3.2 Discrete-Time Signals - Problems

The purpose of this document is to give you a number of problems to solve regarding the folowing topics: linearity, time-invariance, memory/causality, and stability. They involve analysing the a system

$$y[n] = \mathcal{T}\left(x[n]\right), \tag{1}$$

where $x[n]$ is the input, $\mathcal{T}(\cdot)$ is some operator applied to the input, and $y[n]$ is the output response.

Linearity

Determine which of the systems below are **linear**.

(Q1-1)

$$y[n] = \log(x[n]) \tag{2}$$

A: If the system is homogeneous then

$$y[n] = \mathcal{T}(cx[n]) = c\mathcal{T}(x[n]) \tag{3}$$

for any input $x[n]$ and for all complex constants c. The system $y[n] = \log(x[n])$ is **not homogeneous** because the response of the system to $x_1[n] = cx[n]$ is

$$y_1[n] = \log(x_1[n]) = \log(cx[n]) = \log(c) + \log(x[n]) \neq c\log(x[n]). \tag{4}$$

To be additive, the response to $x[n] = x_1[n] + x_2[n]$ must be $y[n] = y_1[n] + y_2[n]$. For this system we have

$$\mathcal{T}(x[n]) = \mathcal{T}(x_1[n] + x_2[n]) = \log(x_1[n] + x_2[n]) \neq \log(x_1[n]) + \log(x_2[n]). \tag{5}$$

Thus, the system is **not additive**. Overall the system is **non-linear**.

(Q1-2)

$$y[n] = 6x[n+2] + 4x[n+1] + 2x[n] \tag{6}$$

A: If the system is homogeneous than

$$y[n] = \mathcal{T}(cx[n]) = c\mathcal{T}(x[n]) \tag{7}$$

Thus for $x_1[n] = c\,x[n]$ we get

$$y_1[n] = 6x_1[n+2] + 4x_1[n+1] + 2x_1[n] = c6x[n+2] + c4x[n+1] + c2x[n] = c(6x[n+2] + 4x[n+1] + 2x[n]) =$$

Thus, the system is **homogeneous**.

To be additive, the response to $x[n] = x_1[n] + x_2[n]$ must be $y[n] = y_1[n] + y_2[n]$. For this system we have

$$\begin{aligned}
\mathcal{T}(x_1[n] + x_2[n]) &= 6(x_1[n+2] + x_2[n+2]) + 4(x_1[n+1] + x_2[n+1]) + 2(x_1[n] + x_2[n]) && \text{(9a)}\\
&= (6x_1[n+2] + 4x_1[n+1] + 2x_1[n]) + (6x_2[n+2] + 4x_2[n+1] + 2x_2[n]) && \text{(9b)}\\
&= y_1[n] + y_2[n] && \text{(9c)}
\end{aligned}$$

Thus, the system is **additive**. Overall, because the system is **additive and homogeneous** it is **linear**.

(Q1-3)

$$y[n] = 6x[n+2] + 4x[n+1] + 2x[n] + 1 \tag{10}$$

A: If the system is homogeneous than

$$y[n] = \mathcal{T}(cx[n]) = c\mathcal{T}(x[n]) \tag{11}$$

Thus for $x_1[n] = c\,x[n]$ we get

$$
\begin{aligned}
y_1[n] &= 6x_1[n+2] + 4x_1[n+1] + 2x_1[n] + 1, && (12a)\\
&= c6x[n+2] + c4x[n+1] + c2x[n] + 1, && (12b)\\
&= c(6x[n+2] + 4x[n+1] + 2x[n]) + 1. && (12c)
\end{aligned}
$$

However,

$$
cy[n] = c(6x[n+2] + 4x[n+1] + 2x[n] + 1) \tag{13}
$$

Thus, the system is **not homogeneous**.

