Petri Nets

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Brief History

Introduced in 1962 by Carl Adam Petri.

Introduced as a model of concurrency in the setting of parallel and distributed computing.

Biological and chemical systems are very often inherently concurrent.

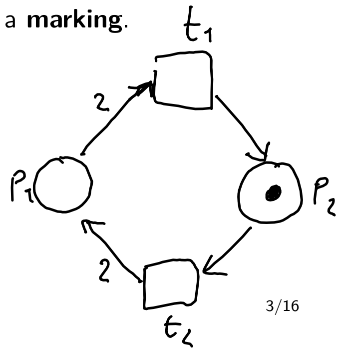
Definition - syntax

A directed bipartite graph with round vertices (**places**) and square vertices (**transitions**).

The edges (**arcs**) are labelled by a **weight** function.

The places may contain **tokens**, representing available resources.

The distribution of tokens over all places constitutes a **marking**.



$$(\overbrace{P, T, F, W}^{\text{Petri net structure } N}, M_0)$$

P ... finite set of places
T ... finite set of transitions

$F \subseteq P \times T \cup T \times P$

$W : T \to \mathbb{N}$

$M_0 : P \to \mathbb{N}_0$

$M_0 = (2, 0)$

(N, M_0)

$\forall x \in P \cup T \qquad {}^{\bullet}x = \{ y \in P \cup T \mid (y, x) \in F \}$

"preset"

$\forall x \in P \cup T \qquad x^{\bullet} = \{ y \in P \cup T \mid (x, y) \in F \}$

"poset"

Definition -semantics

A transition is **enabled** in a marking, if each of the input places has at least as many tokens as indicated by the arc weights.

An enabled transition may **fire**, removing tokens from the source places and producing tokens in the target places, according to the weight function.

$t \in T$ is enabled in $M \iff \forall p \in {}^\bullet t,\ M(p) \geq W(p,t)$

$M \vdash t$ (non-standard)

Firing $t \in T$ in M leads to M' $\qquad M \xrightarrow{t} M'$

$$\forall p \in P \quad M'(p) = \begin{cases} M(p) - W(p,t) + W(t,p) & \text{if } p \in {}^\bullet t \cap t^\bullet \\ M(p) + W(t,p) & \text{if } p \in t^\bullet \setminus {}^\bullet t \\ M(p) - W(p,t) & \text{if } p \in {}^\bullet t \setminus t^\bullet \\ M(p) & \text{if } p \notin {}^\bullet t \cup t^\bullet \end{cases}$$

${}^\bullet t = \emptyset$ "source"

$t^\bullet = \emptyset$ "sink"

Reachability

A sequence of transitions which can be fired in order is called a **firing sequence**.

A marking is **reachable**, if there exists a firing sequence producing it.

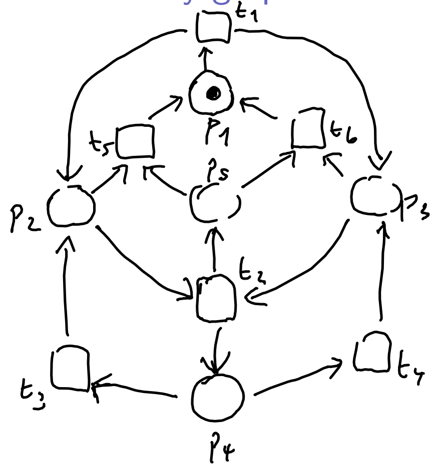
$\sigma = (t_1, \ldots, t_k)$

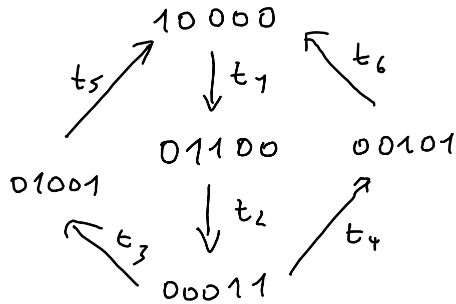
$M \xrightarrow{t_1} M_1 \xrightarrow{t_2} \ldots \xrightarrow{t_k} M_k \qquad M \xrightarrow{\sigma} M_k$

Set of reachable markings $R(N, M_0) =$

$= \{ M \mid \exists \sigma, M_0 \rightarrow M \}$

Reachability graph



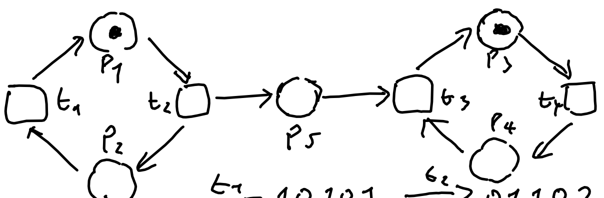


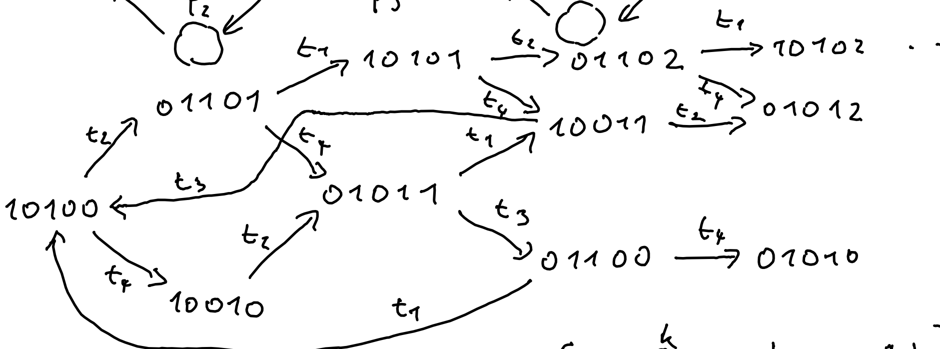
safe (1-bounded)

