

3 Dimensional Physical Pendulum

Chapter 1

3-Dimensional Pendulum

Theorem 1.0.1 (Steiner's Theorem).

Let m be the mass of a body and let I_{ij} be its inertia tensor in a coordinate system with origin in the centre of mass of the body. Then the inertia tensor of the body at $r \in \mathbb{R}^3$ has the form

$$\begin{aligned} J(r)_{ij} &= I_{ij} + m(\|r\|^2 \delta_{ij} - r_i r_j) \\ &= (I + m(\|r\|^2 \text{id} - r \otimes r))_{ij}. \end{aligned}$$

Lemma 1.0.2.

Let $q \in \mathbb{R}^3$ be a pseudo-vector. Then there is a unique rotation matrix associated to this pseudo-vector

$$R(q) = \exp(L_q),$$

where $L_q = L_i q^i$ and L_i are the generators of the Lie algebra $\mathfrak{so}(3)$.

Remark 1.1.

Note that this map is not injective, i.e. $R(q) = R(q + 2k\pi)$, $\forall k \in \mathbb{Z}$.

Lemma 1.0.3.

Let $J(r)$ be the inertia tensor of a body. Then the inertia tensor of the body rotated by $R(q)$ is

$$\begin{aligned} J(q)_{ij} &= R(q)^k{}_i J(r)_{kl} R(q)^l{}_j \\ &= (R(q)^\top \cdot J(r) \cdot R(q))_{ij}. \end{aligned}$$

Lemma 1.0.4 (Kinetic Energy).

The kinetic energy of a rotating body is

$$\begin{aligned} T(q, \dot{q}) &= \frac{1}{2} \dot{q}^i J(q)_{ij} \dot{q}^j \\ &= \frac{1}{2} \dot{q}^\top \cdot J(q) \cdot \dot{q}, \end{aligned}$$

where $\dot{q} = \frac{dq}{dt}$.

Remark 1.2.

Note that $\frac{dR(q)}{dt} = \dot{R}(q) = R(q)L_{\dot{q}} = L_{\dot{q}}R(q) \in T_{R(q)}(\text{SO}(3))$, where $L_{\dot{q}} = L_i \dot{q}^i \in \mathfrak{so}(3)$ is the angular velocity tensor.

Lemma 1.0.5 (Potential Energy).

Let $r(q) \in \mathbb{R}^3$ be the position of the centre of mass of a body. The potential energy of the body is

$$\begin{aligned} V(q) &= mg_i r(q)^i \\ &= mg^\top \cdot r(q). \end{aligned}$$

Definition 1.0.1 (Rayleigh Dissipation Function).

The Rayleigh dissipative function is a function describing the half rate of energy dissipation of a system caused by forces proportional to the velocity of the system. It has the form

$$\begin{aligned} G(\dot{q}) &= \frac{1}{2} \dot{q}^i C_{ij} \dot{q}^j \\ &= \frac{1}{2} \dot{q}^\top \cdot C \cdot \dot{q}. \end{aligned}$$

The equations of motion for any physical system with friction force linearly proportional to the velocity are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} + \frac{\partial G}{\partial \dot{q}^i} = 0,$$

where $L(q, \dot{q}) = T(q, \dot{q}) - V(q)$ is the Lagrangian not depending on time explicitly. Expanding the absolute derivative with respect to time and writing out L as $T - V$ one gets

$$\frac{\partial T}{\partial \dot{q}^j \partial \dot{q}^i} \ddot{q}^j + \frac{\partial T}{\partial q^k \partial \dot{q}^i} \dot{q}^k - \frac{\partial T}{\partial q^i} + \frac{\partial V}{\partial q^i} + \frac{\partial G}{\partial \dot{q}^i} = 0.$$

Solving for \ddot{q} gives

$$\ddot{q}^j = \left(\frac{\partial T}{\partial \dot{q}^j \partial \dot{q}^i} \right)^{-1} \left(-\frac{\partial T}{\partial q^k \partial \dot{q}^i} \dot{q}^k + \frac{\partial T}{\partial q^i} - \frac{\partial V}{\partial q^i} - \frac{\partial G}{\partial \dot{q}^i} \right).$$