To be additive, the response to $x[n] = x_1[n] + x_2[n]$ must be $y[n] = y_1[n] + y_2[n]$. For this system we have

$$
\begin{aligned}
\mathcal{T}(x_1[n] + x_2[n]) &= 6(x_1[n+2] + x_2[n+2]) + 4(x_1[n+1] + x_2[n+1]) + 2(x_1[n] + x_2[n]) + 1\\
&= (6x_1[n+2] + 4x_1[n+1] + 2x_1[n]) + (6x_2[n+2] + 4x_2[n+1] + 2x_2[n]) + 1\\
&\neq y_1[n] + y_2[n] = (6x_1[n+2] + 4x_1[n+1] + 2x_1[n] + 1) + (6x_2[n+2] + 4x_2[n+1] + 2x_2[n] +\\
&= (6x_1[n+2] + 4x_1[n+1] + 2x_1[n]) + (6x_2[n+2] + 4x_2[n+1] + 2x_2[n]) + 2
\end{aligned}
$$

Thus, the system is **not additive**. Thus, because the system is both **not additive and homogeneous**, it is a **non-linear** system.

(Q1-4)

$$
y[n] = \sin\left(\frac{n\pi}{2}\right) x[n] \tag{15}
$$

A: Let's examine homogeneity and additivity at the same time. Let $y_1[n]$ and $y_2[n]$ the outputs to $x_1[n]$ and $x_2[n]$, respectively. The response of the system to

$$
x[n] = a\,x_1[n] + b\,x_2[n] \tag{16}
$$

is

$$
\begin{aligned}
y[n] &= \sin\left(\frac{n\pi}{2}\right) x[n] && (17a)\\
&= \sin\left(\frac{n\pi}{2}\right) (a\,x_1[n] + b\,x_2[n]) && (17b)\\
&= \sin\left(\frac{n\pi}{2}\right) a\,x_1[n] + \sin\left(\frac{n\pi}{2}\right) bx_2[n] && (17c)\\
&= a\,y_1[n] + b\,y_2[n] && (17d)
\end{aligned}
$$

where

$$
y_1[n] = \sin\left(\frac{n\pi}{2}\right) x_1[n] \quad \text{and} \quad y_2[n] = \sin\left(\frac{n\pi}{2}\right) x_2[n]. \tag{18}
$$

Thus, this system is both **homogeneous** and **additive**, and is **linear**.

Time Invariance

Determine which of the systems below are **shift-invariant**:

(Q2-1)

$$
y[n] = x[n] + x[n-1] + x[n-2] \tag{19}
$$

A: Let $y[n]$ be the response of the system to an arbitrary input $x[n]$. To test for shift invariance we want to compare the shifted response $y[n-n_0]$ to the shifted input $x[n-n_0]$. With

$$
y[n] = x[n] + x[n-1] + x[n-2], \tag{20}
$$

we have, for the shifted response $n \to n - n_0$:

$$
y[n-n_0] = x[n-n_0] + x[n-1-n_0] + x[n-2-n_0]. \tag{21}
$$

Now, the response of the system to $x_1[n] = x[n-n_0]$ is

$$
\begin{aligned}
y_1[n] &= x_1[n] + x_1[n-1] + x_1[n-2], && (22a)\\
&= x[n-n_0] + x[n-n_0-1] + x[n-n_0-2], && (22b)\\
&= y[n-n_0]. && (22c)
\end{aligned}
$$

Because $y_1[n] = y[n-n_0]$, the system is **time invariant**.

(Q2-2)

$$
y[n] = \sum_{k=-\infty}^{n} x[k] \tag{23}
$$

A: Let's shift $n \to n - n_0$ in the equation above:

$$
y[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k] \tag{24}
$$

Now, let's define a system

$$
y_1[n] = \sum_{k=-\infty}^{n} x_1[k] \tag{25}
$$

where $x_1[n] = x[n-n_0]$. The response of the system to the shifted input is

$$
\begin{aligned}
y_1[n] &= \sum_{k=-\infty}^{n} x_1[k], && (26a)\\
&= \sum_{k=-\infty}^{n} x[k-n_0], && (26b)\\
&= \sum_{l+n_0=-\infty}^{n} x[l], \quad l = k - n_0 && (26c)\\
&= \sum_{l=-\infty}^{n-n_0} x[l], && (26d)\\
&= y[n-n_0]. && (26e)
\end{aligned}
$$

Thus, because $y_1[n] = y[n-n_0]$ the system is **shift-invariant**.

