Juraj's Notes

on

3 Dimensional Physical Pendulum

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Chapter 1

3-Dimensional Pendulum

Theorem 1.0.1 (Steiner's Theorem)**.**
Let m be the mass of a body and let I_{ij} be its inertia tensor in a coordinate system with origin in the centre of mass of the body. Then the inertia tensor of the body at $r \in \mathbb{R}^3$ has the form

$$\begin{aligned} J(r)_{ij} &= I_{ij} + m(\|r\|^2 \delta_{ij} - r_i r_j) \\ &= \big(I + m(\|r\|^2 \operatorname{id} - r \otimes r)\big)_{ij} \,. \end{aligned}$$

Lemma 1.0.2.
Let $q \in \mathbb{R}^3$ be a pseudo-vector. Then there is a unique rotation matrix associated to this pseudo-vector

$$R(q) = \exp(L_q)\,,$$

where $L_q = L_i q^i$ and L_i are the generators of the Lie algebra $\mathfrak{so}(3)$.

Remark 1.1.
Note that this map is not injective, i.e. $R(q) = R(q + 2k\pi)$, $\forall k \in \mathbb{Z}$.

Lemma 1.0.3.
Let $J(r)$ be the inertia tensor of a body. Then the inertia tensor of the body rotated by $R(q)$ is

$$\begin{aligned} J(q)_{ij} &= R(q)^k{}_i J(r)_{kl} R(q)^l{}_j \\ &= \big(R(q)^\intercal \cdot J(r) \cdot R(q)\big)_{ij} \,. \end{aligned}$$

Lemma 1.0.4 (Kinetic Energy)**.**
The kinetic energy of a rotating body is

$$\begin{aligned} T(q, \dot q) &= \frac{1}{2} \dot q^i J(q)_{ij} \dot q^j \\ &= \frac{1}{2} \dot q^\intercal \cdot J(q) \cdot \dot q \,, \end{aligned}$$

where $\dot q = \frac{\mathrm{d}q}{\mathrm{d}t}$.

Remark 1.2.
Note that $\frac{\mathrm{d}R(q)}{\mathrm{d}t} = \dot R(q) = R(q) L_{\dot q} = L_{\dot q} R(q) \in \mathrm{T}_{R(q)}\big(\mathrm{SO}(3)\big)$, where $L_{\dot q} = L_i \dot q^i \in \mathfrak{so}(3)$ is the angular velocity tensor.

Lemma 1.0.5 (Potential Energy)**.**
Let $r(q) \in \mathbb{R}^3$ be the position of the centre of mass of a body. The potential energy of the body is

$$\begin{aligned} V(q) &= m g_i r(q)^i \\ &= m g^\intercal \cdot r(q) \,. \end{aligned}$$

Definition 1.0.1 (Rayleigh Dissipation Function)**.**
The Rayleigh dissipative function is a function describing the half rate of energy dissipation of a system caused by forces proportional to the velocity of the system. It has the form

$$\begin{aligned} G(\dot q) &= \frac{1}{2} \dot q^i C_{ij} \dot q^j \\ &= \frac{1}{2} \dot q^\intercal \cdot C \cdot \dot q \,. \end{aligned}$$

The equations of motion for any physical system with friction force linearly proportional to the velocity are

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot q^i} - \frac{\partial L}{\partial q^i} + \frac{\partial G}{\partial \dot q^i} = 0\,,$$

where $L(q, \dot q) = T(q, \dot q) - V(q)$ is the Lagrangian not depending on time explicitly. Expanding the absolute derivative with respect to time and writing out L as $T - V$ one gets

$$\frac{\partial T}{\partial \dot q^j \partial \dot q^i} \ddot q^j + \frac{\partial T}{\partial q^k \partial \dot q^i} \dot q^k - \frac{\partial T}{\partial q^i} + \frac{\partial V}{\partial q^i} + \frac{\partial G}{\partial \dot q^i} = 0\,.$$

Solving for $\ddot q$ gives

$$\ddot q^j = \Big(\frac{\partial T}{\partial \dot q^j \partial \dot q^i}\Big)^{-1} \Big(- \frac{\partial T}{\partial q^k \partial \dot q^i} \dot q^k + \frac{\partial T}{\partial q^i} - \frac{\partial V}{\partial q^i} - \frac{\partial G}{\partial \dot q^i} \Big)\,.$$

Now, assuming that the physical system is a 3-dimensional pendulum with

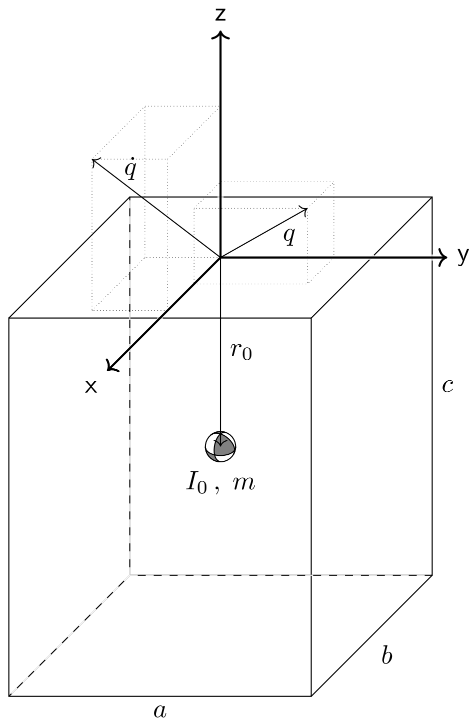

Figure 1.1: A non-rotated pendulum, i.e. the frame of the pendulum corresponds to the base frame.

