Elements of Fourier Analysis

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Contents

1 Preliminaries

1.1 Convolution

In this section we study the convolution operation on $\mathbb{R}^n$. If a function f is defined on $U \subset \mathbb{R}^n$, we can replace it by its natural zero extension $f : \mathbb{R}^n \to \mathbb{R}$ which assigns $f(x) = 0$ for $x \notin U$.

Definition 1.1 (Convolution)**.** Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable functions. Define the bad set as

$$E(f,g) := \left\{ x \in \mathbb{R}^n : \int_{\mathbb{R}^n} |f(x-y)g(y)|\, dy = \infty \right\}.$$

The *convolution* of f and g is the function $f * g : \mathbb{R}^n \to \mathbb{R}$ defined by

$$(f * g)(x) = \begin{cases} \int_{\mathbb{R}^n} f(x-y)g(y)\, dy, & x \notin E(f,g), \\ 0, & x \in E(f,g). \end{cases}$$

Remark. Define $F : \mathbb{R}^{2n} \to \mathbb{R}, (x,y) \mapsto f(x)$ and $G : \mathbb{R}^{2n} \to \mathbb{R}, (x,y) \mapsto g(y)$. Then both F and G are measurable functions on $\mathbb{R}^{2n}$, as well as their product $F \cdot G : (x,y) \mapsto f(x)g(y)$. Given linear transformation $T(x,y) = (x-y, y)$, the composition $H = (F \cdot G) \circ T : (x,y) \mapsto f(x-y)g(y)$ is measurable. By Tonelli's theorem, the function $x \mapsto \int_{\mathbb{R}^n} |H(x,y)|\, dy$ is measurable, and $E(f,g)$ is a Lebesgue measurable set.

Clearly, the convolution operation is both commutative and associative, i.e. $f * g = g * f$, and $(f * g) * h = f * (g * h)$. Furthermore, the distributivity of convolution with respect to functional addition immediately follows, i.e. $f * (g + h) = f * g + f * h$.

Proposition 1.2 (Properties of convolution)**.** *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable functions.*

(i) If $f, g \in L^1(\mathbb{R}^n)$, then the bad set $E(f,g)$ is of measure zero. Moreover, $f * g \in L^1(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^m} (f * g)\, dm = \int_{\mathbb{R}^n} f\, dm \int_{\mathbb{R}^n} g\, dm. \tag{1.1}$$

(ii) If $f \in C_u(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in C_u(\mathbb{R}^n)$.

(iii) If $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in L^p(\mathbb{R}^n)$, and

$$\|f * g\|_{L^p} \le \|f\|_{L^p} \|g\|_{L^1}.$$

Proof. (i) Define the measurable function $H(x,y) \mapsto f(x-y)g(y)$ on $\mathbb{R}^{2n}$. By Tonelli's theorem,

$$\int_{\mathbb{R}^{2n}} |H|\, dm = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|\, |g(y)|\, dx \right) dy = \|f\|_{L^1} \|g\|_{L^1}.$$

Hence $H : \mathbb{R}^{2n} \to \mathbb{R}$ is integrable. By Fubini's theorem, for a.e. $x \in \mathbb{R}^n$, $y \mapsto H(x,y)$ is integrable, hence $m(E(f,g)) = 0$. Furthermore, the function $f * g : x \mapsto \int_{\mathbb{R}^n} H(x,y)\, dy$ is also integrable, that is, $f * g \in L^1(\mathbb{R}^n)$. The equation (1.1) follows from Fubini's theorem.

(ii) Given $\epsilon > 0$. By uniform continuity of f, there exists $\eta > 0$ such that $|f(x) - f(x')| < \epsilon / \|g\|_{L^1}$ for all $|x - x'| < \eta$, . As a result, for all $x, x' \in \mathbb{R}^n$ such that $|x - x'| < \eta$, we have

$$|(f * g)(x) - (f * g)(x')| \le \int_{\mathbb{R}^n} |f(x-y) - f(x'-y)|\, |g(y)|\, dy < \epsilon.$$

(iii) is a special case of the following proposition. □

Proposition 1.3 (Young's convolution inequality)**.** *Given $r \in [1,\infty]$ and Hölder r-conjugates $p,q \in [1,\infty]$, i.e. $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then the bad set $E(f,g)$ is of measure zero, and we have*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

Remark. Note that

$$r=\frac{pq}{p+q-pq} \geq 1 \quad \Leftrightarrow \quad \frac{pq}{p+q} \geq \frac{1}{2} \quad \Leftrightarrow \quad p \geq \frac{q}{2q-1} \quad \Leftrightarrow \quad q \geq \frac{p}{2p-1},$$

and

$$r<\infty \quad \Leftrightarrow \quad p+q>pq \quad \Leftrightarrow \quad p<\frac{q}{q-1} \quad \Leftrightarrow \quad q<\frac{p}{p-1}.$$

Proof. We first bound $f*g$. By applying generalized Hölder's inequality on $\frac{1}{r}+\frac{r-p}{pr}+\frac{r-q}{qr}=1$, we have

$$\begin{aligned}
|(f*g)(x)| &\leq \int_{\mathbb{R}^n} |f(x-y)|\,|g(y)|\,dy = \int_{\mathbb{R}^n} \left(|f(x-y)|^p|g(y)|^q\right)^{1/r} |f(x-y)|^{\frac{r-p}{r}}|g(y)|^{\frac{r-q}{r}}\,dy \\
&\leq \left(\int_{\mathbb{R}^n} |f(x-y)|^p|g(y)|^q\,dy\right)^{1/r} \left(\int_{\mathbb{R}^n} |f(x-y)|^p\,dy\right)^{\frac{r-p}{pr}} \left(\int_{\mathbb{R}^n} |g(y)|^q\,dy\right)^{\frac{r-q}{qr}} \\
&= \left(\int_{\mathbb{R}^n} |f(x-y)|^p|g(y)|^q\,dy\right)^{1/r} \|f\|_{L^p}^{\frac{r-p}{r}}\,\|g\|_{L^q}^{\frac{r-q}{r}}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(x-y)|\,|g(y)|\,dy\right)^r dx &\leq \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f(x-y)|^p|g(y)|^q\,dy\,dx\right)\|f\|_{L^p}^{r-p}\,\|g\|_{L^q}^{r-q} \\
&\leq \|f\|_{L^p}^{r-p}\,\|g\|_{L^q}^{r-q}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(x-y)|^p\,dx\right)|g(y)|^q\,dy = \|f\|_{L^p}^r\,\|g\|_{L^q}^r,
\end{aligned}$$

where we use Fubini's theorem in the second inequality. From the last display, we have $m(E(f,g))=0$, and $\|f*g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$. □

Remark. If $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, and $g \in L^q(\mathbb{R}^n)$ is compactly supported, then $f*g \in L^r_{\mathrm{loc}}(\mathbb{R}^n)$.

Proposition 1.4 (Convolution of compactly supported functions)**.** *Let $f,g:\mathbb{R}^n \to \mathbb{R}$.*

(i) If $f,g \in L^1(\mathbb{R}^n)$, then $\operatorname{supp}(f*g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g} := \overline{\{x+y : x \in \operatorname{supp} f, y \in \operatorname{supp} g\}}$. Furthermore, if both f and g are compactly supported on $\mathbb{R}$, then $f*g$ is also compactly supported. In this case, $\operatorname{supp}(f*g) \subset \operatorname{supp} f + \operatorname{supp} g$.

(ii) Let $1 \leq p \leq \infty$, and let $k \in \mathbb{N}_0$. If $f \in C_c^k(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then $f*g \in C_u^k(\mathbb{R}^n)$. Furthermore, differentiation commutes with convolution, i.e., $\partial^\alpha(f*g)=\partial^\alpha f*g$, for all $|\alpha| \leq k$.

(iii) Let $1 \leq p \leq \infty$. If $f \in C_c^\infty(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then $f*g \in C_u^\infty(\mathbb{R}^n)$. Similarly, differentiation commutes with convolution, i.e., $\partial^\alpha(f*g)=\partial^\alpha f*g$ for multi-indices α.

Remark. Here is a slight modification of assertions (ii) and (iii):

(ii') Let $1 \leq p \leq \infty$, and let $k \in \mathbb{N}_0$. If $f \in C_c^k(\mathbb{R}^n)$ and $g \in C_c(\mathbb{R}^n)$, then $f*g \in C_c^k(\mathbb{R}^n)$. Furthermore, differentiation commutes with convolution, i.e.,

$$\partial^\alpha(f*g)=\partial^\alpha f*g, \qquad \forall|\alpha| \leq k,$$

(iii') Let $1 \leq p \leq \infty$. If $f \in C_c^\infty(\mathbb{R}^n)$ and $g \in C_c(\mathbb{R}^n)$, then $f*g \in C_c^\infty(\mathbb{R}^n)$. Similarly, differentiation commutes with convolution, i.e., $\partial^\alpha(f*g)=\partial^\alpha f*g$ for multi-indices α.

Proof. (i) Let $f, g \in L^1(\mathbb{R}^n)$, and take any $x \in \mathbb{R}^n$. Then

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy = \int_{(x-\operatorname{supp} f)\cap \operatorname{supp} g} f(x-y)g(y)\,dy.$$

For $x \notin \operatorname{supp} f + \operatorname{supp} g$, we have $(x - \operatorname{supp} f) \cap \operatorname{supp} g = \emptyset$, which implies $(f * g)(x) = 0$. Hence

$$(f * g)(x) \neq 0 \ \Rightarrow x \in \operatorname{supp} f + \operatorname{supp} g \ \Rightarrow \ \operatorname{supp}(f * g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}.$$

If $f, g \in C_c(\mathbb{R}^n)$, then $\operatorname{supp} f$ and $\operatorname{supp} g$ are compact in $\mathbb{R}^n$. Define $\phi(x, y) = x + y$, which is a continuous map on $\mathbb{R}^n \times \mathbb{R}^n$. Then $\operatorname{supp} f + \operatorname{supp} g = \phi(\operatorname{supp} f \times \operatorname{supp} g)$ is also compact. Consequently, $\operatorname{supp} f + \operatorname{supp} g$ is closed, and its closed subset $\operatorname{supp}(f * g)$ is also compact. which implies $f * g \in C_c(\mathbb{R}^n)$.

(ii) *Step I:* We first show the case $k = 0$. Let $q = p/(p-1)$. Note that f is continuous and compact supported, then $m(\operatorname{supp} f) < \infty$, f is uniformly continuous, and $\|f\|_\infty = \max_{x\in\operatorname{supp} f} |f(x)| < \infty$. By Hölder's inequality,

$$\int_{\mathbb{R}^n} |f(x-y)|\,|g(y)|\,dy \leq \|f\|_{L^q}\|g\|_{L^p} \leq m\big(\operatorname{supp} f\big)^{1/q}\|f\|_\infty\|g\|_{L^p} < \infty, \quad x \in \mathbb{R}^n.$$

Then $f * g$ is well-defined on $\mathbb{R}^n$. To show uniform continuity of $f * g$, we fix $\epsilon > 0$ and let η be such that $|x - x'| < \eta$ implies $|f(x) - f(x')| < \epsilon$. Then

$$|(f * g)(x) - (f * g)(x')| = \left|\int_{\mathbb{R}^n} [f(x-y) - f(x'-y)]\,g(y)\,dy\right| \leq 2m\big(\operatorname{supp} f\big)^{1/q}\|g\|_{L^p}\,\epsilon.$$

Step II: We prove the case $k = 1$. It suffices to show the interchangeability of derivative and integral. Given any quantity $h > 0$, we have

$$\frac{(f * g)(x + he_i) - (f * g)(x)}{h} = \int_{\mathbb{R}^n} \frac{f(x + he_i - y) - f(x-y)}{h} g(y)\,dy. \tag{1.2}$$

Since $f \in C_c^1(\mathbb{R}^n)$, by Lagrange's mean value theorem, there exists $\xi \in [0, 1]$ such that

$$\left|\frac{f(x + he_i - y) - f(x-y)}{h}\right| = |\partial_{x_i} f(x + \xi he_i - y)|, \tag{1.3}$$

Note that $\partial_{x_i} f$ is also continuous and compactly supported on $\mathbb{R}^n$, the RHS of (1.3) is bounded by $\|\partial_{x_i} f\|_\infty$, and the integrand in (1.2) is dominated by an integrable function $\|\partial_{x_i} f\|_\infty g$. Using Lebesgue's dominate convergence theorem, we have

$$\lim_{h\to 0} \int_{\mathbb{R}^n} \frac{f(x + he_i - y) - f(x-y)}{h} g(y)\,dy = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}(x-y)g(y)\,dy.$$

Therefore $\partial_{x_i}(f * g) = \partial_{x_i} f * g$. Since $\partial_{x_i} f \in C_c(\mathbb{R}^n)$, we have $\partial_{x_i}(f * g) \in C_u(\mathbb{R}^n)$, and $f * g \in C_u^1(\mathbb{R}^n)$.

Step III: Use induction. Suppose our conclusion holds for $C_c^{k-1}(\mathbb{R}^n)$. For each $f \in C_c^k(\mathbb{R}^n) \subset C_c^{k-1}(\mathbb{R}^n)$, $\partial^{k-1} f \subset C_c^1(\mathbb{R}^n)$. By Step II, for any $|\alpha| = k - 1$,

$$\partial^{\alpha+e_i}(f * g) = \partial_{x_i}(\partial^\alpha(f * g)) = \partial_{x_i}(\partial^\alpha f * g) = (\partial^{\alpha+e_i} f) * g,$$

which is uniformly continuous on $\mathbb{R}^n$. Hence $f * g \in C_u^k(\mathbb{R}^n)$.

(iii) Note that $C_c^\infty(\mathbb{R}^n) = \bigcap_{k=0}^\infty C_c^k(\mathbb{R}^n)$, we have $\partial^\alpha(f * g) = \partial^\alpha f * g$ for all $\alpha \in \mathbb{N}_0^n$. Following Step II, $\partial^\alpha f \in C_c(\mathbb{R}^n)$ implies $\partial^\alpha(f * g) \in C_u(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$. Hence $f * g \in \bigcap_{k=0}^\infty C_u^k(\mathbb{R}^n) = C_u^\infty(\mathbb{R}^n)$. $\square$

Translation operators. Let X be a vector space, let Y^X be the set of functions $f: X \to Y$, and let s be a vector in X. The *translation operator* $\tau_s : Y^X \to Y^X$ is defined as

$$(\tau_s f)(x) = f(x-s), \quad \text{for } f \in Y^X.$$

The following proposition gives a description of the continuity of $(\tau_s)_{s\in X}$ in C_c and L^p spaces.

Proposition 1.5. *Let $1 \le p < \infty$.*

(i) For any $f \in C_c(\mathbb{R}^n)$, $\tau_s f \to f$ uniformly and in L^p-norm as $s \to 0$.

(ii) For any $f \in L^p(\mathbb{R}^n)$, $\tau_s f \to f$ in L^p-norm as $s \to 0$.

Proof. Let $f \in C_c(\mathbb{R}^n)$, and let $B_1 = \{x \in \mathbb{R}^n : |x| \le 1\}$ be the compact unit ball in $\mathbb{R}^n$. The collection of functions $\{\tau_s f : |s| \le 1\}$ has a common support

$$K = \bigcup_{|s|\le 1} \operatorname{supp}(\tau_s f) = \operatorname{supp} f + B_1 = \{x + y : x \in \operatorname{supp} f, y \in B_1\}.$$

Since the addition operation is continuous, K is also a compact subset of $\mathbb{R}^n$.

By uniform continuity of f, given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $|x - y| < \delta$. Hence $\tau_s f \to f$ uniformly as $s \to 0$. Moreover, for any s with $|s| < |\min(\delta, 1)|$, we have

$$\|\tau_s f - f\|_{L^p}^p = \int_K |f(x-s) - f(x)|^p dx \le \mu(K)\, \epsilon^p.$$

Since $\mu(K) < \infty$, and ϵ is arbitrary, we conclude that $\|\tau_s f - f\|_{L^p} \to 0$ as $s \to 0$.

Now we assume $f \in L^p(\mathbb{R}^n)$, and fix $\epsilon > 0$. Since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, there exists $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_\infty < \epsilon/3$. Choose δ such that $\|\tau_s g - g\|_{L^p} < \epsilon/3$ for all $|s| < \delta$. Then for all $|s| < \delta$,

$$\|\tau_s f - f\|_{L^p} \le \|\tau_s f - \tau_s g\|_{L^p} + \|\tau_s g - g\|_{L^p} + \|g - f\|_{L^p} = 2\|f - g\| + \|\tau_s g - g\|_{L^p} < \epsilon.$$

Therefore, $\lim_{s\to 0} \|\tau_s f - f\|_{L^p} = 0$ for all $f \in L^p(\mathbb{R}^n)$. ☐

Proposition 1.6 (Mollification)**.** *Let $\phi \in L^1(\mathbb{R}^n)$, with $\int_{\mathbb{R}^n} \phi\, dx = a$. Given $t > 0$, define*

$$\phi_t(x) = t^{-n}\phi\left(\frac{x}{t}\right). \tag{1.4}$$

(i) If f is bounded and uniformly continuous, $f * \phi_t \to af$ uniformly as $t \to 0$.

(ii) If $f \in C(\mathbb{R}^n)$, then $f * \phi_t \to af$ uniformly on every compact subset of $\mathbb{R}^n$ as $t \downarrow 0$.

(iii) If $f \in L^p(\mathbb{R}^n)$, $f * \phi_t \to af$ in $L^p(\mathbb{R}^n)$ as $t \downarrow 0$.

Proof. Using the decomposition $\phi = \phi^+ - \phi^-$, we may assume $\phi \ge 0$ on $\mathbb{R}^n$. We further assume $a = 1$ by replacing ϕ by ϕ/a if necessary. Then

$$(f * \phi_t)(x) - f(x) = \int_{\mathbb{R}^n} (f(x-y) - f(x))\phi_t(y)\, dy = \int_{\mathbb{R}^n} (\tau_y f - f)(x)\phi_t(y)\, dy.$$

For any $\epsilon > 0$, we fix $M > 0$ such that $\int_{|x|>M} \phi(x)\, dx < \epsilon$. Then $\int_{|x|>Mt} \phi_t(x)\, dx < \epsilon$.

(i) By uniform continuity of f, we have $\tau_y f \to f$ uniformly as $y \to 0$.

$$\begin{aligned}\sup_{x\in\mathbb{R}^n} |(f * \phi_t)(x) - f(x)| &\le \sup_{x\in\mathbb{R}^n} \int_{|y|\le Mt} (\tau_y f - f)(x)\phi_t(y)\, dy + \sup_{x\in\mathbb{R}^n} \int_{|y|>Mt} (\tau_y f - f)(x)\phi_t(y)\, dy \\ &\le \sup_{|y|\le Mt} \|\tau_y f - f\|_\infty + 2\|f\|_\infty \epsilon \to 2\|f\|_\infty \epsilon, \quad \text{as } t \downarrow 0.\end{aligned}$$

(ii) It suffices to take the supremum over compact sets $\{|x| \le N\},\ N = 1, 2, \cdots$ instead of $\mathbb{R}^n$ in (i), and use the uniform continuity and boundedness of $f \in C(\mathbb{R}^n)$ on compact sets.

(iii) By Jensen's inequality, Fubini's theorem and Proposition 1.5 (ii),

$$\begin{aligned}
&\int_{\mathbb{R}^n} |(f * \phi_t)(x) - f(x)|^p \, dx = \int_{\mathbb{R}^n} \left| \int_{|y| \le Mt} (\tau_y f - f)(x)\phi_t(y)\, dy + \int_{|y|>Mt} (\tau_y f - f)(x)\phi_t(y)\, dy \right|^p dx \\
&\le C_p \int_{\mathbb{R}^n} \int_{|y| \le Mt} |\tau_y f(x) - f(x)|^p \, \phi_t(y)\, dy\, dx + C_p \int_{\mathbb{R}^n} \int_{|y|>Mt} |\tau_y f(x) - f(x)|^p \, \phi_t(y)\, dy\, dx \\
&\le C_p \int_{|y| \le Mt} \left(\int_{\mathbb{R}^n} |\tau_y f(x) - f(x)|^p \, dx \right) \phi_t(y)\, dy + C_p \|f\|_{L^p}^p \int_{|y|>Mt} \phi_t(y)\, dy \\
&\le C_p \sup_{|y| \le Mt} \|\tau_y f - f\|_{L^p}^p + C_p \|f\|_{L^p}^p \epsilon \to C_p \|f\|_{L^p}^p \epsilon, \quad \text{as } t \downarrow 0,
\end{aligned}$$

where C_p is the notation for aconstant depending only on p and may vary across lines. Then let $\epsilon \downarrow 0$. ☐

When we establish the density arguments of C_c^∞ functions, the above result is very useful.

Proposition 1.7. *For $1 \le p < \infty$, $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.*

Proof. By the first assertion in Proposition 1.6, $C_c^\infty(\mathbb{R}^n)$ is dense in $C_c(\mathbb{R})$ in $\|\cdot\|_1$ norm. Since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, the result follows. ☐

Proposition 1.8. *For $1 \le p < \infty$, $C_c^\infty(\mathbb{R}^n)$ is dense in $C_0(\mathbb{R}^n)$.*

Proof. By the second assertion in Proposition 1.6, $C_c^\infty(\mathbb{R}^n)$ is dense in $C_c(\mathbb{R})$ in $\|\cdot\|_\infty$ norm. Since $C_0(\mathbb{R}^n)$ is the closure of $C_c(\mathbb{R}^n)$ in $\|\cdot\|_\infty$ norm, the result follows. ☐

Aside from the convergence in L^p-norm discussed in Proposition 1.6, we are also interested in the pointwise convergence property of mollification $f * \phi_\epsilon$.

Proposition 1.9 (Mollification)**.** *Assume $\phi \in L^1(\mathbb{R}^n)$ satisfies $|\phi(x)| \le C(1+|x|)^{-n-\gamma}$ for some $C, \gamma > 0$, and $\int_{\mathbb{R}^n} \phi\, dx = a$. Define ϕ_ϵ as in (1.4). Let $1 \le p \le \infty$. If $f \in L^p(\mathbb{R}^n)$, then $(f * \phi_\epsilon)(x) \to af(x)$ as $\epsilon \to 0$ for every Lebesgue point x of f.*

Proof. If x is a Lebesgue point of f, we have

$$\lim_{r \to 0^+} \frac{1}{r^n} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0.$$

For any $\epsilon > 0$, we choose $\delta > 0$ such that $\int_{B(x,r)} |f(y) - f(x)|\, dy < r^n \epsilon$ for all $r \le \delta$, and set

$$I_1 = \int_{|y|<\delta} |f(x-y) - f(x)|\, |\phi_t(y)|\, dy, \quad I_2 = \int_{|y| \ge \delta} |f(x-y) - f(x)|\, |\phi_t(y)|\, dy.$$

We claim that I_1 is bounded by $A\epsilon$, where A is independent of t, and $I_2 \to 0$ as $t \to 0$. Since

$$|(f * \phi_t)(x) - af(x)| \le I_1 + I_2,$$

we will have

$$\limsup_{t \to 0^+} |(f * \phi_t)(x) - af(x)| \le A\epsilon,$$

Since $\epsilon > 0$ is arbitrary, the proof will be completed.

To estimate I_1, let N be the integer such that $2^N \le \delta/t < 2^{N+1}$, if $\delta/t \ge 1$, and $N = 0$ if $\delta/t < 1$. We view the ball $|y| < \delta$ as the union of the annuli $2^{-k}\delta \le |y| < 2^{1-k}\delta$, $1 \le k \le N$ and the ball $|y| < 2^{-N}\delta$. On the k^{th} annulus we use the estimate

$$|\phi_t(y)| = \frac{1}{t^n}\left|\phi\left(\frac{y}{t}\right)\right| \le Ct^{-n}\left|\frac{y}{t}\right|^{-n-\gamma} \le Ct^{-n}\left(\frac{2^{-k}\delta}{t}\right)^{-n-\gamma}$$

and in the ball $|y| < 2^{-N}\delta$, we use the estimate $|\phi_t(y)| \le Ct^{-n}$. Thus

$$\begin{aligned}
I_1 &\le \sum_{k=1}^{N} Ct^{-n}\left(\frac{2^{-k}\delta}{t}\right)^{-n-\gamma}\int_{2^{-k}\delta\le|y|<2^{1-k}\delta}|f(x-y)-f(x)|\,dy + Ct^{-n}\int_{|y|<2^{-N}\delta}|f(x-y)-f(x)|\,dy \\
&\le C\epsilon\sum_{k=1}^{N}(2^{1-k}\delta)^n t^{-n}\left(\frac{2^{-k}\delta}{t}\right)^{-n-\gamma} + C\epsilon(2^{-N}\delta)^n t^{-n} = 2^nC\epsilon\left(\frac{\delta}{t}\right)^{-\gamma}\sum_{k=1}^{N}2^{k\gamma} + C\epsilon\left(\frac{2^{-N}\delta}{t}\right)^n \\
&= 2^nC\epsilon\left(\frac{\delta}{t}\right)^{-\gamma}\frac{2^{(N+1)\gamma}-2^\gamma}{2^\gamma-1} + C\epsilon\left(\frac{2^{-N}\delta}{t}\right)^n \le \underbrace{2^nC\left(\frac{2^\gamma}{2^\gamma-1}+1\right)}_{=:A}\epsilon.
\end{aligned}$$

As for I_2, if q is the conjugate exponent to p and χ is the characteristic function of the set $\{y \in \mathbb{R}^n : |y| \ge \delta\}$,

$$I_2 \le \int_{|y|\ge\delta}(|f(y-x)| - |f(x)|)\,|\phi_t(y)|\,dy \le \|f\|_{L^p}\|\chi\phi_t\|_{L^q} + |f(x)|\,\|\chi\phi_t\|_{L^1}.$$

If $q = \infty$,

$$\|\chi\phi_t\|_{L^\infty} \le Ct^{-n}\left(1+\frac{\delta}{t}\right)^{-n-\gamma} = \frac{Ct^\delta}{(t+\delta)^{n+\gamma}} \le \frac{Ct^\delta}{\delta^{n+\gamma}},$$

which converges to 0 as $t \to 0$. If $1 \le q < \infty$, we switch to the sphere coordinates:

$$\begin{aligned}
\|\chi\phi_t\|_{L^q} &= \int_{|y|\ge\delta} t^{-nq}\left|\phi\left(\frac{y}{t}\right)\right|^q dy = \int_{|z|\ge\delta/t} t^{n(1-q)}\,|\phi(z)|^q\,dz \\
&\le C_n t^{n(1-q)}\int_{\delta/t}^{\infty} r^{n-1}C(1+r)^{-(n+\gamma)q}\,dr \\
&\le C_nCt^{n(1-q)}\int_{\delta/t}^{\infty} r^{n-1-(n+\gamma)q}\,dr \\
&= C_nCt^{n(1-q)}\frac{(\delta/t)^{n-(n+\gamma)q}}{(n+\gamma)q-n} = \frac{C_nC\delta^{n-(n+\gamma)q}t^{\gamma q}}{(n+\gamma)q-n},
\end{aligned}$$

which also converges to 0 as $t \to 0$. Therefore $I_2 \to 0$ as $t \to 0$, and we are done. □

Finally we see an application of the mollification.

Proposition 1.10 (C^∞-Urysohn lemma)**.** *Let $U \subset \mathbb{R}^n$ be an open set, and let $K \subset U$ be a compact set. There exists $f \in C_c^\infty(U)$ such that $0 \le f \le 1$, and $f = 1$ on K.*

Proof. Since K is compact and U is open, we take $0 < \epsilon < d(K, U^c)$. Define

$$V = \left\{x \in U : d(x,K) \le \frac{\epsilon}{3}\right\}, \quad \text{and} \quad W = \left\{x \in U : d(x,K) < \frac{2\epsilon}{3}\right\}.$$

Then V is a compact set, W is an open set, and $K \subset V^\circ \subset V \subset W \subset \overline{W} \subset U$. By Urysohn's lemma, there exists $g \in C_c(W)$ such that $0 \le g \le 1$ and $g = 1$ on V. Now we choose $\phi \in C_c^\infty(\mathbb{R}^n)$ such that ϕ is supported on the closed ball $\overline{B(0, \frac{\epsilon}{3})}$ and $\int_{\mathbb{R}^n}\phi(x)\,dx = 1$. Then $f = g * \phi$ is the desired function. □

1.2 The Schwartz Space

Definition 1.11 (Schwartz space)**.** The *Schwartz space* consists of all C^∞-functions, which, together with their derivatives, vanishes at infinity faster than any power of $|x|$. More precisely, for any $f \in C^\infty(\mathbb{R}^n)$, any nonnegative integer N and any multi-index $\alpha \in \mathbb{N}_0^n$, define the norm

$$\|f\|_{(N,\alpha)} = \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha f(x)|.$$

The Schwartz space is

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \|f\|_{(N,\alpha)} < \infty \text{ for all } N \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^n \right\}.$$

Remark. For any $\phi \in C_c^\infty(\mathbb{R}^n)$, all its derivatives are also C_c^∞, and

$$\|\phi\|_{(N,\alpha)} \leq \sup_{x\in\operatorname{supp}\phi} (1+|x|)^N \|\partial^\alpha \phi\|_\infty < \infty.$$

Therefore, we have $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.

Proposition 1.12. *The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space under the topology induced by norms $\|\cdot\|_{(N,\alpha)}$.*

Proof. It suffices to show the completeness of $\mathcal{S}(\mathbb{R}^n)$. Let (f_k) be a Cauchy sequence in $\mathcal{S}(\mathbb{R}^n)$, which implies that $\|f_k - f_m\|_{(N,\alpha)} \to 0$ as $k, m \to \infty$ for all $N \in \mathbb{N}_0$ and all multi-indices $\alpha \in \mathbb{N}_0^n$. In particular, for each α, the sequence $(\partial^\alpha f_k)$ converges uniformly to a function g_α. We denote by $e_j = (0, \cdots, \underset{j\text{-th}}{1}, 0, \cdots, 0)$. Then

$$f_k(x + he_j) - f_k(x) = \int_0^h \frac{\partial f_k}{\partial x_j}(x + te_j)\, dt.$$

Letting $k \to \infty$ and apply dominated convergence theorem, we obtain $g_0(x+he_j) - g_0(x) = \int_0^h g_{e_j}(x+te_j)\, dt$, which implies that $\partial_{x_j} g_0 = g_{e_j}$ by the fundamental theorem of calculus. An inductive argument on $|\alpha|$ implies $D^\alpha g_0 = g_\alpha$. Then $\|f_k - g_0\|_{(N,\alpha)} \to 0$ for all $N \in \mathbb{N}_0$ and all $\alpha \in \mathbb{N}_0^n$. $\square$

Proposition 1.13 (Characterization of Schwartz space)**.** *Let $f \in C^\infty(\mathbb{R}^n)$. The following are equivalent:*

(i) $f \in \mathcal{S}(\mathbb{R}^n)$;

(ii) For all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, the function $x^\beta \partial^\alpha f$ is bounded;

(iii) For all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, the function $\partial^\alpha(x^\beta f)$ is bounded.

Proof. To show (i) ⇒ (ii), note that $|x|^\beta \leq (1+|x|)^N$ for $|\beta| \leq N$. On the other hand, if (ii) holds, we fix an order $N \in \mathbb{N}$ and a multi-index $\alpha \in \mathbb{N}_0^n$, and take

$$\delta_N = \min\left\{ \sum_{j=1}^n |x_j|^N : |x|^2 = \sum_{j=1}^n |x_j|^2 = 1 \right\} > 0.$$

By homogeneity, we have $\sum_{j=1}^n |x_j|^N \geq \delta_N |x|^N$ for all $x \in \mathbb{R}^n$, and

$$(1+|x|)^N \leq 2^N \left(1 + |x|^N\right) \leq 2^N \left(1 + \frac{1}{\delta_N} \sum_{j=1}^n |x_j|^N \right) \leq \frac{2^N}{\delta_N} \sum_{|\beta|\leq N} |x^\beta|.$$

Hence (ii) ⇒ (i). The equivalence of (ii) and (iii) follows from the fact that each $\partial^\alpha(x^\beta f)$ is a linear combination of terms of the form $x^\delta \partial^\gamma f$ and vice versa, by the product rule. $\square$

Proposition 1.14. *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then $f * g \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, if two sequences $f_j \to f$ and $g_j \to g$ in $\mathcal{S}(\mathbb{R}^n)$, then $f_j * g_j \to f * g$ in $\mathcal{S}(\mathbb{R}^n)$.*

Proof. (i) By Proposition 1.4 (iii'), we have $f * g \in C^\infty(\mathbb{R}^n)$. Furthermore, since

$$1 + |x| \leq 1 + |x - y| + |y| \leq (1 + |x - y|)(1 + |y|), \tag{1.5}$$

we have for all order $N \in \mathbb{N}_0$ and multi-index $\alpha \in \mathbb{N}_0^n$ that

$$\begin{aligned}(1 + |x|)^N |\partial^\alpha (f * g)(x)| &\leq \int_{\mathbb{R}^n} (1 + |x - y|)^N |\partial^\alpha f(x - y)| (1 + |y|)^N |g(y)| \, dy \\ &\leq \|f\|_{(N,\alpha)} \|g\|_{(N+n+1,0)} \int_{\mathbb{R}^n} (1 + |y|)^{-n-1} \, dy \\ &\leq \|f\|_{(N,\alpha)} \|g\|_{(N+n+1,0)} \int_0^\infty \frac{C_n}{1 + r^2} \, dr < \infty,\end{aligned}$$

where C_n is some constant depends only on the dimension n. Hence $f * g \in \mathcal{S}(\mathbb{R}^n)$.

(ii) Using the above estimate, for all $N \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$, we also have

$$\begin{aligned}&\|f_j * g_j - f * g\|_{(N,\alpha)} \leq \|f * (g_j - g)\|_{(N,\alpha)} + \|(f_j - f) * g\|_{(N,\alpha)} + \|(f_j - f) * (g_j - g)\|_{(N,\alpha)} \\ &\quad \leq \frac{C_n \pi}{2} \left[\|f\|_{(N,\alpha)} \|g_j - g\|_{(N+n+1,0)} + \|f_j - f\|_{(N,\alpha)} \|g\|_{(N+n+1,0)} + \|f_j - f\|_{(N,\alpha)} \|g_j - g\|_{(N+n+1,0)} \right],\end{aligned}$$

which converges to 0 as $j \to \infty$. Hence $f_j * g_j \to f * g$. □

Proposition 1.15. *$\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ and in $C_0(\mathbb{R}^n)$.*

Proof. Since $\mathcal{S}(\mathbb{R}^n) \supset C_c^\infty(\mathbb{R}^n)$, the result follows from Propositions 1.7 and 1.8. □

2 Fourier Transform

2.1 Motivation: Fourier Series

In this part, we study the periodic functions on $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{C}$ is said to be 2π-*periodic*, if

$$f(x + 2\pi\kappa) = f(x)$$

for all $x \in \mathbb{R}^n$ and all $\kappa \in \mathbb{Z}^n$. According to periodicity, every 2π-periodic function f is completely determined by its values on the cube $[0, 2\pi)^n$. Hence we may regard f as a function on the quotient space

$$\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n = \{x + 2\pi\mathbb{Z}^n : x \in \mathbb{R}^n\}.$$

We call $\mathbb{T}^n$ the n-dimensional torus. For measure-theoretic purposes, we identify $\mathbb{T}^n$ with the cube $Q = [0, 2\pi)^n$, and the Lebesgue measure on $\mathbb{T}^n$ is induced by Lebesgue measure on Q. In particular, $m(\mathbb{T}^n) = m(Q) = (2\pi)^n$. Functions on $\mathbb{T}^n$ maybe considered as periodic functions on $\mathbb{R}^n$ or as functions Q, depending on the context.

Theorem 2.1. *The functions $(e^{i\kappa\cdot x})_{\kappa\in\mathbb{Z}^n}$ form an orthogonal basis of $L^2(\mathbb{T}^n)$.*

Proof. Let $\mathcal{A}$ be the set of all finite linear combinations of $e^{i\kappa\cdot x}$. Then $\mathcal{A}$ is a self-adjoint algebra that separates points and vanishes at no points of $\mathbb{T}^n$. Since $\mathbb{T}^n$ is compact, by Stone-Weierstrass theorem, $\mathcal{A}$ is dense in $C(\mathbb{T}^n)$ in the supremum norm, and hence in L^2-norm. Since $C(\mathbb{T}^n)$ is dense in $L^2(\mathbb{T}^n)$, the result follows. $\square$

The Fourier series of a periodic function is then defined by its expansion under the orthogonal basis.

Definition 2.2. If $f \in L^2(\mathbb{T}^n)$, we define its Fourier transform $\widehat{f} : \mathbb{Z}^n \to \mathbb{C}$ by

$$\widehat{f}(\kappa) = \frac{\langle f, e^{i\kappa\cdot x}\rangle_{L^2}}{\langle e^{i\kappa\cdot x}, e^{i\kappa\cdot x}\rangle_{L^2}} = \frac{1}{(2\pi)^n}\int_Q f(x)e^{-i\kappa\cdot x}\,dx, \tag{2.1}$$

and we call the series $\sum_{\kappa\in\mathbb{Z}^n}\widehat{f}(\kappa)e^{i\kappa\cdot x}$ the Fourier series of f.

Remark. (I) According to Theorem 2.1, the Fourier series of a function $f \in L^2(\mathbb{T}^n)$ converges to f in L^2. Consequently, we have the Parseval's equality:

$$\|\widehat{f}\|_{\ell^2}^2 := \sum_{\kappa\in\mathbb{Z}^n}|\widehat{f}(\kappa)|^2 = \frac{1}{(2\pi)^n}\|f\|_{L^2}^2.$$

Hence the Fourier transform $\mathcal{F}$ maps $L^2(\mathbb{T}^n)$ onto $\ell^2(\mathbb{Z}^n)$.

(II) In fact, the definition (2.1) of Fourier transform makes sense if $L^1(\mathbb{T}^n)$, and $|\widehat{f}(\kappa)| \le (2\pi)^{-n}\|f\|_{L^1}$. Hence the Fourier transform $\mathcal{F}$ is a bounded linear map from $L^1(\mathbb{T}^n)$ to $\ell^\infty(\mathbb{Z}^n)$.

Theorem 2.3 (Convolution Theorem)**.** *Let $f, g \in L^1(\mathbb{T}^n)$. Then*

$$\widehat{f * g} = (2\pi)^n\widehat{f}\,\widehat{g}.$$

Proof. By Young's convolution inequality [Proposition 1.3], $f * g \in L^1(\mathbb{T}^n)$. By Fubini's theorem,

$$\begin{aligned}\widehat{(f * g)}(\kappa) &= \frac{1}{(2\pi)^n}\int_Q\int_Q f(x-y)g(y)e^{-i\kappa\cdot x}\,dy\,dx = \int_Q\left(\frac{1}{(2\pi)^n}\int_Q f(x-y)e^{-i\kappa\cdot(x-y)}\,dx\right)g(y)e^{-i\kappa\cdot y}\,dy\\ &= \widehat{f}(\kappa)\int_Q g(y)e^{-i\kappa\cdot y}\,dy = (2\pi)^n\widehat{f}(\kappa)\,\widehat{g}(\kappa).\end{aligned}$$

Thus we finish the proof. $\square$

2.2 Fourier Transform on $L^1(\mathbb{R}^n)$

Definition 2.4 (Fourier transform and inversion)**.** For $f \in L^1(\mathbb{R}^n)$, we define its *Fourier transform* by

$$(\mathcal{F}f)(\omega) = \widehat{f}(\omega) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i\omega\cdot x}\, dx, \quad \omega \in \mathbb{R}^n,$$

and its *inverse Fourier transform* by

$$(\mathcal{F}^{-1}f)(x) = f^\vee(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\omega) e^{i\omega\cdot x}\, d\omega, \quad x \in \mathbb{R}^n.$$

Remark. By definition, both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are linear operators. That is, for all $f, g \in L^1(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{C}$,

$$\mathcal{F}(\alpha f + \beta g) = \alpha \mathcal{F} f + \beta \mathcal{F} g, \quad \mathcal{F}^{-1}(\alpha f + \beta g) = \alpha \mathcal{F}^{-1} f + \beta \mathcal{F}^{-1} g.$$

Also, we have $f^\vee(x) = \widehat{f}(-x)$. In the sequel, we first consider the Fourier transform.

