Distortion risk measures in portfolio optimization

Miloš Kopa[1], Juraj Zelman[2]

Abstract. The paper deals with mean-risk problems where the risk is modeled by a distortion measure. This measure could be seen as a generalization of Conditional Value-at-Risk or Expected shortfall. If the associated distortion function is concave the measure is coherent. We analyze several distortion measures for different choices of a concave distortion function. First, assuming a discrete distribution of returns, we identify the efficient frontier. Then we compute the portfolio maximizing reward-risk ratio. Finally, we compare the results for various distortion measures among each other.

Keywords: portfolio optimization, distortion risk measure, efficient frontier, performance ratio

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1 Introduction

Historically, distortion risk measures have their roots in the dual theory of choice under uncertainty proposed by [13] and were later developed by the axiomatic approach in [11]. The idea behind the distortion risk measure is the transformation of the given probability measure in order to quantify the tail risk more accurately and therefore give more weight to higher risk events. The motivation for distorting a probability measure arose from numerous studies on risk perception, such as the work [4], who observed that people evaluate risk as a non-linear distorted function rather than a linear function of the probabilities. Originally, distortion risk measures found their application in the insurance problems. For example, [10] presented an approach to insurance pricing using the proportional hazards transform. However, due to the relation between insurance and investment risks, distortion risk measures started to be also used in the investment context and portfolio selection problems (see for example [9]). Perhaps interesting could be a relation to stochastic dominance which is an attractive tool for random returns comparisons in various applications, see e.g. [7], [3], or [5] for recent applications of stochastic dominance in pension fund management.

The remainder of this paper is structured as follows. Section 2 presents a notation and basic properties of the distortion risk measures. It is followed by a formulation of reward-risk ratio model based on distortion measures of risk in Section 3. Empirical study is presented in Section 4 and the paper is concluded in Section 5.

2 Distortion risk measures

In the whole text, we assume that $\mathcal{X}$ is a set of random variables on a probability space $(\Omega, \mathcal{F}, P)$. A random variable $X \in \mathcal{X}$ represents a loss random variable (typically, positive values are associated with losses and negative values represent gains) of some financial asset over a time interval of length $T \in \mathbb{R}_+$..

Definition 1. ([2]) *Suppose that $g : [0,1] \to [0,1]$ is a non-decreasing function such that $g(0) = 0$ and $g(1) = 1$ (also known as the **distortion function**) and $X \in \mathcal{X}$ with a distribution function $F_X(x)$. Then, the **distortion risk measure** associated with the distortion function g is defined as*

$$\rho_g(X) = -\int_{-\infty}^{0} [1 - g(1 - F_X(x))]dx + \int_{0}^{\infty} g(1 - F_X(x))dx,$$

provided that at least one of the integrals is finite.

When we define the **decumulative distribution function** (also known as the **survival function**) $S_X(x) = 1 - F_X(x) = P(X > x)$ and we use it instead of the distribution function, we obtain

[1] Charles University, Faculty of Mathematics and Physics, Department of Probability and Mathematical Statistics, Sokolovská 83, 186 75 Prague 8, Czech Republic, kopa@karlin.mff.cuni.cz
[2] Charles University, Faculty of Mathematics and Physics, Department of Probability and Mathematical Statistics, Sokolovská 83, 186 75 Prague 8, Czech Republic, zelman.juraj@gmail.com

$$\rho_g(X) = -\int_{-\infty}^{0} [1 - g(S_X(x))]dx + \int_{0}^{\infty} g(S_X(x))dx.$$

The interpretation of this definition is that the distortion measure represents the expectation of a new random variable with re-weighted probabilities. In some cases, such as problems related to insurance or capital requirements, it is appropriate to assume that the random variable $X \in \mathcal{X}$ is non-negative. In this case, when $X \in \mathcal{X}$ is a non-negative random variable, then ρ_g reduces to

$$\rho_g(X) = \int_{0}^{\infty} g(S_X(x))dx.$$

The class of distortion risk measures is prospective, because distortion measures, in the general case, fulfill the conditions of monotonicity, positive homogeneity and translation invariance.

Theorem 1. *([8]) (Monotonicity) Suppose that $X, Y \in \mathcal{X}$ and $X \leq Y$. Then $\rho_g(X) \leq \rho_g(Y)$.*
(Positive homogeneity) For a distortion risk measure ρ_g, $X \in \mathcal{X}$ and $\lambda \geq 0 : \rho_g(\lambda X) = \lambda \rho_g(X)$.
(Translation invariance) For a distortion risk measure ρ_g and $X \in \mathcal{X}$ it holds that $\forall c \in \mathbb{R} : \rho_g(X + c) = \rho_g(X) + c$.