Now, assuming that the physical system is a 3-dimensional pendulum with

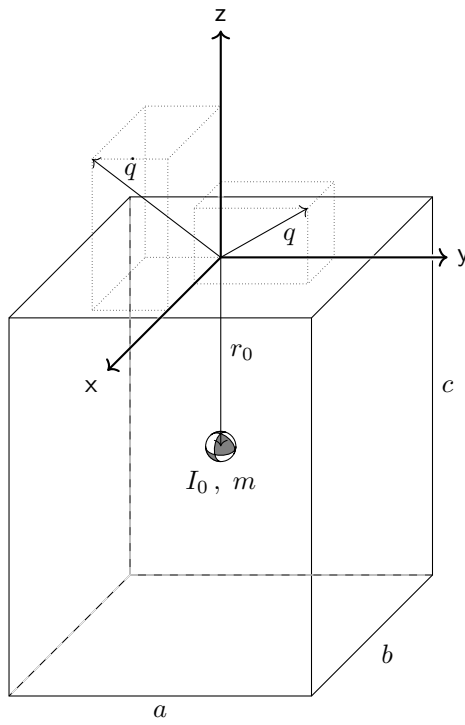


Figure 1.1: A non-rotated pendulum, i.e. the frame of the pendulum corresponds to the base frame.

- $q \in \mathbb{R}^3$ being a pseudo-vector describing rotation (c.f. lemma 1.0.2),
- $J(q)$ being the moment of inertia (c.f. lemma 1.0.3),
- $T(q, \dot{q})$ being the kinetic energy (c.f. lemma 1.0.4),
- $V(q)$ being the potential energy (c.f. lemma 1.0.5),

one has the following partial derivations

$$\begin{aligned}
 \frac{\partial R}{\partial q^i} &= R \cdot L_i = L_i \cdot R, \\
 \frac{\partial J(q)}{\partial q^i} &= L^\top_i \cdot J(q) + J(q) \cdot L_i = [J(q), L_i], \\
 \frac{\partial T}{\partial \dot{q}^i} &= J(q)_{il} \dot{q}^l = (J(q) \cdot \dot{q})_i, \\
 \frac{\partial T}{\partial \dot{q}^j \partial \dot{q}^i} &= J(q)_{ij}, \\
 \frac{\partial T}{\partial q^k \partial \dot{q}^i} &= \frac{\partial J(q)_{il}}{\partial q^k} \dot{q}^l = [J(q), L_k]_{il} \dot{q}^l, \\
 \frac{\partial T}{\partial q^i} &= \frac{1}{2} \frac{\partial J(q)_{kl}}{\partial q^i} \dot{q}^k \dot{q}^l = \frac{1}{2} [J(q), L_i]_{kl} \dot{q}^k \dot{q}^l = \frac{1}{2} \dot{q}^\top \cdot [J(q), L_i] \cdot \dot{q}, \\
 \frac{\partial V}{\partial q^i} &= mg^\top \cdot \frac{\partial R(q)}{\partial q^i} \cdot r = mg^\top \cdot L_i \cdot r(q).
 \end{aligned}$$

Assuming $G(\dot{q}) = \frac{1}{2} c \|\dot{q}\|^2$ (c.f. definition 1.0.1) gives

$$\frac{\partial G}{\partial \dot{q}^i} = c \dot{q}_i.$$

Finally, plugging it all together yields

$$\begin{aligned}
 \ddot{q}^j &= (J(q)^{-1})^{ji} \left(-[J(q), L_k]_{il} \dot{q}^k \dot{q}^l + \frac{1}{2} [J(q), L_i]_{kl} \dot{q}^k \dot{q}^l - mg^\top \cdot L_i \cdot r(q) - c \dot{q}_i \right) \\
 &= (J(q)^{-1})^{ji} \left(-\frac{1}{2} ([J(q), L_k]_{il} + [J(q), L_l]_{ik} - [J(q), L_i]_{kl}) \dot{q}^k \dot{q}^l - mg^\top \cdot L_i \cdot r(q) - c \dot{q}_i \right) \\
 &= (J(q)^{-1})^{ji} \left(-\Gamma_{ikl} \dot{q}^k \dot{q}^l - mg^\top \cdot L_i \cdot r(q) - c \dot{q}_i \right),
 \end{aligned}$$

where $\Gamma_{ikl} = \frac{1}{2} ([J(q), L_k]_{il} + [J(q), L_l]_{ik} - [J(q), L_i]_{kl})$ are the Christoffel symbols of the first kind.

