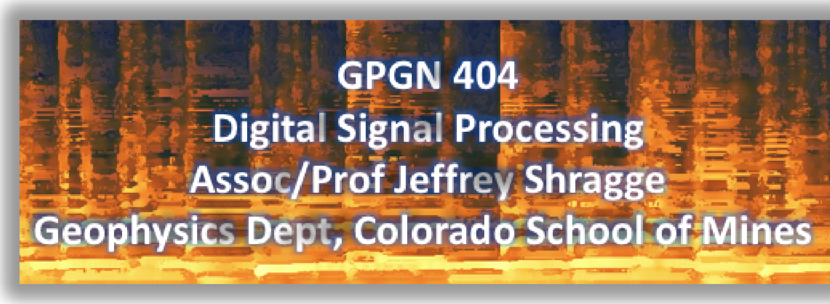
Out[1]: The raw code for this Jupyter notebook is by default hidden for easier reading. To toggle on/off the raw code, click here.



Module 12: Introduction to Z-transforms

To date we have looked at a number of transforms (e.g. Fourier Transform, Discrete Fourier Fourier) that are useful approaches for examining signals in a different domain. The purpose of this set of notes is to introduce a powerful **generalization** of these: the **Z-transform**. Studying this transform will allow us to introduce new tools for filtering signals (e.g., low-pass, high-pass, band-pass and band-reject filters).

Z-transforms

The Z-Transform (https://en.wikipedia.org/wiki/Z-transform) is used to convert a **discrete-time** signal (i.e., a sequence of real or complex numbers) into a complex frequency domain representation. The Z-Transform, here written symbolically as $\mathcal{Z}[\cdot]$, is commonly given as the **bilateral** or **two-sided** transformation of a discrete time-series $x[n]$ into the formal power series (https://en.wikipedia.org/wiki/Formal_power_series) $X(z)$ defined as:

$$X(z) = \mathcal{Z}\left\{x[n]\right\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \tag{1}$$

where n is an integer and z is in general a complex number $z = re^{i\phi} = r\left(\cos\phi + i\sin\phi\right)$. For clarity let's explicitly put this expression into the function:

$$X(z) = \mathcal{Z}\left\{x[n]\right\} = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-i\phi n}, \tag{2}$$

In some cases - and particularly for causal signals (i.e., $x[n] = 0$ for $n < 0$), one often finds the **unilateral or one-sided Z-Transform**:

$$X(z) = \mathcal{Z}\left\{x[n]\right\} = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} x[n]r^{-n}e^{-i\phi n}, \tag{3}$$

which is commonly used for evaluating the unit impulse response of a discrete-time **causal** system.

Connection with Discrete Fourier Transform

An interesting immediate question is "What is the connection between the Z-transform and the DFT we studied previously?". The answer is the that Z-transform **reduces** to the DFT for scenarios obeying the following three conditions:

1. A signal where $x[n] = 0$ for $n < 0$ and $n > N$;
2. A ϕ defined as $\phi = 2\pi k/N$; and

3. A r value defined as $r = 1$.

Given these three conditions, we **exactly** recover the **Discrete Fourier Transform (DFT)**. Thus, the **Z-Transform** can be seen as a generalization of the **DFT**. It turns out that this generalization is very powerful and leads to a whole range of tools that are useful for filtering!

Geophysical Definition: In geophysics the Z-transform is commonly described as a power series in z as opposed to z^{-1}. While the two equations are equivalent, the do result in a number of changes that I will mention these below. Because most of the DSP literature uses the former definition, I will use this terminology in this section.

Example 1 - Unit Delay

Q: What happens when we take the Z-transform of signal $x[n]$ ($n \geq 0$) that has been delayed by k samples (i.e., $x[n-k]$)?