- $q \in \mathbb{R}^3$ being a pseudo-vector describing rotation (c.f. lemma 1.0.2),
- $J(q)$ being the moment of inertia (c.f. lemma 1.0.3),
- $T(q, \dot{q})$ being the kinetic energy (c.f. lemma 1.0.4),
- $V(q)$ being the potential energy (c.f. lemma 1.0.5),

one has the following partial derivations

$$
\begin{aligned}
\frac{\partial R}{\partial q^i} &= R \cdot L_i = L_i \cdot R\,, \\
\frac{\partial J(q)}{\partial q^i} &= L^{\mathsf{T}}{}_i \cdot J(q) + J(q) \cdot L_i = [J(q), L_i]\ , \\
\frac{\partial T}{\partial \dot{q}^i} &= J(q)_{il} \dot{q}^l = \big(J(q) \cdot \dot{q}\big)_i\,, \\
\frac{\partial T}{\partial \dot{q}^j \partial \dot{q}^i} &= J(q)_{ij}\,, \\
\frac{\partial T}{\partial q^k \partial \dot{q}^i} &= \frac{\partial J(q)_{il}}{\partial q^k} \dot{q}^l = [J(q), L_k]_{il}\, \dot{q}^l\,, \\
\frac{\partial T}{\partial q^i} &= \frac{1}{2} \frac{\partial J(q)_{kl}}{\partial q^i} \dot{q}^k \dot{q}^l = \frac{1}{2} [J(q), L_i]_{kl}\, \dot{q}^k \dot{q}^l = \frac{1}{2} \dot{q}^{\mathsf{T}} \cdot [J(q), L_i] \cdot \dot{q}\,, \\
\frac{\partial V}{\partial q^i} &= m g^{\mathsf{T}} \cdot \frac{\partial R(q)}{\partial q^i} \cdot r = m g^{\mathsf{T}} \cdot L_i \cdot r(q)\,.
\end{aligned}
$$

Assuming $G(\dot{q}) = \frac{1}{2} c \left\|\dot{q}\right\|^2$ (c.f. definition 1.0.1) gives

$$
\frac{\partial G}{\partial \dot{q}^i} = c \dot{q}_i\,.
$$

Finally, plugging it all together yields

$$
\begin{aligned}
\ddot{q}^j &= \big(J(q)^{-1}\big)^{ji} \Big(- [J(q), L_k]_{il}\, \dot{q}^k \dot{q}^l + \frac{1}{2} [J(q), L_i]_{kl}\, \dot{q}^k \dot{q}^l - m g^{\mathsf{T}} \cdot L_i \cdot r(q) - c \dot{q}_i \Big) \\
&= \big(J(q)^{-1}\big)^{ji} \Big(- \frac{1}{2} \big([J(q), L_k]_{il} + [J(q), L_l]_{ik} - [J(q), L_i]_{kl} \big) \dot{q}^k \dot{q}^l - m g^{\mathsf{T}} \cdot L_i \cdot r(q) - c \dot{q}_i \Big) \\
&= \big(J(q)^{-1}\big)^{ji} \Big(- \Gamma_{ikl} \dot{q}^k \dot{q}^l - m g^{\mathsf{T}} \cdot L_i \cdot r(q) - c \dot{q}_i \Big)\,,
\end{aligned}
$$

where $\Gamma_{ikl} = \frac{1}{2} \big([J(q), L_k]_{il} + [J(q), L_l]_{ik} - [J(q), L_i]_{kl} \big)$ are the Christoffel symbols of the first kind.

Chapter 2

Robotic Arm with Revolute Joints

$$R_i = \exp(L_{q_i}) = \exp(L_j q_i^j) = \exp(L_{\hat{q}_i} \theta_i)$$

$$\frac{\mathrm{d}R_i}{\mathrm{d}t} = L_{\dot{q}_i} \cdot R_i = L_j (\dot{q}_i^j) \cdot R_i = (L_{\hat{q}_i} \dot{\theta}_i) \cdot R_i$$

$$L_x = \begin{pmatrix} & & \\ & & -1 \\ & 1 & \end{pmatrix}, \quad L_y = \begin{pmatrix} & & 1 \\ & & \\ -1 & & \end{pmatrix}, \quad L_z = \begin{pmatrix} & -1 & \\ 1 & & \\ & & \end{pmatrix}$$

$$R = \begin{pmatrix} R_1 & R_1 R_2 & R_1 R_2 R_3 & R_1 R_2 R_3 R_4 & R_1 R_2 R_3 R_4 R_5 & R_1 R_2 R_3 R_4 R_5 R_6 & R_1 R_2 R_3 R_4 R_5 R_6 R_7 \\ & R_2 & R_2 R_3 & R_2 R_3 R_4 & R_2 R_3 R_4 R_5 & R_2 R_3 R_4 R_5 R_6 & R_2 R_3 R_4 R_5 R_6 R_7 \\ & & R_3 & R_3 R_4 & R_3 R_4 R_5 & R_3 R_4 R_5 R_6 & R_3 R_4 R_5 R_6 R_7 \\ & & & R_4 & R_4 R_5 & R_4 R_5 R_6 & R_4 R_5 R_6 R_7 \\ & & & & R_5 & R_5 R_6 & R_5 R_6 R_7 \\ & & & & & R_6 & R_6 R_7 \\ & & & & & & R_7 \end{pmatrix}$$

$$\partial_1 R = \begin{pmatrix} (dR_1) & (dR_1) R_2 & (dR_1) R_2 R_3 & (dR_1) R_2 R_3 R_4 & (dR_1) R_2 R_3 R_4 R_5 & (dR_1) R_2 R_3 R_4 R_5 R_6 & (dR_1) R_2 R_3 R_4 R_5 R_6 R_7 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix}$$