Chapter 2

Robotic Arm with Revolute Joints

$$\begin{aligned}
 R_i &= \exp(L q_i) = \exp(L_j q_i^j) = \exp(L_{\hat{q}_i} \theta_i) \\
 \frac{dR_i}{dt} &= L_{\dot{q}_i} \cdot R_i = L_j(\dot{q}_i^j) \cdot R_i = (L_{\hat{q}_i} \dot{\theta}_i) \cdot R_i \\
 L_x &= \begin{pmatrix} & & \\ & -1 & \\ 1 & & \end{pmatrix}, \quad L_y = \begin{pmatrix} & 1 & \\ & & \\ -1 & & \end{pmatrix}, \quad L_z = \begin{pmatrix} & -1 & \\ & & \\ & & \end{pmatrix} \\
 R &= \begin{pmatrix} R_1 & R_1 R_2 & R_1 R_2 R_3 & R_1 R_2 R_3 R_4 & R_1 R_2 R_3 R_4 R_5 & R_1 R_2 R_3 R_4 R_5 R_6 & R_1 R_2 R_3 R_4 R_5 R_6 R_7 \\ R_2 & R_2 R_3 & R_2 R_3 R_4 & R_2 R_3 R_4 R_5 & R_2 R_3 R_4 R_5 R_6 & R_2 R_3 R_4 R_5 R_6 R_7 \\ R_3 & R_3 R_4 & R_3 R_4 R_5 & R_3 R_4 R_5 R_6 & R_3 R_4 R_5 R_6 R_7 \\ R_4 & R_4 R_5 & R_4 R_5 R_6 & R_4 R_5 R_6 R_7 \\ R_5 & R_5 R_6 & R_5 R_6 R_7 \\ R_6 & R_6 R_7 \\ R_7 \end{pmatrix} \\
 \partial_1 R &= \begin{pmatrix} (dR_1) & (dR_1)R_2 & (dR_1)R_2R_3 & (dR_1)R_2R_3R_4 & (dR_1)R_2R_3R_4R_5 & (dR_1)R_2R_3R_4R_5R_6 & (dR_1)R_2R_3R_4R_5R_6R_7 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{pmatrix} \\
 \partial_2 R &= \begin{pmatrix} R_1(dR_2) & R_1(dR_2)R_3 & R_1(dR_2)R_3R_4 & R_1(dR_2)R_3R_4R_5 & R_1(dR_2)R_3R_4R_5R_6 & R_1(dR_2)R_3R_4R_5R_6R_7 \\ (dR_2) & (dR_2)R_3 & (dR_2)R_3R_4 & (dR_2)R_3R_4R_5 & (dR_2)R_3R_4R_5R_6 & (dR_2)R_3R_4R_5R_6R_7 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \\
 \partial_3 R &= \begin{pmatrix} R_1 R_2(dR_3) & R_1 R_2(dR_3)R_4 & R_1 R_2(dR_3)R_4R_5 & R_1 R_2(dR_3)R_4R_5R_6 & R_1 R_2(dR_3)R_4R_5R_6R_7 \\ R_2(dR_3) & R_2(dR_3)R_4 & R_2(dR_3)R_4R_5 & R_2(dR_3)R_4R_5R_6 & R_2(dR_3)R_4R_5R_6R_7 \\ (dR_3) & (dR_3)R_4 & (dR_3)R_4R_5 & (dR_3)R_4R_5R_6 & (dR_3)R_4R_5R_6R_7 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix} \\
 \partial_4 R &= \begin{pmatrix} R_1 R_2 R_3(dR_4) & R_1 R_2 R_3(dR_4)R_5 & R_1 R_2 R_3(dR_4)R_5R_6 & R_1 R_2 R_3(dR_4)R_5R_6R_7 \\ R_2 R_3(dR_4) & R_2 R_3(dR_4)R_5 & R_2 R_3(dR_4)R_5R_6 & R_2 R_3(dR_4)R_5R_6R_7 \\ R_3(dR_4) & R_3(dR_4)R_5 & R_3(dR_4)R_5R_6 & R_3(dR_4)R_5R_6R_7 \\ (dR_4) & (dR_4)R_5 & (dR_4)R_5R_6 & (dR_4)R_5R_6R_7 \\ & & & \\ & & & \\ & & & \end{pmatrix} \\
 \partial_5 R &= \begin{pmatrix} R_1 R_2 R_3 R_4(dR_5) & R_1 R_2 R_3 R_4(dR_5)R_6 & R_1 R_2 R_3 R_4(dR_5)R_6R_7 \\ R_2 R_3 R_4(dR_5) & R_2 R_3 R_4(dR_5)R_6 & R_2 R_3 R_4(dR_5)R_6R_7 \\ R_3 R_4(dR_5) & R_3 R_4(dR_5)R_6 & R_3 R_4(dR_5)R_6R_7 \\ R_4(dR_5) & R_4(dR_5)R_6 & R_4(dR_5)R_6R_7 \\ (dR_5) & (dR_5)R_6 & (dR_5)R_6R_7 \\ & & \\ & & \end{pmatrix} \\
 \partial_6 R &= \begin{pmatrix} R_1 R_2 R_3 R_4 R_5(dR_6) & R_1 R_2 R_3 R_4 R_5(dR_6)R_7 \\ R_2 R_3 R_4 R_5(dR_6) & R_2 R_3 R_4 R_5(dR_6)R_7 \\ R_3 R_4 R_5(dR_6) & R_3 R_4 R_5(dR_6)R_7 \\ R_4 R_5(dR_6) & R_4 R_5(dR_6)R_7 \\ R_5(dR_6) & R_5(dR_6)R_7 \\ (dR_6) & (dR_6)R_7 \\ & \end{pmatrix} \\
 \partial_7 R &= \begin{pmatrix} R_1 R_2 R_3 R_4 R_5 R_6(dR_7) \\ R_2 R_3 R_4 R_5 R_6(dR_7) \\ R_3 R_4 R_5 R_6(dR_7) \\ R_4 R_5 R_6(dR_7) \\ R_5 R_6(dR_7) \\ R_6(dR_7) \\ (dR_7) \end{pmatrix}
 \end{aligned}$$