A: Let's evaluate this using the definition of the Z-transform.

$$\begin{array}{rcll} \mathcal{Z}\{x[n-k]\} & = & \sum_{n=0}^{\infty} x[n-k]z^{-n} & \\ & = & \sum_{j=-k}^{\infty} x[j]z^{-n} & j = n-k \\ & = & \sum_{j=-k}^{\infty} x[j]z^{-(j+k)} & \\ & = & \sum_{j=-k}^{\infty} x[j]z^{-j}z^{-k} & \\ & = & z^{-k}\sum_{j=-k}^{\infty} x[j]z^{-j} & \\ & = & z^{-k}\sum_{j=0}^{\infty} x[j]z^{-j} & x[j]=0, j<0 \\ & = & z^{-k}X(z) & \end{array} \tag{4}$$

where $X(z)$ is the Z-transform of $x[n]$. Thus, z^{-k} can be interpreted as an operator that **delays** a sequence by k samples!

A consequence of this interpretation is that it can be used to represent regularly sampled time-series. For example, if we have a sequence of numbers $x[n] = [2, 4, 6, 8, 10]$ for $n = [0, 1, 2, 3, 4]$ then we can think about representing this sequence as the following $X(z)$ time-series:

$$X(z) = 2z^0 + 4z^1 + 6z^2 + 8z^3 + 10z^4, \tag{5a}$$

where the nth term in the time series corresponds to the power z^n.

Linear Constant Cofficient Difference Equations (LCCDE)

Previously in the class we mentioned linear constant coefficient difference equations (LCCDE) in the context of linear time-invariant (LTI) systems. In any LTI system, its input $x[n]$ and output $y[n]$ can be related via a Nth order linear constant coefficient [difference equation (https://en.wikipedia.org/wiki/Linear_difference_equation)](https://en.wikipedia.org/wiki/Linear_difference_equation):

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{5b}$$

Let's explore what happens when we apply the Z-transform concept that we discussed above. Apply this to both side of equation 5 yields the following result:

$$\mathcal{Z}\left[\sum_{k=0}^{N} a_k y[n-k]\right] = \mathcal{Z}\left[\sum_{k=0}^{M} b_k x[n-k]\right]. \tag{6a}$$

$$\sum_{k=0}^{N} a_k \mathcal{Z}\left[y[n-k]\right] = \sum_{k=0}^{M} b_k \mathcal{Z}\left[x[n-k]\right]. \tag{6b}$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z). \tag{6c}$$

Or realizing that $Y(z)$ and $X(z)$ do not depend on k we write:

$$Y(z)\sum_{k=0}^{N} a_k z^{-k} = X(z)\sum_{k=0}^{M} b_k z^{-k}. \tag{7}$$

Let's now rewrite this equation by dividing both sides of equation 7 by $X(z)$ and by the series on the left. This results in the following expression:

$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}, \tag{8a}$$

where $H(z)$ is known as the [transfer function (https://en.wikipedia.org/wiki/Transfer_function)](https://en.wikipedia.org/wiki/Transfer_function) and effectively describes how the input and output are related in Z-space:

$$Y(z) = H(z)X(z). \tag{8b}$$

Applying $H(z)$ may also be thought of as applying a **filtering** operation.

We have seen something like this before when we discussed the convolution theorem. In the case where the Z-transform becomes a Fourier Transform the operaion in equation 8b effectively represents the **convolution theorem in the frequency domain**. Thus, this may be thought of as an extension of the convolution theorem to the Z-transform.

FIR and IIR systems

There are two classes of transfer function that are commonly applied in digital signal processing:

1. A system with coefficients $a_k = 0, k \geq 1$ does not have feedback, and is called a **nonrecursive filter** or a [finite-impulse response (FIR) (https://en.wikipedia.org/wiki/Finite_impulse_response)](https://en.wikipedia.org/wiki/Finite_impulse_response) filter.
2. A system with coefficients $a_k \neq 0, k \geq 1$ is said to have **feedback** since the current output value depends on the *previous* output values. A filter exhibiting such a characteristic is called both a **recursive filter** or an [infinite-impulse response (IIR) (https://en.wikipedia.org/wiki/Infinite_impulse_response)](https://en.wikipedia.org/wiki/Infinite_impulse_response) filter.

