Notes on Introductory Point-Set Topology

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1 Open and Closed Sets

1.1 Open and Closed Sets in Topological Spaces

Definition 1.1 (Topology and open sets). Let X be a nonempty open set. A *topology* on X is a collection $\mathscr{T}$ of nonempty subsets of X, called *open sets*, such that
(i) any union of open sets is open,
(ii) any finite intersection of open sets is open, and
(iii) both X and $\emptyset$ are open.
A set X together with a topology $\mathscr{T}$ on it is called a *a topological space*, denoted by $(X, \mathscr{T})$. Without ambiguity, we drop $\mathscr{T}$ and say X is a topological space.

Examples of topological spaces.

- (Euclidean Space). Let $X = \mathbb{R}^n$. A subset U of X is open if for every $\mathbf{x} \in U$, there exists $\delta > 0$ such that the ball $O(\mathbf{x}, \delta) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\|_2 < \delta\}$ lies entirely in U.
- (Discrete topology). Let X be a non-empty set. Every subset of X is an open set.
- (Subspace topology/induced topology). Let Y be a non-empty subset of X. A subset U of Y is open if there exists an open set O in X such that $U = O \cap Y$.

Definition 1.2 (Neighborhood). Let X be a topological space. Given a point $x \in X$, a subset N of X is called a *neighborhood* of x, if we can find an open set O in X such that $x \in O \subseteq N$.

By definition, an open set $O \subseteq X$ is a neighborhood of each of its points. Conversely, if O is a neighborhood of each of its points, we can find an open N_x for each $x \in O$ such that $x \in N_x \subseteq O$. Then $O = \bigcup_{x \in O} N_x$ as a union of open sets is itself open.

Proposition 1.3 (Properties of neighborhoods). Let X be a topological space, and x is a point in X. Then the following statements hold:
(i) x lies in each of its neighborhood.
(ii) The intersection of two neighborhoods of x is itself a neighborhood of x.
(iii) If N is a neighborhood of x and $N \subseteq M \subseteq X$, then M is a neighborhood of x.
(iv) If N is a neighborhood of x, then $\mathring{N} = \{y \in N : N \text{ is a neighborhood of } y\}$ is also a neighborhood of x.

Proof. The first three statements are trivial. We prove the fourth statement. Let N be a neighborhood of x, then there exists an open set O in X such that $x \in O \subseteq N$. Since O, as an open set, is a neighborhood of each of its points, we have $O \subseteq \mathring{N}$, which concludes the proof. $\square$

Remark. The four properties in Proposition 1.3 form an alternative construction of a topological space. More specifically, let X be a non-empty sets, for each point $x \in X$ we define the collection of its neighborhoods as satisfying (i)-(iv). A subset O of X is called an open set if it is a neighborhood of each of its point. Then we can verify that the collection of open sets in X satisfies Definition 1.1.

Definition 1.4 (Closed sets). A subset A of a topological space X is said to be a *closed* set in X, if its complement $X \backslash A$ is open.

Remark. Combining Definition 1.1 and Definition 1.4, it is clear that any intersection of closed sets, any finite union of closed sets, the entire space X and the empty set $\emptyset$ are closed. To characterize closed sets in a topological space, we introduce the following definition.

Definition 1.5 (Limit points/accumulation points). Let A be a subset of a topological space X. A point $p \in X$ is called a *limit point* of A if every neighborhood of p contains at least one point of $A \backslash \{p\}$.

Theorem 1.6 (Characterization of closed sets). A set is closed if and only if it contains all its limit points.

Proof. Let A be a closed set in a topological space X. Then its complement $X \backslash A$, being an open set, is a neighborhood of each of its points. Then any $x \in X \backslash A$ is not a limit point of A, and A contains all its limit points. Conversely, if A contains all of its limit points, then for each $x \in X \backslash A$, there exists a neighborhood of x lying in $X \backslash A$. Therefore $X \backslash A$ is a neighborhood of each of its points. □

Definition 1.7 (Closure). Let A be a subset of a topological space X. The union of A and all its limit points, denoted by $\overline{A}$, is called the *closure* of A.

Theorem 1.8. Let A be a subset of a topological space X. Then $\overline{A}$ is the intersection of all closed sets in X that contains A. In other words, $\overline{A}$ is the smallest closed set that contains A.

