{02}(x, y) := -x \le 0, \ g_{03}(x, y) := y - 2 \le 0, \ g_{04}(x, y) := x - 2 \le 0, \ g_{05}(x, y) := -y \le 0\}$,

$T_1 = \{(x, y) : g_{11}(x, y) := -y \le 0, \ g_{12}(x, y) := -2x + y \le 0, \ g_{13}(x, y) := x - 1 \le 0\}$,
$T_2 = \{(x, y) : g_{21}(x, y) := -x - y + 1 \le 0, \ g_{22}(x, y) := -x + y - 1 \le 0, \ g_{23}(x, y) := y - 2 \le 0, \ g_{24}(x, y) := 2x - y - 2 \le 0\}$.

With $d^1_{01} = d^1_{02} = d^1_{05} = 0, \ d^1_{03} = 2, \ d^1_{04} = 1$ and $d^2_{02} = d^2_{05} = 0, \ d^2_{01} = -1, \ d^2_{03} = d^2_{04} = 2$, then (12) is as follows:

[1] $\lceil x\rceil$ denotes the smallest integer not smaller than x.

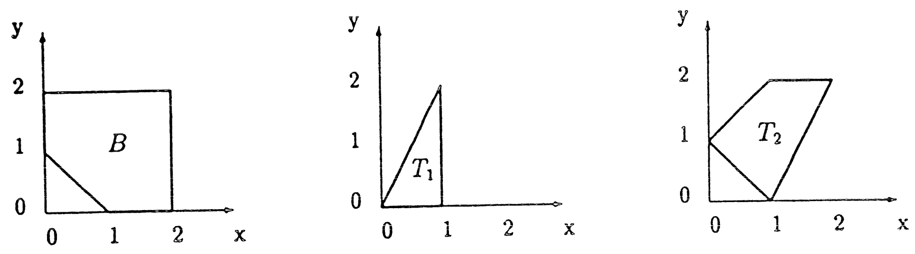

Figure 1 Region B and given objects T_1 and T_2

$$
\begin{aligned}
-x_1 - y_1 + 1 &\le M(1-a_1), & -x_2 - y_2 &\le M(1-a_2),\\
-x_1 &\le M(1-a_1), & -x_2 &\le M(1-a_2),\\
y_1 &\le M(1-a_1), & y_2 &\le M(1-a_2),\\
x_1 - 1 &\le M(1-a_1), & x_2 &\le M(1-a_2),\\
-y_1 &\le M(1-a_1), & -y_2 &\le M(1-a_2);\\
& & a_1, a_2 &\in \{0,1\}.
\end{aligned}
$$

Condition (13) has the form (because of $d_{11}^2 = d_{12}^2 = -2$, $d_{13}^2 = 0$ and $d_{21}^1 = -3$, $d_{22}^1 = -1$, $d_{23}^1 = d_{24}^1 = 0$):

$$
\begin{aligned}
y_2 - y_1 + 2 &\le M(1-u_{11}^2), & (x_1 - x_2) + (y_1 - y_2) + 2 &\le M(1-u_{21}^1),\\
2(x_2 - x_1) - (y_2 - y_1) + 2 &\le M(1-u_{12}^2), & (x_1 - x_2) + (y_1 - y_2) + 2 &\le M(1-u_{22}^1),\\
-(x_2 - x_1) + 1 &\le M(1-u_{13}^2), & -(y_1 - y_2) + 2 &\le M(1-u_{23}^1),\\
& & -2(x_1 - x_2) + (y_1 - y_2) + 2 &\le M(1-u_{24}^1),
\end{aligned}
$$

$$
\begin{gathered}
u_{11}^2 + u_{12}^2 + u_{13}^2 + u_{21}^1 + u_{22}^1 + u_{23}^1 + u_{24}^1 \ge 1,\\
u_{11}^2, u_{12}^2, u_{13}^2, u_{21}^1, u_{22}^1, u_{23}^1, u_{24}^1 \in \{0,1\}.
\end{gathered}
$$

The objective function

$$
z = c_1 a_1 + c_2 a_2 \qquad (c_1 > 0, c_2 > 0)
$$

has the maximum value $c_1 + c_2$ and the optimal solution is as follows:

$$
\begin{gathered}
a_1 = a_2 = 1, \quad u_{12}^2 = u_{24}^1 = 1,\\
u_{11}^2 = u_{13}^2 = u_{21}^1 = u_{22}^1 = u_{23}^1 = 0,\\
(x_1, y_1) = (1, 0), \quad (x_2, y_2) = (0, 0).
\end{gathered}
$$

Remark : Instead of the use of 7 0/1-variables u^i_{ik}, it is sufficient to define $3 = \lceil \log_2(k_1 + k_2) \rceil$ 0/1-variables.

