

# 1 Introduction

Organo-metallic frameworks or MOFs are materials consisting of one or multiple central metallic ions and surrounding organic ligands. They are crystalline materials with very high porosity. Internal surface area in some cases exceeds 6000  $\text{m}^2/\text{g}$  [1]. Thanks to their chemical properties they are widely used in the fields of biochemistry, catalysis and electrochemistry. They are especially useful in clean energy application as storage for gasses (eg. hydrogen) or energy through absorption/adsorption process [1]. They can also be used as gas separation medium, as second harmonic generators in nonlinear optics, some of them also display interesting ferroelectric properties. Their usage is so broad thanks to numerous combinations of metallic ions and organic ligands [? ? ]. Many chemical properties stem from unpaired electrons, which are often found in such materials. Ions commonly found in MOFs are  $\text{Cu}(\text{II})$ ,  $\text{Ni}(\text{II})$ ,  $\text{Pb}(\text{II})$ . Such ions have unpaired electron(s) in their d orbitals and one can clearly see their effects on  $^{13}\text{C}$  and  $^1\text{H}$  nmr spectra, one of the most common tools for characterization of the molecular and electronic structure of organic molecules. Materials presented in this work are paramagnetic. This means they exhibit weak attraction to the external magnetic field, which is a consequence of unpaired electrons in their structure. The effects of the latter are easily recognized in nmr spectra by the large paramagnetic shifts they cause.

Nmr spectra of organic materials usually feature chemical shifts caused by induced currents which in turn are caused by external magnetic field [2]. In paramagnetic materials, this is not the only contribution to the total shifts visible in spectra. An important interaction, not present in diamagnetic materials, is interaction between unpaired electrons and nuclei. Such interaction can cause large paramagnetic shifts also called hyperfine shifts. Typical for such spectra are also widened spectral lines. These large shifts make interpretation of spectra more difficult [3]. A useful tool to help with the interpretation are first-principle quantum calculations. Large growth of computational power in recent years have enabled more accurate calculations and calculations performed on a more complex systems. However, calculations of hyperfine constants and total chemical shifts for paramagnetic materials / mofs have not yet been systematically tested and documented in literature.

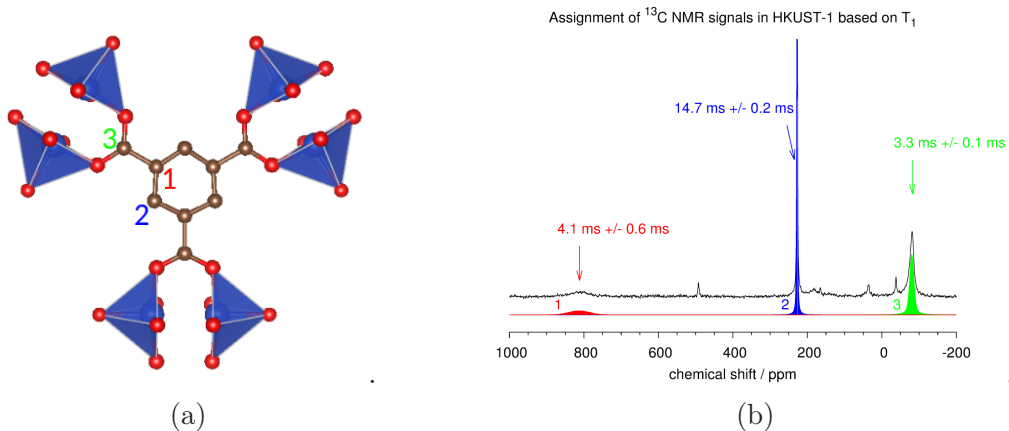


Figure 1: An example of mof material named Hkust-1. It is a crystalline material. Image (a) depicts a part of the structure used for calculation of nmr parameters of atoms marked with 1, 2 and 3. (b)  $\text{C}^{13}$ -nmr spectra of atoms marked on image (a).

In this work we will present the most common approach to calculation of nmr parameters of paramagnetic materials, which is based on density functional theory (dft). First we will present basics of dft, which is the most important part of the whole procedure. Values of nmr parameters completely depend on electronic configuration, consequently it's accurate calculation is of a paramount importance.

## 2 Calculation of electronic wavefunction

Accurate calculation of electronic structure has always been a challenge. It quickly became apparent that direct use of Schrödinger equation is not a realistic prospect, except for some small molecules, as the time consumed to solve it grows exponentially [4] as a function of electron number at a given accuracy level. With the development of computers, different numerical approximations for computation of electronic wave function and optimization of molecular structure have emerged. One of the most successful methods has been density functional theory (dft from now on), which has been known for roughly 50 years. Through the years dft has evolved and today it represents one of the main tools for calculation of electronic structure especially for complex molecules and crystals.

