3.2 Discrete-Time Signals - Problems

The purpose of this document is to give you a number of problems to solve regarding the folowing topics: linearity, time-invariance, memory/causality, and stability. They involve analysing the a system

$$y[n] = \mathcal{T}\left(x[n]\right),$$

where $x[n]$ is the input, $\mathcal{T}(\cdot)$ is some operator applied to the input, and $y[n]$ is the output response.

Linearity

Determine which of the systems below are **linear**.

(Q1-1)

$$y[n] = \log(x[n])$$

A: If the system is homogeneous then

$$y[n] = \mathcal{T}(cx[n]) = c\mathcal{T}(x[n])$$

for any input $x[n]$ and for all complex constants c. The system $y[n] = \log(x[n])$ is **not homogeneous** because the response of the system to $x_1[n] = cx[n]$ is

$$y_1[n] = \log(x_1[n]) = \log(cx[n]) = \log(c) + \log(x[n]) \neq c\log(x[n])$$

.

To be additive, the response to $x[n] = x_1[n] + x_2[n]$ must be $y[n] = y_1[n] + y_2[n]$. For this system we have

$$\mathcal{T}(x[n]) = \mathcal{T}(x_1[n] + x_2[n]) = \log(x_1[n] + x_2[n]) \neq \log(x_1[n]) + \log(x_2[n])$$

Thus, the system is **not additive**. Overall the system is **non-linear**.

(Q1-2)

$$y[n] = 6x[n+2] + 4x[n+1] + 2x[n]$$

A: If the system is homogeneous than

$$y[n] = \mathcal{T}(cx[n]) = c\mathcal{T}(x[n])$$

Thus for $x_1[n] = c\,x[n]$ we get

$$y_1[n] = 6x_1[n+2] + 4x_1[n+1] + 2x_1[n] = c6x[n+2] + c4x[n+1] + c2x[n] = c(6x[n+2] + 4x[n+1] + 2x[n]) = cy[n]$$

.

Thus, the system is **homogeneous**.

To be additive, the response to $x[n] = x_1[n] + x_2[n]$ must be $y[n] = y_1[n] + y_2[n]$. For this system we have

$$\begin{aligned}\mathcal{T}(x_1[n] + x_2[n]) &= 6(x_1[n+2] + x_2[n+2]) + 4(x_1[n+1] + x_2[n+1]) + 2(x_1[n] + x_2[n]) \\ &= (6x_1[n+2] + 4x_1[n+1] + 2x_1[n]) + (6x_2[n+2] + 4x_2[n+1] + 2x_2[n]) \\ &= y_1[n] + y_2[n]\end{aligned}$$

Thus, the system is **additive**. Overall, because the system is **additive and homogeneous** it is **linear**.

(Q1-3)

$$y[n] = 6x[n+2] + 4x[n+1] + 2x[n] + 1$$

A: If the system is homogeneous than

$$y[n] = \mathcal{T}(cx[n]) = c\mathcal{T}(x[n])$$

Thus for $x_1[n] = c\,x[n]$ we get

$$\begin{aligned}y_1[n] &= 6x_1[n+2] + 4x_1[n+1] + 2x_1[n] + 1 \\ &= c6x[n+2] + c4x[n+1] + c2x[n] + 1 \\ &= c(6x[n+2] + 4x[n+1] + 2x[n]) + 1\end{aligned}$$

.

However,

$$cy[n] = c(6x[n+2] + 4x[n+1] + 2x[n] + 1)$$

Thus, the system is **not homogeneous**.

To be additive, the response to $x[n] = x_1[n] + x_2[n]$ must be $y[n] = y_1[n] + y_2[n]$. For this system we have

$$\begin{aligned}\mathcal{T}(x_1[n] + x_2[n]) &= 6(x_1[n+2] + x_2[n+2]) + 4(x_1[n+1] + x_2[n+1]) + 2(x_1[n] + x_2[n]) + 1 \\ &= (6x_1[n+2] + 4x_1[n+1] + 2x_1[n]) + (6x_2[n+2] + 4x_2[n+1] + 2x_2[n]) + 1 \\ &\neq y_1[n] + y_2[n]\end{aligned}$$

Thus, the system is **not additive**. Overall, because the system is **not additive and homogeneous**, it is **not linear**.

(Q1-4)

$$y[n] = \sin\left(\frac{n\pi}{2}\right) x[n]$$

A: Let's examine homogeneity and additivity at the same time. Let $y_1[n]$ and $y_2[n]$ the outputs to $x_1[n]$ and $x_2[n]$, respectively. The response of the system to

$$x[n] = a\, x_1[n] + b\, x_2[n]$$

is

$$\begin{aligned}y[n] &= \sin\left(\frac{n\pi}{2}\right) x[n] \\ &= \sin\left(\frac{n\pi}{2}\right) (a\, x_1[n] + b\, x_2[n]) \\ &= \sin\left(\frac{n\pi}{2}\right) a\, x_1[n] + \sin\left(\frac{n\pi}{2}\right) b x_2[n] \\ &= a\, y_1[n] + b\, y_2[n]\end{aligned}$$

Thus, this system is both **homogeneous** and **additive**, and is **linear**.