Proof. We first prove that $\overline{A}$ is closed. For every $x \in X \backslash \overline{A}$, we can find an open neighborhood O of x such that O does not intersect with A. If O contains a limit point of A, denoted by p, then O as a neighborhood of p contains a point of A, a contradiction! Hence $O \subseteq X \backslash \overline{A}$, showing $\overline{A}$ is closed. Now let $B \supseteq A$ be a closed set in X. It suffices to show that any limit point p of A is contained in B. To see this, suppose $p \notin B$. Since $X \backslash B$ is open, it is a neighborhood of p. Then $X \backslash B$ contains at least one point in A, again a contradiction! □

The following conclusion immediately follows from Theorem 1.8.

Corollary 1.9. A set is closed if and only if it is equal to its closure.

Definition 1.10 (Interior). Let A be a subset of a topological space X. The *interior* of A, denoted by $\mathring{A}$, is the union of all subsets of A that are open in X. A point that is in $\mathring{A}$ is an *interior point* of A.

It is clear that a set is open if and only if it is equal to its interior. We can also check that a point x lies in $\mathring{A}$ if and only if A is a neighborhood of x, which is consistent with the notation we use in Proposition 1.3.

Definition 1.11 (Frontier). Let A be a subset of a topological space X. We define the *frontier* of A as the intersection of its closure and the closure of its complements, $\partial A := \overline{A} \cap \overline{(X \backslash A)}$.

Proposition 1.12. Let A be a subset of a topological space X. Then $\mathring{A} \cap \partial A = \emptyset$, and $\mathring{A} \cup \partial A = \overline{A}$.

Proof. Let $\{O_\lambda : \lambda \in \Lambda\}$ be the collection of all open subsets of A. Then $\{X \backslash O_\lambda : \lambda \in \Lambda\}$ is the collection of all closed sets in X that contains $X \backslash A$. By Definition 1.10 and Theorem 1.8, we have $\mathring{A} = \bigcup_{\lambda \in \Lambda} O_\lambda$, and $\overline{(X \backslash A)} = \bigcap_{\lambda \in \Lambda} (X \backslash O_\lambda) = X \backslash \mathring{A}$. Hence $\mathring{A} \cap \partial A = \emptyset$, and $\partial A = \overline{A} \cap (X \backslash \mathring{A}) = \overline{A} \backslash \mathring{A}$. □

Remark. In the above proof, we obtain an alternative characterization of the interior: $\mathring{A} = X \backslash \overline{(X \backslash A)}$

Proposition 1.13 Let A and B be two subsets of a topological space X. The following statements hold:
(i) $(A \cup B)^\circ \supseteq \mathring{A} \cup \mathring{B}$; (ii) $(A \cap B)^\circ = \mathring{A} \cap \mathring{B}$; (iii) $(\mathring{A})^\circ = \mathring{A}$.
(iv) $\overline{A \cup B} = \overline{A} \cup \overline{B}$; (v) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$; (vi) $\overline{\overline{A}} = \overline{A}$;

Proof. Applying (i) and (ii) to $X \backslash A$ and $X \backslash B$ yields (v) and (iv), respectively. The result (iii) holds because $\mathring{A}$ is open, and (vi) holds because $\overline{A}$ is closed. Hence it remains to show (i) and (ii).
(i): $x \in \mathring{A} \cup \mathring{B} \;\Leftrightarrow\;$ either A or B is a neighborhood of $x \Rightarrow\; A \cup B$ is a neighborhood of $x \;\Leftrightarrow\; x \in (A \cup B)^\circ$.
(ii): $x \in (A \cap B)^\circ \;\Leftrightarrow\; A \cap B$ is a neighborhood of $x \;\Leftrightarrow\;$ both A, B are neighborhoods of $x \;\Leftrightarrow\; x \in \mathring{A} \cup \mathring{B}$. □

Remark. The equality does not necessarily holds in (i) and (v). As a counterexample of (i), consider $A = [-1, 0]$ and $B = [0, 1]$ in $X = \mathbb{R}$.

1.2 Density and Separability

Definition 1.14 (Dense sets). Let A be a subset of a topological space X. A subset D of X is said to be *dense* in A if $A \subseteq \overline{D}$. Specifically, D is a *dense set* if $\overline{D} = X$.

Remark. D is also called an *everywhere dense set* when $\overline{D} = X$.

Proposition 1.15. Let A be a subset of a topological space X. Then A is dense if and only if it intersects with every nonempty open set in X.