### 2.1 Hamiltonian

The first step in formulation of the problem is to define hamiltonian which describes motion of nuclei and electrons. Non-relativistic hamiltonian describing the interaction of nuclei and electrons can be written as follows:

$$\hat{H} = \hat{T}_n + \hat{T}_e + \hat{W}_{n-n} + \hat{W}_{e-e} + \hat{W}_{n-e} + \hat{V}_{ext}, \quad (1)$$

where  $T_n$  and  $T_e$  are kinetic energies of nuclei and electrons respectively,  $W_{n-n}$ ,  $W_{e-e}$  and  $W_{e-n}$  represent nuclei-nuclei, electron-electron and electron-nuclei interaction terms.  $V_{ext}$  is strictly multiplicative external potential. Ground state of such a system is given by the solution of the time independent Schrödinger equation:

$$\hat{H}\psi = \epsilon_0\psi_0, \quad (2)$$

where index 0 denotes the solution with the lowest energy. In general  $\psi_0$  depends on  $3N$  coordinates, where  $N$  is total number of particles. This means that systems with more than e.g. 10 atoms are very computationally demanding. It is common practice to reduce the dimensionality of the problem by employing Born-Oppenheimer approximation in which the motion of nuclei is separated from motion of electrons, sometimes nuclei are even fixed. Thus, from now on we will restrict ourselves to hamiltonians describing only the motion of electrons:

$$\hat{H} = \hat{T}_e + \hat{W}_{e-e} + \hat{W}_{n-e} + \hat{V}_{ext}. \quad (3)$$

The number of electrons  $N$  for a small molecules, like water, is of the order  $\sim 10$ . In medium sized molecules with  $\sim 50 - 100$  atoms, the number can grow to a few hundred, while in large molecules, like proteins, the number can grow into thousands and ten-thousands. As one can imagine, solving a system of coupled differential equations with such huge number of coordinates ( $3N$ ) is computationally an almost impossible task. This is the reason for development of approaches which, while still being sufficiently accurate, offer faster computational times. DFT represents one of the most successful methods to solving such systems.

## 2.2 Density functional theory - dft

Dft is a method, which allows us to map many-electron problem to an effectively single electron problem. It replaces all electron wave function with particle density. The core of dft lies in Kohn-Sham theorems. These two theorems ensure that stationary many-particle systems are fully characterized by their ground state particle density. This means that given a group of electrons, one only has to know ground state electron density to be able to tell all other properties of the system. For non-degenerate case the latter is uniquely determined by ground state many-particle wave function, which in turn is uniquely determined by the external potential. For a simple non-degenerate case we will prove these two theorems. Let us consider hamiltonian of the form:

$$\hat{H} = \hat{T} + \hat{W} + \hat{V}_{ext}, \quad (4)$$

where  $T$  is kinetic energy,  $W$  is inter-particle interaction and  $V_{ext}$  is external potential determined up to a constant. Let  $V_{ext}$  be such potential that ground state  $\psi_0$  is non-degenerate. Consider now the set of all  $H$  of the form (4), which differ only in  $V_{ext}$ , with non-degenerate ground states. Since kinetic and inter-particle interaction terms are the same for all  $H$  in such set, the latter can be represented by the set of all non-equivalent potentials:

$$\mathcal{V} = \{V_{ext}; \quad V \text{ determined up to multiplication factor and a constant}; \\ |\psi_0\rangle \text{ exists and is non-degenerate}\} \quad (5)$$

According to the above definition we can define the set of all corresponding ground state-densities determined up to phase as:

$$\mathcal{G} = \{\psi_0; \quad \psi_0 \text{ ground state corresponding to a potential from } V; \\ \psi_0 \sim \psi_0 e^{i\phi} \} \quad (6)$$

The map from  $\mathcal{V}$  to  $\mathcal{G}$  is surjective by definition. What we would like to know is, if it is also injective (4), i.e., can a single  $\psi_0$  be a ground state for two non-equivalent potentials? Suppose now that  $\psi_0$  is a ground state for two non-equivalent potentials  $V_{ext}$  and  $V'_{ext}$ .