Theorem 2.5 (Riemann-Lebesgue lemma)**.** *The Fourier transform $\mathcal{F}$ maps $L^1(\mathbb{R}^n)$ into $C_0(\mathbb{R}^n)$.*

Proof. Fix $f \in L^1(\mathbb{R}^n)$. By definition, for all $\omega \in \mathbb{R}^n$,

$$|\widehat{f}(\omega)| \le (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f(x)|\, dx.$$

Hence $\widehat{f}$ is bounded, and

$$\|\widehat{f}\|_\infty \le (2\pi)^{-n/2} \|f\|_{L^1}. \tag{2.2}$$

To show continuity of $\widehat{f}$, use dominated convergence theorem:

$$\begin{aligned}
\lim_{h\to 0} f(\omega + h) - f(\omega) &= (2\pi)^{-n/2} \lim_{h\to 0} \int \underbrace{f(x) e^{-ix\cdot\omega} \left(e^{-ix\cdot h} - 1\right)}_{\text{dominated by } 2|f| \in L^1(\mathbb{R}^n)} dx \\
&= (2\pi)^{-n/2} \int f(x) e^{-ix\cdot\omega} \lim_{h\to 0} \left(e^{-ix\cdot h} - 1\right) dx = 0.
\end{aligned}$$

Hence $\widehat{f}$ is a bounded continuous function. It remains to show that $\widehat{f}(\omega) \to 0$ as $|\omega| \to \infty$. Note that

$$\begin{aligned}
\widehat{f}(\omega) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\omega}\, dx &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f\left(x + \frac{\omega\pi}{|\omega|^2}\right) e^{-i\left(x + \frac{\omega\pi}{|\omega|^2}\right)\cdot\omega}\, dx \\
= -(2\pi)^{-n/2} \int_{\mathbb{R}^n} f\left(x + \frac{\omega\pi}{|\omega|^2}\right) e^{-ix\cdot\omega}\, dx.
\end{aligned}$$

By averaging,

$$\begin{aligned}
|\widehat{f}(\omega)| &= \frac{(2\pi)^{-n/2}}{2} \left| \int_{\mathbb{R}^n} \left(f(x) - f\left(x + \frac{\omega\pi}{|\omega|^2}\right)\right) e^{-ix\cdot\omega}\, dx \right| \\
&\le \frac{(2\pi)^{-n/2}}{2} \int_{\mathbb{R}^n} \left| f(x) - f\left(x + \frac{\omega\pi}{|\omega|^2}\right)\right| dx \\
&= \frac{(2\pi)^{-n/2}}{2} \|f - \tau_h f\|_{L^1}, \quad \text{where } h = -\frac{\omega\pi}{|\omega|^2}.
\end{aligned}$$

By translation continuity, the last display converges to 0 as $|\omega| \to \infty$. ☐

Remark. By (2.2), the Fourier transform $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ is a bounded linear operator.

Proposition 2.6 (Properties of Fourier transform)**.** *Let $f, g \in L^1(\mathbb{R}^n)$.*

(i) $\int_{\mathbb{R}^n} \widehat{f}(x) g(x)\, dx = \int_{\mathbb{R}^n} f(x)\widehat{g}(x)\, dx$.

(ii) $(\overline{f})^\wedge = \overline{f^\vee}$, *and* $(\overline{f})^\vee = \overline{f^\wedge}$.

(iii) (Translation/Modulation) Let $\xi \in \mathbb{R}^n$. *Then*

$$\widehat{(\tau_\xi f)}(\omega) = e^{-i\omega\cdot\xi}\widehat{f}(\omega), \quad \text{and} \quad \widehat{e^{i\xi\cdot x} f} = \tau_\xi \widehat{f}.$$

(iv) (Linear transformation) If $T : \mathbb{R}^n \to \mathbb{R}^n$ *is an invertible linear transformation, and* $S = (T^*)^{-1}$ *is its inverse transpose, then*

$$\widehat{f \circ T} = |\det T|^{-1}\, \widehat{f} \circ S.$$

In particular, if T is a rotation matrix, i.e. $T^*T = TT^* = \mathrm{Id}$, then $\widehat{f\circ T} = \widehat{f} \circ T$; if $Tx = t^{-1}x$ is a dilation, then $\widehat{(f \circ T)}(\omega) = t^n \widehat{f}(t\omega)$.

Proof. (i) By Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^n} \widehat{f}(x) g(x)\, dx &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(\omega) e^{-i\omega\cdot x}\, d\omega \right) g(x)\, dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\omega) g(x) e^{-i\omega\cdot x}\, dx\, d\omega = \int_{\mathbb{R}^n} f(\omega)\widehat{g}(\omega)\, d\omega.\end{aligned}$$

(ii) We only prove the first identity (the second is similar):

$$\int_{\mathbb{R}^n} \overline{f(x)} e^{-i\omega\cdot x}\, dx = \overline{\int_{\mathbb{R}^n} f(x) e^{i\omega\cdot x}\, dx} = \overline{f^\vee(x)}.$$

(iii) By definition,

$$\widehat{(\tau_\xi f)}(\omega) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x-\xi) e^{-i\omega\cdot x}\, dx = \frac{1}{(2\pi)^{n/2}} e^{-i\omega\cdot\xi} \int_{\mathbb{R}^n} f(x-\xi) e^{-i\omega\cdot(x-\xi)}\, dx = e^{i\omega\cdot\xi}\widehat{f}(\omega),$$

and

$$\widehat{(e^{i\xi\cdot x} f)}(\omega) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi\cdot x} f(x) e^{-i\omega\cdot x}\, dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i(\omega-\xi)\cdot x}\, dx = \widehat{f}(\omega - \xi).$$

(iv) By definition,

$$\begin{aligned}\widehat{(f\circ T)}(\omega) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(Tx) e^{i\omega\cdot x}\, dx \\ &= \frac{1}{(2\pi)^{n/2}} \frac{1}{|\det T|} \int_{\mathbb{R}^n} f(y) e^{i\omega\cdot T^{-1}y}\, dy \\ &= \frac{1}{(2\pi)^{n/2}} \frac{1}{|\det T|} \int_{\mathbb{R}^n} f(y) e^{iS\omega\cdot y}\, dy = \frac{\widehat{f}(S\omega)}{|\det T|}.\end{aligned}$$

Thus we finish the proof. □

Remark. Let $\epsilon > 0$. Recall our notation that $\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon})$, we have

$$\widehat{\phi_\epsilon}(\omega) = \widehat{\phi}(\epsilon\omega).$$

Moreover, if we let $g(x) = f(-x)$,

$$\widehat{g}(x) = \widehat{f}(-x) = f^\vee(x).$$

Next we discuss the relation between Fourier transform and differentiation.

Proposition 2.7 (Differentiation). *Let $k \in \mathbb{N}_0$ and $f \in L^1(\mathbb{R}^n)$.*

(i) If $x^\alpha f \in L^1(\mathbb{R}^n)$ for all multi-indices $|\alpha| \le k$, then $\widehat{f} \in C^k(\mathbb{R}^n)$, and

$$\partial^\alpha \widehat{f} = [(-ix)^\alpha f]^\frown$$

(ii) If $f \in C^k(\mathbb{R}^n)$, $\partial^\alpha f \in L^1(\mathbb{R}^n)$ for all multi-indices $|\alpha| \le k$, and $\partial^\alpha f \in C_0(\mathbb{R}^n)$ for all $|\alpha| \le k-1$, then

$$\widehat{\partial^\alpha f}(\omega) = (i\omega)^\alpha \widehat{f}(\omega).$$

Proof. (i) Let $F(x,\omega) = f(x)e^{-i\omega\cdot x}$. Then

$$\frac{\partial F}{\partial \omega_j}(x,\omega) = -ix_j f(x)e^{-i\omega\cdot x}, \quad j = 1,2,\cdots,n.$$

Fix $j \in \{1,2,\cdots,n\}$. Note that when h is near 0, we have

$$\left|\frac{F(x,\omega+he_j) - F(x,\omega)}{h}\right| = \left|\frac{\mathrm{e}^{-ihx_j}-1}{h}\right| |f(x)| \le 2|x_j f(x)|.$$

Since $x_j f \in L^1(\mathbb{R}^n)$, by dominated convergence theorem,

$$\begin{aligned}
\lim_{h\to 0}\frac{\widehat{f}(\omega+he_j)-\widehat{f}(\omega)}{h} &= \frac{1}{(2\pi)^{n/2}}\lim_{h\to 0}\int_{\mathbb{R}^n}\frac{F(x,\omega+he_j)-F(x,\omega)}{h}\,dx \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\lim_{h\to 0}\frac{F(x,\omega+he_j)-F(x,\omega)}{h}\,dx \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} -ix_j f(x)e^{-i\omega\cdot x}\,dx = \widehat{-ix_j f}.
\end{aligned}$$

(ii) Consider $|\alpha| = 1$. Since $\partial^\alpha f \in L^1(\mathbb{R}^n)$ and $f \in C_0(\mathbb{R}^n)$, use Fubini's theorem and integrate by parts:

$$\begin{aligned}
\widehat{\frac{\partial f}{\partial x_j}}(\omega) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\frac{\partial f}{\partial x_j}(x)e^{-i\omega\cdot x}\,dx = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\left(\int_{-\infty}^{\infty}\frac{\partial f}{\partial x_j}(x)e^{-i\omega_j x_j}\,dx_j\right)e^{-i\omega_{-j}\cdot x_{-j}}\,dx_{-j} \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\left(f(x)e^{-i\omega_j x_j}\Big|_{x_j=-\infty}^{x_j=\infty} + i\omega_j\int_{-\infty}^{\infty} f(x)e^{-i\omega_j x_j}\,dx_j\right)e^{-i\omega_{-j}\cdot x_{-j}}\,dx_{-j} \\
&= \frac{i\omega_j}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x)e^{-i\omega\cdot x}\,dx = i\omega_j\widehat{f}(\omega).
\end{aligned}$$

Hence we prove the case $k = |\alpha| = 1$ for (i) and (ii). The general case follows from induction on $|\alpha|$. $\square$

Theorem 2.8 (Convolution Theorem). *Let $f, g \in L^1(\mathbb{R}^n)$. Then*

$$\widehat{f * g} = (2\pi)^{n/2}\widehat{f}\,\widehat{g}.$$

Proof. By Young's convolution inequality [Proposition 1.3], $f * g \in L^1(\mathbb{R}^n)$. By Fubini's theorem,

$$\begin{aligned}
\widehat{(f*g)}(\omega) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)g(y)e^{-i\omega\cdot x}\,dy\,dx = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x-y)e^{-i\omega\cdot(x-y)}g(y)e^{-i\omega\cdot y}\,dy\,dx \\
&= \int_{\mathbb{R}^n}\left(\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x-y)e^{-i\omega\cdot(x-y)}\,dx\right)g(y)e^{-i\omega\cdot y}\,dy \\
&= \widehat{f}(\omega)\int_{\mathbb{R}^n} g(y)e^{-i\omega\cdot y}\,dy = (2\pi)^{n/2}\widehat{f}(\omega)\,\widehat{g}(\omega).
\end{aligned}$$

Thus we finish the proof. $\square$

We compute the Fourier transform of a function.

Lemma 2.9. *Define the function* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *by* $\varphi(x) = e^{-\frac{|x|^2}{2}}$. *Then* $\varphi = \widehat{\varphi} = \varphi^\vee$.

Proof. For all $\omega \in \mathbb{R}^n$,

$$\begin{aligned}
\widehat{\varphi}(\omega) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} e^{-ix\cdot\omega}\, dx = \prod_{j=1}^n \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x_j^2/2} e^{-ix_j\omega_j}\, dx_j \right) \\
&= \prod_{j=1}^n \left(\frac{e^{-\omega_j^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x_j + i\omega_j)^2/2}\, dx_j \right) = \prod_{j=1}^n \left(\frac{e^{-\omega_j^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x_j^2/2}\, dx_j \right) \\
&= \prod_{j=1}^n e^{-\omega_j^2/2} = e^{-\frac{|\omega|^2}{2}}.
\end{aligned}$$

Hence $\widehat{\varphi} = \varphi$. The case $\varphi^\vee = \varphi$ is similar. $\square$

Now we discuss how to recover a function f from its Fourier transform $\widehat{f}$.

Theorem 2.10 (Fourier inversion theorem). *Let* $f \in L^1(\mathbb{R}^n)$. *If* $\widehat{f} \in L^1(\mathbb{R}^n)$, *then* $(\widehat{f})^\vee = f$ *a.e.*.

Proof. We take the function φ in Lemma 2.9. Consider the function

$$f^t(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(t\omega)\widehat{f}(\omega) e^{i\omega\cdot x}\, d\omega = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(t\omega) f(y) e^{i\omega\cdot(x-y)}\, dy\, d\omega.$$

Since $0 \le \varphi \le 1$ is bounded, $|\varphi(t\omega)\widehat{f}(\omega)| \le \widehat{f}(\omega)$. Since $\widehat{f} \in L^1(\mathbb{R}^n)$, by dominated convergence theorem,

$$\lim_{t\to 0} f^t(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \lim_{t\to 0} \varphi(t\omega)\widehat{f}(\omega) e^{i\omega\cdot x}\, d\omega = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{f}(\omega) e^{i\omega\cdot x}\, d\omega = (\widehat{f})^\vee(x), \quad \forall x \in \mathbb{R}^n.$$

On the other hand, if we show that $f^t \to f$ in L^1 as $t \to 0$, the result follows. By Fubini's theorem,

$$\begin{aligned}
f^t(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi(t\omega) f(y) e^{i\omega\cdot(x-y)}\, dy\, d\omega \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(t\omega) e^{i\omega\cdot(x-y)}\, d\omega \right) f(y)\, dy \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(\frac{t^{-d}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(\xi) f(y) e^{i\frac{\xi}{t}\cdot(x-y)}\, d\xi \right) dy \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} t^{-d} \varphi\left(\frac{x-y}{t}\right) f(y)\, dy = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi_t\,(x-y)\, f(y)\, dy.
\end{aligned}$$

By Proposition 1.6, $\varphi_t * f \to (2\pi)^{n/2} f$ in L^1. Thus we complete the proof. $\square$

Remark. We also have $\mathcal{F} f^\vee = f$ *a.e.* under the same assumption. To show this, let $g(x) = f(-x)$. Then

$$(\widehat{g})^\vee(x) = (\mathcal{F}^{-1} f^\vee)(x) = (\mathcal{F} f^\vee)(-x).$$

Since $(\widehat{g})^\vee = g$ a.e. and $g(x) = f(-x)$, the result follows.

Corollary 2.11. *If* $f \in L^1(\mathbb{R}^n)$ *and* $\widehat{f} = 0$ *a.e.*, *then* $f = 0$ *a.e.*.

Proof. Clearly $\widehat{f} = 0 \in L^1(\mathbb{R}^n)$. Then $f = (\widehat{f})^\vee = 0$. Here all equalities hold in a.e. sense. $\square$

Remark. Also, if $f \in L^1(\mathbb{R}^n)$ and $f^\vee = 0$ *a.e.*, then $f = 0$ *a.e.*.

2.3 Fourier Transform on $L^2(\mathbb{R}^n)$

In this section, we extend the Fourier transform to the space $L^2(\mathbb{R}^n)$.

Theorem 2.12. *The Fourier transform $\mathcal{F}$ is an isomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ onto itself.*

Proof. Take $f \in \mathcal{S}(\mathbb{R}^n)$. By Proposition 1.13 (i), $x^\beta \partial^\alpha f \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. By Proposition 2.7 (i), $\widehat{f} \in C^\infty(\mathbb{R}^n)$, and

$$\widehat{x^\beta \partial^\alpha f} = i^{|\beta|} \partial^\beta \widehat{(\partial^\alpha f)} = i^{|\alpha|+|\beta|} \partial^\beta (\omega^\alpha \widehat{f}).$$

Since $x^\beta \partial^\alpha f \in L^1(\mathbb{R}^n)$, we have $\partial^\beta(\omega^\alpha \widehat{f}) \in C_0(\mathbb{R}^n)$, which is bounded. By Proposition 1.13 (ii), $\widehat{f} \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, since $\int_{\mathbb{R}^n} (1+|x|)^{-n-1}\, dx < \infty$, by Hölder's inequality,

$$\left\|\partial^\beta(\omega^\alpha \widehat{f})\right\|_\infty = \left\|\widehat{x^\beta \partial^\alpha f}\right\|_\infty \leq \left\|x^\beta \partial^\alpha f\right\|_{L^1} \leq C \left\|(1+|x|)^{n+1} x^\beta \partial^\alpha f\right\|_\infty \leq C \|f\|_{(|\beta|+n+1,\alpha)}.$$

Following the proof of Proposition 1.13, we have $\|\widehat{f}\|_{(N,\alpha)} \leq C_{N,\alpha} \sum_{|\gamma|\leq|\alpha|} \|f\|_{(N+n+1,\gamma)}$. Hence the Fourier transform $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R}^n)$ continuously into itself. On the other hand, since $f^\vee(x) = \widehat{f}(-x)$, the inverse Fourier transform $\mathcal{F}^{-1}$ also maps the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ into itself. By Fourier inversion theorem [Theorem 2.10], these maps are inverse to each other on $\mathcal{S}(\mathbb{R}^n)$. Hence we complete the proof. $\square$

Theorem 2.13 (Plancherel)**.** *$\mathcal{F}$ extends from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to a unitary isomorphism on $L^2(\mathbb{R}^n)$.*

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, and let $h = \overline{\widehat{g}}$. Then

$$\widehat{h}(\omega) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \overline{\widehat{g}(x)} e^{-i\omega\cdot x}\, dx = \frac{1}{(2\pi)^{n/2}} \overline{\int_{\mathbb{R}^n} \widehat{g}(x) e^{i\omega\cdot x}\, dx} = \overline{(\widehat{g})^\vee(\omega)}$$

By Fourier inversion theorem, we have $\widehat{h} = \overline{g}$. Hence

$$\begin{aligned}
\langle f, g\rangle_{L^2} &= \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}^n} f(x)\widehat{h}(x)\, dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x) h(\omega) e^{-i\omega\cdot x}\, d\omega\, dx \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x) e^{-i\omega\cdot x}\, dx\right) h(\omega)\, d\omega \qquad \text{(By Fubini's theorem)} \\
&= \int_{\mathbb{R}^n} \widehat{f}(\omega) h(\omega)\, d\omega = \int_{\mathbb{R}^n} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}\, d\omega = \langle \widehat{f}, \widehat{g}\rangle_{L^2}.
\end{aligned}$$

Hence $\mathcal{F}|_{\mathcal{S}(\mathbb{R}^n)}$ preserves the L^2 inner product. Now for each $f \in L^2(\mathbb{R}^n)$, since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we can take a sequence $f_k \in \mathcal{S}(\mathbb{R}^n)$ with $f_k \to f$ in L^2. Then $(\widehat{f_k})$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$:

$$\lim_{k,j\to\infty} \|\widehat{f_k} - \widehat{f_j}\|_{L^2} = \lim_{k,j\to\infty} \|\widehat{f_k - f_j}\|_{L^2} = \lim_{k,j\to\infty} \|f_k - f_j\|_{L^2} = 0.$$

This sequence converges to a limit $\widehat{f} = \mathcal{F}f \in L^2(\mathbb{R}^n)$. If $g_k \in \mathcal{S}(\mathbb{R}^n)$ with $g_k \to f$ in L^2, we have

$$\|\widehat{g} - \widehat{f}\|_{L^2} = \lim_{k\to\infty} \|\widehat{g}_k - \widehat{f}_k\|_{L^2} = \lim_{k\to\infty} \|g_k - f_k\|_{L^2} \leq \lim_{k\to\infty} \|g_k - f\|_{L^2} + \lim_{k\to\infty} \|f - f_k\|_{L^2} = 0.$$

Hence the limit does not depend on the choice of the sequence (f_k), and the transform $\widehat{f} = \mathcal{F}f$ is well-defined. Furthermore, for all $f, g \in L^2(\mathbb{R}^n)$,

$$\langle f, g\rangle_{L^2} = \langle \widehat{f}, \widehat{g}\rangle_{L^2}.$$

Hence $\mathcal{F}$ is a unitary isomorphism on $L^2(\mathbb{R}^n)$. $\square$

Remark. Likewise, $\mathcal{F}^{-1}$ also extends from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to a unitary isomorphism on $L^2(\mathbb{R}^n)$.

Corollary 2.14. *Let $f \in L^2(\mathbb{R}^n)$. Then $(\widehat{f})^\vee = (f^\vee)^\wedge = f$ in L^2.*

Proof. Take a sequence $f_k \in \mathcal{S}(\mathbb{R}^n)$ with $f_k \to f$ in L^2. Then $\widehat{f_k} \to \widehat{f}$ in L^2, and $f_k = (\widehat{f_k})^\vee \to (\widehat{f})^\vee$ in L^2. The proof for the identity $(f^\vee)^\wedge$ is similar. $\square$

Similarly we have the convolution theorem for L^2-functions.

Proposition 2.15. *If $f, g \in L^2(\mathbb{R}^n)$, then*

$$\widehat{f}\,\widehat{g} = (2\pi)^{-n/2}\,\widehat{f * g}, \quad \text{and} \quad \widehat{fg} = (2\pi)^{-n/2}\widehat{f} * \widehat{g}.$$

Proof. By Plancherel's theorem and Hölder's inequality, we have $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}^n)$, and $\widehat{f}\,\widehat{g} \in L^1(\mathbb{R}^n)$. We fix $x \in \mathbb{R}^n$, and set $h_x(y) = \overline{g(x-y)}$. Then

$$\widehat{h_x}(\omega) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \overline{g(x-y)}e^{-i\omega\cdot y}\,dy = (2\pi)^{-n/2}\overline{\int_{\mathbb{R}^n} g(x-y)e^{-i\omega\cdot(x-y)}\,dy\, e^{i\omega\cdot x}} = \overline{\widehat{g}(\omega)}e^{-i\omega\cdot x}.$$

Since $\mathcal{F}$ is unitary in $L^2(\mathbb{R}^n)$,

$$(f*g)(x) = \int_{\mathbb{R}^n} f(y)\overline{h_x(y)}\,dy = \int_{\mathbb{R}^n} \widehat{f}(\omega)\overline{\widehat{h_x}(\omega)}\,d\omega = \int_{\mathbb{R}^n} \widehat{f}(\omega)\widehat{g}(\omega)e^{i\omega x}\,d\omega = (2\pi)^{n/2}(\widehat{f}\,\widehat{g})^\vee(x).$$

For the second statement,

$$\begin{aligned}
\widehat{fg}(\omega) &= (2\pi)^{-n/2}(f^\vee * g^\vee)^{\widehat{}}(\omega) = (2\pi)^{-n/2}(f^\vee * g^\vee)(-\omega)\\
&= (2\pi)^{-n/2}\int_{\mathbb{R}^n} f^\vee(-\omega-\xi)g^\vee(\xi)\,d\xi\\
&= (2\pi)^{-n/2}\int_{\mathbb{R}^n} \widehat{f}(\omega+\xi)\widehat{g}(-\xi)\,d\xi = (2\pi)^{-n/2}(\widehat{f} * \widehat{g})(\omega).
\end{aligned}$$

Thus we complete the proof. $\square$

Thus we extend the domain of the Fourier transform from $L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. For more general case $f \in L^p(\mathbb{R}^n)$, where $1 \le p \le 2$, we have the following estimate for its Fourier transform.

Theorem 2.16 (Hausdorff-Young)**.** *Let $p \in [1,2]$, and let $q \in [2,\infty]$ be the conjugate exponent to p. For every $f \in L^p(\mathbb{R}^n)$, we have $\widehat{f} \in L^q(\mathbb{R}^n)$, and*

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \le (2\pi)^{-n\left(\frac{1}{p}-\frac{1}{2}\right)}\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. The Fourier transform $f \mapsto \widehat{f}$ is a linear mapping from $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. Note

$$\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)} \le (2\pi)^{-n/2}\|f\|_{L^1(\mathbb{R}^n)} \quad \text{for } f \in L^1(\mathbb{R}^n).$$

On the other hand, by the Plancherel theorem,

$$\|\widehat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} \quad \text{for } f \in L^2(\mathbb{R}^n),$$

We fix $t = 2(1-\frac{1}{p})$, so $\frac{1}{p} = \frac{1-t}{1} + \frac{t}{2}$. By Riesz-Thorin Theorem, for the conjugate $\frac{1}{q} = \frac{1-t}{\infty} + \frac{t}{2} = \frac{p-1}{p}$,

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \le (2\pi)^{-n(1-t)/2}\|f\|_{L^p(\mathbb{R}^n)} = (2\pi)^{-n\left(\frac{1}{p}-\frac{1}{2}\right)}\|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n).$$

Thus we complete the proof. $\square$

Also, we have an approximation formula for Fourier transform in L^p.

Corollary 2.17 (Carleson)**.** *Let $p \in (1,2]$, and let $q \in [2,\infty)$ be the conjugate exponent to p. If $f \in L^p(\mathbb{R}^n)$, then we have*

$$\lim_{N\to\infty} \int_{\mathbb{R}^n} \left| \widehat{f}(\omega) - \frac{1}{(2\pi)^{n/2}} \int_{|x|\le N} f(x)e^{-i\omega\cdot x}\,dx \right|^q d\omega,$$

i.e. the limit holds in L^q sense.

Proof. We choose $f_N = f\chi_{\{|x|\le N\}}$, which is in $L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ by Hölder's inequality, and converges in L^p to f as $N \to \infty$, by monotone convergence theorem. By Hausdorff-Young theorem, $\widehat{f}_N \to \widehat{f}$ in L^q. ☐

Finally, we discuss the problem of recovering a function from its Fourier transform under minimal conditions. The key idea is to replace the possibly divergent integral $\int_{\mathbb{R}^n} \widehat{f}(\omega)e^{i\omega\cdot x}\,d\omega$ by $\int_{\mathbb{R}^n} \widehat{f}(\omega)\Phi(t\omega)e^{i\omega\cdot x}\,d\omega$, where $\Phi(t\cdot)$ is a continuous function vanishing fast enough at infinity to make the integral converge.

Theorem 2.18. *Let $\Phi \in L^1(\mathbb{R}^n)\cap C_0(\mathbb{R}^n)$ with $\Phi(0) = 1$ and $\phi = \Phi^\vee \in L^1(\mathbb{R}^n)$. Given $f \in L^1(\mathbb{R}^n)+L^2(\mathbb{R}^n)$, for every $t > 0$, define*

$$f^t(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(\omega)\Phi(t\omega)e^{i\omega\cdot x}\,d\omega, \quad x \in \mathbb{R}^n.$$

(i) If $p \in [1,\infty)$ and $f \in L^p(\mathbb{R}^n)$, then every $f^t \in L^p(\mathbb{R}^n)$, and $\|f^t - f\|_{L^p(\mathbb{R}^n)} \to 0$ as $t \downarrow 0$.
(ii) If f is bounded and uniformly continuous, so is every f^t, and $f^t \to f$ uniformly as $t \downarrow 0$.
(iii) If $f \in C(\mathbb{R}^n)$, so is every f^t, and $f^t \to f$ uniformly on every compact subset of $\mathbb{R}^n$ as $t \downarrow 0$.
(iv) In addition, assume that $|\phi(\omega)| \le C(1+|\omega|)^{-n-\epsilon}$ for some $C,\epsilon > 0$. Then $f^t(x) \to f(x)$ as $t \downarrow 0$ for every Lebesgue point x of f.

Proof. We let $f = f_1 + f_2$, where $f_1 \in L^1(\mathbb{R}^n)$ and $f_2 \in L^2(\mathbb{R}^n)$. Since $\widehat{f_1} \in C_0(\mathbb{R}^n)$ and $\widehat{f_2} \in L^2(\mathbb{R}^n)$, and $\Phi \in L^1(\mathbb{R}^n)\cap C_0(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, the functions f^t are well-defined. If we define $\phi_t(\omega) = t^{-n}\phi(t^{-1}\omega)$, we have $\Phi(t\omega) = \widehat{\phi}_t(\omega)$ by the Fourier inversion theorem and Proposition 2.6 (iv).

Since $\Phi, \phi \in L^1(\mathbb{R}^n)$, we have $\widehat{f_1}\Phi(t\cdot) \in L^1(\mathbb{R}^n)$ and $f_1 * \phi_t \in L^1(\mathbb{R}^n)$. By Theorem 2.8,

$$(f_1 * \phi_t)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f_1 * \phi_t}(\omega)e^{-i\omega\cdot x}\,d\omega = \int_{\mathbb{R}^n} \widehat{f_1}(\omega)\Phi(t\omega)e^{-i\omega\cdot x}\,d\omega$$

By Plancherel's theorem, $\widehat{f_2}, \phi_t \in L^2(\mathbb{R}^n)$, and $\widehat{f_2}\Phi(t\cdot), f_2 * \phi_t \in L^1(\mathbb{R}^n)$ by the Cauchy-Schwarz inequality. By Proposition 2.15, we have

$$(f_2 * \phi_t)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f_2 * \phi_t}(\omega)e^{-i\omega\cdot x}\,d\omega = \int_{\mathbb{R}^n} \widehat{f_2}(\omega)\Phi(t\omega)e^{-i\omega\cdot x}\,d\omega.$$

Therefore $f^t = (2\pi)^{-n/2} f * \phi_t$. Since $\int_{\mathbb{R}^n} \phi(\omega)\,d\omega = (2\pi)^{n/2}\Phi(0) = (2\pi)^{n/2}$, the conclusion then follows from Propositions 1.5 and 1.9. ☐

Remark. An example of functions Φ that can be applied in the above theorem is the Gaussian radial basis function $\varphi(\omega) = e^{-|\omega|^2/2}$ defined in Lemma 2.9. Another example is the function $\Phi(\omega) = e^{-\beta|\omega|}$, whose inverse Fourier transform is the *Poisson kernel*

$$\Phi^\vee(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)\beta}{(\pi(\beta^2 + |x|^2))^{\frac{n+1}{2}}}.$$

A detailed deduction is given in Example 2.28.

2.4 Fourier Transform of Radial Functions and Hankel Transform

Bessel functions. Consider the Bessel's differential equation about function $y(z)$:

$$z^2y'' + zy' + (z^2 - \nu^2)y = 0. \tag{2.3}$$

The *Bessel function of the first kind* of order $\nu \in \mathbb{C}$ solves this equation:

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{2m+\nu}, \quad z \in \mathbb{C}\backslash\{0\},$$

where the power in this definition is given by $z^\nu = e^{\nu \log z}$, where $\log z$ is chosen to be the principal branch of the logarithm, i.e. $-\pi < \arg(z) \leq \pi$. The Bessel function $J_\nu(z)$ is holomorphic in $\mathbb{C}\backslash(-\infty, 0]$ for every $\nu \in \mathbb{C}$.

- When $\nu \notin \mathbb{Z}$, the Bessel functions $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent, and the general solution of the Bessel's equation is
$$y(z) = \gamma_1 J_\nu(z) + \gamma_2 J_{-\nu}(z), \quad \gamma_1, \gamma_2 \in \mathbb{C}.$$
- When $\nu = n \in \mathbb{Z}$, the Bessel function J_n has an analytic extension to $\mathbb{C}$. Furthermore, using the property that $1/\Gamma(-n) = 0$ for nonnegative integers $n = 0, 1, 2, \cdots$, we have
$$J_{-n}(z) = (-1)^n J_n(z), \quad n \in \mathbb{N}_0.$$
- To get a solution of (2.3) when $\nu = n \in \mathbb{Z}$ that is linearly independent of from $J_{\pm\nu}$, we introduce the *Bessel function of the second kind* of order $\nu \in \mathbb{C}$, which is defined as
$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}, \quad \nu \notin \mathbb{Z}, \quad \textit{and} \quad Y_n(z) = \lim_{\nu \notin \mathbb{Z},\, \nu \to n} Y_\nu(z), \quad n \in \mathbb{Z}.$$

  The Bessel function $Y_n(z)$ solves (2.3) when $\nu = n \in \mathbb{Z}$.

Proposition 2.19. *Let $\nu \in \mathbb{C}$, and let $J_\nu(z)$ be the Bessel function of the first kind.*

(i) The following recursive formulae hold:

$$J_{\nu-1}(z) = \frac{dJ_\nu}{dz} + \frac{\nu}{z}J_\nu(z), \quad \text{and} \quad J_{\nu+1}(z) = -\frac{dJ_\nu}{dz} + \frac{\nu}{z}J_\nu(z).$$

(ii) In particular,

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin(z), \quad \text{and} \quad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos(z).$$

Remark. Combining the two assertions, one can recurrently derive Bessel functions of half integer orders.

Proof. (i) The first formula follows from the following identity:

$$\frac{d}{dz}\left[z^\nu J_\nu(z)\right] = \sum_{m=0}^{\infty} \frac{(-1)^m(2m+2\nu)}{\Gamma(m+1)\Gamma(\nu+m+1)} \frac{z^{2m+2\nu-1}}{2^{2m+\nu}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(\nu+m)} \frac{z^{2m+2\nu-1}}{2^{2m+\nu-1}} = z^\nu J_{\nu-1}(z).$$

Similarly, the second formula follows from the following identity:

$$\begin{aligned}\frac{d}{dz}\left[z^{-\nu} J_\nu(z)\right] &= \sum_{m=0}^{\infty} \frac{(-1)^m(2m)}{\Gamma(m+1)\Gamma(\nu+m+1)} \frac{z^{2m-1}}{2^{2m+\nu}} = \sum_{m=1}^{\infty} \frac{(-1)^m}{\Gamma(m)\Gamma(\nu+m+1)} \frac{z^{2m-1}}{2^{2m+\nu-1}} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{\Gamma(m+1)\Gamma(\nu+m+2)} \frac{z^{2m+1}}{2^{2m+\nu+1}} = -z^{-\nu}J_{\nu+1}(z).\end{aligned}$$

(ii) Note that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Then

$$\begin{aligned} J_{\frac{1}{2}}(z) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+\frac{3}{2})} \left(\frac{z}{2}\right)^{2m+\frac{1}{2}} = \sqrt{\frac{2}{z}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)\cdots\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \left(\frac{z}{2}\right)^{2m+1} \\ &= \sqrt{\frac{2}{z}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!\sqrt{\pi}} z^{2m+1} = \sqrt{\frac{2}{\pi z}} \sin(z), \end{aligned}$$

and

$$\begin{aligned} J_{-\frac{1}{2}}(z) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+\frac{1}{2})} \left(\frac{z}{2}\right)^{2m-\frac{1}{2}} = \sqrt{\frac{2}{z}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right)\cdots\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \left(\frac{z}{2}\right)^{2m} \\ &= \sqrt{\frac{2}{z}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!\sqrt{\pi}} z^{2m} = \sqrt{\frac{2}{\pi z}} \cos(z). \end{aligned}$$

Therefore we complete the proof. □

The Bessel functions are related to the integral of plane wave functions on the sphere.

Proposition 2.20 (Sphere integral form of the Bessel functions of the first kind)**.** *Let $n \geq 2$, and denote by $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ the unit sphere in $\mathbb{R}^n$. Then*

$$\int_{S^{n-1}} e^{i\omega\cdot x}\, dS(\omega) = (2\pi)^{\frac{n}{2}} |x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(|x|). \tag{2.4}$$

The proof of this result requires some technical lemmata. We first introduce a type of special integrals.

Lemma 2.21. *For each $n, m \in \mathbb{N}_0$,*

$$\int_0^{\pi} \sin^n\theta \cos^{2m}\theta\, d\theta = \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(m+\frac{n}{2}+1\right)}.$$

In particular,

$$\int_0^{\pi} \sin^n\theta\, d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

Proof. (i) We begin from the second integral. Let $I_n = \int_0^{\pi} \sin^n\theta\, d\theta$. To begin with, we have $I_0 = \pi$ and $I_1 = 2$. For $n \geq 2$, compute I_n recurrently:

$$I_n = -\int_0^{\pi} \sin^{n-1}\theta\, d\cos\theta = \int_0^{\pi} (n-1)\sin^{n-2}\theta\cos^2\theta\, d\theta = (n-1)(I_{n-2} - I_n).$$

Hence $I_n = \frac{n-1}{n} I_{n-2}$. By induction, for any $n \in \mathbb{N}_0$,

$$I_{2k+1} = \frac{2k}{2k+1} \cdot \frac{2k-2}{2k-1} \cdot \cdots \cdot \frac{2}{3} \cdot I_1 = \frac{\Gamma(k+1)\sqrt{\pi}}{\Gamma\left(k+\frac{3}{2}\right)},$$

and

$$I_{2k} = \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdot \cdots \cdot \frac{1}{2} \cdot I_0 = \frac{\Gamma\left(k+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(k+1)}.$$

The first result is obtained by summarizing the last two identities.

(ii) Let $I_{n,m} = \int_0^\pi \sin^n\theta\cos^{2m}\theta\, d\theta$. Then

$$\begin{aligned}
I_{n,m} &= \int_0^\pi \sin^n\theta\cos^{2m-1}\theta\, d\sin\theta = -\int_0^\pi \sin\theta\, d\left(\sin^n\theta\cos^{2m-1}\theta\right)\\
&= -n\int_0^\pi \sin^n\theta\cos^{2m}\theta\, d\theta + (2m-1)\int_0^\pi \sin^{n+2}\theta\cos^{2m-2}\theta\, d\theta\\
&= -nI_{n,m} + (2m-1)(I_{n,m-1} - I_{n,m}) = (1-2m-n)I_{n,m} + (2m-1)I_{n,m-1}.
\end{aligned}$$

Hence $I_{n,m} = \frac{2m-1}{2m+n} I_{n,m-1}$. By induction,

$$\begin{aligned}
I_{n,m} &= \frac{2m-1}{2m+n}\cdot\frac{2m-3}{2m+n-2}\cdot\cdots\cdot\frac{1}{n+2}\cdot I_{n,0}\\
&= \frac{2^m\Gamma\left(m+\frac{1}{2}\right)/\sqrt{\pi}}{2^m\Gamma\left(m+\frac{n}{2}+1\right)/\Gamma\left(\frac{n}{2}+1\right)}\frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(m+\frac{n}{2}+1\right)}.
\end{aligned}$$

Therefore the first result holds. □

Lemma 2.22. *Let $n \geq 2$. The surface area of unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is $\sigma_{n-1} = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$.*

Proof. Using the spherical coordinates, and by Lemma 2.21, we have

$$\sigma_{n-1} = \int_{S^{n-1}} dS(x) = \int_0^\pi \sigma_{n-2}\sin^{n-2}\theta\, d\theta = \frac{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}}{\Gamma(n/2)}\sigma_{n-2}.$$

Since $\sigma_1 = 2\pi$ and $\Gamma(1) = 1$, the result follows by induction. □

Proof of Proposition 2.20. Let $r = |x|$. Since $\int_{S^{n-1}} e^{i\omega\cdot x}\, dS(\omega)$ is radial about x, we take $x = (r, 0, \cdots, 0)$:

$$\int_{S^{n-1}} e^{i\omega\cdot x}\, dS(\omega) = \int_{S^{n-1}} e^{ir\omega_1}\, dS(\omega). \tag{2.5}$$

For $\omega \in S^{n-1}$, let $\theta = \arccos(\langle\omega, e_1\rangle)$, where $e_1 = (1, 0, \cdots, 0)$. Then $\cos\theta = \omega_1$, and $\sin\theta = \sqrt{\omega_2^2 + \cdots + \omega_n^2}$. Switching to the spherical coordinates, we have

$$\begin{aligned}
\int_{S^{n-1}} e^{i\omega\cdot x}\, dS(\omega) &= \int_{S^{n-1}} e^{ir\omega_1}\, dS(\omega) = \int_0^\pi e^{ir\cos\theta}\sigma_{n-2}\sin^{n-2}\theta\, d\theta\\
&= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\int_0^\pi e^{ir\cos\theta}\sin^{n-2}\theta\, d\theta.
\end{aligned}\tag{2.6}$$

We compute the last integral by expanding the exponent and integrating term by term:

$$\begin{aligned}
\int_0^\pi e^{ir\cos\theta}\sin^{n-2}\theta\, d\theta &= \sum_{k=0}^\infty \frac{(ir)^k}{k!}\int_0^\pi \cos^k\theta\sin^{n-2}\theta\, d\theta = \sum_{m=0}^\infty \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(m+\frac{n}{2}\right)}\frac{(ir)^{2m}}{(2m)!}\\
&= \sum_{m=0}^\infty \frac{(2m-1)!!\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}{2^m\Gamma\left(m+\frac{n}{2}\right)}\frac{(ir)^{2m}}{(2m)!} = \sum_{m=0}^\infty \frac{(-1)^m\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}{m!\,\Gamma\left(m+\frac{n}{2}\right)}\left(\frac{r}{2}\right)^{2m},
\end{aligned}\tag{2.7}$$

where the odd terms vanish on $[0, \pi]$, and the even terms follow from Lemma 2.21. Combining (2.6) and (2.7), we obtain

$$\int_{S^{n-1}} e^{i\omega\cdot x}\, dS(\omega) = 2\pi^{\frac{n}{2}}\sum_{m=0}^\infty \frac{(-1)^m}{m!\,\Gamma\left(m+\frac{n}{2}\right)}\left(\frac{r}{2}\right)^{2m} = (2\pi)^{\frac{n}{2}} r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r).$$

Thus we complete the proof. □

We now turn to the Laplace transforms of some specific functions involving Bessel functions.

