GPGN 404
Digital Signal Processing
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Module 3: Complex Numbers

This module focuses on some of the fundamental mathematical skills that are required for the material in this course. Starting from elementary arithmatic, we will refresh our memory of the complex plane; complex exponentials, and multiplication/division and roots; and finally complex functions and the concept of analycity.

Motivation for Being in the Complex Plane

In many instances in geophysical digitial signal processing we apply integral transforms (https://en.wikipedia.org/wiki/Integral_transform) that project our geophysical data from one domain to another. The reason why we do this is that many of the problems we are trying to solve (e.g., filtering out unwanted components, signal-noise separation, etc) are easier to solve in the transform domain. Ideally, the integral transform is **invertible** in the sense that it has an **inverse transform** that allows you to bring your (filtered, separated, etc) data back to the original domain without losing any of its information content.

Forward and inverse integral transforms

Mathematically, we may write the **integral transform** as the following:

$$(Tg)(u) = \int_{t_1}^{t_2} g(t)\, K(t,u)\, \mathrm{d}t, \tag{1}$$

where the input is a function $g(t)$ of variable t, and the output is another function Tg of transform variable u. Often we symbolize the output of this transform as $\widehat{G}(u)$. There are numerous useful integral transforms, where each is specified by a choice of the integration kernel function (https://en.wikipedia.org/wiki/Integral_transform) $K(u,t)$ of two variables - t and u - as well as the limits applied - t_1 and t_2.

Some kernels have an associated inverse kernel $K^{-1}(t,u)$, which roughly speaking, delivers an **inverse transform**:

$$g(t) = \int_{u_1}^{u_2} (Tg)(u)\, K^{-1}(t,u)\, \mathrm{d}u. \tag{2}$$

when applied with the appropriate limits u_1 and u_2. There are many different integral transforms (https://en.wikipedia.org/wiki/Integral_transform#Table_of_transforms) that are used across a whole range of the physical sciences and mathematics.

Fourier and Inverse Fourier Transform Pair

The most important set of integral transforms that you will likely experience over your career is the Forward and Inverse Fourier Transform pair (denoted $\mathcal{F}$ and $\mathcal{F}^{-1}$). By defining $u = f$, we see that the kernel functions for these transforms, $K_{\mathcal{F}}(f,t)$ and $K^{-1}_{\mathcal{F}^{-1}}(f,t)$, are given by:

$$K_{\mathcal{F}}(f,t) = e^{-i2\pi ft} \quad \text{and} \quad K^{-1}_{\mathcal{F}^{-1}}(f,t) = e^{i2\pi ft}. \tag{3}$$

In this course, we will sometimes use the **angular frequency** definition $\omega = 2\pi f$ so that we may write

$$K_{\mathcal{F}}(\omega,t) = e^{-i\omega t} \quad \text{and} \quad K^{-1}_{\mathcal{F}^{-1}}(\omega,t) = e^{i\omega t}. \tag{4}$$

When we include the limits $t_1 = u_1 = -\infty$ and $t_2 = u_2 = \infty$ we may write:

$$\widehat{G}(\omega) \equiv (\mathcal{F}g)(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-i\omega t}\, \mathrm{d}t \quad \text{and} \quad g(t) = \int_{-\infty}^{\infty} (\mathcal{F}g)(\omega)\, e^{i\omega t}\, \mathrm{d}\omega = \int_{-\infty}^{\infty} \widehat{G}(\omega)\, e^{i\omega t}\, \mathrm{d}\omega. \tag{5}$$

The above Fourier Transform process looks pretty straightforward to compute. However, there are a number of fundamental questions that need to be addressed:

(1) Does the formulation work for all $g(t)$?

- **NO!** There are some restrictions on the types of $g(t)$ to which we can apply the Fourier Transform. For example, does it have "finite" (good!) or "infinite" (bad!) energy in the signal?

(2) Is it possible to specify some $g(t)$ for which the Fourier Transform "fails"?

- **YES!** It is actually very easy to specify a function that would "fail" - e.g., $g(t) = e^{-t}(1-t)^{-1}$ around $t = 1$. This is quite common when we are using designer functions (i.e., "filters") to do specific filtering tasks.

(3) Are we able to analyze these types of signals?