Theorem 2. *([12]) The distortion risk measure $\rho_g(X)$ is sub-additive*

$$\rho_g(X + Y) \leq \rho_g(X) + \rho_g(Y),$$

if and only if g is a concave distortion function.

Summarizing, a distortion risk measure $\rho_g(X)$ is coherent iff g is a concave distortion function.

In the following example, we present the representations of risk measures Value-at-Risk (VaR) and Expected Shortfall (ES) as distortion risk measures.

Example 1. Suppose that $X \in \mathcal{X}$, $\alpha \in (0, 1)$. If

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 - \alpha \\ 1 & \text{if } 1 - \alpha \leq x \leq 1. \end{cases}$$

then $VaR_\alpha(X) = \rho_g(X)$ and if

$$g(x) = min\left(\frac{x}{1-\alpha}, 1\right), \text{ where } x \in [0, 1].$$

then, $ES_\alpha(X) = \rho_g(X)$.

Another example of distortion risk measure includes the **Proportional Hazard (PH) transform** proposed by [10] as a new risk-adjusted premium for insurance risk pricing. This measure has a distortion function

$$g(x) = x^{1/\gamma}, \quad x \in [0, 1], \gamma \geq 1. \tag{1}$$

Consequently, we define the **PH-transform measure** as:

$$\rho_{PH}(X) = \int_{0}^{\infty} S_X(x)^{1/\gamma} dx, \quad \gamma \geq 1,$$

where $S_X(x) = 1 - F_X(x)$ is defined as previously.

As we can see from the definition of the distortion function g of the PH transform, this function is concave and therefore, the PH-transform measure satisfies the sub-additivity property. As [10] mentions, this is an important property as it does not provide any advantage to policy-holders when splitting the risk of their positions into pieces.

Another well known examples of distortion functions which generate coherent risk measures include:

- The **Wang transform** ([11])

$$g_\lambda(x) = \Phi(\Phi^{-1}(x) + \lambda) \quad \text{for } x \in [0,1], \lambda \geq 0,$$

where Φ is the standard normal distribution function.
- The **MINVAR** distortion function ([1])

$$g(x) = 1 - (1-x)^{1+\lambda} \quad \text{for } x \in [0,1], \lambda \geq 0. \tag{2}$$

- The **MINMAXVAR** distortion function ([1])

$$g(x) = 1 - (1 - x^{1/(1+\lambda)})^{1+\lambda} \quad \text{for } x \in [0,1], \lambda \geq 0.$$

3 Reward-risk ratio

Suppose that we have a discrete real random variable Y, representing losses (in percent), with possible values $y_1, \dots, y_m \in \mathbb{R}$, where $y_1 \leq y_2 \leq \cdots \leq y_m$. As we need to separate these values to negative and non-negative, assume that the index $k \in \{0, \dots, m\}$ is such that values $y_1, y_2, \dots, y_k$ are negative and $y_{k+1}, \dots, y_m$ are non-negative (where for $k = 0$ we understand that all values are non-negative and for $k = m$ are all negative). For the simplicity, we assume that $\forall i \in \{1, \dots, m\} : P(Y = y_i) = \frac{1}{m}$. Then, we know that its cumulative distribution function is $F_Y(y) = \frac{1}{m}\sum_{i=1}^m 1_{\{y_i \leq y\}}$, where 1_A denotes an indicator function of a set A. This means that $F_Y(y)$ is constant on intervals $(-\infty, y_1), [y_1, y_2), \dots, [y_m, \infty)$. Thus, from Definition 1 of a distortion measure ρ_g, we can derive that

$$\rho_g(Y) = -\sum_{i=1}^{k-1}(y_{i+1} - y_i)\left[1 - g\left(1 - \frac{i}{m}\right)\right] + y_k\left[1 - g\left(1 - \frac{k}{m}\right)\right] +$$

$$+y_{k+1}g\left(1 - \frac{k}{m}\right) + \sum_{i=k+1}^{m-1}(y_{i+1} - y_i)g\left(1 - \frac{i}{m}\right) = y_1 + \sum_{i=1}^{m-1}(y_{i+1} - y_i)g\left(1 - \frac{i}{m}\right).$$

Therefore, to compute distortion risk measure ρ_g for a discrete random variable Y, it is sufficient to have all the possible values y_i, where $i \in \{1, \dots, m\}$, ordered. We do not need to differentiate between non-negative and negative values. Now we can focus on the formulation of the reward-risk optimization problem.