Proposition 2.23. *(i) For every $\nu > -1$ and $r > 0$,*

$$\int_0^\infty J_\nu(x) x^{\nu+1} e^{-rx}\,dx = \frac{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right) r}{\sqrt{\pi}(1+r^2)^{\nu+\frac{3}{2}}}. \tag{2.8}$$

(ii) For every $r > 0$,

$$\int_0^\infty J_0(x) e^{-rx}\,dx = \frac{1}{\sqrt{1+r^2}}. \tag{2.9}$$

Proof. (i) For $0 < r < 1$ and $\mu > 0$, the Taylor series of $(1+r)^{-\mu}$ is

$$(1+r)^{-\mu} = \sum_{m=0}^\infty \frac{(-1)^m \Gamma(\mu+m)}{m!\,\Gamma(\mu)} r^m.$$

Replacing r by $1/r^2$, we have

$$\frac{r^{2\mu}}{(1+r^2)^\mu} = \sum_{m=0}^\infty \frac{(-1)^m \Gamma(\mu+m)}{m!\,\Gamma(\mu)} r^{-2m}, \quad r > 1. \tag{2.10}$$

Hence the right hand side of (2.8) is

$$\begin{aligned}
\frac{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right) r}{\sqrt{\pi}(1+r^2)^{\nu+\frac{3}{2}}} &= \frac{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right) r^{-2\nu-2}}{\sqrt{\pi}} \sum_{m=0}^\infty \frac{(-1)^m \Gamma(\nu+\frac{3}{2}+m)}{m!\,\Gamma\left(\nu+\frac{3}{2}\right)} r^{-2m} \\
&= \sum_{m=0}^\infty \frac{(-1)^m 2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}+m\right)}{\Gamma(m+1)\sqrt{\pi}} r^{-2m-2\nu-2} \\
&= \sum_{m=0}^\infty \frac{(-1)^m \Gamma\left(2\nu+2m+2\right)}{2^{2m+\nu}\Gamma(m+1)\Gamma\left(\nu+m+1\right)} r^{-2m-2\nu-2},
\end{aligned} \tag{2.11}$$

where the last equality follows from Legendre's duplication formula. Now we turn to the integral. By Sterling's formula, there exists a constant c_ν depending only on $\nu > -1$ such that $\Gamma\left(\nu+m+1\right) \geq \frac{m!}{c_\nu}$. Then

$$\begin{aligned}
\sum_{m=1}^\infty \frac{x^{2m+2\nu+1}e^{-rx}}{2^{2m+\nu}\Gamma(m+1)\Gamma\left(\nu+m+1\right)} &\leq \frac{c_\nu}{2^\nu} x^{2\nu+1} e^{-rx} \sum_{m=1}^\infty \frac{x^{2m}}{(2^m m!)^2} \\
&\leq \frac{c_\nu}{2^\nu} x^{2\nu+1} e^{-rx} \sum_{m=1}^\infty \frac{x^{2m}}{(2m)!} \leq \frac{c_\nu}{2^\nu} x^{2\nu+1} e^{-(r-1)x},
\end{aligned}$$

which is absolutely integrable. Using dominated convergence theorem, we can interchange infinite summation and integral in the left hand side of (2.8):

$$\begin{aligned}
\int_0^\infty J_\nu(x) x^{\nu+1} e^{-rx}\,dx &= \sum_{m=1}^\infty \frac{(-1)^m}{2^{2m+\nu}\Gamma(m+1)\Gamma\left(\nu+m+1\right)} \int_0^\infty x^{2m+2\nu+1} e^{-rx}\,dx \\
&= \sum_{m=1}^\infty \frac{(-1)^m r^{-2m-2\nu-2}}{2^{2m+\nu}\Gamma(m+1)\Gamma\left(\nu+m+1\right)} \int_0^\infty y^{-2m-2\nu-1} e^{-y}\,dy \\
&= \sum_{m=0}^\infty \frac{(-1)^m \Gamma\left(2\nu+2m+2\right)}{2^{2m+\nu}\Gamma(m+1)\Gamma\left(\nu+m+1\right)} r^{-2m-2\nu-2},
\end{aligned}$$

which is consistent with (2.11). Hence the identity (2.8) holds for $r > 1$. Finally, since both sides of (2.11) is analytic in the region $\mathrm{Re}(r) > 0$ and $|\mathrm{Im}(r)| < 1$, the case $0 < r \leq 1$ follows from analytic continuation.

(ii) Interchanging summation and integration gives

$$\int_0^\infty J_0(x)e^{-rx}\,dx = \sum_{m=0}^\infty \frac{(-1)^m}{2^{2m}(m!)^2}\int_0^\infty x^{2m}e^{-rx}\,dx = \sum_{m=0}^\infty \frac{(-1)^m(2m)!}{2^{2m}(m!)^2} r^{-2m-1}.$$

On the other hand, we set $\nu = 1/2$ in (2.10) to obtain

$$\frac{1}{\sqrt{1+r^2}} = \sum_{m=0}^\infty \frac{(-1)^m\Gamma(m+\frac{1}{2})}{m!\,\Gamma(\frac{1}{2})} r^{-2m-1} = \sum_{m=0}^\infty \frac{(-1)^m(2m)!}{2^{2m}(m!)^2} r^{-2m-1}.$$

Combining the last two displays, we obtain the desired result. □

Now we study the Fourier transform of radial functions on $\mathbb{R}^n$. A function $F : \mathbb{R}^n \to \mathbb{C}$ is said to be radial, if there exists a function f such that $F(x) = f(|x|)$ for all $x \in \mathbb{R}^n$.

Definition 2.24 (Hankel transform)**.** Let $\nu \geq -\frac{1}{2}$. We define the *Hankel transform of order* ν of a function $f \in L^2((0,\infty), r\,dr)$ by

$$(H_\nu f)(\lambda) = \int_0^\infty r f(r) J_\nu(\lambda r)\,dr, \quad \lambda > 0.$$

The Hankel transform of order $\frac{n}{2} - 1$ is related to the Fourier transform of radial functions in $\mathbb{R}^n$.

Theorem 2.25. *Let $F \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ be a radial function, i.e. $F(x) = f(|x|)$ for $x \in \mathbb{R}^n$. Then the Fourier transform $\widehat{F}$ is also radial, i.e. $\widehat{F}(\omega) = \phi(|\omega|)$, with*

$$\phi(\lambda) = \lambda^{1-\frac{n}{2}} \int_0^\infty r^{\frac{n}{2}} f(r) J_{\frac{n}{2}-1}(\lambda r)\,dr$$

In other words, $|\omega|^{\frac{n}{2}-1}\widehat{F}(\omega)$ coincides the Hankel transform of order $\frac{n}{2}-1$ of $r^{\frac{n}{2}-1}f(r)$

Proof. For the case $n = 1$, we have $J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos(z)$ by Proposition 2.19. Since $F : \mathbb{R} \to \mathbb{C}$ is even,

$$\begin{aligned}\widehat{F}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(x)e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(x)\left(\cos(\omega x) - i\sin(\omega x)\right)dx \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(r)\cos(|\omega| r)\,dr = |\omega|^{\frac{1}{2}}\int_0^\infty \sqrt{r} f(r) J_{-\frac{1}{2}}(|\omega| r)\,dr.\end{aligned}$$

For the case $n \geq 2$, we switch to sphere coordinates and use Proposition 2.20:

$$\begin{aligned}\widehat{F}(\omega) &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} F(x)e^{-i\omega\cdot x}\,dx = (2\pi)^{-\frac{n}{2}}\int_0^\infty r^{n-1}\int_{S^{n-1}} f(r|x|)e^{-ir\omega\cdot x}\,dS(x)\,dr \\ &= (2\pi)^{-\frac{n}{2}}\int_0^\infty r^{n-1}f(r)\left(\int_{S^{n-1}} e^{-ir\omega\cdot x}\,dS(x)\right)dr \\ &= (2\pi)^{-\frac{n}{2}}\int_0^\infty r^{n-1}f(r)\cdot(2\pi)^{\frac{n}{2}}(r|\omega|)^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(r|\omega|)\,dr \\ &= |\omega|^{1-\frac{n}{2}}\int_0^\infty r^{\frac{n}{2}}f(r)J_{\frac{n}{2}-1}(r|\omega|)\,dr.\end{aligned}$$

Then we conclude the proof. □

Remark. In particular, taking $n = 2$, we know that the Hankel transform of order 0 coincides the Fourier transformation of radial function in $\mathbb{R}^2$.

2.5 Application in Partial Differential Equations

Fourier transform and differential operators. Consider the Laplacian operator:

$$\Delta : C^2(\mathbb{R}^n) \to C(\mathbb{R}^n), \qquad \Delta f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}.$$

For the plane wave function $f(x) = e^{i\omega\cdot x}$, we have

$$\Delta e^{i\omega\cdot x} = \sum_{j=1}^{n} (i\omega_j)^2 e^{i\omega\cdot x} = -|\omega|^2 e^{i\omega\cdot x}.$$

In other words, the function $e^{i\omega\cdot x}$ is an eigenfunction of Δ, with eigenvalue $-|\omega|^2$. Furthermore, under regularity conditions [See Proposition 2.7], we have

$$\widehat{\Delta f}(\omega) = \sum_{j=1}^{n} (i\omega_j)^2 \widehat{f}(\omega) = -|\omega|^2 \widehat{f}(\omega).$$

This identity shows that *the Fourier transform diagonalizes the Laplacian* Δ. In other words, the Laplacian is nothing more than an explicit multiplier when viewed using the Fourier transform.

Example 2.26 (Heat equation with Dirichlet boundary condition)**.** Consider the heat equation about the time-varying function $u(x,t)$, which is defined on $\mathbb{R}^n \times \mathbb{R}_+$:

$$\begin{cases} u_t = \Delta_x u & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R}^n \times \{t=0\}, \\ \lim_{|x|\to\infty} u(x,t) = 0 & \text{for } t \in [0,\infty), \end{cases} \tag{2.12}$$

where the initial function $f \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$.

Solution. We let $\widehat{u}(\omega,t) = \int_{\mathbb{R}^n} u(x,t) e^{-i\omega\cdot x}\, dx$ be the Fourier transform of u with respect to x. Applying Fourier transform on both the heat equation and the initial condition, we get the initial value problem:

$$\begin{cases} \widehat{u}_t = -|\omega|^2 \widehat{u}, \\ \widehat{u}(\omega,0) = \widehat{f}(\omega). \end{cases}$$

The solution of this problem is given by $\widehat{u}(\omega,t) = \widehat{f}(\omega) e^{-|\omega|^2 t}$. To recover u, we employ the inverse Fourier transform and convolution theorem [Theorem 2.8]:

$$u(x,t) = \mathcal{F}^{-1}\left(\widehat{f}(\omega) e^{-|\omega|^2 t}\right) = (2\pi)^{-n/2} f * \mathcal{F}^{-1}(e^{-|\omega|^2 t}).$$

It remains to compute the inverse Fourier transform of $e^{-|\omega|^2 x}$:

$$\begin{aligned} \mathcal{F}^{-1}(e^{-|\omega|^2 t})(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|\omega|^2 t} e^{i\omega\cdot x}\, d\omega = \prod_{j=1}^{n} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\omega_j^2 t + i\omega_j x_j}\, d\omega_j \\ &= \prod_{j=1}^{n} e^{-\frac{x_j^2}{4t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\omega_j\sqrt{t} - \frac{ix_j}{2\sqrt{t}}\right)^2}\, d\omega_j \\ &= \prod_{j=1}^{n} \frac{1}{\sqrt{2t}} e^{-\frac{x_j^2}{4t}} = \frac{1}{(2t)^{n/2}} e^{-\frac{|x|^2}{4t}}. \end{aligned}$$

Hence the solution of problem (2.12) is given by

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|y-x|^2}{4t}} f(y)\, dy. \qquad \square$$

Remark. We write the heat kernel by

$$\Phi_t(x) = \begin{cases} \delta(x), & t = 0, \\ (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}, & t > 0. \end{cases}$$

Then the solution of problem (2.12) can be represented as $u = \Phi_t * f$.

Example 2.27 (Heat equation with a source)**.** Consider the heat equation about the time-varying function $u(x,t)$, which is defined on $\mathbb{R}^n \times \mathbb{R}_+$:

$$\begin{cases} u_t = \Delta_x u + S(x,t) & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R}^n \times \{t=0\}, \\ \lim_{|x|\to\infty} u(x,t) = 0 & \text{for } t \in [0,\infty), \end{cases} \tag{2.13}$$

where the source $S(x,t) \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ for every t, and the initial function $f \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$.

Solution. Similar to the case without the source $S(x,t)$, we apply Fourier transform on both the equation and the initial condition to get an initial value problem:

$$\begin{cases} \widehat{u}_t = -|\omega|^2 \widehat{u} + \widehat{S}(\omega,t), \\ \widehat{u}(\omega,0) = \widehat{f}(\omega). \end{cases}$$

We solve this problem by multiplying by a factor $e^{|\omega|^2 t}$:

$$\frac{\partial}{\partial t}\left(e^{|\omega|^2 t}\widehat{u}\right) = e^{|\omega|^2 t}\left(\widehat{u}_t + |\omega|^2 \widehat{u}\right) = e^{|\omega|^2 t}\widehat{S}(\omega,t).$$

Hence

$$e^{|\omega|^2 t}\widehat{u}(\omega,t) = \widehat{f}(\omega) + \int_0^t e^{|\omega|^2 \tau}\widehat{S}(\omega,\tau)\, d\tau,$$

and

$$\widehat{u}(\omega,t) = e^{-|\omega|^2 t}\widehat{f}(\omega) + \int_0^t e^{-|\omega|^2 (t-\tau)}\widehat{S}(\omega,\tau)\, d\tau.$$

Applying inverse Fourier transform, we obtain the solution of (2.13):

$$u(x,t) = \int_{\mathbb{R}^n} \Phi_t(x-y) f(y)\, dt + \int_0^t \int_{\mathbb{R}^n} \Phi_{t-\tau}(x-y) S(y,t)\, dy\, d\tau. \qquad \square$$

Example 2.28 (Laplace equation in the upper half space)**.** Consider the Laplace equation about the function $u(x,y)$ in the upper half space $\mathbb{R}^n \times \mathbb{R}_+$:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R}^n \times \{t=0\}, \\ \lim_{|x|\to\infty} u(x,y) = 0 & \text{and } \lim_{y\to\infty} u(x,y) = 0, \end{cases} \tag{2.14}$$

where the function $f \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$.

Solution. We write the Laplace equation as $u_{yy} = -\Delta_x u$, and apply Fourier transform on variable x. Then we get the initial value problem:

$$\begin{cases} \widehat{u}_{yy} = |\omega|^2 \widehat{u}, \\ \widehat{u}(\omega, 0) = \widehat{f}(\omega), \\ \lim_{y\to\infty} \widehat{u}(\omega, y) = 0 \end{cases}$$

Since u is vanishing as $y \to \infty$, the solution to this problem is $\widehat{u}(\omega, y) = e^{-|\omega|y}\widehat{f}(\omega)$. Hence the solution to (2.14) is

$$u(x, y) = \frac{1}{(2\pi)^{n/2}} \cdot \mathcal{F}^{-1}(e^{-|\omega|y}) * f.$$

We compute inverse Fourier transform of $e^{-|\omega|y}$:

$$\begin{aligned} \mathcal{F}^{-1}(e^{-|\omega|y}) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|\omega|y} e^{i\omega\cdot x}\, d\omega = \frac{1}{(2\pi)^{n/2}} \int_0^\infty \int_{\partial B(x,r)} e^{-|\omega|y} e^{i\omega\cdot x}\, dS(\omega)\, dr \\ &= \frac{1}{(2\pi)^{n/2}} \int_0^\infty \int_{S^{n-1}} e^{-ry} e^{ir\xi\cdot x} r^{n-1}\, dS(\xi)\, dr = \int_0^\infty r^{\frac{n}{2}} e^{-ry} |x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r|x|)\, dr \quad \text{(By Proposition 2.20)} \\ &= |x|^{-n} \int_0^\infty \rho^{\frac{n}{2}} e^{-\rho\frac{y}{|x|}} J_{\frac{n}{2}-1}(\rho)\, d\rho = \frac{2^{\frac{n}{2}}\Gamma\left(\frac{n+1}{2}\right) y}{\sqrt{\pi}\,(|x|^2 + y^2)^{\frac{n+1}{2}}}. \quad \text{(By Proposition 2.23)} \end{aligned}$$

Then

$$u(x, y) = \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{\mathbb{R}^n} \frac{y}{(|x - z|^2 + y^2)^{\frac{n+1}{2}}} f(z)\, dz. \qquad \square$$

Remark. We define the *Poisson kernel* by

$$P(x, y) = c_n \frac{y}{(|x|^2 + y^2)^{\frac{n+1}{2}}}, \quad \text{where } c_n = \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right).$$

Then the solution of problem (2.12) can be represented as $u(\cdot, y) = P(\cdot, y) * f$.

Example 2.29 (Wave equation with Dirichlet boundary condition)**.** Consider the wave equation about the time-varying function $u(x, t)$, which is defined on $\mathbb{R}^n \times \mathbb{R}_+$:

$$\begin{cases} u_{tt} = \Delta_x u & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = f(x),\ u_t(x, 0) = g(x) & \text{on } \mathbb{R}^n \times \{y = 0\}, \\ \lim_{|x|\to\infty} u(x, t) = 0 & \text{for } \ t \in [0, \infty). \end{cases} \tag{2.15}$$

where the functions $f, g \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$.

Solution. Applying Fourier transform with respect to the variable $x \in \mathbb{R}^n$, we get the initial value problem

$$\begin{cases} \widehat{u}_{tt} = -|\omega|^2 \widehat{u}, \\ \widehat{u}(\omega, 0) = \widehat{f}(\omega), \quad \widehat{u}_t(\omega, 0) = \widehat{g}(\omega). \end{cases} \tag{2.16}$$

The solution to (2.16) is

$$\widehat{u}(\omega, t) = \widehat{f}(\omega)\cos(|\omega|t) + \widehat{g}(\omega)\frac{\sin(|\omega|t)}{|\omega|}.$$

We write $W_t(x) = \frac{1}{(2\pi)^{n/2}} \mathcal{F}^{-1}\left(\frac{\sin(|\omega|t)}{|\omega|}\right)$. By convolution theorem, the solution to problem 2.15 is

$$u(\cdot, t) = \frac{\partial}{\partial t} W_t * f + W_t * g. \qquad \square$$

Example 2.30 (Transport equation)**.** Consider the following transport equation with constant coefficients:

$$\begin{cases} u_t - b \cdot \nabla_x u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R}^n \times \{t=0\}, \\ \lim_{|x|\to\infty} u(x,t) = 0 & \text{for } \ t \in [0,\infty), \end{cases} \tag{2.17}$$

where the velocity $b \in \mathbb{R}^n$ is a constant vector, and $f \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$.

Solution. We apply Fourier transform with respect to the variable x:

$$\begin{cases} \widehat{u}_t = ib \cdot \omega \widehat{u}, \\ \widehat{u}(\omega,0) = \widehat{f}(\omega). \end{cases}$$

Then $\widehat{u}(\omega,t) = e^{itb\cdot\omega}\widehat{f}(\omega)$, and the solution to problem (2.17) is

$$u(x,t) = \mathcal{F}^{-1}\left[e^{itb\cdot\omega}\widehat{f}(\omega)\right] = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{f}(\omega) e^{i(x+tb)\cdot\omega}\, d\omega = f(x+tb). \qquad \square$$

Example 2.31 (Linearized Korteweg–De Vries equation)**.** Consider the equation about $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C}$.

$$\begin{cases} u_t + u_{xxx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R} \times \{t=0\}, \\ \lim_{|x|\to\infty} u(x,t) = 0 & \text{for } \ t \in [0,\infty). \end{cases} \tag{2.18}$$

Solution. We apply Fourier transform with respect to the variable x:

$$\begin{cases} \widehat{u}_t - i\omega^3 \widehat{u} = 0, \\ \widehat{u}(\omega,0) = \widehat{f}(\omega). \end{cases}$$

Then $\widehat{u}(\omega,t) = e^{i\omega^3 t}\widehat{f}(\omega)$, and u is recovered by taking the inverse Fourier transform of $\widehat{u}$. By convolution theorem, $u = G(\cdot,t) * f$, where $G(\cdot,t)$ is the inverse Fourier transform of $e^{i\omega^3 t}$ up to a factor $1/\sqrt{2\pi}$:

$$G(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega^3 t} e^{i\omega x}\, d\omega.$$

We compute the function G by constructing an ordinary differential equation for it. Fix $t = \frac{1}{3}$, and consider the function $g(x) = G(x, \frac{1}{3})$. Then

$$\widehat{g}(\omega) = \frac{1}{\sqrt{2\pi}} e^{i\omega^3/3}.$$

By Proposition 2.7,

$$g'' = \mathcal{F}^{-1}(-\omega^2 \widehat{g}) = -\frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left(\omega^2 e^{i\omega^3/3}\right), \quad \text{and} \quad xg = -i\mathcal{F}^{-1}(\widehat{g}') = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left(\omega^2 e^{i\omega^3/3}\right).$$

Hence the function g satisfies the *Airy equation* $g'' - xg = 0$. Since our solution should vanish at infinity, we take the solution $g(x) = \mathrm{Ai}(x)$. For general $t > 0$, applying change of variable gives

$$G(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{1}{3}i\left(\sqrt[3]{3t}\omega\right)^3} e^{i\omega x}\, d\omega = \frac{1}{\sqrt[3]{3t}} \mathrm{Ai}\left(\frac{x}{\sqrt[3]{3t}}\right).$$

The solution to the problem (2.18) is $u(\cdot,t) = G(\cdot,t) * f$.

3 Distribution Theory

3.1 Topology on $C_c^\infty(U)$

The Fréchet space $\mathcal{D}(K)$. Let K be a compact set of $\mathbb{R}^n$. The space $C_c^\infty(K)$ is defined to be all C^∞ functions on $\mathbb{R}^n$ whose support is compact and contained in K. This is a *Fréchet space* with topology $\mathscr{T}_K$ defined by norms

$$\|\phi\|_{K,N} = \sup_{x\in K, |\alpha|\le N} |\partial^\alpha \phi(x)|, \ N \in \mathbb{N}_0.$$

That is, a local base for this topology at $\phi \in C_c^\infty(K)$ is the family of sets

$$U_{K,N}^\epsilon(\phi) = \{\psi \in C_c^\infty(K) : \|\psi - \phi\|_{K,N} < \epsilon\},$$

where $N \in \mathbb{N}_0$ and $\epsilon > 0$. Indeed, we only need to define the base sets

$$U_{K,N}^\epsilon = \{\psi \in C_c^\infty(K) : \|\psi\|_{K,N} < \epsilon\}, \quad N \in \mathbb{N}_0, \ \epsilon > 0$$

at 0, and take $\phi + U_{K,N}^\epsilon$ to be the base sets at ϕ. The Fréchet space $C_c^\infty(K)$ is metrizable by setting

$$d_K(\phi,\psi) = \sum_{N=1}^\infty \frac{1}{2^N} \frac{\|\phi-\psi\|_{K,N}}{1+\|\phi-\psi\|_{K,N}}, \quad \phi,\psi \in C_c^\infty(K).$$

We denote by $\mathcal{D}(K)$ the space $C_c^\infty(K)$ endowed with the topology $\mathscr{T}_K$. In $\mathcal{D}(K)$, every sequence (ϕ_k) converges to ϕ if and only if $\partial^\alpha\phi_k \to \partial^\alpha \phi$ uniformly for all multi-indices α.

Construct a base for a topology on $C_c^\infty(U)$. For an open set $U \subset \mathbb{R}^n$, the space $C_c^\infty(U)$ is defined to be the set of C^∞ functions whose support is compact and contained in U. Indeed, $C_c^\infty(U)$ can be viewed as the union of spaces $C_c^\infty(K)$ as K ranges over all compact subsets of U.

To construct a topology on $C_c^\infty(U)$, let $\mathscr{B}_0$ be the family of all balanced[1], convex sets $V \subset C_c^\infty(U)$ such that $V \cap C_c^\infty(K) \in \mathscr{T}_K$ for all compact $K \subset U$. We can show that $\mathscr{B}_0$ is nonempty. For example, let

$$V_N^\epsilon = \left\{\psi \in C_c^\infty(U) : \sup_{x\in U, |\alpha|\le N} |\partial^\alpha\psi(x)| < \epsilon\right\}. \tag{3.1}$$

Then V_N^ϵ is balanced, convex, and $V_N^\epsilon \cap C_c^\infty(K) = U_{K,N}^\epsilon \in \mathscr{T}_K$. We then define

$$\mathscr{B} = \{\phi + V : \phi \in C_c^\infty(U), V \in \mathscr{B}_0\}. \tag{3.2}$$

The sets in $\mathscr{B}$ gives an appropriate topology on $C_c^\infty(U)$.

Theorem 3.1. *The family $\mathscr{B}$ defined in* (3.2) *is a base for a locally convex Hausdorff topology $\mathscr{T}$ on $C_c^\infty(U)$ that turns $C_c^\infty(U)$ into a topological vector space.*

Remark. We write for $\mathcal{D}(U)$ the topological space $(C_c^\infty(U), \mathscr{T})$, whose elements are called *testing functions*.

Proof. **Step I.** We first verify that $\mathscr{B}$ is a base for a topology on $C_c^\infty(U)$. It suffices to verify the following two conditions:

(i) For each $\phi \in C_c^\infty(U)$ there exists $U \in \mathscr{B}$ such that $\phi \in U$;

(ii) For each $U_1, U_2 \in \mathscr{B}$ with $U_1 \cap U_2 \neq \emptyset$ and each $\phi \in U_1 \cap U_2$, there exists $V \in \mathscr{B}$ such that $V \ni \phi$ and $V \subset U_1 \cap U_2$. In other words, $\mathscr{B}$ is closed under finite intersection operation.

[1]A subset E of a vector space X is *balanced* if $tx \in E$ for all $x \in E$ and $|t| \le 1$.

• For (i), we let $\phi \in C_c^\infty(U)$, $N \in \mathbb{N}_0$ and $\epsilon > 0$. The set V_N^ϵ defined in (3.1) is in $\mathscr{B}_0$, and $\phi + V_N^\epsilon \in \mathscr{B}$.

• For (ii), we let $\phi_1, \phi_2 \in C_c^\infty(U)$ and $V_1, V_2 \in \mathscr{B}_0$ be such that $(\phi_1 + V_1) \cap (\phi_2 + V_2) \neq \emptyset$. We fix any

$$\phi \in (\phi_1 + V_1) \cap (\phi_2 + V_2),$$

and take a compact set $K \subset U$ such that K contains the supports of ϕ_1, ϕ_2 and ϕ. Then for $j = 1, 2$, we have

$$\phi - \phi_j \in V_j \cap C_c^\infty(K) \in \mathscr{T}_K.$$

Using the continuity of scalar multiplication in $C_c^\infty(K)$, we may find $0 < \alpha < 1$, such that

$$\phi - \phi_j \in (1-\alpha)(V_j \cap C_c^\infty(K)) \subset (1-\alpha)V_j, \quad j = 1, 2.$$

By convexity of the sets V_j, we have

$$\phi - \phi_j + \alpha V_j = (1-\alpha)V_j + \alpha V_j = V_j, \quad j = 1, 2,$$

so that $\phi + \alpha V_j \in \phi_j + V_j$ for $j = 1, 2$, and

$$\phi + \alpha(V_1 \cap V_2) \subset (\phi_1 + V_1) \cap (\phi_2 + V_2).$$

Hence $\mathscr{B}$ is a base for a topology $\mathscr{T}$ given by all unions of members of $\mathscr{B}$.

Step II. Next we verify that $C_c^\infty(U)$ is a topological vector space under $\mathscr{T}$.

• To prove the continuity of scalar multiplication at a point $(t_0, \phi_0) \in \mathbb{C} \times C_c^\infty(U)$, we notice that each neighborhood of $t_0\phi_0$ contains some $t_0\phi_0 + V$, where $V \in \mathscr{B}_0$. Let $K = \operatorname{supp}(\phi_0)$. Then $\phi_0 \in \mathcal{D}(K)$. By continuity of scalar multiplication in $\mathcal{D}(K)$, we may find $\gamma > 0$ so small that

$$\gamma\phi_0 \in \frac{1}{2}\left(V \cap C_c^\infty(K)\right) \subset \frac{1}{2}V.$$

Fix $s = \dfrac{1}{2(|t_0| + \gamma)}$. Then for every $|t - t_0| < \gamma$ and $\phi \in \phi_0 + sV$,

$$t\phi - t_0\phi_0 = t(\phi - \phi_0) + (t - t_0)\phi \in tsV + \frac{1}{2}V \subset \frac{1}{2}V + \frac{1}{2}V = V,$$

where we use the fact that V is convex and balanced. Therefore $t\phi \in t_0\phi_0 + V$ for every $|t - t_0| < \gamma$ and $\phi \in \phi_0 + sV$, which proves the continuity of scalar multiplication.

• To prove the continuity of addition at a point $(\phi_1, \phi_2) \in C_c^\infty(U) \times C_c^\infty(U)$, we consider a neighborhood $\phi_1 + \phi_2 + V$ of $\phi_1 + \phi_2$, where $V \in \mathscr{B}_0$. The convexity of V implies that

$$\left(\phi_1 + \frac{1}{2}V\right) + \left(\phi_2 + \frac{1}{2}V\right) = \phi_1 + \phi_2 + V.$$

Since $V \cap \mathcal{D}(K) \in \mathscr{T}_K$ for all compact $K \subset U$, and since the scalar multiplication is continuous in $\mathcal{D}(K)$, we have $\frac{1}{2}V \cap \mathcal{D}(K) \in \mathscr{T}_K$ for all compact $K \subset U$, and $\frac{1}{2}V \in \mathscr{B}_0$. Hence

$$\phi_1 + \frac{1}{2}V, \ \phi_2 + \frac{1}{2}V \in \mathscr{B},$$

and the addition operation is continuous.

Step III. Finally, to prove that $(C_c^\infty(U), \mathscr{T})$ is a Hausdorff space, we take $\phi_1 \neq \phi_2$ from $C_c^\infty(U)$ and define

$$V = \left\{ \psi \in C_c^\infty(U) : \sup_{x \in U} |\psi(x)| < \frac{1}{2} \sup_{x \in U} |\phi_1(x) - \phi_2(x)| \right\}.$$

In view of (3.1), we have $V \in \mathscr{B}_0$. If $\phi \in (\phi_1 + V) \cap (\phi_2 + V)$, we have

$$\begin{aligned} \sup_{x \in U} |\phi_1(x) - \phi_2(x)| &\leq \sup_{x \in U} |\phi(x) - \phi_1(x)| + \sup_{x \in U} |\phi(x) - \phi_2(x)| \\ &< \frac{1}{2} \sup_{x \in U} |\phi_1(x) - \phi_2(x)| + \frac{1}{2} \sup_{x \in U} |\phi_1(x) - \phi_2(x)| = \sup_{x \in U} |\phi_1(x) - \phi_2(x)|, \end{aligned}$$

a contradiction! Hence $(\phi_1 + V) \cap (\phi_2 + V) = \emptyset$, and we finish the proof. $\square$

We now show that the topology $\mathscr{T}$, when restricted to $\mathcal{D}(K)$, for some compact set $K \subset U$, does not produce more open sets than the ones in $\mathscr{T}_K$.

Proposition 3.2. *Let $U \subset \mathbb{R}^n$ be an open set. For every compact set $K \subset U$, the topology on $\mathcal{D}(K)$ coincide with the relative topology of $\mathcal{D}(K)$ as a subspace of $\mathcal{D}(U)$.*

Proof. Fix a compact set $K \subset U$ and let $W \in \mathscr{T}$. We claim $W \cap \mathcal{D}(K) \in \mathscr{T}_K$. We may assume $W \cap \mathcal{D}(K)$ is nonempty, otherwise the claim is clear. Let $\phi \in W \cap \mathcal{D}(K)$. Since $\mathscr{B}$ is a base for $\mathscr{T}$, we take $V \in \mathscr{B}_0$ such that $\phi + V \subset W$. Then $\phi + (V \cap \mathcal{D}(K)) \subset W \cap \mathcal{D}_K$, and $\phi + (V \cap \mathcal{D}(K)) \in \mathscr{T}_K$ since $\phi \in \mathcal{D}(K)$ and $V \cap \mathcal{D}(K) \in \mathscr{T}_K$. Hence every point of $W \cap \mathcal{D}(K)$ is in the interior with respect to $\mathscr{T}_K$, and $W \cap \mathcal{D}(K) \in \mathscr{T}_K$.

Conversely, let $W \subset \mathscr{T}_K$. We claim that $W = V \cap \mathcal{D}(K)$ for some open $V \in \mathscr{T}$. Since the family of sets $U_{K,N}^\epsilon$ is a local base for the topology $\mathscr{T}_K$, for each $\phi \in W$, we may find $N_\phi \in \mathbb{N}_0$ and $\epsilon_\phi > 0$ such that $\phi + U_{K,N_\phi}^{\epsilon_\phi} \subset W$. Let $V_{N_\phi}^{\epsilon_\phi}$ be defined as in (3.1). Then

$$(\phi + V_{N_\phi}^{\epsilon_\phi}) \cap \mathcal{D}(K) = \phi + U_{K,N_\phi}^{\epsilon_\phi} \subset W,$$

and $\phi + V_{N_\phi}^{\epsilon_\phi} \in \mathscr{B}$. Therefore $V = \bigcup_{\phi \in W} (\phi + V_{N_\phi}^{\epsilon_\phi})$ is a set in $\mathscr{T}$ with the desired property. $\square$

Proposition 3.3. *Let $U \subset \mathbb{R}^n$ be an open set. If $W \subset \mathcal{D}(U)$ is topologically bounded, there exists a compact set $K \subset U$ such that $W \subset \mathcal{D}(K)$.*

Proof. Assume that W is not contained in $\mathcal{D}(K)$ for any compact $K \subset U$. We take an increasing sequence (K_j) of compact sets such that $K_j \subset \mathring{K}_{j+1}$ for all $j \in \mathbb{N}$ and $U = \bigcup_{j=1}^\infty K_j$. Then we may find for each $j \in \mathbb{N}$ a function $\phi_j \in W$ and a point $x_j \in K_{j+1} \backslash K_j$ such that $\phi_j(x_j) \neq 0$. Define

$$V = \left\{ \phi \in \mathcal{D}(U) : |\phi(x_j)| < \frac{1}{j} |\phi_j(x_j)| \text{ for all } j \in \mathbb{N} \right\}.$$

Since each compact set $K \subset U$ contains only finitely many x_j, we have $V \cap \mathcal{D}(K) \in \mathscr{T}_K$, and so $V \subset \mathscr{T}$. Since W is topologically bounded, there exists $t > 0$ such that $W \subset tV$. If an integer $N \geq t$, we have $\phi_N(x_N) \neq 0$, and $t^{-1} |\phi_N(x_N)| \geq N^{-1} |\phi_N(x_N)|$. Hence $t^{-1} \phi_N \notin V$, and $\phi_N \notin tV$. However $\phi_N \in W \subset tV$, which yields a contradiction. Hence there exists a compact $K \subset U$ with $\mathcal{D}(K) \supset W$. $\square$

The topology on $\mathcal{D}(U)$ is complete, and convergent sequence in $\mathcal{D}(U)$ can be explicitly characterized.

Proposition 3.4. *Let $U \subset \mathbb{R}^n$. The space $\mathcal{D}(U)$ is complete. Furthermore, a sequence (ϕ_j) in $\mathcal{D}(U)$ converges to $\phi \in \mathcal{D}(U)$ if and only if*

(i) there exists a compact set $K \subset U$ such that $(\phi_j) \subset \mathcal{D}(K)$, and

(ii) $\lim_{j \to \infty} \partial^\alpha \phi_j = \partial^\alpha \phi$ uniformly on K for each multi-index $\alpha \in \mathbb{N}_0^n$.

Proof. Let (ϕ_j) be a Cauchy sequence in $\mathcal{D}(U)$. Then (ϕ_j) is topologically bounded, and by Proposition 3.3, there exists a compact set $K \subset U$ such that $(\phi_j) \subset \mathcal{D}(K)$. By Proposition 3.2, we obtain a Cauchy sequence (ϕ_j) in $\mathcal{D}(K)$. Therefore, for every $N \in \mathbb{N}_0$ and every $\epsilon > 0$, there exists M such that

$$\sup_{x \in K, |\alpha| \leq N} |\phi_j(x) - \phi_k(x)| < \epsilon$$

for all $j, k \geq M$. Consequently, for every multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$, the Cauchy sequence $\{\partial^\alpha \phi_j\}$ converges uniformly in K to a continuous function $\psi_\alpha \in C_c(K)$. An inductive argument using the fundamental theorem of calculus shows that $\partial^\alpha \psi_0 = \psi_\alpha$ for every multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$. Given the arbitrariness of $N \in \mathbb{N}_0$, we conclude that $\psi_0 \in \mathcal{D}(K)$ and that the sequence (ϕ_j) converges to ψ_0 with respect to $\mathscr{T}$. Hence the space $\mathcal{D}(U)$ is complete.

Conversely, if a sequence (ϕ_j) in $\mathcal{D}(U)$ satisfies conditions (i) and (ii), it converges to ϕ in $\mathcal{D}(K)$. By Proposition 3.3, it also converges to ϕ in $\mathcal{D}(U)$. □

Now we discuss the continuous mappings on $\mathcal{D}(U)$.

Proposition 3.5. *Let $U \subset \mathbb{R}^n$ be an open set, X a locally convex topological vector space, and $T : \mathcal{D}(U) \to X$ a linear operator. The following properties are equivalent:*

(i) T is continuous.

(ii) T is bounded, i.e. it sends topologically bounded sets of $\mathcal{D}(U)$ into topologically bounded sets of X.

(iii) If (ϕ_j) converges to ϕ in $\mathcal{D}(U)$, then $\lim_{j\to\infty} T\phi_j = T\phi$.

(iv) The restriction of T to $\mathcal{D}(K)$ is continuous for every compact set $K \subset U$.

If $X = \mathbb{C}$, the following statement is also equivalent to above all:

(v) For every compact set $K \subset U$, there exists an integer $N \in \mathbb{N}_0$ and a constant $c_K > 0$ such that $|T\phi| \leq c_K \|\phi\|_{K,N}$ for all $\phi \in \mathcal{D}(K)$.

Proof. We prove that (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i).

• (i) ⇒ (ii). Suppose that $T : \mathcal{D} \to X$ is continuous, and $W \subset \mathcal{D}(U)$ is a topologically bounded set. If V is a neighborhood of 0 in X, then $T^{-1}(V)$ is a neighborhood of 0 in $\mathcal{D}(X)$, and there exists $t > 0$ such that $W \subset tT^{-1}(V)$. Consequently $T(W) \subset tV$. Hence $T(W)$ is also topologically bounded.