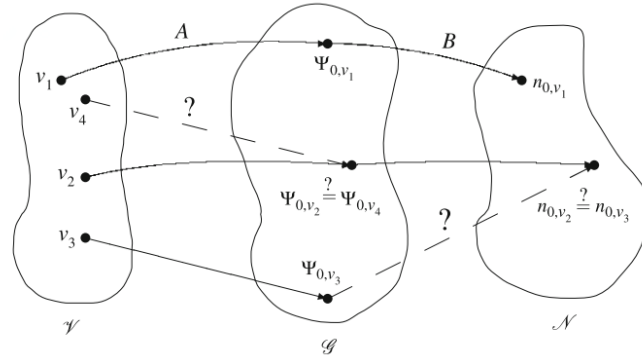


Figure 2: Bijection between the set of potentials, their corresponding ground states and ground state densities. Existence of such bijection is proven by Kohn-Sham theorems and proves that many-particle system is uniquely determined by it's ground state particle density [5].

$$\hat{H} |\psi_0\rangle = (\hat{T} + \hat{W} + \hat{V}_{ext}) |\psi_0\rangle = \epsilon_0 |\psi_0\rangle, \quad (7)$$

$$\hat{H}' |\psi_0\rangle = (\hat{T} + \hat{W} + \hat{V}'_{ext}) |\psi_0\rangle = \epsilon'_0 |\psi_0\rangle, \quad (8)$$

subtracting above equation yields:

$$(\hat{V}_{ext} - \hat{V}'_{ext}) |\psi_0\rangle = (\epsilon_0 - \epsilon'_0) |\psi_0\rangle. \quad (9)$$

due to multiplicative nature of potentials, we can just divide the whole equation by  $|\psi_0\rangle$  and obtain:

$$(\hat{V}_{ext} - \hat{V}'_{ext}) = (\epsilon_0 - \epsilon'_0), \quad (10)$$

which is a contradiction, since  $V_{ext}$  and  $V'_{ext}$  must differ for more than a constant. Similarly one can show that two different ground state wave functions, corresponding to two different external potentials, cannot lead to the same ground state densities. To see this we compare ground state energies and rewrite them using ground state densities, which are supposed to be the same for both wave functions:

$$\begin{aligned} \langle \psi_0 | \hat{H} | \psi_0 \rangle &= \epsilon_0 < \langle \psi'_0 | \hat{H} | \psi'_0 \rangle = \langle \psi'_0 | \hat{H} + \hat{V}'_{ext} - \hat{V}'_{ext} | \psi'_0 \rangle = \\ &\epsilon'_0 + \langle \psi'_0 | \hat{V}_{ext} - \hat{V}'_{ext} | \psi'_0 \rangle. \end{aligned} \quad (11)$$

Rewriting this in terms of densities and taking into account equivalence of  $H$  and  $H'$  yields:

$$\epsilon_0 < \epsilon'_0 + \int (V_{ext}(\mathbf{r}) - V'_{ext}(\mathbf{r})) n(\mathbf{r}) d\mathbf{r} \quad (12)$$

and

$$\epsilon'_0 < \epsilon_0 + \int (V'_{ext}(\mathbf{r}) - V_{ext}(\mathbf{r})) n(\mathbf{r}) d\mathbf{r}. \quad (13)$$

Upon subtracting both equations one obtains a contradiction:

$$\epsilon_0 + \epsilon'_0 < \epsilon_0 + \epsilon'_0, \quad (14)$$

which proves that for hamiltonians which yield non-degenerate ground states, each ground state leads to a different ground state particle density. Ground state particle densities form a set where each density corresponds to a single wave function from  $\mathcal{G}$ :

$$\mathcal{N} = \{n; n = \langle \psi | \hat{n} | \psi \rangle, \psi \in \mathcal{G}\}, \quad (15)$$

where  $\hat{n}$  is quantum mechanical particle density operator.

Extension of this simple proof to hamiltonians with degenerate ground states is possible by replacing ground state wave functions by linear span of degenerate ground states. Thus, in degenerate case one obtains bijection between external potential, set of linear spans, each belonging to a certain external potential and a set of sets of ground state densities. Special treatment is necessary also for systems containing magnetic fields, where one can separate hamiltonian into spin up and spin down hamiltonian of the form (4). In the latter case one obtains bijections between the set of pairs  $(\mathbf{A}(\mathbf{r}), V_{ext}(\mathbf{r}))$ ,  $(\psi_\uparrow, \psi_\downarrow)$  and  $(n(\mathbf{r}), \mathbf{m}(\mathbf{r}))$ .