4.2.3 Modeling with Hodographs

The description of the non-overlapping of two objects $T_i(v^i)$ and $T_j(v^j)$ is an essential part when handling packing problems. In the previous section, therefore the position of corner points of $T_j(v^j)$ with respect to bounding hyperplains of $T_i(v^i)$, and reversely, is used. A second possibility consists in characterizing the beginning of the overlapping of T_i by $T_j(v^j)$. With other words, which are the allocation points v_j of $T_j(v^j)$ such that $\text{int}\, T_i \cap \text{int}\, T_j(v^j) = \emptyset$ and $T_i \cap T_j(v^j) \neq \emptyset$ (T_i is considered to be in normalized position). The set of such allocation points is called (outer) hodograph of the dense allocation of T_i and T_j (see also in [6] and [8]). The **hodograph** of two convex polygons T_i and T_j is the boundary of a convex polygon H_{ij} defined according to

$$H_{ij} = \{(x, y) \in R^2 : ((x, y) + T_j) \cap T_i \neq \emptyset\}.$$

Hence, H_{ij} can also be described using closed half-plains:

$$H_{ij} := \{(x, y) \in R^2 : h^{ij}_k(x, y) \leq 0, \quad k = 1, \ldots, k_{ij}\}. \tag{15}$$

On the other hand, H_{ij} can be characterized by hyperplains given T_i and T_j.

Statement 2: *Let T_i and T_j be convex polygons and let $g_{ik} = a_{ik}x + b_{ik}y + c_{ik}$, $k = 1, \ldots, k_i$, and $g_{jk} = a_{jk}x + b_{jk}y + c_{jk}$, $k = 1, \ldots, k_j$, be the bounding hyperplains of T_i and T_j. Then it holds with $p = (p_x, p_y)$:*

$$\begin{aligned} H_{ij} = \{(x, y) \in R^2 : g_{ik}(x, y) + \min_{p \in T_j} g_{ik}(p_x, p_y) - c_{ik} \leq 0, \quad k = 1, \ldots, k_i, \\ -g_{jk}(x, y) + \min_{p \in T_i} g_{jk}(p_x, p_y) + c_{jk} \leq 0, \quad k = 1, \ldots, k_j\} \end{aligned} \tag{16}$$

Proof: Let $v = (x, y) \in H_{ij}$. Then, per definition, $T_j(v) \cap T_i \neq \emptyset$. Hence, for any bounding hyperplain of $T_j(v)$ there exists at least one element $(\bar{x}, \bar{y}) \in T_i$ such that $g_{jk}(\bar{x} - x, \bar{y} - y) \leq 0$. Moreover:

$$-g_{jk}(x, y) + \min_{p \in T_i} g_{jk}(p_x, p_y) + c_{jk} \leq 0, \quad k = 1, \ldots, k_j.$$

Since $(v + T_j) \cap T_i \neq \emptyset$ if and only if $T_j \cap (T_i - v) \neq \emptyset$, one has:

$$g_{ik}(x, y) + \min_{p \in T_j} g_{ik}(p_x, p_y) - c_{ik} \leq 0, \quad k = 1, \ldots, k_i \qquad \blacksquare$$

Hence, the functions h^{ij}_k for (15) are chosen from the functions used in the inequalities in (16). The non-overlapping condition for $T_i(v^i)$ and $T_j(v^j)$ requires, using (15), that at least one of the inequalities

$$h_k^{ij}(x_j - x_i, y_j - y_i) \geq 0, \quad k = 1, \ldots, k_{ij}, \tag{17}$$

is fulfilled. Since the non-overlapping of T_j and T_i is ensured if T_i and T_j are non-overlapping. $\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} k_{ij} =: K$ inequalities of the type (17) are to consider. Since the set H_{ij} has at most $k_i + k_j$ corner points (compare in [8], page 67), that is $k_{ij} \leq k_i + k_j$, it follows $K \leq (m-1) \sum_{i=1}^{m} k_i$. Hence, in the case $k_{ij} < k_i + k_j$, the consideration of K inequalities of the type (17) is more favourable than to consider $(m-1) \sum_{i=1}^{m} k_i$ inequalities of type (8) resp. (9).

To guarantee that at least one of the k_{ij} inequalities is fulfilled, $\lceil \log_2(k_{ij}) \rceil$ 0/1-variables u_p^{ij}, $p = 1, \ldots, v$, with $v = v_{ij} := \lceil \log_2(k_{ij}) \rceil$ are necessary. The k-th inequality

$$h_k^{ij}(x_j - x_i, y_j - y_i) + M(\sum_{p=1}^{v} \alpha_p^k + \sum_{p=1}^{v} (-1)^{\alpha_p^k} u_p^{ij}) \geq 0, \tag{18}$$

(where $\alpha_1^k, \ldots, \alpha_v^k$ are the coefficients of the binary representation of $2^v - k$, i.e. $\sum_{p=1}^{v} 2^{v-p} \alpha_p^k = 2^v - k,$) is non-trivial if and only if $u_p^{ij} = \alpha_p^k$ for $p = 1, \ldots, v$. If $2^v > k_{ij}$ then the inequality

$$\sum_{p=1}^{v} 2^{v-p} u_p^{ij} \geq 2^v - k_{ij}$$

ensures that only these values $(u_1^{ij}, \ldots, u_v^{ij})$ are feasible which are required.

