3.2 Discrete-Time Signals - Problems

The purpose of this document is to give you a number of problems to solve regarding the folowing topics: linearity, time-invariance, memory/causality, and stability. They involve analysing the a system

$$y[n] = \mathcal{T}\left(x[n]\right),$$

where $x[n]$ is the input, $\mathcal{T}(\cdot)$ is some operator acting on the input, and $y[n]$ is the output response.

Linearity

Determine which of the systems below are **linear**.

(Q1-1)

$$y[n] = \log(x[n])$$

A: If the system is homogeneous then

$$y[n] = \mathcal{T}(cx[n]) = c\mathcal{T}(x[n])$$

for any input $x[n]$ and for all complex constants c. The system $y[n] = \log(x[n])$ is **not homogeneous** because the response of the system to $x_1[n] = cx[n]$ is

$$y_1[n] = \log(x_1[n]) = \log(cx[n]) = \log(c) + \log(x[n]) \neq c\log(x[n])$$

.

To be additive, the response to $x[n] = x_1[n] + x_2[n]$ must be $y[n] = y_1[n] + y_2[n]$. For this system we have

$$\mathcal{T}(x[n]) = \mathcal{T}(x_1[n] + x_2[n]) = \log(x_1[n] + x_2[n]) \neq \log(x_1[n]) + \log(x_2[n])$$

Thus, the system is **not additive**. Overall the system is **non-linear**.

(Q1-2)

$$y[n] = 6x[n+2] + 4x[n+1] + 2x[n]$$

(Q1-3)

$$y[n] = 6x[n+2] + 4x[n+1] + 2x[n] + 1$$

(Q1-4)

$$y[n] = \sin\left(\frac{n\pi}{2}\right)x[n]$$

Time Invariance

Determine which of the systems below are **shift-invariant**:

(Q2-1)

$$y[n] = x[n] + x[n-1] + x[n-2]$$

A: Let $y[n]$ be the response of the system to an arbitrary input $x[n]$. To test for shift invariance we want to compare the shifted response $y[n-n_0]$ to the shifted input $x[n-n_0]$. With

$$y[n] = x[n] + x[n-1] + x[n-2]$$

we have, for the shifted response

$$y[n-n_0] = x[n-n_0] + x[n-1-n_0] + x[n-2-n_0]$$

Now, the response of the system to $x_1[n] = x[n-n_0]$ is

$$\begin{aligned} y_1[n] &= x_1[n] + x_1[n] + x_2[n] \\ &= x[n-n_0] + x_1[n-n_0] + x_2[n-n_0] \end{aligned}$$

Because $y_1[n] = y[n-n_0]$, the system is **time invariant**.

(Q2-2)

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

(Q2-3)

$$y[n] = x[n^2]$$

(Q2-4)

$$y[n] = x(-n)$$

Causal / Memory

Determine which of the following systems are **causal**, and which have **memory**?

(Q3-1)

$$y[n] = x^2[n]u[n]$$

The system $y[n] = x^2[n]u[n]$ is **memoryless** because the response of the system at time n depends only on the input at time n and no other values of the input. Therefore, this system is **causal**.

(Q3-2)

$$y[n] = x[|n|]$$

(Q3-3)

$$y[n] = x[n] + x[n-3] + x[n-1]$$

(Q3-4)

$$y[n] = x[n] - x[n^2 - n]$$

Stability

Determine which of the following systems are **stable**:

(Q4-1)

$$y[n] = x^2[n]$$

Supposed that $x[n]$ a bounded and finite input such that $|x[n]| < M$. Then it follows that the output $y[n] = x^2[n]$ is also bounded and finite because:

$$|y[n]| = |x^2[n]| < M^2.$$

Therefore, this system is **stable**.

(Q4-2)

$$y[n] = \frac{e^{-x[n]}}{x[n-1]}$$

(Q4-3)

$$y[n] = \cos(x[n])$$

(Q4-4)

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$