- **YES!** Just because a signal is not well defined on the unit circle in the complex plane does not mean we have to give up. We have to expand our ability to analyze signals in complex plane beyond the unit circle. Doing so leads to some course material that requires us to be comfortable with complex numbers and some of the fundamental principles of complex analysis. Hopefully, you have encountered these topics in your previous courses and just need a short review.

Complex Numbers: A Review

You have probably encountered the expression $i = \sqrt{-1}$ with the imaginary unit many times before - likely first with certain solutions to the quadratic equation:

$$ax^2 + bx + c = 0, \tag{6}$$

given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{7}$$

When $b^2 < 4ac$ and we can find no real number equal to x. The solutions in this case are termed **complex numbers**. We have also assumed the existence of certain arithmetic operations on complex number that largely parallel our experience with real numbers (generally due to some sort of *mathematical symmetry*). However, we do not need to delve into such a detailed discussion for this course.

Elementary Arithmetic

We first consider simple powers of the imaginary unit $i = \sqrt{-1}$:

- $i^0 = 1$
- $i^1 = i$
- $i^2 = \sqrt{-1}\sqrt{-1} = -1$
- $i^3 = i^2 i = -i$
- $i^4 = i^2 i^2 = (-1)(-1) = 1$
- $i^5 = i$

> Note: squaring i causes an effective positive rotation of the output. Applying a square root or other fractional power will naturally cause a negative rotation of the output.

When performing the operations of addition, subtraction, multiplication and division we treat i as though it were an algebraic constant. For example, if $z_1 = 3 - 2i$ and $z_2 = 2 - i$ then:

$$\begin{aligned}
\text{Addition}: \quad & z_1 + z_2 = (3 + 2i) + (2 - i) = 5 + i \\
\text{Subtraction}: \quad & z_1 - z_2 = (3 + 2i) - (2 - i) = 1 + 3i \\
\text{Multiplication}: \quad & z_1 z_2 = (3 + 2i)(2 - i) = 6 - 3i + 4i - 2i^2 = 6 + i + 2 = 8 + i
\end{aligned} \tag{9}$$

Division requires remembering a simple trick. To compute $\frac{z_1}{z_2} = \frac{3+2i}{2-i}$ we first multiply both the numerator and denominator by the **complex conjugate** (i.e., if $z = a + ib$ then $\overline{z} = \overline{a + ib} = a - ib$ where $[a, b] \in \mathfrak{R}$ and the overbar indicates a complex conjugate) of the term in the denominator. Thus,

$$\text{Division}: \quad \frac{z_1}{z_2} = \frac{3 + 2i}{2 - i} = \frac{(3 + 2i)(2 + i)}{(2 - i)(2 + i)} = \frac{4 + 7i}{5}. \tag{10}$$

This process is also called **rationalizing the denominator**.

We often want to extract out just the real or imaginary components of a complex number, which is written notationally using the **real and imaginary operators**, $\mathfrak{R}$ and $\mathfrak{I}$, respectively. Thus, if $z = a + ib$ we have the following:

$$\mathfrak{R}(z) = a \tag{10a}$$

and

$$\mathfrak{I}(z) = b \tag{10b}$$

Note that the result of either operation is a **real number**.

QUICK PROBLEMS:

(1) Let $z = 3 - 3i$ and $w = 1 + \sqrt{3}i$. Find the result of the following operations:

(a) $z + w$

(b) $z - w$

(c) $z\,w$

(d) $z\,/\,w$

Let's test our answers using the **sympy** package (symbolic python):

```
Q1a: z+w= 4.0 - 3.0*I + 1.0*sqrt(3)*I
Q1b: z-w= 2.0 - 3.0*I - 1.0*sqrt(3)*I
Q1c: z*w= (1 + 1.0*sqrt(3)*I)*(3.0 - 3.0*I)
Q1d: z/w= (3.0 - 3.0*I)/(1 + 1.0*sqrt(3)*I)
```

(2) Let $z = a + ib$ and $w = c + id$, where $[a, b, c, d] \in \mathcal{R}$ (i.e. are real numbers). Show that:

(a) $\overline{z + w} = \overline{z} + \overline{w}$

(b) $\overline{z - w} = \overline{z} - \overline{w}$

(c) $\overline{z\,w} = \overline{z}\,\overline{w}$

(d) $\overline{z / w} = \overline{z} / \overline{w}$

Let's again test our answers using python:

```
Test 2a
z1=conj(z+w)       = conjugate(a) - 1.0*I*conjugate(b) + conjugate(c) - 1.0*I*conjugate(d)
z2=conj(z)+conj(w)= conjugate(a) - 1.0*I*conjugate(b) + conjugate(c) - 1.0*I*conjugate(d)
Does z1=z2? True

Test 2b
z1=conj(z-w)       = conjugate(a) - 1.0*I*conjugate(b) - conjugate(c) + 1.0*I*conjugate(d)
z2=conj(z)-conj(w)= conjugate(a) - 1.0*I*conjugate(b) - conjugate(c) + 1.0*I*conjugate(d)
Does z1=z2? True

Test 2c
z1=conj(z*w)       = (conjugate(a) - 1.0*I*conjugate(b))*(conjugate(c) - 1.0*I*conjugate(d))
z2=conj(z)*conj(w)= (conjugate(a) - 1.0*I*conjugate(b))*(conjugate(c) - 1.0*I*conjugate(d))
Does z1=z2? True

Test 2d
z1=conj(z/w)       = (conjugate(a) - 1.0*I*conjugate(b))/(conjugate(c) - 1.0*I*conjugate(d))
z2=conj(z)/conj(w)= (conjugate(a) - 1.0*I*conjugate(b))/(conjugate(c) - 1.0*I*conjugate(d))
Does z1=z2? True
```

The Complex Plane

We are familiar with the real line as a geometric device for visualizing real numbers. Since the complex number $z = x + iy$ is actually two dimensional, it is not surprising that many consider complex numbers as points on a plane. For example, the point $z = 3 + 4i$ can be represented on the complex plane by:

```
/Users/jeffreyshragge/miniconda3/lib/python3.5/site-packages/matplotlib/figure.py:98: MatplotlibDeprecationWarnin
g:
Adding an axes using the same arguments as a previous axes currently reuses the earlier instance.  In a future ver
sion, a new instance will always be created and returned.  Meanwhile, this warning can be suppressed, and the futu
re behavior ensured, by passing a unique label to each axes instance.
  "Adding an axes using the same arguments as a previous axes "
```
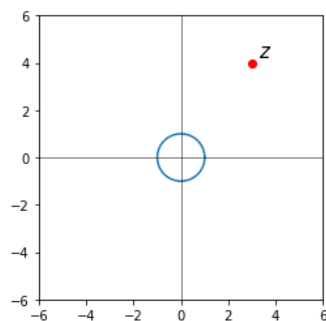
Polar representation of complex numbers

This point can be thought as having Cartesian coordinates $[x, iy] = [3, 4]$ or as the vector drawn from the origin to this point. The complex number $z = 3 + 4i$ can also be defined by **polar coordinate** variables, r and θ, which represent the vector magnitude from the origin of the complex plane and the angle made with respect to the x-axis, respectively. Recall that for $z = x + iy$ we have the following forward and inverse coordinate transformations

Forward (Cartesian to Polar)

$$r = \sqrt{x^2 + y^2}, \tag{11a}$$

$$\theta = \operatorname{atan}\left(\frac{y}{x}\right). \tag{11b}$$

> Note: this is sometimes called atan2 in programming languages and requires two inputs - *atan2(y,x)* to put the angle in the correct quadrant. Using an *atan* function (if defined) will give you the angle in the first quadrant, which is probably not what you are looking for!

Inverse (Polar to Cartesian)

$$x = r\cos\theta, \tag{11c}$$
$$y = r\sin\theta. \tag{11d}$$

Thus, for the example $z = 3 + 4i$ we have $r = \sqrt{3^2 + 4^2} = 5$ and $\theta = \operatorname{atan}(4/3) = 0.93$ radians. As in analytic geometry θ is not unique and we can add to θ any multiple of 2π radians:

$$\theta = 0.93 + 2\pi n, \quad n = 0, \pm1, \pm2, \ldots \tag{11e}$$

We often call r the **modulus** of z and denote it by $|z|$. The angle θ is called the **argument** of z and is denoted $arg(z) = \theta$ or sometimes just $\angle(z) = \theta$. It would come as no surprise that complex numbers can be represented by vectors, and that the geometric description of addition and subtraction of vectors is similar to that you would have experienced in other courses.

