Risk minimization using distortion risk measures via linear programming

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Abstract

The paper deals with risk minimizing portfolios using distortion risk measures. First, the paper presents new formulation of coherent distortion risk measures via linear programming under assumption of equiprobable realizations of random returns. Second, the formulation is employed in the portfolio selection problem where a coherent distortion risk measure is minimized. Finally, in the empirical application, the sensitivity analysis with respect to a parameter of risk aversion of these risk minimizing portfolios is presented for a special type of distortion function – proportional hazard transform. The analysis includes in-sample and out-of-sample performance of the optimal portfolios with a focus on return and risk.

Keywords

Distortion risk measures, portfolio selection, linear programming

JEL Classification: D81, G11

1. Introduction

Investors typically seek for investments with high mean returns. Starting from a seminal paper of Markowitz (1952), a risk measuring and controlling has become another important issue in decision making. The risk of the investment can be modelled in various ways using e.g. risk measures, deviation functions or utility functions. Combining risk minimization with mean maximization leads to mean-risk portfolio selection problems which could be formulated in various different ways. If the investor wants only to minimize risk of the portfolio the portfolio optimization problems simplifies a lot when using risk measures. Especially, coherent risk measures (see Artzner et al. 1999) are the most popular ones because of their convexity.

In this paper, we investigate a special type of risk measures called distortion risk measures. These measures are very flexible in capturing the risk preferences of the decision maker. They are generated by a distortion function and cumulative distribution function of the returns. We present a new representation of distortion risk measures under assumption of discrete distribution of returns with equiprobable realizations. The main advantage of that is easy and straightforward implementation in mean-risk models which turn to be linear programs. One of the attractive properties of coherent distortion risk measures (generated by concave distortion function) is its consistency with the second order stochastic dominance. The basics of stochastic dominance go back to 1960th, see Quirk and Saposnik (1962), Hadar and Russell (1969), Hanoch and Levy (1969), Rothschild and Stiglitz (1970) and Whitmore (1970). However, as demonstrated in e.g. Moriggia et al. (2019), Kabasinskas et al. (2020) or Kopa et al. (2021), it is still an useful tool in the decision making or portfolio optimization.

In the empirical part of the paper, we apply the new representation of distortion measures to the risk minimizing portfolio selection problem using data on 10 representative industry portfolios from Kenneth French library. The goal of the empirical study is to present the

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sensitivity analysis of this portfolio selection problem considering Proportional hazard transform (PHT) as the distortion function.

The remainder of this paper is structured as follows. Section 2 presents the basics of distortion measures and new representation. Section 3 formulates the risk minimizing portfolio selection problem using the representation. Section 4 presents the in-sample and out-of-sample results of the problem, including sensitivity analysis with respect to the parameter of PHT . The paper is summarized and concluded in Section 5.

2. Distortion risk measures

Let random variable $X \in \mathcal{X}$ represent a random loss of investment (portfolio) and let $F_X(x)$ be its cumulative distribution function. Then a risk measure is a functional of $X \in \mathcal{X}$ which assigns a real number to $X \in \mathcal{X}$. At present, the most popular risk measures are Value at Risk (VaR) and Conditional Value at Risk (CVaR). Distortion risk measures could be seen as their generalizations:

Definition 1 [Dhaene et al. 2012]
*Suppose that $g: [0,1] \to [0,1]$ is a non-decreasing function such that $g(0) = 0$ and $g(1) = 1$ (also known as the **distortion function**). Then, the **distortion risk measure** associated with the distortion function g is defined as*

$$\rho_g(X) = -\int_{-\infty}^{0} [1 - g(1 - F_X(x))]dx + \int_{0}^{\infty} g(1 - F_X(x))dx,$$

provided that at least one of the integrals is finite.

In this paper, we focus only on the coherent distortion risk measures.

Lemma 1 [Wirch and Hardy 1999, Sereda et al. 2010]
The distortion risk measure $\rho_g(X)$ is coherent if and only if g is a concave distortion function.