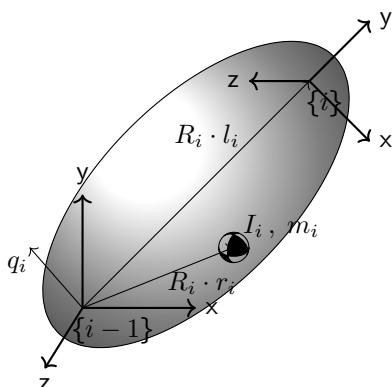


Figure 2.1: A link and link parameters.

$$\tilde{L}_c = \begin{pmatrix} L_{c1}^0 & & & & & & & & \\ & L_{c2}^1 & & & & & & & \\ L_{c2}^0 & & & & & & & & \\ & L_{c3}^1 & L_{c2}^2 & & & & & & \\ L_{c3}^0 & & L_{c4}^2 & & & & & & \\ & L_{c4}^1 & & L_{c3}^3 & & & & & \\ L_{c4}^0 & & & & L_{c4}^3 & & & & \\ & L_{c5}^1 & L_{c2}^4 & & & L_{c5}^4 & & & \\ L_{c5}^0 & & L_{c5}^2 & L_{c3}^5 & & & & & \\ & L_{c6}^1 & & L_{c6}^3 & L_{c4}^5 & & L_{c5}^6 & & \\ L_{c6}^0 & & L_{c2}^5 & & L_{c4}^6 & & & L_{c2}^6 & \\ & L_{c7}^1 & L_{c2}^6 & & & L_{c4}^7 & & & L_{c6}^6 \\ L_{c7}^0 & & & L_{c2}^7 & & & & & \end{pmatrix}$$

$$L_c = \begin{pmatrix} L_{c_1^0} & & & & & & & & \\ & R_1 L_{c_2^1} & & & & & & & \\ & & R_1 L_{c_3^1} & & & & & & \\ & & & R_1 R_2 L_{c_3^2} & & & & & \\ & & & & R_1 R_2 L_{c_4^2} & & & & \\ & & & & & R_1 R_2 R_3 L_{c_4^3} & & & \\ & & & & & & R_1 R_2 R_3 R_4 L_{c_5^4} & & \\ & & & & & & & R_1 R_2 R_3 R_4 L_{c_6^4} & \\ & & & & & & & & R_1 R_2 R_3 R_4 R_5 L_{c_6^5} & \\ & & & & & & & & & R_1 R_2 R_3 R_4 R_5 L_{c_7^5} & \\ & & & & & & & & & & R_1 R_2 R_3 R_4 R_5 R_6 L_{c_7^6} \end{pmatrix}$$