To help you visualize the unit circle, here is a diagram with various key angles:
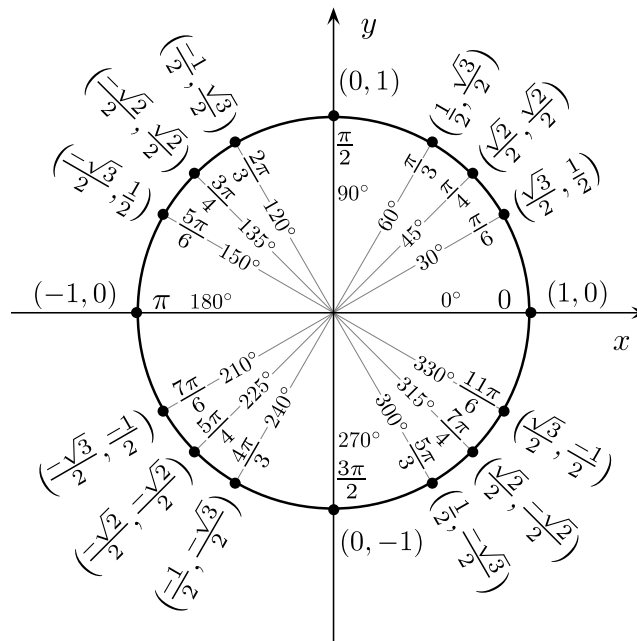

Figure 2. Illustrating various key angles in unit circle. The values at these angles would be good to memorize for this course.

QUICK PROBLEMS

(3) Express the following numbers in the form $re^{i\theta}$ with $r > 0$ and $0 \le \theta < 2\pi$.

- $5i$
- $-7 + 7i$
- $-4 - 4\sqrt{3}i$
- -1

(4) Express the following numbers in the form $z = x + iy$

- $2e^{i3\pi}$
- $e^{i13\pi/4}$
- $7e^{-i\pi/6}$
- $e^{-i\pi/2}$

Regions in the Complex Plane

In this course we will sometimes discuss **regions within the complex plane**. For example:

- (1) a disk contained within the unit circle centered at $c = 1 + 2i$
- (2) the complement of the disk in (1)
- (3) the right half plane
- (4) the upper half plane

Let's again use the sympy library to plot out these regions (though it is difficult to visually see if the sets are open or closed).
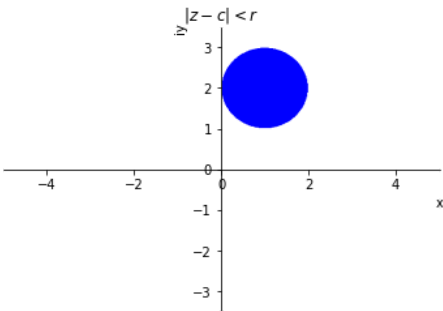

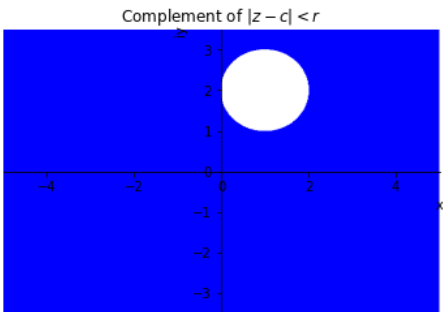**Figure 3. Plot showing a disk contained within the unit circle centered at $c = 1 + 2i$.**

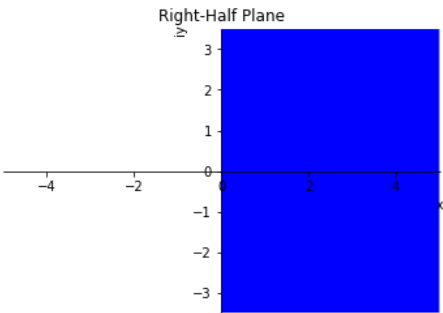**Figure 4. Plot showing the complement of the disk shown in Figure 3.**

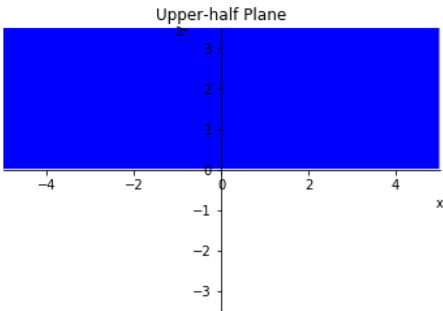**Figure 5. Plot showing the right-half plane.**

Figure 6. Plot showing the upper-half plane.