Proof. Suppose that A intersects with every nonempty open set in X. It suffices to show that for any $x \in X \backslash A$, x is a limit point of A. This is clear because every neighborhood of x, containing a nonempty open set in X, intersects with A. Conversely, let O be a nonempty open set in X, and let A be dense in O. Choose $x \in O$. The conclusion is clear if $x \in A$, so it remains to prove the case $x \notin A$. Since A is dense in O, x is a limit point of A. Hence O as a neighborhood of x contains at least one point of A, which concludes the proof. $\square$

Proposition 1.16. Let A be a dense set in a topological space X. Then for every nonempty open set $O \subseteq X$, $A \cap O$ is dense in O.

Proof. Choose $x \in O$, we want to show that $x \in \overline{A \cap O}$. It suffices to prove the case $x \notin A$. Let N be an arbitrary neighborhood of x. By Proposition 1.3 (ii), $N \cap O$ is a neighborhood of x, and contains at least one point of A. Hence $N \cap (O \cap A) \neq \emptyset$, and x is a limit point of $A \cap O$. $\square$

Definition 1.17 (Basis). A *basis* for a topological space X is a collection $\mathscr{B}$ of open sets such that every open set in X is a union of members of $\mathscr{B}$. Elements of $\mathscr{B}$ are called *basic open sets.*

Theorem 1.18. Let $\mathscr{B}$ be a nonempty collection of subsets of a set X. If the intersection of any finite number of members of $\mathscr{B}$ is always in $\mathscr{B}$, and if $\bigcup_{\beta \in \mathscr{B}} \beta = X$, then $\mathscr{B}$ is a basis for a topology on X.

Proof. Let $\mathscr{T}$ be the collection of all unions of members of β. Then $\mathscr{T}$ forms a topology on X. (To see this, just check the three conditions in Definition 1.1.) $\square$

Definition 1.19 (Second countable space). A topological space is said to be a *second countable* space if it has a countable basis.

Definition 1.20 (Separable space). A topological space is *separable* if it has a countable dense subset.

Theorem 1.21 A second countable topological space is separable.

Proof. Let X be a second countable topological space with a basis $\mathscr{B} = \{B_n : n \in \mathbb{N}\}$, and without loss of generality let each B_n be nonempty since empty sets can be discarded. By Axiom of Choice, there exists a map ϕ defined on $\mathscr{B}$ such that for any $B_n \in \mathscr{B}$, $\phi(B_n) \in B_n$.

Define countable set $A = \{f(B_n) : n \in \mathbb{N}\}$, then X is separable if we can show that A is dense in X. By Proposition 1.15, it remains to show that A intersects with every nonempty set in X. Let O be an arbitrary nonempty open set in X. Then there exists B_n such that $B_n \subseteq O$, and $A \ni \phi(B_n) \in B_n \subseteq O$. $\square$

1.3 Subspace Topology

Definition 1.22 (Subspace topology). Let Y be a non-empty subset of a topological space $(X, \mathscr{T}_X)$. Let $\mathscr{T}_Y = \{O \cap Y : O \in \mathscr{T}_X\}$ be the collection of open sets in Y, then $\mathscr{T}_Y$ defines a topology on Y, which is called the *subspace topology.* The topological space $(Y, \mathscr{T}_Y)$ is also called a *subspace.* Without ambiguity we can drop $\mathscr{T}_Y$ and say Y is a subspace of X.

Proposition 1.23. If Y is a subspace of X, and Z is a subspace of Y, then Z is a subspace of X.

Proof. By definition, we have $Z \subseteq Y \subseteq X$, and $\mathscr{T}_Y = \{O \cap Y : O \in \mathscr{T}_X\}$, $\mathscr{T}_Z = \{O' \cup Z : O' \in \mathscr{T}_Y\}$.
$\forall O' \cap Z \in \mathscr{T}_Z$, $\exists O \in \mathscr{T}_X$ such that $O' \cap Z = (O \cap Y) \cap Z = O \cap Z$.
And $\forall O \in \mathscr{T}_X$, we have $O \cap Z = O \cap (Y \cap Z) = O' \cap Z \in \mathscr{T}_Z$, where $O' := O \cap Y \in \mathscr{T}_Y$.
Hence $\mathscr{T}_Z = \{O \cup Z : O \in \mathscr{T}_X\}$, and Z is a subspace of X. $\square$

Proposition 1.24 (Closed sets in a subspace). Let Y be a subspace of X. Then a subset of Y is closed if and only if it is the intersection of Y with a closed set in X.