$$-c^0 = L_c \cdot \dot{q}$$

$$L_c @ \dot{q} = \begin{pmatrix} L_{c_1^0} \dot{q}_1 & & & & & & & & \\ & R_1 L_{c_2^1} \dot{q}_2 & & & & & & & \\ & & R_1 L_{c_3^1} \dot{q}_2 & & & & & & \\ & & & R_1 R_2 L_{c_3^2} \dot{q}_3 & & & & & \\ & & & & R_1 R_2 L_{c_4^2} \dot{q}_3 & & & & \\ & & & & & R_1 R_2 R_3 L_{c_4^3} \dot{q}_4 & & & \\ & & & & & & R_1 R_2 R_3 R_4 L_{c_5^4} \dot{q}_5 & & \\ & & & & & & & R_1 R_2 R_3 R_4 L_{c_6^4} \dot{q}_5 & & \\ & & & & & & & & R_1 R_2 R_3 R_4 R_5 L_{c_6^5} \dot{q}_6 & & \\ & & & & & & & & & R_1 R_2 R_3 R_4 R_5 L_{c_7^5} \dot{q}_6 & & \\ & & & & & & & & & & R_1 R_2 R_3 R_4 R_5 R_6 L_{c_7^6} \dot{q}_7 \end{pmatrix}$$

$$M = \begin{pmatrix} m_1 \mathbb{1}_3 & & & & & & \\ & m_2 \mathbb{1}_3 & & & & & \\ & & m_3 \mathbb{1}_3 & & & & \\ & & & m_4 \mathbb{1}_3 & & & \\ & & & & m_5 \mathbb{1}_3 & & \\ & & & & & m_6 \mathbb{1}_3 & \\ & & & & & & m_7 \mathbb{1}_3 \end{pmatrix}$$

$$\begin{aligned} T^{tra} &= \sum_i m_i \dot{c}_i^0 \cdot \dot{c}_i^0 \\ &= \dot{\theta}^T \cdot \dot{q}^T @ (L_c^T M L_c) @ \dot{q} \cdot \dot{\theta} \\ &= \dot{\theta}^T \cdot (L_c @ \dot{q})^T M (L_c @ \dot{q}) \cdot \dot{\theta} \end{aligned}$$

$$L_c^T = \begin{pmatrix} L_{c_1^0}^T & & & & & & & & \\ & L_{c_2^1}^T R_1^T & & & & & & & \\ & & L_{c_3^1}^T R_1^T & & & & & & \\ & & & L_{c_3^2}^T R_2^T R_1^T & & & & & \\ & & & & L_{c_4^2}^T R_2^T R_1^T & & & & \\ & & & & & L_{c_5^4}^T R_3^T R_2^T R_1^T & & & \\ & & & & & & L_{c_6^4}^T R_4^T R_3^T R_2^T R_1^T & & \\ & & & & & & & L_{c_6^5}^T R_5^T R_4^T R_3^T R_2^T R_1^T & \\ & & & & & & & & L_{c_7^6}^T R_6^T R_5^T R_4^T R_3^T R_2^T R_1^T \end{pmatrix}$$

$$S_1 = m_1 \begin{pmatrix} L_{c_1^0}^T L_{c_1^0} & & & & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}$$

$$S_2 = m_2 \begin{pmatrix} L_{c_2^1}^T L_{c_2^1} & & & & & & \\ & L_{c_2^1}^T R_1^T L_{c_2^1} & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}$$

$$S_3 = m_3 \begin{pmatrix} L_{c_3^1}^T L_{c_3^1} & & & & & & \\ & L_{c_3^1}^T R_1^T L_{c_3^1} & & & & & \\ & & L_{c_3^2}^T R_2^T L_{c_3^2} & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}$$

$$S_4 = m_4 \begin{pmatrix} L_{c_4^2}^T L_{c_4^2} & & & & & & \\ & L_{c_4^2}^T R_1^T L_{c_4^2} & & & & & \\ & & L_{c_4^3}^T R_2^T L_{c_4^3} & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}$$

$$S_5 = m_5 \begin{pmatrix} L_{c_5^4}^T L_{c_5^4} & & & & & & \\ & L_{c_5^4}^T R_1^T L_{c_5^4} & & & & & \\ & & L_{c_5^5}^T R_2^T L_{c_5^5} & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}$$