Assume that we have $m \in \mathbb{N}$ time periods (e.g. weeks) numbered $1, \dots, m$ and $n \in \mathbb{N}$ financial assets $1, \dots, n$. Let $l = (l_{ij})_{i=1,j=1}^{n,m} \in \mathbb{R}^{n \text{ x } m}$, $m, n \in \mathbb{N}$ be a matrix, where l_{ij} represents a concrete realization of a loss of i-th financial asset at time j. Suppose that $w = (w_1, \dots, w_n)^T \in \mathbb{R}^n$ denotes weights of a portfolio associated to our financial assets such that $w^T e = 1$ and $\forall i \in \{1, \dots, n\} : w_i \geq 0$ (we do not allow short sales).

For a given vector of weights w, we can calculate a vector of losses for this portfolio as $l_p = (w^T l)^T \in \mathbb{R}^m$. Equivalently the j-th position of the vector l_p is equal to re-weighted sum of assets' losses at time j or $\sum_{i=1}^n w_i l_{ij}$. However, as we see from the previous part, where we derived the formula for distortion measure, to calculate values of risk measure for different portfolios, we need to first re-order the values of l_p. Therefore, in our optimization problem we need to define a permutation matrix $P = (p_{i,j})_{i=1,j=1}^{m,m}$ consisting of 0 and 1 such that the sum in every row and column is equal to 1. Then, we can define a new vector $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ such that it has the same values as l_p, but its values are ordered from the lowest to the highest. Let Y denote a discrete loss random variable with the possible values $y_1, \dots, y_m$, defined at the beginning of this section. We will denote the expected value of its returns (or negative losses of Y) as $\mu(-Y)$.

If we define a variable R representing the reciprocal value of a distortion reward-risk ratio (minimization over a reciprocal value of a reward-risk ratio is equivalent to maximization of a reward-risk ratio), we can formulate the distortion reward-risk optimization problem as

$$
\begin{aligned}
\underset{w}{\text{minimize}} \quad & R \\
\text{subject to} \quad & \rho_g(\tilde{Y}) = \mu(\tilde{Y}_-) \times R \\
& \tilde{l}_p = (w^T \tilde{l})^T \\
& P\tilde{l}_p = \tilde{y} \quad \text{, where } P = (p_{i,j})_{i=1,j=1}^{m,m} \\
& \sum_{i=1}^{m} p_{ij} = 1 \quad \forall j \in \{1, \dots, m\} \\
& \sum_{j=1}^{m} p_{ij} = 1 \quad \forall i \in \{1, \dots, m\} \\
& p_{ij} \in \{0, 1\} \quad \forall i, j \in \{1, \dots, m\} \\
& \tilde{y}_1 \leq \tilde{y}_2 \leq \dots \leq \tilde{y}_m \\
& w^T e = 1 \\
& w \geq 0.
\end{aligned} \tag{3}
$$

where loss matrix l is substituted by gross losses $\tilde{l}$ by adding one (e.g. the value 1,1 represents 10% loss and value 0,9 represents 10% return). and $\tilde{Y}_-$ denotes gross returns of $\tilde{Y}$ (representing gross losses).

4 Empirical study

To demonstrate our model, we selected ten stocks (A1 - A10), which are traded at stock exchanges NYSE and Nasdaq, see Table 1. We restrict to a smaller sample of weekly adjusted closing prices ranging from 2020-12-21 to 2021-02-22. A smaller sample was selected due to the computational complexity of our model, which leads to a non-linear mixed-integer optimization problem.

Asset	**Company**	**Ticker**	**GICS Sector**
A1	Microsoft Corp.	MSFT	Information Technology
A2	Intel Corp.	INTC	
A3	Goldman Sachs Group	GS	Financials
A4	BlackRock	BLK	
A5	Alphabet Inc.	GOOGL	Communication Services
A6	AT&T Inc.	T	
A7	Amazon.com, Inc.	AMZN	Consumer Discretionary
A8	Johnson & Johnson	JNJ	Health Care
A9	General Electric	GE	Industrials
A10	Exxon Mobil Corp.	XOM	Energy

Table 1: Selected assets and their corresponding GICS sectors

In our implementation, we focused on two distortion risk measures. The Proportional Hazard transform (defined in (1)) for two different parameters $\gamma = 2$ and $\gamma = 5$ and the MINVAR distortion risk measure (defined in (2)) for two parameters $\lambda = 1$ and $\lambda = 4$. For better illustration of the position of the portfolio with the highest reward-risk ratio, we present it with resulting efficient frontiers in Figures 2a and 2b and Tables 2 and 3 with allocations of the optimal portfolios.