Proof. For the "if" statement, let $K = B \cap Y$, where B is closed in X. Then $Y \backslash K = Y \backslash B = Y \cap (X \backslash B)$ is open in Y, because $X \backslash B$ is open in X. Therefore K is closed in Y.

For the "only if" statement, suppose K is closed in Y. Then we know that $Y \backslash K$ is open in Y, and $\exists$ open O in X such that $Y \backslash K = O \cap Y$. Hence $(X \backslash O) \cap Y = Y \backslash (O \cap Y) = K$, which concludes the proof. $\square$

Lemma 1.25 (Open and closed subspaces). Let Y be a subspace of X such that Y is open (closed) in X, and A be a subset of Y. Then A is open (closed) in Y if and only if A is open (closed) in X.

Proof. By Definition 1.22 and Proposition 1.24. $\square$

Proposition 1.26 (Closures and interiors in a subspace). Let Y be a subspace of X, and A be a subset of Y. Denoted by $\overline{A}_X$ and $\overline{A}_Y$ the closure of A in X and in Y, respectively, and similarly $\mathring{A}_X$ and $\mathring{A}_Y$ the interiors. Then: (i) $\overline{A}_Y = \overline{A}_X \cap Y$, (ii) $\mathring{A}_Y \supseteq \mathring{A}_X$.

Proof. (i) Let $\{B_\lambda : \lambda \in \Lambda\}$ be the collection of all closed subsets of X that contains A. By Proposition 1.24, $\{B_\lambda \cap Y : \lambda \in \Lambda\}$ is the collection of all closed subsets of Y that contains A, and (i) follows from Theorem 1.8.
(ii) We only show the case $\mathring{A}_X \neq \emptyset$. If $a \in \mathring{A}_X$, then $\exists$ an open set O in X such that $a \in O \subseteq \mathring{A}_X$. Since $O \cap Y$ is open in Y, and we have $a \in O \cap Y \subseteq A$, A is a neighborhood of a in subspace Y. Hence $a \in \mathring{A}_Y$. $\square$

Remark. In Proposition 1.26, the equality in (ii) does not necessarily holds. As a counterexample, consider Euclidean spaces $X = \mathbb{R}^2$ and $Y = \mathbb{R}$, $A = (0, 1) \subseteq Y$. Then $\mathring{A}_X = \emptyset$, $\mathring{A}_Y = A$.

2 Continuity

2.1 Continuous Functions

Definition 2.1 (Continuous functions). Let X and Y be two topological spaces. A function $f : X \to Y$ is said to be *continuous* if the inverse image of each open set in Y is open in X, i.e., for each open set $O \subseteq Y$, the inverse image $f^{-1}(O) := \{x \in X : f(x) \in O\}$ is open in X.

Proposition 2.2 (Neighborhood characterization of continuity). Let X and Y be two topological spaces. A function $f : X \to Y$ is continuous if and only if for each point $x \in X$ and each neighborhood N of $f(x)$ in Y, the inverse image $f^{-1}(N)$ is a neighborhood of x in X.

Proof. *"If" part:* Let O be an open set in Y. Then for every $x \in f^{-1}(O)$, O is a neighborhood of $f(x)$ in Y. By our assumption, $f^{-1}(O)$ is a neighborhood of x in X.
"Only if" part: Let $x \in X$ and N be a neighborhood of $f(x)$ in Y. Then $\exists$ an open set O such that $f(x) \in O \subseteq N$. Since $f^{-1}(O)$ is open, $f^{-1}(N) \supseteq f^{-1}(O) \ni x$ is a neighborhood of x in X. $\square$

Remark. In some literature, Proposition 2.2 is also used as the definition of continuous functions.

Theorem 2.3. The composition of two continuous functions is continuous.

Proof. Let X, Y, Z be topological spaces, and $f : X \to Y,\ g : Y \to Z$ be continuous functions. Let O be an open set in Z, then $g^{-1}(O)$ is open in Y, and $f^{-1}g^{-1}(O)$ is open in X. Since $(g \circ f)^{-1}(O) = f^{-1}g^{-1}(O)$, we conclude that $g \circ f : X \to Z$ is continuous. $\square$

Theorem 2.4. Let X and Y be two topological spaces, and $f : X \to Y$ be a continuous function. Let $A \subset X$ have the subspace topology. Then the restriction $f|_A : A \to Y$ is continuous.