As discussed in Kopa and Zelman (2021), the most commonly used distortion measures are VaR and CVaR and those with the following distortion measures:

- **Proportional Hazard transform (PHT):** $g(x) = x^{1/\gamma}, \quad x \in [0,1], \gamma \geq 1.$ (1)
- The **Wang transform**: $g_\lambda(x) = \Phi(\Phi^{-1}(x) + \lambda)$ for $x \in [0,1], \lambda \geq 0,$ where Φ is the standard normal distribution function.
- The **MINVAR** distortion function; $g(x) = 1 - (1 - x)^{1+\lambda}$ for $x \in [0,1], \lambda \geq 0.$
- The **MINMAXVAR** distortion function: $g(x) = 1 - (1 - x^{\frac{1}{1+\lambda}})^{1+\lambda}$ for $x \in [0,1], \lambda \geq 0.$

Since now, let us assume that random loss variable $X \in \mathcal{X}$ has a discrete distribution with equiprobable realizations $x_i,\ i \in \{1, \dots, m\}: P(X = x_i) = \frac{1}{m}$. Moreover, assume that the realizations are ordered from the smallest to the largest one, that is: $x_1 \leq x_2 \leq \cdots \leq x_m$. Then Kopa and Zelman (2021) proved that:

$$\rho_g(X) = x_1 + \sum_{i=1}^{m-1} (x_{i+1} - x_i) g\left(1 - \frac{i}{m}\right).$$

Now, let $x_0 = 0$ and then

$$\rho_g(X) = \sum_{i=0}^{m-1} (x_{i+1} - x_i) g\left(1 - \frac{i}{m}\right)$$

because $g(1) = 1$, see Definition 1. Using Abel summation lemma we get:

$$\rho_g(X) = x_m g\left(1 - \frac{m-1}{m}\right) - x_0 g(1) - \sum_{i=1}^{m-1} x_i \left(g\left(1 - \frac{i}{m}\right) - g\left(1 - \frac{i-1}{m}\right)\right)$$

and since the second term on right hand side is zero and $g\left(1 - \frac{m}{m}\right) = 0$ we may simplify it to:

$$\rho_g(X) = \sum_{i=1}^{m} x_i \left(g\left(1 - \frac{i-1}{m}\right) - g\left(1 - \frac{i}{m}\right)\right)$$

Finally, let $G_i = g\left(1 - \frac{i-1}{m}\right) - g\left(1 - \frac{i}{m}\right)$. Then

$$\rho_g(X) = \sum_{i=1}^{m} x_i G_i \tag{2}$$

Lemma 2:
If $g(x)$ is a concave function than $G_i \leq G_{i+1}$.

Proof: Since $g(x)$ is a concave function one directly has:

$$g\left(1 - \frac{i}{m}\right) \geq \frac{g\left(1 - \frac{i-1}{m}\right) + g\left(1 - \frac{i+1}{m}\right)}{2}$$

and: consequently:

$$g\left(1 - \frac{i}{m}\right) - g\left(1 - \frac{i+1}{m}\right) \geq g\left(1 - \frac{i-1}{m}\right) - g\left(1 - \frac{i}{m}\right)$$

what completes the proof.
Now we are ready to present the new represenatin of coherent distortion measures via linear programming where we use the properties of double stochastic matrices.

Theorem 1:
Let $g(x)$ be a concave function and let y_j be a realization of random loss $X \in \mathcal{X}$ in time period j, such that $P(X = y_j) = \frac{1}{m}$, j=1,...,m. Then

$$\rho_g(X) = \max_{z,\mathbf{W}} \sum_{i=1}^{m} z_i G_i \tag{3}$$
$$\text{s.t.} - z_i + \sum_{j=1}^{m} w_{ij} y_j = 0, \; i=1\ldots m$$

$$\sum_{j=1}^{m} w_{ij} = 1, \; i=1\ldots m$$
$$\sum_{i=1}^{m} w_{ij} = 1, \; j=1\ldots m$$
$$w_{ij} \geq 0, \; i,j=1\ldots m.$$

Proof:
Since $G_i \leq G_{i+1}$, see Lemma 2, optimal value of z_m equals to the largest loss, that is x_m. This is due to the fact that the objective is maximization and thanks to properties of double stochastic matrix $W = \{w_{ij}\}_{i,j}$. Similarly, for the optimal values of all other z_i we have: $z_i^* = x_i$, i=1…m, what completes the proof.

