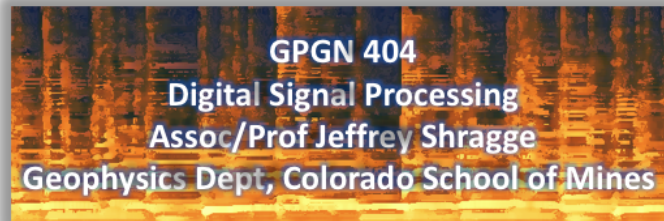
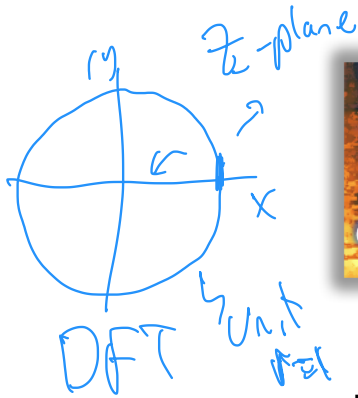


Out[2]: The raw code for this Jupyter notebook is by default hidden for easier reading. To toggle on/off the raw code, click [here](#).



$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

$n = \text{time}$   
 $k = \text{frequency}$   
Fourier kernel

## Module 12: Introduction to Z-transforms

To date we have looked at a number of transforms (e.g. Fourier Transform, Discrete Fourier Fourier) that are useful approaches for examining signals in a different domain. The purpose of this set of notes is to introduce a powerful **generalization** of these: the **Z-transform**. Studying this transform will allow us to introduce new tools for filtering signals (e.g., low-pass, high-pass, band-pass and band-reject filters).

### Z-transforms

The **Z-Transform** (<https://en.wikipedia.org/wiki/Z-transform>) is used to convert a **discrete-time** signal (i.e., a sequence of real or complex numbers) into a complex frequency domain representation. The Z-Transform, here written symbolically as  $\mathcal{Z}[\cdot]$ , is commonly given as the **bilateral** or **two-sided** transformation of a discrete time-series  $x[n]$  into the **formal power series** ([https://en.wikipedia.org/wiki/Formal\\_power\\_series](https://en.wikipedia.org/wiki/Formal_power_series))  $X(z)$  defined as:

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad (1)$$

where  $n$  is an integer and  $z$  is in general a complex number  $z = re^{j\phi} = r(\cos \phi + j \sin \phi)$ . For clarity let's explicitly put this expression into the function:

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\phi n}, \quad (2)$$

In some cases - and particularly for causal signals (i.e.,  $x[n] = 0$  for  $n < 0$ ), one often finds the **unilateral or one-sided Z-Transform**:

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} x[n]r^{-n}e^{-j\phi n}, \quad (3)$$

which is commonly used for evaluating the unit impulse response of a discrete-time **causal** system.

### Connection with Discrete Fourier Transform

An interesting immediate question is "What is the connection between the Z-transform and the DFT we studied previously?". The answer is the that Z-transform **reduces** to the DFT for scenarios obeying the following three conditions:

1. A signal where  $x[n] = 0$  for  $n < 0$  and  $n > N$ ;
2. A  $\phi$  defined as  $\phi = 2\pi k/N$ ; and
3. A  $r$  value defined as  $r = 1$ .

Given these three conditions, we **exactly** recover the **Discrete Fourier Transform (DFT)**. Thus, the **Z-Transform** can be seen as a generalization of the **DFT**. It turns out that this generalization is very powerful and leads to a whole range of tools that are useful for filtering!

**Geophysical Definition:** In geophysics the Z-transform is commonly described as a power series in  $z$  as opposed to  $z^{-1}$ . While the two equations are equivalent, the do result in a number of changes that I will mention these below. Because most of the DSP literature uses the former definition, I will use this terminology in this section.

## Example 1 - Unit Delay

**Q:** What happens when we take the Z-transform of signal  $x[n]$  ( $n \geq 0$ ) that has been delayed by  $k$  samples (i.e.,  $x[n - k]$ )?