• (ii) ⇒ (iii). We may assume $(\phi_j) \to 0$ by replacing (ϕ_j) with $(\phi_j - \phi)$. By Proposition 3.4, there exists a compact set K such that $(\phi_j) \subset \mathcal{D}(K)$, and $d_K(\phi_j, 0) \to 0$ as $j \to \infty$.

Let $B = \{\phi \in \mathcal{D}(K) : d_K(\phi, 0) < 1\}$ be the unit ball in $\mathcal{D}(K)$ centered at 0. If T is bounded, the set $T(B)$ is topologically bounded. Then for any neighborhood V of 0 in X, there exists $t > 0$ such that $T(B) \subset tV$, so $T(t^{-1}B) \subset V$. Since $d_K(\phi_j, 0) \to 0$ as $j \to 0$, there exists N such that $\phi_j \in t^{-1}(B)$ for all $j \geq N$. Hence $(T\phi_j)$ is eventually in V, and $T\phi_j$ converges to 0.

• (iii) ⇒ (iv). Fix a compact set $K \subset U$. If (ϕ_j) is a sequence in $\mathcal{D}(K)$ such that $d_K(\phi_j, 0) \to 0$ as $j \to \infty$, by Proposition 3.4, we have $\phi_j \to 0$ in $\mathcal{D}(U)$, and $T\phi = \lim_{j\to\infty} T\phi_j$ by property (iii). Hence the restriction of T to $\mathcal{D}(K)$ is continuous at 0. By linearity, the restriction is continuous.

• (iv) ⇒ (i). For every neighborhood V of 0 in X and every compact set $K \subset U$, the restriction of T to $\mathcal{D}(K)$ is continuous at zero, and $T^{-1}(V) \cap \mathcal{D}(K) \in \mathscr{T}_K$. Since K is arbitrary, $T^{-1}(V) \in \mathscr{T}$. Therefore, T is continuous at 0 and, by linearity, everywhere in $\mathcal{D}(U)$.

• (iv) ⇔ (v). Let $X = \mathbb{C}$. Assume that (iv) holds and fix a compact $K \subset U$. By continuity of $T|_{\mathcal{D}(K)}$ at the origin, there exists $N \in \mathbb{N}_0$ and $\epsilon > 0$ such that $U^\epsilon_{K,N} \subset T^{-1}(\{|z| < 1\})$, that is, $|T\phi| < 1$ for all $\phi \in \mathcal{D}(K)$ with $\|\phi\|_{K,N} < \epsilon$. If $\phi \in \mathcal{D}(K)$ and $\phi \neq 0$, then $\|\phi\|_{K,N} \neq 0$, and by linearity of T, we have $|T\phi| \leq \frac{2}{\epsilon}\|\phi\|_{K,N}$. Conversely, if (v) holds, for any $\delta > 0$, by taking $\epsilon > 0$ sufficiently small, we have $|T\phi| < \delta$ for all $\phi \in U^\epsilon_{K,N}$. Hence the restriction $T|_{\mathcal{D}(K)}$ is continuous. □

Proposition 3.6. *Let U and U' be open subsets of $\mathbb{R}^n$, and $T : \mathcal{D}(U) \to \mathcal{D}(U')$ a linear operator. The following properties are equivalent:*

(i) T is continuous;

(ii) for each compact set $K \subset U$, there exists a compact set $K' \subset U'$ such that $T(\mathcal{D}(K)) \subset \mathcal{D}(K')$, and the restriction $T : \mathcal{D}(K) \to \mathcal{D}(K')$ is continuous.

Proof. (ii) $\Rightarrow$ (i) is a special case of the implication (iv) $\Rightarrow$ (i) in Proposition 3.5. To prove (i) $\Rightarrow$ (ii), we let $T : \mathcal{D}(U) \to \mathcal{D}(U')$ be a continuous linear operator and fix a compact set $K \subset U$. According to the implication (i) $\Rightarrow$ (iv) in Proposition 3.5, the restriction of T to $\mathcal{D}(K)$ is continuous. If we can show that $T(\mathcal{D}(K)) \subset \mathcal{D}(K')$ for some compact $K' \subset U'$, the proof will be completed by Proposition 3.2.

Assume that $T(\mathcal{D}(K))$ is not contained in $\mathcal{D}(K')$ for any compact $K' \subset U'$. Take an increasing sequence (K_j') of compact sets such that $K_j' \subset \mathring{K}_{j+1}'$ for all $j \in \mathbb{N}$ and $U' = \bigcup_{j=1}^{\infty} K_j'$. Then we may find for each $j \in \mathbb{N}$ a function $\phi_j \in \mathcal{D}(U')$ and a point $x_j \in K_{j+1}' \backslash K_j'$ such that $d_K(\phi_j, 0) = 1$ and $(T\phi_j)(x_j) \neq 0$. Since (ϕ_j) is topologically bounded in $\mathcal{D}(U)$, by Proposition 3.5 (ii), $(T\phi_j)$ is topologically bounded in $\mathcal{D}(U')$, and by Proposition 3.3, there exists $K' \subset U'$ such that $(\phi_j) \subset \mathcal{D}(K')$, which is contradiction! $\square$

3.2 Distributions

Motivation. Let $f \in L^p(\mathbb{R}^n)$, where $1 < p \leq \infty$. For $q = \frac{p}{p-1}$, we define $T_f : L^q(\mathbb{R}^n) \to \mathbb{C}$ by

$$T_f g = \int_{\mathbb{R}^n} f(x)g(x)\,dx, \quad g \in L^q(\mathbb{R}^n).$$

The Riesz representation theorem states that the map $f \mapsto T_f$ is an isometric isomorphism of $L^p(\mathbb{R}^n)$ onto the dual space $L^q(\mathbb{R}^n)^*$ of $L^q(\mathbb{R}^n)$. In other words, $f \in L^p(\mathbb{R}^n)$ is completely determined by its action as a bounded linear functional on $L^q(\mathbb{R}^n)$. On the other hand, by Lebesgue differentiation theorem,

$$\lim_{r \to 0^+} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\,dy = f(x), \quad \text{for } a.e.\ x \in \mathbb{R}^n,$$

where $B(x,r)$ is the (open) ball of radius r about x, and m is the Lebesgue measure. Hence if we take $g = m(B(x,r))^{-1}\chi_{B(x,r)}$, we can recover the pointwise value of f for almost every $x \in \mathbb{R}^n$ as $r \to 0$. Thus, we lose nothing by thinking of f as a linear mapping from $L^q(\mathbb{R}^n)$ to $\mathbb{C}$ rather than a map from $\mathbb{R}^n$ to $\mathbb{C}$.

The idea of distribution follows by allowing $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and requiring $\phi \in \mathcal{D}(\mathbb{R}^n)$. The map T_f defines a linear functional on $\mathcal{D}(\mathbb{R}^n)$, and the pointwise values of f can be recovered a.e. by a similar approach of Theorem 1.9. Nevertheless, there are also linear functionals on $\mathcal{D}(\mathbb{R}^n)$ that are not of the form T_f.

Definition 3.7 (Distribution)**.** Let U be an open subset of $\mathbb{R}^n$. A *distribution* on U is a continuous linear functional on $\mathcal{D}(U)$. The space of all distributions on U is denoted by $\mathcal{D}'(U)$. We equip $\mathcal{D}'(U)$ with the weak* topology, i.e. the neighborhoods of $T_0 \in \mathcal{D}'(U)$ is generated by the sets

$$U^{\epsilon}_{f_1,\cdots,f_m}(T_0) = \left\{T \in \mathcal{D}'(U) : |Tf_j - T_0 f_j| < \epsilon,\ j = 1, 2, \cdots, m\right\},$$

where $\epsilon > 0$, $m \in \mathbb{N}$ and $f_1, \cdots, f_m \in C_c^{\infty}(U)$. Furthermore, a sequence $T_j \to T$ in the weak* topology if and only if $T_j f \to Tf$ for all $f \in C_c^{\infty}(U)$.

Notations. If $F \in \mathcal{D}'(U)$ and $\phi \in C_c^{\infty}(U)$, we use the pairing notation $\langle F, \phi\rangle$ for the value of F evaluated at the point ϕ. Sometimes it is helpful to pretend that a distribution $F \in \mathcal{D}'(U)$ is a function on U even when it really is not, and to write $\int_U F(x)\phi(x)\,dx$ instead of $\langle F, \phi\rangle$.

We shall use a tilde to denote the reflection of a function in the origin: $\widetilde{\phi}(x) = \phi(-x)$.

Example 3.8. Following are some examples of distribution on an open set $U \subset \mathbb{R}^n$:

- Every function $f \in L^1_{\text{loc}}(U)$ defines a distribution on U, namely, the functional $\phi \mapsto \int f\phi\, dx$. Clearly, two functions that are equal a.e. define the same distribution, since they are identified in $L^1_{\text{loc}}(U)$.
- Every Radon measure μ on U defines a distribution $\phi \mapsto \int \phi\, d\mu$.
- For a point $x_0 \in U$ and a multi-index $\alpha \in \mathbb{N}_0^n$, the map $\phi \mapsto \partial^\alpha \phi(x_0)$ defines a distribution that does not arise from a function.
- In particular, when $U = \mathbb{R}^n$, $\alpha = 0$ and $x = 0$, this distribution arise from a measure μ which is the point mass at the origin 0. We call this distribution the *Dirac δ-function*, denoted by δ:

  $$\langle \delta, \phi \rangle = \phi(0), \quad \phi \in C_c^\infty(\mathbb{R}^n).$$

  It can be heuristically written as

  $$\delta(x) = \begin{cases} \infty, & x = 0, \\ 0, & x \neq 0, \end{cases}$$

  and we write $\int_{\mathbb{R}^n} \delta(x)\phi(x)\, dx = \phi(0)$.

We have the following approximation for Dirac δ-function.

Proposition 3.9. *Assume $f \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} f(x)\, dx = 1$. Define*

$$f_t(x) = \frac{1}{t^n} f\left(\frac{x}{t}\right), \quad t > 0.$$

Then $f_t \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$ as $t \to 0$.

Proof. If $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle f_t, \phi \rangle = \int_{\mathbb{R}^n} f_t(x)\phi(x)\, dx = \int_{\mathbb{R}^n} f_t(x)\widetilde{\phi}(-x)\, dx = (f_t * \widetilde{\phi})(0),$$

which converges to $\widetilde{\phi}(0) = \phi(0) = \langle \delta, \phi \rangle$ as $t \to 0$ by Proposition 1.6. $\square$

Let $F \in \mathcal{D}'(U)$ be a distribution on an open set $U \subset \mathbb{R}^n$. For an open set $V \subset U$, we say $F = 0$ on V if $\langle F, \phi \rangle = 0$ for all $\phi \in C_c^\infty(V)$ (for example, if $F \in L^1_{\text{loc}}(U)$, it means that $F = 0$ a.e. on V). Since a function in $C_c^\infty(V_1 \cup V_2)$ need not to be supported in either V_1 or V_2, it is not so clear that $F = 0$ on both V_1 and V_2 implies $F = 0$ on $V_1 \cup V_2$. Nevertheless, it is true:

Proposition 3.10. *Let $(V_\alpha)_{\alpha \in A}$ be a collection of open subsets of U, and $V = \bigcup_{\alpha \in A} V_\alpha$. If $F \in \mathcal{D}'(U)$ and $F = 0$ on each V_α, then $F = 0$ on V.*

Proof. If $\phi \in C_c^\infty(V)$, by compactness, there exist finitely many $\alpha_1, \cdots, \alpha_m \in A$ such that $\text{supp}(\phi) \subset \bigcup_{j=1}^m V_{\alpha_j}$. Take a smooth partition of unity $(\psi_j)_{j=1}^m$, i.e. $\text{supp}(\psi_j) \subset V_{\alpha_j}$ for each j and $\sum_{j=1}^m \psi_j = 1$ on $\text{supp}(\phi)$. Then

$$\langle F, \phi \rangle = \sum_{j=1}^m \langle F, \phi\psi_j \rangle = 0.$$

Hence $F = 0$ on V. $\square$

Remark. (I) According to this proposition, we can take a maximal open set W on which $F = 0$, namely the union of all open sets on which $F = 0$. Its complement $U \backslash W$ is called the *support of F*.

(II) More generally, we say two distributions $F, G \in \mathcal{D}'(V)$ *agree* on an open set $V \subset U$ if $F - G = 0$ on V. According to this proposition, if two distributions agree on each member of a collection of open sets, they also agree on the union of those open sets.

Operations on distributions. Let $U \subset \mathbb{R}^n$ be an open set, and $F \in \mathcal{D}'(U)$.

(i) (Product). If $\psi \in C^\infty(U)$, we define the *product* ψF to be

$$\langle \psi F, \phi \rangle = \langle F, \psi\phi \rangle, \quad \phi \in \mathcal{D}(U).$$

For any compact $K \subset U$ and any sequence $\phi_j \in C_c^\infty(K)$ that converges to ϕ in $\mathcal{D}(K)$, since $\psi\phi_j \to \psi\phi$ and $F|_{\mathcal{D}(K)}$ is continuous, we have $\langle F, \psi\phi_j \rangle \to \langle F, \psi\phi \rangle$. Hence $\psi F \in \mathcal{D}'(U)$.

(ii) (Translation). If $y \in \mathbb{R}^n$ and $F \in L^1_{\mathrm{loc}}(U)$,

$$\int_{U+y} F(x-y)\phi(x)\,dx = \int_U F(x)\phi(x+y)\,dx, \quad \phi \in \mathcal{D}(U+y).$$

Similarly, for $F \in \mathcal{D}'(U)$, we define the *translated distribution* $\tau_y F$ to be

$$\langle \tau_y F, \phi \rangle = \langle F, \tau_{-y}\phi \rangle, \quad \phi \in \mathcal{D}(U+y).$$

Then $\tau_y F \in \mathcal{D}'(U+y)$. In particular, the point mass at y is $\tau_y \delta$.

(iii) (Composition with linear map). If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation and $F \in L^1_{\mathrm{loc}}(U)$,

$$\int_U F(Tx)\phi(x)\,dx = |\det(T)|^{-1} \int_{T^{-1}(U)} F(y)\phi(T^{-1}y)\,dy, \quad \phi \in \mathcal{D}(T^{-1}(U)).$$

Similarly, for $F \in \mathcal{D}'(U)$, we define the *composition* $F \circ T$ to be

$$\langle F \circ T, \phi \rangle = |\det(T)|^{-1} \langle F, \phi \circ T^{-1} \rangle, \quad \phi \in \mathcal{D}(T^{-1}(U)).$$

Then $F \circ T \in \mathcal{D}'(T^{-1}(U))$. In particular, if $Tx = -x$, we define the *reflection of F in the origin* by

$$\langle \widetilde{F}, \phi \rangle = \langle F, \widetilde{\phi} \rangle, \quad \phi \in \mathcal{D}^\infty(-U).$$

(iv) (Convolution). Given $\psi \in C_c^\infty(\mathbb{R}^n)$, let $V = \{x : x - y \in U \text{ for all } y \in \mathrm{supp}(\psi)\}$. If $F \in L^1_{\mathrm{loc}}(U)$,

$$(F * \psi)(x) = \int_U F(y)\psi(x-y)\,dy = \int_U F(y)(\tau_x \widetilde{\psi})(y)\,dy, \quad x \in V, \tag{3.3}$$

and by Fubini's theorem, if $\phi \in C_c^\infty(V)$,

$$\int_V (F * \psi)(x)\phi(x)\,dx = \int_U \int_V F(y)\widetilde{\psi}(y-x)\phi(x)\,dx\,dy = \int_U F(y)(\phi * \widetilde{\psi})(y)\,dy. \tag{3.4}$$

For $F \in \mathcal{D}'(U)$, we have two approaches to define the *convolution* $F * \psi$:

• As in (3.3), define function

$$(F * \psi)(x) = \langle F, \tau_x \widetilde{\psi} \rangle, \quad x \in V.$$

• Analogous to (3.4), define $F * \psi$ be the mapping

$$\langle F * \psi, \phi \rangle = \langle F, \phi * \widetilde{\psi} \rangle, \quad \phi \in \mathcal{D}(V).$$

If $K \subset V$ is compact and $(\phi_j) \subset C_c^\infty(K)$ is a sequence converging to ϕ in $\mathcal{D}(K)$, we have

$$\partial^\alpha(\phi_j * \widetilde{\psi}) = (\partial^\alpha \phi_j) * \widetilde{\psi} \to (\partial^\alpha \phi) * \widetilde{\psi} = \partial^\alpha(\phi * \widetilde{\psi})$$

uniformly for all multi-indices $\alpha \in \mathbb{N}_0^n$. Hence $(F * \psi)|_{\mathcal{D}(K)}$ is continuous, and $F * \psi \in \mathcal{D}'(V)$.

The following proposition shows that the two definitions of the convolution $F * \psi$ coincide. Furthermore, the distribution as a function on U is infinitely differentiable.

Proposition 3.11. *Let $U \subset \mathbb{R}^n$ be open. Given $\psi \in C_c^\infty(\mathbb{R}^n)$, let $V = \{x : x - y \in U \text{ for all } y \in \operatorname{supp}(\psi)\}$. For $F \in \mathcal{D}'(U)$, define $(F * \psi)(x) = \langle F, \tau_x \widetilde{\psi} \rangle$ for all $x \in V$. Then*

(i) $F * \psi \in C^\infty(V)$, and

$$\partial^\alpha (F * \psi) = F * (\partial^\alpha \psi)$$

for all multi-indices $\alpha \in \mathbb{N}_0^n$;

(ii) For all $\phi \in C_c^\infty(V)$,

$$\int_V (F * \psi)(x) \phi(x) \, dx = \langle F, \phi * \widetilde{\psi} \rangle.$$

Proof. If $x \in V$, by Proposition 1.5, we have $\tau_{x+s}\widetilde{\psi} \to \tau_x \widetilde{\psi}$ uniformly as $s \to 0$, and the same holds for all partial derivatives. Then $\tau_{x+s}\widetilde{\psi} \to \tau_x \widetilde{\psi}$ in $\mathcal{D}(U)$ as $s \to 0$. By continuity of F on $\mathcal{D}(U)$ we have that $\langle F, \tau_x \widetilde{\psi} \rangle$ is continuous in x. Furthermore, for any $j = 1, 2, \cdots, n$, we have

$$\left| \frac{\psi(x + he_j - y) - \psi(x - y)}{h} - \partial_j \psi(x - y) \right| \leq \sup_{t \in \mathbb{R} : |t| < |h|} \left| \partial_j \psi(x + te_j - y) - \partial_j \psi(x - y) \right|.$$

For any $\epsilon > 0$, by uniform continuity of $\partial_j \psi$, there exists a constant $\eta > 0$ independent of x and y such that the last bound is less than ϵ whenever $|h| < \eta$. Hence the difference quotient

$$\frac{\tau_{x+he_j}\widetilde{\psi} - \tau_x \widetilde{\psi}}{h} \to \tau_x \widetilde{\partial_j \psi} \tag{3.5}$$

uniformly as $h \to 0$. Since the same conclusion of difference quotient holds for all partial derivatives, the convergence (3.5) also holds in $\mathcal{D}(U)$. Therefore

$$\partial_j (F * \psi)(x) = \lim_{h \to 0} \frac{\langle F, \tau_{x+he_j}\widetilde{\psi} \rangle - \langle F, \tau_x \widetilde{\psi} \rangle}{h} = \langle F, \tau_x \widetilde{\partial_j \psi} \rangle = (F * \partial_j \psi)(x).$$

By induction on $|\alpha|$, we have $F * \psi \in C^\infty(V)$, and $\partial^\alpha (F * \psi) = F * \partial^\alpha \psi$. To prove the second result, we note that $\psi, \phi \in C_c^\infty(\mathbb{R}^n)$. Then we approximate the convolution $\phi * \widetilde{\psi}$ by Riemann sums:

$$(\phi * \widetilde{\psi})(x) = \int_{\mathbb{R}^n} \widetilde{\psi}(x - y) \phi(y) \, dy = \lim_{\epsilon \to 0^+} S_\epsilon(x) := \lim_{\epsilon \to 0^+} \epsilon^n \sum_{\kappa \in \mathbb{Z}^n} \widetilde{\psi}(x - \epsilon\kappa) \phi(\epsilon\kappa),$$

where there are finitely many nonzero terms when κ runs over $\mathbb{Z}^n$. The Riemann sums S_ϵ are supported in a common compact subset of U, and converges to $\phi * \widetilde{\psi}$ uniformly as $\epsilon \to 0$. Also, for all multi-indices α,

$$\partial^\alpha S_\epsilon = \epsilon^n \sum_{\kappa \in \mathbb{Z}^n} \partial^\alpha \widetilde{\psi}(x - \epsilon\kappa) \phi(\epsilon\kappa)$$

converges uniformly to $\partial^\alpha (\phi * \widetilde{\psi})$. Hence $S_\epsilon \to \phi * \widetilde{\psi}$ in $\mathcal{D}(U)$, and

$$\langle F, \phi * \widetilde{\psi} \rangle = \lim_{\epsilon \to 0^+} \langle F, S_\epsilon \rangle = \lim_{\epsilon \to 0^+} \epsilon^n \sum_{\kappa \in \mathbb{Z}^n} \phi(\epsilon\kappa) \langle F, \tau_{\epsilon\kappa} \widetilde{\psi} \rangle = \int_V \phi(x) \langle F, \tau_x \widetilde{\psi} \rangle \, dx = \int_V (F * \psi)(x) \phi(x) \, dx.$$

Hence the two definitions of $F * \psi$ are equivalent. ☐

Next we show that although distributions can be highly singular objects, they can always be approximated by compactly supported smooth functions in the weak* topology.

Theorem 3.12. *For any open set $U \subset \mathbb{R}^n$, the space $C_c^\infty(U)$ is dense in $\mathcal{D}'(U)$ in the weak* topology.*

To prove this theorem we need some technical lemma.

Lemma 3.13. *Assume that $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \psi(x)\,dx = 1$. Let $\psi_t(x) = t^{-n}\psi(t^{-1}x)$ for $t > 0$.*

(i) Given any neighborhood U of $\operatorname{supp}(\phi)$, we have $\operatorname{supp}(\phi * \psi_t) \subset U$ for $t > 0$ sufficiently small.

(ii) $\phi * \psi_t \to 0$ in $\mathcal{D}(\mathbb{R}^n)$ as $t \to 0$.

Proof. If $\operatorname{supp}(\psi) \subset \{x \in \mathbb{R}^n : |x| < R\}$, then $\operatorname{supp}(\phi * \psi_t)$ is contained in the set

$$V = \{x \in \mathbb{R}^n : d(x, \operatorname{supp}(\phi)) < tR\}.$$

When $t < R^{-1} d(\operatorname{supp}(\phi), U^c)$, the support of $\phi * \psi_t$ is contained in U. Moreover, by Propositions 1.3 and 1.6, $\partial^\alpha(\phi * \psi_t) = (\partial^\alpha \phi) * \psi_t \to \partial^\alpha t$ uniformly as $t \to 0$, and the second result follows. □

Proof of Theorem 3.12. Assume $F \in \mathcal{D}'(U)$. We first approximate F by distributions supported on compact subsets of U, then approximate the latter by functions in $C_c^\infty(U)$.

• Let (V_j) be a sequence of precompact open subsets of U increasing to U. For each j, by C^∞-Urysohn lemma [Proposition 1.10], we take $\zeta_j \in C_c^\infty(U)$ such that $\zeta_j = 1$ on $\overline{V}_j$. Given $\phi \in C_c^\infty(U)$, for j sufficiently large we have $\operatorname{supp}(\phi) \subset V_j$, and

$$\langle F, \phi \rangle = \langle F, \zeta_j \phi \rangle = \langle \zeta_j F, \phi \rangle.$$

Hence $\zeta_j F \to F$ in the weak* topology as $j \to \infty$.

• Let ψ and (ψ_t) be defined as in Lemma 3.13. Then $\phi * \widetilde{\psi}_t \to \phi$ in $\mathcal{D}(\mathbb{R}^n)$ as $t \to 0$. On the other hand, by Proposition 3.11, we have $(\zeta_j F) * \psi_t \in C^\infty(\mathbb{R}^n)$, and

$$\langle (\zeta_j F) * \psi_t, \phi \rangle = \langle \zeta_j F, \phi * \widetilde{\psi}_t \rangle \to \langle \zeta_j F, \phi \rangle \quad \text{as } t \to 0.$$

Hence $(\zeta_j F) * \psi_t \to \zeta_j F$ in $\mathcal{D}'(\mathbb{R}^n)$. Observing that $\operatorname{supp}(\zeta_j) \subset V_k$ for some k, if $\operatorname{supp}(\phi) \cap \overline{V}_k = \emptyset$, we have $\operatorname{supp}(\phi * \widetilde{\psi}_t) \cap \overline{V}_k = \emptyset$ for $t > 0$ sufficiently small, by Lemma 3.13, and $\langle (\zeta_j F) * \psi_t, \phi \rangle = \langle F, \zeta_j(\phi * \widetilde{\psi}_t) \rangle = 0$. Hence $\operatorname{supp}((\zeta_j F) * \psi_t) \subset \overline{V}_k \subset U$, and $(\zeta_j F) * \psi_t \in C_c^\infty(U)$ for j large enough and t small enough. □

Derivatives of distributions. Let U be an open subset of $\mathbb{R}^n$. If $f \in C_c^\infty(U)$, for any multi-index $\alpha \in \mathbb{N}_0^n$,

$$\int_U (\partial^\alpha f)(x)\phi(x)\,dx = (-1)^{|\alpha|} \int_U f(x)(\partial^\alpha \phi)(x)\,dx, \quad \phi \in C_c^\infty(U).$$

This is the integration by parts formula, where the boundary term vanishes since f is compactly supported. Generally, for $F \in \mathcal{D}'(U)$, we can define a linear functional $\partial^\alpha F$ on $C_c^\infty(U)$ by

$$\langle \partial^\alpha F, \phi \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \phi \rangle, \quad \phi \in C_c^\infty(U).$$

For any compact $K \subset U$ and any sequence $(\phi_j) \subset C_c^\infty(K)$ that converges to ϕ in $\mathcal{D}(K)$, by continuity of F, we have $\langle F, \partial^\alpha \phi_j \rangle \to \langle F, \partial^\alpha \phi \rangle$ as $j \to \infty$. Hence $\partial^\alpha F|_{\mathcal{D}(K)}$ is continuous, and $\partial^\alpha F \in \mathcal{D}'(U)$.

The distribution $\partial^\alpha F$ is called the α^{th} *derivative of* F. Moreover, if $F_j \to F$ in $\mathcal{D}'(U)$, we have $\langle \partial^\alpha F_j, \phi \rangle = \langle F_j, \partial^\alpha \phi \rangle \to \langle F, \partial^\alpha \phi \rangle = \langle \partial^\alpha F, \phi \rangle$ for each $\phi \in C_c^\infty(U)$, and $\partial^\alpha F_j \to \partial^\alpha F$ in $\mathcal{D}'(U)$. Therefore, the differentiation operator $\partial^\alpha : \mathcal{D}'(U) \to \mathcal{D}'(U)$ is a continuous linear map with respect to the weak* topology.

In particular, for any locally integrable function $\psi \in L^1_{\mathrm{loc}}(U)$, we can define its derivatives of arbitrary order even if it is not differentiable in the classical sense. To be specific, we define $\langle T_\psi, \phi \rangle = \int_U \psi(x)\phi(x)\,dx$. The derivative $\partial^\alpha T_\psi$ of the distribution T_ψ is called the α^{th} *distributional derivative of* ψ, denoted by $\partial^\alpha \psi$. Following are some examples of distributional derivatives.

Jump discontinuity. For simplicity, we first consider the functions on $\mathbb{R}$. Differentiating functions with jump discontinuities leads to δ-singularities. The simplest example is the *Heaviside step function* $H = \chi_{[0,\infty)}$, for which we have

$$\langle H', \phi \rangle = -\langle H, \phi' \rangle = -\int_0^\infty \phi'(x)\, dx = \phi(0) = \langle \delta, \phi \rangle, \quad \phi \in C_c^\infty(\mathbb{R}).$$

Hence the first distributional derivative of H is the Dirac function δ. More generally, for any $x \in \mathbb{R}$, the distributional derivative of the step function $\tau_x H = \chi_{[x,\infty)}$ is $\tau_x \delta$, which is the point mass at x.

If f is piecewise continuously differentiable on $\mathbb{R}$, f only has jump discontinuities at $x_1 < x_2 < \cdots < x_m$, and its pointwise derivative f' is in $L^1_{\text{loc}}(\mathbb{R})$. Then

$$\begin{aligned}
\langle f', \phi \rangle = -\langle f, \phi' \rangle &= -\sum_{j=0}^{m} \int_{x_j}^{x_{j+1}} f(x)\phi'(x)\, dx \\
&= -\sum_{j=0}^{m} \left[f(x_{j+1}^-)\phi(x_{j+1}) - f(x_j^+)\phi(x_j) - \int_{x_j}^{x_{j+1}} \frac{df}{dx}(y)\phi(y)\, dy \right] \\
&= \int_{-\infty}^{\infty} f'(y)\phi(y)\, dy + \sum_{j=1}^{m} \phi(x_j) \left[f(x_j^+) - f(x_j^-) \right]
\end{aligned}$$

Therefore, the distributional derivative of f is given by

$$Df = f' + \sum_{j=1}^{m} \left[f(x_j^+) - f(x_j^-) \right] \tau_{x_j} \delta.$$

Generalized Heaviside step function.

3.3 Compactly Supported Distributions

The C^∞ topology. Let $U \subset \mathbb{R}^n$ be an open set. The C^∞ topology on the space $C^\infty(U)$ of all smooth functions on U is the topology of uniform convergence of functions, together with all their derivatives, on compact subsets of U. This topology can be defined by a countable family of seminorms as follows. Let (V_m) be an increasing sequence of precompact open subsets of U whose union is U. For each $m \in \mathbb{N}$ and each multi-index $\alpha \in \mathbb{N}_0^n$, define the seminorm

$$\|f\|_{[m,\alpha]} = \sup_{x \in \overline{V}_m} |\partial^\alpha f(x)|. \tag{3.6}$$

With the topology induced by the family of these seminorms, $C^\infty(U)$ is a Fréchet space. Furthermore, a sequence (f_j) converges to f in $C^\infty(U)$ if and only if $\|f_j - f\|_{[m,\alpha]} \to 0$ for all $m \in \mathbb{N}, \alpha \in \mathbb{N}_0^n$, if and only if $\partial^\alpha f_j \to \partial^\alpha f$ uniformly on compact sets for all $\alpha \in \mathbb{N}_0^n$.

Proposition 3.14. *Let $U \subset \mathbb{R}^n$ be an open set. The space $C_c^\infty(U)$ is dense in $C^\infty(U)$.*

Proof. We take the sequence (V_m) as in (3.6). By C^∞-Urysohn lemma [Theorem 1.10], for each m, we take $\psi_m \in C_c^\infty(U)$ with $\psi_m = 1$ on $\overline{V}_m$. If $\phi \in C^\infty(U)$, for all multi-indices $\alpha \in \mathbb{N}_0^n$, we have $\|\psi_m \phi - \phi\|_{[m_0,\alpha]} = 0$ for all indices $m \geq m_0$. Hence $\psi_m \phi \in C_c^\infty(U)$ converges to ϕ in the C^∞ topology. $\square$

If U is an open subset of $\mathbb{R}^n$, we denote by $\mathcal{E}'(U)$ the space of all distributions on U whose support is a compact subset of U.

Theorem 3.15. *Let $U \subset \mathbb{R}^n$ be an open set.*

(i) If $F \in \mathcal{E}'(U)$, then F extends uniquely to a continuous linear functional on $C^\infty(U)$.

(ii) If G is a continuous linear functional on $C^\infty(U)$, then $G|_{C_c^\infty(U)} \in \mathcal{E}'(U)$.

To summarize, $\mathcal{E}'(U)$ equals the dual space of $C^\infty(U)$.

Proof. (i) If $F \in \mathcal{E}'(U)$, take $\psi \in C_c^\infty(U)$ such that $\psi = 1$ on $\operatorname{supp}(F)$, and define the linear functional G on $C^\infty(U)$ by $\langle G, \phi \rangle := \langle F, \psi\phi \rangle$. Since F is continuous on $\mathcal{D}(\operatorname{supp}(\psi))$, and the topology of the latter is defined by the norms $\phi \mapsto \|\partial^\alpha \phi\|_\infty$, there exists $C > 0$ and $N \in \mathbb{N}$ such that

$$|\langle G, \phi \rangle| = |\langle F, \psi\phi \rangle| \leq C \sum_{|\alpha| \leq N} \|\partial^\alpha(\psi\phi)\|_\infty$$

for all $\phi \in C^\infty(U)$. By the product rule, if we choose m large enough so that $\overline{V}_m \supset \operatorname{supp}(\psi)$,

$$|\langle G, \phi \rangle| \leq C' \sum_{|\alpha| \leq N} \sup_{x \in \operatorname{supp}(\psi)} |\partial^\alpha \phi(x)| \leq \ C' \sum_{|\alpha| \leq N} \|\phi\|_{[m,\alpha]}.$$

Hence G is continuous on $C^\infty(U)$. By Proposition 3.14, the continuous extension G of F is unique.

(ii) On the other hand, if G is a continuous linear functional on $C^\infty(U)$, there exist constants $m, N \in \mathbb{N}$ and $C > 0$ such that

$$|\langle G, \phi \rangle| \leq C \sum_{|\alpha| \leq N} \|\phi\|_{[m,\alpha]}$$

for all $\phi \in C^\infty(U)$. Since $\|\phi\|_{[m,\alpha]} \leq \|\partial^\alpha \phi\|_\infty$, the functional G is continuous on $\mathcal{D}(K)$ for each compact $K \subset U$, and $G|_{C_c^\infty(U)} \in \mathcal{D}'(U)$. Moreover, if $\operatorname{supp}(\phi) \cap \overline{V}_m = \emptyset$, we have $\langle G, \phi \rangle = 0$, and $\operatorname{supp}(G) \subset \overline{V}_m$. Therefore $G|_{C_c^\infty(U)} \in \mathcal{E}'(U)$, concluding the proof. $\square$

Remark. In fact, one can easily check that the operations of multiplication by C^∞ functions, translation, composition by invertible linear maps and differentiation, as is discussed in the last section, all preserves the class of $\mathcal{E}'(U)$. The case of convolution is a bit more complicated.

3.4 Tempered Distributions and Fourier Transform

Definition 3.16 (Tempered distributions)**.** A *tempered distribution (on $\mathbb{R}^n$)* is a continuous linear functional on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. The space of tempered distribution is denoted by $\mathcal{S}'(\mathbb{R}^n)$. As usual, we equip $\mathcal{S}'(\mathbb{R}^n)$ with the weak* topology.

The following proposition helps to understand the relation of distributions and tempered distributions.

Proposition 3.17. *The space $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$.*

Proof. We fix $\phi \in \mathcal{S}(\mathbb{R}^n)$, which is to be approximated. We take $\psi \in C_c^\infty(\mathbb{R}^n, [0,1])$ such that $\psi(0) = 1$, and let $\psi^t(x) = \psi(tx)$ for $t > 0$. Given any $N \in \mathbb{N}$ and $\epsilon > 0$, we can choose a compact $K \subset \mathbb{R}^n$ such that $(1 + |x|)^N |\phi(x)| < \epsilon$ for all $x \notin K$. Then $\psi^t(x) \to 1$ uniformly on K as $t \to 0$, and

$$\lim_{t\to 0} \|\psi^t \phi - \phi\|_{(N,0)} \leq \sup_{x\notin K} (1 + |x|)^N |\psi^t(x)\phi(x) - \phi(x)| < \epsilon.$$

Since N and ϵ are arbitrary, $\|\psi^t\phi - \phi\|_{(N,0)} \to 0$ as $t \to 0$ for all $N \in \mathbb{N}_0$. For the terms involving derivatives, by the product rule,

$$(1 + |x|)^N \partial^\alpha(\psi^t\phi - \phi) = (1 + |x|)^N (\psi^t\partial^\alpha\phi - \partial^\alpha\phi) + R_t(x),$$

where the remainder R_t is a sum of terms involving derivatives of ψ^t. Since

$$\left|\partial^\beta\psi^t(x)\right| = t^{|\beta|} \left|\partial^\beta\psi(tx)\right| \leq C_\beta t^{|\beta|},$$

we have $\|R_t\|_\infty \leq Ct \to 0$ as $t \to 0^+$. An analogue of the preceding argument shows that $\|\psi^t\phi - \phi\|_{(N,\alpha)} \to 0$ as $t \to 0$. Hence $\psi^t\phi \in C_c^\infty(\mathbb{R}^n)$ converges to ϕ in $\mathcal{S}(\mathbb{R}^n)$, which completes the proof. ☐

Remark. Since the convergence in $\mathcal{D}(\mathbb{R}^n)$ implies the convergence in $\mathcal{S}(\mathbb{R}^n)$, if $F \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution, the restriction of F to $C_c^\infty(\mathbb{R}^n)$ is also continuous. Hence $F|_{C_c^\infty(\mathbb{R}^n)}$ is a distribution. Furthermore, by Proposition 3.17, the restriction $F|_{C_c^\infty(\mathbb{R}^n)}$ determines $F \in \mathcal{S}'(\mathbb{R}^n)$ uniquely. Thus we may identify $\mathcal{S}'(\mathbb{R}^n)$ with the sets of all distributions on $\mathbb{R}^n$ that extends continuously from $C_c^\infty(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

Example 3.18. Following are some examples of tempered distributions on $\mathbb{R}^n$.

- Every compactly supported distribution is tempered.
- If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} (1 + |x|)^{-N} |f(x)|\, dx < \infty$ for some $N \in \mathbb{N}_0$, then f is tempered, since

$$\left|\int_{\mathbb{R}^n} f(x)\phi(x)\, dx\right| \leq \left\|(1 + |x|)^{-N} f\right\|_{L^1} \left\|(1 + |x|)^N \phi\right\|_\infty \leq C\|\phi\|_{(N,0)}, \quad \phi \in \mathcal{S}(\mathbb{R}^n).$$

- Given $\omega \in \mathbb{R}^n$, the *plane wave* function

$$B_\omega(x) = e^{i\omega\cdot x}, \quad x \in \mathbb{R}^n$$

  is a tempered distribution on $\mathbb{R}^n$. This distribution is related to the Fourier transform:

$$\langle B_\omega, \phi\rangle = \widehat{\phi}(-\omega) \quad \phi \in \mathcal{S}(\mathbb{R}^n).$$

- In fact, the exponential function $f(x) = e^{\beta\cdot x}$ on $\mathbb{R}^n$ is tempered if and only if β is purely imaginary. We assume $\beta = \gamma + i\omega$ with $\delta, \omega \in \mathbb{R}^n$. If $\gamma \neq 0$, we take $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x)\, dx = 1$ and let

$$\phi_m(x) = e^{-\beta\cdot x}\psi(x - m\gamma).$$

  Then $\phi_m \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ as $m \to \infty$, but $\int_{\mathbb{R}^n} f\phi_m\, dx = \int_{\mathbb{R}^n} \psi\, dx = 1$.

In the following examples, we let $F \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution.

- The derivative $\partial^\alpha F$ is also a tempered distribution. Indeed, $\phi_j \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$ implies

$$\langle \partial^\alpha F, \phi_j \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \phi_j \rangle \to (-1)^{|\alpha|} \langle F, \partial^\alpha \phi \rangle = \langle \partial^\alpha F, \phi \rangle.$$

- If $y \in \mathbb{R}^n$, the translated distribution $\tau_y F$, defined by

$$\langle \tau_y F, \phi \rangle = \langle F, \tau_{-y} \phi \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^n),$$

  is also a tempered distribution.
- If T is an invertible linear mapping on $\mathbb{R}^n$, the composition $F \circ T$, defined by

$$\langle F \circ T, \phi \rangle = |\det T|^{-1} \langle F, \phi \circ T^{-1} \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^n),$$

  is also a tempered distribution.
- Let $\psi \in C^\infty(\mathbb{R}^n)$ be a *slowly increasing* function. That is, ψ and all its derivatives have at most polynomial growth at infinity, i.e. for every multi-index α there exist $C_\alpha > 0$ and $N_\alpha \in \mathbb{N}_0$ such that

$$|\partial^\alpha \psi(x)| \le C_\alpha (1 + |x|)^{N_\alpha}.$$

  For any $\phi \in \mathcal{S}(\mathbb{R}^n)$, since ψ is slowly increasing, we have

$$\sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha \psi(x)| \left|\partial^\beta \phi(x)\right| \le \sup_{x \in \mathbb{R}^n} C_\alpha (1 + |x|)^{N + N_\alpha} \left|\partial^\beta \phi(x)\right| < \infty,$$

  and $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$. Then the product ψF, defined by

$$\langle \psi F, \phi \rangle = \langle F, \psi \phi \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^n),$$

  is a tempered distribution. In particular, every slowly increasing function is a tempered distribution.