Despite of the fact that (3) is linear program and this LP represenation could be used in portfolio selection problems directly, we will rather proceed with a dual expression. The reason is that terms "$w_{ij}y_j$" would make the portfolio selection problem non-linear and non-convex, so computationally more demanding than LP problems.

Theorem 2:
Let $g(x)$ be a concave function and let y_j be a realization of random loss $X \in \mathcal{X}$ in time period j, such that $P(X = y_j) = \frac{1}{m}$, j=1,...,m. Then

$$\rho_g(X) = \min_{\boldsymbol{c},\boldsymbol{d}} \sum_{j=1}^{m} \mathrm{c}_j + \sum_{i=1}^{m} d_i \qquad (4)$$
$$\text{s.t. } -G_i y_j + \mathrm{c}_j + d_i \geq 0, \quad i,j = 1,..,m.$$

Proof:
The dual problém to (3) is:

$$\rho_g(X) = \min_{\boldsymbol{c},\boldsymbol{d}} \sum_{j=1}^{m} \mathrm{c}_j + \sum_{i=1}^{m} d_i$$
$$\text{s.t. } b_i y_j + \mathrm{c}_j + d_i \geq 0, \quad i,j = 1,..,m \qquad [w_{ij}]$$
$$-b_i = G_i \qquad [z_i]$$

and substituting last equations into inequalities eliminates variables b_i and completes the proof.

The advantage of (4) is that values y_j are not multiplicated by any variable what enables to formulate the risk minimizing portfolio selection problem as linear program what we demonstrate in the next session.

3. Distortion risk measures in portfolio selection problems

Let us consider a random vector $\mathbf{r} = (\mathrm{r}_1, \mathrm{r}_2, \ldots, \mathrm{r}_\mathrm{N})$ of returns of N assets in m equiprobable scenarios. The returns of the assets for the various scenarios are collected in matrix:

$$\mathbf{X} = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^\mathrm{m} \end{pmatrix}$$

where $\mathbf{x}^\mathrm{t} = (\mathrm{x}_1^\mathrm{t}, \mathrm{x}_2^\mathrm{t}, \ldots, \mathrm{x}_\mathrm{N}^\mathrm{t})$ is the t-th row of matrix **X**. A vector of portfolio weights is denoted by $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_\mathrm{N})'$. In this paper, we exclude short sales, that is, the set of all feasible portfolios Λ can be characterized as follows:

$$\Lambda = \{\boldsymbol{\lambda} \in R^N \,|\, \textstyle\sum_{n=1}^{N} \lambda_n = 1, \quad \lambda_n \geq 0, \ n = 1,2, \ldots, N\}.$$

In this notation, the loss of portfolio $\boldsymbol{\lambda}$ is - $\mathbf{r}^{\mathrm{T}}\boldsymbol{\lambda}$ with equiprobable realizations: $-\mathbf{x}^{\mathrm{t}}\boldsymbol{\lambda}, \mathrm{t} = 1,..,m$ which play the role of vector losses y_j in Section 2. Therefore, employing (4), the distortion risk minimizng problem is formulated as follows:

$$\rho_g(X) = \min_{\boldsymbol{c},\boldsymbol{d},\boldsymbol{\lambda}} \sum_{j=1}^{m} \mathrm{c}_j + \sum_{i=1}^{m} d_i \qquad (5)$$
$$\text{s.t. } G_i \mathbf{x}^{\mathrm{j}} \boldsymbol{\lambda} + \mathrm{c}_j + d_i \geq 0, \quad i,j = 1,..,m$$
$$\boldsymbol{\lambda} \in \Lambda.$$

4. Empirical study

4.1 Data description

We consider ten industry representative portfolios from the Kenneth French library as the base assets in our empirical study. The data of daily returns are divided into two parts: in-sample period (1.1.2018 – 31.12.2020) and out-of-sample period (1.1.2021 – 30.6.2021). Descriptive statistics of both datasets are summarized in Table 1 and Table 2.