**A:** Let's evaluate this using the definition of the Z-transform.

$$\begin{aligned}
 \mathcal{Z}\{x[n - k]\} &= \sum_{n=0}^{\infty} x[n - k] z^{-n} \\
 &= \sum_{j=-k}^{\infty} x[j] z^{-n} \quad j = n - k \\
 &= \sum_{j=-k}^{\infty} x[j] z^{-(j+k)} \\
 &= \sum_{j=-k}^{\infty} x[j] z^{-j} z^{-k} \\
 &= z^{-k} \sum_{j=-k}^{\infty} x[j] z^{-j} \\
 &= z^{-k} \sum_{j=0}^{\infty} x[j] z^{-j} \quad x[j] = 0, j < 0 \\
 &= z^{-k} X(z)
 \end{aligned} \tag{4}$$

where  $X(z)$  is the Z-transform of  $x[n]$ . Thus,  $z^{-k}$  can be interpreted as an operator that **delays** a sequence by  $k$  samples!

## Linear Constant Coefficient Difference Equations (LCCDE)

Previously in the class we mentioned linear constant coefficient difference equations (LCCDE) in the context of linear time-invariant (LTI) systems. In any LTI system, its input  $x[n]$  and output  $y[n]$  can be related via a Nth order linear constant coefficient [difference equation](https://en.wikipedia.org/wiki/Linear_difference_equation) ([https://en.wikipedia.org/wiki/Linear\\_difference\\_equation](https://en.wikipedia.org/wiki/Linear_difference_equation)):

$$\sum_{k=0}^N a_k y[n - k] = \sum_{k=0}^M b_k x[n - k]. \tag{5}$$

Let's explore what happens when we apply the Z-transform concept that we discussed above. Apply this to both side of equation 5 yields the following result:

$$\mathcal{Z} \left[ \sum_{k=0}^N a_k y[n - k] \right] = \mathcal{Z} \left[ \sum_{k=0}^M b_k x[n - k] \right]. \tag{6a}$$

$$\sum_{k=0}^N a_k \mathcal{Z}\{y[n - k]\} = \sum_{k=0}^M b_k \mathcal{Z}\{x[n - k]\}. \tag{6b}$$

$$\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k z^{-k} X(z). \tag{6c}$$

Or realizing that  $Y(z)$  and  $X(z)$  do not depend on  $k$  we write:

$$Y(z) \sum_{k=0}^N a_k z^{-k} = X(z) \sum_{k=0}^M b_k z^{-k}. \tag{7}$$

Let's now rewrite this equation by dividing both sides of equation 7 by  $X(z)$  and by the series on the left. This results in the following expression:

$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}, \tag{8a}$$

where  $H(z)$  is known as the [transfer function](https://en.wikipedia.org/wiki/Transfer_function) ([https://en.wikipedia.org/wiki/Transfer\\_function](https://en.wikipedia.org/wiki/Transfer_function)) and effectively describes how the input and output are related in Z-space:

$$Y(z) = H(z)X(z). \tag{8b}$$

Applying  $H(z)$  may also be thought of as applying a **filtering** operation.

We have seen something like this before when we discussed the convolution theorem. In the case where the Z-transform becomes a Fourier Transform the operation in equation 8b effectively represents the **convolution theorem in the frequency domain**. Thus, this may be thought of as an extension of the convolution theorem to the Z-transform.

## FIR and IIR systems

There are two classes of transfer function that are commonly applied in digital signal processing:

1. A system with coefficients  $a_k = 0, k \geq 1$  does not have feedback, and is called a **nonrecursive filter** or a [finite-impulse response \(FIR\)](https://en.wikipedia.org/wiki/Finite_impulse_response) ([https://en.wikipedia.org/wiki/Finite\\_impulse\\_response](https://en.wikipedia.org/wiki/Finite_impulse_response)) filter.
2. A system with coefficients  $a_k \neq 0, k \geq 1$  is said to have **feedback** since the current output value depends on the *previous* output values. A filter exhibiting such a characteristic is called both a **recursive filter** or an [infinite-impulse response \(IIR\)](https://en.wikipedia.org/wiki/Infinite_impulse_response) ([https://en.wikipedia.org/wiki/Infinite\\_impulse\\_response](https://en.wikipedia.org/wiki/Infinite_impulse_response)) filter.

$$\begin{aligned}
 & a_0 \neq 0, a_k = 0 \quad k \neq 0 \\
 & a_k \neq 0 \quad k \geq 1 \\
 & a_0 \neq 0
 \end{aligned}$$

## Fundamental Theorem of Algebra

The [Fundamental Theorem of Algebra](https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra) ([https://en.wikipedia.org/wiki/Fundamental\\_theorem\\_of\\_algebra](https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra)) states that

"Every non-zero, single-variable, degree  $n$  polynomial with complex coefficients has, counted with multiplicity, exactly  $n$  complex roots. The equivalence of the two statements can be proven through the use of successive polynomial division."