Fundamental Theorem of Algebra

The [Fundamental Theorem of Algebra (https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra)](https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra) states that

"Every non-zero, single-variable, degree n polynomial with complex coefficients has, counted with multiplicity, exactly n complex roots. The equivalence of the two statements can be proven through the use of successive polynomial division."

What is the implication here? This means that a n-order polynomial expanded about some complex value z_0,

$$p(z) = \sum_{k=0}^{n} c_k (z-z_0)^k = c_0 + c_1(z-z_0)^1 + c_2(z-z_0)^2 + \cdots + c_n(z-z_0)^n, \tag{9}$$

can be rewritten as

$$p(z) = a_0 \Pi_{k=1}^{n}(z-z_k) = a_0(z-z_1)(z-z_2)\cdots(z-z_n). \tag{10}$$

where the Π represents Pi notation [e.g., $\Pi_{k=3}^{5} k = (3)(4)(5) = 60$]. Thus, we can write the above transfer function $H(z)$ as:

$$H(z) \equiv \frac{Y(z)}{X(z)} = \frac{p_0 \Pi_{k=1}^{M}(z-p_k)}{q_0 \Pi_{k=1}^{N}(z-q_k)}. \tag{11}$$

You may (should!) be concerned about what's going on in the denominator. In particular, what is happening whenever $z = q_k$: division by zero! Thus, we must examine the **stability** of these transfer functions as well as determine in which region of the complex plane these series converge.

Poles and Zeros

Starting from the $H(z)$ defined in equation 11, let's examine a few different important values.

Zeros: are the values of z for which $H(z) = 0$. These occur when by $z = p_k$ in equation 11. These **do not cause** any stability issues; however, they cause certain parts of the output spectrum to be equal to zero.

Poles: are the values of z for which $H(z) = \infty$. These occur when $z = q_k$ in equation 11. These **do cause** stability issues since division by zero will cause an infinite output!

An important concept is that by specifying the locations of where we put p_k and q_k we can design filters (e.g., low-pass, high-pass, band-pass) that will take our input data $x[n]$ and give us the desired output signals $y[n]$ (e.g., low-passed, high-passed or band-passed data).

Q: How many zeros and poles are there in equation 11?

A:: There are M zeros and N poles.

Thinking about convergence

Let's first refresh ourselves about the concept of convergence and (infinite) geometric series. We can start with the following N term **finite geometric series**:

$$y = \sum_{n=0}^{N-1} ax^n = a\left(\frac{1-x^N}{1-x}\right), \tag{11a}$$

where we have used a [geometric sum (https://en.wikipedia.org/wiki/Geometric_progression#Geometric_series)](https://en.wikipedia.org/wiki/Geometric_progression#Geometric_series) to evaluate on the right hand side. There are no issues here in terms of stability (i.e., $y < \infty$) even at $x = 1$ because this just results in a sum of $y = aN$.

What happens when we now consider an **infinite geometric series**?

$$y = \sum_{n=0}^{\infty} ax^n = a\left(\frac{1-x^\infty}{1-x}\right), \tag{11b}$$

Clearly, there is now a different concern because for $y < \infty$ we now have to put restrictions on $|x| \leq 1$ because otherwise x^∞ will lead to $y = \infty$. The restriction $-1 \leq x \leq 1$ is thus the **region of convergence** of this infinite sum along the real x axis.

What happens if we now have the follow **infinite geometric series**?

$$y = \sum_{n=0}^{\infty} (ax)^n = a\left(\frac{1-(ax)^\infty}{1-(ax)}\right). \tag{11c}$$

In this case we see that $|ax| < 1$ or better yet $|a||x| < 1$. Thus, for this to be stable (i.e., $y < \infty$) we need to have $|x| < |a|^{-1}$.