Since there exists bijection between ground state wave functions and ground state densities, one can formally rewrite ground state wave function as functional of ground state particle density:

$$|\psi'_0\rangle = |\psi'_0[n]\rangle \quad (16)$$

and using above formula one can also rewrite operators in terms of ground state density. As an example, let us rewrite ground state energy as functional of ground state particle density:

$$E[n_0] = \langle \psi_0[n_0] | \hat{H} | \psi_0[n_0] \rangle, \quad (17)$$

for which one can find minimum energy principle:  $E[n_0] < E[n]$  whenever  $n$  belongs to  $\mathcal{N}$ . This is an obvious consequence of wave function functional  $|\psi[n]\rangle$ , which is only defined for densities which are in  $\mathcal{N}$ . Thus, one has to ask himself whether every nonnegative normalizable function  $n(\mathbf{r})$  is in  $\mathcal{N}$ . The answer is no. A density from  $\mathcal{N}$  has a corresponding potential  $V_{ext}$  such that it minimizes energy functional of the form 17 and consequently they are called V-representable densities.

In practice, one does not need deep knowledge about such mathematical definitions of functionals. The reason for this is the discretization of space into grid points. On final grid for any strictly positive particle density ( $n(\mathbf{r}) > 0$ ), which is compatible with Pauli principle, there exists a potential for which the density represents ground state density and is thus contained in  $\mathcal{N}$  [5].

### 3 DFT in practice

Kohn-Sham theorems unfortunately tell nothing about the explicit dependence of energy functional on density. Nonetheless, we would still like to use Kohn-Sham theorems, to construct numerical scheme where particle density has the central role. We will concentrate on construction of energy functional  $E[n(\mathbf{r})]$ , which should approximate  $F[n(\mathbf{r})]$  as well as possible.

## 4 Kohn-Sham equations

Kohn-Sham (KS) equations represent a standard and most common approach to dft. They are based on energy functional dependent only on electron density  $n(\mathbf{r})$ . To introduce them in an understandable and consistent fashion, let's start with a non-interacting system.

### 4.1 Noninteracting system

Hamiltonian of  $N$ -particle non-interacting system can be written in the following way:

$$\hat{H} = \hat{W}_k + \hat{V}_{ext}, \quad (18)$$

where  $V_{ext}$  is an external potential of multiplicative nature. It is well known the solution to this problem can be written in the form of Slater determinant:

$$|\Phi_0\rangle = \frac{1}{\sqrt{N!}} \det \begin{bmatrix} \phi_1(\mathbf{r}_1, \sigma_1) & \phi_2(\mathbf{r}_1, \sigma_1) & \phi_3(\mathbf{r}_1, \sigma_1) & \dots & \phi_N(\mathbf{r}_1, \sigma_1) \\ \phi_1(\mathbf{r}_2, \sigma_2) & \phi_2(\mathbf{r}_2, \sigma_2) & \phi_3(\mathbf{r}_2, \sigma_2) & \dots & \phi_N(\mathbf{r}_2, \sigma_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_1(\mathbf{r}_N, \sigma_N) & \phi_2(\mathbf{r}_N, \sigma_N) & \phi_3(\mathbf{r}_N, \sigma_N) & \dots & \phi_N(\mathbf{r}_N, \sigma_N) \end{bmatrix},$$

which, when inserted into equation (18) effectively produces single particle problem:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}_{ext}(\mathbf{r})\right)\psi(\mathbf{r}) = \epsilon_i\psi(\mathbf{r}). \quad (19)$$

Thus, using Slater determinant we have effectively converted many particle problem with  $3N$  coordinates to a single particle problem. Of course the particles in this case do not feel each other and motion of each particle should not depend on other particles, so such a breakdown is completely natural.

The ground state of such a system is obtained using  $N/2$  lowest states by putting 2 electrons into each state. Calculation of kinetic energy and particle density for such a system is also straight forward. As we can see, using Slater determinant as ansatz for the solution of noninteracting hamiltonian offers simple expressions for ground state wavefunction, kinetic energy and particle density calculation. Particle density corresponding to such a wave function can be written using the following expression:

$$n_0(\mathbf{r}) = \sum_{\sigma=\uparrow,\downarrow} \sum_i \Theta(\epsilon_F - \epsilon_i) |\phi_i(\mathbf{r}, \sigma)|^2. \quad (20)$$