Table 1: Basic descriptive statistics of daily returns (in %) – in sample period

	mean	st. dev.	min	max	skewness	kurtosis
NoDur	0.029	1.288	-9.870	7.450	-0.757	14.139
Durbl	0.163	2.287	-14.430	15.030	-0.334	8.049
Manuf	0.044	1.603	-11.110	10.830	-0.515	11.515
Enrgy	-0.039	2.473	-19.730	16.000	-0.301	12.272
HiTec	0.114	1.740	-13.180	10.690	-0.501	9.730
Telcm	0.052	1.352	-9.080	9.060	-0.530	10.975
Shops	0.087	1.390	-10.610	7.050	-0.721	10.126
Hlth	0.061	1.363	-9.740	6.980	-0.410	8.313
Utils	0.037	1.545	-11.610	11.760	-0.051	18.257
Other	0.045	1.723	-13.380	12.240	-0.563	14.600

Table 2: Basic descriptive statistics of daily returns (in %) – out-of-sample period

	mean	st. dev.	min	max	skewness	kurtosis
NoDur	0.072	0.761	-1.980	2.420	-0.043	0.620
Durbl	0.093	2.633	-6.380	11.510	0.574	2.175
Manuf	0.123	1.036	-2.840	2.680	-0.137	0.438
Enrgy	0.345	2.063	-5.030	4.860	-0.046	-0.371
HiTec	0.129	1.343	-3.510	3.560	-0.258	0.289
Telcm	0.042	0.950	-3.100	2.530	-0.200	1.226
Shops	0.096	0.876	-2.840	2.260	-0.512	1.010
Hlth	0.076	0.872	-2.910	1.870	-0.424	0.444
Utils	0.053	0.947	-2.750	2.650	-0.340	0.618
Other	0.148	1.065	-3.100	3.190	-0.071	0.500

4.2 Results

As the distortion function we consider only PHT but with several parameters γ. To fulfil the assumptions of Theorem 2, we consider only $\gamma \geq 1$. Note that if $\gamma = 1$ then the distortion measure is expected value and, hence, the optimal portfolio invest everything in the second asset which is the most profitable one. The optimal portfolios of (5) for in-sample period and for considered parameters γ are summarized in Table 3.

Table 3: Compositions of optimal portfolios – in sample period

γ	NoDur	Durbl	Manuf	Enrgy	HiTec	Telcm	Shops	Hlth	Utils	Other
1		1								
1.5	0.143					0.17	0.336	0.25	0.101	
2	0.31					0.221	0.161	0.308		
3	0.319					0.352		0.329		
4	0.313					0.474		0.213		
5	0.258					0.596		0.146		
10						0.91		0.09		

We can see that as the risk aversion expressed by parameter γ increases the portfolio is more concentrated in least risky assets. Moreover, for very large values of parameter γ, (almost) everything is invested in the asset with the highest minimal return. This is due to the fact that investors with extremely large risk aversions want to hedge against the worst realizations – smallest returns.

Finally, we present in Table 4 mean returns and distortion risk measures of the optimal portfolios from Table 3. For the sake of comparison, we express the risk using PHT distortion measure with $\gamma = 2$.

Table 4: Risk – return performance (in %) of optimal portfolios – in sample & out-of-sample period

in sample \ γ	1	1.5	2	3	4	5	10
Mean return	0.1631	0.0614	0.0535	0.0478	0.0468	0.0474	0.0527
Risk	1.8063	1.1001	1.0919	1.0958	1.1008	1.1083	1.1383
out of sample \ γ	1	1.5	2	3	4	5	10
Mean return	0.0927	0.0743	0.0707	0.0630	0.0588	0.0549	0.0454
Risk	1.5835	0.4787	0.4687	0.4856	0.5034	0.5305	0.6285

5. Conclusions

In this paper we deal with portfolio selection problems where the only objective is to minimize the risk of the portfolio. The risk is measured by distortion risk measures which could be seen as generalizations of Value-at-Risk and Conditional Value-at-Risk. A special attention is paid to distortion measures which are generated by concave distortion functions because these risk measures are coherent. Firstly, we derive a new representation of the coherent distortion functions in the form of linear program. Secondly, we apply it to the risk minimizing portfolio selection problems. Finally, we demonstrate the technique in the empirical study where we limit our attention on the PHT distortion functions and associated distortion measures.