Region of Convergence

The region of convergence (ROC) indicates when Z-transforms of a sequence converges. Generally, there exists some z such that

$$|X(z)| = \left|\sum_{k=-\infty}^{\infty} x[n]z^{-n}\right| \rightarrow \infty \tag{12}$$

where the Z-transform **does not converge**. The set of values for z for which $X(z)$ converges,

$$|X(z)| = \left|\sum_{k=-\infty}^{\infty} x[n]z^{-n}\right| \leq \sum_{k=-\infty}^{\infty} \left|x[n]z^{-n}\right| < \infty \tag{13}$$

is called the ROC. The ROC must be specified along with $X(z)$ in order for the Z-transform to be considered "complete".

Assuming that $x[n]$ is of infinite length, let's decompose $X(z)$ as the following:

$$X(z) = X_-(z) + X_+(z), \tag{14}$$

where these two contributions are the anticausal ($X_-(z)$) and causal ($X_+(z)$) components of $X(z)$:

$$X_-(z) = \sum_{n=-\infty}^{-1} x[n]z^{-n} \tag{15a}$$

$$X_+(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \tag{15b}$$

where it is clear that the sum of equations 15a-b satisfy equation 14.

Convergence of $X_+(z)$

For a series to converge, the series

$$X_+(z) = x[0]z^{-0} + x[1]z^{-1} + \cdots + x[n]z^{-n} + \ldots \tag{16a}$$

$$= f_0(z) + f_1(z) + \cdots + f_n(z) + \ldots \tag{16b}$$

has to satisfy the following [ratio test (https://en.wikipedia.org/wiki/Ratio_test)](https://en.wikipedia.org/wiki/Ratio_test) behaviour:

$$\lim_{n\to\infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right| < 1. \tag{17}$$

Thus, assuming that the input sequence $x[n]$ converges to a value finite value R_+

$$\lim_{n\to\infty} \left| \frac{x[n+1]}{x[n]} \right| = R_+ \tag{18}$$

then $X_+[z]$ will converge if

$$\lim_{n\to\infty} \left| \frac{x[n+1]z^{-n-1}}{x[n]z^{-n}} \right| = \lim_{n\to\infty} \left| \frac{x[n+1]}{x[n]} \right| \left| z^{-1} \right| < 1 \tag{19}$$

This implies that

$$|z| > \lim_{n\to\infty} \left| \frac{x[n+1]}{x[n]} \right| = R_+. \tag{20}$$

That is, the ROC for $X_+(z)$ is

$$|z| > R_+. \tag{21}$$

Convergence of $X_-(z)$

Assuming that the sequence converges to a non-infinite value R_-

$$\lim_{m\to\infty} \left| \frac{x[-m-1]}{x[-m]} \right| = R_-, \tag{22}$$

the anticausal components will conver if

$$\lim_{m\to\infty} \left| \frac{x[-m-1]z^{m+1}}{x[-m]z^{m}} \right| = \lim_{m\to\infty} \left| \frac{x[-m-1]}{x[-m]} \right| |z| < 1. \tag{23}$$

This implies that

$$|z| > \lim_{m\to\infty} \left| \frac{x[-m]}{x[-m-1]} \right| = R_-. \tag{24}$$

Thus, the ROC for $X_-(z)$ is

$$|z| < R_-. \tag{25}$$

Combining the Results

The ROC for a infinite sequence, $X(z)$ is given by

$$R_+ < |z| < R_-. \tag{26}$$

Note that if $R_- < R_+$ then there is **no ROC** and $X(z)$ **does not exist**.