Next, we study the convolution of tempered distributions with Schwartz functions.

Proposition 3.19. *If $F \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$, the function $(F * \psi)(x) = \langle F, \tau_x \widetilde{\psi} \rangle$ is a slowly increasing C^∞ function, and we have*

$$\langle F, \phi * \widetilde{\psi} \rangle = \int_{\mathbb{R}^n} (F * \psi)(x) \phi(x) \, dx, \quad \phi \in \mathcal{S}(\mathbb{R}^n). \tag{3.7}$$

Proof. The conclusion $F * \psi \in C^\infty(\mathbb{R}^n)$ is established as in Proposition 3.11. By continuity of F on $\mathcal{S}(\mathbb{R}^n)$, there exist $m, N \in \mathbb{N}$ and $C > 0$ such that

$$|\langle F, \phi \rangle| \le C \sum_{|\alpha| \le N} \|\phi\|_{(m,\alpha)} \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Then for every $x \in \mathbb{R}^n$, by (1.5),

$$\begin{aligned} |(F * \psi)(x)| \le \left|\langle F, \tau_x \widetilde{\psi} \rangle\right| &\le C \sum_{|\alpha| \le N} \sup_{y \in \mathbb{R}^n} (1 + |y|)^m |\partial^\alpha \psi(x - y)| \\ &\le C (1 + |x|)^m \sum_{|\alpha| \le N} \sup_{y \in \mathbb{R}^n} (1 + |x - y|)^m |\partial^\alpha \psi(x - y)| \le C (1 + |x|)^m \sum_{|\alpha| \le N} \|\psi\|_{(m,\alpha)}. \end{aligned}$$

The same coclusion holds if ψ is replaced by $\partial^\beta \psi$. Hence $F * \psi$ is a slowly increasing C^∞ function.

Using Proposition 3.17, we take sequences $(\phi_j), (\psi_j) \subset C_c^\infty(\mathbb{R}^n)$ such that $\phi_j \to \phi$ and $\psi_j \to \psi$ in $\mathcal{S}(\mathbb{R}^n)$. By Proposition 3.11,

$$\langle F, \phi_j * \widetilde{\psi}_j \rangle = \int_{\mathbb{R}^n} (F * \psi_j)(x)\phi_j(x)\,dx.$$

For the left side, by Proposition 1.14, $\phi_j * \widetilde{\psi}_j \to \phi * \widetilde{\psi}$ in $\mathcal{S}(\mathbb{R}^n)$, and $\langle F, \phi_j * \widetilde{\psi}_j \rangle \to \langle F, \phi * \widetilde{\psi} \rangle$ by continuity of F in $\mathcal{S}(\mathbb{R}^n)$. For the right side, the preceding estimate shows that $|F * \psi_j(x)| \le C_1(1+|x|)^m$ with C_1 and m independent of j, and similarly $|\phi_j(x)| \le C_2(1+|x|)^{-m-n-1}$. Hence

$$\int_{\mathbb{R}^n} (F * \psi_j)(x)\phi_j(x)\,dx \to \int_{\mathbb{R}^n} (F * \psi)(x)\phi(x)\,dx$$

by the dominated convergence theorem. Then we conclude the proof of (3.7). □

One of the main reasons for studying tempered distributions is to extend the Fourier transform. Recalling that the Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto itself [Theorem 2.12]. For $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \widehat{f}(x)g(x)\,dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)g(x)e^{-ix\cdot y}\,dx\,dy = \int_{\mathbb{R}^n} f(y)\widehat{g}(y)\,dy$$

Therefore, we can extend the Fourier transform to $\mathcal{S}'(\mathbb{R}^n)$ by defining

$$\langle \widehat{F}, \phi \rangle = \langle F, \widehat{\phi} \rangle, \quad F \in \mathcal{S}'(\mathbb{R}^n),\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

Proposition 3.20. *The Fourier transform of a tempered distribution F on $\mathbb{R}^n$ is also a tempered distribution. Furthermore, the mapping $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), F \mapsto \widehat{F}$ is continuous.*

Proof. Let $(\phi_j) \subset \mathcal{S}(\mathbb{R}^n)$ be a sequence such that $\phi_j \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$. By continuity of the Fourier transform and F on $\mathcal{S}(\mathbb{R}^n)$, $\widehat{\phi}_j \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$, and

$$\langle \widehat{F}, \phi \rangle = \langle F, \widehat{\phi} \rangle = \lim_{j\to\infty} \langle F, \widehat{\phi}_j \rangle = \lim_{j\to\infty} \langle \widehat{F}, \phi_j \rangle.$$

Hence $\widehat{F} \in \mathcal{S}'(\mathbb{R}^n)$. On the other hand, for sequence $(F_j) \subset \mathcal{S}'(\mathbb{R}^n)$ with $F_j \to F$ in $\mathcal{S}'(\mathbb{R}^n)$,

$$\langle \widehat{F}, \phi \rangle = \langle F, \widehat{\phi} \rangle = \lim_{j\to\infty} \langle F_j, \widehat{\phi} \rangle = \lim_{j\to\infty} \langle \widehat{F}_j, \phi \rangle, \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n).$$

Therefore $\widehat{F}_j \to \widehat{F}$ in $\mathcal{S}'(\mathbb{R}^n)$, and the mapping $F \mapsto \widehat{F}$ is continuous in $\mathcal{S}'(\mathbb{R}^n)$. □

Proposition 3.21 (Properties of Fourier transform)**.** *Let $F \in \mathcal{S}'(\mathbb{R}^n)$.*

(i) (Translation/Modulation) Let $\xi \in \mathbb{R}^n$. Then

$$\widehat{\tau_\xi F} = B_{-\xi}\widehat{F}, \quad \textit{and} \quad \widehat{B_\xi F} = \tau_\xi \widehat{F},$$

where $B_\xi(x) = e^{i\xi\cdot x}$ is the plane wave function.

(ii) (Linear transofrmation) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear mapping, and $S = (T^*)^{-1}$ is its inverse transpose, then

$$\widehat{F \circ T} = |\det T|^{-1}\,\widehat{F} \circ S.$$

(iii) (Derivative) For every $\alpha \in \mathbb{N}_0^n$, $\widehat{\partial^\alpha F} = (i\omega)^\alpha \widehat{F}$ and $\partial^\alpha \widehat{F} = [(-ix)^\alpha F]^\wedge$.

(iv) (Convolution) For $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\widehat{F * \psi} = (2\pi)^{n/2}\widehat{\psi}\widehat{F}, \quad \textit{and} \quad \widehat{\psi F} = (2\pi)^{-n/2}\widehat{\psi} * \widehat{F}.$$

Proof. (i) For the first identity, note that for every $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\langle \widehat{\tau_\xi F}, \phi \right\rangle = \left\langle \tau_\xi F, \widehat{\phi} \right\rangle = \left\langle F, \tau_{-\xi}\widehat{\phi} \right\rangle = \left\langle F, (e^{-i\xi\cdot x}\phi)^\wedge \right\rangle = \langle \widehat{F}, e^{-i\xi\cdot x}\phi \rangle = \langle B_{-\xi}\widehat{F}, \phi \rangle.$$

The second identity follows the same approach.

(ii) For every $\phi \in \mathcal{S}(\mathbb{R}^n)$, the definition of $F \circ T$ gives

$$\langle \widehat{F \circ T}, \phi \rangle = \langle F \circ T, \widehat{\phi} \rangle = |\det T|^{-1} \langle F, \widehat{\phi} \circ T^{-1} \rangle$$

On the other hand, by Proposition 2.6 (iv),

$$|\det T|^{-1} \langle \widehat{F} \circ S, \phi \rangle = \langle \widehat{F}, \phi \circ T^* \rangle = \langle F, \widehat{\phi \circ T^*} \rangle = |\det T^*|^{-1} \langle F, \widehat{\phi} \circ T^{-1} \rangle.$$

These two results coincide.

(iii) For the first identity, note that for every $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \widehat{\partial^\alpha F}, \phi \rangle = \langle \partial^\alpha F, \widehat{\phi} \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \widehat{\phi} \rangle = (-1)^{|\alpha|} \left\langle F, [(-ix)^\alpha \phi]^\wedge \right\rangle = (-1)^{|\alpha|} \langle \widehat{F}, (-ix)^\alpha \phi \rangle = \langle (i\omega)^\alpha \widehat{F}, \phi \rangle.$$

The second identity follows the same approach.

(iv) For every $\phi \in \mathcal{S}(\mathbb{R}^n)$, by Proposition 3.19,

$$\langle \widehat{F * \psi}, \phi \rangle = \langle F * \psi, \widehat{\phi} \rangle = \langle F, \widehat{\phi} * \widetilde{\psi} \rangle.$$

On the other hand, by Proposition 2.15,

$$\begin{aligned} \langle \widehat{\psi}\widehat{F}, \phi \rangle = \langle \widehat{F}, \phi\widehat{\psi} \rangle &= (2\pi)^{-n/2} \left\langle F, (\phi^\vee * \psi)^\wedge \right\rangle = (2\pi)^{-n/2} \langle \widehat{F}, \phi^\vee * \psi \rangle \\ &= (2\pi)^{-n/2} \left\langle \widehat{F}, \widetilde{\phi^\vee} * \widetilde{\psi} \right\rangle = (2\pi)^{-n/2} \langle F, \widehat{\phi} * \widetilde{\psi} \rangle. \end{aligned}$$

Comparing the above two results, we have $\widehat{F * \psi} = (2\pi)^{n/2} \widehat{\psi}\widehat{F}$. The second result is similar. ◻

Similarly, we extend the inverse Fourier transform to $\mathcal{S}'(\mathbb{R}^n)$ by defining

$$\langle F^\vee, \phi \rangle = \langle F, \phi^\vee \rangle, \quad F \in \mathcal{S}'(\mathbb{R}^n),\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

Then $F \mapsto F^\vee$ is also a continuous linear mapping from $\mathcal{S}'(\mathbb{R}^n)$ itself. Furthermore, we can extend the Fourier inversion theorem formula to tempered distributions:

$$\langle (\widehat{F})^\vee, \phi \rangle = \langle \widehat{F}, \phi^\vee \rangle = \langle F, (\phi^\vee)^\wedge \rangle = \langle F, \phi \rangle,$$

so that $(\widehat{F})^\vee = F$, and similarly $(F^\vee)^\wedge = F$. Hence *the Fourier transform is an isomorphism on* $\mathcal{S}'(\mathbb{R}^n)$.

In addition, for a tempered distribution $F \in \mathcal{S}'(\mathbb{R}^n)$, we can relate its inverse Fourier transform to the Fourier transform of its reflection $\widetilde{F} = F \circ (-\operatorname{Id})$.

Proposition 3.22. *Let $F \in \mathcal{S}'(\mathbb{R}^n)$. Then $\widehat{F} = (\widetilde{F})^\vee$ and $F^\vee = (\widetilde{F})^\wedge$.*

Proof. For every $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle \widehat{F}, \phi \rangle = \langle F, \widehat{\phi} \rangle = \langle F, (\phi^\vee)^\sim \rangle = \langle \widetilde{F}, \phi^\vee \rangle = \langle (\widetilde{F})^\vee, \phi \rangle.$$

Hence $\widehat{F} = (\widetilde{F})^\vee$. The other formula follows by applying this conclusion on $\widetilde{F}$. ◻

Next, we study the Fourier transform of compactly supported distributions $F \in \mathcal{E}'(\mathbb{R}^n)$, which admits an explicit formula.

Proposition 3.23. *If $F \in \mathcal{E}'(\mathbb{R}^n)$, then $\widehat{F}$ is a slowly increasing C^∞ function, which is explicitly given by*

$$\widehat{F}(\omega) = (2\pi)^{-n/2}\langle F, B_{-\omega}\rangle, \quad \omega \in \mathbb{R}^n,$$

where $B_\omega(x) = e^{i\omega\cdot x}$ is the plane wave function.

Proof. We define the function

$$g(\omega) = \langle F, B_{-\omega}\rangle, \quad \omega \in \mathbb{R}^n.$$

We fix $\omega \in \mathbb{R}^n$. For each $\alpha \in \mathbb{N}_0^\alpha$ and $N > 0$,

$$\sup_{|x|\le N}\left|\frac{\partial^\alpha B_{-\omega-he_j}(x) - \partial^\alpha B_{-\omega}(x)}{h} + ix_j\partial^\alpha B_{-\omega}(x)\right| \le \sup_{|x|\le N}\left|(-i\omega)^\alpha\left[\frac{e^{-i(\omega+he_j)\cdot x} - e^{-i\omega\cdot x}}{h} + ix_je^{-i\omega\cdot x}\right]\right|$$
$$\le |\omega|^{|\alpha|}\sup_{|x|\le N}\left|\frac{e^{-ihx_j}-1}{h} + ix_j\right| \to 0, \quad \text{as } h \to 0.$$

Therefore $(B_{-\omega-he_j} - B_{-\omega})/h$ converges to the function $x \mapsto -ix_je^{-i\omega\cdot x}$ in $C^\infty(\mathbb{R}^n)$. Then

$$\lim_{h\to 0}\frac{g(\omega+he_j)-g(\omega)}{h} = \lim_{h\to 0}\left\langle F, \frac{B_{-\omega-he_j} - B_{-\omega}}{h}\right\rangle = \langle F, -ix_jB_{-\omega}\rangle.$$

By induction on α, we have

$$\partial^\alpha g(\omega) = \langle F, \partial^\alpha B_{-\omega}\rangle = (-i)^{|\alpha|}\langle F, x^\alpha e^{-i\omega\cdot x}\rangle.$$

Moreover, by Theorem 3.15 and continuity of F on $C^\infty(\mathbb{R}^n)$, there exist $m, N \in \mathbb{N}$ and $C > 0$ such that

$$|\partial^\alpha g(\omega)| \le C\sum_{|\beta|\le N}\sup_{|x|\le m}\left|\partial_x^\beta(x^\alpha e^{-i\omega\cdot x})\right| \le C'(1+m)^{|\alpha|}(1+|\omega|)^N.$$

Therefore $g \in C^\infty(\mathbb{R}^n)$ is slowly incresing.

Now it remains to show that $g = (2\pi)^{n/2}\widehat{F}$. By Proposition 3.17, it suffices to show that for $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} g(x)\phi(x)\,dx = \langle F, \widehat{\phi}\rangle.$$

Since $g\phi \in C_c^\infty(\mathbb{R}^n)$, as in the proof of Proposition 3.11, the left side can be approximated by the Riemann sum $\epsilon^n\sum_{\kappa\in\mathbb{Z}^n} g(\epsilon\kappa)\phi(\epsilon\kappa)$ as $\epsilon \downarrow 0$. For the right side, note that by uniform continuity of function $x \mapsto \phi(x)e^{-i\omega\cdot x}$ and its derivatives, for all $\alpha \in \mathbb{N}_0^n$ and $N > 0$,

$$\sup_{|x|\le N}\left|\partial_\omega^\alpha\left[\epsilon^n\sum_{\kappa\in\mathbb{Z}^n}\frac{\phi(\epsilon\kappa)}{(2\pi)^{n/2}}e^{-i\omega\cdot\epsilon\kappa}\right] - \partial_\omega^\alpha\widehat{\phi}(\omega)\right| = \sup_{|x|\le N}\left|\epsilon^n\sum_{\kappa\in\mathbb{Z}^n}\frac{(-i\epsilon\kappa)^\alpha\phi(\epsilon\kappa)}{(2\pi)^{n/2}}e^{-i\omega\cdot\epsilon\kappa} - [(-ix)^\alpha\phi]^\wedge(\omega)\right| \to 0.$$

Hence the Riemann sum $S_\epsilon = (2\pi)^{-n/2}\epsilon^n\sum_{\kappa\in\mathbb{Z}^n}\phi(\epsilon\kappa)B_{-\epsilon\kappa}$ converges to $\widehat{\phi}$ in $C^\infty(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^n} g(x)\phi(x)\,dx = \lim_{\epsilon\downarrow 0}\epsilon^n\sum_{\kappa\in\mathbb{Z}^n} g(\epsilon\kappa)\phi(\epsilon\kappa) = \lim_{\epsilon\downarrow 0}\epsilon^n\sum_{\kappa\in\mathbb{Z}^n}\langle F, B_{-\epsilon\kappa}\rangle\,\phi(\epsilon\kappa) = \lim_{\epsilon\downarrow 0}\langle F, S_\epsilon\rangle = \langle F, \widehat{\phi}\rangle.$$

Therefore $g = \widehat{F}$, and we complete the proof. $\square$

Now we focus on some concrete examples.

Example 3.24. (i) The Fourier transforms of linear combinations of δ and its derivatives are polynomials:

$$\left(\sum_{|\alpha|\le m} c_\alpha \partial^\alpha \delta\right)^\wedge = (2\pi)^{-n/2}\left(\sum_{|\alpha|\le m} c_\alpha i^{|\alpha|}\omega^\alpha\right).$$

More generally,

$$\left(\sum_{|\alpha|\le m} c_\alpha \partial^\alpha \tau_{\xi_\alpha}\delta\right)^\wedge = (2\pi)^{-n/2}\left(\sum_{|\alpha|\le m} c_\alpha i^{|\alpha|}\omega^\alpha e^{-i\xi_\alpha\cdot\omega}\right).$$

(ii) The Fourier transforms of polynomials are linear combinations of δ and its derivatives:

$$\left(\sum_{|\alpha|\le m} c_\alpha x^\alpha\right)^\wedge = (2\pi)^{n/2}\left(\sum_{|\alpha|\le m} c_\alpha i^{|\alpha|}\partial^\alpha\delta\right).$$

More generally,

$$\left(\sum_{|\alpha|\le m} c_\alpha x^\alpha e^{i\xi_\alpha\cdot x}\right)^\wedge = (2\pi)^{n/2}\left(\sum_{|\alpha|\le m} c_\alpha i^{|\alpha|}\partial^\alpha\tau_{\xi_\alpha}\delta\right).$$

Proof. (i) By Proposition 3.23, for the Dirac distribution $\delta : \phi \mapsto \phi(0)$, we have

$$(2\pi)^{n/2}\,\widehat{\tau_\xi\delta}(\omega) = \langle \tau_\xi\delta, B_{-\omega}\rangle = B_{-\omega}(\xi) = e^{-i\omega\cdot\xi}, \quad \omega\in\mathbb{R}^n.$$

More generally, for each $\alpha\in\mathbb{N}_0^n$ and $\xi\in\mathbb{R}^n$,

$$\begin{aligned}(2\pi)^{n/2}\,\widehat{\partial^\alpha\tau_\xi\delta}(\omega) &= \langle \partial^\alpha\tau_\xi\delta, B_{-\omega}\rangle = (-1)^{|\alpha|}\langle \tau_\xi\delta, \partial^\alpha B_{-\omega}\rangle \\ &= (-1)^{|\alpha|}\langle \tau_\xi\delta, (-i\omega)^\alpha B_{-\omega}\rangle = (i\omega)^\alpha B_{-\omega}(\xi) = (i\omega)^\alpha e^{-i\xi\cdot\omega}.\end{aligned}$$

In particular, we have

$$\widehat{\partial^\alpha\delta}(\omega) = (2\pi)^{-n/2} i^{|\alpha|}\omega^\alpha.$$

(ii) For each $\alpha\in\mathbb{N}_0^n$ and $\xi\in\mathbb{R}^n$, by Proposition 3.22 and the conclusion in (i),

$$(x^\alpha e^{i\xi\cdot x})^\wedge = \left[(-x)^\alpha e^{-i\xi\cdot x}\right]^\vee = i^{|\alpha|}\left[(ix)^\alpha e^{-i\xi\cdot x}\right]^\vee = (2\pi)^{n/2} i^{|\alpha|}\partial^\alpha\tau_\xi\delta.$$

In particular, we have

$$\widehat{x^\alpha} = (2\pi)^{n/2} i^{|\alpha|}\partial^\alpha\delta.$$

□

Remark. Heuristically, we may write

$$\tau_\xi\delta(x) = (2\pi)^{-n/2}\,\widehat{B}_\xi(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(\xi-x)\cdot y}\,dy = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(x-\xi)\cdot y}\,dy.$$

Although this is not an essential pointwise equality, it express the fact that the Fourier transform of the plane wave function B_ξ is $\tau_\xi\delta$. In the sense of integral, for all $\phi\in C_c^\infty(\mathbb{R}^n)$, we integrate both sides with $\phi\in\mathcal{S}(\mathbb{R}^n)$ and reverse the order of integration to obtain

$$(2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi(x)e^{i(\xi-x)\cdot y}\,dx\,dy = \int_{\mathbb{R}^n}\delta(x-\xi)\phi(x)\,dx.$$

The integral on the left is $(\widehat{\phi})^\vee(\xi)$, and the integral on the right equals $\phi(\xi)$.

Convolution and differential operators. The convolution also commutes with distributional differential operators: for each $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$
\begin{aligned}
\langle \partial^\alpha (F * \psi), \phi \rangle &= (-1)^{|\alpha|} \langle F * \psi, \partial^\alpha \phi \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \phi * \widetilde{\psi} \rangle \\
&= \begin{cases} (-1)^{|\alpha|} \langle F, \partial^\alpha (\phi * \widetilde{\psi}) \rangle = \langle \partial^\alpha F, \phi * \widetilde{\psi} \rangle = \langle \partial^\alpha F * \psi, \phi \rangle, \\ (-1)^{|\alpha|} \langle F, \phi * \partial^\alpha \widetilde{\psi} \rangle = \langle F, \phi * \widetilde{\partial^\alpha \psi} \rangle = \langle F * \partial^\alpha \psi, \phi \rangle. \end{cases}
\end{aligned}
$$

Therefore $\partial^\alpha (F * \psi) = \partial^\alpha F * \psi = F * \partial^\alpha \psi$.

Example 3.25 (Heat equation)**.** Consider the following function on $\mathbb{R}^n \times \mathbb{R}$:

$$
G(x,t) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}} \chi_{(0,\infty)}(t), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}.
$$

By Lemma 2.9, the Fourier transform of G is

$$
\widehat{G}(\omega, \rho) = (2\pi)^{-\frac{n+1}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} G(x,t) e^{-i\omega \cdot x} e^{-i\rho t} \, dx \, dt = (2\pi)^{-\frac{n+1}{2}} \int_0^\infty e^{-t|\omega|^2} e^{-i\rho t} \, dt = \frac{(2\pi)^{-\frac{n+1}{2}}}{i\rho + |\omega|^2}.
$$

The Proposition 3.21 implies

$$
(2\pi)^{-\frac{n+1}{2}} = \left(i\rho + |\omega|^2\right) \widehat{G} = [(\partial_t - \Delta_x) G]^\wedge.
$$

Using Example 3.24, we know that G is the Green function of the heat equation:

$$
(\partial_t - \Delta_x) G = \delta.
$$

Therefore the function $u = G * \psi$ satisfies

$$
(\partial_t - \Delta_x) u = [(\partial_t - \Delta_x) G] * \psi = \delta * \psi = \psi.
$$

Every distribution is indeed, at least locally, a linear combination of derivatives of continuous functions. We can prove this conclusion using the Fourier transform.

Proposition 3.26. *(i) For every $F \in \mathcal{E}'(\mathbb{R}^n)$, there exist $N \in \mathbb{N}$, constants c_α for $|\alpha| \le N$, and $f \in C_0(\mathbb{R}^n)$ such that*

$$
F = \sum_{|\alpha| \le N} c_\alpha \partial^\alpha f.
$$

(ii) For every $F \in \mathcal{D}'(U)$ and precompact set V with $\overline{V} \subset U$, there exist $N \in \mathbb{N}$, constants c_α for $|\alpha| \le N$, and $f \in C_0(\mathbb{R}^n)$ above such that

$$
F = \sum_{|\alpha| \le N} c_\alpha \partial^\alpha f.
$$

Proof. (i) By Proposition 3.23, if $F \in \mathcal{E}'(\mathbb{R}^n)$, then $\widehat{F}$ is slowly increasing, and for large enough $M \in \mathbb{N}$, the function $g(\xi) = (1 + |\xi|^2)^{-M} \widehat{F}(\xi)$ is in L^1. By Riemann-Lebesgue lemma, the function $f := g^\vee \in C_0(\mathbb{R}^n)$, and

$$
\widehat{F} = (1 + |\xi|^2)^M \widehat{f} = \left[\left(\mathrm{Id} - \sum_{j=1}^n \partial_j^2 \right)^M f \right]^\wedge.
$$

Therefore $F = (\mathrm{Id} - \Delta)^M f$ in $\mathcal{S}'(\mathbb{R}^n)$, and also in $\mathcal{E}'(\mathbb{R}^n)$.

(ii) We choose $\psi \in C_c^\infty(U)$ such that $\psi = 1$ on $\overline{V}$, and then apply (i) to ψF. $\square$

3.5 Sobolev Spaces

In this subsection we discuss the smoothness of functions and distributions. For a nonnegative integer k, we equip the Sobolev space $H^k(\mathbb{R}^n)$ of all functions $f \in L^2(\mathbb{R}^n)$ whose distributional derivatives $\partial^\alpha f \in L^2(\mathbb{R}^n)$ for all multi-indices $|\alpha| \le k$ with inner product and norm

$$\|f\|_{(k)}^2 = \sum_{|\alpha|\le k} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}^2, \quad \langle f, g\rangle_{(k)} = \sum_{|\alpha|\le k} \int_{\mathbb{R}^n} \partial^\alpha f(x)\,\overline{\partial^\alpha g(x)}\,dx, \quad f, g \in H^k(\mathbb{R}^n).$$

In fact, it is more convenient to use an equivalent inner product defined in terms of Fourier transform.

Proposition 3.27. *Let $k \in \mathbb{N}_0$. Then a function $f \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if $(1+|\omega|^2)^{k/2}\widehat{f}$ belongs to $L^2(\mathbb{R}^n)$. In addition, there exists $C > 1$ such that*

$$C^{-1}\|f\|_{(k)} \le \big\|(1+|\omega|^2)^{k/2}\widehat{f}\big\|_{L^2(\mathbb{R}^n)}^2 \le C\|f\|_{(k)}, \quad \textit{for all } f \in H^k(\mathbb{R}^n).$$

Proof. Step I. If $f \in H^k(\mathbb{R}^n)$, we have $\partial^\alpha f \in L^2(\mathbb{R}^n)$ and $\widehat{\partial^\alpha f} = (i\omega)^\alpha \widehat{f} \in L^2(\mathbb{R}^n)$ for all $|\alpha| \le k$. If we choose $\alpha = \ell e_j$ for each $\ell = 1, \cdots, k$ and $j = 1, \cdots, n$, we obtain

$$\int_{\mathbb{R}^n} |\omega_j|^{2\ell} |\widehat{f}(\omega)|^2\,d\omega = \int_{\mathbb{R}^n} \big|\partial_{x_j}^\ell f(x)\big|^2\,dx.$$

Hence there exist constants $C_1, C_2 > 0$ such that

$$\int_{\mathbb{R}^n} |\omega|^{2\ell} |\widehat{f}(\omega)|^2\,d\omega = C_1 \sum_{|\alpha|\le \ell} \int_{\mathbb{R}^n} \big|\partial^\alpha f(x)\big|^2\,dx, \quad \ell = 1, 2, \cdots, k,$$

and that

$$\int_{\mathbb{R}^n} (1+|\omega|^2)^k |\widehat{f}(\omega)|^2\,d\omega = C_2 \sum_{|\alpha|\le k} \int_{\mathbb{R}^n} \big|\partial^\alpha f(x)\big|^2\,dx = C_2\|f\|_{(k)}^2.$$

Hence $(1+|\omega|^2)^{k/2}\widehat{f} \in L^2(\mathbb{R}^n)$.

Step II. Conversely, if $(1+|\omega|^2)^{k/2}\widehat{f} \in L^2(\mathbb{R}^n)$ and $|\alpha| \le k$, we have

$$\big\|(i\omega)^\alpha \widehat{f}\big\|_{L^2(\mathbb{R}^n)}^2 \le \int_{\mathbb{R}^n} |\omega|^{2|\alpha|} |\widehat{f}(\omega)|^2\,d\omega \le \big\|(1+|\omega|^2)^{k/2}\widehat{f}\big\|_{L^2(\mathbb{R}^n)}^2.$$

We then define $f_\alpha = \big[(i\omega)^\alpha \widehat{f}\big]^\vee \in L^2(\mathbb{R}^n)$. Then for each $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \phi(x)\overline{f_\alpha(x)}\,dx = \int_{\mathbb{R}^n} \widehat{\phi}(\omega)\overline{(i\omega)^\alpha \widehat{f}(\omega)}\,d\omega = \int_{\mathbb{R}^n} \widehat{\partial^\alpha \phi}(\omega)\overline{\widehat{f}(\omega)}\,d\omega = \int_{\mathbb{R}^n} \partial^\alpha \phi(x)\overline{f(x)}\,dx.$$

Hence $f_\alpha = \partial^\alpha f \in L^2(\mathbb{R}^n)$ for all $|\alpha| \le k$, and $f \in H^k(\mathbb{R}^n)$. We note that

$$\|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}^2 = \big\|(i\omega)^\alpha \widehat{f}\big\|_{L^2(\mathbb{R}^n)}^2 \le \big\|(1+|\omega|^2)^{k/2}\widehat{f}\big\|_{L^2(\mathbb{R}^n)}^2,$$

and $\|f\|_{(k)} \le C_3\big\|(1+|\omega|^2)^{k/2}\widehat{f}\big\|_{L^2(\mathbb{R}^n)}^2$ for some constant $C_3 > 0$. Thus we complete the proof. ☐

More generally, we fix $s \in \mathbb{R}$. Then the function $\omega \mapsto (1+|\omega|^2)^{s/2}$ is slowly increasing, since for each multi-index $\alpha \in \mathbb{N}_0^\alpha$, the derivative $\partial^\alpha(1+|\omega|^2)^{s/2}$ is of the form

$$\partial^\alpha(1+|\omega|^2)^{\frac{s}{2}} = \sum_{k=1}^{|\alpha|} P_k(\omega)(1+|\omega|^2)^{\frac{s}{2}-k} \le C_\alpha(1+|\omega|)^{N_\alpha}, \quad (P_k) \text{ are polynomials with } \deg(P_k) \le |\alpha|.$$

Lemma 3.28. *Let $\Lambda_s : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be*

$$\Lambda_s f = \left[(1+|\omega|^2)^{s/2}\widehat{f}\right]^\vee, \quad f \in \mathcal{S}'(\mathbb{R}^n).$$

Then Λ_s defines an isomorphism on $\mathcal{S}'(\mathbb{R}^n)$, and $\Lambda_s^{-1} = \Lambda_{-s}$.

Proof. Let $f_k \to f$ in $\mathcal{S}'(\mathbb{R}^n)$. Then

$$\langle \Lambda_s f_k, \phi\rangle = \left\langle \widehat{f_k}, (1+|\omega|^2)^{s/2}\phi^\vee \right\rangle \to \left\langle \widehat{f}, (1+|\omega|^2)^{s/2}\phi^\vee \right\rangle = \langle \Lambda_s f, \phi\rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^n),$$

and $\Lambda_s f_k \to \Lambda_s f$. Hence Λ_s is continuous on $\mathcal{S}'(\mathbb{R}^n)$. Furthermore,

$$\langle \Lambda_s\Lambda_{-s} f, \phi\rangle = \left\langle \widehat{\Lambda_{-s} f}, (1+|\omega|^2)^{s/2}\phi^\vee \right\rangle = \left\langle (1+|\omega|^2)^{-s/2}\widehat{f}, (1+|\omega|^2)^{s/2}\phi^\vee \right\rangle = \langle \widehat{f}, \phi^\vee\rangle = \langle f, \phi\rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^n),$$

which implies $\Lambda_s\Lambda_{-s} = \mathrm{Id}$. Similarly $\Lambda_s\Lambda_{-s} = \mathrm{Id}$. Hence $\Lambda_s^{-1} = \Lambda_{-s}$ is also continuous. $\square$

Definition 3.29 (Sobolev spaces)**.** For each $s \in \mathbb{R}$, define the Sobolev space H^s by

$$H^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \Lambda_s f \in L^2(\mathbb{R}^n) \right\} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : (1+|\omega|^2)^{s/2}\widehat{f} \in L^2(\mathbb{R}^n) < \infty \right\},$$

and we define an inner product and norm by

$$\|f\|_{H^s(\mathbb{R}^n)} = \|\Lambda_s f\|_{L^2(\mathbb{R}^n)} = \left[\int_{\mathbb{R}^n} |\widehat{f}(\omega)|^2(1+|\omega|^2)^s \, d\omega\right]^{1/2},$$
$$\langle f, g\rangle_{H^s(\mathbb{R}^n)} = \langle \Lambda_s f, \Lambda_s g\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}(1+|\omega|^2)^s \, d\omega.$$

Remark. For the case $s = 0$, we have $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ and $\|\cdot\|_{H^0(\mathbb{R}^n)} = \|\cdot\|_{L^2(\mathbb{R}^n)}$ by Plancherel's theorem.

Proposition 3.30. *Let $s \in \mathbb{R}$. The following statements about Sobolev spaces hold:*

(i) $H^s(\mathbb{R}^n)$ is a Hilbert space, and $\mathcal{S}(\mathbb{R}^n)$ is a dense subspace of $H^s(\mathbb{R}^n)$.

(ii) For $t < s$, $H^s(\mathbb{R}^n)$ is a dense subspace of $H^t(\mathbb{R}^n)$ in the topology of $H^t(\mathbb{R}^n)$, and $\|\cdot\|_{H^t(\mathbb{R}^n)} \le \|\cdot\|_{H^s(\mathbb{R}^n)}$.

(iii) For every $t \in \mathbb{R}$, Λ_t is a unitary isomorphism from $H^s(\mathbb{R}^n)$ to $H^{s-t}(\mathbb{R}^n)$.

(iv) For every $\alpha \in \mathbb{N}_0^n$, the differential operator $\partial^\alpha : H^s(\mathbb{R}^n) \to H^{s-|\alpha|}(\mathbb{R}^n)$ is a bounded linear map.

Proof. (i) By the definition of $\langle\cdot,\cdot\rangle_{H^s(\mathbb{R}^n)}$, the Fourier transform is a unitary isomorphism from $H^s(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n, (1+|\omega|^2)^{s/2}\, d\omega)$, which is a Hilbert space. If (f_j) is a Cauchy sequence in $H^s(\mathbb{R}^n)$, $(\widehat{f_j})$ is also a Cauchy sequence in $L^2(\mathbb{R}^n, (1+|\omega|^2)^{s/2} d\omega)$, which converges to an element $g \in L^2(\mathbb{R}^n, (1+|\omega|^2)^{s/2} d\omega)$. Then $(1+|\omega|^2)^{s/2} g \in L^2(\mathbb{R}^n)$, and $g^\vee \in H^s(\mathbb{R}^n)$ satisfies $\|g^\vee - f_j\|_{L^2(\mathbb{R}^n)} = \|g - \widehat{f_j}\|_{H^s(\mathbb{R}^n)} \to 0$. This implies $f_j \to g^\vee$ in $H^s(\mathbb{R}^n)$, and $H^s(\mathbb{R}^n)$ is complete.

For each $f \in H^s(\mathbb{R}^n)$, the function $(1+|\omega|)^{s/2}\widehat{f}$ is in $L^2(\mathbb{R}^n)$. By Proposition 1.15, we take a sequence $(g_j) \subset \mathcal{S}(\mathbb{R}^n)$ such that $g_j \to (1+|\omega|)^{s/2}\widehat{f}$ in $L^2(\mathbb{R}^n)$. Then $f_j = \left[(1+|\omega|)^{-s/2} g_j\right]^\vee \in \mathcal{S}(\mathbb{R}^n)$ satisfies

$$\|f_j - f\|_{H^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \left| \frac{g_j(\omega)}{(1+|\omega|)^{s/2}} - \widehat{f}(\omega) \right|^2 (1+|\omega|^2)^s \, d\omega = \int_{\mathbb{R}^n} \left| g_j(\omega) - (1+|\omega|^2)^{s/2}\widehat{f}(\omega) \right|^2 \, d\omega \to 0.$$

(ii) Since $H^s(\mathbb{R}^n) \supset \mathcal{S}(\mathbb{R}^n)$, by (i), $H^s(\mathbb{R}^n)$ is a dense subspace of $H^t(\mathbb{R}^n)$. The estimate $\|\cdot\|_{H^t(\mathbb{R}^n)} \le \|\cdot\|_{H^s(\mathbb{R}^n)}$ is clear from the definition of H^s-norms.

(iii) By definition, for all $f, g \in H^s(\mathbb{R}^n)$,

$$\langle \Lambda_t f, \Lambda_t g\rangle_{H^{s-t}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}(1+|\omega|^2)^s \, d\omega = \langle f, g\rangle_{H^s(\mathbb{R}^n)}.$$

(iv) Note that $|\omega^\alpha| \le (1+|\omega|^2)^{|\alpha|/2}$. Then for each $f \in H^s(\mathbb{R}^n)$,

$$\begin{aligned}\|\partial^\alpha f\|^2_{H^{s-|\alpha|}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left|\widehat{\partial^\alpha f}(\omega)\right|^2 (1+|\omega|)^{s-|\alpha|}\,d\omega = \int_{\mathbb{R}^n} \left|(i\omega)^\alpha \widehat{f}(\omega)\right|^2 (1+|\omega|)^{s-|\alpha|}\,d\omega \\ &\le \int_{\mathbb{R}^n} \left|\widehat{f}(\omega)\right|^2 (1+|\omega|)^s\,d\omega = \|f\|^2_{H^s(\mathbb{R}^n)}.\end{aligned}$$

Thus we complete the proof. □

Remark. According to (iii), $H^s(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ for all $s \ge 0$. However, for $s < 0$, the elements of $H^s(\mathbb{R}^n)$ are generally not functions. For example, the point mass $\delta \in H^s(\mathbb{R}^n)$ if and only if $s < -n/2$, because the Fourier transform $\widehat{\delta}$ is a constant function, and $\int_{\mathbb{R}^n} (1+|\omega|)^s\,d\omega < \infty$ if and only if $s < -n/2$.

Theorem 3.31 (Riesz representation theorem)**.** *Let $s \in \mathbb{R}$. The duality between $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ induces a unitary isomorphism from $H^{-s}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)^*$, written $H^s(\mathbb{R}^n)^* \simeq H^{-s}(\mathbb{R}^n)$. To be precise,*

(i) for every $f \in H^{-s}(\mathbb{R}^n)$, the functional $\phi \mapsto \langle f, \phi\rangle$ on $\mathcal{S}(\mathbb{R}^n)$ extends to a continuous linear functional on $H^s(\mathbb{R}^n)$ with operator norm equal to $\|f\|_{H^{-s}(\mathbb{R}^n)}$, and

(ii) every element of $H^s(\mathbb{R}^n)^*$ is identified in this manner.

