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Juraj Zelman

Risk and ratio measures in portfolio optimization

Department of Probability and Mathematical Statistics

Supervisor of the bachelor thesis: doc. RNDr. Ing. Miloš Kopa, Ph.D.
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Title: Risk and ratio measures in portfolio optimization

Author: Juraj Zelman

Department: Department of Probability and Mathematical Statistics

Supervisor: doc. RNDr. Ing. Miloš Kopa, Ph.D., Department of Probability and Mathematical Statistics

Abstract: This thesis is focused on distortion risk measures and distortion reward-risk ratios. Firstly, we summarize the properties of these measures related to coherency axioms and stochastic dominance. We present the proofs and explain the assumptions for consistency of distortion risk measures with stochastic dominance. Furthermore, their relation to Value-at-Risk and Expected Shortfall is explained, and numerous examples of these measures are presented. Then, the basics of the theory of distortion reward-risk ratios are provided. The main theoretical result of this thesis is the proposition of the distortion reward-risk optimization model. We propose this model with the assumption of a discrete loss random variable that has realizations with equal probabilities. Lastly, we analyze and discuss the results and limitations of our implementation in the specialized optimization software GAMS on real financial data. As it turns out, the class of distortion risk measures is prospective because it allows us to re-weight probabilities in the distribution and to model the risk-aversion preferences.

Keywords: risk measures, ratio measures, portfolio optimization

Contents

Introduction

In the financial industry and portfolio selection problems, the trade-off between risk and reward is usually the main topic of interest. Since the introduction of the Markowitz mean-variance model (Markowitz [1959]), numerous quantitative methods that allow us to maximize returns efficiently have been proposed. However, various past financial events have demonstrated the necessity and the importance of thorough risk measurement, particularly in the area of quantitative trading and hedge fund strategies.

The theory of risk measures, which is the main topic of this thesis, offers numerous methods on how to measure and estimate risks. In this thesis, we focus on the class of distortion risk measures (Wang [2000]). The class of distortion risk measures is prospective, as some of these measures allow us to assign higher probabilities to events with low probability but extreme losses. This is of great importance, as the impact of such events often reaches far beyond the financial industry.

This thesis is organized as follows. In Chapter 1, we introduce the theory of risk measures on two widely-known risk measures: Value-at-Risk (Acerbi and Tasche [2002a]) and Expected Shortfall (Acerbi and Tasche [2002b]). We present their properties and closed formulas under the assumption of normally distributed loss random variables. Then, we introduce the theory of distortion risk measures. We propose their properties related to coherency and stochastic dominance. In this chapter, we also present several examples of distortion risk measures, some of which we later analyze on the financial data from the S&P 500 index.

In the following chapter, Chapter 2, we extend the theory of risk measures to the studies of reward-risk ratios, which allow us to construct portfolios with the highest reward per unit of risk. We introduce the Sharpe ratio (Sharpe [1966]) and the general form of a distortion reward-risk ratio (Cheridito and Kromer [2013]).

The main theoretical result of this thesis is the proposition of a general reward-risk optimization framework for distortion risk measures, which can be found in Chapter 3 (to our knowledge, a similar result has not yet been published in the literature). We propose this model with the assumption of a discrete loss random variable with equal probabilities of its realizations.

In Chapter 4, we summarize the results of our implementation of the reward-risk optimization model, which was implemented in the specialized optimization software GAMS. We illustrate the effect of distortion risk measures on the shape of efficient frontiers and compare optimal portfolios for various distortion functions with a special focus on the portfolios with the highest reward-risk ratio.

1. Risk Measures

In the whole thesis, we assume that $\mathcal{X}$ is a set of random variables on a probability space $(\Omega, \mathcal{F}, P)$ such as the space of all essencially bounded random variables $L^\infty(\Omega, \mathcal{F}, P)$ (Rudin [1987]). A random variable $X \in \mathcal{X}$ represents a loss random variable (typically, positive values are associated with losses and negative values represent gains) of some financial asset over a time interval of length $T \in \mathbb{R}_+$. Furthermore, we denote for $X, Y \in \mathcal{X}$ inequalities of random variables almost surely $X \leq Y \iff X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$. For $X \in \mathcal{X}$ we denote its cumulative distribution function as $F_X(x) = P(X \leq x)$.

In this chapter, we introduce the theory of risk measures. Firstly, we recall coherency axioms and two widely known risk measures called Value-at-Risk and Expected Shortfall. We present their properties as well as their closed formulas under the assumption of a normally distributed loss random variable. Then, we focus on the theory of distortion risk measures. We explain their relation to coherent risk measures and risk measures Value-at-Risk and Expected Shortfall. Furthermore, we prove some of their properties related to stochastic dominance and present numerous examples of these measures. This chapter begins with the definition of a general risk measure.

Definition 1. *(Artzner et al. [1999]) We define a **risk measure** as a functional $\rho : \mathcal{X} \to \mathbb{R}$.*

In other words, a risk measure is a mapping that assigns a numerical value to elements of a set of loss random variables. The purpose of such functional is usually to determine the amount of capital that is at risk or that should be kept in reserve. However, as we will see in the section related to the optimization of distortion risk measures, this interpretation does not always hold. Sometimes these values only provide us with a form of ranking on random variables.

One of the major milestones in risk measurement was the proposition of the first axioms of risk measurement, summarized in Artzner et al. [1999]. Risk measures obeying these axioms are known as coherent risk measures.

Definition 2. *(Artzner et al. [1999]) The four coherency axioms are defined as:*

(i) ***Translation invariance:*** *For all random variables $X \in \mathcal{X}$ and every constant $\lambda \in \mathbb{R}$ it follows*

$$\rho(X + \lambda) = \rho(X) + \lambda.$$

(ii) ***Subadditivity:*** *For all random variables X and Y from $\mathcal{X}$ it holds*

$$\rho(X + Y) \leq \rho(X) + \rho(Y).$$

(iii) ***Positive homogeneity:*** *For all $\lambda \geq 0$ and all random variables $X \in \mathcal{X}$:*

$$\rho(\lambda X) = \lambda \rho(X).$$

(iv) ***Monotonicity:*** *For all random variables X and Y from $\mathcal{X}$ such that $X \leq Y$ it follows*

$$\rho(X) \leq \rho(Y).$$

A risk measure ρ is called ***coherent****, if it satisfies all four axioms of translation invariance, subadditivity, positive homogeneity and monotonicity.*

These properties (adjusted for losses) have been introduced due to their financial interpretation, which is explained below. However, some of them are subject to criticism as every class of risk measures has its drawbacks and limitations. If we understand a risk measure as a functional that determines the amount of capital that is at risk; then their financial interpretation is as follows.

The property of translation invariance states that if the losses increase by a given constant, the risk increases proportionally by the same deterministic quantity. If we choose $\alpha = -\rho(X)$ or in other words, we invest in our portfolio a capital investment with the volume equal to $\rho(X)$ (a value that is usually interpreted as a capital requirement), then the risk of the whole position is equal to zero.

The axiom of subadditivity reflects the idea that risk can be reduced by diversification or, as was stated by Artzner et al. [1999] “a merger does not create extra risk”. This means that the portfolio’s risk should not exceed the sum of risks of its components. This property also allows risk managers to choose suitable constraints for different trading desks in order to ensure that the total risk of their positions does not exceed the given upper bound.

Positive homogeneity implies from a financial viewpoint that a linear increase of the loss results in a linear increase in risk by the same proportion. However, as Föllmer and Schied [2002] note, the portfolio’s risk does not necessarily increase linearly with the portfolio’s size in some cases. Therefore, they suggest relaxing this condition.

The monotonicity axiom intuitively implies that if one financial instrument has the loss X, which is smaller or equal to the loss Y of the second instrument, then it requires less risk capital. Moreover, positions where $X \leq 0$ almost surely do not require any risk capital.

1.1 Value-at-Risk

In this section, we define one of the well-known risk measures used in the financial industry. This measure is called Value-at-Risk (VaR). VaR replaced volatility as a superior measure of risk and still remains in use due to its simple operational implementation. Firstly, we define upper and lower quantiles.

Definition 3. *(Acerbi and Tasche [2002a]) Let $X \in \mathcal{F}$ and $\alpha \in (0,1)$ be some confidence level. We define:*

$q_\alpha(X) = inf\{x \in \mathbb{R} : P(X \leq x) \geq \alpha\}$ *as the lower α-quantile of X,*

$$q^{\alpha}(X) = inf\{x \in \mathbb{R} : P(X \leq x) > \alpha\} \text{ as the upper } \alpha\text{-quantile of } X.$$

Based on this definition, we define Value-at-Risk as the lower α-quantile of X.

Definition 4. *(McNeil et al. [2015]) Given some confidence level $\alpha \in (0,1)$ and $X \in \mathcal{X}$, we define **Value-at-Risk** of a loss X at a confidence level α as*

$$VaR_{\alpha}(X) = q_{\alpha}(X) = inf\{x \in \mathbb{R} : P(X > x) \leq 1 - \alpha\}.$$

The interpretation of Definition 4 is as follows: Suppose that we have a portfolio with the given loss $X \in \mathcal{X}$ and we choose the confidence level $\alpha \in (0,1)$. Typical choices of α include 0.95 or 0.99, although other values of this parameter are also in use. Then, $VaR_{\alpha}(X)$ is the smallest value x such that the probability of the loss X greater than x does not exceed $1-\alpha$. Or in other words, $VaR_{\alpha}(X)$ is the α-quantile of the loss distribution. For this reason, $VaR_{\alpha}(X)$ is categorized as a quantile risk measure.

Some of the important properties of $VaR_{\alpha}(X)$ are proved in the next theorem.

Theorem 1. *(Artzner et al. [1999]) Let $\alpha \in (0,1)$ and $X, Y \in \mathcal{X}$. The risk measure $VaR_{\alpha}(X)$ is translation invariant, positively homogeneous and monotone.*

Proof.

1. From Definition 4 we get

$$VaR_{\alpha}(X + \lambda) = inf\{x \in \mathbb{R} : P(X + \lambda > x) \leq 1 - \alpha\}.$$

   By a simple substitution $u = x - \lambda$ we obtain

$$\begin{aligned} VaR_{\alpha}(X + \lambda) &= inf\{u + \lambda \in \mathbb{R} : P(X > u) \leq 1 - \alpha\} = \\ &= inf\{u \in \mathbb{R} : P(X > u) \leq 1 - \alpha\} + \lambda = \\ &= VaR_{\alpha}(X) + \lambda, \end{aligned}$$

   what proves the translation invariance property.

2. Firstly, if $\lambda = 0$ we immediately obtain $VaR_{\alpha}(0X) = 0$. Now suppose that $\lambda > 0$. Then

$$VaR_{\alpha}(\lambda X) = inf\{x \in \mathbb{R} : P(\lambda X > x) \leq 1 - \alpha\} =$$

   Again, by substitution $u = \frac{x}{\lambda}$ we get

$$\begin{aligned} VaR_{\alpha}(\lambda X) &= inf\{\lambda u \in \mathbb{R} : P(X > u) \leq 1 - \alpha\} = \\ &= \lambda inf\{u \in \mathbb{R} : P(X > u) \leq 1 - \alpha\} = \\ &= \lambda VaR_{\alpha}(X), \end{aligned}$$

   what was to be shown for positive homogeneity.

3. The condition $\forall \omega \in \Omega : X(\omega) \leq Y(\omega)$ implies that $\forall x \in \mathbb{R} : P(X > x) \leq P(Y > x)$. Therefore

$$VaR_\alpha(X) = inf\{x \in \mathbb{R} : P(X > x) \leq 1 - \alpha\} \leq$$

$$\leq inf\{x \in \mathbb{R} : P(Y > x) \leq 1 - \alpha\} = VaR_\alpha(Y),$$

what proves the monotonicity property and, therefore, the whole theorem.

□

The following theorem explains one of the VaR's drawbacks, the fact that VaR does not fulfill the sub-additivy property.

Theorem 2. *(Artzner et al. [1999]) Let $\alpha \in (0, 1)$ and $X, Y \in \mathcal{X}$. Then, VaR_α violates the sub-additivity property*

$$VaR_\alpha(X + Y) \nleq VaR_\alpha(X) + VaR_\alpha(Y).$$

*Therefore, VaR_α is **not** a coherent risk measure.*

As we have seen from the previous theorem, Value-at-Risk is a risk measure that violates the sub-additive property and therefore, the risk of a portfolio could be potentially larger than the sum of risks of its components.