The density of electrons is just a sum over all occupied orbitals. Now let's remember Kohn-Sham theorems, which state that ground state density is unique functional of the ground state wave function:

$$|\psi\rangle = |\psi(n(\mathbf{r}))\rangle. \quad (21)$$

One can show that there exists an even stronger connection; every  $\phi_i$  is a uniquely determined by ground state density. One can see this by considering a single particle problem using the same potential  $V_{ext}$  as found in eq. (18) and then gradually adding particles, thus:

$$|\phi_i\rangle = |\phi_i(n(\mathbf{r}))\rangle. \quad (22)$$

Using this relations we can define HK functional:

$$E_s[n(\mathbf{r})] = \langle\psi[n(\mathbf{r})]| T |\psi(n(\mathbf{r}))\rangle + \int d\mathbf{r} n(\mathbf{r}) V_{ext}(\mathbf{r}), \quad (23)$$

where

$$T[n(\mathbf{r})] = \sum_i \Theta(\epsilon_F - \epsilon_i) \sum_{\sigma=\uparrow,\downarrow} \int \phi_\sigma^*(\mathbf{r}) \frac{-i\hbar\nabla^2}{2m} \phi_\sigma(\mathbf{r}), \quad (24)$$

where we have implicitly used  $|\phi_i(\mathbf{r})\rangle = |\phi[n(\mathbf{r})]\rangle$ . In practice this is not necessary, since functions  $\phi(\mathbf{r})$  are known - they are direct result of calculation.

Now we would like to use a similar construction to solve hamiltonian 17. Using solution ansatz in the form of Slater determinant offers straight forward calculation of kinetic energy and particle density. Using Slater determinant leads to differential equations for  $\phi_k$  wavefunctions. Modern computational packages instead of solving differential equations use large sets of basis functions, which in the end lead to over determined set of algebraic equations. Solution is then given by populating lowest energy orbitals until all electrons are allocated.

First we have to ask ourselves if it is at all possible to convert interacting  $N$  particle problem ( $3N$  coordinates) to an effectively single particle (3 coordinates) problem. By this we mean if there exists a noninteracting system that has the

same ground state electron density as original interacting system. It turns out that for every  $N$ -electron interacting system with ground state density  $n_0(\mathbf{r})$  there also exists a noninteracting system with exactly the same ground state density. The question still remains what kind of external potential one should use to produce the same electron density.

Intuitively, one would expect that potential belonging to effective single particle hamiltonian should reflect the properties, geometry and potentials found in given many particle system. Hamiltonian naturally contains kinetic energy term, but it should also somehow contain inter particle interactions. Usually Kohn-Sham hamiltonian consists of kinetic energy term, Hartree inter particle interaction, external potential and exchange-correlation functional [5]:

$$E[n(\mathbf{r})] = T[n(\mathbf{r})] + E_H[n(\mathbf{r})] + E_{ext}[n(\mathbf{r})] + E_{xc}[n(\mathbf{r})]. \quad (25)$$

Hartree term accounts for Coulomb repulsion:

$$E_H[n(\mathbf{r})] = \int n(\mathbf{r})w(\mathbf{r}, \mathbf{r}')n(\mathbf{r}')d\mathbf{r}'d\mathbf{r}. \quad (26)$$

The above expression is not the same as the one we get from Hartree-Fock equations and includes self-repulsion. Dft treats exchange term, also arising from Coulomb potential separately. Computational packages often employ different approximation and optimizations for faster calculation of this term [6]. Functional belonging to external potential is well known:

$$E_{ext}[n(\mathbf{r})] = \int n(\mathbf{r})V_{ext}(\mathbf{r})d\mathbf{r}. \quad (27)$$

Unfortunately exchange-correlation functional is much less well known. It is defined by the equation (25) and contains all inter-particle interactions not contained in  $T[n(\mathbf{r})]$ . Hartree (??) and external potential energy functionals (27) are exact functionals, e.g. if  $\phi_i(\mathbf{r})$  are exact wave functions of electrons in a given system, then these two functionals will produce correct Coulomb repulsion energy and correct energy of particles in external potential. The same can not be said for kinetic energy functional. The difference between exact kinetic energy of interacting system and  $T[n(\mathbf{r})]$  is the exchange energy we are looking for:

$$E_{xc}^{\psi_0} - E_{xc}[n(\mathbf{r})] = \langle \psi_0 | \hat{T} | \psi_0 \rangle - \langle \Phi_0 | \hat{T} | \Phi_0 \rangle, \quad (28)$$

where  $|\psi_0\rangle$  is the true ground state of interacting system.  $|\psi\rangle_0$  and  $|\Phi\rangle_0$  may yield completely different electronic states, but the aim here is to find such  $|\Phi_0\rangle$  that both functions yield the same electron density. \*check page 62 and add here\*

Our goal is to minimize hamiltonian of the form (25) using Slater determinant as ansatz for many electron system and density calculation. By considering (27) and (24) one can write KS equation:

$$\left( \hat{T} + v_{ext} + E_H[n(\mathbf{r})] + E_{xc}[n(\mathbf{r})] \right) |\phi\rangle_i = \epsilon_i \phi_i, \quad (29)$$

where density  $n(\mathbf{r})$  has to be calculated according to equation (20). The functions  $\phi_i$  are produced by directly solving differential equation. They are usually determined by inserting ansatz in the form of gaussian basis functions from which one can then