The characterization of the non-overlapping of T_i and T_j using (17) yields in general a reducing of the number of restrictions. A similar result can be achieved if the inner hodograph is used to characterize the containment of T_i within B.

The **inner hodograph** is defined to be the boundary of the set

$$H_{0i} = \{(x, y) \in R^2 : (x, y) + T_i \subset B\}.$$

H_{0i} is the set of all feasible allocation points (x, y) of T_i which possibly is empty. H_{0i} is a convex polygon if B and T_i are convex polygons. If k_{0i} denotes the number of corner points of H_{0i} then $k_{0i} \leq k_0$ holds. Therefore, H_{0i} can be described as follows:

$$H_{0i} = \{(x, y) \in R^2 : h_k^{0i}(x, y) \leq 0,\ k = 1, \ldots, k_{0i}\}.$$

Hence, $T_i(v^i) \subset B$ if and only if

$$h_k^{0i}(x_i, y_i) \leq 0, \quad k = 1, \ldots, k_{0i}. \tag{19}$$

In the case $k_{0i} < k_0$, the system of inequalities (19) contains fewer inequalities than the system (10) does.

Using the inequalities of type (18) and (19) the model (11)–(13) is equivalent to:

$$z = \sum_{1=1}^{m} c_i a_i \to \max \tag{20}$$

subject to

$$h_k^{0i}(x_i, y_i) \leq M(1 - a_i), \quad k = 1, \dots, k_{0i}, \quad i = 1, \dots, m; \tag{21}$$
$$a_i \in \{0, 1\}, \quad i = 1, \dots, m;$$

$$h_k^{ij}(x_j - x_i, y_j - y_i) + M\Big(\sum_{p=1}^{v_{ij}} \alpha_p^{ijk} + \sum_{p=1}^{v_{ij}} (-1)^{\alpha_p^{ijk}} u_p^{ij}\Big) \geq 0, \tag{22}$$

with $v_{ij} = \lceil \log_2(k_{ij}) \rceil$; $i = 1, \dots, m-1$, $k = 1, \dots, k_{ij}$, $j = i+1, \dots, m$;

$$\sum_{p=1}^{v_{ij}} 2^{v_{ij}-p} u_p^{ij} \geq 2^{v_{ij}} - k_{ij}, \quad \text{if } 2^{v_{ij}} > k_{ij}; \tag{23}$$

$$u_1^{ij}, \dots, u_{v_{ij}}^{ij} \in \{0, 1\}, \quad i = 1, \dots, m-1, \ j = i+1, \dots, m.$$

The model (20)–(23) contains $2m$ continuous, $m + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} v_{ij}$ 0/1-variables and $\sum_{i=1}^{m} k_{0i} + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} k_{ij}$ inequalities and at most furthermore $m(m-1)/2$ inequalities. Since

$$\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} v_{ij} = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \lceil \log_2(k_{ij}) \rceil < \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} k_{ij} \leq m-1) \sum_{i=1}^{m} k_i,$$

in the model above are used essentially fewer 0/1-variables in comparison to model (11)–(13). Because of $\sum_{i=1}^{m} k_{0i} \leq m k_0$, also fewer inequalities are needed in general.

4.2.4 Example

The region and objects of the example in 4.2.2 are used. In Fig 2 the hodograph of T_1 and T_2 and the set H_{12} are shown.

It holds:

$$\begin{aligned} H_{12} = \{(x, y) \in R^2 : \ & h_1^{12}(x, y) = -2x + y - 2 \leq 0, \\ & h_2^{12}(x, y) = x + y - 2 \leq 0, \\ & h_3^{12}(x, y) = x - 1 \leq 0, \\ & h_4^{12}(x, y) = x - y - 2 \leq 0, \\ & h_5^{12}(x, y) = -y - 2 \leq 0\}. \end{aligned}$$

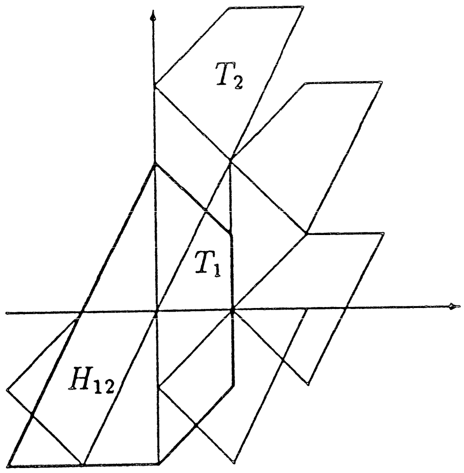

Figure 2 Hodograph of T_1 and T_2 and H_{12}

Because of $v = \lceil \log_2(5) \rceil = 3$, three 0/1-variables u_1, u_2, u_3 are needed to describe the condition $\text{int}\ (\ T_1(v^1)\ ) \cap \text{int}\ (\ T_2(v^2)\ ) = \emptyset$.