In the following part, we derive the exact formula for Value-at-Risk with the assumption of normally or t_n-distributed loss random variable.

Theorem 3. *(McNeil et al. [2015]) Suppose that $X \in \mathcal{X}$ is normally distributed with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. Then*

$$VaR_\alpha(X) = \mu + \sigma\Phi^{-1}(\alpha), \tag{1.1}$$

where Φ denotes the standard normal distribution function and $\Phi^{-1}(\alpha)$ is the α-quantile of Φ.

Proof. We provide proof based on the idea from (McNeil et al. [2015]). From the definition of VaR_α and by equivalent manipulations we obtain

$$VaR_\alpha(X) = inf\{x \in \mathbb{R} : P(X > x) \leq 1 - \alpha\} =$$

$$= inf\{x \in \mathbb{R} : P\left(\frac{X - \mu}{\sigma} > \frac{x - \mu}{\sigma}\right) \leq 1 - \alpha\} =$$

$$= inf\{x \in \mathbb{R} : P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) \geq \alpha\}.$$

Because $X \sim \mathcal{N}(\mu, \sigma^2)$, we have $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$ and therefore we can rewrite

$$VaR_\alpha(X) = inf\{x \in \mathbb{R} : \Phi\left(\frac{x - \mu}{\sigma}\right) \geq \alpha\} =$$

$$= inf\{x \in \mathbb{R} : x \geq \mu + \sigma\Phi^{-1}(\alpha)\} = \mu + \sigma\Phi^{-1}(\alpha),$$

what proves the statement. □

In the previous theorem, we assumed that the losses are normally distributed. However, the choice of suitable distribution for losses is one of the most discussed topics in the financial industry. Since the observation of Mandelbrot [1963], who argued that the assumption of normal distribution for some asset returns could underestimate tail risks and suggested alternative classes of distributions with heavier tails, many classes of heavy-tailed distributions that assign higher probabilities to extreme events have been proposed.

One of such distributions with similar properties to the normal distribution but heavier tails is the t_n-distribution (for low values of n). According to Stoyanov et al. [2011], t_n-distribution is probably one of the most commonly used heavy-tailed distributions due to its computational simplicity.

Therefore, a similar result to Theorem 3 but for the t_n-distribution can be derived. In this case, we will assume that the loss X has a generalized t_n-distribution. The generalized t_n-distribution is defined as $t_n(a, b) = a + bt_n$, where t_n denotes the standard t_n distribution and $a, b \in \mathbb{R}$ are chosen scaling constants. In our case, this distribution fulfills $E[X] = \mu$ and $var(X) = n\sigma^2/(n-2)$, provided that $n > 2$. Then we have

$$VaR_\alpha(X) = \mu + \sqrt{\frac{n-2}{n}}\sigma {t_n}^{-1}(\alpha), \tag{1.2}$$

where ${t_n}^{-1}(\alpha)$ denotes the α-quantile of the standard t_n-distribution.

From equations (1.1) and (1.2), we see the computational advantage of VaR_α. This computational simplicity is one of the benefits that this risk measure provides. Another strength of VaR_α is its universality and use across different classes of securities.

On the other hand, these advantages are compensated by several drawbacks. We have already seen that VaR is not a sub-additive risk measure. Furthermore, numerous financial events where extreme losses occurred, such as the collapse of LTCM in 1998 or the US subprime mortgage crisis in 2007-2008, raised the awareness of the limitation of VaR_α to capture rare events with extreme losses. It is obvious from the definition of quantile that VaR_α does not take into consideration extreme losses with probability lower than $1 - \alpha$. Therefore, alternative risk measures which capture the tail risks more accurately are studied.

1.2 Expected Shortfall

The risk measure Expected Shortfall was proposed as an alternative to VaR and was constructed in order to deal with the deficiencies of VaR, mostly the subadditivity property. This measure has numerous variants with different names such as conditional value at risk, tail conditional expectation or worst conditional expectation. However, as Acerbi and Tasche [2002a] proposed, most definitions of Expected Shortfall lead to the same results under the assumption of continuous loss distributions. We define Expected Shortfall in the following definition.

Definition 5. *(Acerbi and Tasche [2002b]) Let $X \in \mathcal{X}$ be a random variable, $E[|X|] < \infty$ and $\alpha \in (0, 1)$. We define **Expected Shortfall** at a confidence level*

α *as*

$$ES_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\gamma(X)\,d\gamma,$$

where $VaR_\gamma(X)$ *denotes Value-at-Risk.*

It must be noted that this definition is adjusted for loss random variables. The interpretation of Definition 5 is the following. The risk measure Expected Shortfall is closely related to $VaR_\alpha(X)$ and can be interpreted as an average over all possible losses that exceed or are equal to $VaR_\alpha(X)$. As we mentioned before, in comparison to VaR_α, ES_α fulfills the subadditivity property.

Theorem 4. *(Acerbi and Tasche [2002a]) Let* $E[|X|] < \infty$*. Then, the risk measure* $ES_\alpha(X)$ *is monotone, subadditive, positive homogeneous and translation invariant. Therefore,* $ES_\alpha(X)$ *is a coherent risk measure.*

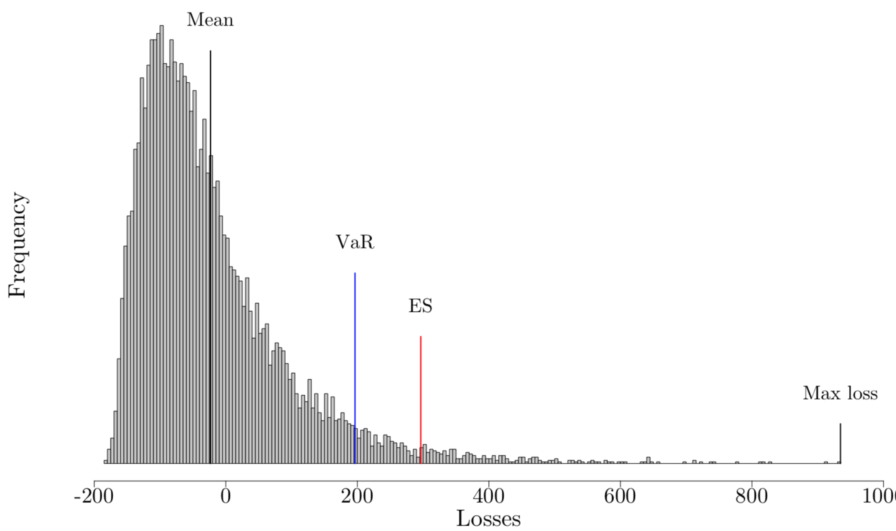


Figure 1.1: Value-at-Risk and Expected Shortfall for a possible realization of a loss random variable X.

In a similar way to VaR_α, closed formulas for ES_α under the assumption of a loss with a normal or t_n-distribution can be obtained. The following two theorems summarize these formulas.

Theorem 5. *(McNeil et al. [2015]) Suppose that* $X \in \mathcal{X}$ *is a random variable with a normal distribution with mean* $\mu \in \mathbb{R}$ *and variance* $\sigma^2 > 0$*. Fix* $\alpha \in (0, 1)$*. Then*

$$ES_\alpha(X) = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}, \tag{1.3}$$

where ϕ *is the density of the standard normal distribution and* $\Phi^{-1}(\alpha)$ *is the* α*-quantile of the standard normal distribution function.*

Proof. We provide proof based on the idea from (McNeil et al. [2015]). Firstly, we recall from Theorem 3 that when the loss X is normally distributed it holds that $VaR_\alpha(X) = \mu + \sigma\Phi^{-1}(\alpha)$. Therefore, we can rewrite from Definition 5

$$ES_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\gamma(X)\,d\gamma = \frac{1}{1-\alpha}\int_\alpha^1 (\mu + \sigma\Phi^{-1}(\gamma))\,d\gamma =$$

$$= \mu + \frac{1}{1-\alpha}\int_\alpha^1 \sigma\Phi^{-1}(\gamma)\,d\gamma.$$

Now we will calculate the last integral by substituting $\gamma = \Phi(u)$. Hence we have $u = \Phi^{-1}(\gamma)$ and $d\gamma = \phi(u)du$, where Φ is the distribution function of the standard normal distribution with the density ϕ and Φ^{-1} is the inverse of the standard normal distribution. Thus, we obtain

$$\int_\alpha^1 \sigma\Phi^{-1}(\gamma)\,d\gamma = \int_{\Phi^{-1}(\alpha)}^{\Phi^{-1}(1)} \sigma u\phi(u)\,du = \int_{\Phi^{-1}(\alpha)}^{\infty} \frac{\sigma u}{\sqrt{2\pi}} exp\left(-\frac{u^2}{2}\right) du.$$

By solving the last integral, we get

$$\int_{\Phi^{-1}(\alpha)}^{\infty} \frac{\sigma u}{\sqrt{2\pi}} exp\left(-\frac{u^2}{2}\right) du = \frac{\sigma}{\sqrt{2\pi}}\int_{\Phi^{-1}(\alpha)}^{\infty} u exp\left(-\frac{u^2}{2}\right) du =$$

$$= \frac{\sigma}{\sqrt{2\pi}}\left[-exp\left(-\frac{u^2}{2}\right)\right]_{\Phi^{-1}(\alpha)}^{\infty} = \frac{\sigma}{\sqrt{2\pi}} exp\left(-\frac{[\Phi^{-1}(\alpha)]^2}{2}\right) = \sigma\phi(\Phi^{-1}(\alpha)).$$

Therefore, from the previous result, we obtain

$$ES_\alpha(X) = \mu + \frac{1}{1-\alpha}\int_\alpha^1 \sigma\Phi^{-1}(\gamma)\,d\gamma = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha},$$

what we wanted to prove. □

Similarly to the previous theorem, we can derive a formula for Expected Shortfall for a loss with a t_n-distribution.

Theorem 6. *(Norton et al. [2019]) Suppose the loss random variable $X \in \mathcal{X}$ has a generalized t_n-distribution $t_n(\mu, \sigma)$ such that $E[X] = \mu$ and $var(X) = n\sigma^2/(n-2)$, where $n > 2$. Then*

$$ES_\alpha(X) = \mu + \sqrt{\frac{n-2}{n}}\sigma\frac{[n + (t_n^{-1}(\alpha))^2]}{n-1}\frac{\tau_n(t_n^{-1}(\alpha))}{1-\alpha}, \tag{1.4}$$

where t_n denotes the distribution function and τ_n the density of the standard t_n-distribution.

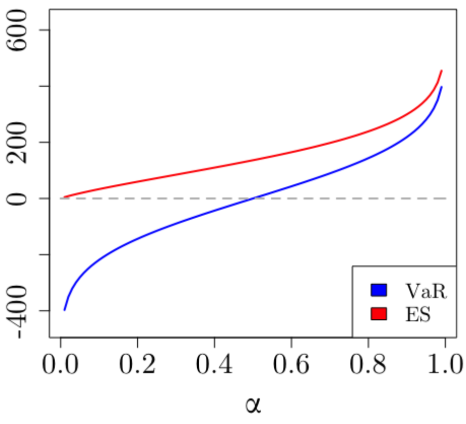
In order to better illustrate the sensitivity of ES to the severity of losses exceeding VaR_α, we propose a concrete example on daily losses of a particular stock using the idea from McNeil et al. [2015]. Suppose that the current value of our position is $V_t = 15000$ and that daily log returns X_{t+1} have mean 0 and standard deviation $\sigma = 0.181/\sqrt{253}$. We assume the average number of trading days is equal to 253. Moreover, we assume that the stock has an annualized volatility equal to 18.1% what is the average annualized volatility of the stock index S&P500. Therefore, we can calculate the loss of our portfolio as $L_{t+1}^{\Delta} = -V_t X_{t+1}$.

In this example, we compare values of VaR_α and ES_α for different confidence levels α and for two different distributions. In the first case we have a loss with a normal distribution, and in the second case we will have a t_n-distribution with $n = 4$ degrees of freedom scaled to have the standard deviation equal to σ. As we stated in the section behind Theorem 3, a t_n-distribution with $n = 4$ degrees of freedom is a distribution with heavier tails than a normal distribution, meaning that higher absolute values have higher probabilities. Values of VaR_α in a normal or a t_n model can be calculated by (1.1) or (1.2) respectively. Values of ES_α are given by (1.3) and (1.4). We summarize our computations in Table 1.1 and illustrate them in Figure 1.2.