(Q2-3)

$$
y[n] = x[n^2] \tag{27}
$$

A: The system is **not shift-invariant**, which may be demonstrated with a simple counterexample. Note that if $x[n] = \delta[n]$ then the response will be $y[n] = \delta[n^2] = \delta[n]$. However, if $x_1[n] = \delta[n-2]$, then the response will be $y_1[n] = x_1[n^2] = \delta[n^2 - 2] = 0$, which is not equal to $y[n-2]$. Thus, the system is **not shift-invariant**.

(Q2-4)

$$
y[n] = x[-n] \tag{28}
$$

A: Let $x[n]$ be the input and $y[n] = x[-n]$ be the response. If we shift the index according to $n \to n - n_0$, we obtain:

$$
y[n-n_0] = x[-(n-n_0)] = x[-n+n_0]. \tag{29}
$$

However, if we consider the shifted input $x_1[n] = x[n-n_0]$, we find that the response is

$$
y_1[n] = x_1[-n] = x[-n-n_0] \neq y[n-n_0]. \tag{30}
$$

Therefore, because $y_1[n] \neq y[n-n_0]$ this system is **shift-variant**.

Causal / Memory

Determine which of the following systems are **causal**, and which have **memory**?

(Q3-1)

$$y[n] = x^2[n]u[n] \tag{31}$$

A: The system $y[n] = x^2[n]u[n]$ is **memoryless** because the response of the system at time n depends only on the input at time n and no other values of the input. Therefore, this system is **causal**.

(Q3-2)

$$y[n] = x[|n|] \tag{32}$$

A: The output $y[n] = x[|n|]$ is an example of a non-causal system. This can be observed when looking at the response of the system for $n < 0$. For example, for $n = -1$ we get $y[-1] = x[1]$ and for n=1 we get $y[1] = x[1]$. Therefore the output of the system is dependent on both future and current input. Therefore, the system is **anticausal**.

(Q3-3)

$$y[n] = x[n] + x[n-3] + x[n-1] \tag{33}$$

A: This system is **causal** because all of the input required to compute the response at n are at or before n.

(Q3-4)

$$y[n] = x[n] - x[n^2 - n] \tag{34}$$

A: Let's evaluate a few values to see what the response is:

$$\begin{aligned} y[-1] &= x[-1] - x[2] && \text{(35a)} \\ y[0] &= x[0] - x[0] && \text{(35b)} \\ y[1] &= x[1] - x[0] && \text{(35c)} \\ y[2] &= x[2] - x[2] && \text{(35d)} \\ y[3] &= x[3] - x[6] && \text{(35e)} \end{aligned}$$

Thus, the output $y[n]$ will always depend on the current input $x[n]$ as well as future input $x[m]$ where $m > n$. Thus, the system is **anticausal**.

Stability

Determine which of the following systems are **stable**:

(Q4-1)

$$y[n] = x^2[n] \tag{36}$$

A: Let input signal $x[n]$ be bounded with $|x[n]| < M$. Then it follows that the output $y[n] = x^2[n]$ may be bounded by

$$|y[n]| = |x^2[n]| < M^2. \tag{37}$$

Therefore, this system is **stable**.

(Q4-2)

$$y[n] = \frac{e^{-x[n]}}{x[n-1]} \tag{38}$$

A: This system is clearly **not guaranteed to be stable** because one can devise examples where the denominator goes to zero. For example, note that the response of the system to a unit impulse $x[n] = \delta[n]$ is infinite for all values of n except $n = 1$.

(Q4-3)

$$y[n] = \cos(x[n]) \tag{39}$$

A: For input $x[n]$, we find that the output response is bounded: $|y[n]| = |\cos(x[n])| \leq 1$. Thus, the system is **stable**.

(Q4-4)

$$y[n] = \sum_{k=-\infty}^{n} x[k] \tag{40}$$

A: This system is a numerical integrator and is **stable**. For any bounded input series $\sum_{k=-\infty}^{\infty} |x[k]| < M$ we know that any output partial sum of these terms will also be bounded by $\sum_{k=-\infty}^{n} x[k] < M_1 < M$.

In [ ]:

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