$$
\begin{aligned}
-2(x_2 - x_1) + y_2 - y_1 - 2 + M(3 - u_1 - u_2 - u_3) &\geq 0, \\
x_2 - x_1 + y_2 - y_1 - 2 + M(2 - u_1 - u_2 + u_3) &\geq 0, \\
x_2 - x_1 - 1 + M(2 - u_1 + u_2 - u_3) &\geq 0, \\
x_2 - x_1 - (y_2 - y_1) - 2 + M(1 - u_1 + u_2 + u_3) &\geq 0, \\
-(y_2 - y_1) - 2 + M(2 + u_1 - u_2 - u_3) &\geq 0,
\end{aligned}
$$

and

$$4u_1 + 2u_2 + u_3 \geq 2^3 - 5 = 3.$$

The last inequality secures that exact one of the first five inequalities is non-trivial.

4.3. PACKING OF ARBITRARY POLYGONS

4.3.1 Modeling

For problems of packing convex objects it was shown in section 4.2 that the containment of T_i within B can be described in a good manner by using the inner hodograph, and in a similarly good way the non-overlapping of T_i and T_j can be handled using the hodograph of these two objects. Now, these ideas will be translated to the case of arbitrary polygonal objects.

4.3.1.1 Containment of T_i within B

The set

$$H_{0i} := \{ (x,y) \in R^2 : (x,y) + T_i \subset B \}$$

is the set of all feasible allocation points (x,y) of T_i. In contrast to the convex case, H_{0i} is in general not connected (see the example in 4.3.2). But H_{0i} is always a union of convex polygons. Hence, if H_{0i} consists of k_{0i} convex polygons then it can be described as follows:

$$H_{0i} = \Big\{ (x,y) \in R^2 : \big((h_{11}^{0i}(x,y) \le 0) \wedge \cdots \wedge (h_{1l_1^{0i}}^{0i}(x,y) \le 0)\big)$$

$$\vee \cdots \vee \quad \big((h_{k_{0i}1}^{0i}(x,y) \le 0) \ \wedge \cdots \wedge \ (h_{k_{0i}l_{k_{0i}}^{0i}}^{0i}(x,y) \le 0)\big) \Big\}$$

$$= \Big\{ (x,y) \in R^2 : \bigvee_{j=1}^{k_{0i}} \bigwedge_{l=1}^{l_j^{0i}} (h_{jl}^{0i}(x,y) \le 0) \Big\}.$$

Moreover, $T_i(v^i) \subset B$ if and only if $v^i = (x_i, y_i)$ is an element of one of the k_{0i} convex polygons of H_{0i}. Or, if (x_i, y_i) fufills one of the k_{0i} systems of inequalities

$$h_{j1}^{0i}(x_i,y_i) \le 0, \ldots, h_{jl_j^{01}}^{0i}(x_i,y_i) \le 0, \quad j = 1, \ldots, k_{0i}, \tag{24}$$

then $T_i(x_i, y_i)$ is contained in B.
As demonstrated in 4.2.3 the "or"-conditions can be handled by using of 0/1-variables. Let $\lceil \log_2(k_{0i}) \rceil = v_{0i}$ and let $(\alpha_1^{0ij}, \ldots, \alpha_{v_{0i}}^{0ij})$ be the binary coefficients of $2^{v_{0i}} - j$ in accordance to $\sum_{p=1}^{v_{0i}} 2^{v_{0i}-p} \alpha_p^{0ij} = 2^{v_{0i}} - j, \quad j = 1, \ldots, k_{0i}$.

Using the 0/1-variables $u_1^{0i}, \ldots, u_{v_{0i}}^{0i}$ the following system of inequalities results from (24):

$$h_{jl}^{0i}(x_i,y_i) \le M \left(\sum_{p=1}^{v_{0i}} \alpha_p^{0ij} + \sum_{p=1}^{v_{0i}} (-1)^{\alpha_p^{0ij}} u_p^{0i} \right), \quad l = 1, \ldots, l_j^{0i}, \tag{25}$$

$$u_p^{0i} \in \{0,1\}, \quad p = 1, \ldots, v_{0i}.$$

Only if $u_p^{0i} = \alpha_p^{0ij}, p = 1, \ldots, v_{0i}$, then (25) is non-trivial. M has to be sufficiently large, e.g. $M \ge \max_{i=1,\ldots,m} \max_{l=1,\ldots,l_j^{0i}} \max_{(x,y)\in H_{0i}} h_{jl}^{0i}(x,y)$.