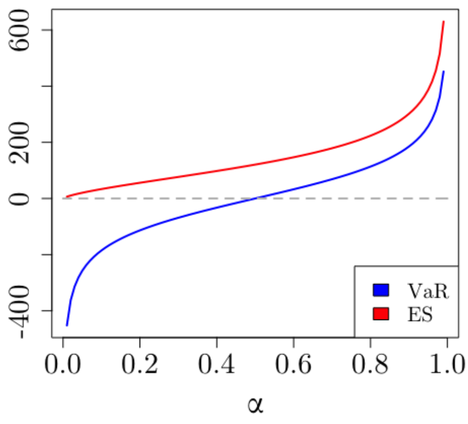
α	0.90	0.95	0.975	0.99	0.995
VaR_α (normal)	218.7	280.8	334.5	397.1	439.7
VaR_α (t_n)	185.1	257.3	335.1	452.2	555.7
ES_α (normal)	299.6	352.1	399.0	454.9	493.6
ES_α (t_n)	301.7	386.6	482.0	630.1	763.4

Table 1.1: Values of VaR_α and ES_α under the assumption of a loss with a normal and t_n distribution

What makes this example interesting is the comparison of values of VaR_α between both distributions. As was mentioned, the t_n-distribution is a distribution with heavier tails than the normal distribution. Intuitively, we would probably assume that this fact would be reflected on higher values of VaR_α with the assumption of the t_n distribution. This effect is visible only on values above the confidence level 0.99. Thus, this illustrates one of the drawbacks of VaR_α. In comparison to VaR_α, the risk measure Expected Shortfall captures the tail risk of the t_n-distributed loss in this example more accurately.



(a) VaR_α and ES_α (normal)



(b) VaR_α and ES_α (t_n)

Figure 1.2: VaR_α and ES_α for different values of α and losses with different distributions

1.3 Distortion Risk Measures

In the following section, we introduce the class of distortion risk measures. We recall some of their known characteristics and then show their relation to coherent risk measures known from the previous section. At the end of this chapter, we explain that VaR and ES, which were defined previously, can be represented as distortion risk measures. We also present some other known examples of these measures.

Historically, distortion risk measures have their roots in the dual theory of choice under uncertainty proposed by Yaari [1987] and were later developed by the axiomatic approach of Wang [2000]. The idea behind the distortion risk measure is the transformation of the given probability measure in order to quantify the tail risk more accurately and therefore give more weight to higher risk events.

The motivation for distorting a probability measure arose from numerous studies on risk perception, such as the work of Kahneman and Tversky [1979], who observed that people evaluate risk as a non-linear distorted function rather than a linear function of the probabilities. The function $g(S(x))$, defined after the following definition, can be thus interpreted as a risk-adjusted decumulative distribution function.

Definition 6. *(Dhaene et al. [2012]) Suppose that $g : [0,1] \to [0,1]$ is a non-decreasing function such that $g(0) = 0$ and $g(1) = 1$ (also known as the **distortion function**) and $X \in \mathcal{X}$ with a distribution function $F_X(x)$. Then, the **distortion risk measure** associated with the distortion function g is defined as*

$$\rho_g(X) = -\int_{-\infty}^{0} [1 - g(1 - F_X(x))]dx + \int_{0}^{\infty} g(1 - F_X(x))dx,$$

provided that at least one of the integrals is finite.

When we define the **decumulative distribution function** (also known as the **survival function**) $S_X(x) = 1 - F_X(x) = P(X > x)$ and we use it instead of the distribution function, we obtain

$$\rho_g(X) = -\int_{-\infty}^{0} [1 - g(S_X(x))]dx + \int_{0}^{\infty} g(S_X(x))dx.$$

The interpretation of this definition is that the distortion measure represents the expectation of a new random variable with re-weighted probabilities.

Originally, distortion risk measures found their application in the insurance problems. For example, Wang [1995] presented an approach to insurance pricing using the proportional hazards transform. However, due to the relation between insurance and investment risks, distortion risk measures started to be also used in the investment context and portfolio selection problems (see for example Van der Hoek and Sherris [2001]).

In some cases, such as problems related to insurance or capital requirements, it is appropriate to assume that the random variable $X \in \mathcal{X}$ is non-negative. In this case, when $X \in \mathcal{X}$ is a non-negative random variable, then ρ_g reduces to

$$\rho_g(X) = \int_{0}^{\infty} g(S_X(x))dx.$$

1.3.1 Properties of distortion risk measures

As was stated in the previous section about coherent risk measures, in order to develop a risk measure with desired properties, some axiomatic conditions should be proposed. The class of distortion risk measures is prospective, because distortion measures, in the general case, fulfill the conditions of monotonicity, positive homogeneity and translation invariance. These three properties are the scope of the following theorems.

Theorem 7. *(Monotonicity) Suppose that $X, Y \in \mathcal{X}$ and $X \leq Y$. Then $\rho_g(X) \leq \rho_g(Y)$.*

Proof. Proof of this theorem follows immediately from the properties of distribution functions. We have $X \leq Y \implies \forall x \in \mathbb{R} : F_X(x) = P(X \leq x) \geq P(Y \leq x) = F_Y(x)$. Thus $\forall x \in \mathbb{R} : S_X(x) \leq S_Y(x)$. Due to the fact, that g is a non-decreasing function, we obtain $\forall x \in \mathbb{R} : g(S_X(x)) \leq g(S_Y(x))$. Therefore, we get from Definition 6 that $\rho_g(X) \leq \rho_g(Y)$. $\square$

Theorem 8. *(Positive homogeneity) For a distortion risk measure ρ_g, $X \in \mathcal{X}$ and $\lambda \geq 0$:*

$$\rho_g(\lambda X) = \lambda \rho_g(X)$$

Proof. Firstly, assume that $\lambda = 0$. Then λX has a degenerate distribution function

$$F_{\lambda X}(x) = \begin{cases} 1 & \text{when x} \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\rho_g(\lambda X) = 0 = \lambda \rho_g(X),$$

because

$$1 - g(1 - F_{\lambda X}(x)) = 0 \text{ for } x \in (-\infty, 0) \text{ and } g(1 - F_{\lambda X}(x)) = 0 \text{ for } x \in [0, \infty).$$

Now suppose that $\lambda > 0$. Then it holds

$$S_{\lambda X}(x) = P(\lambda X > x) = P\left(X > \frac{x}{\lambda}\right) = S_X\left(\frac{x}{\lambda}\right).$$

Thus, we obtain

$$\rho_g(\lambda X) = -\int_{-\infty}^{0} [1 - g(S_{\lambda X}(x))]dx + \int_{0}^{\infty} g(S_{\lambda X}(x))dx =$$

$$= -\int_{-\infty}^{0} [1 - g(S_X\left(\frac{x}{\lambda}\right))]dx + \int_{0}^{\infty} g(S_X\left(\frac{x}{\lambda}\right))dx.$$

By substituting $u = \frac{x}{\lambda}$ and $du = \frac{1}{\lambda}dx$ we get

$$\rho_g(\lambda X) = -\lambda \int_{-\infty}^{0} [1 - g(S_X(u))]du + \lambda \int_{0}^{\infty} g(S_X(u))du =$$

$$= \lambda \rho_g(X),$$

what was to be shown. $\square$

Theorem 9. *(Sereda et al. [2010]) (Translation invariance)*
For a distortion risk measure ρ_g and $X \in \mathcal{X}$ it holds that

$$\forall c \in \mathbb{R} : \rho_g(X + c) = \rho_g(X) + c.$$

Proof. We see from the properties of a decumulative distribution function that

$$S_{X+c}(x) = P(X + c > x) = P(X > x - c) = S_X(x - c).$$

Then, we have

$$\rho_g(X + c) = -\int_{-\infty}^{0}[1 - g(S_{X+c}(x))]dx + \int_{0}^{c} g(S_{X+c}(x))dx + \int_{c}^{\infty} g(S_{X+c}(x))dx =$$

$$= -\int_{-\infty}^{0}[1 - g(S_X(x - c)]dx + \int_{0}^{c} g(S_X(x - c))dx + \int_{c}^{\infty} g(S_X(x - c))dx.$$

Now, by substituting $x = c + u, dx = du$ we obtain

$$\rho_g(X + c) = -\int_{-\infty}^{-c}[1 - g(S_X(u)]du + \int_{-c}^{0} g(S_X(u))du + \int_{0}^{\infty} g(S_X(u))du =$$

$$= -\int_{-\infty}^{0}[1 - g(S_X(u)]du + \int_{0}^{\infty} g(S_X(u))du + \int_{-c}^{0} 1du =$$

$$= \rho_g(X) + c.$$

□

Moreover, distortion risk measures fulfill an additional property related to their dual distortion function.

Theorem 10. *(Sereda et al. [2010]) For a distortion risk measure ρ_g and $X \in \mathcal{X}$ it holds that*

$$\rho_g(-X) = -\rho_{\tilde{g}}(X),$$

where $\tilde{g}(x) = 1 - g(1 - x)$.

Proof. Firstly, we will rewrite from the definition

$$S_{-X}(x) = P(-X > x) = P(X < -x) = 1 - S_X(-x) - P(X = -x).$$

Now, we can calculate

$$\rho_g(-X) = -\int_{-\infty}^{0}[1 - g(S_{-X}(x))]dx + \int_{0}^{\infty} g(S_{-X}(x))dx =$$

$$= -\int_{-\infty}^{0}[1-g(1-S_X(-x)-P(X = -x))]dx+\int_{0}^{\infty} g(1-S_X(-x)-P(X = -x))dx.$$

By using the substitution $u = -x$ and $du = -dx$ we obtain

$$\rho_g(-X) = \int_{\infty}^{0}[1 - g(1 - S_X(u))]du - \int_{0}^{-\infty} g(1 - S_X(u))du =$$

$$= \int_{-\infty}^{0}[1 - \tilde{g}(S_X(u))]du - \int_{0}^{\infty} \tilde{g}(S_X(u))du = -\rho_{\tilde{g}}(X),$$

what proves this property. □

Distortion risk measures in general case do not fulfill the additive property. This fact is shown in the following theorem.

Theorem 11. *In general, for $X, Y \in \mathcal{X}$:*

$$\rho_g(X+Y) \neq \rho_g(X) + \rho_g(Y).$$

Thus, in general, distortion risk measures are not additive.

Proof. We will prove this theorem by counterexample using the idea from Sereda et al. [2010]. Assume that $X, Y \in \mathcal{X}$ are two discrete iid losses with the distribution $P(X = 3) = P(X = -3) = \frac{1}{2}$. Suppose that the joint distribution of X and Y is defined as follows:

$$P(3,3) = P(-3,3) = P(3,-3) = P(-3,-3) = \frac{1}{4},$$

where $P(a,b)$ denotes $P(X = a, Y = b)$. Furthermore, let $g = x^2$. Then, we obtain

$$\rho_g(X) = -\int_{-3}^{0} [1 - \left(1 - \frac{1}{2}\right)^2] dx + \int_0^3 \left(1 - \frac{1}{2}\right)^2 dx = -\frac{3}{2}.$$

Similarly, for $X + Y$ we have that

$$\rho_g(X+Y) = -\int_{-6}^{0} [1 - \left(1 - \frac{1}{4}\right)^2] dx + \int_0^6 \left(1 - \frac{3}{4}\right)^2 dx = -\frac{9}{4}.$$

Therefore, we proved the statement because

$$\rho_g(X+Y) = -\frac{9}{4} \neq -\frac{3}{2} - \frac{3}{2} = \rho_g(X) + \rho_g(Y).$$

□

In the last theorem of this section, we state the important property of one subgroup of distortion risk measures. As could be noticed, we proved that distortion risk measures obey 3 out of 4 axiomatic properties proposed in the previous section. The following theorem is related to the last property of sub-additivity and therefore explains the relationship between distortion risk measures and coherent risk measures.

Theorem 12. *The distortion risk measure $\rho_g(X)$ is sub-additive*

$$\rho_g(X+Y) \leq \rho_g(X) + \rho_g(Y),$$

if and only if g is a concave distortion function. Therefore, concave distortion risk measures are coherent risk measures.

Proof. The proof is given in Wirch and Hardy [2001], Theorem 2.2. □

More properties of distortion risk measures follow from the construction of the Choquet integral and can be found in Denneberg [1994]. In the following part, we summarize another group of properties, results related to the stochastic ordering.