Note that there is a difference between an **open set** and a **closed set**. An open set is a 2D region with the boundary points removed. For example, the inequality $|z| < r$ would be an open set while $|z| \leq r$ would be a closed set.

QUICK PROBLEMS:

(5) What would the following regions look like:

- An annulus defined by $r < |z - c| < R$

- A infinite open sector defined by $\alpha < arg(z) < \beta$

- A finite open sector defined by $\alpha < arg(z) < \beta$ and $r < |z| < R$

(6) Describe the following regions:

- $|3z - 6| \leq 9$

- $\left| \frac{z-1}{1+z} \right| < 1$

- $|z| \leq |1 - z|$

The Complex Exponential

In previous courses you would have seen that functions of e^x, $\sin \theta$ and $\cos \theta$ can be expanded in power series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \mathcal{O}(x^5) \tag{12}$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} + \mathcal{O}(x^{11}) \tag{13}$$

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \mathcal{O}(x^{10}) \tag{14}$$

where, e.g., $\mathcal{O}(x^5)$ means "terms of order x^5 and greater", and the ! operator is factorial. If we now set $x = i\theta$ and substitute this into e^x we get:

$$\begin{aligned} e^{i\theta} &= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \mathcal{O}\left(x^6\right) \\ &= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \mathcal{O}\left(x^6\right) \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \mathcal{O}\left(x^6\right) \right) + i \left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \mathcal{O}\left(x^7\right) \right) \\ &= \cos \theta + i\sin \theta. \end{aligned} \tag{15}$$

This is **Euler's formula** that defines $e^{i\theta}$ for any given real-valued θ as a vector of unit length making angle θ with the real axis. Alternatively, the Euler Formulae for the cos θ and sin θ functions in terms of complex exponentials are given by:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{16}$$

Note that multiplying $e^{i\theta}$ by r gives $re^{i\theta}$ which is a complex number having modulus r and argument θ. Thus, $z = re^{i\theta}$ is convenient when we wish to display the **polar coordinates** of z and the notation $z = x + iy$ is convenient when we want to display the **Cartesian components** of z. For example, all of the following are equivalent representations:

$$1 + i = \sqrt{2}e^{i\pi/4} = -\sqrt{2}e^{i5\pi/4} = -\sqrt{2}e^{-i3\pi/4}. \tag{17}$$

Geometric aspects of Multiplication and Division

Based on the rules from the study of exponents involving real numbers, it is fairly straightforward to suppose that the following operations involving imaginary exponents will hold: $e^{ia}e^{ib}$, e^{ia}/e^{ib} and $(e^{ia})^b$ where $[a, b] \in \Re$.

To illustrate this, let's define $z = re^{i\theta}$ and $w = se^{i\gamma}$ where $[r, s, \theta, \gamma] \in \Re$. We can then specify the following multiplication and division operations:

Multiplication: We obtain the following:

$$z\,w = r\,e^{i\theta} s\,e^{i\gamma} = r\,s\,e^{i\theta}\,e^{i\gamma} = r\,s\,e^{i(\theta+\gamma)}. \tag{18}$$

The last argument tells us that the modulus is $|z\,w| = |z|\,|w|$ and the argument is $arg(zw) = arg(z) + arg(w)$.

Division: We obtain the following:

$$\frac{z}{w} = \frac{r\,e^{i\theta}}{s\,e^{i\gamma}} = \frac{r}{s}\,e^{i(\theta-\gamma)}, \tag{19}$$

where the modulus is given by $|z\,/\,w| = |z|\,/\,|w|$ and the argument by $arg(z\,/\,w) = arg(z) - arg(w)$.

QUICK PROBLEMS:

(7) Prove **de Moivre's Theorem**:

$$(\cos\,\theta + i\sin\,\theta)^n = \cos\,n\theta + i\sin\,n\theta, \tag{20}$$

where n is an integer. (Hint: it's straightforward with Euler's Identity. You can also do this more formally through an inductive proof that does not rely on Euler's Identity.)

(8) By squaring $e^{i\theta} = \cos\theta + i\sin\theta$, find formulas for $\cos 2\theta$ and $\sin 2\theta$ in terms of $\cos\theta$ and $\sin\theta$.