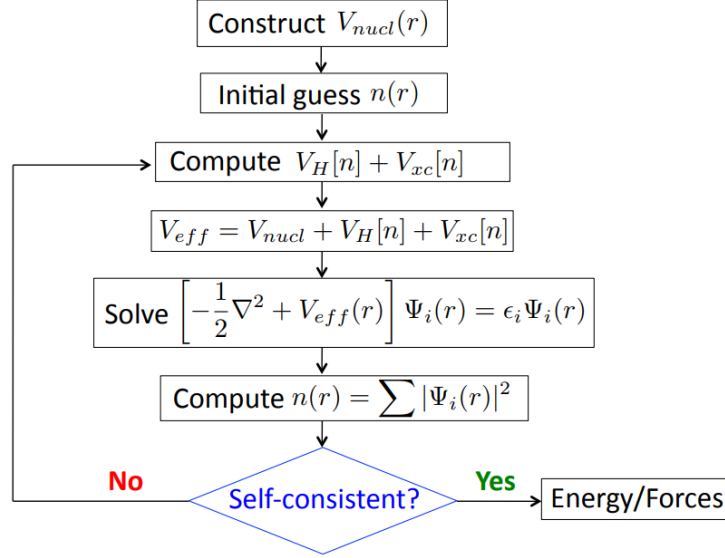


Figure 3: Self consistent scheme depicting the procedure in which Kohn Sham equations are solved. One starts positioning nuclei into desired positions. The next step is to construct potential of nuclei and in parallel one can also construct starting approximation for electronic wave function. The latter is then used for constructing all other functionals of density. After all functionals have been established one can solve KS equations, construct new density and repeat the described procedure until the change in density is not small enough.

construct correct single particle states. Multiplying equation by  $\langle \phi |_j$  from the left side yields generalized eigenvalue problem:

$$\langle \phi_j | \left( \hat{T} + v_{ext}(\mathbf{r}) + E_H[n(\mathbf{r})] + E_{xc}[n(\mathbf{r})] \right) | \phi_i \rangle = \epsilon_i \langle \phi_j | \phi_i \rangle. \quad (30)$$

System obviously has to be solved in a self consistent fashion 3. One starts with the positioning of nuclei into desired positions and construction of nuclei potentials. In parallel with the last step starting electron orbitals  $\phi_i^0$  are constructed. Usually they are written as a series of Gaussian functions. Electron density  $n^0$  can then be quickly calculated from constructed orbitals. The next step is to calculate functionals  $E_H[n(\mathbf{r})]$  and  $E_{xc}[n(\mathbf{r})]$ . In the next step equations are solved. The solution are new orbitals  $\phi_i^1$  and from them new density  $n^1$  is calculated. The latter is then used to construct new  $E_H[n(\mathbf{r})]$  and  $E_{xc}[n(\mathbf{r})]$  which are again used to solve KS equation from which one gets new orbitals  $\phi_i^3$ . The procedure is repeated until the change in density or energy is not small enough.

When given a certain system functionals  $T$ ,  $E_H$  and  $V_{ext}$  are precisely determined.  $E_{xc}$  is not. It is thus the determining factor for how well the KS equations describe our system. We will present various functionals used today in one of the next chapters.

## 4.2 Degenerate ground state

So far we have talked little about the problem of degeneracy of KS states. In general degeneracy is not a problem, except at Fermi level. When there are more KS states than there are electrons, several possible ground states and thus also



particle densities can be constructed. In such cases density matrix arising from several Slater determinants is constructed:

$$\hat{D} = \sum_i d_i |\Phi_i\rangle \langle \Phi_i|. \quad (31)$$

Density belonging to such ground state is just a weighted sum of densities corresponding to each Slater determinant  $n(\mathbf{r}) = \sum_i d_i n_i(\mathbf{r})$ , where  $n_i(\mathbf{r}) = \sum_j \phi_{ij}(\mathbf{r})$ . Index  $i$  runs over all possible Slater determinants one can construct from given degenerate states and index  $j$  runs over all states in each Slater determinant. The sum of coefficients  $d_k$  is the trace of density matrix and has to be 1. The choice of coefficients  $d_k$  is not trivial. They have to be constructed in such a way that the new particle density does not break degeneracy. When a new degenerate state emerges in scf procedure (fig. 3), it may significantly affect particle density and resulting potentials and consequently break or destroy convergence of the procedure. An example is a boron atom where 2p orbitals are degenerate. Only the choice of  $d_{2p^0} = d_{2p^{-1}} = d_{2p^1} = 1/3$  leads to spherically symmetric potential and preservation of degeneracy[5].