1.3.2 Stochastic dominance

The stochastic dominance approach is one of the concepts related to investment decision-making under uncertainty. It is a form of stochastic ordering which allows us to partially order random variables, in our case, loss random variables. In the following definition, we define two such rankings, known as first- and second-order stochastic dominance.

Definition 7. *(Wirch and Hardy [2001]) Assume that $X, Y \in \mathcal{X}$ are losses with decumulative distribution functions S_X and S_Y. Then, we say X* ***first-order stochastically dominates (FSD)*** *Y, if $\forall x \in \mathbb{R} : S_X(x) \leq S_Y(x)$. We denote $Y \preceq_{1st} X$.*

We also say that X is ***second-order stochastically dominant (SSD)*** *over Y, if and only if*

$$\forall x \in \mathbb{R} : \int_x^\infty S_X(t)dt \leq \int_x^\infty S_Y(t)dt.$$

We denote this property by $Y \preceq_{2nd} X$.

Firstly, we must note that this is a definition for loss random variables (where positive values represent losses) and is a modification of the definition in the original text. This definition is an equivalent reformulation for profit-and-loss random variables (where positive values represent returns).

Furthermore, the definition states the following: In the case of first-order stochastic dominance, loss random variables with lower realizations (or lower losses) are preferred. However, second-order stochastic does not only take into account the scale of losses but also their volatility. Thus, it also allows us to capture the risk-aversion of investors.

In the following theorem, we prove that distortion risk measures are consistent with first-order stochastic dominance.

Theorem 13. *Suppose that g is a distortion function and ρ_g is the associated distortion risk measure. Let $X, Y \in \mathcal{X}$ such that $Y \preceq_{1st} X$. Then, $\rho_g(X) \leq \rho_g(Y)$.*

Proof. This proof follows from the definition of first-order stochastic dominance. We have $\forall x \in \mathbb{R} : S_X(x) \leq S_Y(x)$ and a non-decreasing distortion function g. Therefore, it holds $\forall x \in \mathbb{R}$:

$$S_X(x) \leq S_Y(x) \implies g(S_X(x)) \leq g(S_Y(x)) \implies \rho_g(X) \leq \rho_g(Y).$$

□

In order to obtain second-order stochastic dominance, an additional assumption for a distortion function g is needed.

Theorem 14. *(Wirch and Hardy [2001]) Let g be a concave distortion function with the associated distortion risk measure ρ_g. Assume that $X, Y \in \mathcal{X}$ are such that $Y \preceq_{2nd} X$. Then, $\rho_g(X) \leq \rho_g(Y)$.*

Proof. This proof is based on the proof from Wirch and Hardy [2001], but we extend this result for negative losses. It must also be noted that our definition of second-order stochastic dominance slightly differs from the one in the original text. Furthermore, due to Müller [1996], it is sufficient to prove the theorem for non-decreasing concave distortion risk measures where the decumulative functions cross only once.

Let us assume that $E[X] \leq E[Y]$, $Y \preceq_{2nd} X$ and t_0 denotes the crossing point such that

$$
\begin{aligned}
S_X(t) \geq S_Y(t) \quad & \text{for } t < t_0 \\
S_X(t) \leq S_Y(t) \quad & \text{for } t \geq t_0.
\end{aligned}
$$

Now we will define a new decumulative distribution function,

$$
S_Z(t) = \max\{S_X(t), S_Y(t)\} = \begin{cases} S_X(t) & \text{for } t < t_0 \\ S_Y(t) & \text{for } t \geq t_0. \end{cases}
$$

We defined $S_Z(t)$ because in the first step, we will calculate $\rho_g(Z) - \rho_g(X)$ and $\rho_g(Z) - \rho_g(Y)$. Then, we will estimate these integrals, and by combining both inequalities, we obtain that $\rho_g(Y) - \rho_g(X) \geq 0$.

Therefore, in the first part of the proof, we estimate $\rho_g(Z) - \rho_g(X)$. Firstly, suppose that $t_0 \geq 0$. From the definition of distortion risk measures and $S_Z(t)$, we have

$$
\rho_g(Z) - \rho_g(X) = \int_{t_0}^{\infty} [g(S_Y(t)) - g(S_X(t))]dt
$$

and for $t_0 < 0$, we can also calculate that

$$
\begin{aligned}
\rho_g(Z) - \rho_g(X) &= \int_{t_0}^{0} [1 - g(S_X(t))] - [1 - g(S_Y(t))]dt + \int_{0}^{\infty} [g(S_Y(t)) - g(S_X(t))]dt \\
&= \int_{t_0}^{\infty} [g(S_Y(t)) - g(S_X(t))]dt.
\end{aligned}
$$

In order to use the assumption of second-order stochastic dominance, we need to estimate the difference in this integral. For $t > t_0$ it holds that $S_Z(t_0) \geq S_Y(t) \geq S_X(t)$. If for all $t > t_0 : S_X(t) = S_Y(t)$, then $g(S_Y(t)) - g(S_X(t)) = 0$. Now suppose that for some $t > t_0 : S_Z(t_0) \geq S_Y(t) > S_X(t)$. For some $\epsilon \geq 0$, it holds, due to the monotonicity of g, that

$$
\frac{g(S_Y(t)) - g(S_X(t)) - \epsilon}{S_Y(t) - S_X(t)} = \frac{g(S_Z(t_0)) - g(S_X(t))}{S_Z(t_0) - S_X(t)},
$$

and therefore, we can rewrite

$$
\begin{aligned}
g(S_Y(t)) - g(S_X(t)) &\geq (S_Y(t) - S_X(t)) \frac{g(S_Z(t_0)) - g(S_X(t))}{S_Z(t_0) - S_X(t)} \\
&\geq (S_Y(t) - S_X(t)) g'_+(S_Z(t_0)),
\end{aligned}
$$

where g'_+ denotes a right-side derivative.

Thus, for $t > t_0$, we obtain the following inequality

$$\rho_g(Z) - \rho_g(X) \geq g'_+(S_Z(t_0)) \int_{t_0}^{\infty} [S_Y(t) - S_X(t)]dt. \tag{1.5}$$

In the second part of the theorem, which goes similarly to the first part, we estimate $\rho_g(Z) - \rho_g(Y)$. For $t_0 \geq 0$, we obtain that

$$\rho_g(Z)-\rho_g(Y) = \int_{-\infty}^{0} -[1-g(S_X(t))]+[1-g(S_Y(t))]dt+\int_{0}^{t_0} [g(S_X(t))-g(S_Y(t))]dt$$

$$= \int_{-\infty}^{t_0} [g(S_X(t)) - g(S_Y(t))]dt.$$

When $t_0 < 0$, we have

$$\rho_g(Z) - \rho_g(Y) = \int_{-\infty}^{t_0} [g(S_X(t)) - g(S_Y(t))]dt.$$

In this case, for $t < t_0 : S_X(t) \geq S_Y(t) \geq S_Z(t_0)$. If $S_Y(t) = S_X(t)$ for all $t < t_0$ then $\rho_g(Z) - \rho_g(Y) = 0$. Now assume that for some $t < t_0 : S_X(t) > S_Y(t) \geq S_Z(t_0)$. Then, for some $\epsilon \geq 0$, it holds

$$\frac{g(S_X(t)) - g(S_Y(t)) + \epsilon}{S_X(t) - S_Y(t)} = \frac{g(S_X(t)) - g(S_Z(t_0))}{S_X(t) - S_Z(t_0)},$$

and therefore, we can again rewrite

$$g(S_X(t)) - g(S_Y(t)) \leq (S_X(t) - S_Y(t))\frac{g(S_X(t)) - g(S_Z(t_0))}{S_X(t) - S_Z(t_0)}$$

$$\leq (S_X(t) - S_Y(t))g'_-(S_Z(t_0)),$$

where g'_- denotes a left-side derivative. Thus, for $t < t_0$, we have the second inequality

$$\rho_g(Z) - \rho_g(Y) \leq g'_-(S_Z(t_0)) \int_{-\infty}^{t_0} [S_X(t) - S_Y(t)]dt. \tag{1.6}$$

In the last part of the proof, by subtracting inequalities (1.5) and (1.6), we obtain

$$\rho_g(Y) - \rho_g(X) \geq min\{g'_+(S_Z(t_0)), g'_-(S_Z(t_0))\} \int_{-\infty}^{\infty} [S_Y(t) - S_X(t)]dt \geq 0,$$

where the right side of the first inequality is non-negative due to the assumption of second-order stochastic dominance and the fact that g is non-decreasing. In summary, we proved that

$$\rho_g(Y) \geq \rho_g(X).$$

□

1.3.3 Examples of distortion risk measures

In the previous parts of this thesis, we studied two widely used risk measures VaR and Expected Shortfall. Moreover, these risk measures can also be represented as distortion risk measures. In the following two theorems, we propose these representations and explain the relationship between VaR or ES and distortion risk measures. In the following theorem, we define the distortion function related to VaR.

Theorem 15. *(Ambrož [2011]) Suppose that $X \in \mathcal{X}$, $\alpha \in (0,1)$ and*

$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1-\alpha \\ 1 & \text{if } 1-\alpha \leq x \leq 1. \end{cases}$$

Then, VaR_α can be represented as a distortion risk measure associated with the distortion function g, meaning that

$$VaR_\alpha(X) = \rho_g(X).$$

Proof. The function g fulfills all conditions in Definition 6 and therefore is a well defined distortion function. We recall from Definition 4 that we have $VaR_\alpha(X) = q_\alpha(X) = q_\alpha$. From the definition of quantile, we obtain

$$\forall x \in (-\infty, q_\alpha) : g(S_X(x)) = 1, \text{ because } S_X(x) \geq 1-\alpha,$$

and also

$$\forall x \in (q_\alpha, \infty) : g(S_X(x)) = 0, \text{ because } S_X(x) < 1-\alpha.$$

Thus, we have for $q_\alpha \geq 0$

$$\rho_g(X) = -\int_{-\infty}^{0} [1 - g(S_X(x))]dx + \int_{0}^{\infty} g(S_X(x)))dx =$$
$$= -\int_{-\infty}^{0} 0dx + \int_{0}^{q_\alpha} 1dx + \int_{q_\alpha}^{\infty} 0dx = q_\alpha = VaR_\alpha(X).$$

Similarly, for $q_\alpha < 0$ we get

$$\rho_g(X) = -\int_{-\infty}^{0} [1 - g(S_X(x))]dx + \int_{0}^{\infty} g(S_X(x)))dx =$$
$$= -\int_{-\infty}^{q_\alpha} 0dx - \int_{q_\alpha}^{0} 1dx + \int_{0}^{\infty} 0dx = q_\alpha = VaR_\alpha(X).$$

Therefore, we have proved that $VaR_\alpha(X) = \rho_g(X)$. □

In the second example, we define the distortion function that is related to Expected Shortfall and thus show that ES_α can be represented as a distortion risk measure.

Theorem 16. *(Ambrož [2011]) Let $X \in \mathcal{X}$, $\alpha \in (0,1)$ and*

$$g(x) = min\left(\frac{x}{1-\alpha}, 1\right), \text{ where } x \in [0,1].$$

Then, Expected Shortfall can be represented as a distortion risk measure with the distortion function g, meaning that

$$ES_\alpha(X) = \rho_g(X).$$

Proof. Similarly to the proof of the previous theorem, we see that function g meets all conditions in Definition 6 and thus is a distortion function. From the definition of Expected Shortfall, properties of quantile and S_X, we can rewrite

$$ES_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\gamma(X)d\gamma = q_\alpha + \frac{1}{1-\alpha}\int_{q_\alpha}^\infty S_X(x)dx,$$

where $q_\alpha = q_\alpha(X)$ and S_X is a decumulative distribution function. Furthermore, we can conclude that

$$\forall x \in (-\infty, q_\alpha) : g(S_X(x)) = 1, \text{ because } S_X(x) \geq 1-\alpha,$$

and also

$$\forall x \in (q_\alpha, \infty) : g(S_X(x)) = \frac{S_X(x)}{1-\alpha}, \text{ because } S_X(x) < 1-\alpha.$$

For $q_\alpha \geq 0$ we have

$$\begin{aligned}
\rho_g(X) &= -\int_{-\infty}^0 [1-g(S_X(x))]dx + \int_0^\infty g(S_X(x)))dx = \\
&= -\int_{-\infty}^0 0dx + \int_0^{q_\alpha} 1dx + \int_{q_\alpha}^\infty \frac{S_X(x)}{1-\alpha}dx = \\
&= q_\alpha + \frac{1}{1-\alpha}\int_{q_\alpha}^\infty S_X(x)dx = ES_\alpha(X).
\end{aligned}$$

It remains to prove the assertion for $q_\alpha < 0$. In this case, we get

$$\begin{aligned}
\rho_g(X) &= -\int_{-\infty}^0 [1-g(S_X(x))]dx + \int_0^\infty g(S_X(x)))dx = \\
&= -\int_{-\infty}^{q_\alpha} 0dx - \int_{q_\alpha}^0 \left[1 - \frac{S_X(x)}{1-\alpha}\right]dx + \int_0^\infty \frac{S_X(x)}{1-\alpha}dx = \\
&= q_\alpha + \frac{1}{1-\alpha}\int_{q_\alpha}^\infty S_X(x)dx = ES_\alpha(X).
\end{aligned}$$

We have thus proved that $ES_\alpha(X) = \rho_g(X)$. ☐

Another example of distortion risk measure includes the **Proportional Hazard (PH) transform** proposed by Wang [1995] as a new risk-adjusted premium for insurance risk pricing. This measure has a distortion function

$$g(x) = x^{1/\gamma}, \quad x \in [0,1], \gamma \geq 1. \tag{1.7}$$

Consequently, we define the **PH-transform measure** as:

$$\rho_{PH}(X) = \int_0^\infty S_X(x)^{1/\gamma}dx, \quad \gamma \geq 1,$$

where $S_X(x) = 1 - F_X(x)$ is defined as previously.