(9) Find formulas for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Roots of a Complex Number

Suppose we are asked for the cube root of $8i$. We would seek seek z such that $z^3 = 8i$. First, we recognize that the **principle value** of $i = e^{i\pi/2}$. Recognizing that we can have $2\pi n$ multiples of this (where n is integer) allows us to write:

$$8i = 8e^{i(\pi/2 + 2\pi n)}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{21}$$

Thus, we can enforce the cube root on both sides:

$$\begin{aligned}
z^3 &= 8e^{i(\pi/2 + 2\pi n)} \\
z &= \left(8e^{i(\pi/2 + 2\pi n)}\right)^{\frac{1}{3}} \\
z &= 8^{\frac{1}{3}}\left(e^{i(\pi/2 + 2\pi n)}\right)^{\frac{1}{3}} \\
z &= 2e^{i(\pi/6 + 2\pi/3n)}, \quad n = 0, \pm 1, \pm 2, \ldots
\end{aligned} \tag{22}$$

It looks like there are an infinite number of different values! However, let's look at this in complex plane using an Argand Diagram (http://mathworld.wolfram.com/ArgandDiagram.html). In the example below, I have defined an *argand(a)* function that will plot the location of a given set of points z on the complex plane and then draw the vector from the origin. In this example, I am plotting the points in the above example from $n = 0$ to $n = \pm 50$.
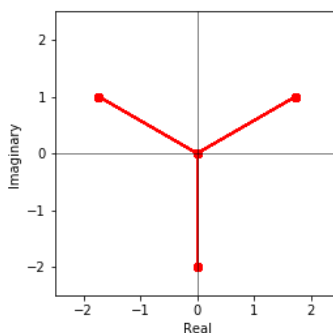

Figure 7. Plotting the three cube roots of $z = 8i$ using the defined Argand function.

In fact, as illustrated in Figure 7, there are only three distinct answers! This should not be surprising since $z^3 = 8i$ is a cubic equation (https://en.wikipedia.org/wiki/Cubic_function) and therefore we should expect exactly three solutions.

QUICK PROBLEMS

(10) Solve the following equations and show the answers on an Argand diagram:

- $z^2 + 4 = 0$
- $z^3 - 8 = 0$
- $z^4 = 81(1 + i)/\sqrt{2}$
- $z^3 + i = 0$
- $z^6 = -64$

Complex Functions

Complex analysis is concerned with complex functions that are differentiable in some defined domain. However, we should define what we mean by **complex function**. A function f on a complex domain S assigns to every $z \in S$ a complex number w called the value of f at z:

$$w = f(z). \tag{23}$$

For example, $w = f(z) = z^2 + 3z$ is a complex function defined for all of z, such that the domain S is the whole complex plane. Generally speaking, w is itself a complex number, $w = u + iv$, where both u and v can be dependent on $z = x + iy$. Thus, we write:

$$w = f(z) = u(x, y) + iv(x, y) \tag{24}$$

where $u(x, y)$ and $v(x, y)$ are functions of two real variables.

EXAMPLE

Q1: If $w = f(z) = z^2$, what are the associated $u(x, y)$ and $v(x, y)$?

A1: Because $z = x + iy$ we may write the following:

$$w = z^2 = (x + iy)^2 = x^2 + 2ixy + (iy)^2 = x^2 - y^2 + 2ixy$$

Therefore, by inspection we see that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

Limits and Continuity

A function $f(z)$ is said to have the **limit** l as z approaches some z_0:

$$\lim_{z \to z_0} f(z) = l, \tag{25}$$

if and only if f is defined in a neighbourhood of z_0 and if the values of f are "close" to l for all z "close" to z_0. More precisely, in the disk $|z - z_0| < \delta$ we have $|f(z) - l| < \epsilon$. This is similar to how limits are defined in the calculus of 1D real variables; however, one big difference is that z may approach z_0 **from any direction** (i.e., in θ, not just from the left (-) and right (+) along the real axis).

A function $f(z)$ is said to be **continuous** at $z = z_0$ if and only if $f(z_0)$ is defined and

$$\lim_{z \to z_0} f(z) = f(z_0). \tag{26}$$

$f(z)$ is said to be **continuous in a domain** if it is continuous at each point in the domain.

EXAMPLE

Q1: Is $f(z) = \sum_{n=1}^{10} z^n$ continuous at $z = 0$?

A1: Let's look at the following limit:

$$\lim_{z \to 0} \sum_{n=1}^{10} z^n = \lim_{r \to 0} \sum_{n=1}^{10} (re^{i\theta})^n = \lim_{r \to 0} \sum_{n=1}^{10} r^n e^{in\theta} = 0.$$

Thus, because the limit approaches $l = 0$, the function is continuous at $z = 0$.