### 4.3 Exchange and correlation functionals

Exchange and correlation functionals are functionals which try to account for exchange and correlation interactions between particles/electrons. They can be of a different forms and in general one can roughly divide them into 4 groups [7] - Lda, Gga, meta-Gga and Hybrid functionals. The first three group all have in common that they explicitly depend on density of particles. This causes a self-interaction error, which causes excessive delocalization of electrons. As a solution to these problems hybrid functionals, which contain exact Hartree-Fock exchange, have been proposed.

#### 4.3.1 Lda

Local density approximation (lda) are functionals which depend only on density of particles (electrons) [7]. First functional of such form dates back into the year 1930, when the exchange interaction for uniform gas was discovered [7]:

$$E_{xc}^{lda}[n] = -\frac{3}{4} \left( \frac{3}{\pi} \right)^{1/3} \int n(\mathbf{r})^{4/3} d\mathbf{r}. \quad (32)$$

Today, there exist multiple other lda exchange functionals, but for most cases they are not very useful. Their only advantage are fast computational times. For every other purpose gga and hybrid functionals are much better suited.

#### 4.3.2 Gga

Generalized gradient approximation (gga) functionals depend not only on density of particles but also on its gradient [7]. Most commonly, gga functionals are build upon (32) [7]:

$$E_{xc}^{gga} = \int n(\mathbf{r})^{4/3} F(x); \quad x = |\nabla n(\mathbf{r})|/n(\mathbf{r})^{4/3}. \quad (33)$$

One of most commonly used gga functional is PBE functional [7] [7]:

$$E_{xc}^{pbe} = - \int n(\mathbf{r})^{4/3} \left[ \frac{3}{4} \left( \frac{3}{\pi} \right)^{1/3} + \frac{\mu s}{1 + \mu s^2/\kappa} \right]; \quad s = x/(2(3\pi/2)^{1/3}). \quad (34)$$

Gga functionals offer acceptable accuracy at fast computational times and are most commonly used for approximate calculation, before starting more accurate and more time consuming calculation using hybrid or meta-gga functionals.

#### 4.3.3 Meta-gga functionals

Since gga functionals have their shortcomings, meta-gga functionals were formed in belief that adding higher derivatives will improve accuracy [7]. Meta-gga functionals are built upon gga functional form and add terms containing higher order derivatives of particle density. Functionals are formulated according to the following equation [7]:

$$E_{xc}^{MGGA} = \int \rho^{4/3} F(\rho(\mathbf{r}) \nabla \rho(\mathbf{r}), \nabla^2 \rho(\mathbf{r}), \tau(\mathbf{r})); \quad \tau = \frac{1}{2} \sum_i |\nabla \phi_i(\mathbf{r})|^2 \quad (35)$$

Meta-gga functionals have higher computational cost than gga functionals. Unfortunately, as it turned out, they are not significantly better than gga functionals and are thus not so popular.

#### 4.3.4 Hybrid functionals

On the contrary to meta-gga functionals, hybrid functionals are much more successful [7]. These functionals are not a continuation of lda, gga, meta-gga chain. Instead, they incorporate exact Hartree-Fock exchange term [7]:

$$E_{xc}^{hf} = \sum_{i,j,\sigma} \int \frac{\phi_{i\sigma}^*(\mathbf{r}) \phi_{j\sigma}(\mathbf{r}) \phi_{j\sigma}^*(\mathbf{r}') \phi_{i\sigma}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r} d\mathbf{r}'. \quad (36)$$

Such functionals do not break Kohn Sham formalism since the wave functions  $\phi_i(\mathbf{r})$  are unique functional of densities shown in previous chapter.

### 4.4 Van der Waals dispersion

Dft does not take into account effects caused by charge fluctuations. These Van der Waals effects can be taken care of with different methods. The most straightforward way is simply to add a special non-local functional (VdW-DF method). A similar approach is to use highly parameterized functionals, like metahybrid functionals (called DFs). Another way is to add dispersion-corrected atom-centered potentials (DCACPs). Recently, the most promising seems to be DFT-D method [9], which is just a sum of terms  $CR^{-6}$  over all atom pairs. As it is generally known, for large distances Van der Waals potential should decay as  $R^{-6}$ . It is obvious that DFT-D does obey this rule as also does VdW-DF method. The other two unfortunately do not obey it. As a consequence DF and DCACP usually cause underbinding.