Proof. (i) For every $f \in H^{-s}(\mathbb{R}^n)$, the element $(1+|\omega|)^{-s/2}\widehat{f} \in L^2(\mathbb{R}^n)$. Note that $\widehat{f}(\omega) = f^\vee(-\omega)$. By the Cauchy-Schwarz inequality, for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}|\langle f,\phi\rangle| &= |\langle f^\vee, \widehat{\phi}\rangle| = \left|\int_{\mathbb{R}^n} f^\vee(\omega)\widehat{\phi}(\omega)\,d\omega\right| \\ &\le \left|\int_{\mathbb{R}^n} |f^\vee(\omega)|^2(1+|\omega|^2)^{-s}\,d\omega\right|^{1/2} \left|\int_{\mathbb{R}^n} |\widehat{\phi}(\omega)|^2(1+|\omega|^2)^{s}\,d\omega\right|^{1/2} = \|f\|_{H^{-s}(\mathbb{R}^n)}\|\phi\|_{H^s(\mathbb{R}^n)}.\end{aligned}$$

Since $\mathcal{S}(\mathbb{R}^n)$ is a dense subspace of $H^s(\mathbb{R}^n)$, the functional $\phi \mapsto \langle f,\phi\rangle$ extends continuously to $H^s(\mathbb{R}^n)$, with the operator norm at most $\|f\|_{H^{-s}(\mathbb{R}^n)}$. Indeed, for every $\phi \in H^s(\mathbb{R}^n)$, we take a sequence $g_j \in \mathcal{S}(\mathbb{R}^n)$ with $g_j \to (1+|\omega|^2)^{s/2}\widehat{\phi}$ in $L^2(\mathbb{R}^n)$. Then $(1+|\omega|^2)^{-s/2}g_j \to \widehat{\phi}$ in $L^2(\mathbb{R}^n, (1+|\omega|^2)^s\,d\omega)$, and

$$\begin{aligned}\langle f,\phi\rangle &= \langle f^\vee, \widehat{\phi}\rangle = \left\langle (1+|\omega|^2)^{-s/2} f^\vee, (1+|\omega|^2)^{s/2}\widehat{\phi}\right\rangle \\ &= \lim_{j\to\infty} \left\langle (1+|\omega|^2)^{-s/2} f^\vee, g_j\right\rangle = \lim_{j\to\infty}\left\langle f^\vee, (1+|\omega|^2)^{-s/2} g_j\right\rangle = \langle f^\vee, \widehat{\phi}\rangle.\end{aligned}$$

We take $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\widehat{\phi}(\omega) = (1+|\omega|^2)^{-s}\overline{f^\vee(\omega)} \in \mathcal{S}(\mathbb{R}^n)$. Then $\phi \in H^s(\mathbb{R}^n)$, and

$$\langle f,\phi\rangle = \left\langle f^\vee, (1+|\omega|^2)^{-s}\overline{f^\vee}\right\rangle = \int_{\mathbb{R}^n} |\widehat{f}(-\omega)|^2(1+|\omega|^2)^{-s}\,d\omega = \|f\|^2_{H^{-s}(\mathbb{R}^n)} = \|f\|_{H^{-s}(\mathbb{R}^n)}\|\phi\|_{H^s(\mathbb{R}^n)}.$$

Hence the operator norm of $\langle f,\cdot\rangle$ equals $\|f\|_{H^{-s}(\mathbb{R}^n)}$.

(ii) If $G \in H^s(\mathbb{R}^n)^*$, then $G \circ \mathcal{F}^{-1} \in L^2(\mathbb{R}^n, (1+|\omega|^2)^s\,d\omega)^*$, and we identify it with $g \in L^2(\mathbb{R}^n, (1+|\omega|^2)^s\,d\omega)$. Then for every $\phi \in H^s(\mathbb{R}^n)$, the Fourier transform $\widehat{\phi} \in L^2(\mathbb{R}^n, (1+|\omega|^2)^s\,d\omega)$, and

$$G(\phi) = G \circ \mathcal{F}^{-1}(\widehat{\phi}) = \int_{\mathbb{R}^n} g(\omega)\widehat{\phi}(\omega)(1+|\omega|^2)^s\,d\omega.$$

Then $G(\phi) = \langle f,\phi\rangle$ where $f = \left[(1+|\omega|^2)^s g(\omega)\right]^\wedge$. Note that

$$\|f\|^2_{H^{-s}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\widehat{f}(\omega)|(1+|\omega|^2)^{-s}\,d\omega = \int_{\mathbb{R}^n} |g(\omega)|(1+|\omega|^2)^s\,d\omega.$$

Hence $f \in H^{-s}(\mathbb{R}^n)$, and we complete the proof. □

Now we study the smoothness of Sobolev functions. For $k \in \mathbb{N}_0$, we denote

$$C_0^k(\mathbb{R}^n) = \left\{ f \in C^k(\mathbb{R}^n) : \partial^\alpha f \in C_0(\mathbb{R}^n) \text{ for all } |\alpha| \le k \right\}.$$

In the following, if possible, we always identify a function and its derivatives by their continuous version.

Theorem 3.32 (Sobolev Embedding Theorem)**.** *Let $k \in \mathbb{N}_0$ and $s > 0$.*

(i) If $s > k + \frac{n}{2}$, then $\widehat{\partial^\alpha f} \in L^1(\mathbb{R}^n)$ for every $f \in H^s(\mathbb{R}^n)$ and every multi-index $|\alpha| \le k$. Furthermore, there exists a constant C depending only on n and $k - s$ such that

$$\|\widehat{\partial^\alpha f}\|_{L^1(\mathbb{R}^n)} \le C \|f\|_{H^s(\mathbb{R}^n)}, \quad f \in H^s(\mathbb{R}^n).$$

(ii) If $s > k + \frac{n}{2}$, $H^s(\mathbb{R}^n) \subset C_0^k(\mathbb{R}^n)$, and the inclusion map is continuous.

(iii) Conversely, if $H^s(\mathbb{R}^n) \subset C_0^k(\mathbb{R}^n)$, then $s > k + \frac{n}{2}$.

Proof. (i) By the Cauchy-Schwarz inequality, for every $|\alpha| \le k$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|\widehat{\partial^\alpha f}(\omega)\right| d\omega = \int_{\mathbb{R}^n} \left|\omega^\alpha \widehat{f}(\omega)\right| d\omega \le \int_{\mathbb{R}^n} \left|(1+|\omega|^2)^{k/2} \widehat{f}(\omega)\right| d\omega \\
&\quad \le \left[\int_{\mathbb{R}^n} (1+|\omega|^2)^{k-s} d\omega\right]^{1/2} \left[\int_{\mathbb{R}^n} (1+|\omega|^2)^s \left|\widehat{f}(\omega)\right|^2 d\omega\right]^{1/2} = \left[\int_{\mathbb{R}^n} (1+|\omega|^2)^{k-s} d\omega\right]^{1/2} \|f\|_{H^s(\mathbb{R}^n)}.
\end{aligned}$$

Since $k - s < -n/2$, the first factor is finite.

(ii) By the Fourier inversion theorem and the Riemann-Lebesgue lemma, $\partial^\alpha f \in C_0(\mathbb{R}^n)$. We identify f with its C_0 version. We consider the mollification $f^t = f * \phi_t \in C_c^\infty(\mathbb{R}^n)$. Then for every $j = 1, \cdots, n$,

$$\begin{aligned}
\partial_{x_j} f^t(x) = (\partial_{x_j} \phi_t * f)(x) &= \int_{\mathbb{R}^n} \partial_{x_j} \phi_t(x-y) f(y)\, dy = -\int_{\mathbb{R}^n} \partial_{y_j} \phi_t(x-y) f(y)\, dy \\
&= \int_{\mathbb{R}^n} \phi_t(x-y) \partial_{y_j} f(y)\, dy = (\phi_t * \partial_{x_j} f)(x), \quad x \in \mathbb{R}.
\end{aligned}$$

By Proposition 1.5, $f^t \to f$ and $\partial_{x_j} f^t \to \partial_{x_j} f$ on every compact ball $\{x \in \mathbb{R}^n : |x| \le N\}$, $N > 0$. Then for every $x \in \mathbb{R}^n$ and every $|h| > 0$,

$$f(x+he_j) - f(x) = \lim_{t \downarrow 0} \left[f^t(x+he_j) - f^t(x)\right] = \lim_{t \downarrow 0} \left[\int_0^h \partial_{x_j} f^t(x+re_j)\, dr\right] = \int_0^h \partial_{x_j} f(x+re_j)\, dr.$$

By induction on $|\alpha|$, the true derivatives of f exist and equal the distributional derivatives. Furthermore,

$$\|f\|_{C_0^k(\mathbb{R}^n)} = \sum_{|\alpha| \le k} \|\partial^\alpha f\|_\infty \le (2\pi)^{-n/2} \sum_{|\alpha| \le k} \|\widehat{\partial^\alpha f}\|_{L^1(\mathbb{R}^n)} \le C \|f\|_{H^s(\mathbb{R}^n)}.$$

Hence the inclusion map is continuous.

(iii) Assume $H^s(\mathbb{R}^n) \subset C_0^k(\mathbb{R}^n)$. We first show that the inclusion map $\mathrm{Id} : H^s(\mathbb{R}^n) \to C_0^k(\mathbb{R}^n)$ is continuous. By the closed graph theorem, it suffices to show that $\mathrm{Id} : H^s(\mathbb{R}^n) \to C_0^k(\mathbb{R}^n)$ has closed graph. We consider a sequence $(f_j) \subset H^s(\mathbb{R}^n)$ such that $f_j \to f$ in $H^s(\mathbb{R}^n)$ and $f_j \to g$ in $C_0^k(\mathbb{R}^n)$. If $f \ne g$, there would exist $\epsilon, \delta > 0$ and $x_0 \in \mathbb{R}^n$ such that $\inf_{x \in B(x_0,\delta)} |f(x) - g(x)| > \epsilon$. Since $f_j \to g$ in $C_0^k(\mathbb{R}^n)$, one could find $N \in \mathbb{N}$ such that $\inf_{x \in B(x_0,\delta)} |f_j(x) - f(x)| > \epsilon/2$ for all $j \ge N$. Then

$$\int_{\mathbb{R}^n} \left|\widehat{f}_j(\omega) - \widehat{f}(\omega)\right|^2 (1+|\omega|^2)^s\, d\omega \ge \int_{\mathbb{R}^n} \left|\widehat{f}_j(\omega) - \widehat{f}(\omega)\right|^2 d\omega = \|f_j - f\|_{L^2(\mathbb{R}^n)} \ge C_n \delta^n \epsilon^2$$

for all $j > N$. This is a contradiction to the fact $f_j \to f$ in $H^s(\mathbb{R}^n)$.

Consequently, there exists $C > 0$ depending on n, s and k such that $\|f\|_{C_0^k(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}$. Then for every $|\alpha| \leq k$ and every $\phi \in H^s(\mathbb{R}^n) \subset C_0^k(\mathbb{R}^n)$,

$$|\langle \partial^\alpha \delta, f\rangle| = |\langle \delta, \partial^\alpha f\rangle| = |\partial^\alpha f(0)| \leq \|f\|_{C_0^k(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}.$$

Hence $\partial^\alpha \delta \in H^s(\mathbb{R}^n)^* \simeq H^{-s}(\mathbb{R}^n)$ for all $|\alpha| \leq k$. Since the Fourier transform of $\partial^k_{\omega_1}\delta$ is $\omega \mapsto (i\omega_1)^k$,

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|\omega_1|^{2k}}{(1+|\omega|^2)^s}\, d\omega &= \frac{1}{n}\int_{\mathbb{R}^n} \sum_{j=1}^n \frac{|\omega_j|^{2k}}{(1+|\omega|^2)^s}\, d\omega \geq C_1 \int_{\mathbb{R}^n} \frac{|\omega|^{2k}}{(1+|\omega|^2)^s}\, d\omega \\ &= C_2 \int_0^\infty \frac{r^{2k+n-1}}{(1+r^2)^s}\, dr \geq \frac{C_2}{2^s}\int_1^\infty r^{n+2k-2s-1} dr.\end{aligned}$$

where $C_1, C_2 > 0$ are constants depending only on n and k. If $s \leq k + \frac{n}{2}$, the above integral is infinite, and $\partial^k_{\omega_1}\delta \notin H^{-s}(\mathbb{R}^n)$, which is a contradiction. Therefore $s > k + \frac{n}{2}$. $\square$

Following is an immediate corollary of the above theorem.

Corollary 3.33. *If $f \in H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, then $f \in C_0^\infty(\mathbb{R}^n)$.*

Next, we show that multiplication by suitably smooth functions preserves the H^s spaces.

Theorem 3.34. *Suppose that $\psi \in C_0(\mathbb{R}^n)$ and that the Fourier transformation $\widehat{\psi}$ satisfies*

$$\int_{\mathbb{R}^n} (1+|\omega|^2)^{a/2}|\widehat{\psi}(\omega)|\, d\omega < \infty \quad \textit{for some } a > 0.$$

Then the map $M_\psi(f) = \psi f$ is a bounded linear operator on $H^s(\mathbb{R}^n)$ for $|s| \leq a$.

Proof. Since Λ_s is a unitary isomorphism from $H^s(\mathbb{R}^n)$ to $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, it is equivalent to show that $\Lambda_s M_\psi \Lambda_{-s}$ is a bounded linear operator on $L^2(\mathbb{R}^n)$. Note that

$$\begin{aligned}(\Lambda_s M_\psi \Lambda_{-s} f)^\wedge(\omega) &= (1+|\omega|^2)^{s/2}(M_\psi\Lambda_{-s}f)^\wedge(\omega) = (2\pi)^{-n/2}(1+|\omega|^2)^{s/2}\left(\widehat{\psi} * \widehat{\Lambda_{-s}f}\right)(\omega) \\ &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} (1+|\omega|^2)^{s/2}(1+|\xi|^2)^{-s/2}\widehat{\psi}(\omega-\xi)\widehat{f}(\xi)\, d\xi.\end{aligned}$$

Note that for all $\omega, \xi \in \mathbb{R}^n$,

$$1+|\omega|^2 \leq 1 + (|\xi| + |\omega-\xi|)^2 \leq 1 + 2|\xi|^2 + 2|\omega-\xi|^2 \leq 2\left(1+|\xi|^2\right)\left(1+|\omega-\xi|^2\right).$$

Similarly,

$$1+|\xi|^2 \leq 2\left(1+|\omega|^2\right)\left(1+|\xi-\omega|^2\right).$$

The last two estimates imply

$$(1+|\omega|^2)^s(1+|\xi|^2)^{-s} \leq 2^{|s|}\left(1+|\omega-\xi|^2\right)^{|s|} \quad \text{for all } \xi, \omega \in \mathbb{R}^n. \tag{3.8}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}&|(\Lambda_s M_\psi \Lambda_{-s} f)^\wedge(\omega)| \leq C_{n,s}\int_{\mathbb{R}^n} \left(1+|\omega-\xi|^2\right)^{|s|/2}\left|\widehat{\psi}(\omega-\xi)\right|\left|\widehat{f}(\xi)\right| d\xi \\ &\leq C_{n,s}\left[\int_{\mathbb{R}^n} \left(1+|\omega-\xi|^2\right)^{|s|/2}\left|\widehat{\psi}(\omega-\xi)\right| d\xi\right]^{1/2}\left[\int_{\mathbb{R}^n} \left(1+|\omega-\xi|^2\right)^{|s|/2}\left|\widehat{\psi}(\omega-\xi)\right|\left|\widehat{f}(\xi)\right|^2 d\xi\right]^{1/2} \\ &\leq C_{n,s}\left[\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{|s|/2}\left|\widehat{\psi}(\xi)\right| d\xi\right]^{1/2}\left[\int_{\mathbb{R}^n} \left(1+|\omega-\xi|^2\right)^{|s|/2}\left|\widehat{\psi}(\omega-\xi)\right|\left|\widehat{f}(\xi)\right|^2 d\xi\right]^{1/2}.\end{aligned}$$

Using the Plancherel theorem and Tonelli's theorem,

$$\begin{aligned}
\|\Lambda_s M_\psi \Lambda_{-s} f\|_{L^2(\mathbb{R}^n)}^2 &= \|(\Lambda_s M_\psi \Lambda_{-s} f)^\wedge\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \left|(\Lambda_s M_\psi \Lambda_{-s} f)^\wedge(\omega)\right|^2 \, d\omega \\
&\leq C_{n,s}^2 \left[\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{|s|/2} \left|\widehat{\psi}(\xi)\right| d\xi\right] \left[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(1+|\omega-\xi|^2\right)^{|s|/2} \left|\widehat{\psi}(\omega-\xi)\right| \left|\widehat{f}(\xi)\right|^2 d\xi \, d\omega\right] \\
&\leq C_{n,s}^2 \left[\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{|s|/2} \left|\widehat{\psi}(\xi)\right| d\xi\right]^2 \|\widehat{f}\|_{L^2(\mathbb{R}^n)}^2 \leq C_{n,s}^2 \left[\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{a/2} \left|\widehat{\psi}(\xi)\right| d\xi\right]^2 \|f\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned}$$

Therefore, $\Lambda_s M_\psi \Lambda_{-s}$ is a bounded linear operator on $L^2(\mathbb{R}^n)$, and the result follows. $\square$

Corollary 3.35. *Let $\psi \in \mathcal{S}(\mathbb{R}^n)$, the map M_ψ is a continuous linear operator on $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.*

Finally we introduce a compact embedding theorem for Sobolev spaces. For a compact set $K \subset \mathbb{R}^n$, define

$$H_K^s(\mathbb{R}^n) = \{f \in H^s(\mathbb{R}^n) : \operatorname{supp}(f) \subset K\}.$$

Recall that an operator is compact if it maps bounded sets in its domain to precompact sets in its range.

Theorem 3.36 (Rellich's lemma)**.** *Let $t < s$. Then $H_K^s(\mathbb{R}^n)$ is **compactly embedded** in $H^t(\mathbb{R}^n)$, i.e. the inclusion map $\mathrm{Id} : H_K^s(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$ is compact.*

Proof. We need to prove that every bounded sequence of distributions $(f_m) \subset H_K^s(\mathbb{R}^n)$ has a subsequence (f_{m_j}) that converges in $H^t(\mathbb{R}^n)$. By Proposition 3.23, every $\widehat{f_m}$ is a slowly increasing C^∞ function. We take $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\phi = 1$ on a neighborhood of K. Then $f_m = \psi f_m$, and $\widehat{f_m} = \widehat{\psi} * \widehat{f_m} \in C(\mathbb{R}^n)$, where the convolution is pointwise defined. By the estimate (3.8),

$$\begin{aligned}
(1+|\omega|^2)^{s/2}|\widehat{f}_m(\omega)| &= (1+|\omega|^2)^{s/2} \int_{\mathbb{R}^n} \left|\widehat{\psi}(\omega-\xi)\right| \left|\widehat{f}_m(\xi)\right| d\xi \\
&\leq 2^{|s|/2} \int_{\mathbb{R}^n} (1+|\omega-\xi|^2)^{|s|/2} \left|\widehat{\psi}(\omega-\xi)\right| (1+|\xi|^2)^{s/2} \left|\widehat{f}_m(\xi)\right| d\xi \\
&\leq 2^{|s|/2} \|\psi\|_{H^{|s|}(\mathbb{R}^n)} \|f_m\|_{H^s(\mathbb{R}^n)} \leq 2^{|s|/2} \|\psi\|_{H^{|s|}(\mathbb{R}^n)} \sup_{k\in\mathbb{N}} \|f_k\|_{H^s(\mathbb{R}^n)}.
\end{aligned}$$

Similarly, since $\partial_{\omega_j}(\widehat{\phi} * f_m) = (\partial_{\omega_j}\widehat{\phi}) * f_m \in C(\mathbb{R}^n)$, the function $(1+|\omega|)^{s/2}|\partial_{\omega_j}\widehat{f}_m|$ is bounded by a constant depending on s, ψ and $\sup_{k\in\mathbb{N}} \|f_k\|_{H^s(\mathbb{R}^n)}$. Similar to the proof of Theorem 3.32,. In particular, the $\widehat{f}_m$'s and their first derivatives are uniformly bounded on compact sets. By the mean value theorem and the Arzelà-Ascoli theorem, a subsequence $(\widehat{f}_{m_j})$ converges uniformly on compact sets.

Now we show that (f_{m_j}) is Cauchy in $H^t(\mathbb{R}^n)$ for all $t < s$. In fact, for any $R > 0$,

$$\begin{aligned}
\left\|f_{m_j} - f_{m_k}\right\|_{H^t(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} \left|\widehat{f}_{m_j}(\omega) - \widehat{f}_{m_k}(\omega)\right|^2 (1+|\omega|^2)^t \, d\omega \\
&= \int_{|\omega|\leq R} \left|\widehat{f}_{m_j}(\omega) - \widehat{f}_{m_k}(\omega)\right|^2 (1+|\omega|^2)^t \, d\omega + \int_{|\omega|>R} \left|\widehat{f}_{m_j}(\omega) - \widehat{f}_{m_k}(\omega)\right|^2 (1+|\omega|^2)^t \, d\omega \\
&\leq C_n R^n (1+R^2)^{|t|} \sup_{|\omega|\leq R} \left|\widehat{f}_{m_j}(\omega) - \widehat{f}_{m_k}(\omega)\right|^2 + (1+R^2)^{t-s} \int_{|\omega|>R} \left|\widehat{f}_{m_j}(\omega) - \widehat{f}_{m_k}(\omega)\right|^2 (1+|\omega|^2)^s \, d\omega \\
&\leq C_n R^n (1+R^2)^{|t|} \sup_{|\omega|\leq R} \left|\widehat{f}_{m_j}(\omega) - \widehat{f}_{m_k}(\omega)\right|^2 + 2(1+R^2)^{t-s} \sup_{m\in\mathbb{N}} \|f_m\|_{H^s(\mathbb{R}^n)}.
\end{aligned}$$

Given any $\epsilon > 0$, since $t - s < 0$, we can choose $R > 0$ so large that the second term is less than $\epsilon/2$. With $R > 0$ fixed, the first term is less than $\epsilon/2$ for sufficiently large j and k. Then (f_{m_j}) is a Cauchy sequence that converges in $H^t(\mathbb{R}^n)$, and the proof is complete. $\square$

3.6 Application: Elliptic Partial Differential Equations

Before we proceed, we discuss the local smoothness of distributions. If $U \subset \mathbb{R}^n$ is an open set, the *localized Sobolev space* $H^s_{\text{loc}}(U)$ is the set of all distributions $f \in \mathcal{D}'(U)$ such that for every precompact open set $V \Subset U$, there exists $g \in H^s(\mathbb{R}^n)$ such that $g = f$ on V, i.e. $\langle f - g, \phi \rangle =$ for all $\phi \in C_c^\infty(V)$.

Proposition 3.37. *A distribution $f \in \mathcal{D}'(U)$ is in $H^s_{\text{loc}}(U)$ if and only if $\phi f \in H^s(\mathbb{R}^n)$ for every $\phi \in C_c^\infty(U)$.*

Proof. If $f \in H^s_{\text{loc}}(U)$ and $\phi \in C_c^\infty(U)$, we have $f = g$ on a precompact open subset $V \supset \operatorname{supp}(\phi)$, and $\phi f = \phi g \in H^s(\mathbb{R}^n)$ by Corollary 3.35. Conversely, given any precompact open subset $V \Subset U$, we take $\phi \in C_c^\infty(U)$ with $\phi = 1$ on $\overline{V}$ on a neighborhood of V by C^∞-Urysohn lemma. Then $\phi f \in H^s(\mathbb{R}^n)$ and $f = \phi f$ on V, and we complete the proof. □

Remark. We can apply Proposition 3.27 in addition to derive another characterization of Sobolev spaces with nonnegative integer exponent. In particular, for every $k \in \mathbb{N}_0$, a function $f \in L^1_{\text{loc}}(U)$ is in $H^k_{\text{loc}}(U)$ if and only if $\partial^\alpha f \in L^2_{\text{loc}}(U)$ for every $|\alpha| \leq k$.

Next, we consider a constant-coefficient differential operator $L = P(\partial) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$, where P can be viewed as a polynomial of degree no more than m. We assume that m is the true order of L, i.e. that $c_\alpha \neq 0$ for some $|\alpha| = m$. The *principle symbol* P_m of L is the sum of the top-order terms:

$$P_m(\omega) = \sum_{|\alpha|=m} c_\alpha \omega^\alpha.$$

The operator $P(\partial)$ is said to be *elliptic* if $P_m(\omega) \neq 0$ for all nonzero $\omega \in \mathbb{R}^m$. Intuitively, ellipticity means that $P(\partial)$ is genuinely m-th order in all directions. For example, the Laplacian Δ is elliptic on $\mathbb{R}^n$, but the heat operator $\partial_t - \Delta_x$ and the wave operator $\partial_t^2 - \Delta_x$ are not elliptic on $\mathbb{R}^{n+1}$.

Lemma 3.38. *Let $P(\partial)$ be a constant-coefficient differential operator of order m. Then $P(\partial)$ is eliptic if and only if there exists $C, R > 0$ such that $|P(i\omega)| \geq C|\omega|^m$ for all $\omega \in \mathbb{R}^n$ with $|\omega| \geq R$.*

Proof. If $P(\partial)$ is elliptic, the minimum value of the principal symbol on the unit sphere satisfies

$$M_1 = \inf_{\omega \in \mathbb{R}^n : |\omega|=1} |P_m(\omega)| > 0.$$

By homogeneity of P_m of degree m, $|P_m(i\omega)| = |i^m P_m(\omega)| \geq M_1 |\omega|^m$ for all $\omega \in \mathbb{R}^n$. Meanwhile, since $P - P_m$ is of order less than m, there exists M_2 such that $|P(i\omega) - P_m(i\omega)| \leq M_2 |\omega|^{m-1}$. Hence

$$|P(i\omega)| \geq |P_m(i\omega)| + |P(i\omega) - P_m(i\omega)| \geq M_1|\omega|^m - M_2|\omega|^{m-1} \geq \frac{M_1}{2}|\omega|^m, \quad \text{for all } |\omega| \geq \frac{2M_2}{M_1}.$$

Conversely, if P is not elliptic, there exists $\omega_0 \neq 0$ such that $P_m(i\omega_0) = i^m P_m(\omega_0) = 0$, and $|P(i\omega)| \leq C|\omega|^{m-1}$ for every scalar multiple ω of ω_0. □

Lemma 3.39. *Let $L = P(\partial)$ be elliptic of order m. If $u \in H^s(\mathbb{R}^n)$ and $Lu \in H^s(\mathbb{R}^n)$, then $u \in H^{s+m}(\mathbb{R}^n)$.*

Proof. By our hypothesis, $(1 + |\omega|^2)^{s/2} \widehat{u} \in L^2(\mathbb{R}^n)$ and $(1 + |\omega|^2)^{s/2} P(i\omega) \widehat{u} \in L^2(\mathbb{R}^n)$. By Lemma 3.38, for some $R \geq 1$, we have

$$(1 + |\omega|^2)^{m/2} \leq 2^m |\omega|^m \leq C_1 2^m |P(i\omega)| \quad \text{for all } |\omega| \geq R.$$

Therefore

$$\int_{\mathbb{R}^n} (1 + |\omega|^2)^{s+m} |\widehat{u}(\omega)|^2 \, d\omega \leq (1 + R^2)^m \int_{|\omega| \leq R} (1 + |\omega|^2)^s |\widehat{u}(\omega)|^2 \, d\omega + C_2 \int_{|\omega| > R} (1 + |\omega|^2)^s |P(i\omega)\widehat{u}(\omega)|^2 \, d\omega,$$

which is finite. Therefore $u \in H^{s+m}(\mathbb{R}^n)$. □

Regularity of distributional solutions. Let U be an open subset of $\mathbb{R}^n$. Given a distribution solution $u \in \mathcal{D}'(U)$ to partial differential equation

$$Lu = f \quad \text{in } U,$$

we are often interested in the smoothness of u. In fact, if u is smooth enough, the equation $Lu = f$ holds in the classical sense. The regularity theory studies how the smoothness of f affects the property of u.

Theorem 3.40 (Elliptic Regularity)**.** *Suppose that L is a constant-coefficient elliptic differential operator of order m, and U is an open subset of $\mathbb{R}^n$.*

(i) If $Lu \in H^s_{\mathrm{loc}}(U)$ for some $s \in \mathbb{R}$, then $u \in H^{s+m}_{\mathrm{loc}}(U)$.

(ii) If $Lu \in C^\infty(U)$, then $u \in C^\infty(U)$.

Proof. (i) By Proposition 3.37, we need to show that if $Lu \in H^s_{\mathrm{loc}}(U)$ and $\phi \in C_c^\infty(U)$, then $\phi f \in H^{s+m}(\mathbb{R}^n)$. We fix a precompact open set V with $\operatorname{supp}\phi \subset V \subset \overline{V} \subset U$, and choose $\psi \in C_c^\infty(U)$ with $\psi = 1$ on $\overline{V}$ by C^∞-Urysohn lemma. Then $\psi u \in \mathcal{E}'(\mathbb{R}^n)$, and by Proposition 3.23, $\psi u \in H^t(\mathbb{R}^n)$. By decreasing t we may assume that $s + m - t$ equals a positive integer k. We fix $\psi_0 = \psi$, $\psi_k = \phi$, and take recursively $\psi_1, \cdots, \psi_{k-1}$ such that $\psi_j = 1$ on a neighborhood of $\operatorname{supp}(\phi)$ and $\operatorname{supp}(\psi_j) \subset \{\psi_{j-1} = 1\}^\circ$. Hence $\psi_j\psi_{j-1} = \psi_j$ for all j.

We shall prove by induction that $\psi_j u \in H_{t+j}$. It is clear that $\psi_0 u = \psi u \in H^t(\mathbb{R}^n)$. Then we assume that $\psi_{j-1}u \in H^{t+j-1}(\mathbb{R}^n)$, where $1 \le j \le k$. By the product rule, $L(\psi_j u) - \psi_j Lu$ can be written as

$$\begin{aligned} L(\psi_j u) - \psi_j Lu &= L(\psi_j\psi_{j-1}u) - \psi_j L(\psi_{j-1}u) \\ &= \sum_{1\le|\alpha|\le m}\ \sum_{0\le|\beta|\le m-1} c_{\alpha,\beta}\partial^\alpha\psi_j\partial^\beta(\psi_{j-1}u), \qquad \text{where } (c_{\alpha,\beta}) \text{ are constants.} \end{aligned}$$

These terms involve only derivatives of ψ_j, which vanishes on the open set where ψ_j is constant. Meanwhile, since $\psi_{j-1}u \in H^{t+j-1}(\mathbb{R}^n)$, Proposition 3.30 implies that $\partial^\beta(\psi_{j-1}u) \in H^{t+j-m}(\mathbb{R}^n)$ for all $|\beta| \le m-1$. By Corollary 3.35, $L(\psi_j u) - \psi_j Lu \in H^{t+j-m}(\mathbb{R}^n)$, and $\psi_j Lu \in H^s(\mathbb{R}^n)$. Then their summation

$$L(\psi_j u) \in H^{t+j-m}(\mathbb{R}^n) + H^s(\mathbb{R}^n) = H^{t+j-m}(\mathbb{R}^n).$$

Again by Corollary 3.35 $\psi_j u = \psi_j\psi_{j-1}u \in H^{t+j-m}(\mathbb{R}^n)$. By Lemma 3.39, $\psi_j u \in H^{t+j}(\mathbb{R}^n)$. Taking $j = k$, we have $\phi u = \psi_k u \in H^{t+k}(\mathbb{R}^n) = H^{s+m}(\mathbb{R}^n)$. Since $\phi \in C_c^\infty(U)$ is arbitrary, $u \in H^{s+m}_{\mathrm{loc}}(U)$.

(ii) If $Lu \in C^\infty(U)$, we have $\phi Lu \in C_c^\infty(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ for all $\phi \in C_c^\infty(U)$ and all $s \in \mathbb{R}$. By Corollary 3.33 $Lu \in H^s_{\mathrm{loc}}(U)$ for all $s \in \mathbb{R}$. By (i), $u \in H^s_{\mathrm{loc}}(U)$ for all $s \in \mathbb{R}$, and $\phi u \in H^s(\mathbb{R}^n)$ for all $\phi \in C_c^\infty(U)$. For every precompact open $V \subset \overline{V} \subset U$, we may take $\phi \in C_c^\infty(U)$ with $\phi = 1$ on $\overline{V}$ by C^∞-Urysohn lemma to conclude that $u|_V \in C^\infty(V)$. Since V is arbitrary, we have $u \in C^\infty(U)$. $\square$

Following are some applications of the elliptic regularity theorem.

Corollary 3.41 (Weyl's lemma)**.** *Let $U \subset \mathbb{R}^n$ be open. Every distribution solution u of the Laplacian equation*

$$\Delta u = 0 \quad \textit{in } U$$

is in $C^\infty(U)$ and satisfies $\Delta u = 0$ pointwise in U.

4 Positive Definite Functions and Native Spaces

Motivation: Scattered Data Approximation. Let $\Omega \subset \mathbb{R}^n$ be a set containing at least N distinct points. Given data points $(x_1, y_1), \cdots, (x_N, y_N) \in \Omega \times \mathbb{R}$, our goal is to reconstruct a function $f \in C(\Omega)$ such that $y_j = f(x_j)$ for all $j = 1, \cdots, N$. The idea is to fix a finite dimensional subset $\mathcal{V} \subset C(\Omega)$ that allows a sufficiently good approximation of the full space.

Definition 4.1 (Haar Space)**.** Suppose that $\Omega \subset \mathbb{R}^n$ is a set containing at least N points. Let $\mathcal{V} \subset C(\Omega)$ be an N-dimensional vector space of functions. Then $\mathcal{V}$ is called a *Haar space*, if for any $x_1, \cdots, x_N \in \Omega$ and any $y_1, \cdots, y_N \in \mathbb{R}$, there exists *exactly* one function $u \in \mathcal{V}$ such that $y_j = u(x_j)$ for all $j = 1, \cdots, N$.

Proposition 4.2. *Let $\mathcal{V} \subset C(\Omega)$ be an N-dimensional subspace. The following statements are equivalent:*

(i) $\mathcal{V}$ is an N-dimensional Haar space;

(ii) Every $u \in \mathcal{V}\backslash\{0\}$ has at most $N-1$ zeros;

(iii) For any distinct points $x_1, \cdots, x_N \in \Omega$ and any basis $\{b_1, \cdots, b_N\}$ of $\mathcal{V}$,

$$\det \begin{pmatrix} b_1(x_1) & b_1(x_2) & \cdots & b_1(x_N) \\ b_2(x_1) & b_2(x_2) & \cdots & b_2(x_N) \\ \vdots & \vdots & \ddots & \vdots \\ b_N(x_1) & b_N(x_2) & \cdots & b_N(x_N) \end{pmatrix} \neq 0.$$

Proof. (i) $\Rightarrow$ (ii). If $\mathcal{V}$ is an N-dimensional Haar space and $u \in \mathcal{V}\backslash\{0\}$ has N zeros, we fix the N zeros $x_1, \cdots, x_N \in \Omega$. Since both u and the zero function both interpolate zero on these N points, by the uniqueness argument in the definition of Haar space, $u \equiv 0$, which is a contradiction to $u \in \mathcal{V}\backslash\{0\}$.

(ii) $\Rightarrow$ (iii). If there exist distinct points $x_1, \cdots, x_N \in \Omega$ and basis $\{b_1, \cdots, b_N\}$ of $\mathcal{V}$ such that $\det(B) = 0$ with $B = [b_j(x_k)]_{j,k=1}^N$, there exists $\xi \in \mathbb{R}^N\backslash\{0\}$ with $B\xi = 0$, i.e. $\sum_{k=1}^N \xi_k b_k(x_j) = 0$ for each $j = 1, \cdots, N$. Therefore, the function

$$u = \sum_{k=1}^N \xi_k b_k \in \mathcal{V}\backslash\{0\}$$

has N zeros. If (ii) holds, u is identically zeros, which is a contradiction to the fact $\xi \neq 0$.

(iii) $\Rightarrow$ (i). Since $u = \sum_{k=1}^N \xi_k b_k \in \mathcal{V}$, the interpolation condition on data points $\{(x_j, y_j)\}_{j=1}^N$ becomes

$$\begin{cases} y_1 = u(x_1) = \xi_1 b_1(x_1) + \xi_2 b_2(x_1) + \cdots + \xi_N b_N(x_1), \\ y_2 = u(x_2) = \xi_1 b_1(x_2) + \xi_2 b_2(x_2) + \cdots + \xi_N b_N(x_2), \\ \quad \vdots \\ y_N = u(x_N) = \xi_1 b_1(x_N) + \xi_2 b_2(x_N) + \cdots + \xi_N b_N(x_N). \end{cases}$$

If the third property holds, a unique solution $(\xi_1, \cdots, \xi_N) \in \mathbb{R}^N$ exists since $[b_j(x_k)]_{j,k=1}^N$ is nonsingular. $\square$

Example 4.3. We consider the space $\pi_{N-1}(\mathbb{R}) = \text{span}\{1, x, \cdots, x^{N-1}\} \subset C(\mathbb{R})$ of polynomials of degree at most $N-1$. For any distinct points $x_1 < x_2 < \cdots < x_N$, the Vandermonde determinant

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{pmatrix} = \prod_{1 \le j \le k \le N} (x_k - x_j) \neq 0.$$

Then $\pi_{N-1}(\mathbb{R})$ is an N-dimensional Haar space.

We consider the interpolation for multivariate cases. The next result shows that Haar spaces do not exist in higher space dimensions.

Theorem 4.4 (Haar-Mairhuber-Curtis). *Assume that $U \subset \mathbb{R}^n$ is an open set, where $n \geq 2$. Then there exists no Haar space on U of dimension $N \geq 2$.*

Proof. Suppose that $\mathcal{V} = \text{span}\{b_1, \cdots, b_N\} \subset C(U)$ is a Haar space on U. Since U is an open set, there exists a ball $B(x_0, \delta) \subset U$. Next, we choose two continuous curves $x_j : [0,1] \to B(x_0, \delta)$, $j = 1, 2$ such that $x_1(0) = x_2(1)$, $x_2(0) = x_1(1)$. We then fix $x_3, \cdots, x_N \in B(x_0, \delta) \backslash \bigcup_{t \in [0,1]} \{x_1(t), x_2(t)\}$, which is possible since the dimension $n \geq 2$. Then on the one hand, since $\mathcal{V}$ is a Haar space on U, the function

$$D(t) = \det \begin{pmatrix} b_1(x_1(t)) & b_1(x_2(t)) & b_1(x_3) & \cdots & b_1(x_N) \\ b_2(x_1(t)) & b_2(x_2(t)) & b_2(x_3) & \cdots & b_2(x_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_N(x_1(t)) & b_N(x_2(t)) & b_N(x_3) & \cdots & b_N(x_N) \end{pmatrix}, \quad t \in [0,1]$$

is a continuous function that does not change sign. On the other hand, we have $D(1) = -D(0)$ because only the first two rows of the involved matrices are exchanged. Then D must change signs, which is a contradiction. $\square$

4.1 Positive Definite Functions

According to the Haar-Mairhuber-Curtis theorem, in high-dimensional spaces, one cannot interpolate multiple data points using functions from a finite-dimensional subspace. Given a fixed function $\Phi : U \to \mathbb{R}$, one simple way to form an interpolant is

$$u(x) = \sum_{j=1}^{N} \xi_j \Phi(x - x_j), \tag{4.1}$$

where the coefficients $(\xi_j)_{j=1}^N$ are determined by the interpolation conditions $u(x_1) = y_1, \cdots, u(x_N) = y_N$, which implies

$$\begin{cases} \xi_1 \Phi(x_1 - x_1) + \xi_2 \Phi(x_1 - x_2) + \cdots + \xi_N \Phi(x_1 - x_N) = y_1, \\ \xi_1 \Phi(x_2 - x_1) + \xi_2 \Phi(x_2 - x_2) + \cdots + \xi_N \Phi(x_2 - x_N) = y_2, \\ \qquad \vdots \\ \xi_1 \Phi(x_N - x_1) + \xi_2 \Phi(x_N - x_2) + \cdots + \xi_N \Phi(x_N - x_N) = y_N. \end{cases}$$

To interpolate arbitrary data points, we may require an invertible interpolation matrix

$$A_\Phi(x_1, \cdots, x_N) = [\Phi(x_j - x_k)]_{1 \leq j,k \leq N}.$$

For numerical stability, we would be happy to impose more conditions on the interpolation matrix, such as positive definiteness.

Definition 4.5. A continuous function $\Phi : \mathbb{R}^n \to \mathbb{C}$ is said to be *positive semidefinite*, if for all $N \in \mathbb{N}$, all pairwise distinct data points $x_1, \cdots, x_N \in \mathbb{R}^n$ and all $\xi_1, \cdots, \xi_N \in \mathbb{C}$, the quadratic form

$$\sum_{j,k=1}^{N} \xi_j \bar{\xi}_k \Phi(x_j - x_k)$$

is nonnegative. The function Φ is called *positive definite*, if the quadratic form is positive for all $\xi \in \mathbb{C}^N \backslash \{0\}$.