In analogy to previous models 0/1-variables a_i are used defined in (2). Now, $T_i(v^i) \subset B, i = 1, \ldots, m$, can be modelled as follows:

$$h_{jl}^{0i}(x_i, y_i) - M\left(\sum_{p=1}^{v_{0i}} \alpha_p^{0ij} + \sum_{p=1}^{v_{0i}} (-1)^{\alpha_p^{0ij}} u_p^{0i}\right) \leq M(1 - a_i), \tag{26}$$

$$l = 1, \ldots, l_j^{0i}, \quad j = 1, \ldots, k_{0i},$$

$$u_p^{0i} \in \{0, 1\}, \quad p = 1, \ldots, v_{0i} = \lceil \log_2(k_{0i}) \rceil,$$

and, if $2^{v_{0i}} > k_{0i}$ then additionally

$$\sum_{p=1}^{v_{0i}} 2^{v_{0i}-p} u_p^{0i} \geq 2^{v_{0i}} - k_{0i},$$

$$a_i \in \{0, 1\}, \quad i = 1, \ldots, m.$$

4.3.1.2 Non-overlapping of T_i and T_j

The hodograph of the dense packing of T_j and T_i (in normalized position) is again defined to be the boundary of the set

$$H_{ij} := \{(x, y) \in R^2 : ((x, y) + T_j) \cap T_i \neq \emptyset\}.$$

The set H_{ij} is, in contrast to H_{0i}, always connected, but not necessarily convex. The description of non-convex polygons T can be done in different ways e.g. as a union of convex polygons. Here, another possibility is chosen. The basis is the convex hull conv (T) of T. Any sequence of bounding edges of T between two neighbourhood corner points of conv (T) can be described by the corresponding straight lines resp. half-plains. After suitable transformations of the linear systems of inequalities which are connected by "$\wedge$"- and "$\vee$"-relations, T can be represented in the following form:

$$T = \Big\{ (x, y) \in R^2 : \Big((g_{11}(x, y) \leq 0) \vee \cdots \vee (g_{1l_1}(x, y) \leq 0) \Big)$$

$$\wedge \cdots \wedge \quad \Big((g_{k1}(x, y) \leq 0) \vee \cdots \vee (g_{kl_k}(x, y) \leq 0) \Big) \Big\}.$$

Hence, $(x, y) \notin$ int(T) if and only if

$$\Big((g_{11}(x, y) \geq 0) \quad \wedge \cdots \wedge \quad ((g_{1l_1}(x, y) \geq 0) \Big)$$

$$\vee \cdots \vee \quad \Big((g_{k1}(x, y) \geq 0) \quad \wedge \cdots \wedge \quad ((g_{kl_k}(x, y) \geq 0) \Big).$$

Let

$$T = H_{ij} = \Big\{ (x,y) \in R^2 : \big((h^{ij}_{11}(x,y) \leq 0) \quad \vee \cdots \vee \quad (h^{ij}_{1l^{1j}_1}(x,y) \leq 0) \big)$$
$$\wedge \cdots \wedge \quad \big((h^{ij}_{k_{ij}1}(x,y) \leq 0) \quad \vee \cdots \vee \quad (h^{ij}_{k_{ij}l^{ij}_{k_{ij}}}(x,y) \leq 0) \big) \Big\} .$$

Then, the non-overlapping of $T_i(v^i)$ and $T_j(v^j)$ is equivalent to:

$$\Big((h^{ij}_{11}(x_j - x_i, y_j - y_i) \geq 0) \quad \wedge \cdots \wedge \quad (h^{ij}_{1l^{1j}_1}(x_j - x_i, y_j - y_i) \geq 0) \Big)$$
$$\vee \cdots \vee \quad \Big((h^{ij}_{k_{ij}1}(x_j - x_i, y_j - y_i) \geq 0) \quad \wedge \cdots \wedge \quad (h^{ij}_{k_{ij}l^{ij}_{k_{ij}}}(x_j - x_i, y_j - y_i) \geq 0) \Big)$$
$$= \bigvee_{r=1}^{k_{ij}} \bigwedge_{l=1}^{l^{ij}_r} (h^{ij}_{rl}(x_j - x_i, y_j - y_i) \geq 0),$$

i.e., if (x_j, y_j) fulfills one of the above systems of inequalities then $T_i(v^i)$ and $T_j(v^j)$ do not overlap each other.

The handling of the "$\vee$"-conditions can be done by using 0/1-variables in a similar way as in 4.3.1.1 for the description of H_{0i}. It results k_{ij} systems of inequalities of the type (25). Hence, the non-overlapping of $T_i(v^i)$ and $T_j(v^j), i = 1, \ldots, m-1, j = i+1, \ldots, m$, can be characterized as follows:

$$h^{ij}_{r1}(x_j - x_i, y_j - y_i) + M \left(\sum_{p=1}^{v_{ij}} \alpha^{ijr}_p + \sum_{p=1}^{v_{ij}} (-1)^{\alpha^{ijr}_p} u^{ij}_p \right) \geq 0, l = 1, \ldots, l^{ij}_r, \quad (27)$$

$$r = 1, \ldots, k_{ij}, u^{ij}_1, \ldots, u^{ij}_{v_{ij}} \in \{0,1\}, i = 1, \ldots, m-1, j = i+1, \ldots, m,$$

and if $2^{v_{ij}} - k_{ij} > 0$ then additionally

$$\sum_{p=1}^{v_{ij}} 2^{v_{ij}-p} u^{ij}_p \geq 2^{v_{ij}} - k_{ij}$$

where $v_{ij} = \lceil \log_2(k_{ij}) \rceil$, $\sum_{p=1}^{v_{ij}} 2^{v_{ij}-p} \alpha^{ijr}_p = 2^{v_{ij}} - r$. The constant M has to be sufficiently large, e.g.