$$\partial_2 R = \begin{pmatrix} & R_1 (dR_2) & R_1 (dR_2) R_3 & R_1 (dR_2) R_3 R_4 & R_1 (dR_2) R_3 R_4 R_5 & R_1 (dR_2) R_3 R_4 R_5 R_6 & R_1 (dR_2) R_3 R_4 R_5 R_6 R_7 \\ & (dR_2) & (dR_2) R_3 & (dR_2) R_3 R_4 & (dR_2) R_3 R_4 R_5 & (dR_2) R_3 R_4 R_5 R_6 & (dR_2) R_3 R_4 R_5 R_6 R_7 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix}$$

$$\partial_3 R = \begin{pmatrix} & & R_1 R_2 (dR_3) & R_1 R_2 (dR_3) R_4 & R_1 R_2 (dR_3) R_4 R_5 & R_1 R_2 (dR_3) R_4 R_5 R_6 & R_1 R_2 (dR_3) R_4 R_5 R_6 R_7 \\ & & R_2 (dR_3) & R_2 (dR_3) R_4 & R_2 (dR_3) R_4 R_5 & R_2 (dR_3) R_4 R_5 R_6 & R_2 (dR_3) R_4 R_5 R_6 R_7 \\ & & (dR_3) & (dR_3) R_4 & (dR_3) R_4 R_5 & (dR_3) R_4 R_5 R_6 & (dR_3) R_4 R_5 R_6 R_7 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix}$$

$$\partial_4 R = \begin{pmatrix} & & & R_1 R_2 R_3 (dR_4) & R_1 R_2 R_3 (dR_4) R_5 & R_1 R_2 R_3 (dR_4) R_5 R_6 & R_1 R_2 R_3 (dR_4) R_5 R_6 R_7 \\ & & & R_2 R_3 (dR_4) & R_2 R_3 (dR_4) R_5 & R_2 R_3 (dR_4) R_5 R_6 & R_2 R_3 (dR_4) R_5 R_6 R_7 \\ & & & R_3 (dR_4) & R_3 (dR_4) R_5 & R_3 (dR_4) R_5 R_6 & R_3 (dR_4) R_5 R_6 R_7 \\ & & & (dR_4) & (dR_4) R_5 & (dR_4) R_5 R_6 & (dR_4) R_5 R_6 R_7 \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix}$$

$$\partial_5 R = \begin{pmatrix} & & & & R_1 R_2 R_3 R_4 (dR_5) & R_1 R_2 R_3 R_4 (dR_5) R_6 & R_1 R_2 R_3 R_4 (dR_5) R_6 R_7 \\ & & & & R_2 R_3 R_4 (dR_5) & R_2 R_3 R_4 (dR_5) R_6 & R_2 R_3 R_4 (dR_5) R_6 R_7 \\ & & & & R_3 R_4 (dR_5) & R_3 R_4 (dR_5) R_6 & R_3 R_4 (dR_5) R_6 R_7 \\ & & & & R_4 (dR_5) & R_4 (dR_5) R_6 & R_4 (dR_5) R_6 R_7 \\ & & & & (dR_5) & (dR_5) R_6 & (dR_5) R_6 R_7 \\ & & & & & & \\ & & & & & & \end{pmatrix}$$

$$\partial_6 R = \begin{pmatrix} & & & & & R_1 R_2 R_3 R_4 R_5 (dR_6) & R_1 R_2 R_3 R_4 R_5 (dR_6) R_7 \\ & & & & & R_2 R_3 R_4 R_5 (dR_6) & R_2 R_3 R_4 R_5 (dR_6) R_7 \\ & & & & & R_3 R_4 R_5 (dR_6) & R_3 R_4 R_5 (dR_6) R_7 \\ & & & & & R_4 R_5 (dR_6) & R_4 R_5 (dR_6) R_7 \\ & & & & & R_5 (dR_6) & R_5 (dR_6) R_7 \\ & & & & & (dR_6) & (dR_6) R_7 \\ & & & & & & \end{pmatrix}$$

$$\partial_7 R = \begin{pmatrix} & & & & & & R_1 R_2 R_3 R_4 R_5 R_6 (dR_7) \\ & & & & & & R_2 R_3 R_4 R_5 R_6 (dR_7) \\ & & & & & & R_3 R_4 R_5 R_6 (dR_7) \\ & & & & & & R_4 R_5 R_6 (dR_7) \\ & & & & & & R_5 R_6 (dR_7) \\ & & & & & & R_6 (dR_7) \\ & & & & & & (dR_7) \end{pmatrix}$$

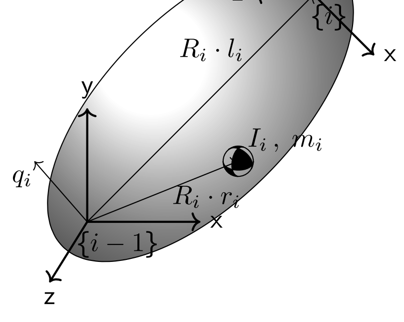

Figure 2.1: A link and link parameters.