In fact, we call a function positive semidefinite if the associated interpolation matrices are positive definite, and positive semidefinite if the associated interpolation matrices are positive semidefinite.

Proposition 4.6. *Suppose that $\Phi : \mathbb{R}^n \to \mathbb{C}$ is a positive semidefinite function. Then all properties hold:*

(i) $\Phi(0) \geq 0$.

(ii) $\Phi(-x) = \overline{\Phi(x)}$ *for each* $x \in \mathbb{R}^n$.

(iii) Φ *is bounded, and* $|\Phi(x)| \leq \Phi(0)$ *for all* $x \in \mathbb{R}^n$. *Consequently,* $\Phi \equiv 0$ *if and only if* $\Phi(0) = 0$.

Proof. (i) We take $N = 1$ and $\xi_1 = 1$ in the definition.

(ii) We take $N = 2$, $\xi = (1, \lambda)$, $x_1 = 0$ and $x_2 = x$ to get

$$(1 + |\lambda|^2)\Phi(0) + \lambda\Phi(x) + \overline{\lambda}\Phi(-x) \geq 0.$$

We set $\lambda = 1$ and $\lambda = i$, respectively, to obtain $\Phi(x) + \Phi(-x) + 2\Phi(0) \geq 0$ and $2\Phi(0) + i(\Phi(x) - \Phi(-x)) \geq 0$, which implies $\mathrm{Im}(\Phi(x) + \Phi(-x)) = 0$ and $\mathrm{Re}(\Phi(x) - \Phi(-x)) = 0$. Hence $\Phi(-x) = \overline{\Phi(x)}$.

(iii) As in (ii), the matrix

$$\begin{bmatrix} \Phi(0) & \Phi(x) \\ \Phi(-x) & \Phi(0) \end{bmatrix} \succeq 0,$$

and the determinant $\Phi(0)^2 - \Phi(x) \cdot \Phi(-x) = \Phi(0)^2 - |\Phi(x)|^2 \geq 0$. Hence $|\Phi(x)| \leq \Phi(0)$. $\square$

Remark. A real-valued function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is positive semidefinite if and only if Φ is even, and for all $N \in \mathbb{N}$, all pairwise distinct data points $x_1, \cdots, x_N \in \mathbb{R}^n$ and all $\xi_1, \cdots, \xi_N \in \mathbb{R}$. the quadratic form

$$\sum_{j,k=1}^{N} \xi_j \xi_k \Phi(x_j - x_k) \geq 0.$$

Proposition 4.7. *Supose that* $\Phi, \Psi : \mathbb{R}^n \to \mathbb{C}$ *are two positive semidefinite functions.*

(i) For all $\lambda_1, \lambda \geq 0$, *the function* $\lambda_1\Phi + \lambda_2\Psi$ *is positive semidefinite. In addition, if one of* Φ, Ψ *is positive definite and the corresponding coefficient is positive, then* $\lambda_1\Phi + \lambda_2\Psi$ *is positive definite.*

(ii) (Schur product theorem) The pointwise product $\Phi \cdot \Psi$ *is positive semidefinite. In addition, if* Φ *and* Ψ *are positive definite, so is* $\Phi \cdot \Psi$.

Proof. (i) It suffices to verify the definition.

(ii) We fix $x_1, x_2, \cdots, x_N \in \mathbb{R}^n$ and $\xi_1, \xi_2, \cdots, \xi_N \in \mathbb{C}$. Let $A = (A_{jk})_{1 \leq j,k \leq N} = (\Phi(x_j - x_k))_{1 \leq j,k \leq N}$, and $B = (B_{jk})_{1 \leq j,k \leq N} = (\Psi(x_j - x_k))_{1 \leq j,k \leq N}$, which are both positive semidefinite. It suffices to show that the Hadamard product $A \circ B$ is also positive semidefinite.

Since A and B are both positive semidefinite, we consider the eigendecompositions $A = \sum_{j=1}^{N} \lambda_j \upsilon_j \upsilon_j^*$ and $B = \sum_{k=1}^{N} \mu_k \zeta_k \zeta_k^*$, where $\lambda_j, \mu_k \geq 0$ are eigenvalues. Then the Hadamard product

$$A \circ B = \left(\sum_{j=1}^{N} \lambda_j \upsilon_j \upsilon_j^* \right) \circ \left(\sum_{k=1}^{N} \mu_k \zeta_k \zeta_k^* \right) = \sum_{j,k=1}^{N} \lambda_j \mu_k (\upsilon_j \circ \zeta_k)(\upsilon_j \circ \zeta_k)^*.$$

For each $\xi \in \mathbb{C}^N$, we have

$$\xi^* \left(A \circ B \right) \xi = \sum_{j,k=1}^{N} \lambda_j \mu_k \xi^* (\upsilon_j \circ \zeta_k) \left(\upsilon_j \circ \zeta_k \right)^* \xi = \sum_{j,k=1}^{N} \lambda_j \mu_k \left(\sum_{m=1}^{N} \xi_m \upsilon_{j,m} \zeta_{k,m} \right)^2 \geq 0. \tag{4.2}$$

Hence $A \circ B$ is positive semidefinite. Moreover, if A and B are positive definite, then $\lambda_j, \mu_k > 0$ for all j, k. If $\xi \neq 0$, there exists j such that $\xi^* \upsilon_j \neq 0$, and $\xi \circ \upsilon_j \neq 0$. Likewise there exists k such that $(\xi \circ \upsilon_j)^* \zeta_k \neq 0$. Therefore the quadratic form (4.2) is positive, and $A \circ B$ is positive definite. $\square$

4.2 Bochner's Theorem

Motivation. If a function $\Phi \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ has an integrable Fourier transform $\widehat{\Phi} \in L^1(\mathbb{R}^n)$, by the Fourier inversion theorem, one can recover Φ from the Fourier transform:

$$\Phi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{\Phi}(\omega) e^{i\omega \cdot x} \, d\omega, \quad x \in \mathbb{R}^n.$$

Then the quadratic form involving Φ is

$$\begin{aligned} \sum_{j,k=1}^{N} \xi_j \bar{\xi}_k \Phi(x_j - x_k) &= (2\pi)^{-n/2} \sum_{j,k=1}^{N} \xi_j \bar{\xi}_k \int_{\mathbb{R}^n} \widehat{\Phi}(\omega) e^{i\omega \cdot (x_j - x_k)} \, d\omega \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{\Phi}(\omega) \left| \sum_{j=1}^{N} \xi_j e^{i\omega \cdot x_j} \right|^2 d\omega. \end{aligned}$$

Consequently, if the Fourier transform $\widehat{\Phi}$ is nonnegative and nonnegative, then Φ is positive semidefinite. For a complete characterization, we need to replace the measure with Lebesgue density $\widehat{\Phi}(\omega)$ by a more general Borel measure μ on $\mathbb{R}^n$.

Proposition 4.8. *A continuous function $\Phi : \mathbb{R}^n \to \mathbb{C}$ is positive semidefinite if and only if Φ is bounded and satisfies*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(x - y) \gamma(x) \overline{\gamma(y)} \, dx \, dy \geq 0 \tag{4.3}$$

for all test functions $\gamma \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Suppose that Φ is a positive semidefinite function. Then Φ is bounded, and for every $\gamma \in \mathcal{S}$, the integral (4.3) is well-defined. Moreover, for any $\epsilon > 0$, there exists a cube $Q \subset \mathbb{R}^n$ such that

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(x - y) \gamma(x) \overline{\gamma(y)} \, dx \, dy - \int_Q \int_Q \Phi(x - y) \gamma(x) \overline{\gamma(y)} \, dx \, dy \right| < \frac{\epsilon}{2}.$$

Since $(x, y) \mapsto \Phi(x - y) \gamma(x) \overline{\gamma(y)}$ is continuous, the double integral over $Q \times Q$ is the limit of Riemann sums, and there exist points $x_1, \cdots, x_N \in \mathbb{R}^n$ and weights $w_1, \cdots, w_N > 0$ such that

$$\left| \int_Q \int_Q \Phi(x - y) \gamma(x) \overline{\gamma(y)} \, dx \, dy - \sum_{j=1}^{N} \Phi(x_j - x_k) \gamma(x_j) w_j \overline{\gamma(x_k)} w_k \right| < \frac{\epsilon}{2}.$$

The last two estimates imply that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(x - y) \gamma(x) \overline{\gamma(y)} \, dx \, dy > \sum_{j=1}^{N} \Phi(x_j - x_k) w_j \gamma(x_j) \overline{w_k} \overline{\gamma(x_k)} - \epsilon.$$

Since Φ is positive semidefinite, we let $\epsilon \downarrow 0$ to conclude that the integral (4.3) is nonnegative.

Lemma 4.9. *For each $m > 0$, define $\varphi_m(x) = (2m)^{n/2} e^{-m|x|^2}$. The following statements hold:*

(i) $\widehat{\varphi}_m(\omega) = e^{-\frac{|\omega|^2}{4m}}$;

(ii) $\widehat{\widehat{\varphi}}_m = \varphi_m$;

(iii) If $\Phi \in C(\mathbb{R}^n)$ is slowly increasing, we have

$$\Phi(x) = \lim_{m \uparrow \infty} (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Phi(y) \varphi_m(y - x) \, dy, \quad x \in \mathbb{R}^n.$$

Proof. (i) By Lemma 2.9, the Fourier transform of $\varphi = \varphi_{1/2}$ is itself. By Proposition 2.6 (iv),

$$\widehat{\varphi}_m(\omega) = (2m)^{n/2}(\varphi \circ \sqrt{2m}\,\mathrm{Id})^\wedge(\omega) = \widehat{\varphi}\left(\frac{\omega}{\sqrt{2m}}\right) = e^{-\frac{|\omega|^2}{4m}}.$$

(ii) Using (i) twice:

$$\widehat{\widehat{\varphi}}_m = \left(e^{-\frac{|\omega|^2}{4m}}\right)^\wedge = (2m)^{n/2}\widehat{\varphi}_{\frac{1}{4m}} = (2m)^{n/2}e^{-m|x|^2} = \varphi_m.$$

(iii) We first consider the case $x = 0$. Note that $\int_{\mathbb{R}^n} \varphi_m(y)\,dy = (2\pi)^{n/2}$. By (i),

$$(2\pi)^{-n/2}\int_{\mathbb{R}^n} \Phi(y)\varphi_m(y)\,dy - \Phi(0) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} [\Phi(y) - \Phi(0)]\,\varphi_m(y)\,dy.$$

For any $\epsilon > 0$, there exists $\delta > 0$ such that $|\Phi(y) - \Phi(0)| < \epsilon/2$ for all $|y| \le \delta$. Since Φ is slowly increasing, there exists $p \in \mathbb{N}_0$ and $C_p > 0$ such that $|\Phi(y)| \le C_p(1+|y|)^p$ for all $y \in \mathbb{R}^n$. Then

$$\begin{aligned}\left|\int_{\mathbb{R}^n} [\Phi(y) - \Phi(0)]\,\varphi_m(y)\,dy\right| &\le \int_{|y|\le\delta} |\Phi(y) - \Phi(0)|\,\varphi_m(y)\,dy + C_p\int_{|y|>\delta} \varphi_m(y)\,(1+|y|)^p\,dy \\ &\le \frac{\epsilon}{2}\int_{|y|\le\delta} \varphi_m(y)\,dy + C_{p,\delta}\int_{|y|>\delta} |y|^p\varphi_m(y)\,dy \\ &\le \frac{(2\pi)^{n/2}\epsilon}{2} + C_{p,\delta}\int_{|y|>\sqrt{2m}\delta} \left|\frac{y}{\sqrt{2m}}\right|^p e^{-|y|^2/2}\,dy \\ &\le \frac{(2\pi)^{n/2}\epsilon}{2} + (2m)^{-p/2}C_{p,\delta}\int_{|y|>\sqrt{2m}\delta} |y|^p\,e^{-|y|^2/2}\,dy < (2\pi)^{n/2}\epsilon\end{aligned}$$

for large enough m. The case $x \neq 0$ follows by replacing Φ with $\Phi(\cdot + x)$ in the previous case. $\square$

Now we continue the proof of Proposition 4.8. Conversely, if Φ is bounded and (4.3) holds for all $\gamma \in \mathcal{S}(\mathbb{R}^n)$, we can rewrite the double integral as

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(x-y)\gamma(x)\overline{\gamma(y)}\,dx\,dy &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(x)\gamma(x+y)\overline{\gamma(y)}\,dx\,dy = \int_{\mathbb{R}^n} \Phi(x)\left[\int_{\mathbb{R}^n} \gamma(x+y)\overline{\gamma(y)}\,dy\right]dx \\ &= \int_{\mathbb{R}^n} \Phi(x)\left[\int_{\mathbb{R}^n} \gamma(x-y)\overline{\gamma(-y)}\,dy\right]dx = \int_{\mathbb{R}^n} \Phi(x)(\gamma * \widetilde{\gamma})(x)\,dx,\end{aligned}$$

where we use the notation $\widetilde{\gamma}(x) = \overline{\gamma(-x)}$. Next, given $x_1, \cdots, x_N \in \mathbb{R}^n$ and $\xi_1, \cdots, \xi_N \in \mathbb{C}$, we take

$$\gamma(x) = \gamma_m(x) = (2\pi)^{-n/2}\sum_{j=1}^N \xi_j\varphi_{2m}(x - x_j), \quad x \in \mathbb{R}^n,$$

and

$$\widehat{\gamma}_m(\omega) = (2\pi)^{-n/2}\sum_{j=1}^N \xi_j e^{-i\omega\cdot x_j} e^{-\frac{|\omega|^2}{8m}}, \quad \omega \in \mathbb{R}^n.$$

By the convolution formula [Theorem 2.8],

$$\begin{aligned}(\gamma_m * \widetilde{\gamma}_m)^\wedge(\omega) &= (2\pi)^{n/2}|\widehat{\gamma}_m|^2(\omega) = (2\pi)^{-n/2}\left|\sum_{j=1}^N \xi_j e^{-i\omega\cdot x_j}\right|^2 e^{-\frac{|\omega|^2}{4m}} \\ &= (2\pi)^{-n/2}\sum_{j,k=1}^N \xi_j\overline{\xi}_k e^{-i\omega\cdot(x_j - x_k)}\widehat{\varphi}_m(\omega) = (2\pi)^{-n/2}\left(\sum_{j,k=1}^N \xi_j\overline{\xi}_k\tau_{x_j - x_k}\varphi_m\right)^\wedge(\omega).\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{j,k=1}^{N} \xi_j\overline{\xi}_k\Phi(x_j-x_k) &= (2\pi)^{-n/2}\sum_{j,k=1}^{N}\xi_j\overline{\xi}_k \lim_{m\uparrow\infty}\int_{\mathbb{R}^n}\Phi(y)\varphi_m(y-(x_j-x_k))\,dy \\ &= \lim_{m\uparrow\infty}\int_{\mathbb{R}^n}\Phi(y)(\gamma_m*\widetilde{\gamma}_m)(y)\,dy = \lim_{m\uparrow\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Phi(x-y)\gamma_m(x)\overline{\gamma_m(y)}\,dx\,dy\geq 0.\end{aligned}$$

Thus we complete the proof of Proposition 4.8. $\square$

We have the following characterization of positive semidefinite functions.

Theorem 4.10 (Bochner's Theorem)**.** *A continuous function $\Phi:\mathbb{R}^n\to\mathbb{C}$ is positive semidefinite if and only if it is the Fourier-Stieltjes transform of a finite nonnegative Borel measure μ on $\mathbb{R}^n$, i.e.*

$$\Phi(x)=\widehat{\mu}(x)=(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{-i\omega\cdot x}\,d\mu(\omega),\quad x\in\mathbb{R}^n. \tag{4.4}$$

Proof. Step I. Given a finite nonnegative Borel measure μ on $\mathbb{R}^n$, let Φ be defined as in (4.4). Since Φ is the Fourier transform of a finite measure, it is continuous. For all $\xi\in\mathbb{C}^N$ and $x_1,\cdots,x_N\in\mathbb{R}^n$,

$$\sum_{j,k=1}^{N}\xi_j\overline{\xi}_k\Phi(x_j-x_k)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\sum_{j,k=1}^{N}\xi_j\overline{\xi}_k e^{i\omega\cdot(x_k-x_j)}\,d\mu(\omega)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\left|\sum_{j=1}^{N}\xi_j e^{-i\omega\cdot x_j}\right|^2 d\mu(\omega)\geq 0.$$

Step II. Let $\psi\in\mathcal{S}(\mathbb{R}^n)$ with $f=\psi^\vee\in\mathcal{S}(\mathbb{R}^n)$. By the convolution formula [Theorem 2.8],

$$\begin{aligned}0&\leq(2\pi)^{-n/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Phi(x-y)f(x)\overline{f(y)}\,dx\,dy=(2\pi)^{-n/2}\int_{\mathbb{R}^n}\Phi(x)(f*\widetilde{f})(x)\,dx\\ &=\int_{\mathbb{R}^n}\Phi(x)(|\widehat{f}|^2)^\vee(x)\,dx=\int_{\mathbb{R}^n}\Phi(x)(|\psi|^2)^\vee(x)\,dx.\end{aligned}$$

Next, for any nonnegative $g\in C_c^\infty(\mathbb{R}^n)$, we define $\psi_\epsilon=\sqrt{g+\epsilon^2\varphi}$, where $\varphi(x)=e^{-|x|^2/2}$. Then $\psi_\epsilon\in\mathcal{S}(\mathbb{R}^n)$ since it coincides $\epsilon\varphi(x/2)$ when evaluated at $x\in\mathbb{R}^n$ with sufficiently large $|x|$. By the above conclusion,

$$\int_{\mathbb{R}^n}\Phi(x)g^\vee(x)\,dx+\epsilon^2\int_{\mathbb{R}^n}\Phi(x)\varphi(x)\,dx=\int_{\mathbb{R}^n}\Phi(x)\left(|\psi_\epsilon|^2\right)^\vee(x)\,dx\geq 0.$$

Since Φ is bounded and $\varphi=\widehat{\varphi}$ is integrable, we let $\epsilon\downarrow 0$ and conclude that

$$Tg:=\int_{\mathbb{R}^n}\Phi(x)g^\vee(x)\,dx\geq 0,$$

which forms a positive linear functional $T:C_c^\infty(\mathbb{R}^n)\to\mathbb{C}$, i.e. it takes nonnegative functions to nonnegative (real) values. This extends uniquely to a positive linear functional $T:C_c(\mathbb{R}^n)\to\mathbb{C}$ by mollification. By Riesz representation theorem, there exists a nonnegative Borel measure μ on $\mathbb{R}^n$ such that

$$Tg=\int_{\mathbb{R}^n}g(x)\,d\mu(x),\quad g\in C_c(\mathbb{R}^n).$$

Step III. We fix a nonnegative function $g\in L^1(\mathbb{R}^n)\cap C(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}g(x)\,dx=1$ and $\widehat{g}\in C_c(\mathbb{R}^n)$ is nonnegative. To achieve this, we can take a nonnegative bump $\phi\in C_c^\infty(\mathbb{R}^n)$ and define $g=(\phi*\widetilde{\phi})^\vee$ with appropriate normalization. Let $g_m(x)=m^n g(mx)$, where $m>0$. Then $\widehat{g}_m(x)=\widehat{g}(x/m)$, and

$$\int_{\mathbb{R}^n}\Phi(x)g_m(x)\,dx=T\,\widehat{g}_m=\int_{\mathbb{R}^n}\widehat{g}_m(x)\,d\mu(x)=\int_{\mathbb{R}^n}\widehat{g}\left(\frac{x}{m}\right)\,d\mu(x).$$

Note that $\widehat{g}(x/m) \to (2\pi)^{-n/2}$ as $m \uparrow \infty$, by Fatou's lemma,

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} d\mu(x) \le \liminf_{m\to\infty} \int_{\mathbb{R}^n} \widehat{g}\left(\frac{x}{m}\right) d\mu(x) = \liminf_{m\to\infty} \int_{\mathbb{R}^n} \Phi(x) g_m(x)\, dx = \Phi(0).$$

Hence the total mass of μ is bounded by $(2\pi)^{n/2}\Phi(0)$. Finally, by Proposition 1.6,

$$\Phi(x) = \lim_{m\uparrow\infty} \int_{\mathbb{R}^n} \Phi(y) g_m(x-y)\, dy = \lim_{m\uparrow\infty} \int_{\mathbb{R}^n} \widehat{g}_m(-\omega) e^{-i\omega\cdot x}\, d\mu(\omega) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\omega\cdot x}\, d\mu(\omega),$$

where the last inequality follows from the dominated convergence theorem and the finiteness of μ. $\square$

Recalling the interpolation problem, we are more interested in positive definite than in positive semi-definite functions. Now we discuss some sufficient conditions for a function to be positive definite.

Theorem 4.11. *A positive semidefinite function $\Phi : \mathbb{R}^n \to \mathbb{C}$ is positive definite if the support of the Borel measure μ in the representation* (4.4) *contains an open subset.*

Proof. We denote by U the open set contained in the support of μ. Then $\mu(U) > 0$. According to Step I in the proof of Theorem 4.10, for any $x_1, \cdots, x_N \in \mathbb{R}^n$, if $\xi \in \mathbb{C}^N$ satisfies $\sum_{j,k=1}^N \xi_j \overline{\xi}_k \Phi(x_j - x_k) = 0$, we have

$$\sum_{j=1}^N \xi_j e^{-i\omega\cdot x_j} = 0 \quad \text{for all } \omega = (\omega_1, \cdots, \omega_n) \in U.$$

Applying the analytic continuation in each coordinate of ω, we derive that $\sum_{j=1}^N \xi_j e^{-i\omega\cdot x_j} = 0$ for all $\omega \in \mathbb{R}^n$. Then for any test function $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$0 = \sum_{j=1}^N \xi_j e^{-i\omega\cdot x_j} \widehat{\psi}(\omega) = \left(\sum_{j=1}^N \xi_j \tau_{x_j} \psi\right)^{\wedge} (\omega) \quad \text{for all } \omega \in \mathbb{R}^n.$$

Then $\sum_{j=1}^N \xi_j \psi(x - x_j) = 0$ for all $x \in \mathbb{R}^n$. We take $\psi \in C_c^\infty(\mathbb{R}^n)$ with support contained in $B(0, \epsilon/2)$, where $\epsilon = \min_{j\ne k} |x_j - x_k|$, and $f(0) = 1$. This implies

$$0 = \sum_{j=1}^N \xi_j \psi(x_k - x_j) = \xi_k \psi(0) = \xi_k \quad \text{for all } k = 1, \cdots, N.$$

Hence $\xi = 0$, and Φ is positive definite. $\square$

We can use Bochner's characterization to construct positive definite functions by choosing the Borel measure μ. This is even simpler if μ has a Lebesgue density $\rho(x)\, dx$.

Corollary 4.12. *Let $\rho \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. If Φ is nonnegative and not identically zero, then*

$$\Phi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \rho(\omega) e^{-i\omega\cdot x}\, d\omega, \quad x \in \mathbb{R}^n$$

defines a positive definite function.

Proof. We define a Borel measure μ by

$$\mu(A) = \int_A \rho(x)\, dx, \quad A \in \mathscr{B}(\mathbb{R}^n).$$

Then the support of μ equals the support of ρ. Since ρ is continuous and not identically zero, its support contains an open set, and hence the Fourier transform of ρ is positive definite by the Theorem 4.11. $\square$

Example 4.13 (Gaussian radial basis function)**.** For each $m > 0$, the function

$$\varphi_m(x) = (2m)^{n/2} e^{-m|x|^2}, \quad x \in \mathbb{R}^n$$

is positive definite on $\mathbb{R}^n$.

Proof. By Lemma 4.9,

$$\varphi_m(x) = \widehat{\widehat{\varphi}}_m(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{\varphi}_m(\omega) e^{-i\omega\cdot x}\, d\omega = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{|\omega|^2}{4m}} e^{-i\omega\cdot x}\, d\omega, \quad x \in \mathbb{R}^n.$$

Then φ_m is a positive definite function by Corollary 4.12. ☐

The Gaussian example gives rise to a criterion for verifying positive definiteness.

Theorem 4.14. *Let $\Phi \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Then Φ is positive definite if and only if Φ is bounded and the Fourier transform $\widehat{\Phi}$ is nonnegative and not identically zero.*

Proof. Step I. We assume that Φ is bounded and has nonnegative and nonvanishing Fourier transform. Then by the monotone convergence theorem and Lemma 4.9,

$$\begin{aligned}
\int_{\mathbb{R}^n} \widehat{\Phi}(\omega)\, d\omega &= \lim_{m\uparrow\infty} \int_{\mathbb{R}^n} \widehat{\Phi}(\omega)\widehat{\varphi}_m(\omega)\, d\omega \\
&= (2\pi)^{-n/2} \lim_{m\uparrow\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(x)\widehat{\varphi}_m(\omega) e^{-i\omega\cdot x}\, dx\, d\omega \\
&= \lim_{m\uparrow\infty} \int_{\mathbb{R}^n} \Phi(\omega)\widehat{\widehat{\varphi}}_m(\omega)\, d\omega \\
&= \lim_{m\uparrow\infty} \int_{\mathbb{R}^n} \Phi(\omega)\varphi_m(\omega)\, d\omega = (2\pi)^{n/2}\Phi(0).
\end{aligned}$$

Then $\widehat{\Phi} \in L^1(\mathbb{R}^n)$, and by the Fourier inversion formula,

$$\Phi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{\Phi}(\omega) e^{i\omega\cdot x}\, d\omega = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{\Phi}(-\omega) e^{-i\omega\cdot x}\, d\omega.$$

By Corollary 4.12, Φ is positive definite.

Step II. Conversely, if $\Phi \in L^1(\mathbb{R}^n)$ is a positive definite function, Φ is bounded, and by Theorem 4.10, Φ is the Fourier transform of a nonnegative Borel measure μ on $\mathbb{R}^n$,

$$\begin{aligned}
\widehat{\Phi}(\omega) &= \lim_{m\uparrow\infty} (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{\Phi}(\xi)\varphi_m(\xi - \omega)\, d\xi \\
&= \lim_{m\uparrow\infty} (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Phi(y)\widehat{\varphi}_m(y) e^{-i\omega\cdot y}\, dy \\
&= \lim_{m\uparrow\infty} (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} e^{-i\xi\cdot y} d\mu(\xi)\right] \widehat{\varphi}_m(y) e^{-i\omega\cdot y}\, dy \\
&= \lim_{m\uparrow\infty} (2\pi)^{-n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \widehat{\varphi}_m(y) e^{-i(\xi+\omega)\cdot y}\, dy\, d\mu(\xi) \\
&= \lim_{m\uparrow\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi_m(\xi + \omega)\, d\mu(\xi) \geq 0.
\end{aligned}$$

Then the Fourier transform $\widehat{\Phi}$ is nonnegative. We proceed as in Step I to conclude $\|\widehat{\Phi}\|_{L^1} = (2\pi)^{n/2}\Phi(0) > 0$. Therefore $\widehat{\Phi}$ does not vanish identically. ☐

Remark. As is shown in our proof, if $\Phi \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is a positive definite function, the nonnegative Fourier transform $\widehat{\Phi} \in L^1(\mathbb{R}^n)$.

Following are two applications of this criterion.

Corollary 4.15. *Let $\phi_1, \cdots, \phi_n \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ be positive definite functions on $\mathbb{R}$. Then the product*

$$\Phi(x) = \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n), \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n$$

also defines a positive definite function on $\mathbb{R}^n$.

Proof. Since the univariate functions $\phi_1, \cdots, \phi_n \in C(\mathbb{R}) \cap L^1(\mathbb{R})$, the function $\Phi \in C_c(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Moreover, its n-variate Fourier transform is

$$\widehat{\Phi}(\omega) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Phi(x) e^{-i\omega\cdot x}\, dx = \prod_{j=1}^{n} (2\pi)^{-1/2} \int_{\mathbb{R}} \phi_j(x) e^{-i\omega_j x_j}\, dx_j = \prod_{j=1}^{n} \widehat{\phi}_j(\omega_j).$$

By Theorem 4.14, the univariate Fourier transforms $\widehat{\phi}_1, \cdots, \widehat{\phi}_n$ are nonnegative and not identically zero, and the n-variate Fourier transform $\widehat{\Phi} = \widehat{\phi}_1\widehat{\phi}_2 \cdots \widehat{\phi}_n$ also possesses these properties. Hence Φ is positive definite. $\square$

Proposition 4.16. *Let $n \in \mathbb{N}$ and $s > n/2$. Then the function*

$$\Phi(x) = (1 + |x|^2)^{-s}, \quad x \in \mathbb{R}^n$$

is positive definite, and its Fourier transform is

$$\widehat{\Phi}(\omega) = \frac{2^{1-s}}{\Gamma(s)} |\omega|^{s-\frac{n}{2}} K_{\frac{n}{2}-s}(|\omega|), \quad \omega \in \mathbb{R}^n,$$

where K_ν is the modified Bessel function of the second kind:

$$K_\nu(r) = \int_0^\infty e^{-r\cosh t} \cosh(\nu t)\, dt, \quad \nu \in \mathbb{C},\ r > 0.$$

Proof. By the definition of $\Gamma(s)$,

$$\Phi(x) = \frac{(1+|x|^2)^{-s}}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t}\, dt = \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} e^{-(1+|x|^2)u}\, du,$$

where we substitute $t = (1 + |x|^2)u$ in the last expression. Since $s > n/2$, the function $\Phi \in L^1(\mathbb{R}^n)$, and

$$\begin{aligned}
\widehat{\Phi}(\omega) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Phi(x) e^{-i\omega\cdot x}\, dx \\
&= \frac{(2\pi)^{-n/2}}{\Gamma(s)} \int_{\mathbb{R}^n} \int_0^\infty u^{s-1} e^{-(1+|x|^2)u} e^{-i\omega\cdot x}\, du\, dx \\
&= \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} e^{-u} \left[(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-u|x|^2} e^{-i\omega\cdot x}\, dx \right] du \\
&= \frac{2^{-n/2}}{\Gamma(s)} \int_0^\infty u^{s-\frac{n}{2}-1} e^{-u} e^{-\frac{|\omega|^2}{4u}}\, du,
\end{aligned}$$

where we use Lemma 4.9 in the last expression. On the other hand, for every $a > 0$,

$$\begin{aligned}
K_\nu(r) &= \int_0^\infty e^{-r\cosh t} \cosh(\nu t)\, dt = \frac{1}{2} \int_{-\infty}^\infty e^{-r\cosh t} e^{\nu t}\, dt \\
&= \frac{1}{2} \int_{-\infty}^\infty e^{-\frac{r}{2}(e^t + e^{-t})} e^{\nu t}\, dt = \frac{1}{2a^\nu} \int_0^\infty e^{-\frac{r}{2}\left(\frac{u}{a} + \frac{a}{u}\right)} u^{\nu-1}\, du. \qquad \text{(change the variable } u = ae^t\text{)}
\end{aligned}$$

Setting $\nu = s - \frac{n}{2}$, $r = |\omega|$ and $a = |\omega|/2$, we have

$$K_{\frac{n}{2}-s}(|\omega|) = \frac{2^{s-\frac{n}{2}-1}}{|\omega|^{s-\frac{n}{2}}} \int_0^\infty u^{s-\frac{n}{2}-1} e^{-u} e^{-\frac{|\omega|^2}{4u}} \, du = \frac{2^{s-1}\Gamma(s)}{|\omega|^{s-\frac{n}{2}}} \widehat{\Phi}(\omega), \quad |\omega| > 0.$$

The continuity shows that $\widehat{\Phi}(0) = 2^{-n/2}\Gamma(s - \frac{n}{2})/\Gamma(s)$. Thus we complete the proof. □

Finally, we consider the differentiability of positive definite functions.

Theorem 4.17. *If a positive definite function $\Phi : \mathbb{R}^n \to \mathbb{C}$ is C^{2k} in some neighborhood of the origin, then $\Phi \in C^{2k}(\mathbb{R}^n)$, and*

$$\partial^\alpha \Phi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} (-i\omega)^\alpha e^{-i\omega\cdot x} \, d\mu(\omega), \quad x \in \mathbb{R},$$

where μ is a nonnegative Borel measure μ on $\mathbb{R}^n$ satisfying (4.4).

Proof. For every $\gamma \in \mathcal{S}(\mathbb{R}^n)$, by (4.4),

$$\int_{\mathbb{R}^n} \Phi(x)\gamma(x) \, dx = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \gamma(x) e^{-i\omega\cdot x} \, dx \, d\mu(\omega) = \int_{\mathbb{R}^n} \widehat{\gamma}(\omega) \, d\mu(\omega).$$

Next, we take a nonnegative function $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x) \, dx = 1$ and support in $B(0,1) \subset \mathbb{R}^n$, and let $\phi_m(x) = m^n \phi(mx)$ for each $m > 0$. We take m so large that Φ is C^{2k} in the neighborhood $B(0, m^{-1})$, which is contained in the support of ϕ_m. Then

$$\int_{\mathbb{R}^n} \left(1 + |\omega|^2\right)^k \widehat{\phi}_m(\omega) \, d\mu(\omega) = \int_{\mathbb{R}^n} \Phi(x)(1-\Delta)^k \phi_m(x) \, dx = \int_{\mathbb{R}^n} (1-\Delta)^k \Phi(x) \phi_m(x) \, dx,$$

where the integral is indeed over $B(0, m^{-1})$. Since

$$\widehat{\phi}_m(\omega) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \phi_m(x) e^{-i\omega\cdot x} \, dx = (2\pi)^{-n/2} \int_{|x|\le 1} \phi(x) \left[\cos\left(\frac{\omega\cdot x}{m}\right) - i \sin\left(\frac{\omega\cdot x}{m}\right)\right] dx \to (2\pi)^{-n/2},$$

by Fatou's Lemma, we let $m \uparrow 0$ to obtain

$$\int_{\mathbb{R}^n} (1 + |\omega|^2)^k \, d\mu(\omega) \le (2\pi)^{n/2} (1-\Delta)^k \Phi(0).$$

Hence for each $|\alpha| \le 2k$, the function $\omega \mapsto |\omega^\alpha|$ is integrable with respect to μ. Note that

$$\left|e^{i\theta} - 1\right| = \left|2ie^{i\theta} \sin\frac{\theta}{2}\right| \le 2\left|\sin\frac{\theta}{2}\right| \le |\theta| \quad \text{for all } \theta \in \mathbb{R}.$$

By the dominated convergence theorem,

$$\frac{\partial \Phi}{\partial x_j}(x) = (2\pi)^{-n/2} \lim_{h\downarrow 0} \int_{\mathbb{R}^n} \frac{e^{-ih\omega_j} - 1}{h} \cdot e^{-i\omega\cdot x} \, d\mu(\omega) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} (-i\omega_j) e^{-i\omega\cdot x} \, d\mu(\omega).$$

By induction on α, for all $|\alpha| \le 2k$, we obtain that

$$\partial^\alpha \Phi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} (-i\omega)^\alpha e^{-i\omega\cdot x} \, d\mu(\omega).$$

Thus we complete the proof. □

4.3 Reproducing Kernel Hilbert Spaces

Motivation. By the Sobolev embedding theorem, for $s > n/2$, the Sobolev space $H^s(\mathbb{R}^n)$ is

$$H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) \cap C_0(\mathbb{R}^n) : (1 + |\cdot|^2)^{s/2} \widehat{f} \in L^2(\mathbb{R}^n) \right\}.$$

In other words, every equivalent class in $H^s(\mathbb{R}^n)$ has a continuous representitive. Recalling that the inner product in $H^s(\mathbb{R}^n)$ by

$$\langle f, g \rangle_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}(1 + |\omega|^2)^s \, d\omega.$$

Since the order $s > n/2$, the weight function is related to a positive definite function $\Phi : \mathbb{R}^n \to \mathbb{C}$ defined by $\widehat{\Phi}(\omega) = (2\pi)^{-n/2}(1 + |\omega|^2)^{-s}$. By Proposition 4.16, Φ is given by

$$\Phi(x) = \frac{2^{1-s-\frac{n}{2}}\pi^{-\frac{n}{2}}}{\Gamma(s)} |x|^{s-\frac{n}{2}} K_{\frac{n}{2}-s}(|x|), \quad x \in \mathbb{R}^n.$$

Since $(1 + |\omega|^2)^{s/2}\widehat{\Phi} \in L^2(\mathbb{R}^n)$, we have $\Phi \in H^s(\mathbb{R}^n)$, and

$$\langle f, \Phi(\cdot - x) \rangle_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \widehat{f}(\omega)\overline{\widehat{\Phi}(\omega)e^{-i\omega\cdot x}}(1 + |\omega|^2)^s \, d\omega = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(\omega)e^{i\omega\cdot x} \, d\omega = f(x).$$

By continuity of elements in $H^s(\mathbb{R}^n)$, the above identity holds pointwise in $\mathbb{R}^n$. This is called the *reproducing property*, which reveals the role of the function Φ in the Sobolev space $H^s(\mathbb{R}^n)$.

Kernel functions. Let Ω be a nonempty set, and let $\Phi : \Omega \times \Omega \to \mathbb{C}$ be a bivariate function. Similar to (4.1), an alternative way to construct a function interpolating points $(x_1, y_1), \cdots, (x_N, y_N) \in \Omega \times \mathbb{C}$ is

$$u = \sum_{j=1}^{N} \xi_j \Phi(\cdot, x_j).$$

To avoid confusion, we call such a bivariate function Φ a *kernel*. We say that $\Phi : \Omega \times \Omega \to \mathbb{C}$ is *a positive semidefinite kernel*, if for all pairwise distinct $x_1, \cdots, x_N \in \Omega$ and all $\xi_1, \cdots, \xi_N \in \mathbb{C}$,

$$\sum_{j,k=1}^{N} \xi_j \overline{\xi}_k \Phi(x_j, x_k) \geq 0.$$

The kernel Φ is called *positive definite*, if the quadratic form is positive for all $\xi \in \mathbb{C}^N \backslash \{0\}$. Similar to Proposition 4.6, one can show that a positive semidefinite function has the following properties:

(i) $\Phi(x, x) \geq 0$ for each $x \in \Omega$.

(ii) $\Phi(x, y) = \overline{\Phi(y, x)}$ for each $x, y \in \Omega$.

(iii) $|\Phi(x, y)| \leq \sqrt{\Phi(x, x)\Phi(y, y)}$ for all $x, y \in \Omega$.

Now we introduce the reproducing kernel for Hilbert spaces of functions.

Definition 4.18. Let $\mathcal{F}$ be a real Hilbert space of functions $f : \Omega \to \mathbb{R}$. A bivariate function $\Phi : \Omega \times \Omega \to \mathbb{R}$ is called a *reproducing kernel* for $\mathcal{F}$, if

(i) $\Phi(\cdot, x) \in \mathcal{F}$ for every $x \in \Omega$, and

(ii) $\langle f, \Phi(\cdot, x) \rangle_{\mathcal{F}} = f(x)$ for every $f \in \mathcal{F}$ and every $x \in \Omega$.

Remark. The reproducing kernel of a Hilbert space must be unique. In fact, if both Φ_1 and Φ_2 are reproducing kernels of $\mathcal{F}$, the second property implies $\langle f, \Phi_1(\cdot, x) - \Phi_2(\cdot, x) \rangle$ for every $f \in \mathcal{F}$ and every $x \in \Omega$. Setting $f = \Phi_1(\cdot, x) - \Phi_2(\cdot, x)$ for a fixed $x \in \Omega$ demonstrates the uniqueness.

Theorem 4.19 (Characterization of reproducing kernel Hilbert space)**.** *Let $\mathcal{F}$ be a real Hilbert space of functions $f : \Omega \to \mathbb{R}$. Then $\mathcal{F}$ has a reproducing kernel if and only if the point evaluation functionals are continuous on $\mathcal{F}$, i.e. $\delta_x \in \mathcal{F}^*$ for all $x \in \Omega$.*

Proof. If every point evaluation functional $\delta_x \in \mathcal{F}^*$, by the Riesz representation theorem, there exists $\Phi_x \in \mathcal{F}$ such that $f(x) = \delta_x(f) = \langle f, \Phi_x \rangle$. Hence the function $\Phi(\cdot, x) := \Phi_x(\cdot)$ is a reproducing kernel of $\mathcal{F}$.