What is the implication here? This means that a  $n$ -order polynomial expanded about some complex value  $z_0$ ,

$$p(z) = \sum_{k=0}^n c_k (z - z_0)^k = c_0 + c_1 (z - z_0)^1 + c_2 (z - z_0)^2 + \dots + c_n (z - z_0)^n,$$

can be rewritten as

$$p(z) = a_0 \prod_{k=1}^n (z - z_k) = a_0 (z - z_1)(z - z_2) \dots (z - z_n).$$

where the  $\Pi$  represents Pi notation [e.g.,  $\Pi_{k=3}^5 k = (3)(4)(5) = 60$ ]. Thus, we can write the above transfer function  $H(z)$  as:

$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{p_0 \prod_{k=1}^M (z - p_k)}{q_0 \prod_{k=1}^N (z - q_k)}.$$

You may (should!) be concerned about what's going on in the denominator. In particular, what is happening whenever  $z = q_k$ : division by zero! Thus, we must examine the **stability** of these transfer functions as well as determine in which region of the complex plane these series converge.

## Poles and Zeros

Starting from the  $H(z)$  defined in equation 11, let's examine a few different important values.

**Zeros:** are the values of  $z$  for which  $H(z)=0$ . These occur when  $z = p_k$  in equation 11. These **do not cause** any stability issues; however, they cause certain parts of the output spectrum to be equal to zero.

**Poles:** are the values of  $z$  for which  $H(z) = \infty$ . These occur when  $z = q_k$  in equation 11. These **do cause** stability issues since division by zero will cause an infinite output!

An important concept is that by specifying the locations of where we put  $p_k$  and  $q_k$  we can design filters (e.g., low-pass, high-pass, band-pass) that will take our input data  $x[n]$  and give us the desired output signals  $y[n]$  (e.g., low-passed, high-passed or band-passed data).

**Q:** How many zeros and poles are there in equation 11?

$M$  zeros,  $N$  poles

$$y = \sum_{n=0}^{\infty} a_n x^n$$

(11a)

## Thinking about convergence

Let's first refresh ourselves about the concept of convergence and (infinite) geometric series. We can start with the following  $N$  term **finite geometric series**:

$$y = \sum_{n=0}^{N-1} a x^n = a \left( \frac{1 - x^N}{1 - x} \right),$$

where we have used a [geometric sum](https://en.wikipedia.org/wiki/Geometric_progression#Geometric_series) ([https://en.wikipedia.org/wiki/Geometric\\_progression#Geometric\\_series](https://en.wikipedia.org/wiki/Geometric_progression#Geometric_series)) to evaluate on the right hand side. There are no issues here in terms of stability (i.e.,  $y < \infty$ ) even at  $x = 1$  because this just results in a sum of  $y = aN$ .

What happens when we now consider an **infinite geometric series**?

$$y = \sum_{n=0}^{\infty} a x^n = a \left( \frac{1 - x^{\infty}}{1 - x} \right),$$

Clearly, there is now a different concern because for  $y < \infty$  we now have to put restrictions on  $|x| \leq 1$  because otherwise  $x^{\infty}$  will lead to  $y = \infty$ . The restriction  $-1 \leq x \leq 1$  is thus the **region of convergence** of this infinite sum along the real  $x$  axis.

What happens if we now have the follow **infinite geometric series**?

$$y = \sum_{n=0}^{\infty} (ax)^n = a \left( \frac{1 - (ax)^{\infty}}{1 - (ax)} \right).$$

In this case we see that  $|ax| < 1$  or better yet  $|a||x| < 1$ . Thus, for this to be stable (i.e.,  $y < \infty$ ) we need to have  $|x| < |a|^{-1}$ .



$$|x| < |a|^{-1}$$

## Region of Convergence

The region of convergence (ROC) indicates when Z-transforms of a sequence converges. Generally, there exists some  $z$  such that

$$|X(z)| = \left| \sum_{k=-\infty}^{\infty} x[k]z^{-k} \right| \rightarrow \infty \quad (12)$$

where the Z-transform **does not converge**. The set of values for  $z$  for which  $X(z)$  converges,

$$|X(z)| = \left| \sum_{k=-\infty}^{\infty} x[k]z^{-k} \right| \leq \sum_{k=-\infty}^{\infty} |x[k]z^{-k}| < \infty \quad (13)$$

is called the ROC. The ROC must be specified along with  $X(z)$  in order for the Z-transform to be considered "complete".