$$
R^T = \begin{pmatrix}
R_1^T \\
R_2^T R_1^T & R_2^T \\
R_3^T R_2^T R_1^T & R_3^T R_2^T & R_3^T \\
R_4^T R_3^T R_2^T R_1^T & R_4^T R_3^T R_2^T & R_4^T R_3^T & R_4^T \\
R_5^T R_4^T R_3^T R_2^T R_1^T & R_5^T R_4^T R_3^T R_2^T & R_5^T R_4^T R_3^T & R_5^T R_4^T & R_5^T \\
R_6^T R_5^T R_4^T R_3^T R_2^T R_1^T & R_6^T R_5^T R_4^T R_3^T R_2^T & R_6^T R_5^T R_4^T R_3^T & R_6^T R_5^T R_4^T & R_6^T R_5^T & R_6^T \\
R_7^T R_6^T R_5^T R_4^T R_3^T R_2^T R_1^T & R_7^T R_6^T R_5^T R_4^T R_3^T R_2^T & R_7^T R_6^T R_5^T R_4^T R_3^T & R_7^T R_6^T R_5^T R_4^T & R_7^T R_6^T R_5^T & R_7^T R_6^T & R_7^T
\end{pmatrix}
$$

$$
\begin{aligned}
\theta &= \begin{pmatrix}\theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & \theta_6 & \theta_7\end{pmatrix}^T \\
\hat{q} &= \begin{pmatrix}\hat{q}_1 & \hat{q}_2 & \hat{q}_3 & \hat{q}_4 & \hat{q}_5 & \hat{q}_6 & \hat{q}_7\end{pmatrix}^T \\
q &= \begin{pmatrix}q_1 & q_2 & q_3 & q_4 & q_5 & q_6 & q_7\end{pmatrix}^T \\
&= \hat{q} * \theta
\end{aligned}
$$

$$
\begin{aligned}
\dot{q} &= \begin{pmatrix}\dot{q}_1 & \dot{q}_2 & \dot{q}_3 & \dot{q}_4 & \dot{q}_5 & \dot{q}_6 & \dot{q}_7\end{pmatrix}^T \\
&= \hat{q} * \dot{\theta}
\end{aligned}
$$

$$
\omega = R^T \cdot \dot{q}
$$

$$
I = \begin{pmatrix}
I_1 \\
& I_2 \\
& & I_3 \\
& & & I_4 \\
& & & & I_5 \\
& & & & & I_6 \\
& & & & & & I_7
\end{pmatrix}
$$

$$
\begin{aligned}
T^{rot} &= \dot{q}^T \cdot R \cdot I \cdot R^T \cdot \dot{q} \\
&= \dot{\theta}^T \cdot \hat{q}^T @ R \cdot I \cdot R^T @ \hat{q} \cdot \dot{\theta} \\
&= \dot{\theta}^T \cdot (R^T @ \hat{q})^T \cdot I \cdot (R^T @ \hat{q}) \cdot \dot{\theta} \\
&= \dot{\theta}^T \cdot \hat{q}^T @ \left(R \cdot I \cdot R^T\right) @ \hat{q} \cdot \dot{\theta}
\end{aligned}
$$

$$
C = \begin{pmatrix}
r_1 & l_1 & l_1 & l_1 & l_1 & l_1 & l_1 \\
& r_2 & l_2 & l_2 & l_2 & l_2 & l_2 \\
& & r_3 & l_3 & l_3 & l_3 & l_3 \\
& & & r_4 & l_4 & l_4 & l_4 \\
& & & & r_5 & l_5 & l_5 \\
& & & & & r_6 & l_6 \\
& & & & & & r_7
\end{pmatrix}
$$

$$
c = R \cdot C = \begin{pmatrix}
c_1^0 & c_2^0 & c_3^0 & c_4^0 & c_5^0 & c_6^0 & c_7^0 \\
& c_2^1 & c_3^1 & c_4^1 & c_5^1 & c_6^1 & c_7^1 \\
& & c_3^2 & c_4^2 & c_5^2 & c_6^2 & c_7^2 \\
& & & c_4^3 & c_5^3 & c_6^3 & c_7^3 \\
& & & & c_5^4 & c_6^4 & c_7^4 \\
& & & & & c_6^5 & c_7^5 \\
& & & & & & c_7^6
\end{pmatrix}
$$

$$
\begin{aligned}
c_1^0 &= R_1 r_1 \\
c_2^0 &= R_1 l_1 + R_1 R_2 r_2 \\
c_3^0 &= R_1 l_1 + R_1 R_2 l_2 + R_1 R_2 R_3 r_3 \\
c_4^0 &= R_1 l_1 + R_1 R_2 l_2 + R_1 R_2 R_3 l_3 + R_1 R_2 R_3 R_4 r_4 \\
c_5^0 &= R_1 l_1 + R_1 R_2 l_2 + R_1 R_2 R_3 l_3 + R_1 R_2 R_3 R_4 l_4 + R_1 R_2 R_3 R_4 R_5 r_5 \\
c_6^0 &= R_1 l_1 + R_1 R_2 l_2 + R_1 R_2 R_3 l_3 + R_1 R_2 R_3 R_4 l_4 + R_1 R_2 R_3 R_4 R_5 l_5 + R_1 R_2 R_3 R_4 R_5 R_6 r_6 \\
c_7^0 &= R_1 l_1 + R_1 R_2 l_2 + R_1 R_2 R_3 l_3 + R_1 R_2 R_3 R_4 l_4 + R_1 R_2 R_3 R_4 R_5 l_5 + R_1 R_2 R_3 R_4 R_5 R_6 l_6 + R_1 R_2 R_3 R_4 R_5 R_6 R_7 r_7
\end{aligned}
$$

$$
\begin{aligned}
c_1^0 &= R_1 r_1 \\
c_2^0 &= R_1 (l_1 + R_2 r_2)\,, \qquad\qquad c_2^1 = R_2 r_2 \\
c_3^0 &= R_1 [l_1 + R_2 (l_2 + R_3 r_3)]\,, \quad c_3^1 = R_2 (l_2 + R_3 r_3)\,, \quad c_3^2 = R_3 r_3 \\
c_4^0 &= R_1 (l_1 + R_2 [l_2 + R_3 (l_3 + R_4 r_4)]) \\
c_5^0 &= R_1 [l_1 + R_2 (l_2 + R_3 [l_3 + R_4 (l_4 + R_5 r_5)])] \\
c_6^0 &= R_1 \left(l_1 + R_2 [l_2 + R_3 (l_3 + R_4 [l_4 + R_5 (l_5 + R_6 r_6)])]\right) \\
c_7^0 &= R_1 \left[l_1 + R_2 \left(l_2 + R_3 [l_3 + R_4 (l_4 + R_5 [l_5 + R_6 (l_6 + R_7 r_7)])]\right)\right]
\end{aligned}
$$