QUICK PROBLEMS

(11) Find out (and give a reason) whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function is equal to:

- $\dfrac{\Im(z)}{(1 - |z|)}$

- $\dfrac{\Im(z^2)}{|z|^2}$

(Hint: Use the polar notation, assert $r \to 0$ and see if this is valid for all θ).

Derivative of complex function

Provided that the limit exists, the **derivative** of a complex function f at a point z_0 is written $f'(z_0)$ and is defined by

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{28}$$

Then $f(z)$ is said to be **differentiable** at z_0. If we write $\Delta z = z - z_0$ then we can change the limit argument and denominator as follows:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z_0 + \Delta z) - f(z_0)}{z - z_0}. \tag{29}$$

Note that differentiability of f at z_0 requires along whatever path z approaches z_0 the expressions always approaches a certain value and that all these values are equal.

EXAMPLES

Q1: Is the function $f(z) = z^2$ differentiable for all z? If yes, what is its derivative?.

A1: Yes, the function $f(z) = z^2$ is differentiable for all z and has the derivative $f'(z) = 2z$ because:

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + \Delta z^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z. \tag{30}$$

Q2: Is the function $f(z) = \overline{z}$ differentiable for all z? If yes, what is its derivative?.

A2: No, the function $f(z) = \overline{z}$ is not differentiable because:

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}. \tag{31}$$

If $\Delta y = 0$ and we approach z_0 along Δx then this expression is equal to 1; however, if $\Delta x = 0$ and we approach z_0 along Δy then the expression is -1. Thus, the limit does not exist and $\overline{z}$ is not differentiable.

QUICK PROBLEMS

(12) Are the following functions differentiable? If so, what is the derivative?:

- $f(z) = \left(z^3 + i\right)^2$

- $f(z) = \frac{i}{(1-z)^2}$

Analytic Functions

Complex analysis is concerned with the theory and application of analytic functions (https://en.wikipedia.org/wiki/Analytic_function) - i.e., functions that are differentiable in some domain such that we can "do calculus". An **analytic function** is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points in D. The function $f(z)$ is said to be **analytic** at a point $z = z_0$ in D if f(z) is analytic in a neighbourhood of z_0.

Importantly, non-negative integer powers $\{1, z, z^2, z^3, \dots\}$ are analytic in the complex plane and so are polynomials (https://en.wikipedia.org/wiki/Polynomial) of the form:

$$f(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \dots + c_n z^n = \sum_{n=0}^{N} c_n z^n. \tag{32}$$

The quotient of two polynomials $g(z)$ and $h(z)$:

$$f(z) = \frac{g(z)}{h(z)} \tag{33}$$

is called a rational function (https://en.wikipedia.org/wiki/Rational_function). Rational function $f(z)$ is analytic except at points where $h(z) = 0$ [also called the poles of $f(z)$]. We also can assume that many common factors of of g and h cancel out. Rational functions will be quite important in the context of designing filters in the later sections of this course (Modules 6 and 7).

Cauchy-Riemann Equations

Finally, we are going to conclude this overview of complex numbers with a brief discussion of the Cauchy-Riemann equations (https://en.wikipedia.org/wiki/Cauchy%E2%80%93Riemann_equations), which are arguably the most important equations in complex analysis (https://en.wikipedia.org/wiki/Complex_analysis). These equations provide a test criterion for the analycity of a complex function $w = f(z) = u(x, y) + iv(x, y)$. Roughly, f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{34a}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{34b}$$

everywhere in domain D.

Q1: Show that $f(z) = z^2$ is analytic for all z.

A1: First, we observe that $z^2 = x^2 - y^2 + i2xy$ and thus $u = x^2 - y^2$ and $v = 2xy$. We then apply the Cauchy-Riemann test:

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = -2y, \tag{35a}$$

$$\frac{\partial v}{\partial x} = 2y, \qquad \frac{\partial v}{\partial y} = 2x. \tag{35b}$$

Thus, $f(z) = z^2$ satifies the Cauchy-Riemann equations and is analytic for all of z.

QUICK PROBLEMS

(13) Determine whether the following functions are analytic:

- $f(z) = z^4$

- $f(z) = e^{2x} \left(\cos y + i \sin y \right)$