As we can see from the definition of the distortion function g of the PH transform, this function is concave and therefore, Theorem 12 implies that the PH-transform measure satisfies the sub-additivity property. As Wang [1995] mentions, this is an important property as it does not provide any advantage to policy-holders when splitting the risk of their positions into pieces.

Another well known examples of distortion functions include:

- The **Wang transform** (Wang [2000])

$$g_\lambda(x) = \Phi(\Phi^{-1}(x) + \lambda) \quad \text{for } x \in [0,1], \lambda \geq 0,$$

where Φ is the standard normal distribution function. The parameter λ can be adjusted in order to inflate the probabilities of high losses.

- The **MINVAR** distortion function (Cherny and Madan [2009])

$$g(x) = 1 - (1-x)^{1+\lambda} \quad \text{for } x \in [0,1], \lambda \geq 0. \tag{1.8}$$

- The **MINMAXVAR** distortion function (Cherny and Madan [2009])

$$g(x) = 1 - (1 - x^{1/(1+\lambda)})^{1+\lambda} \quad \text{for } x \in [0,1], \lambda \geq 0.$$

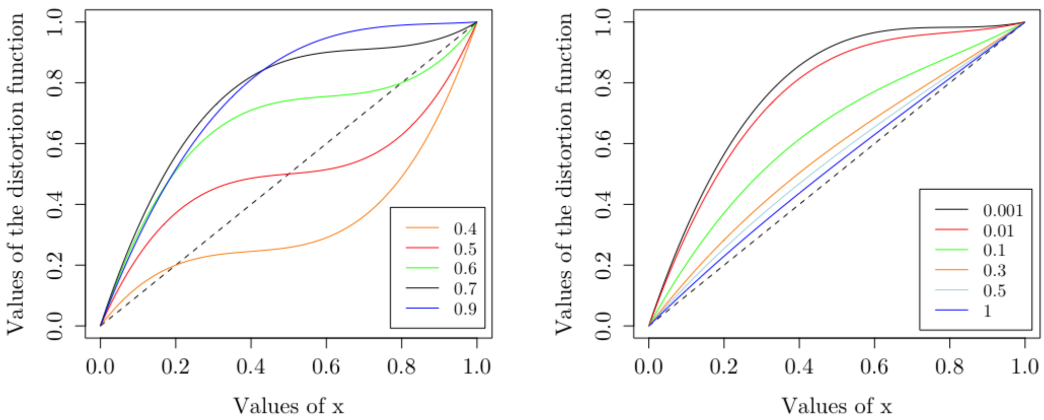
In all three cases, the positivity of the parameter λ ensures that the distortion function is concave.

In the last example, we will illustrate the effect of distortion risk measures on the example of the function presented by Guegan and Hassani [2015]. Suppose that

$$g_\delta(x) = a\left[\frac{x^3}{6} - \frac{\delta}{2}x^2 + \left(\frac{\delta^2}{2} + \beta\right)x\right], \tag{1.9}$$

where $a = (\frac{1}{6} - \frac{\delta}{2} + (\frac{\delta^2}{2} + \beta))^{-1}, x \in [0,1], \delta \in [0,1]$ and $\beta \in \mathbb{R}$.

This function was constructed in order to modify the given distribution and obtain a bimodal distribution; hence we choose a third degree polynomial function.



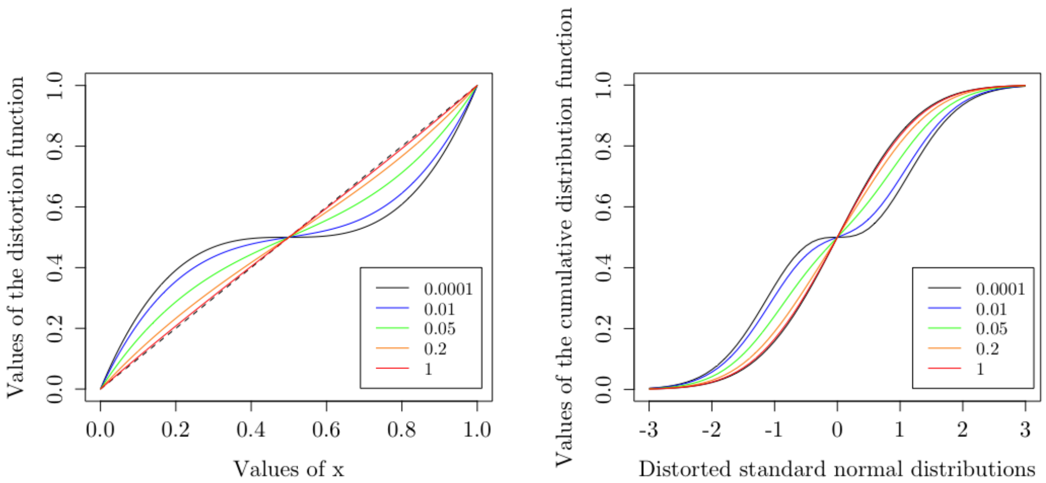
(a) Different values of δ ($\beta = 0.005$)

(b) Different values of β ($\delta = 0.8$)

Figure 1.3: The distortion functions introduced in equation (1.9) for different choices of δ and β

In Figure 1.3, we illustrate the effect of this distortion measure for different values of δ and β. In this Figure, we can see a better illustration of the possibilities that distortion measures provide. Firstly, in Figure 1.3a there are several distortion functions with different choices of parameters δ with the fixed parameter $\beta = 0.005$. As can be noticed, this parameter influences the position of the saddle points and thus creates a convex and concave part in the interval $[0, 1]$. We should also notice that the effect of saddle point decreases for higher values of δ.

On the other hand, Figure 1.3b illustrates the effect of the second parameter β on several presented distortion functions with the fixed parameter δ equal to 0.8. We might notice that the closer the β is to 0, the more significant would be the effect of g on the distribution of losses. Moreover, when β approaches 1 then the distortion function tends to the mapping of identity.



(a) Distortion functions

(b) Distorted standard normal distributions

Figure 1.4: The distortion effect on the standard normal distribution for different values of β (with fixed $\delta = 0.5$)

Similarly to the previous Figure 1.3a, in Figure 1.4a, we have a simulation of distortion functions for numerous parameters β. However, in this case, we fix the parameter δ to be equal to 0.5. As was mentioned previously, in this graph, we also observe the effect of convex and concave parts. Specifically, for lower values of β (0.0001 and 0.01), we obtain noticable separate parts, one concave part for $x \in [0, 0.5]$ and second convex part for $x \in [0.5, 1]$.

Distortion measures with the same parameters as in Figure 1.4a are also shown in Figure 1.4b. However, in this Figure, we illustrate their effect on the distribution function of the standard normal distribution. If the value of β approaches 1 then the distorted distribution tends to the standard normal distribution. A more interesting phenomenon can be observed for values of β approaching 0. In this case, we might notice that the distorted measure associated with the distortion function g_δ assigns smaller probabilities to the middle part of the distribution and shifts more weight to the tail parts of the distribution. This is an important example of the benefit that distortion risk measures provide.

2. Reward-Risk Ratios

In the previous chapter, we studied the theory of risk measurement. We introduced one approach to partial risk ordering, the concept of stochastic dominance. This approach is based on the axiomatic model of risk-averse preferences. However, in practice, the optimizations problems that arise from this theory are usually not easy to solve. Moreover, it is often not sufficient to only minimize risk. We often also aim to maximize our return. Therefore, we extend the theory of risk to studies of reward-risk ratios. The main source of this short chapter is the article from Cheridito and Kromer [2013].

The concept of reward-risk ratios gives us another approach to the partial ordering of risks. In this theory, the choice of the optimal portfolio is given by two criteria - portfolio risk and the expected portfolio return. This choice is based on the value of a reward-risk ratio (RRR), which is generally defined as

$$R(X) = \frac{\mu(-X)^+}{\rho(X)^+}, \tag{2.1}$$

where $X \in \mathcal{X}$, $\mu : \mathcal{X} \to \mathbb{R}$ denotes a reward measure, $\rho : \mathcal{X} \to \mathbb{R}$ is a risk measure and x^+ denotes the positive part $x^+ = \max\{0, x\}$. If we understand $0/0 = 0$ and $\forall x \in \mathbb{R}_+ \setminus \{0\} : x/0 = \infty$, then this ratio is well defined for all cases. For example, if the value of a reward measure is negative and the value of a risk measure is positive, the ratio is equal to 0. Similarly, for the positive value of a reward measure and the negative value of a risk measure, the ratio is equal to ∞. This is a general theoretical construction that allows us to capture different cases. However, in practice, we often work with loss random variables for which the values of reward and risk measures are positive. In the following definition, we define some of the interesting properties of RRR.

Definition 8. *(Cheridito and Kromer [2013]) Properties of reward-risk measures:*

(i) ***(Monotonicity)*** *For all $X, Y \in \mathcal{X}$, $X \leq Y$, we have $R(X) \geq R(Y)$*

(ii) ***(Quasi-concavity)*** *For all $X, Y \in \mathcal{X}$ and $\lambda \in \mathbb{R}$ satisfying $0 \leq \lambda \leq 1$ is $R(\lambda X + (1-\lambda)Y) \geq \min(R(X), R(Y))$*

(iii) ***(Scale-invariance)*** *For any $X \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+ \setminus 0$ such that $\lambda X \in \mathcal{X}$ it holds $R(\lambda X) = R(X)$*

(iv) ***(Distribution-based)*** *$R(X)$ only depends on the distribution of X*

The following theorem gives us the sufficient conditions for properties defined in Definition 8 and hence explains the connection between the properties of risk measures and reward-risk ratios.

Theorem 17. *(Cheridito and Kromer [2013]) Let R be of the form (2.1).*

1. *If $\mu(-X) \geq \mu(-Y)$ and $\rho(X) \leq \rho(Y)$ for all $X, Y \in \mathcal{X}$ such that $X \leq Y$, then R satisfies the monotonicity property*

2. *If μ is concave and ρ convex, then R satisfies the quasi-concavity property.*

3. *If $\rho(\lambda X) = \lambda\rho(X)$ and $\mu(\lambda X) = \lambda\mu(X)$ for all $X \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+ \setminus 0$ such that $\lambda X \in \mathcal{X}$, then R satisfies the scale-invariance property.*

4. *If μ and ρ satisfy the distribution-based property, then so does R.*

2.1 Sharpe Ratio

Suppose that $X \in \mathcal{X}$ denotes a loss random variable. We can define $\mu(-X) := E[-X]$. Then, $\mu(-X)$ is equal to the expected value of negative losses (or equivalently to the expected value of returns). Furthermore, assume that the risk measure is defined as the standard deviation of negative losses, $\sigma(X) := \|(-X) - E[-X]\|_2$. We obtain one of the widely known ratios

$$SR(X) = \frac{E[-X]^+}{\sigma(X)^+}, \tag{2.2}$$

known as the **Sharpe ratio**. This ratio was first presented in Sharpe [1966] and is the extension of the Markowitz model from Markowitz [1959]. However, in comparison to the original text, we present it in (2.2) in the form that corresponds to the generalized form of a reward-risk ratio in (2.1). This ratio allows us to find the optimal portfolio, for which is the expected return for a unit of risk (in this case, the standard deviation) maximized. In the optimization part of this thesis, we choose the Sharpe ratio as our benchmark ratio against distortion risk ratios.

