


Linear Algebra

Least Squares Problems^{1 2}

¹Steven J. Leon, *Linear Algebra with Applications*, 8th ed, Pearson Prentice Hall

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Application 1. Hooke's law states that the force applied to a spring is proportional to the distance that the spring is stretched. Thus, if F is the force applied and x is the distance that the spring has been stretched, then $F = kx$. The proportionality constant k is called the *spring constant*.

Some physics students want to determine the spring constant for a given spring. They apply forces of 3, 5, and 8 pounds, which have the effect of stretching the spring 4, 7, and 11 inches, respectively. Using Hooke's law, they derive the following system of equations:

$$4k = 3$$

$$7k = 5$$

$$11k = 8$$

The system is clearly inconsistent, since each equation yields a different value of k . Rather than use any one of these values, the students decide to compute the least squares solution of the system:

$$(4, 7, 11) \begin{pmatrix} 4 \\ 7 \\ 11 \end{pmatrix} (k) = (4, 7, 11) \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}$$

$$186k = 135$$

$$k \approx 0.726$$

Problem 1. Find the least squares solution of the system

$$\begin{aligned}x_1 + x_2 &= 3 \\ -2x_1 + 3x_2 &= 1 \\ 2x_1 - x_2 &= 2\end{aligned}$$

Sol:

$$\begin{aligned}\begin{pmatrix} 1 & -2 & 2 \\ 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 & -2 & 2 \\ 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} 9 & -7 \\ -7 & 11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 5 \\ 4 \end{pmatrix}\end{aligned}$$

The least squares solution of the system is $\left(\frac{83}{50}, \frac{71}{50}\right)^T$

Problem 2. Find the constant function that is the least squares fit to the following data

x	0	1	2	3
$f(x)$	1	0	1	2

$$f(x) = c \Rightarrow \begin{cases} c=1 \\ c=0 \\ c=1 \\ c=2 \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (c) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$(1, 1, 1, 1) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} (c) = (1, 1, 1, 1) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$c = \frac{1}{4}(1 + 0 + 1 + 2) = 1 \quad (\text{mean arithmetic value})$$

Problem 3. Find the linear polynomial that is the least squares fit to the following data

x	0	1	2	3
$f(x)$	1	0	1	2

$$f(x) = c_1 + c_2x \Rightarrow \begin{cases} c_1 = 1 \\ c_1 + c_2 = 0 \\ c_1 + 2c_2 = 1 \\ c_1 + 3c_2 = 2 \end{cases} \Rightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} \iff \begin{cases} c_1 = 0.4 \\ c_2 = 0.4 \end{cases}$$

Problem 4. Find the quadratic polynomial that is the least squares fit to the following data

x	0	1	2	3
$f(x)$	1	0	1	2

$$f(x) = c_1 + c_2x + c_3x^2$$

$$\Rightarrow \begin{cases} c_1 = 1 \\ c_1 + c_2 + c_3 = 0 \\ c_1 + 2c_2 + 4c_3 = 1 \\ c_1 + 3c_2 + 9c_3 = 2 \end{cases} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 22 \end{pmatrix} \iff \begin{cases} c_1 = 0.9 \\ c_2 = -1.1 \\ c_3 = 0.5 \end{cases}$$