$$M \geq - \min_{r=1,\ldots,k_{ij}} \min_{q=1,\ldots,l^{ij}_r} \min_{(x,y) \in H_{ij}} h^{ij}_{rq}(x,y).$$

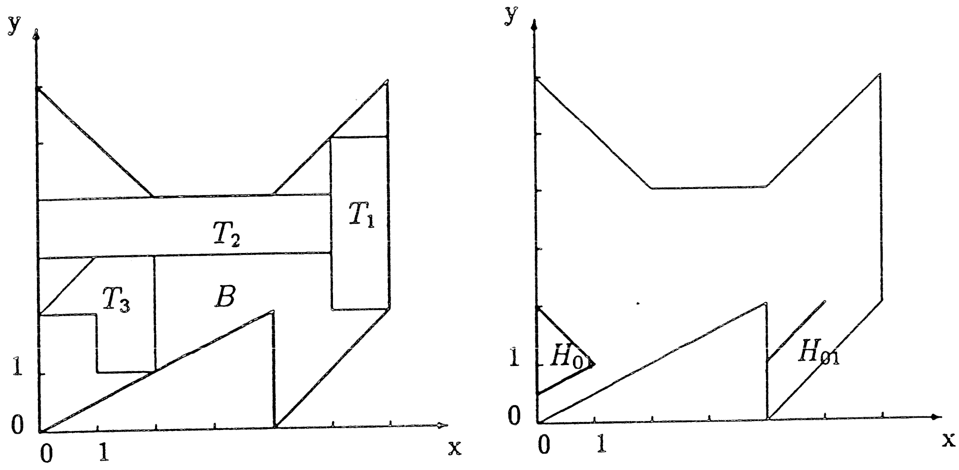

Figure 3 a) Region B and objects T_1, T_2 and T_3; b) set H_{01}

Hence, the problem of packing arbitrary polygons can be written as follows:

$$z = \sum_{i=1}^{m} c_i a_i \to \max \qquad \text{subject to (26) and (27) .}$$

Here, $2m$ continuous, $m + \sum_{i=1}^{m} v_{0i} + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} v_{ij}$ 0/1-variables and $\sum_{i=1}^{m} \sum_{r=1}^{k_{0i}} l_r^{0i} + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \sum_{r=1}^{k_{ij}} l_r^{ij}$ inequalities and at most further $\sum_{i=1}^{m} k_{0i} + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} k_{ij}$ inequalities are used.

The computation of the sets H_{0i} and H_{ij} can be done in analogy to the algorithm given in [8] to compute a hodograph.

4.3.2 Example

Let be given a region B and three objects T_1, T_2 and T_3 as shown in Fig. 3a). Figures 3b), 4 and 5 show the sets H_{01} (not connected), H_{02} (convex), H_{03} (non-convex), H_{12} (convex), H_{13} (non-convex) and H_{23} (non-convex).

It holds:

$$H_{01} = \Big\{ (x,y) \in R^2 : \big[(h_{11}^{01}(x,y) = -x \le 0) \wedge (h_{12}^{01}(x,y) = x + y - 2 \le 0) \wedge$$
$$(h_{13}^{01}(x,y) = x - 2y + 1 \le 0) \big] \vee \big[(h_{21}^{01}(x,y) = -x + 4 \le 0) \wedge$$
$$(h_{22}^{01}(x,y) = x - 3 \le 0) \wedge (h_{23}^{01}(x,y) = -x + y + 3 \le 0) \wedge$$
$$(h_{24}^{01}(x,y) = x - y - 3 \le 0) \big] \Big\}.$$