Proof. Let O be an open set in Y, then $f^{-1}(O)$ is open in X, and $(f|_A)^{-1}(O) = A \cap f^{-1}(O)$ is open in the subspace topology on A. Then $f|_A$ is continuous. $\square$

Remark. The function from X to X which sends each point $x \in X$ to itself is called the *identity map*, denoted by I_X. If we restrict I_X to a subspace A of X, we obtain the *inclusion map*, denoted by $\iota : A \to X$.

Theorem 2.5. Let X and Y be two topological spaces. The following statements are equivalent:

(i) $f : X \to Y$ is continuous.
(ii) If $\mathscr{B}$ is a basis for the topology of Y, then the inverse image of every member of $\mathscr{B}$ is open in X.
(iii) $f(\overline{A}) \subseteq \overline{f(A)}$ for any subset A of X.
(iv) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for any subset B of Y.
(v) The inverse image of each closed set in Y is closed in X.

Proof. (i) $\Rightarrow$ (ii): By Definition 2.1.
(ii) $\Rightarrow$ (iii): Let A be a subset of X. Since $f(A) \subseteq \overline{f(A)}$, it suffices to show $f(x)$ is a limit point of $f(A)$ for $x \in \overline{A} \backslash A$ such that $f(x) \notin f(A)$. If N is a neighborhood of $f(x)$ in Y, then $\exists B \in \mathscr{B}$ such that $f(x) \in B \subseteq N$, and $f^{-1}(B)$ is an open neighborhood of x. Since x is a limit point of A, $f^{-1}(B)$ contains at least one point in A. As a result B and N both contain at least on point in $f(A)$, which concludes the proof.
(iii) $\Rightarrow$ (iv): Let $A = f^{-1}(B)$ in (iii).
(iv) $\Rightarrow$ (v): Let B be a closed set in Y. Then $B = \overline{B}$, and by (iv) $f^{-1}(B) \subseteq \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) = f^{-1}(B)$.
(v) $\Rightarrow$ (i): Let O be an open set in Y. Then by (iv) $f^{-1}(Y \backslash O) = X \backslash f^{-1}(O)$ is closed, and $f^{-1}(O)$ is open. $\square$

Definition 2.6 (Homeomorphism). Let X and Y be two topological spaces. A *homeomorphism* is a function $h : X \to Y$ that is continuous, bijective and that has continuous inverse.

Remark. In Definition 2.6, the condition of continuous inverse is required. Consider function $f : [0,1) \to \{z \in \mathbb{C} : |z| = 1\}$, $x \mapsto \mathrm{e}^{\mathrm{i}2\pi x}$, which is continuous and bijective. The inverse $f^{-1} : \{z \in \mathbb{C} : |z| = 1\} \to [0,1)$, $z \mapsto \frac{1}{2\pi} \arg z$ is not continuous. For example, it maps $\left\{\mathrm{e}^{\mathrm{i}\theta} : \theta \in [0,\pi)\right\}$ to $[0,1/2)$, the inverse image of an open set is not open!

2.2 Metric spaces and Tietze Extension Theorem

Definition 2.7 (Metric). Let X be a nonempty set. A *metric* on X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$ the following conditions are satisfied:
(i) $d(x,y) \geq 0$, and $d(x,y) = 0$ holds if and only if $x = y$;
(ii) $d(x,y) = d(y,x)$; (iii) $d(x,z) + d(z,y) \geq d(x,y)$.

A set X together with a metric d is called a *metric space*, denoted by (X,d). Without ambiguity, we drop d and say X is a metric space.

Given a metric d on a set X, we let $O(x,\epsilon) := \{y : d(x,y) < \epsilon\}$ the *open ball* centered at x of radius $\epsilon > 0$. Then a topology can be induced as follows: a subset U of X is open, if for each $x \in U$, there exists $\epsilon > 0$ such that $O(x,\epsilon)$ is contained in U. This topology satisfies the axioms in Definition 1.1.