Assuming that  $x[n]$  is of infinite length, let's decompose  $X(z)$  as the following:

$$X(z) = X_-(z) + X_+(z), \quad (14)$$

where these two contributions are the anticausal ( $X_-(z)$ ) and causal ( $X_+(z)$ ) components of  $X(z)$ :

$$X_-(z) = \sum_{n=-\infty}^{-1} x[n]z^{-n} \quad (15a)$$

$$X_+(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \quad (15b)$$

where it is clear that the sum of equations 15a-b satisfy equation 14.

### Convergence of $X_+(z)$

For a series to converge, the series

$$\begin{aligned} X_+(z) &= x[0]z^{-0} + x[1]z^{-1} + \dots + x[n]z^{-n} + \dots \\ &= f_0(z) + f_1(z) + \dots + f_n(z) + \dots \end{aligned} \quad \begin{aligned} (16a) \\ (16b) \end{aligned}$$

has to satisfy the following [ratio test](https://en.wikipedia.org/wiki/Ratio_test) ([https://en.wikipedia.org/wiki/Ratio\\_test](https://en.wikipedia.org/wiki/Ratio_test)) behaviour:

$$\lim_{n \rightarrow \infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right| < 1. \quad (17)$$

Thus, assuming that the input sequence  $x[n]$  converges to a value finite value  $R_+$

$$\lim_{n \rightarrow \infty} \left| \frac{x[n+1]}{x[n]} \right| = R_+ \quad (18)$$

then  $X_+[z]$  will converge if

$$\lim_{n \rightarrow \infty} \left| \frac{x[n+1]z^{-n-1}}{x[n]z^{-n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x[n+1]}{x[n]} \right| |z^{-1}| < 1 \quad (19)$$

This implies that

$$|z| > \lim_{n \rightarrow \infty} \left| \frac{x[n+1]}{x[n]} \right| = R_+. \quad (20)$$

That is, the ROC for  $X_+(z)$  is

$$|z| > R_+. \quad (21)$$

### Convergence of $X_-(z)$

Assuming that the sequence converges to a non-infinite value  $R_-$

$$\lim_{m \rightarrow \infty} \left| \frac{x[-m-1]}{x[-m]} \right| = R_-, \quad (22)$$

the anticausal components will converge if

$$\lim_{m \rightarrow \infty} \left| \frac{x[-m-1]z^{m+1}}{x[-m]z^m} \right| = \lim_{m \rightarrow \infty} \left| \frac{x[-m-1]}{x[-m]} \right| |z| < 1. \quad (23)$$

This implies that

$$|z| > \lim_{m \rightarrow \infty} \left| \frac{x[-m]}{x[-m-1]} \right| = R_-. \quad (24)$$

Thus, the ROC for  $X_-(z)$  is

$$|z| < R_-. \quad (25)$$

## Combining the Results

The ROC for a infinite sequence,  $X(z)$  is given by

$$R_+ < |z| < R_- \quad (26)$$

Note that if  $R_- < R_+$  then there is **no ROC** and  $X(z)$  **does not exist**.

Let's look at this graphically:

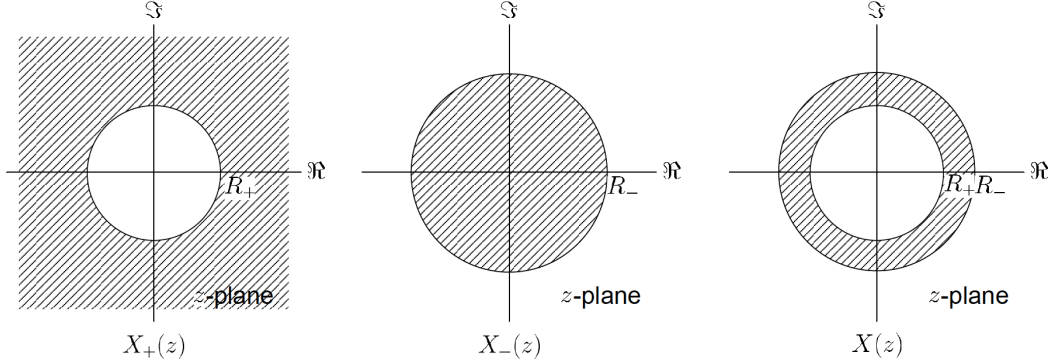


Figure 1. Illustrating the different regions of convergence for  $X_+(z)$ ,  $X_-(z)$  and  $X(z)$ .