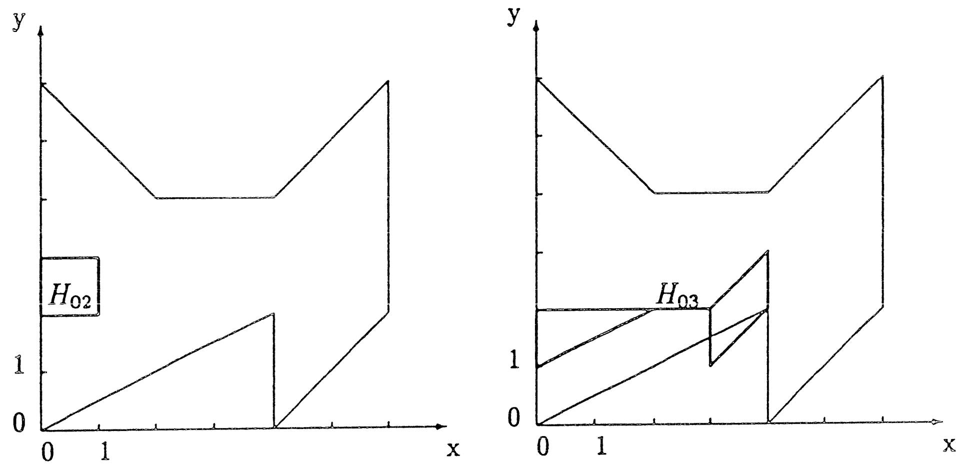

Figure 4 a) H_{02} b) H_{03}

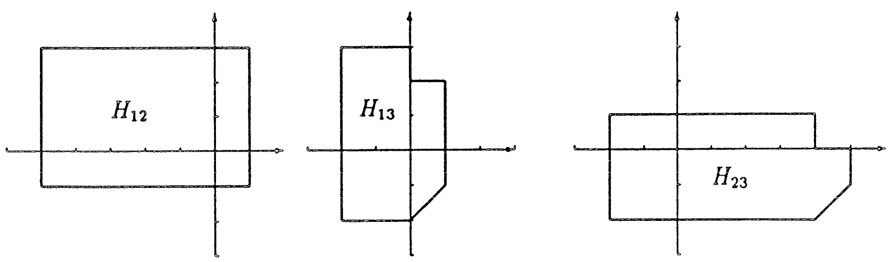

Figure 5 a) H_{12} b) H_{13} c) H_{23}

Since $k_{01} = 2$ only one 0/1-variable u_1^{01} is to introduce. H_{02} is connected, moreover it is convex. An additional 0/1-variable is not necessary. For H_{03} it holds:

$$H_{03} = \Big\{ (x,y) \in R^2 : \big[(h_{11}^{03}(x,y) = -x \leq 0) \wedge (h_{12}^{03}(x,y) = y - 2 \leq 0) \wedge$$
$$(h_{13}^{03}(x,y) = x - 2y + 2 \leq 0) \big] \vee \big[(h_{21}^{03}(x,y) = -x + 2 \leq 0) \wedge$$
$$(h_{22}^{03}(x,y) = x - 3 \leq 0) \wedge (h_{23}^{03}(x,y) = y - 2 \leq 0) \wedge (h_{24}^{03}(x,y) = -y + 2 \leq 0) \big] \vee$$
$$\big[(h_{31}^{03}(x,y) = -x + 3 \leq 0) \wedge (h_{32}^{03}(x,y) = -x + y + 1 \leq 0) \wedge (h_{33}^{03}(x,y) = x - 4 \leq 0) \wedge$$
$$(h_{34}^{03}(x,y) = x - y - 2 \leq 0) \big] \Big\}.$$

Since $k_{03} = 3$, two 0/1-variables u_1^{03}, u_2^{03} are required. Hence, (26) has the following form:

$$\begin{array}{lll}
-x_1 & -M(1 - u_1^{01}) & \leq M(1 - a_1), \\
x_1 + y_1 - 2 & -M(1 - u_1^{01}) & \leq M(1 - a_1), \\
x_1 - 2y_1 + 1 & -M(1 - u_1^{01}) & \leq M(1 - a_1), \\
-x_1 + 4 & -Mu_1^{01} & \leq M(1 - a_1), \\
x_1 - 3 & -Mu_1^{01} & \leq M(1 - a_1), \\
-x_1 + y_1 + 3 & -Mu_1^{01} & \leq M(1 - a_1), \\
x_1 - y_1 - 3 & -Mu_1^{01} & \leq M(1 - a_1),
\end{array}$$

$$\begin{array}{ll}
-x_2 & \leq M(1 - a_2), \\
x_2 - 1 & \leq M(1 - a_2), \\
y_2 - 3 & \leq M(1 - a_2), \\
-y_2 + 2 & \leq M(1 - a_2),
\end{array}$$

$$\begin{array}{lll}
-x_3 & -M(2 - u_1^{03} - u_2^{03}) & \leq M(1 - a_3), \\
y_3 - 2 & -M(2 - u_1^{03} - u_2^{03}) & \leq M(1 - a_3), \\
x_3 - 2y_3 + 2 & -M(2 - u_1^{03} - u_2^{03}) & \leq M(1 - a_3), \\
-x_3 + 2 & -M(1 - u_1^{03} + u_2^{03}) & \leq M(1 - a_3), \\
x_3 - 3 & -M(1 - u_1^{03} + u_2^{03}) & \leq M(1 - a_3), \\
y_3 - 2 & -M(1 - u_1^{03} + u_2^{03}) & \leq M(1 - a_3),
\end{array}$$