Remark. We can check that Definition 2.1 is consistent with the definition of continuity in metric spaces. Let (X, d_X) and (Y, d_Y) be two metric spaces, and $f : X \to Y$ be a function. Suppose $\forall \epsilon > 0$ and $\forall x \in X$, $\exists \delta > 0$ such that $\{x' \in X : d_X(x,x') < \delta\} \subseteq \{x' \in X : d_Y(f(x), f(x')) < \epsilon\}$. Then for every neighborhood N of x, there exists $\epsilon > 0$ such that the open ball $O_Y(f(x),\epsilon)$ is contained in N. By our assumption, $\exists \delta > 0$ such that $O_X(x,\delta) \subseteq f^{-1}(N)$, hence $f^{-1}(N)$ is a neighborhood of x, which implies the continuity of f.

Conversely, suppose f is continuous. Then for any $\epsilon > 0$ and $x \in X$, $f^{-1}\left(O_Y(f(x),\epsilon)\right)$ must be a neighborhood of x in X. As a result, there exists $\delta > 0$ such that $O_X(x,\delta) \subseteq \{x' \in X : d_Y(f(x), f(x')) < \epsilon\}$.

Lemma 2.8. Let A be a subset in a metric space (X,d). Define $d(x,A) = \inf_{y \in A} d(x,y)$. Then the function $x \mapsto d(x,A)$ is continuous on X.

Proof. Let $x \in X$ and let N be a neighborhood of $d(x,A)$ on the real line. Choose a small $\epsilon > 0$ such that $(d(x,A) - \epsilon, d(x,A) + \epsilon) \subseteq N$, and $a \in A$ such that $d(x,a) < d(x,A) + \epsilon/2$. For $z \in O(x,\epsilon/2)$, we have

$$d(z,A) \leq d(z,a) \leq d(z,x) + d(x,a) < d(x,A) + \epsilon.$$

Similarly, we have $d(x,A) < d(z,A) + \epsilon$. Hence $O(x,\epsilon/2)$ is mapped inside $(d(x,A) - \epsilon, d(x,A) + \epsilon) \subseteq N$, and the inverse image of N is a neighborhood of N. Following Proposition 2.2 completes the proof. ☐

Lemma 2.9. Following Lemma 2.8, $d(x,A) = 0$ if and only if $x \in \overline{A}$.

Proof. For the "if" statement, it suffices to show the case $x \in \overline{A} \backslash A$, which implies that x is a limit point of A, and $O(x,\epsilon) \cap A \neq \emptyset$ for all $\epsilon > 0$. Hence $0 \leq d(x,A) = \inf_{y \in A} d(x,y) < \epsilon$ for all $\epsilon > 0$, and $d(x,A) = 0$.

For the "only if" statement, suppose x is neither a point nor a limit point of A. Then there exists $\epsilon > 0$ such that $O(x,\epsilon) \cap A = \emptyset$. As a result, $d(x,A) \geq \epsilon > 0$. Hence $d(x,A) = 0$ only if $x \in \overline{A}$. ☐

Corollary 2.10. Following Lemma 2.8, $d(x,A) = d(x,\overline{A})$ for all $x \in X$.

Proof. Fix $x \in X$. Since $A \subseteq \overline{A}$, it suffices to show $d(x,A) \leq d(x,\overline{A})$. For all $z \in \overline{A}$, by Lemma 2.9, we have $d(x,A) \leq d(x,z) + d(z,A) = d(x,z)$, which completes the proof. ☐

Lemma 2.11 Let A and B be two disjoint closed subsets in a metric space (X, d). Then there exists a continuous $\mathbb{R}$-valued function on X such that $f(A) = \{1\}, f(B) = \{-1\}$ and $f(X\backslash(A \cup B)) = (-1, 1)$.

Proof. Since A and B are closed and disjoint, Lemma 2.9 implies that $d(x, A) + d(x, B) > 0$ for all $x \in X$. Hence we can define

$$f(x) = \frac{d(x, A) - d(x, B)}{d(x, A) + d(x, B)}, \ x \in X$$

which takes on the required values. Moreover, the continuity of f follows from Lemma 2.8. □

Let A be a subspace of topological space X. Given a continuous function $f : A \to \mathbb{R}$, we are interested if we are able to extend f to the whole space X without damage its continuity. More explicitly, we want to find a continuous $\mathbb{R}$-valued function on X such that its restriction on A is f.

Theorem 2.12 (Tietze extension theorem). Any real-valued continuous function defined on a closed subset of a metric space can be extended over the whole space.

Proof. Let A be a closed subset in a metric space X, and $f : A \to \mathbb{R}$ a continuous function.

Step I. Prove the case for bounded f, i.e. $M = \sup_{x \in A} |f(x)| < \infty$.