Conversely, if Φ is a reproducing kernel of $\mathcal{F}$, we have $\delta_x(f) = f(x) = \langle f, \Phi(\cdot, x)\rangle_{\mathcal{F}} \leq \|\Phi(\cdot, x)\|_{\mathcal{F}} \|f\|_{\mathcal{F}}$. Hence the point evaluation functional δ_x is bounded by $\|\delta_x\|_{\mathcal{F}^*} \leq \|\Phi(\cdot, x)\|_{\mathcal{F}}$. □

Proposition 4.20. *Let $\mathcal{F}$ be a real Hilbert space of functions $f : \Omega \to \mathbb{R}$ with reproducing kernel Φ. Then*

(i) $\Phi(x, y) = \langle \Phi(\cdot, x), \Phi(\cdot, y)\rangle_{\mathcal{F}} = \langle \delta_x, \delta_y \rangle_{\mathcal{F}^*}$ for every $x, y \in \Omega$.

(ii) Φ is symmetric and positive semidefinite.

(iii) Φ is positive definite if and only if the point evaluation functionals are linearly independent in $\mathcal{F}^*$.

(iv) If a function sequence $(f_n) \subset \mathcal{F}$ converges to $f \in \mathcal{F}$ in norm, then (f_n) also converges pointwise to f.

Proof. (i) For every $x, y \in \Omega$, it follows by definition that $\langle \Phi(\cdot, y), \Phi(\cdot, x)\rangle = \Phi(x, y)$. By the Riesz representation theorem, $\Phi(\cdot, x) \mapsto \delta_x$ is a unitary isomorphism from $\mathcal{F}$ to $\mathcal{F}^*$, and $\langle \delta_x, \delta_y\rangle_{\mathcal{F}^*} = \langle \Phi(\cdot, x), \Phi(\cdot, y)\rangle_{\mathcal{F}}$.

(ii) & (iii) For pairwise distinct $x_1, \cdots, x_N \in \Omega$ and $\xi \in \mathbb{R}^N \backslash \{0\}$,

$$\sum_{j,k=1}^{N} \xi_j \xi_k \Phi(x_j, x_k) = \sum_{j,k=1}^{N} \xi_j \xi_k \left\langle \delta_{x_j}, \delta_{x_k} \right\rangle = \left\langle \sum_{j=1}^{N} \xi_j \delta_{x_j}, \sum_{k=1}^{N} \xi_k \delta_{x_k} \right\rangle_{\mathcal{F}^*} = \left\| \sum_{j=1}^{N} \xi_j \delta_{x_j} \right\|_{\mathcal{F}^*} \geq 0.$$

The last expression can and will only be zero if the point evaluation functionals are linearly dependent.

(iv) Note that

$$|f_n(x) - f(x)| = |\langle f_n - f, \Phi(\cdot, x)\rangle_{\mathcal{F}}| \leq \|\Phi(\cdot, x)\|_{\mathcal{F}} \|f_n - f\|_{\mathcal{F}}.$$

Hence convergence in norm implies pointwise convergence. □

Next, we study how to construct a reproducing kernel Hilbert space from a positive semidefinite kernel.

Theorem 4.21 (Moore-Aronszajn)**.** *Reproducing kernel Hilbert spaces and positive semidefinite kernel are one-to-one correspondent, i.e., for each symmetric positive semidefinite kernel $\Phi : \Omega \times \Omega \to \mathbb{R}$, there exists a unique real Hilbert space with reproducing kernel Φ.*

Proof. Step I. Let Ω be a non-empty set and $\Phi : \Omega \times \Omega \to \mathbb{R}$ be a positive semidefinite kernel. Define

$$\mathcal{F}_\Phi^0 = \operatorname{span}\{\Phi(\cdot, x) : x \in \Omega\} = \left\{ \sum_{j=1}^{N} \xi_j \Phi(\cdot, x_j) : N \in \mathbb{N},\ \xi_1, \cdots, \xi_N \in \mathbb{R},\ x_1, \cdots, x_N \in \Omega \right\}.$$

We claim that $\mathcal{F}_\Phi^0$ is a pre-Hilbert space with inner product

$$\left\langle \sum_{j=1}^{M} \xi_j \Phi(\cdot, x_j), \sum_{k=1}^{N} \upsilon_k \Phi(\cdot, y_k) \right\rangle_\Phi = \sum_{j=1}^{M} \sum_{k=1}^{N} \xi_j \upsilon_k \Phi(x_j, y_k).$$

We need to show that $\langle \cdot, \cdot \rangle_\Phi$ is indeed an inner product. It is easy to verify that $\langle \cdot, \cdot \rangle_\Phi$ is a semi-inner product, so it remains to show that $\langle f, f\rangle_\Phi = 0$ only if $f = 0$. Assume $\|f\|_\Phi = 0$, and fix $g \in \mathcal{F}_\Phi^0$. Then for all $t \in \mathbb{R}$,

$$0 \leq \|f - tg\|_\Phi^2 \leq -2t\langle f, g\rangle_\Phi + t^2 \langle g, g\rangle_\Phi.$$

If there exists $g \in \mathcal{F}_\Phi^0$ such that $|\langle f, g\rangle_\Phi| > 0$, setting $t = \langle f, g\rangle_\Phi / \langle g, g\rangle_\Phi$ gives a contradiction of the above inequality. Hence $\langle f, g\rangle_{\mathcal{H}_0} = 0$ for all $g \in \mathcal{F}_\Phi^0$. Then $f(x) = \langle f, \Phi(\cdot, x)\rangle_\Phi = 0$ for every $x \in \Omega$, and $f = 0$.

Step II. If (f_j) is a Cauchy sequence in $\mathcal{F}_\Phi^0$, the Cauchy-Schwarz inequality implies

$$|f_j(x) - f_k(x)| = |\langle f_j - f_k, \Phi(\cdot, x)\rangle_\Phi| \le \Phi(x,x)\|f_j - f_k\|_\Phi \to 0 \quad \text{as } j,k \to \infty.$$

Hence $(f_j(x))$ is a Cauchy sequence for all $x \in \Omega$, and we identify (f_j) with the function $f(x) = \lim_{j\to\infty} f_j(x)$. We define function space $\mathcal{F}_\Phi \supset \mathcal{F}_\Phi^0$ by

$$\mathcal{F}_\Phi = \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \text{there exist a Cauchy sequence } (f_j) \text{ in } \mathcal{F}_\Phi^0 \text{ such that } f_j \to f \text{ pointwise} \right\}$$

Then $\mathcal{F}_\Phi$ is a vector space. For each $f \in \mathcal{F}_\Phi$, we define $\|f\|_\Phi = \lim_{j\to\infty} \|f_j\|_\Phi$, where $(f_j) \subset \mathcal{F}_\Phi^0$ is a Cauchy sequence approximating f. We can verify that function $\|\cdot\|_\Phi : \mathcal{F}_\Phi \to [0,\infty)$ is well-defined:

- ∘ *Claim I.* The limit of $(\|f_j\|_\Phi)$ exists for every Cauchy sequence $(f_j) \subset \mathcal{F}_\Phi^0$.
- • Every Cauchy sequence $(f_j) \subset \mathcal{F}_\Phi^0$ is bounded in norm, so we assume $\sup_{j\in\mathbb{N}} \|f_j\|_\Phi \le K < \infty$. Then

$$\begin{aligned} \left|\|f_j\|_\Phi^2 - \|f_k\|_\Phi^2\right| &= |\langle f_j, f_j - f_k\rangle_\Phi + \langle f_j - f_k, f_k\rangle_\Phi| \\ &\le (\|f_j\|_\Phi + \|f_k\|_\Phi)\, \|f_j - f_k\|_\Phi \le 2K\|f_j - f_k\|_\Phi \to 0 \end{aligned}$$

  as $j,k \to \infty$, and $\lim_{j\to\infty} \|f_j\|_\Phi$ exists.
- ∘ *Claim II.* The value of $\|f\|_\Phi$ is independent of the choice of approximating Cauchy sequence $(f_j) \subset \mathcal{F}_\Phi^0$.
- • For two Cauchy sequences $(f_j), (g_j) \subset \mathcal{F}_\Phi^0$ approximating f, we have $f_j - g_j \to 0$ pointwise. Therefore, it suffices to show that for every Cauchy sequence $(f_j) \subset \mathcal{F}$ such that $f_j \to 0$ pointwise, $\|f_j\|_\Phi \to 0$. As in Claim I, we assume $\sup_{j\in\mathbb{N}} \|f_j\|_\Phi \le K < \infty$. Given any fixed $\epsilon > 0$, we take $M_0 \in \mathbb{N}$ such that $\|f_j - f_k\|_\Phi < \epsilon/(2K)$ for all $j,k \ge M_0$. We may increase M_0 to make $f_{M_0} \ne 0$ (otherwise, $f_j = 0$ for all $j \ge M_0$, and the result is clear) and write $f_{M_0} = \sum_{k=1}^N \xi_k \Phi(\cdot, x_k)$ with $\xi_1, \cdots, \xi_N \ne 0$. Next, we take $M_1 \in \mathbb{N}$ such that $|f_j(x_k)| < \epsilon/(2N|\xi_k|)$ for $k = 1, \cdots, N$. Then for all $j \ge \max\{M_0, M_1\}$,

$$\begin{aligned} \|f_j\|_\Phi^2 &\le |\langle f_j, f_j - f_{M_0}\rangle_\Phi| + |\langle f_j, f_{M_0}\rangle_\Phi| \le \|f_j\|_\Phi \|f_{M_0} - f_j\|_\Phi + \sum_{k=1}^N |\xi_k|\, |\langle f_j, \Phi(\cdot, x_k)\rangle_\Phi| \\ &\le A\|f_{M_0} - f_j\|_\Phi + \sum_{k=1}^N |\xi_k|\, |f_j(x_k)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

- ∘ *Claim III.* For a Cauchy sequence $(f_j) \subset \mathcal{F}_\Phi^0$ approximating $f \in \mathcal{F}_\Phi$ pointwise, we have $\|f_j - f\|_\Phi \to 0$. That means, $\mathcal{F}_\Phi^0$ is a dense subspace of $\mathcal{F}_\Phi$.
- • Given any $\epsilon > 0$, there exists $M_2 \in \mathbb{N}$ such that $\|f_j - f_k\|_\Phi < \epsilon/2$ for all $j,k \ge M_2$. Then $(f_j - f_{M_2}) \subset \mathcal{F}_\Phi^0$ converges pointwise to $f - f_{M_2}$, and $\|f_j - f_{M_2}\|_\Phi \to \|f - f_{M_2}\|_\Phi$. Hence for all $j \ge M_2$,

$$\|f_j - f\| \le \|f_j - f_{M_2}\|_\Phi + \|f - f_{M_2}\|_\Phi \le \|f_j - f_{M_2}\|_\Phi + \sup_{k\ge M_2} \|f_k - f_{M_2}\|_\Phi < \epsilon.$$

Following the definition, $\|\cdot\|_\Phi$ is a norm on $\mathcal{F}_\Phi$ and is consistent with its definition on $\mathcal{F}_\Phi^0$. To show that $\mathcal{F}_\Phi$ is complete under $\|\cdot\|_\Phi$, we let (f_j) be a Cauchy sequence in $(\mathcal{F}_\Phi, \|\cdot\|_\Phi)$. Then for each f_j we can pick an approximating sequence $(f_{j,k}) \subset \mathcal{F}_\Phi^0$, and we take N_j such that $\|f_{j,N_j} - f_j\|_\Phi < 1/j$. Given any $\epsilon > 0$, we pick $M_2 \in \mathbb{N}$ such that $\|f_j - f_k\|_\Phi < \epsilon/3$ for all $j,k \ge M_2$. Then for all $j,k \ge \max\{M_2, 3/\epsilon\}$,

$$\|f_{j,N_j} - f_{k,N_k}\|_\Phi \le \|f_{j,N_j} - f_j\|_\Phi + \|f_j - f_k\|_\Phi + \|f_k - f_{k,N_k}\|_\Phi < \epsilon.$$

Therefore $(f_{j,N_j}) \subset \mathcal{F}_\Phi^0$ is a Cauchy sequence, which converges to some $f \in \mathcal{F}_\Phi$ pointwise and in $\|\cdot\|_\Phi$ norm. Since $\|f_j - f\|_\Phi \le \|f_j - f_{j,N_j}\|_\Phi + \|f_{j,N_j} - f\|_\Phi \to 0$ as $j \to \infty$, the space $\mathcal{F}_\Phi$ is complete.

Step III. Note that $\|\cdot\|_\Phi$ satisfies the parallelogram law: for every $f, g \in \mathcal{F}_\Phi$,

$$\|f-g\|_\Phi^2 + \|f+g\|_\Phi^2 = \lim_{j\to\infty}\left(\|f_j - g_j\|_\Phi^2 + \|f_j + g_j\|_\Phi^2\right) = 2\lim_{j\to\infty}\left(\|f_j\|_\Phi^2 + \|g_j\|_\Phi^2\right) = 2\|f\|_\Phi^2 + 2\|g\|_\Phi^2.$$

Then an inner product on $\mathcal{F}_\Phi$ is induced by the polarization identity $\langle f, g\rangle_\Phi = \frac{1}{2}\left(\|f\|_\Phi^2 + \|g\|_\Phi^2 - \|f-g\|_\Phi^2\right)$. For every $f \in \mathcal{F}_\Phi$, every approximating sequence $(f_j) \subset \mathcal{F}_\Phi^0$ and every $x \in \Omega$,

$$\langle f, \Phi(\cdot, x)\rangle_\Phi = \frac{1}{2}\lim_{j\to\infty}\left(\|f_j\|_\Phi^2 + \|\Phi(\cdot,x)\|_\Phi^2 - \|f_j - \Phi(\cdot,x)\|_\Phi^2\right) = \lim_{j\to\infty}\langle f_j, \Phi(\cdot,x)\rangle = \lim_{j\to\infty} f_j(x) = f(x).$$

Therefore Φ is the reproducing kernel for $\mathcal{F}_\Phi$.

Step IV. Now we focus on the uniqueness assertion. If $\mathcal{F}$ is a Hilbert space with reproducing kernel Φ, it contains $\mathcal{F}_\Phi^0$, and the restriction of $\langle\cdot,\cdot\rangle_\mathcal{F}$ on $\mathcal{F}_\Phi^0$ agrees with $\langle\cdot,\cdot\rangle_\Phi$. Since $\mathcal{F}$ is complete, it contains the limits of all $\|\cdot\|_\Phi$-Cauchy sequences in $\mathcal{F}_\Phi^0$, and $\mathcal{F} \supset \mathcal{F}_\Phi$. Still, the restriction of $\langle\cdot,\cdot\rangle_\mathcal{F}$ on $\mathcal{F}_\Phi$ agrees with $\langle\cdot,\cdot\rangle_\Phi$. By completeness of $\mathcal{F}_\Phi$, it is a closed subspace of $\mathcal{F}$. Now for every $f \in \mathcal{F}$, we write the orthogonal decomposition $f = f_0 + f_1$, where $f_0 \in \mathcal{F}_\Phi$ and $f_1 \in \mathcal{F}_\Phi^\perp$. Then $f_1 \perp \mathcal{F}_\Phi^0$, and

$$f(x) = \langle f, \Phi(\cdot,x)\rangle_\mathcal{G} = \langle f_0, \Phi(\cdot,x)\rangle_\mathcal{G} + \langle f_1, \Phi(\cdot,x)\rangle_\mathcal{G} = \langle f_0, \Phi(\cdot,x)\rangle_\mathcal{G} = f_0(x), \quad x \in \Omega.$$

Hence $f = f_0 \in \mathcal{F}_\Phi$, and $\mathcal{F} \supset \Phi = \mathcal{F}$. To summarize, we have $\mathcal{F} = \mathcal{F}_\Phi$, the completion of $\mathcal{F}_\Phi$ under $\|\cdot\|_\Phi$. $\square$

Finally, we discuss the invariance property of spaces and kernels.

Definition 4.22. Let $\mathcal{F}$ be a real Hilbert space of functions $f : \Omega \to \mathbb{R}$. For an invertible transformation $T : \Omega \to \Omega$, we say $\mathcal{F}$ is *invariant under* T, if

(i) $f \circ T \in \mathcal{F}$ for every $f \in \mathcal{F}$, and

(ii) $\langle f \circ T, g \circ T\rangle_\mathcal{F} = \langle f, g\rangle_\mathcal{F}$ for all $f, g \in \mathcal{F}$.

Proposition 4.23. *Let $\mathcal{F}$ be a real Hilbert space of functions $f : \Omega \to \mathbb{R}$ with reproducing kernel Φ. If $\mathcal{F}$ is invariant under invertible transformation $T : \Omega \to \Omega$, we have*

$$\Phi(Tx, Ty) = \Phi(x,y), \quad \textit{for all } x, y \in \Omega.$$

Proof. If Φ is a reproducing kernel of $\mathcal{F}$, for every $f \in \mathcal{F}$ and $y \in \Omega$,

$$f(y) = f \circ T^{-1}(Ty) = \langle f \circ T^{-1}, \Phi(\cdot, Ty)\rangle_\mathcal{F} = \langle f, \Phi(\cdot, Ty) \circ T\rangle_\mathcal{F}$$

By uniqueness of the reproducing kernel, $\Phi(T\cdot, T\cdot)$ coincides the reproducing kernel of $\mathcal{F}$. $\square$

Remark. We let $\Omega = \mathbb{R}^n$, and consider the translations $\{\tau_y : y \in \mathbb{R}^n\}$. If $\mathcal{F}$ is a Hilbert space of functiosn $f : \mathbb{R}^m \to \mathbb{R}$ invariant under the group of translations on $\mathbb{R}^n$, we have

$$\Phi(x,y) = \Phi(\tau_y x, \tau_y y) = \Phi(x-y, 0) =: \Phi_0(x-y).$$

Then the kernel can be written as $\Phi_0(\cdot - \cdot)$ for some positive semidefinite kernel 0. In addition, if $\mathcal{F}$ is invariant under both the translations and the orthogonal transformations, for every $x, y \in \mathbb{R}^n$, we find an orthogonal transformation Q with $Q(x-y) = |x-y|e_1$ to obtain

$$\Phi(x,y) = \Phi(Qx, Qy) = \Phi_0(Q(x-y)) = \Phi_0(|x-y|e_1) = \phi(|x-y|).$$

In this case, the kernel $\Phi(x,y) = \phi(|x-y|)$ is radial.

4.4 Native Spaces for Positive Definite Functions

In this subsection, we assume Ω is a metric space and $\Phi : \Omega \times \Omega \to \mathbb{R}$ is a continuous positive semidefinite kernel. A candidate for a Hilbert space with reproducing kernel Φ is the completion $\mathcal{F}_\Phi$ of the pre-Hilbert space $\mathcal{F}_\Phi^0 = \text{span}\{\Phi(\cdot, x) : x \in \Omega\}$ with respect to the $\|\cdot\|_\Phi$ norm, which contains the equivalent classes of all $\|\cdot\|_\Phi$-Cauchy sequences $\mathbf{f} = (f_j)_{j\in\mathbb{N}}$ in $\mathcal{F}_\Phi^0$, where the equivalence relation is

$$\mathbf{f} = (f_j)_{j\in\mathbb{N}} \sim \mathbf{g} = (g_j)_{j\in\mathbb{N}} \quad \text{if } \lim_{j\to\infty} \|f_j - g_j\|_\Phi = 0.$$

The pre-Hilbert space $\mathcal{F}_\Phi^0$ can be understood as a subspace of its completion $\mathcal{F}_\Phi$ by identifying every $f \in \mathcal{F}_\Phi^0$ with the equivalent class $[(f, f, \cdots)]$. The inner product on $\mathcal{F}_\Phi$ is given by

$$\langle [\mathbf{f}], [\mathbf{g}] \rangle_\Phi = \lim_{j\to\infty} \langle f_j, g_j \rangle_\Phi, \quad \mathbf{f} = (f_j)_{j\in\mathbb{N}}, \mathbf{g} = (g_j)_{j\in\mathbb{N}} \in \mathcal{F}_\Phi.$$

In the last subsection, every abstract element $[\mathbf{f}] \in \mathcal{F}_\Phi$ is identified with the pointwise limit of corresponding Cauchy sequences. In comparison, in this subsection, we obtain a reproducing kernel Hilbert space with the help of the continuous extension of point evaluation functionals on $\mathcal{F}_\Phi^0$ to the completion $\mathcal{F}_\Phi$.

Lemma 4.24. *For every Cauchy sequence $\mathbf{f} = (f_j)_{j\in\mathbb{N}}$ in $\mathcal{F}_\Phi^0$ and every $x \in \Omega$, define*

$$R[\mathbf{f}](x) = \langle [\mathbf{f}], \Phi(\cdot, x) \rangle_\Phi = \lim_{j\to\infty} \langle f_j, \Phi(\cdot, x) \rangle_\Phi.$$

(i) R is a linear map from $\mathcal{F}_\Phi$ into $C(\Omega)$, and $Rf = f$ for every $f \in \mathcal{F}_\Phi^0$.

(iii) The map $R : \mathcal{F}_\Phi \to C(\Omega)$ is injective.

Proof. (i) Clearly R is a function on $\mathcal{F}_\Phi$. For every $x, y \in \Omega$ and every Cauchy sequence $\mathbf{f} = (f_j)_{j\in\mathbb{N}}$ in $\mathcal{F}_\Phi^0$,

$$\begin{aligned} |R[\mathbf{f}](x) - R[\mathbf{f}](y)| = |\langle [\mathbf{f}], \Phi(\cdot, x) - \Phi(\cdot, y) \rangle_\Phi| &= \lim_{j\to\infty} \|f_j\|_\Phi \, \|\Phi(\cdot, x) - \Phi(\cdot, y)\|_\Phi \\ &= \lim_{j\to\infty} \|f_j\|_\Phi \sqrt{\Phi(x, x) + \Phi(y, y) - 2\Phi(x, y)}, \end{aligned}$$

which, by continuity of Φ, converges to 0 as $|x - y| \downarrow 0$. Hence $R[\mathbf{f}]$ is continuous. Furthermore, if $f \in \mathcal{F}_\Phi^0$, we have $Rf(x) = \langle f, \Phi(\cdot, x) \rangle_\Phi = f(x)$ by the reproducing property.

(iii) If $R[\mathbf{f}] = 0$ for some $[\mathbf{f}] = [(f_1, f_2, \cdots)] \in \mathcal{F}_\Phi$, we have $\langle [\mathbf{f}], \Phi(\cdot, x) \rangle_\Phi$ for every $x \in \Omega$, and

$$\lim_{j\to\infty} f_j(x) = \lim_{j\to\infty} \langle f_j, \Phi(\cdot, x) \rangle_\Phi = 0.$$

In the proof of Theorem 4.21 we already show that a Cauchy sequence (f_j) in $\mathcal{F}_\Phi^0$ converges to 0 pointwise if and only if $\|f_j\|_\Phi \to 0$. Hence $\|[\mathbf{f}]\|_\Phi = 0$, and R is injective. □

This technical lemma enables us to define the native Hilbert space of a positive semidefinite kernel Φ.

Theorem 4.25 (Native spaces)**.** *Define the **native Hilbert function space** associated to the symmetric positive semidefinite kernel Φ by*

$$\mathcal{N}_\Phi(\Omega) = R(\mathcal{F}_\Phi).$$

Then $\mathcal{N}_\Phi(\Omega)$ is a Hilbert space of continuous functions $f : \Omega \to \mathbb{R}$ with inner product

$$\langle f, g \rangle_{\mathcal{N}_\Phi(\Omega)} - \langle R^{-1} f, R^{-1} g \rangle_\Phi$$

and reproducing kernel Φ. Furthermore, every Hilbert space of functions $f : \Omega \to \mathbb{R}$ with reproducing kernel Φ coincides the native space $\mathcal{N}_\Phi(\Omega)$, and the inner products are the same.

Proof. Step I. By the definition of $(\mathcal{N}_\Phi(\Omega), \langle\cdot,\cdot\rangle_{\mathcal{N}_\Phi(\Omega)})$, R is a unitary isomorphism between $\mathcal{N}_\Phi(\Omega)$ and $\mathcal{F}_\Phi$, and $\mathcal{N}_\Phi(\Omega)$ is a Hilbert space of continuous functions on Ω. Furthermore, since $\Phi(\cdot,x) \in \mathcal{F}_\Phi^0$ for every $x \in \Omega$, we have $R^{-1}\Phi(\cdot,x) = \Phi(\cdot,x)$. For every $f \in \mathcal{N}_\Phi(\Omega)$,

$$f(x) = R(R^{-1}f)(x) = \langle R^{-1}f, \Phi(\cdot,x)\rangle_\Phi = \langle R^{-1}f, R^{-1}\Phi(\cdot,x)\rangle_\Phi = \langle f, \Phi(\cdot,x)\rangle_{\mathcal{N}_\Phi(\Omega)}.$$

Hence Φ is the reproducing kernel for $\mathcal{N}_\Phi(\Omega)$.

Step II. Let $\mathcal{F}$ be a Hilbert space of real functions on Ω with reproducing kernel Φ. If $f \in \mathcal{N}_\Phi(\Omega)$, we assume $[(f_j)_{j\in\mathbb{N}}] = R^{-1}f \in \mathcal{F}_\Phi$, where (f_j) is a Cauchy sequence in $\mathcal{F}_\Phi^0$. Then f is pointwise given by

$$f(x) = R(R^{-1}f)(x) = \langle R^{-1}f, \Phi(\cdot,x)\rangle_\Phi = \lim_{n\to\infty} f(x), \quad x \in \Omega.$$

Note that (f_j) is also a Cauchy sequence in $\mathcal{F}$, which converges to some $g \in \mathcal{F}$. By Proposition 4.20 (iv), g is pointwise defined by $g(x) = \lim_{j\to\infty} f_j(x)$, $x \in \Omega$, and $f = g \in \mathcal{F}$. Hence $\mathcal{N}_\Phi(\Omega) \subset \mathcal{F}$, and

$$\|f\|_\mathcal{F} = \|g\|_\mathcal{F} = \lim_{j\to\infty} \|f_j\|_\mathcal{F} = \lim_{j\to\infty} \|f_j\|_\Phi = \|R^{-1}f\|_\Phi = \|f\|_{\mathcal{N}_\Phi(\Omega)}.$$

Finally, if $\mathcal{N}_\Phi(\Omega)$ does not equal $\mathcal{F}$, one can find an element $g \in \mathcal{F}\backslash\{0\}$ orthogonal to the closed subspace $\mathcal{N}_\Phi(\Omega)$ of $\mathcal{F}$, which implies $g(x) = \langle g, \Phi(\cdot,x)\rangle_\mathcal{F} = 0$ for every $x \in \Omega$ and contradicts $g \neq 0$. Therefore $\mathcal{N}_\Phi(\Omega) = \mathcal{F}$, and their norms agree with each other. The equivalence of inner products then follows from the polarization identity. □

Remark. We can also skip Step II in the above proof by Moore-Aronszajn theorem [Theorem 4.21].

The uniqness result allows us to give another characterization of the native space in the case where $\Omega = \mathbb{R}^n$ and Φ is translation invariant.

Theorem 4.26. *Let $\Phi \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ be a real-valued positive definite function. Define*

$$\mathcal{G} = \left\{ f \in L^2(\mathbb{R}^n) \cap C(\mathbb{R}^n) : \widehat{\Phi}^{-1/2}\widehat{f} \in L^2(\mathbb{R}^n) \right\},$$

and equip this space with the bilinear form

$$\langle f, g\rangle_\mathcal{G} = (2\pi)^{-n/2} \left\langle \widehat{\Phi}^{-1/2}\widehat{f}, \widehat{\Phi}^{-1/2}\widehat{g} \right\rangle_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \frac{\widehat{f}(\omega)\overline{\widehat{g}(\omega)}}{\widehat{\Phi}(\omega)}\, d\omega.$$

Then $\mathcal{G}$ is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_\mathcal{G}$ and reproducing kernel $\Phi(\cdot - \cdot)$.

Proof. Step I. We first prove that $\mathcal{G}$ is a pre-Hilbert space. By Theorem 4.14, the Fourier transform $\widehat{\Phi} \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ is nonnegative and not identically zero. Then every $f \in \mathcal{G}$ satisfies $\widehat{f} \in L^1(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} |\widehat{f}(\omega)|\, d\omega \leq \left[\int_{\mathbb{R}^n} \frac{|\widehat{f}(\omega)|^2}{\widehat{\Phi}(\omega)}\, d\omega \right]^{1/2} \left[\int_{\mathbb{R}^n} |\widehat{\Phi}(\omega)|\, d\omega \right]^{1/2} < \infty.$$

Since $f \in L^2(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, by Plancherel's theorem and the continuity of f and $(\widehat{f})^\vee$, we may recover f pointwise from its Fourier transform:

$$f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{f}(\omega) e^{i\omega\cdot x}\, d\omega, \quad x \in \mathbb{R}^n. \tag{4.5}$$

Now we prove that $\langle\cdot,\cdot\rangle_\mathcal{G}$ is an inner product on $\mathcal{G}$. Clearly $\langle\cdot,\cdot\rangle$ is $\mathbb{R}$-linear and conjugate symmetric. Then it remains to show that $\langle\cdot,\cdot\rangle$ is real-valued and positive definite. For a real-valued L^2 function f, we have

$\widehat{f}(\omega) = \overline{\widehat{f}(-\omega)}$ a.e.. Then for every $f, g \in \mathcal{G}$,

$$\int_{\mathbb{R}^n} \frac{\widehat{f}(\omega)\overline{\widehat{g}(\omega)}}{\widehat{\Phi}(\omega)}\,d\omega = \frac{1}{2}\int_{\mathbb{R}^n} \frac{\widehat{f}(\omega)\overline{\widehat{g}(\omega)}}{\widehat{\Phi}(\omega)}\,d\omega + \frac{1}{2}\int_{\mathbb{R}^n} \frac{\widehat{f}(-\omega)\overline{\widehat{g}(-\omega)}}{\widehat{\Phi}(-\omega)}\,d\omega$$
$$= \frac{1}{2}\int_{\mathbb{R}^n} \frac{\widehat{f}(\omega)\overline{\widehat{g}(\omega)} + \widehat{f}(-\omega)\overline{\widehat{g}(-\omega)}}{\widehat{\Phi}(\omega)}\,d\omega = \frac{1}{2}\int_{\mathbb{R}^n} \frac{\widehat{f}(\omega)\overline{\widehat{g}(\omega)} + \overline{\widehat{f}(\omega)}\widehat{g}(\omega)}{\widehat{\Phi}(\omega)}\,d\omega = \int_{\mathbb{R}^n} \frac{\mathrm{Re}\big[\widehat{f}(\omega)\overline{\widehat{g}(\omega)}\big]}{\widehat{\Phi}(\omega)}\,d\omega,$$

which is real-valued. As a weighted L^2-inner product, $\langle f, f\rangle_{\mathcal{G}}$ is nonnegative. Furthermore, $\langle f, f\rangle_{\mathcal{G}}$ equals zero if and only if $\widehat{f} = 0$ a.e., and by 4.5 this is equivalent to $f \equiv 0$. Hence $\langle \cdot, \cdot\rangle_{\mathcal{G}}$ is positive definite.

Step II. To prove that $\mathcal{G}$ is a Hilbert space, we only need to show the completeness of $\mathcal{G}$. We fix a Cauchy sequence (f_j) in $\mathcal{G}$. This means that $\widehat{\Phi}^{-1/2}\widehat{f}_j$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$, which converges to some $g \in L^2(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} \left|\sqrt{\widehat{\Phi}(\omega)}g(\omega)\right| d\omega \le \left[\int_{\mathbb{R}^m} |g(\omega)|^2\,d\omega\right]^{1/2} \left[\int_{\mathbb{R}^m} |\widehat{\Phi}(\omega)|\,d\omega\right]^{1/2} < \infty,$$

and

$$\int_{\mathbb{R}^n} \left|\sqrt{\widehat{\Phi}(\omega)}g(\omega)\right|^2 d\omega \le \|\widehat{\Phi}\|_\infty \int_{\mathbb{R}^m} |g(\omega)|^2\,d\omega < \infty.$$

Hence $\widehat{\Phi}^{1/2}g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and the function

$$f(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \sqrt{\widehat{\Phi}(\omega)}g(\omega)e^{i\omega\cdot x}\,d\omega, \quad x \in \mathbb{R}^n$$

is well defined and in $L^2(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$. By Fourier inversion theorem, $\widehat{\Phi}^{-1/2}\widehat{f} \in L^2(\mathbb{R}^n)$, and

$$|f(x) - f_j(x)| \le (2\pi)^{-n/2}\int_{\mathbb{R}^n} \left|\widehat{\Phi}(\omega)^{1/2}g(\omega) - \widehat{f}_j(\omega)\right| d\omega \le (2\pi)^{-n/2}\left\|g - \widehat{\Phi}^{-1/2}f_j\right\|_{L^2(\mathbb{R}^n)} \sqrt{\|\Phi\|_{L^1(\mathbb{R}^n)}},$$

which converges to 0 as $j \to \infty$. Hence f is real-valued, and $f \in \mathcal{G}$. Furthermore,

$$\|f_j - f\|_{\mathcal{G}}^2 = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \frac{|\widehat{f}_j(\omega) - \widehat{f}(\omega)|^2}{\widehat{\Phi}(\omega)}\,d\omega = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \left|\widehat{\Phi}(\omega)^{-1/2}\widehat{f}_j(\omega) - g(\omega)\right|^2 d\omega \to 0.$$

Hence $f_j \to f$ in $\mathcal{G}$, and $\mathcal{G}$ is a Hilbert space.

Step III. Finally we show that $\Phi(\cdot - \cdot)$ is the reproducing kernel of $\mathcal{G}$. Since $\Phi \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is bounded by $\Phi(0)$, it is in $L^2(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. For every $y \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \frac{\left|\widehat{\tau_y\Phi}(\omega)\right|^2}{\widehat{\Phi}(\omega)} = \int_{\mathbb{R}^n} \frac{\left|\widehat{\Phi}(\omega)e^{-i\omega\cdot y}\right|^2}{\widehat{\Phi}(\omega)} = \int_{\mathbb{R}^n} \widehat{\Phi}(\omega)\,d\omega < \infty.$$

Hence $\widehat{\Phi}^{-1/2}(\Phi(\cdot - y))^\wedge \in L^2(\mathbb{R}^n)$, and $\Phi(\cdot - y) \in \mathcal{G}$ for every $y \in \mathbb{R}^n$. For the reproducing property, note that

$$\langle f, \Phi(\cdot - y)\rangle_{\mathcal{G}} = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \frac{\widehat{f}(\omega)\overline{(\tau_y\Phi)^\wedge(\omega)}}{\widehat{\Phi}(\omega)}\,d\omega = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \widehat{f}(\omega)e^{i\omega\cdot y}\,d\omega = f(y).$$

Therefore $\mathcal{G}$ is a Hilbert space with reproducing kernel Φ. □

Remark. By the uniqueness argument in Theorem 4.25, $\mathcal{G}$ is the native space of $\Phi(\cdot - \cdot)$ in $\mathbb{R}^n$, and we use the sloppy notation $\mathcal{N}_\Phi(\mathbb{R}^n) := \mathcal{N}_{\Phi(\cdot - \cdot)}(\mathbb{R}^n) = \mathcal{G}$, and $\langle \cdot, \cdot\rangle_{\mathcal{N}_\Phi(\mathbb{R}^n)} = \langle \cdot, \cdot\rangle_{\mathcal{G}}$. In particular, every $f \in \mathcal{N}_\Phi$ can be recovered from its Fourier transform $\widehat{f} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ by (4.5).

The above result demonstrates that native spaces is a kind of generalization of Sobolev spaces. Recalling that for $s > n/2$, the Sobolev $H^s(\mathbb{R}^n)$ is defined as

$$H^s(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) \cap C(\mathbb{R}^n) : (1 + |\cdot|^2)^{s/2} \widehat{f} \in L^2(\mathbb{R}^n) \right\}.$$

If Φ has an algebraically decaying Fourier transform, then its native space is a Sobolev space.

Corollary 4.27. *Let $s > n/2$, and $\Phi \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ be a positive definite function such that*

$$c_1(1 + |\omega|^2)^{-s} \le \widehat{\Phi}(\omega) \le c_2(1 + |\omega|^2)^{-s} \quad \text{for all } \omega \in \mathbb{R}^n,$$

where $c_1 \le c_2$ are two positive constants. Then the native space $\mathcal{N}_\Phi(\mathbb{R}^n)$ corresponding to Φ coincides with the Sobolev space $H^s(\mathbb{R}^n)$, and the native space norm $\|\cdot\|_{\mathcal{N}_\Phi(\mathbb{R}^n)}$ and the Sobolev norm $\|\cdot\|_{H^s(\mathbb{R}^n)}$ are equivalent.

Remark. By Proposition 4.16, a good candidate of the desired function Φ for constructing Sobolev spaces is the Matérn function

$$\Phi(x) = \frac{2^{1-s}}{\Gamma(s)} |x|^{s-\frac{n}{2}} K_{\frac{n}{2}-s}(|x|), \quad x \in \mathbb{R}^n,$$

whose Fourier transform is $\widehat{\Phi}(\omega) = (1 + |\omega|^2)^{-s}$.

Another important example is the native spaces associated to Gaussian radial basis functions.

Proposition 4.28. *Let $\sigma > 0$, and let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be the Gaussian radial basis function $\Phi(x) = e^{-|x|^2/(2\sigma^2)}$. Then the native space*

$$\mathcal{N}_\Phi(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) \cap C(\mathbb{R}^n) : \widehat{\Phi}^{-1/2} \widehat{f} \in L^2(\mathbb{R}^n) \right\}$$

is continuously embedded into $C^\infty(\mathbb{R}^n)$.

Proof. Note that $\widehat{\Phi}(\omega) = \sigma^n e^{-\sigma^2|\omega|^2/2}$. For every $s \in \mathbb{R}$, the function $\omega \mapsto (1 + |\omega|^2)^s \, \widehat{\Phi}(\omega)$ is bounded on $\mathbb{R}^n$. Then for every $f \in \mathcal{N}_\Phi(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (1 + |\omega|^2)^s |\widehat{f}(\omega)|^2 \, d\omega \le \sup_{\omega \in \mathbb{R}^n} (1 + |\omega|^2)^s \, \widehat{\Phi}(\omega) \left| \int_{\mathbb{R}^n} \frac{|\widehat{f}(\omega)|^2}{\widehat{\Phi}(\omega)} \, d\omega \right| = C \|f\|^2_{\mathcal{N}_\Phi(\mathbb{R}^n)}, \tag{4.6}$$

where C is a constant only depending on σ, n, s. By Corollary 3.33, $f \in C_0^\infty(\mathbb{R}^n)$. Furthermore, if sequence $(f_j) \subset \mathcal{N}_\Phi(\mathbb{R}^n)$ and $f_j \to f$ in $\mathcal{N}_\Phi(\mathbb{R}^n)$, for each $\alpha \in \mathbb{N}_0^n$, we take $s > |\alpha| + \frac{n}{2}$. By Theorem 3.32 and (4.6),

$$\|\partial^\alpha f_j - \partial^\alpha f\|_\infty \le C_1 \|f_j - f\|_{H^s(\mathbb{R}^n)} \le C_2 \|f_j - f\|_{\mathcal{N}_\Phi(\mathbb{R}^n)},$$

where $C_1, C_2 > 0$ are constants depending only on n, s, α and σ. Therefore $\partial^\alpha f_j \to \partial^\alpha f$ uniformly on all compact subsets of $\mathbb{R}^n$, and $f_j \to f$ in the C^∞ topology. Hence the inclusion map $\mathrm{Id} : \mathcal{N}_\Phi(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is continuous, and $\mathcal{N}_\Phi(\mathbb{R}^n)$ is continuously embedded into $C^\infty(\mathbb{R}^n)$. $\square$

References

[1] Folland, G.B. (1999) *Real Analysis: Modern Techniques and Their Applications.* 2nd Edition, John Wiley & Sons, Inc., New York.

[2] Wendland, H. (2004) *Scattered Data Approximation.* Cambridge University Press, Cambridge.