Let's look at this graphically:

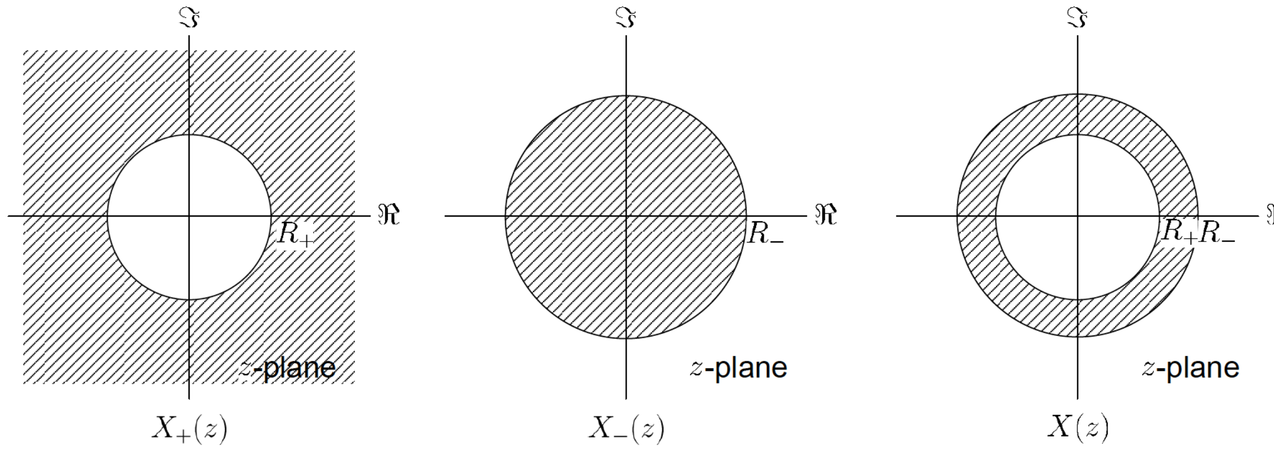


Figure 1. Illustrating the different regions of convergence for $X_+(z)$, $X_-(z)$ and $X(z)$.

Example 2 - Causal geometric sequence

Q: Determine the z-transform of $x[n] = a^n u[n]$ where $u[n]$ is the unit step function.

A: The input function may be written as

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n. \tag{27}$$

According to the above, $X(z)$ will converge if

$$\sum_{n=0}^{\infty} \left|az^{-1}\right|^n < \infty \tag{28}$$

Applying the **ratio test**

$$\lim_{n\to\infty} \left|\frac{f_{n+1}(z)}{f_n(z)}\right| < 1. \tag{29}$$

we get

$$\lim_{n\to\infty} \left|\frac{a^{n+1} z^{-n-1}}{a^n z^{-n}}\right| = \left|az^{-1}\right| < 1. \tag{30}$$

Thus, the convergence condition is that $|a| < |z|$. Thus, within this region we may use a geometric series to evaluate this infinite summation in the region where the series converges:

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1 - \left(az^{-1}\right)^{\infty}}{1 - az^{-1}} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \tag{31}$$

where the second equality is reached in the ROC because $az^{-1} < 1$ which means that $(az^{-1})^{\infty} = 0$; the final step is just multiplying top and bottom by z.

Thus, together with the ROC, the z-transform of $x[n] = a^n u[n]$ is:

$$X(z) = \frac{z}{z - a}, \qquad |a| < |z|. \tag{32}$$

It is clear that $X(z)$ has a **zero** at $z = 0$ and a **pole** at $z = a$.