$$\begin{array}{llll}
-y_3 + 2 & -M(1 - u_1^{03} + u_2^{03}) & \leq & M(1 - a_3), \\
-x_3 + y_3 + 1 & -M(1 + u_1^{03} - u_2^{03}) & \leq & M(1 - a_3), \\
x_3 - y_3 - 2 & -M(1 + u_1^{03} - u_2^{03}) & \leq & M(1 - a_3), \\
-x_3 + 3 & -M(1 + u_1^{03} - u_2^{03}) & \leq & M(1 - a_3), \\
x_3 - 4 & -M(1 + u_1^{03} - u_2^{03}) & \leq & M(1 - a_3), \\
 & 2u_1^{03} + u_2^{03} & \geq & 1.
\end{array}$$

For H_{12} it holds:

$$H_{12} = \Big\{ (x, y) \in R^2 : (h_{11}^{12}(x, y) = -x - 5 \leq 0) \wedge (h_{21}^{12}(x, y) = y - 3 \leq 0) \wedge \\ (h_{31}^{12}(x, y) = x - 1 \leq 0) \wedge (h_{41}^{12}(x, y) = -y - 1 \leq 0) \Big\}.$$

Since $k_{12} = 4$, two 0/1-variables u_1^{12}, u_2^{12} are necessary. Hence, (27) can be written as follows for $i = 1, j = 2$ (non-overlapping of $T_1(v_1)$ and $T_2(v_2)$):

$$\begin{array}{lll}
-(x_2 - x_1) - 5 & +M(2 - u_1^{12} - u_2^{12}) & \geq 0 \\
(y_2 - y_1) - 3 & +M(1 - u_1^{12} + u_2^{12}) & \geq 0 \\
(x_2 - x_1) - 1 & +M(1 + u_1^{12} - u_2^{12}) & \geq 0 \\
-(y_2 - y_1) - 1 & +M(u_1^{12} + u_2^{12}) & \geq 0.
\end{array}$$

Furthermore, it holds:

$$H_{13} = \Big\{ (x, y) \in R^2 : (h_{11}^{13}(x, y) = -x - 2 \leq 0) \wedge (h_{21}^{13}(x, y) = y - 3 \leq 0) \wedge \\ \big[(h_{31}^{13}(x, y) = x \leq 0) \vee (h_{32}^{13}(x, y) = y - 2 \leq 0) \big] \wedge (h_{41}^{13}(x, y) = x - 1 \leq 0) \wedge \\ (h_{51}^{13}(x, y) = x - y - 2 \leq 0) \wedge (h_{61}^{13}(x, y) = -y - 2 \leq 0)] \Big\}.$$

Because of $k_{13} = 6$, variables $u_1^{13}, u_2^{13}, u_3^{13} \in \{0, 1\}$ are to introduce. Hence, the non-overlapping of $T_1(v^1)$ and $T_3(v^3)$ can be described as follows (according to (27) one has $i = 1, j = 3$):

$$
\begin{array}{lll}
-(x_3 - x_1) - 2 & +M(3 - u_1^{13} - u_2^{13} - u_3^{13}) & \geq 0 \\
(y_3 - y_1) - 3 & +M(2 - u_1^{13} - u_2^{13} + u_3^{13}) & \geq 0 \\
x_3 - x_1 & +M(2 - u_1^{13} + u_2^{13} - u_3^{13}) & \geq 0 \\
(y_3 - y_1) - 2 & +M(2 - u_1^{13} + u_2^{13} - u_3^{13}) & \geq 0 \\
x_3 - x_1 - 1 & +M(1 - u_1^{13} + u_2^{13} + u_3^{13}) & \geq 0 \\
x_3 - x_1 - (y_3 - y_1) - 2 & +M(2 + u_1^{13} - u_2^{13} - u_3^{13}) & \geq 0 \\
-(y_3 - y_1) - 2 & +M(1 + u_1^{13} - u_2^{13} + u_3^{13}) & \geq 0 \\
 & 4u_1^{13} + 2u_2^{13} + u_3^{13} & \geq 2.
\end{array}
$$

In a similar way, the corresponding inequalities can be given to secure the non-overlapping of $T_2(v^2)$ and $T_3(v^3)$.

5. GENERALIZATION AND REMARKS

The generalization of the investigations for the two-dimensional case is possible to the three-dimensional case in a direct manner. Especially, the generalization of the packing of rectangular objects in section 4.1 yields the well-known container loading problem. The modeling of the packing problem of convex (three-dimensional) polyhedrons within a convex polyhedron can be done similarly to section 4.2.1 or to section 4.2.3 when using (three-dimensional) inner and outer hodographs.

The dimensions $n = 1, 2, 3$ are to be distinguished because of their practical relevance. Formally, packing problems of higher dimensions can also be defined and modelled as linear mixed-integer optimization problems with 0/1-variables.

In the investigations made only translations of the objects were allowed, but not rotations. Models for that more general case are not linear.

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