$$
V = -g \cdot \sum_i c_i^0 m_i
$$

$$
\begin{aligned}
-\dot{c}_1^0 &= L_{c_1^0} \dot{q}_1 \\
-\dot{c}_2^0 &= L_{c_2^0} \dot{q}_1 + R_1 L_{c_2^1} \dot{q}_2 \\
-\dot{c}_3^0 &= L_{c_3^0} \dot{q}_1 + R_1 L_{c_3^1} \dot{q}_2 + R_1 R_2 L_{c_3^2} \dot{q}_3 \\
-\dot{c}_4^0 &= L_{c_4^0} \dot{q}_1 + R_1 L_{c_4^1} \dot{q}_2 + R_1 R_2 L_{c_4^2} \dot{q}_3 + R_1 R_2 R_3 L_{c_4^3} \dot{q}_4 \\
-\dot{c}_5^0 &= L_{c_5^0} \dot{q}_1 + R_1 L_{c_5^1} \dot{q}_2 + R_1 R_2 L_{c_5^2} \dot{q}_3 + R_1 R_2 R_3 L_{c_5^3} \dot{q}_4 + R_1 R_2 R_3 R_4 L_{c_5^4} \dot{q}_5 \\
-\dot{c}_6^0 &= L_{c_6^0} \dot{q}_1 + R_1 L_{c_6^1} \dot{q}_2 + R_1 R_2 L_{c_6^2} \dot{q}_3 + R_1 R_2 R_3 L_{c_6^3} \dot{q}_4 + R_1 R_2 R_3 R_4 L_{c_6^4} \dot{q}_5 + R_1 R_2 R_3 R_4 R_5 L_{c_6^5} \dot{q}_6 \\
-\dot{c}_7^0 &= L_{c_7^0} \dot{q}_1 + R_1 L_{c_7^1} \dot{q}_2 + R_1 R_2 L_{c_7^2} \dot{q}_3 + R_1 R_2 R_3 L_{c_7^3} \dot{q}_4 + R_1 R_2 R_3 R_4 L_{c_7^4} \dot{q}_5 + R_1 R_2 R_3 R_4 R_5 L_{c_7^5} \dot{q}_6 + R_1 R_2 R_3 R_4 R_5 R_6 L_{c_7^6} \dot{q}_7
\end{aligned}
$$

$$
\tilde{L}_c = \begin{pmatrix}
L_{c_1^0} \\
L_{c_2^0} & L_{c_2^1} \\
L_{c_3^0} & L_{c_3^1} & L_{c_3^2} \\
L_{c_4^0} & L_{c_4^1} & L_{c_4^2} & L_{c_4^3} \\
L_{c_5^0} & L_{c_5^1} & L_{c_5^2} & L_{c_5^3} & L_{c_5^4} \\
L_{c_6^0} & L_{c_6^1} & L_{c_6^2} & L_{c_6^3} & L_{c_6^4} & L_{c_6^5} \\
L_{c_7^0} & L_{c_7^1} & L_{c_7^2} & L_{c_7^3} & L_{c_7^4} & L_{c_7^5} & L_{c_7^6}
\end{pmatrix}
$$

$$L_c = \begin{pmatrix} L_{c_1^0} \\ L_{c_2^0} & R_1 L_{c_2^1} \\ L_{c_3^0} & R_1 L_{c_3^1} & R_1 R_2 L_{c_3^2} \\ L_{c_4^0} & R_1 L_{c_4^1} & R_1 R_2 L_{c_4^2} & R_1 R_2 R_3 L_{c_4^3} \\ L_{c_5^0} & R_1 L_{c_5^1} & R_1 R_2 L_{c_5^2} & R_1 R_2 R_3 L_{c_5^3} & R_1 R_2 R_3 R_4 L_{c_5^4} \\ L_{c_6^0} & R_1 L_{c_6^1} & R_1 R_2 L_{c_6^2} & R_1 R_2 R_3 L_{c_6^3} & R_1 R_2 R_3 R_4 L_{c_6^4} & R_1 R_2 R_3 R_4 R_5 L_{c_6^5} \\ L_{c_7^0} & R_1 L_{c_7^1} & R_1 R_2 L_{c_7^2} & R_1 R_2 R_3 L_{c_7^3} & R_1 R_2 R_3 R_4 L_{c_7^4} & R_1 R_2 R_3 R_4 R_5 L_{c_7^5} & R_1 R_2 R_3 R_4 R_5 R_6 L_{c_7^6} \end{pmatrix}$$