2.2 Distortion Reward-Risk Ratios

Let's assume that we have two distortion functions g, h from Definition 6 such that g is convex and h is concave. Suppose that we have their associated distortion risk measures ρ_g and ρ_h such that $\rho_g(X), \rho_h(X) \in \mathbb{R}$ for all $X \in \mathcal{X}$. The assumption of convexity and concavity implies that ρ_g is concave and ρ_h convex. We will also use this fact in the reformulation of the general optimization problem in Chapter 3. Then, we obtain from Theorem 17 and Definition 6 that the distortion RRR

$$R_{g,h}(X) = \frac{\rho_g(-X)^+}{\rho_h(X)^+}, \quad X \in \mathcal{X},$$

fulfills all four properties from Definition 8.

In the following chapter, where we will optimize portfolios with respect to distortion reward-risk measures, we will choose $g(x) = x$, where $x \in [0, 1]$. This function fulfills all necessary conditions from the definition of a distortion function. In this case, it can be shown that the associated distortion risk measure reduces to $\rho_g(-X) = E[-X]$. Therefore, ρ_g is equal to the expected value of negative losses (or returns). Our distortion reward-risk ratio then reduced to the form

$$R_h(X) = \frac{E[-X]^+}{\rho_h(X)^+}, \quad X \in \mathcal{X}. \tag{2.3}$$

It must be noted that in practical applications of portfolio selection, distortion risk measures might have as well negative values (when assuming that $X \in \mathcal{X}$

is a random variable representing losses). In such cases, to ensure that the risk measure is strictly positive, it is possible to overcome this problem with the suitable transformation of data. We discuss this problem and its solution in Chapter 3.

3. Portfolio Optimization

3.1 Preliminaries

In this chapter, we introduce a reward-risk framework for the optimal portfolio selection problem based on the reward-risk distortion measures defined in the previous section. This approach is based on investors' preferences to maximize returns per unit of risk. Firstly, we will formulate a general reward-risk optimization problem.

Suppose that we have n financial securities, where $n \in \mathbb{N}$. We denote by $L = (L_1, ..., L_n)^T$ a random vector of their losses with expected value $E[L] = (E[L_1], ..., E[L_n])^T$. Let $w = (w_1, \ldots, w_n)^T \in \mathbb{R}^n$ be a vector of portfolio weights, where w_i is the portfolio weight corresponding to the ith asset, such that $w^T e = \sum_{i=1}^n w_i = 1$. The return of the whole portfolio with weights w is given by $r_p(w) = -w^T L = \sum_{i=1}^n -w_i L_i$ and we denote its expected value as $\mu_p(w) = w^T E[-L] = \sum_{i=1}^n w_i E[-L_i]$.

In some cases, the investor faces additional external constraints. For example, non-negative weights (when short sales are not allowed) or restrictions on the proportion of one asset in the whole portfolio. These constraints are often generalized by a matrix $A \in \mathbb{R}^{k \text{ x } n}$, a vector of lower bounds $Vb \in \mathbb{R}^k$, a vector of upper bounds $Ub \in \mathbb{R}^k$ and a relevant inequality including this matrix and bounds in the optimization problem.

To build our portfolio selection framework, we need to establish the following definition.

Definition 9. *(Dupacova et al. [2002]) A portfolio with weights w is called an* ***efficient portfolio*** *(with respect to the mean return μ and risk measure ρ) if there is no other portfolio with weights w^* such that $\sum_{i=1}^n w_i^* = 1, w_i^* \geq 0$ and*

$$[\rho(w^T L)) \geq \rho([w^*]^T L)) \text{ and } \mu_p(w) < \mu_p(w^*)]$$

or

$$[\rho(w^T L) > \rho([w^*]^T L) \text{ and } \mu_p(w) = \mu_p(w^*)].$$

We also define an ***efficient frontier*** *as a subset of $\mathbb{R}^2$ that contains pairs $(\rho(w^T L), \mu_p(w))$ for all efficient portfolios w.*

In other words, this definition states that there does not exist another portfolio with the higher expected return and lower or equal risk or a portfolio such that its expected return is equal, but its risk is lower (in comparison to the efficient portfolio). Furthermore, this model is based on the economic assumptions that investors behave rationally and all necessary information are equally available. We also do not allow short sales in our models.

If we choose the Sharpe ratio defined as (2.2), we can formulate our benchmark optimization problem

$$
\begin{aligned}
\underset{w}{\text{maximize}} \quad & \frac{E[-w^T L]}{\sqrt{w^T \Sigma_p w}} \\
\text{subject to} \quad & w^T e = 1 \\
& Vb \leq Aw \leq Ub,
\end{aligned} \tag{3.1}
$$

where Σ_p denotes the covariance matrix of portfolio's negative losses.

3.2 Distortion Reward-Risk Optimization

In order to be able to formulate and solve a distortion reward-risk optimization problem, we need to derive a formula for a distortion risk measure of a discrete real random variable.

Suppose that we have a discrete real random variable Y, representing losses (in percent), with possible values $y_1, \dots, y_m \in \mathbb{R}$, where $y_1 \leq y_2 \leq \dots \leq y_m$. As we need to separate these values to negative and non-negative, assume that the index $k \in \{0, \dots, m\}$ is such that values $y_1, y_2, \dots, y_k$ are negative and $y_{k+1}, \dots, y_m$ are non-negative (where for $k = 0$ we understand that all values are non-negative and for $k = m$ are all negative). For the simplicity, we assume that $\forall i \in \{1, \dots, m\}$: $P(Y = y_i) = \frac{1}{m}$. Then, we know that its cumulative distribution function is $F_Y(y) = \frac{1}{m} \sum_{i=1}^m \mathbb{1}_{\{y_i \leq y\}}$, where $\mathbb{1}_A$ denotes an indicator function of a set A. This means that $F_Y(y)$ is constant on intervals $(-\infty, y_1), [y_1, y_2), \dots, [y_m, \infty)$. Thus, from Definition 6 of a distortion measure ρ_g, we can derive that

$$
\begin{aligned}
\rho_g(Y) = -\sum_{i=1}^{k-1} (y_{i+1} - y_i) \left[1 - g\left(1 - \frac{i}{m}\right)\right] + y_k \left[1 - g\left(1 - \frac{k}{m}\right)\right] + \\
+ y_{k+1} g\left(1 - \frac{k}{m}\right) + \sum_{i=k+1}^{m-1} (y_{i+1} - y_i) g\left(1 - \frac{i}{m}\right) = \\
= y_1 + \sum_{i=1}^{m-1} (y_{i+1} - y_i) g\left(1 - \frac{i}{m}\right).
\end{aligned}
$$

Therefore, to compute distortion risk measure ρ_g for a discrete random variable Y, it is sufficient to have all the possible values y_i, where $i \in \{1, \dots, m\}$, ordered. We do not need to differentiate between non-negative and negative values. Now we can focus on the formulation of the reward-risk optimization problem.

Assume that we have $m \in \mathbb{N}$ time periods (e.g. weeks) numbered $1, \dots, m$ and $n \in \mathbb{N}$ financial assets $1, \dots, n$. Let $l = (l_{ij})_{i=1,j=1}^{n,m} \in \mathbb{R}^{n \text{ x } m}$, $m, n \in \mathbb{N}$ be a matrix, where l_{ij} represents a concrete realization of a loss of i-th financial asset at time j. Suppose that $w = (w_1, \dots, w_n)^T \in \mathbb{R}^n$ denotes weights of a portfolio associated to our financial assets such that $w^T e = 1$ and $\forall i \in \{1, \dots, n\} : w_i \geq 0$ (we do not allow short sales).

For a given vector of weights w, we can calculate a vector of losses for this portfolio as $l_p = (w^T l)^T \in \mathbb{R}^m$. Equivalently the j-th position of the vector l_p

is equal to re-weighted sum of assets' losses at time j or $\sum_{i=1}^{n} w_i l_{ij}$. However, as we see from the previous part, where we derived the formula for distortion measure, to calculate values of risk measure for different portfolios, we need to first re-order the values of l_p. Therefore, in our optimization problem we need to define a permutation matrix $P = (p_{i,j})_{i=1,j=1}^{m,m}$ consisting of 0 and 1 such that the sum in every row and column is equal to 1. Then, we can define a new vector $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ such that it has the same values as l_p, but its values are ordered from the lowest to the highest. Let Y denote a discrete loss random variable with the possible values $y_1, \ldots, y_m$, defined at the beginning of this section. We will denote the expected value of its returns (or negative losses of Y) as $\mu(-Y)$.

Furthermore, we recall that $Vb \in \mathbb{R}^k$, $A \in \mathbb{R}^{k \times n}$ and $Ub \in \mathbb{R}^k$ are additional financial constraints discussed at the beginning of this chapter. If we define a variable R representing the reciprocal value of a distortion reward-risk ratio (minimization over a reciprocal value of a reward-risk ratio is equivalent to maximization of a reward-risk ratio), we can formulate the distortion reward-risk optimization problem as

$$
\begin{aligned}
\underset{w}{\text{minimize}} \quad & R \\
\text{subject to} \quad & \rho_g(Y) = \mu(-Y) \times R \\
& \mu(-Y) \geq 0 \\
& l_p = (w^T l)^T \\
& P l_p = y \quad \text{, where } P = (p_{i,j})_{i=1,j=1}^{m,m} \\
& \sum_{i=1}^{m} p_{ij} = 1 \quad \forall j \in \{1, \ldots, m\} \\
& \sum_{j=1}^{m} p_{ij} = 1 \quad \forall i \in \{1, \ldots, m\} \\
& p_{ij} \in \{0, 1\} \quad \forall i, j \in \{1, \ldots, m\} \\
& y_1 \leq y_2 \leq \cdots \leq y_m \\
& w^T e = 1 \\
& Vb \leq Aw \leq Ub.
\end{aligned}
\tag{3.2}
$$

However, since distortion risk measures distort the expected value of losses (values of risk can be positive or negative), it might be possible in practical applications to construct portfolios with values of risk close to 0. However, this would degenerate our problem as it would not prefer portfolios with the lowest risk but portfolios with the values of risk closest to 0. Therefore, we have to modify this problem to ensure that the distortion risk measure is strictly positive. In our case, it was sufficient to transform our loss matrix l to gross losses $\tilde{l}$ by adding one (e.g. the value 1,1 represents 10% loss and value 0,9 represents 10% return). This does not affect our results because, as we explained in the first chapter, distortion risk measures are consistent with the properties of translation invariance and positive homogeneity. Therefore, in our case, we do not need any additional restrictions on returns. Thus, we obtain a reformulation of Problem (3.2):

$$
\begin{aligned}
\underset{w}{\text{minimize}} \quad & R \\
\text{subject to} \quad & \rho_g(\tilde{Y}) = \mu(\tilde{Y}_-) \times R \\
& \tilde{l}_p = (w^T \tilde{l})^T \\
& P\tilde{l}_p = \tilde{y} \quad \text{, where } P = (p_{i,j})_{i=1,j=1}^{m,m} \\
& \sum_{i=1}^{m} p_{ij} = 1 \quad \forall j \in \{1, \dots, m\} \\
& \sum_{j=1}^{m} p_{ij} = 1 \quad \forall i \in \{1, \dots, m\} \\
& p_{ij} \in \{0, 1\} \quad \forall i, j \in \{1, \dots, m\} \\
& \tilde{y}_1 \leq \tilde{y}_2 \leq \dots \leq \tilde{y}_m \\
& w^T e = 1 \\
& Vb \leq Aw \leq Ub,
\end{aligned}
\tag{3.3}
$$

where $\tilde{Y}_-$ denotes gross returns of $\tilde{Y}$ (representing gross losses).

4. Results

In this chapter, we propose results of our implementation of the distortion reward-risk optimization model. As a benchmark for the analysis, we will use Markowitz mean-variance model with its extension - the Sharpe ratio. We will compare the shape of efficient frontiers for two distortion risk measures with various parameters representing different levels of risk aversion.