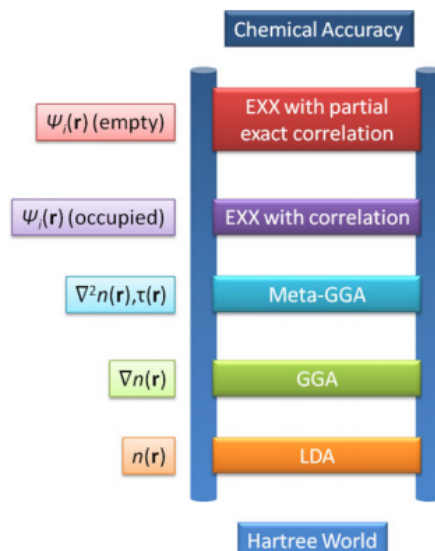


Figure 4: Jacob’s ladder of exchange-correlation functional approximations employed in dft calculations. Hartree world represents starting level where only weak interparticle interaction is present. Lda approxiamtion covers functionals which depend only on particle density. Gga additionally depends on gradient of particle density, while mega-gga incorporates higher order derivatives of particle density and in some cases even kinetic energy of orbitals ( $\tau$ ). Hyper-gga functionals, also called hybrid functionals, stage represents functionals which contain exact exchange calculation (i.e. the one found in Hartree-Fock method). The last level utilizes all Kohn-Sham orbitals to calculate correlation and exchange interaction. This level also accounts for VdW interaction, which is caused by charge fluctuations and is not accounted for in previous stages [8].

## 5 All electron dft calculations

All electron dft calculations are mostly used to calculate electronic structure of a single molecule. The molecule should not be too big, as otherwise the problem may become too computationally expensive. Commonly used open source packages are Orca and Nwchem.

### 5.1 Dft using pseudopotentials

Electronic states of an atom can be divided into three categories; core states, semi-core states and valence states. The latter are the most actively included in formation of bonds. Valence states may be completely deformed once the atom is put into molecules/crystals. Semi-core state are states which do not directly contribute to bonding, but may still be polarized or spatially deformed. Lastly, core states, are highly localized and are assumed to be unaffected by chemical bonding. This means that there should be very little loss of accuracy if core states are replaced by a pseudo wavefunction. Pseudo potentials are constructs which try to replicate effects of core electrons exerted on semi-core, valence electrons and thus reproduce correct chemical and physical properties (bonding energy, bond length, electron localization, magnetization,...).

Of course, pseudo potentials have to be constructed prior to a given calculation

using some other much simpler system. As a consequence, there exists a question of transferability. Is a potential constructed using some reference molecule usable for another molecule. The exact answer is not possible. Many times the only way is to try, especially when transition elements are in question.

## 6 Dft in crystals

Crystals are periodic structures. This fact is also reflected the shape of electronic wave functions, which are Bloch states:

$$\psi_{n,\mathbf{k}}(\mathbf{r}) = u_{n,\mathbf{k}}(\mathbf{r})e^{i\mathbf{k}\mathbf{r}}; \quad u_{n,\mathbf{k}}(\mathbf{r} + \mathbf{R}) = u_{n,\mathbf{k}}(\mathbf{r}), \quad (37)$$

where  $\mathbf{R}$  is a vector in Bravais lattice of a crystal in question. Functions  $u_{n,\mathbf{k}}$  depend on potential at each atomic site. In ideal case, where core potentials are neglected, wave functions are simple plane waves. When potential is weak and reasonably smooth, it can be treated as perturbation. Unfortunately neither of the two conditions is true. Core potential diverges as  $r \rightarrow 0$  and as a consequence the wave functions have a cusp at the origin. For heavier atoms core states have large gradients and can not be represented as plane waves.

The simplest solution is to expand wave functions into series of plane waves. Computationally this is just a discrete fourier transform of a wave function. However, because of already given reasons, the number of plane waves required for such expansion is very large. Thus, even the sum of plane waves is not a suitable representation of core states. For this reason, core states are commonly replaced by pseudo potentials in crystal dft calculations. This approach usually carries *PW calculation* designation.

Although, for many crystals this approach works well it is not good enough especially for transition elements with partially filled d-shells and second row elements. Electron density of transition metals and second row elements still varies widely in spite of use of pseudopotentials. For such cases an improved approach has been developed. A new method, called gaussian augmented plane waves or *GAPW* approach. GAPW basis sets consist of Gaussian functions and plane waves. This approach is suitable for all electron calculations, where core states are expanded in Gaussian functions and valence electrons in plane waves. As a consequence, all electron wave calculations are possible for crystals avoiding pseudo potential inaccuracy issues.

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