$$-\dot{c}^0 = L_c \cdot \dot{q}$$

$$L_c @ \hat{q} = \begin{pmatrix} L_{c_1^0} \hat{q}_1 \\ L_{c_2^0} \hat{q}_1 & R_1 L_{c_2^1} \hat{q}_2 \\ L_{c_3^0} \hat{q}_1 & R_1 L_{c_3^1} \hat{q}_2 & R_1 R_2 L_{c_3^2} \hat{q}_3 \\ L_{c_4^0} \hat{q}_1 & R_1 L_{c_4^1} \hat{q}_2 & R_1 R_2 L_{c_4^2} \hat{q}_3 & R_1 R_2 R_3 L_{c_4^3} \hat{q}_4 \\ L_{c_5^0} \hat{q}_1 & R_1 L_{c_5^1} \hat{q}_2 & R_1 R_2 L_{c_5^2} \hat{q}_3 & R_1 R_2 R_3 L_{c_5^3} \hat{q}_4 & R_1 R_2 R_3 R_4 L_{c_5^4} \hat{q}_5 \\ L_{c_6^0} \hat{q}_1 & R_1 L_{c_6^1} \hat{q}_2 & R_1 R_2 L_{c_6^2} \hat{q}_3 & R_1 R_2 R_3 L_{c_6^3} \hat{q}_4 & R_1 R_2 R_3 R_4 L_{c_6^4} \hat{q}_5 & R_1 R_2 R_3 R_4 R_5 L_{c_6^5} \hat{q}_6 \\ L_{c_7^0} \hat{q}_1 & R_1 L_{c_7^1} \hat{q}_2 & R_1 R_2 L_{c_7^2} \hat{q}_3 & R_1 R_2 R_3 L_{c_7^3} \hat{q}_4 & R_1 R_2 R_3 R_4 L_{c_7^4} \hat{q}_5 & R_1 R_2 R_3 R_4 R_5 L_{c_7^5} \hat{q}_6 & R_1 R_2 R_3 R_4 R_5 R_6 L_{c_7^6} \hat{q}_7 \end{pmatrix}$$

$$M = \begin{pmatrix} m_1 \mathbb{I}_3 \\ & m_2 \mathbb{I}_3 \\ & & m_3 \mathbb{I}_3 \\ & & & m_4 \mathbb{I}_3 \\ & & & & m_5 \mathbb{I}_3 \\ & & & & & m_6 \mathbb{I}_3 \\ & & & & & & m_7 \mathbb{I}_3 \end{pmatrix}$$

$$\begin{aligned} T^{tra} &= \sum_i m_i \dot{c}_i^0 \cdot \dot{c}_i^0 \\ &= \dot{\theta}^T \cdot \hat{q}^T @ (L_c^T M L_c) @ \hat{q} \cdot \dot{\theta} \\ &= \dot{\theta}^T \cdot (L_c @ \hat{q})^T M (L_c @ \hat{q}) \cdot \dot{\theta} \end{aligned}$$

$$L_c^T = \begin{pmatrix} L_{c_1^0}^T & L_{c_2^0}^T & L_{c_3^0}^T & L_{c_4^0}^T & L_{c_5^0}^T & L_{c_6^0}^T & L_{c_7^0}^T \\ & L_{c_2^1}^T R_1^T & L_{c_3^1}^T R_1^T & L_{c_4^1}^T R_1^T & L_{c_5^1}^T R_1^T & L_{c_6^1}^T R_1^T & L_{c_7^1}^T R_1^T \\ & & L_{c_3^2}^T R_2^T R_1^T & L_{c_4^2}^T R_2^T R_1^T & L_{c_5^2}^T R_2^T R_1^T & L_{c_6^2}^T R_2^T R_1^T & L_{c_7^2}^T R_2^T R_1^T \\ & & & L_{c_4^3}^T R_3^T R_2^T R_1^T & L_{c_5^3}^T R_3^T R_2^T R_1^T & L_{c_6^3}^T R_3^T R_2^T R_1^T & L_{c_7^3}^T R_3^T R_2^T R_1^T \\ & & & & L_{c_5^4}^T R_4^T R_3^T R_2^T R_1^T & L_{c_6^4}^T R_4^T R_3^T R_2^T R_1^T & L_{c_7^4}^T R_4^T R_3^T R_2^T R_1^T \\ & & & & & L_{c_6^5}^T R_5^T R_4^T R_3^T R_2^T R_1^T & L_{c_7^5}^T R_5^T R_4^T R_3^T R_2^T R_1^T \\ & & & & & & L_{c_7^6}^T R_6^T R_5^T R_4^T R_3^T R_2^T R_1^T \end{pmatrix}$$

$$S_1 = m_1 \begin{pmatrix} L_{c_1^0}^T L_{c_1^0} \\ & 0 \\ & & 0 \\ & & & 0 \\ & & & & 0 \\ & & & & & 0 \\ & & & & & & 0 \end{pmatrix}$$

$$S_2 = m_2 \begin{pmatrix} L_{c_2^0}^T L_{c_2^0} & L_{c_2^0}^T R_1 L_{c_2^1} \\ L_{c_2^1}^T R_1^T L_{c_2^0} & L_{c_2^1}^T L_{c_2^1} \\ & & 0 \\ & & & 0 \\ & & & & 0 \\ & & & & & 0 \\ & & & & & & 0 \end{pmatrix}$$

$$S_3 = m_3 \begin{pmatrix} L_{c_3^0}^T L_{c_3^0} & L_{c_3^0}^T R_1 L_{c_3^1} & L_{c_3^0}^T R_1 R_2 L_{c_3^2} \\ L_{c_3^1}^T R_1^T L_{c_3^0} & L_{c_3^1}^T L_{c_3^1} & L_{c_3^1}^T R_2 L_{c_3^2} \\ L_{c_3^2}^T R_2^T R_1^T L_{c_3^0} & L_{c_3^2}^T R_2^T L_{c_3^1} & L_{c_3^2}^T L_{c_3^2} \\ & & & 0 \\ & & & & 0 \\ & & & & & 0 \\ & & & & & & 0 \end{pmatrix}$$