4.1 Data Analysis

To demonstrate our model, we selected ten stocks (A1 - A10), which are traded at stock exchanges NYSE and Nasdaq. These stocks were selected to represent various GICS sectors. Their corresponding tickers and GICS sectors are listed in Table A.1. The historical financial data of our selected assets were imported from the Yahoo finance database. This section focuses on the implementation of distortion reward-risk measures, and therefore, we restrict to a smaller sample of weekly adjusted closing prices ranging from 2020-12-21 to 2021-02-22. A smaller sample was selected due to the computational complexity of our model, which leads to a non-linear mixed-integer optimization problem.

In Table 4.1, we present a summary of distributional statistics of selected assets, namely average weekly losses, standard deviation, skewness and kurtosis. These statistics are computed for losses (negative returns) in our chosen period and correspond to selected assets A1-A10. In this table, we can see that the asset A3 is the asset with the highest average returns. In Table 4.2, we illustrate the correlation matrix of selected assets' losses.

Asset	**Mean**	**STD**	**Skew**	**Kurt**
A1	-0,6819	4,0610	-0,3921	-0,7449
A2	-2,6650	4,7336	-0,4433	-0,4911
A3	-2,9216	4,3100	0,6384	0,1735
A4	0,0185	4,6628	-0,0319	-0,9853
A5	-1,7407	4,3618	-1,3716	1,3264
A6	0,3360	2,7173	0,6757	0,3627
A7	0,2898	2,7736	-0,8366	-0,4255
A8	-0,2696	2,7502	0,2278	-0,8388
A9	-1,5568	4,4873	0,1004	-0,9060
A10	-2,7424	5,5100	-0,2532	-0,8491

Table 4.1: A summary of distributional statistics of assets' losses

4.2 Markowitz Model

Our benchmark model, the Markowitz model with the Sharpe ratio optimization, was implemented in Python using the Problem (3.1) and solved by Sequential Least Squares Quadratic Programming (SLSQP). However, in order to be able

1	-0,288	-0,297	0,310	0,730	-0,057	0,758	0,407	-0,283	-0,278
-0,288	1	0,493	0,088	-0,010	0,813	0,033	0,446	0,264	0,404
-0,297	0,493	1	0,662	0,247	0,352	-0,055	0,090	0,749	0,741
0,310	0,088	0,662	1	0,697	0,167	0,510	0,524	0,540	0,464
0,730	-0,010	0,247	0,697	1	0,218	0,849	0,429	0,362	0,395
-0,057	0,813	0,352	0,167	0,218	1	0,267	0,477	0,180	0,494
0,758	0,033	-0,055	0,510	0,849	0,267	1	0,580	-0,019	0,035
0,407	0,446	0,090	0,524	0,429	0,477	0,580	1	-0,008	-0,008
-0,283	0,264	0,749	0,540	0,362	0,180	-0,019	-0,008	1	0,861
-0,278	0,404	0,741	0,464	0,395	0,494	0,035	-0,008	0,861	1

Table 4.2: The correlation matrix of assets' losses

to compare the results with distortion reward-risk ratios, we scale this model for weekly losses. Furthermore, in our implementation we do not allow short sales and do not assume any other financial constraints. The complete source code of our implementation can be found in the attached files.

In Figure 4.1, we present the Markowitz efficient frontier together with highlighted significant portfolios. Namely, the portfolio with the lowest standard deviation and the portfolio with the highest Sharpe ratio. Corresponding allocations of these portfolios can be found in Table 4.3, where we exclude the assets with zero allocation.

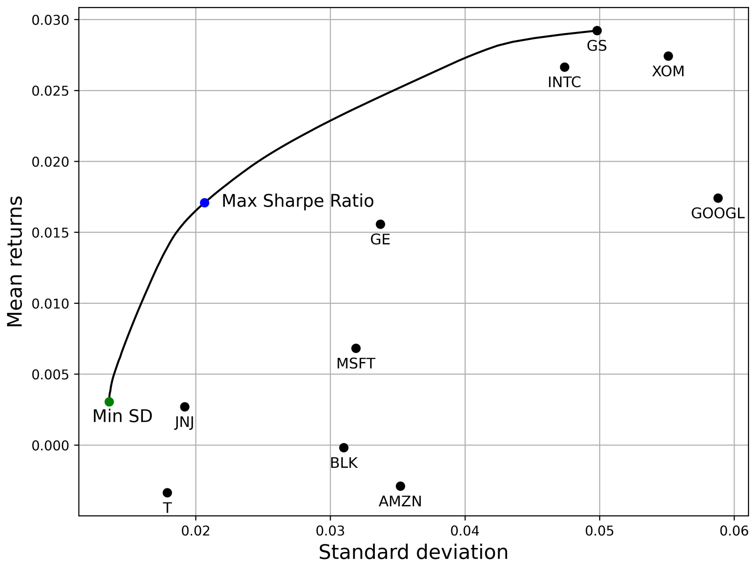


Figure 4.1: The Markowitz efficient frontier scaled for mean weekly returns. The blue point represents the efficient portfolio with respect to Sharpe ratio. Black points represent mean returns and standard deviations of our selected assets.

4.3 Distortion Risk Model

In the previous chapter, we proposed the general distortion reward-risk optimization problem for discrete random variables with the assumption of equal probabilities of their realizations. As can be seen, Problem (3.3) leads to a mixed-integer nonlinear program (MINLP). Due to the complexity of this problem, for the implementation, we had to use the specialized optimization software GAMS with the Branch-And-Reduce Optimization Navigator (BARON) (Tawarmalani and Sahinidis [2005]). All source codes can be found in the attached files. Again, we suppose that short sales are not allowed and do not assume any additional financial constraints.

In our implementation, we focused on two distortion risk measures. The Proportional Hazard transform (defined in (1.7)) for two different parameters $\gamma = 2$ and $\gamma = 5$ and the MINVAR distortion risk measure (defined in (1.8)) for two parameters $\lambda = 1$ and $\lambda = 4$. For better illustration of the position of the portfolio with the highest reward-risk ratio, we present it with resulting efficient frontiers in Figures 4.2 and 4.3 and Tables 4.4 and 4.5 with allocations of the optimal portfolios. Corresponding full allocations of assets for selected levels of returns can be found in Section A.2.

Firstly, we notice that when $\gamma \to 1_+$ or $\lambda \to 0_+$, both the Proportional Hazard transform and the MINVAR distortion function tend to the identity function. From the definition of a distortion risk measure, we immediately obtain that this risk measure is equal to the expected value of the loss random variable $X \in \mathcal{X}$. Therefore, these choices of parameters are not practical due to the fact that the optimization problem degenerates to the problem of finding portfolios with the highest mean return, as the higher mean returns represent lower values of risk.

However, for higher choices of these parameters, we obtain interesting results. As can be seen in Figure 4.2, different choices of parameter γ does not only affect the position of the efficient frontiers but influences their shape as well. This is the result of the shapes of Proportional Hazard functions depicted in Figure A.1a. As we can see, these functions assign higher values especially to lower values of x. Thus, the corresponding risk measure assigns higher probabilities to realizations with the highest losses. This effect is noticeable especially on the portfolios beyond the highest reward-risk ratio portfolio, where risks grow significantly faster than in the previous part of the efficient frontier. Therefore, different choices of parameters allow us to model various levels of risk perception and to construct optimal portfolios with respect to these levels.

Furthermore, from Tables A.2 and A.3 it can also be noticed that different choices of the parameter γ for the PH transform measure do not necessarily lead to different allocations for selected levels of returns. This effect is noticeable for levels of return equal to 2,10%, 2,30%, 2,50% and 2,70%, where we obtain equivalent weights. However, values of risks at these levels of return for different choices of γ differ. This is not surprising, as the PH transform with $\gamma = 5$ assigns higher values to all $x \in (0, 1)$ than for $\gamma = 2$. Moreover, as can be seen from Table 4.4, the optimal portfolios with the lowest risk and the highest reward-risk ratio differ significantly. Not only with respect to their values of risk but regarding their allocations as well.

Similar results are obtained for the MINVAR distortion function. In this case, different choices of parameter λ do not only lead to different values of risk but also to different allocations of optimal portfolios. These differences can be noticed from Table 4.5 and tables in Section A.2. The effect on the shapes of efficient frontiers and their positions is depicted in Figure 4.3. As we can see, the shapes of MINVAR distortion functions from A.1b are translated into the shapes of efficient frontiers. Therefore, in comparison to the PH measure, we also obtain different allocations of optimal reward-risk portfolios.

To conclude, through our formulation of the reward-risk optimization problem with the assumption of concave distortion functions (and therefore associated coherent risk measures, as we explained in Section 1.3.1), we obtain a similar framework for portfolio analysis to the Markowitz model. To be able to implement this model in real financial practice, it would be necessary to compute results for large data samples. The computational aspect is one of the limitations of distortion risk measures, as they, in the general case, lead to mixed-integer nonlinear problems. However, the class of distortion risk measures is prospective as it allows us to retain more information from (empirical) distributions of our assets in values of risk. Furthermore, the main benefit of distortion risk measures is the ability to re-weight probabilities of portfolio distributions, thus influencing shapes of constructed efficient frontiers. This enables us to control different risk aversion levels and allows us to construct optimal portfolios with respect to investors' risk aversion requirements.

The Markowitz mean-variance model										
Return	A1	A2	A3	A6	A8	A9	A10	Risk	SR	Optimum
0,30%	0,209	0	0	0,457	0,158	0,176	0	**1,36%**	0,224	Min Risk
1,71%	0,405	0,258	0,159	0	0	0,175	0,003	2,07%	**0,827**	Max SR

Table 4.3: Optimal portfolios with respect to the Markowitz mean-variance model and their Sharpe ratios scaled to mean weekly returns.

Proportional Hazard transform, $\gamma = 2$							
Return	A1	A2	A3	A10	Risk	RRR	Optimum
1,93%	0,386	0,310	0,024	0,280	**0,992774**	1,026696	Min Risk
2,68%	0	0,860	0	0,140	0,993617	**1,033354**	Max RRR

Proportional Hazard transform, $\gamma = 5$							
Return	A1	A2	A9	A10	Risk	RRR	Optimum
1,28%	0,537	0,071	0,294	0,098	**0,99964**	1,013188	Min Risk
2,54%	0,071	0,759	0	0,170	1,009303	**1,015921**	Max RRR

Table 4.4: Optimal portfolios with respect to the Proportional Hazard transform with corresponding mean returns, risks and reward-risk ratios (RRR).

MINVAR distortion function, $\lambda = 1$							
Return	A1	A2	A3	A10	Risk	RRR	Optimum
1,93%	0,399	0,264	0,187	0,150	**0,993088**	1,026426	Min Risk
2,82%	0	0,401	0,599	0	0,994221	**1,034163**	Max RRR

MINVAR distortion function, $\lambda = 4$								
Return	A1	A2	A3	A9	A10	Risk	RRR	Optimum
1,32%	0,471	0,155	0	0,374	0	**1,0021**	1,0111	Min Risk
1,90%	0,421	0,169	0,211	0	0,200	1,0047	**1,0142**	Max RRR

Table 4.5: Optimal portfolios with respect to the MINVAR distortion function with corresponding mean returns, risks and reward-risk ratios (RRR).

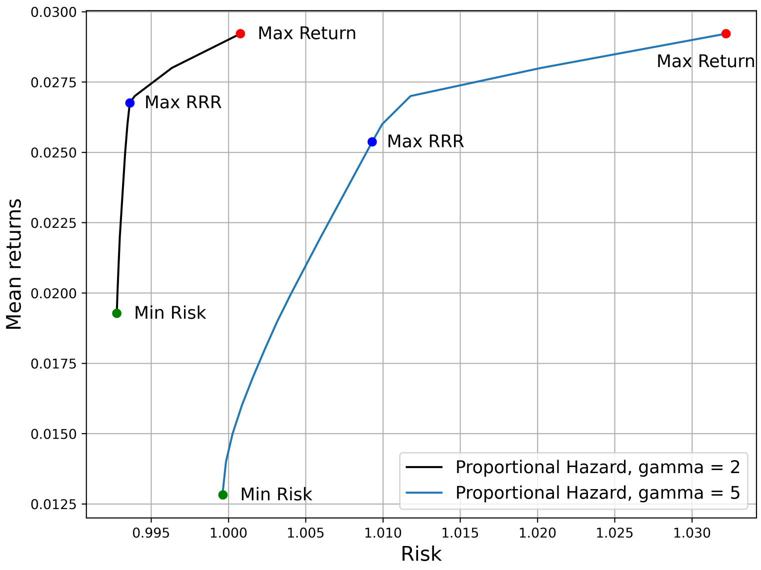


Figure 4.2: The efficient frontiers of the Proportional Hazard transform for two different choices of γ. Portfolios with the highest return, the highest reward-risk ratio and the lowest risk are highlighted. Values are calculated for mean weekly returns.