## Example 2 - Causal geometric sequence

**Q:** Determine the z-transform of  $x[n] = a^n u[n]$  where  $u[n]$  is the unit step function.

**A:** The input function may be written as

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n. \quad (27)$$

According to the above,  $X(z)$  will converge if

$$\sum_{n=0}^{\infty} |az^{-1}|^n < \infty \quad (28)$$

Applying the **ratio test**

$$\lim_{n \rightarrow \infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right| < 1. \quad (29)$$

we get

$$\lim_{n \rightarrow \infty} \left| \frac{a^{n+1} z^{-n-1}}{a^n z^{-n}} \right| = |az^{-1}| < 1. \quad (30)$$

Thus, the convergence condition is that  $|a| < |z|$ . Thus, within this region we may use a geometric series to evaluate this infinite summation in the region where the series converges:

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1 - (az^{-1})^{\infty}}{1 - az^{-1}} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad (31)$$

where the second equality is reached in the ROC because  $az^{-1} < 1$  which means that  $(az^{-1})^{\infty} = 0$ ; the final step is just multiplying top and bottom by  $z$ .

Thus, together with the ROC, the z-transform of  $x[n] = a^n u[n]$  is:

$$X(z) = \frac{z}{z - a}, \quad |a| < |z|. \quad (32)$$

It is clear that  $X(z)$  has a **zero** at  $z = 0$  and a **pole** at  $z = a$ .

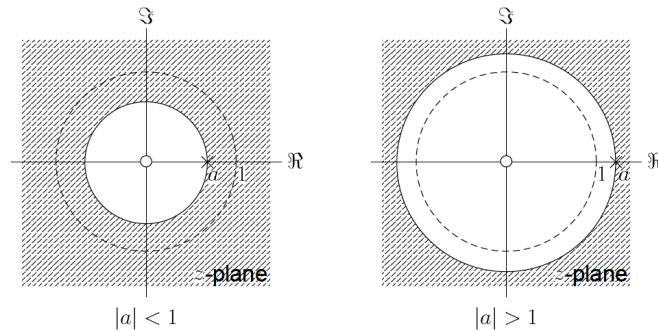


Figure 2. Plotting the ROC of  $|z| > |a|$ . Note that if  $|a| < 1$  then the Discrete Fourier Transform is guaranteed to exist. However, if  $|a| > 1$  then it is not guaranteed to exist (but it may if the input terms in  $u(z)$  fall over faster than  $a^{-1}$ ).

### Example 3 - Anticausal geometric sequence

**Q:** Determine the z-transform of  $x[n] = -b^n u[-n - 1]$ .

**A:** Let's write

$$X(z) = - \sum_{n=-\infty}^{\infty} b^n u[-n - 1] z^{-n} = - \sum_{n=-\infty}^{-1} b^n z^{-n} \quad (33)$$

Let's now identify  $m = -n$  and write

$$X(z) = - \sum_{m=1}^{\infty} b^{-m} z^m = - \sum_{m=1}^{\infty} (b^{-1} z)^m. \quad (34)$$

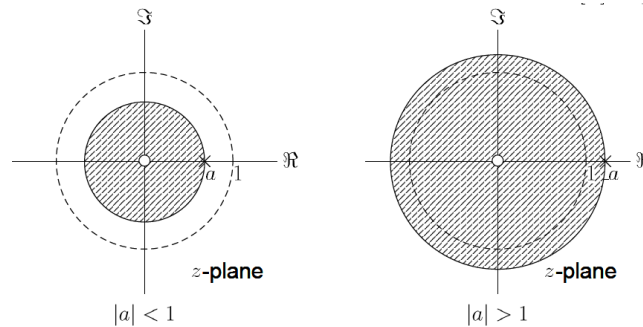
Thus, like the Example 2,  $X(z)$  converges if  $|b^{-1} z| < 1$ , or  $|b| > |z|$ . This gives:

$$X(z) = - \sum_{m=1}^{\infty} (b^{-1} z)^m = - \frac{b^{-1} z (1 - (b^{-1} z)^{\infty})}{1 - b^{-1} z} = - \frac{b^{-1} z}{1 - b^{-1} z} = - \frac{z}{b - z} = \frac{z}{z - b}. \quad (35)$$

Thus, together with the ROC, the z-transform of  $x[n] = -b^n u[-n - 1]$  is:

$$X(z) = \frac{z}{z - b} \quad |b| > |z|. \quad (36)$$

Again, it is clear that  $X(z)$  has a **zero** at  $z = 0$  and a **pole** at  $z = b$ .