In the empirical study, we have analysed the risk-return performance (in-sample and out-of-sample) of the risk minimizing portfolios for various values of parameter γ which can be seen as a parameter of the risk aversion. The higher the parameter is, the stronger the risk aversion is considered. Therefore, as this parameter increases the mean return of the optimal portfolio decreases. However, it is not true for all values of γ, becasue when it is very large, the risk measure is too much conservative – only the smallest return is important. As the consequence, the mean return may increase with increasing γ for large values of γ. Interesingly, this was observed only in the in-sample period but not in the out-of-sample period, where mean return always decreased with increasing γ. It is no surprise that in-sample and out-of-sample risk measured by PHT distortion measure with $\gamma = 2$ is minimal for the optimal portfolio of (5) when considering PHT with $\gamma = 2$.

The new representation of the coherent distortion measures could be attractive also in the multi-stage portfolio optimization problems such as in e.g. Vitali et al. (2017), Zapletal et al. (2020) or Kopa and Rusý (2021).

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References

[1] Artzner, P., Delbaen, F., Eber, J.-M. and Heath, D. (1999). Coherent Measures of Risk. *Mathematical Finance* **9**: 203-228.

[2] Dhaene, J., Kukush, A., Linders, D. and Tang, Q. (2012). Remarks on quantiles and distortion risk measures. *European Actuarial Journal*, **2**, 319–328.

[3] Hadar, J. and Russell, W. R. (1969). Rules for Ordering Uncertain Prospects. *American Economic Review* **59**, 1, 25–34.

[4] Hanoch, G. and Levy, H. (1969). The efficiency analysis of choices involving risk. *Review of Economic Studies* **36**, 335–346.

[5] Kabasinskas, A., Sutiene, K., Kopa, M., Luksys, K. and Bagdonas, K. (2020). Dominance-Based Decision Rules for Pension Fund Selection under Different Distributional Assumptions. *Mathematics*, **8**, n.719.

[6] Kopa, M., Kabasinskas, A. and Sutiene, K. (2021). A stochastic dominance approach to pension-fund selection, *IMA Journal of Management Mathematics*, https://doi.org/10.1093/imaman/dpab002.

[7] Kopa, M., and Rusý, T. (2021). A decision-dependent randomness stochastic program for asset–liability management model with a pricing decision. *Annals of Operations Research*, **299**, 241–271.

[8] Kopa, M., and Zelman, J. (2021). Distortion risk measures in portfolio optimization, Proceedings of 39th International Conference on Mathematical Methods in Economics 2021.

[9] Markowitz, H. M. (1952). Portfolio Selection. *The Journal of Finance* **7**, 1, 77–91.

[10] Moriggia V., Kopa, M. and Vitali S. (2019): Pension fund management with hedging derivatives, stochastic dominance and nodal contamination. *Omega* **87**, 127–141.

[11] Quirk, J. P., and Saposnik, R. (1962). Admissibility and Measurable Utility Functions. *Review of Economic Studies* **29**, 140–146.

[12] Rothschild, M., and Stiglitz, J.E. (1970). Increasing risk: I. A definition. *Journal of Economic Theory* **2**, 225–243.

[13] Sereda, E. N., Bronshtein, E. M., Rachev, S. T., Fabozzi, F. J., Sun, W. and Stoyanov, S. V. (2010). Distortion risk measures in portfolio optimizatio. Handbook of portfolio construction, pp. 649–673.

[14] Vitali, S., Moriggia, V. and Kopa, M. (2017). Optimal pension fund composition for an Italian private pension plan sponsor. *Computational Management Science*, **14**, 135–160.

[15] Whitmore, G.A. (1970). Third-degree Stochastic Dominance. *American Economic Review* **60**, 457–459.

[16] Wirch, J. and Hardy, M. (1999). A synthesis of risk measures for capital adequacy. *Insurance: mathematics and economics*, **25**, 337-347.

[17] Zapletal, F., Šmíd, M. and Kopa, M. (2020). Multi-stage emissions management of a steel company. *Annals of Operations Research*, **292**, 735–751.