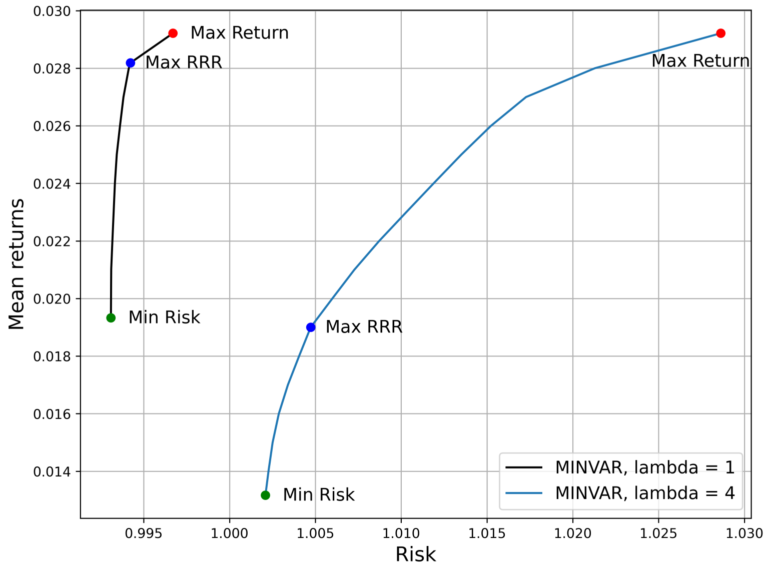


Figure 4.3: The efficient frontiers of the MINVAR distortion measure for two different choices of λ. Portfolios with the highest return, the highest reward-risk ratio and the lowest risk are highlighted. Values are calculated for mean weekly returns.

Conclusion

In this thesis, we studied numerous risk measures. In the first chapter, we presented the properties of two risk measures, Value-at-Risk and Expected Shortfall, that are often used in practice. Furthermore, the theory of distortion risk measures was also provided. We focused on the proofs of properties related to coherency axioms and stochastic dominance. The relation between this class of risk measures and Value-at-Risk and Expected Shortfall was also illustrated.

Then, the theory of general and distortion reward-risk ratios was provided, and their relation to risk measures explained. As we discussed, the framework of reward-risk ratios allows us to construct optimization problems where is reward per unit of risk maximized.

The application of distortion risk measures in optimal portfolio selection problems was presented in Chapter 3. In this chapter, we proposed the main theoretical result of this thesis, the distortion reward-risk optimization model. To our knowledge, a similar result has not yet been published in the literature. We discussed the advantages of using distortion risk measures in portfolio optimization and discussed the limitations of this framework. We managed to implement this theoretical model in the specialized optimization software GAMS. As it turned out, this problem leads to the non-linear mixed-integer optimization and therefore, to be able to fully apply this model in real financial practice, the theory behind optimal solving methods and algorithms has to be further explored.

However, as we have seen on the results computed with real financial data, the class of distortion risk measures is prospective. It allows us not only to re-weight probabilities in the distribution but enables us to translate different levels of risk aversion into portfolio optimization problems.

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List of Abbreviations

α	a confidence level
ES_α	Expected Shortfall at a confidence level α
$E[X]$	expected value of a random variable X
e	an unit vector
$F_X(x)$	a cumulative distribution function of a random variable X
g'_+	a right-side derivative of a function g
g'_-	a left-side derivative of a function g
$\mathbb{1}_A$	an indicator function of a set A
Kurt	Kurtosis
$\mathbb{N}$	a set of natural numbers
$\mathcal{N}(\mu, \sigma^2)$	a normal distribution with mean μ and variance σ^2
Nasdaq	the National Association of Securities Dealers Automated Quotations
NYSE	The New York Stock Exchange
P	a probability measure
ϕ	the density of the standard normal distribution
Φ	the distribution function of the standard normal distribution
$q_\alpha(X)$	a lower α-quantile of a random variable X
$q^\alpha(X)$	an upper quantile of a random variable X
$\mathbb{R}$	a set of real numbers
$\mathbb{R}_+$	a set of non-negative real numbers
ρ	a risk measure
ρ_g	a distortion risk measure associated to a distortion function g
RRR	a reward-risk ratio
S&P 500	The Standard and Poor's 500 stock market index
Skew	Skewness
STD	standard deviation
$S_X(x)$	a decumulative distribution function
t_n	the Student's t-distribution with n degrees of freedom
τ_n	the density of the standard t_n-distribution
VaR_α	Value-at-Risk at a confidence level α
w	portfolio weights
$\mathcal{X}$	a set of loss random variables

A. Attachments

A.1 First Attachment

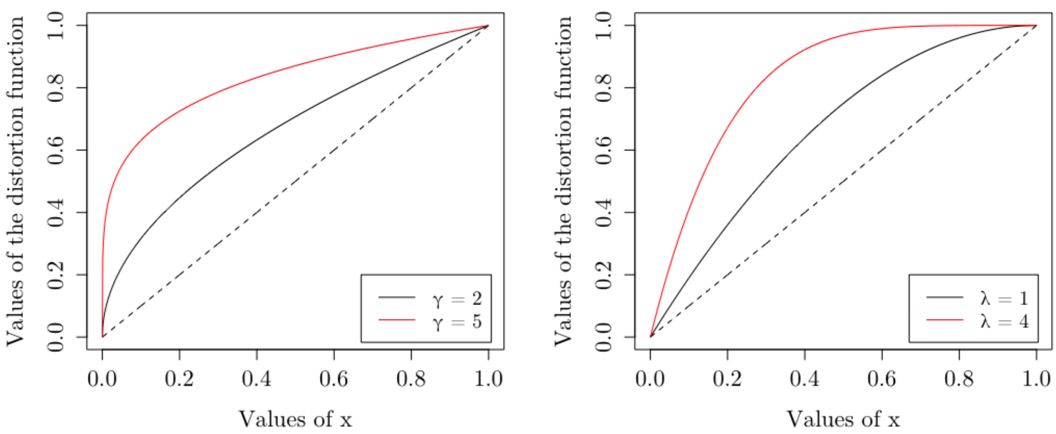
Asset	Company	Ticker	GICS Sector
A1	Microsoft Corp.	MSFT	Information Technology
A2	Intel Corp.	INTC	
A3	Goldman Sachs Group	GS	Financials
A4	BlackRock	BLK	
A5	Alphabet Inc.	GOOGL	Communication Services
A6	AT&T Inc.	T	
A7	Amazon.com, Inc.	AMZN	Consumer Discretionary
A8	Johnson & Johnson	JNJ	Health Care
A9	General Electric	GE	Industrials
A10	Exxon Mobil Corp.	XOM	Energy

Table A.1: Selected assets and their corresponding GICS sectors

A.2 Second Attachment

In the second attachment, we illustrate both distortion risk measures used in our optimization model. These measures are illustrated in Figure A.1.

Corresponding optimal allocations of selected assets with respect to the Proportional Hazard transform (1.7) are presented in Tables A.2 and A.3. Optimal allocations of assets with respect to the MINVAR distortion measure (1.8) are presented in Tables A.4 and A.5.



(a) Proportional Hazard transform (b) MINVAR distortion function

Figure A.1: Selected distortion measures for different parameters

Proportional Hazard transform, $\gamma = 2$													
Return	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	Risk	RRR	Optimum
1,93%	0,386	0,310	0,024	0	0	0	0	0	0	0,280	**0,992774**	1,026696	Min Risk
2,00%	0,347	0,369	0	0	0	0	0	0	0	0,284	0,992818	1,027378	
2,10%	0,295	0,442	0	0	0	0	0	0	0	0,263	0,992892	1,02831	
2,30%	0,193	0,587	0	0	0	0	0	0	0	0,220	0,993085	1,030123	
2,50%	0,090	0,732	0	0	0	0	0	0	0	0,178	0,993317	1,031897	
2,68%	0	0,860	0	0	0	0	0	0	0	0,140	0,993617	**1,033354**	Max RRR
2,70%	0	0,839	0,126	0	0	0	0	0	0	0,035	0,993936	1,033265	
2,80%	0	0,459	0,520	0	0	0	0	0	0	0,021	0,996339	1,031777	
2,92%	0	0	1	0	0	0	0	0	0	0	1,000773	1,028421	Max Return

Table A.2: Optimal portfolios with respect to the distortion risk measure Proportional Hazard transform for $\gamma = 2$, with corresponding mean returns, risks and reward-risk ratios (RRR).

Proportional Hazard transform, $\gamma = 5$													
Return	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	Risk	RRR	Optimum
1,28%	0,537	0,071	0	0	0	0	0	0	0,294	0,098	**0,99964**	1,013188	Min Risk
1,50%	0,603	0,006	0	0	0	0	0	0	0	0,391	1,000269	1,014727	
1,70%	0,500	0,152	0	0	0	0	0	0	0	0,348	1,00158	1,015395	
2,10%	0,295	0,442	0	0	0	0	0	0	0	0,263	1,00503	1,01589	
2,30%	0,193	0,587	0	0	0	0	0	0	0	0,220	1,006971	1,015918	
2,50%	0,090	0,732	0	0	0	0	0	0	0	0,178	1,008937	1,01592	
2,54%	0,071	0,759	0	0	0	0	0	0	0	0,170	1,009303	**1,015921**	Max RRR
2,70%	0	0,839	0,126	0	0	0	0	0	0	0,035	1,011785	1,015037	
2,92%	0	0	1	0	0	0	0	0	0	0	1,032199	0,99711	Max Return

Table A.3: Optimal portfolios with respect to the distortion risk measure Proportional Hazard transform for $\gamma = 5$, with corresponding mean returns, risks and reward-risk ratios (RRR).

MINVAR distortion function, $\boldsymbol{\lambda = 1}$													
Return	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	Risk	RRR	Optimum
1,93%	0,399	0,264	0,187	0	0	0	0	0	0	0,150	**0,993088**	1,026426	Min Risk
2,00%	0,370	0,296	0,241	0	0	0	0	0	0	0,092	0,993094	1,027093	
2,10%	0,327	0,343	0,322	0	0	0	0	0	0	0,008	0,993105	1,028089	
2,30%	0,235	0,372	0,393	0	0	0	0	0	0	0	0,993246	1,029956	
2,50%	0,142	0,401	0,457	0	0	0	0	0	0	0	0,993427	1,031782	
2,70%	0,041	0,432	0,511	0	0,016	0	0	0	0	0	0,993818	1,033388	
2,80%	0	0,417	0,571	0	0,012	0	0	0	0	0	0,994129	1,034071	
2,82%	0	0,401	0,599	0	0	0	0	0	0	0	0,994221	**1,034163**	Max RRR
2,92%	0	0	1	0	0	0	0	0	0	0	0,996703	1,03262	Max Return

Table A.4: Optimal portfolios with respect to the distortion risk measure MINVAR for $\lambda = 1$, with corresponding mean returns, risks and reward-risk ratios (RRR).

MINVAR distortion function, $\boldsymbol{\lambda = 4}$													
Return	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	Risk	RRR	Optimum
1,32%	0,471	0,155	0	0	0	0	0	0	0,374	0	**1,002091**	1,011053	Min Risk
1,50%	0,510	0,110	0,069	0	0	0	0	0	0,165	0,146	1,00251	1,012459	
1,70%	0,419	0,101	0,180	0	0	0	0	0	0,172	0,129	1,003399	1,013555	
1,90%	0,421	0,169	0,211	0	0	0	0	0	0	0,200	1,004728	**1,014207**	Max RRR
2,10%	0,326	0,092	0,205	0	0	0	0	0	0	0,377	1,00727	1,013631	
2,30%	0,218	0,251	0,145	0	0	0	0	0	0	0,386	1,010297	1,012574	
2,50%	0,105	0,488	0,067	0	0	0	0	0	0	0,339	1,013504	1,011343	
2,70%	0	0,650	0,044	0	0	0	0	0	0	0,306	1,017275	1,009559	
2,92%	0	0	1	0	0	0	0	0	0	0	1,028621	1,000579	Max Return

Table A.5: Optimal portfolios with respect to the distortion risk measure MINVAR for $\lambda = 4$, with corresponding mean returns, risks and reward-risk ratios (RRR).