Time Invariance

Determine which of the systems below are **shift-invariant**:

(Q2-1)

$$y[n] = x[n] + x[n-1] + x[n-2]$$

A: Let $y[n]$ be the response of the system to an arbitrary input $x[n]$. To test for shift invariance we want to compare the shifted response $y[n-n_0]$ to the shifted input $x[n-n_0]$. With

$$y[n] = x[n] + x[n-1] + x[n-2]$$

we have, for the shifted response $n \to n - n_0$:

$$y[n-n_0] = x[n-n_0] + x[n-1-n_0] + x[n-2-n_0]$$

Now, the response of the system to $x_1[n] = x[n-n_0]$ is

$$\begin{aligned}y[n-n_0] &= x_1[n] + x_1[n-1] + x_1[n-2] \\ &= y_1[n]\end{aligned}$$

Because $y_1[n] = y[n-n_0]$, the system is **time invariant**.

(Q2-2)

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

A: Let

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

be the response to an arbitrary input $x[n]$. The response of the system to the shifted input $x_1[n] = x[n-n_0]$ is

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=-\infty}^{n} x[k-n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$$

.

Because this is equal to $y[n-n_0]$ the system is **shift-invariant**.

(Q2-3)

$$y[n] = x[n^2]$$

A: The system is **not shift-invariant**, which may be demonstrated with a simple counterexample. Note that if $x[n] = \delta[n]$ then the response will be $y[n] = \delta[n^2] = \delta[n]$. However, if $x_1[n] = \delta[n-2]$, then the response will be $y_1[n] = x_1[n^2] = \delta[n^2 - 2] = 0$, which is not equal to $y[n-2]$. Thus, the system is **not shift-invariant**.

(Q2-4)

$$y[n] = x(-n)$$

A: Let $x[n]$ be the input and $y[n] = x[-n]$ be the response. If we consider the shifted input $x_1[n] = x[n - n_0]$, we find that the response is

$$y_1[n] = x_1[-n] = x[-n - n_0]$$

.

However, if we shift $y[n]$ by n_0 then we obtain

$$y[n - n_0] = x[-(n - n_0)] = x[-n + n_0]$$

which is not equal to $y_1[n]$. Therefore, this system can be shown to be **not shift-invariant**.

Causal / Memory

Determine which of the following systems are **causal**, and which have **memory**?

(Q3-1)

$$y[n] = x^2[n]u[n]$$

A: The system $y[n] = x^2[n]u[n]$ is **memoryless** because the response of the system at time n depends only on the input at time n and no other values of the input. Therefore, this system is **causal**.

(Q3-2)

$$y[n] = x[|n|]$$

A: The output $y[n] = x[|n|]$ is an example of a non-causal system. This can be observed when looking at the repsonse of the system for $n < 0$. For example, for $n = -1$ we get $y[-1] = x[1]$. Therefore the output of the system is dependent on a future input. Therefore the system is **not causal** (and in fact **anticausal**).

(Q3-3)

$$y[n] = x[n] + x[n-3] + x[n-1]$$

A: This system is **causal** because all of the input required to compute the response at n are at or before n.

(Q3-4)

$$y[n] = x[n] - x[n^2 - n]$$

A: This system is **not causal**, which can be demonstrated by looking at $n < 0$. For example, $y[-2] = x[-2] - x[2]$, which is clearly **non-causal**.

Stability

Determine which of the following systems are **stable**:

(Q4-1)

$$y[n] = x^2[n]$$

A: Let $x[n]$ be bounded input with $|x[n]| < M$. Then it follows that the output $y[n] = x^2[n]$ may be bounded by

$$|y[n]| = |x^2[n]| < M^2$$

.

Therefore, this system is **stable**.

(Q4-2)

$$y[n] = \frac{e^{-x[n]}}{x[n-1]}$$

A: This system is clearly **not stable**. For example, note that the response of the system to a unit impulse $x[n] = \delta[n]$ is infinite for all values of n

except $n = 1$.

(Q4-3)

$$y[n] = \cos(x[n])$$

A: For input $x[n]$, we find that the output response is bounded: $|y[n]| = |\cos(x[n])| \leq 1$. Thus, the system is **stable**.

(Q4-4)

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

A: This system is a numerical integrator and is **not stable**. For an example, assume that the input is the step function $x[n] = u[n]$. Thus,

$$y[n] = \sum_{k=-\infty}^{n} x[k] = \sum_{k=-\infty}^{n} u[k] = n + 1$$

Thus, even though the input is bounded, the output is not guaranteed to be bounded and thus the system is **not stable** because $y[n] \to \infty$ as $n \to \infty$.

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