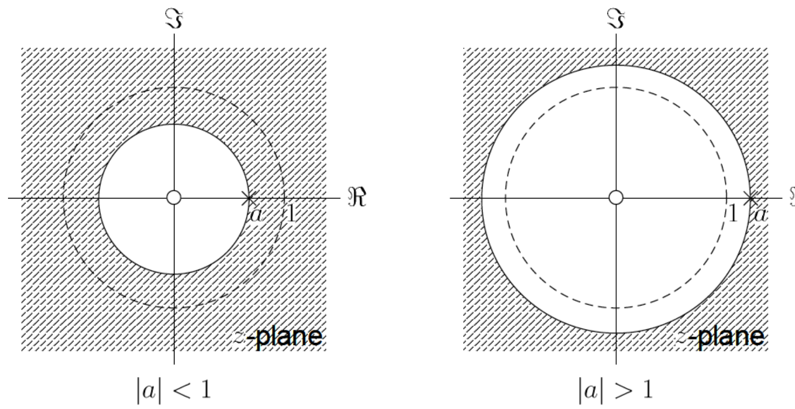


Figure 2. Plotting the ROC of $|z| > |a|$. Note that if $|a| < 1$ then the Discrete Fourier Transform is guaranteed to exist. However, if $|a| > 1$ then it is not guaranteed to exist (but it may if the input terms in $u(z)$ fall over faster than a^{-1}).

Example 3 - Anticausal geometric sequence

Q: Determine the z-transform of $x[n] = -b^n u[-n-1]$.

A: Let's write

$$X(z) = -\sum_{n=-\infty}^{\infty} b^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} b^n z^{-n} \tag{33}$$

Let's now identify $m = -n$ and write

$$X(z) = -\sum_{m=1}^{\infty} b^{-m} z^m = -\sum_{m=1}^{\infty} \left(b^{-1}z\right)^m. \tag{34}$$

Thus, like the Example 2, $X(z)$ converges if $\left|b^{-1}z\right| < 1$, or $|b| > |z|$. This gives:

$$X(z) = -\sum_{m=1}^{\infty} \left(b^{-1}z\right)^m = -\frac{b^{-1}z(1-(b^{-1}z)^{\infty})}{1-b^{-1}z} = -\frac{b^{-1}z}{1-b^{-1}z} = -\frac{z}{b-z} = \frac{z}{z-b}. \tag{35}$$

Thus, together with the ROC, the z-transform of $x[n] = -b^n u[-n-1]$ is:

$$X(z) = \frac{z}{z-b} \quad |b| > |z|. \tag{36}$$

Again, it is clear that $X(z)$ has a **zero** at $z = 0$ and a **pole** at $z = b$.

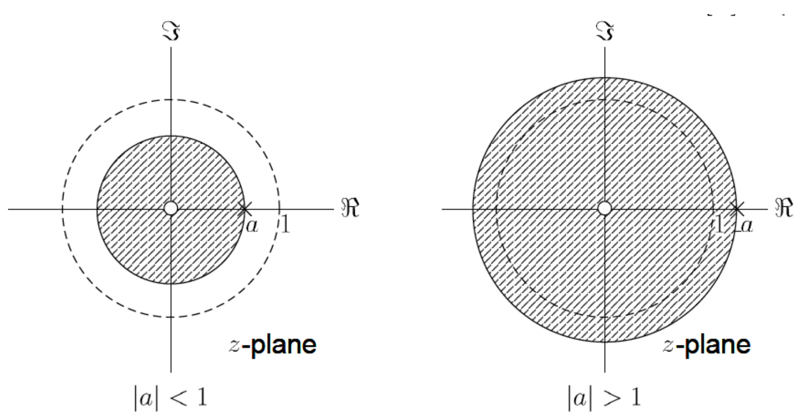


Figure 3. Plotting the ROC of $|b| > |z|$. Note that if $|b| > 1$ then the Discrete Fourier Transform is guaranteed to exist. However, if $|b| < 1$ then it is not guaranteed to exist (but it may if the input terms in $u(z)$ fall over faster than b^{-1}).

Example 4 - Combining Examples 2 and 3

Q: Determine the z-transform of $x[n] = a^n u[n] + b^n u[-n-1]$ where $|a| < |b|$.

A: Employing the previous results we have

$$X(z) = \frac{z}{z-a} - \frac{z}{z-b} \quad |a| < |z| < |b| \tag{37a}$$

$$= \frac{(a-b)z}{(z-a)(z-b)}, \quad |a| < |z| < |b| \tag{37b}$$

Note that there is still one **zero** ($z = 0$) but there are now two **poles** ($z = a$ and $z = b$). Note that the ROC is given by Figure 1c where $a = R_+$ and $b = R_-$.