$$S_4 = m_4 \begin{pmatrix} L_{c_4^0}^T L_{c_4^0} & L_{c_4^0}^T R_1 L_{c_4^1} & L_{c_4^0}^T R_1 R_2 L_{c_4^2} & L_{c_4^0}^T R_1 R_2 R_3 L_{c_4^3} \\ L_{c_4^1}^T R_1^T L_{c_4^0} & L_{c_4^1}^T L_{c_4^1} & L_{c_4^1}^T R_2 L_{c_4^2} & L_{c_4^1}^T R_2 R_3 L_{c_4^3} \\ L_{c_4^2}^T R_2^T R_1^T L_{c_4^0} & L_{c_4^2}^T R_2^T L_{c_4^1} & L_{c_4^2}^T L_{c_4^2} & L_{c_4^2}^T R_3 L_{c_4^3} \\ L_{c_4^3}^T R_3^T R_2^T R_1^T L_{c_4^0} & L_{c_4^3}^T R_3^T R_2^T L_{c_4^1} & L_{c_4^3}^T R_3^T L_{c_4^2} & L_{c_4^3}^T L_{c_4^3} \\ & & & & 0 \\ & & & & & 0 \\ & & & & & & 0 \end{pmatrix}$$

$$S_5 = m_5 \begin{pmatrix} L_{c_5^0}^T L_{c_5^0} & L_{c_5^0}^T R_1 L_{c_5^1} & L_{c_5^0}^T R_1 R_2 L_{c_5^2} & L_{c_5^0}^T R_1 R_2 R_3 L_{c_5^3} & L_{c_5^0}^T R_1 R_2 R_3 R_4 L_{c_5^4} \\ L_{c_5^1}^T R_1^T L_{c_5^0} & L_{c_5^1}^T L_{c_5^1} & L_{c_5^1}^T R_2 L_{c_5^2} & L_{c_5^1}^T R_2 R_3 L_{c_5^3} & L_{c_5^1}^T R_2 R_3 R_4 L_{c_5^4} \\ L_{c_5^2}^T R_2^T R_1^T L_{c_5^0} & L_{c_5^2}^T R_2^T L_{c_5^1} & L_{c_5^2}^T L_{c_5^2} & L_{c_5^2}^T R_3 L_{c_5^3} & L_{c_5^2}^T R_3 R_4 L_{c_5^4} \\ L_{c_5^3}^T R_3^T R_2^T R_1^T L_{c_5^0} & L_{c_5^3}^T R_3^T R_2^T L_{c_5^1} & L_{c_5^3}^T R_3^T L_{c_5^2} & L_{c_5^3}^T L_{c_5^3} & L_{c_5^3}^T R_4 L_{c_5^4} \\ L_{c_5^4}^T R_4^T R_3^T R_2^T R_1^T L_{c_5^0} & L_{c_5^4}^T R_4^T R_3^T R_2^T L_{c_5^1} & L_{c_5^4}^T R_4^T R_3^T L_{c_5^2} & L_{c_5^4}^T R_4^T L_{c_5^3} & L_{c_5^4}^T L_{c_5^4} \\ & & & & & 0 \\ & & & & & & 0 \end{pmatrix}$$

$$S_6 = m_6 \begin{pmatrix} L_{c_6^0}^T L_{c_6^0} & L_{c_6^0}^T R_1 L_{c_6^1} & L_{c_6^0}^T R_1 R_2 L_{c_6^2} & L_{c_6^0}^T R_1 R_2 R_3 L_{c_6^3} & L_{c_6^0}^T R_1 R_2 R_3 R_4 L_{c_6^4} & L_{c_6^0}^T R_1 R_2 R_3 R_4 R_5 L_{c_6^5} \\ L_{c_6^1}^T R_1^T L_{c_6^0} & L_{c_6^1}^T L_{c_6^1} & L_{c_6^1}^T R_2 L_{c_6^2} & L_{c_6^1}^T R_2 R_3 L_{c_6^3} & L_{c_6^1}^T R_2 R_3 R_4 L_{c_6^4} & L_{c_6^1}^T R_2 R_3 R_4 R_5 L_{c_6^5} \\ L_{c_6^2}^T R_2^T R_1^T L_{c_6^0} & L_{c_6^2}^T R_2^T L_{c_6^1} & L_{c_6^2}^T L_{c_6^2} & L_{c_6^2}^T R_3 L_{c_6^3} & L_{c_6^2}^T R_3 R_4 L_{c_6^4} & L_{c_6^2}^T R_3 R_4 R_5 L_{c_6^5} \\ L_{c_6^3}^T R_3^T R_2^T R_1^T L_{c_6^0} & L_{c_6^3}^T R_3^T R_2^T L_{c_6^1} & L_{c_6^3}^T R_3^T L_{c_6^2} & L_{c_6^3}^T L_{c_6^3} & L_{c_6^3}^T R_4 L_{c_6^4} & L_{c_6^3}^T R_4 R_5 L_{c_6^5} \\ L_{c_6^4}^T R_4^T R_3^T R_2^T R_1^T L_{c_6^0} & L_{c_6^4}^T R_4^T R_3^T R_2^T L_{c_6^1} & L_{c_6^4}^T R_4^T R_3^T L_{c_6^2} & L_{c_6^4}^T R_4^T L_{c_6^3} & L_{c_6^4}^T L_{c_6^4} & L_{c_6^4}^T R_5 L_{c_6^5} \\ L_{c_6^5}^T R_5^T R_4^T R_3^T R_2^T R_1^T L_{c_6^0} & L_{c_6^5}^T R_5^T R_4^T R_3^T R_2^T L_{c_6^1} & L_{c_6^5}^T R_5^T R_4^T R_3^T L_{c_6^2} & L_{c_6^5}^T R_5^T R_4^T L_{c_6^3} & L_{c_6^5}^T R_5^T L_{c_6^4} & L_{c_6^5}^T L_{c_6^5} \end{pmatrix}$$