As can be seen in Figure 2a, different choices of parameter γ does not only affect the position of the efficient frontiers but influences their shape as well. This is the result of the shapes of Proportional Hazard functions depicted in Figure 1a. As we can see, these functions assign higher values especially to lower values of x. Thus, the corresponding risk measure assigns higher probabilities to realizations with the highest losses. This effect is noticeable especially on the portfolios beyond the highest reward-risk ratio portfolio, where risks grow significantly faster than in the previous part of the efficient frontier. Therefore, different choices of parameters allow us to model various levels of risk perception and to construct optimal portfolios with respect to these levels. Moreover, as can be seen from Table 2, the optimal portfolios with the lowest risk and the highest reward-risk ratio differ significantly. Not only with respect to their values of risk but regarding their allocations as well. Similar results are obtained

(a) Proportional Hazard transform (b) MINVAR distortion function

Figure 1: Selected distortion measures for different parameters

Proportional Hazard transform, $\gamma = 2$							
Return	A1	A2	A3	A10	Risk	RRR	Optimum
1,93%	0,386	0,310	0,024	0,280	**0,992774**	1,026696	Min Risk
2,68%	0	0,860	0	0,140	0,993617	**1,033354**	Max RRR

Proportional Hazard transform, $\gamma = 5$							
Return	A1	A2	A9	A10	Risk	RRR	Optimum
1,28%	0,537	0,071	0,294	0,098	**0,99964**	1,013188	Min Risk
2,54%	0,071	0,759	0	0,170	1,009303	**1,015921**	Max RRR

Table 2: Optimal portfolios with respect to the Proportional Hazard transform with corresponding mean returns, risks and reward-risk ratios (RRR).

for the MINVAR distortion function. In this case, different choices of parameter λ do not only lead to different values of risk but also to different allocations of optimal portfolios. These differences can be noticed from Table 3. The effect on the shapes of efficient frontiers and their positions is depicted in Figure 2b. As we can see, the shapes of MINVAR distortion functions from 1b are translated into the shapes of efficient frontiers. Therefore, in comparison to the PH measure, we also obtain different allocations of optimal reward-risk portfolios.

MINVAR distortion function, $\lambda = 1$							
Return	A1	A2	A3	A10	Risk	RRR	Optimum
1,93%	0,399	0,264	0,187	0,150	**0,993088**	1,026426	Min Risk
2,82%	0	0,401	0,599	0	0,994221	**1,034163**	Max RRR

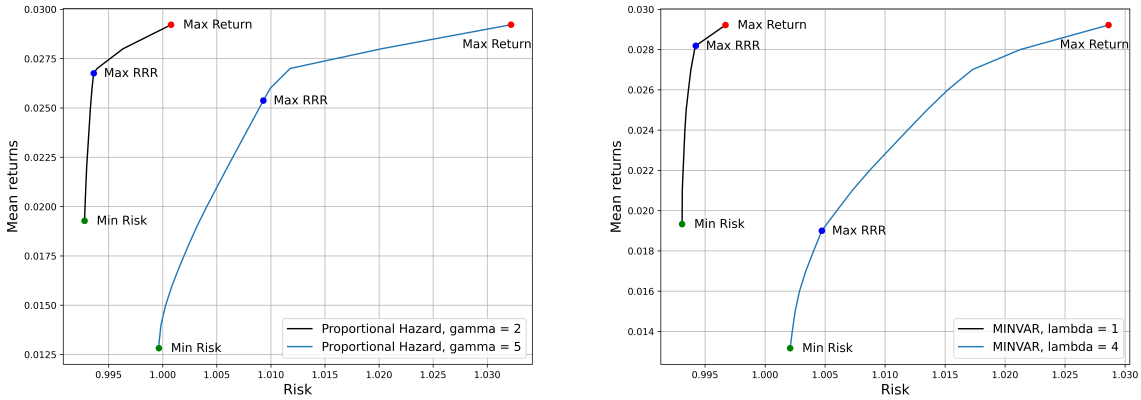
MINVAR distortion function, $\lambda = 4$								
Return	A1	A2	A3	A9	A10	Risk	RRR	Optimum
1,32%	0,471	0,155	0	0,374	0	**1,0021**	1,0111	Min Risk
1,90%	0,421	0,169	0,211	0	0,200	1,0047	**1,0142**	Max RRR

Table 3: Optimal portfolios with respect to the MINVAR distortion function

with corresponding mean returns, risks and reward-risk ratios (RRR).

5 Conclusions

The paper presents a tractable approach to portfolio optimization using distortion risk measures which could be seen as generalizations of Value-at-Risk and Expected Shortfall. In the empirical study, two different formulations (mean-risk and risk-reward ratio) and four different distortion measures are considered. The corresponding efficient frontiers and reward-risk maximizing portfolios are compared. Although the paper presents only static model, the distortion measures could be similarly applied to multistage models with exogenous [14], [15] or endogenous randomness [6].



(a) Proportional Hazard transform

(b) MINVAR distortion function

Figure 2: The efficient frontiers. Portfolios with the highest return, the highest reward-risk ratio and the lowest risk are highlighted

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