**Figure 3. Plotting the ROC of  $|b| > |z|$ . Note that if  $|b| > 1$  then the Discrete Fourier Transform is guaranteed to exist. However, if  $|b| < 1$  then it is not guaranteed to exist (but it may if the input terms in  $u(z)$  fall over faster than  $b^{-1}$ ).**

### Example 4 - Combining Examples 2 and 3

**Q:** Determine the z-transform of  $x[n] = a^n u[n] + b^n u[-n - 1]$  where  $|a| < |b|$ .

**A:** Employing the previous results we have

$$X(z) = \frac{z}{z - a} - \frac{z}{z - b} \quad |a| < |z| < |b| \quad (37a)$$

$$= \frac{(a - b)z}{(z - a)(z - b)}, \quad |a| < |z| < |b| \quad (37b)$$

Note that there is still one **zero** ( $z = 0$ ) but there are now two **poles** ( $z = a$  and  $z = b$ ). Note that the ROC is given by Figure 1c where  $a = R_+$  and  $b = R_-$ .

### Example 5 - Unit Impulse

**Q:** What is the Z-transform of  $x[n] = \delta[n - p]$ .

**A:** We have

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n - p] z^{-n} = z^{-p}, \quad |z| < \infty, \quad (38)$$

which effectively states that as long as  $r^{-p} < \infty$  the Z transform will converge.

### Example 6 - Finite Geometry Series

**Q:** Determine the z-transform of  $x[n]$  which has the form

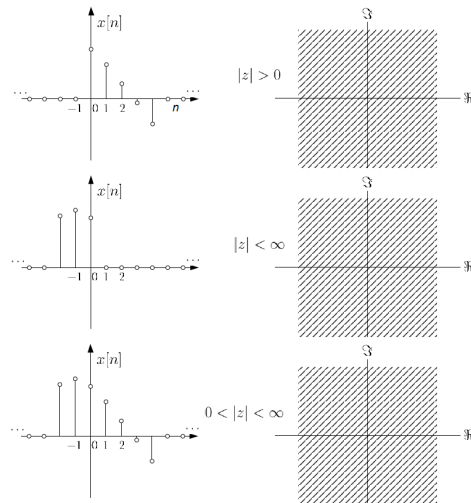
$$x[n] = \begin{cases} a^n & 0 \leq n \leq N - 1 \\ 0 & \text{otherwise} \end{cases} \quad (39)$$

**A:** We have by the geometric series

$$X(z) = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1 - (az^{-1})^N}{1 - az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}, \quad |z| > 0 \quad (40)$$

## Finite- and Infinite-duration sequences

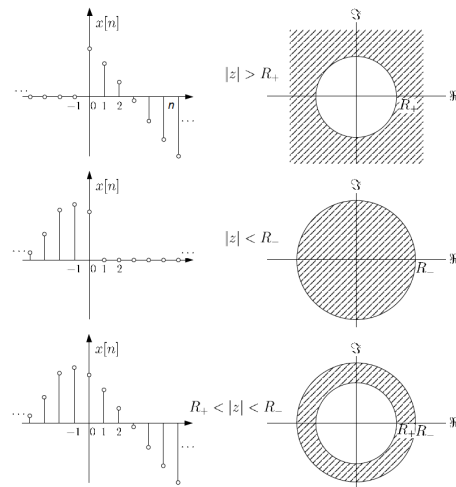
**Finite-duration sequence:** A sequence where values of  $x[n]$  are non-zero only for a finite time interval.



**Figure 4. Illustrating a number of finite-duration sequences and their region of convergence.**

Otherwise  $x[n]$  is an **infinite-duration sequence**. There are a number of different types of these infinite-duration sequences:

- **Right-sided:** If  $x[n] = 0$  for  $n < N_+$  where  $N_+$  is an integer.
- **Left-sided:** If  $x[n] = 0$  for  $n > N_-$  where  $N_-$  is an integer.
- **Two-sided:** Neither a right- nor left-sided sequence.



**Figure 5. Illustrating a number of infinite-duration sequences and their region of convergence.**