$$S_6 = m_6 \begin{pmatrix} L_{c_6^5}^T L_{c_6^5} & & & & & & \\ & L_{c_6^5}^T R_1^T L_{c_6^5} & & & & & \\ & & L_{c_6^6}^T R_2^T L_{c_6^6} & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}$$

$$S_7 = m_7 \left(\begin{array}{cccccccc} L_{c_7^0}^T L_{c_7^0}^T & L_{c_7^0}^T R_1 L_{c_7^1}^T & L_{c_7^0}^T R_1 R_2 L_{c_7^2}^T & L_{c_7^0}^T R_1 R_2 R_3 L_{c_7^3}^T & L_{c_7^0}^T R_1 R_2 R_3 R_4 L_{c_7^4}^T & L_{c_7^0}^T R_1 R_2 R_3 R_4 R_5 L_{c_7^5}^T & L_{c_7^0}^T R_1 R_2 R_3 R_4 R_5 R_6 L_{c_7^6}^T & \\ L_{c_7^1}^T R_1 L_{c_7^0}^T & L_{c_7^1}^T L_{c_7^1}^T & L_{c_7^1}^T R_2 L_{c_7^2}^T & L_{c_7^1}^T R_2 R_3 L_{c_7^3}^T & L_{c_7^1}^T R_2 R_3 R_4 L_{c_7^4}^T & L_{c_7^1}^T R_2 R_3 R_4 R_5 L_{c_7^5}^T & L_{c_7^1}^T R_2 R_3 R_4 R_5 R_6 L_{c_7^6}^T & \\ L_{c_7^2}^T R_2^T R_1^T L_{c_7^0}^T & L_{c_7^2}^T R_2^T L_{c_7^1}^T & L_{c_7^2}^T L_{c_7^2}^T & L_{c_7^2}^T R_3 L_{c_7^3}^T & L_{c_7^2}^T R_3 R_4 L_{c_7^4}^T & L_{c_7^2}^T R_3 R_4 R_5 L_{c_7^5}^T & L_{c_7^2}^T R_3 R_4 R_5 R_6 L_{c_7^6}^T & \\ L_{c_7^3}^T R_3^T R_2^T R_1^T L_{c_7^0}^T & L_{c_7^3}^T R_3^T R_2^T L_{c_7^1}^T & L_{c_7^3}^T R_3^T L_{c_7^2}^T & L_{c_7^3}^T L_{c_7^3}^T & L_{c_7^3}^T R_4 L_{c_7^4}^T & L_{c_7^3}^T R_4 R_5 L_{c_7^5}^T & L_{c_7^3}^T R_4 R_5 R_6 L_{c_7^6}^T & \\ L_{c_7^4}^T R_4^T R_3^T R_2^T R_1^T L_{c_7^0}^T & L_{c_7^4}^T R_4^T R_3^T R_2^T L_{c_7^1}^T & L_{c_7^4}^T R_4^T R_3^T L_{c_7^2}^T & L_{c_7^4}^T R_4^T L_{c_7^3}^T & L_{c_7^4}^T L_{c_7^4}^T & L_{c_7^4}^T R_5 L_{c_7^5}^T & L_{c_7^4}^T R_5 R_6 L_{c_7^6}^T & \\ L_{c_7^5}^T R_5^T R_4^T R_3^T R_2^T R_1^T L_{c_7^0}^T & L_{c_7^5}^T R_5^T R_4^T R_3^T R_2^T L_{c_7^1}^T & L_{c_7^5}^T R_5^T R_4^T R_3^T L_{c_7^2}^T & L_{c_7^5}^T R_5^T R_4^T L_{c_7^3}^T & L_{c_7^5}^T R_5^T L_{c_7^4}^T & L_{c_7^5}^T L_{c_7^5}^T & L_{c_7^5}^T R_6 L_{c_7^6}^T & \\ L_{c_7^6}^T R_6^T R_5^T R_4^T R_3^T R_2^T R_1^T L_{c_7^0}^T & L_{c_7^6}^T R_6^T R_5^T R_4^T R_3^T R_2^T L_{c_7^1}^T & L_{c_7^6}^T R_6^T R_5^T R_4^T R_3^T L_{c_7^2}^T & L_{c_7^6}^T R_6^T R_5^T R_4^T L_{c_7^3}^T & L_{c_7^6}^T R_6^T R_5^T L_{c_7^4}^T & L_{c_7^6}^T R_6^T L_{c_7^5}^T & L_{c_7^6}^T L_{c_7^6}^T & \end{array} \right) /$$