Example 5 - Unit Impulse

Q: What is the Z-transform of $x[n] = \delta[n-p]$.

A: We have

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n-p]z^{-n} = z^{-p}, \quad |z| < \infty, \tag{38}$$

which effectively states that as long as $r^{-p} < \infty$ the Z transform will converge.

Example 6 - Finite Geometry Series

Q: Determine the z-transform of $x[n]$ which has the form

$$x[n] = \begin{cases} a^n & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \tag{39}$$

A: We have by the geometric series

$$X(z) = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1 - \left(az^{-1}\right)^N}{1 - az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}, \quad |z| > 0 \tag{40}$$

Note that when $a = z$ then the first summation term reduces to a sum of N ones.

Finite- and Infinite-duration sequences

Finite-duration sequence: A sequence where values of $x[n]$ are non-zero only for a finite time interval.

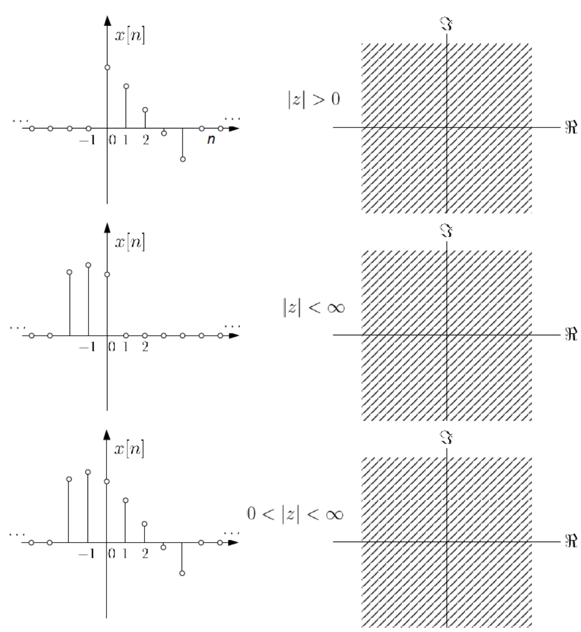


Figure 4. Illustrating a number of *finite-duration* sequences and their region of convergence.

Otherwise $x[n]$ is an **infinite-duration sequence**. There are a number of different types of these infinite-duration sequences:

- **Right-sided**: If $x[n] = 0$ for $n < N_+ < \infty$ where N_+ is an integer.
- **Left-sided**: If $x[n] = 0$ for $n > N_- > -\infty$ where N_- is an integer.
- **Two-sided**: Neither a right- nor left-sided sequence.

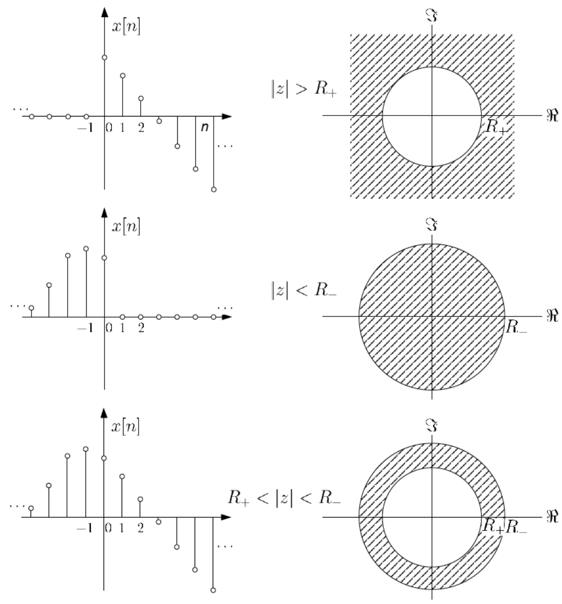


Figure 5. Illustrating a number of *infinite-duration* sequences and their region of convergence.