$M_0: (1,0,0,0,0)$

$M_0: (1,0,1,0,0)$

unbounded





$$\left\{ r + \sum_{i=1}^{k} p_k \cdot \lambda_k \;\middle|\; \lambda_k \in \mathbb{N}_0 \right\}$$

2012, Leroux

Coverability graph

A M is coverable $\Longleftrightarrow$ $\exists M' \in R(N, M_0)$ such that $M' \geq M$

Monotonicity property: $M_1 \leq M_2 \Rightarrow R(N, M_1) \subseteq R(N, M_2)$

ω-markings:

$M: P \to \mathbb{N}_0 \cup \{\omega\}$ $\quad \forall x \in \mathbb{N}_0: \; x < \omega$

$x + \omega = \omega$

$\omega - x = \omega$

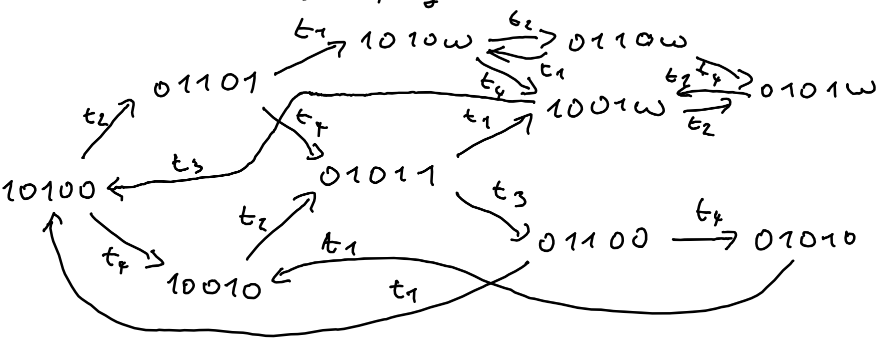
$M_2 \in R(N, M_0)$ $\quad \exists M_1 \in R(N, M_0) \wedge \exists \sigma_2$ such that

$$M_0 \xrightarrow{\sigma_1} M_1 \xrightarrow{\sigma_2} M_2 \quad \wedge \quad M_1 \leq M_2$$

then we introduce an ω-marking M_2' defined as

$$\forall p \in P \quad M_2'(p) = \begin{cases} \omega & \text{if } M_2(p) > M_1(p) \\ M_2(p) & \text{otherwise} \end{cases}$$

This follows from σ_2 being firable from M_2 as well, thus "pumping" tokens into places $M_2(p) > M_1(p)$



Boundedness

A Petri net is **k-bounded** if in every reachable marking, the number of tokens in every place is at most k.

A Petri net is **bounded** if it k-bounded for some $k \in \mathbb{N}$.

1-bounded Petri nets are also known as (1-)**safe** Petri nets.

For $k \in \mathbb{N}$

$p \in P$ is k-bounded $\iff \forall M \in R(N, M_0), M(p) \leq k$

(N, M_0) is k-bounded $\iff \forall p \in P$ p is k-bounded

safe Petri nets are qualitative, as opposed to quantitative.

Deadlocks

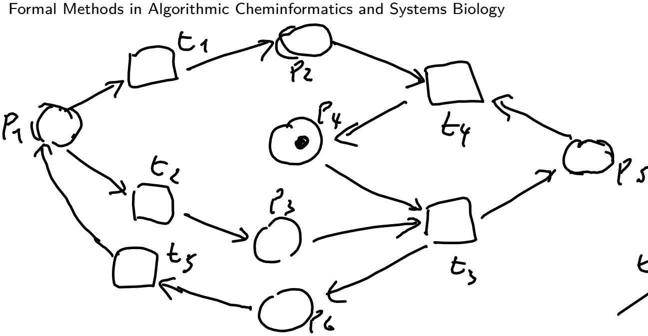
Deadlock is a marking that enables no transitions.

A Petri net is **deadlock-free** if no reachable marking is a deadlock.

$M \in R(N, M_0)$ is a deadlock $\iff \forall t \in T,\ M \not\to t$

(N, M_0) is deadlock-free $\iff \forall M \in R(N, M_0)$ M is not a deadlock

In bio/chem. a deadlock is "steady state"



$M_0 : (1,0,0,1,0,0)$

$100100 \xrightarrow{t_1} \underline{010100}$

$100100 \xrightarrow{t_2} 001100 \xrightarrow{t_3} 000011 \xrightarrow{t_5} 100010$

$100010 \xrightarrow{t_1} 010010 \xrightarrow{t_4} \underline{000100}$

$100010 \xrightarrow{t_2} \underline{001010}$