$$S_7 = m_7 \begin{pmatrix}
L^T_{c_{\frac{0}{7}}} L_{c_{\frac{0}{7}}} & L^T_{c_{\frac{0}{7}}} R_1 L_{c_{\frac{1}{7}}} & L^T_{c_{\frac{0}{7}}} R_1 R_2 L_{c_{\frac{2}{7}}} & L^T_{c_{\frac{0}{7}}} R_1 R_2 R_3 L_{c_{\frac{3}{7}}} & L^T_{c_{\frac{0}{7}}} R_1 R_2 R_3 R_4 L_{c_{\frac{4}{7}}} & L^T_{c_{\frac{0}{7}}} R_1 R_2 R_3 R_4 R_5 L_{c_{\frac{5}{7}}} & L^T_{c_{\frac{0}{7}}} R_1 R_2 R_3 R_4 R_5 R_6 L_{c_{\frac{6}{7}}} \\
L^T_{c_{\frac{1}{7}}} R_1^T L_{c_{\frac{0}{7}}} & L^T_{c_{\frac{1}{7}}} L_{c_{\frac{1}{7}}} & L^T_{c_{\frac{1}{7}}} R_2 L_{c_{\frac{2}{7}}} & L^T_{c_{\frac{1}{7}}} R_2 R_3 L_{c_{\frac{3}{7}}} & L^T_{c_{\frac{1}{7}}} R_2 R_3 R_4 L_{c_{\frac{4}{7}}} & L^T_{c_{\frac{1}{7}}} R_2 R_3 R_4 R_5 L_{c_{\frac{5}{7}}} & L^T_{c_{\frac{1}{7}}} R_2 R_3 R_4 R_5 R_6 L_{c_{\frac{6}{7}}} \\
L^T_{c_{\frac{2}{7}}} R_2^T R_1^T L_{c_{\frac{0}{7}}} & L^T_{c_{\frac{2}{7}}} R_2^T L_{c_{\frac{1}{7}}} & L^T_{c_{\frac{2}{7}}} L_{c_{\frac{2}{7}}} & L^T_{c_{\frac{2}{7}}} R_3 L_{c_{\frac{3}{7}}} & L^T_{c_{\frac{2}{7}}} R_3 R_4 L_{c_{\frac{4}{7}}} & L^T_{c_{\frac{2}{7}}} R_3 R_4 R_5 L_{c_{\frac{5}{7}}} & L^T_{c_{\frac{2}{7}}} R_3 R_4 R_5 R_6 L_{c_{\frac{6}{7}}} \\
L^T_{c_{\frac{3}{7}}} R_3^T R_2^T R_1^T L_{c_{\frac{0}{7}}} & L^T_{c_{\frac{3}{7}}} R_3^T R_2^T L_{c_{\frac{1}{7}}} & L^T_{c_{\frac{3}{7}}} R_3^T L_{c_{\frac{2}{7}}} & L^T_{c_{\frac{3}{7}}} L_{c_{\frac{3}{7}}} & L^T_{c_{\frac{3}{7}}} R_4 L_{c_{\frac{4}{7}}} & L^T_{c_{\frac{3}{7}}} R_4 R_5 L_{c_{\frac{5}{7}}} & L^T_{c_{\frac{3}{7}}} R_4 R_5 R_6 L_{c_{\frac{6}{7}}} \\
L^T_{c_{\frac{4}{7}}} R_4^T R_3^T R_2^T R_1^T L_{c_{\frac{0}{7}}} & L^T_{c_{\frac{4}{7}}} R_4^T R_3^T R_2^T L_{c_{\frac{1}{7}}} & L^T_{c_{\frac{4}{7}}} R_4^T R_3^T L_{c_{\frac{2}{7}}} & L^T_{c_{\frac{4}{7}}} R_4^T L_{c_{\frac{3}{7}}} & L^T_{c_{\frac{4}{7}}} L_{c_{\frac{4}{7}}} & L^T_{c_{\frac{4}{7}}} R_5 L_{c_{\frac{5}{7}}} & L^T_{c_{\frac{4}{7}}} R_5 R_6 L_{c_{\frac{6}{7}}} \\
L^T_{c_{\frac{5}{7}}} R_5^T R_4^T R_3^T R_2^T R_1^T L_{c_{\frac{0}{7}}} & L^T_{c_{\frac{5}{7}}} R_5^T R_4^T R_3^T R_2^T L_{c_{\frac{1}{7}}} & L^T_{c_{\frac{5}{7}}} R_5^T R_4^T R_3^T L_{c_{\frac{2}{7}}} & L^T_{c_{\frac{5}{7}}} R_5^T R_4^T L_{c_{\frac{3}{7}}} & L^T_{c_{\frac{5}{7}}} R_5^T L_{c_{\frac{4}{7}}} & L^T_{c_{\frac{5}{7}}} L_{c_{\frac{5}{7}}} & L^T_{c_{\frac{5}{7}}} R_6 L_{c_{\frac{6}{7}}} \\
L^T_{c_{\frac{6}{7}}} R_6^T R_5^T R_4^T R_3^T R_2^T R_1^T L_{c_{\frac{0}{7}}} & L^T_{c_{\frac{6}{7}}} R_6^T R_5^T R_4^T R_3^T R_2^T L_{c_{\frac{1}{7}}} & L^T_{c_{\frac{6}{7}}} R_6^T R_5^T R_4^T R_3^T L_{c_{\frac{2}{7}}} & L^T_{c_{\frac{6}{7}}} R_6^T R_5^T R_4^T L_{c_{\frac{3}{7}}} & L^T_{c_{\frac{6}{7}}} R_6^T R_5^T L_{c_{\frac{4}{7}}} & L^T_{c_{\frac{6}{7}}} R_6^T L_{c_{\frac{5}{7}}} & L^T_{c_{\frac{6}{7}}} L_{c_{\frac{6}{7}}}
\end{pmatrix},$$