Let $U_1 = \{x \in A : f(x) \geq M/3\}$ and $L_1 = \{x \in A : f(x) \leq -M/3\}$. By Lemma 1.25, U_1 and L_1 are disjoint closed subsets of A in X. By Lemma 2.11, we can find a continuous function $g_1 : X \to [-M/3, M/3]$ such that g_1 takes $M/3$ on U_1, $-M/3$ on L_1, and values in $(-M/3, M/3)$ on $X\backslash(U_1 \cup L_1)$.

Now consider function $f - g_1 : A \to [-2M/3, 2M/3]$. Let $U_2 = \{x \in A : f(x) \geq 2M/9\}$ and $L_2 = \{x \in A : f(x) \leq -2M/9\}$. Applying Lemmas 1.25 and 2.11, we can find a continuous function $g_2 : X \to [-2M/9, 2M/9]$ such that g_2 takes $2M/9$ on U_1, $-2M/9$ on L_1, and values in $(-2M/9, 2M/9)$ on $X\backslash(U_2 \cup L_2)$.

Repeat this procedure, we obtain a sequence of functions $g_n : X \mapsto [-2^{n-1}M/3^n, 2^{n-1}M/3^n]$ such that

- $f - g_1 - \cdots - g_n$ takes values in $[-2^n M/3^n, 2^n M/3^n]$ on A, and
- $|g_n(x)| < 2^{n-1}M/3^n$ on $X\backslash A$.

The series $\sum_{n=1}^{\infty} g_n$ converges uniformly on X by Weierstrass M-test, and we denote it by g. Then g and f agree on A, and g is uniformly bounded by $M \sum_{n=1}^{\infty} 2^{n-1}/3^n = M$. Hence g extends f to the whole space X.

Step II. Prove the case for general continuous f. Choose a homeomorphism $h : \mathbb{R} \mapsto (-1, 1)$ and consider the composition $h \circ f$, which is bounded. Then we extend it to a continuous $\mathbb{R}$-valued function $g : X \to [-1, 1]$ as in Step I. Since g is uniformly bounded by 1, $h^{-1} \circ g$ is well-defined and continuous, and by construction it extends f over X. Thus we complete the proof. □

3 Connectedness

3.1 Connected Spaces

Definition 3.1 (Connected space). A topological space X is connected if it cannot be decomposed as the union of two disjoint nonempty open sets.

By saying a subset Y of X is connected, we mean that Y is connected in its subspace topology.

Proposition 3.2 (Alternative definition of connectedness). The following statements are equivalent:

(i) X is a connected space.

(ii) Any decomposition $X = A \cup B$ of nonempty subsets of X satisfies $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$.

(iii) X cannot be decomposed as the union of two disjoint nonempty closed sets.

(iv) The only sets that are both open and closed in X are $\emptyset$ and X itself.

(v) There exists no onto continuous function from X to a discrete space that contains more than one points.

Proof. (i) $\Rightarrow$ (ii): Assume there exist nonempty subsets A and B of X such that $X = A \cup B$, $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. Then $X = A \cup B \subseteq \overline{A} \cup B = X$, hence $B = X \backslash \overline{A}$ is open in X. Similarly A is open in X. Hence X is the union of two disjoint nonempty open sets A and B, a contradiction!

(ii) $\Rightarrow$ (iii): Assume there exist two nonempty closed sets A and B such that $A \cap B = \emptyset$, $A \cup B = X$, then $A = \overline{A}$ and $B = \overline{B}$, which contradicts (ii).

(iii) $\Rightarrow$ (iv): If there exists a both open and closed subset A in X such that $A \neq X$ and $A \neq \emptyset$, then $X = A \cup (X \backslash A)$ is a decomposition of two disjoint nonempty closed sets.

(iv) $\Rightarrow$ (i): If $X = A \cup B$ is a decomposition of two disjoint nonempty open sets, then A must be a both open and closed in X such that $A \neq \emptyset$ and $A \neq X$.

(i) $\Rightarrow$ (v): Let Y be a discrete space with more than one point and $f : X \to Y$ an onto continuous function. Break up Y as a union $U \cup V$ of two disjoint nonempty open sets. Then $X = (f^{-1}U) \cup (f^{-1}V)$.

(v) $\Rightarrow$ (ii): Assume there exist two nonempty sets A and B such that $A \cap B = \emptyset$, $A \cup B = X$, then both $A = X \backslash \overline{B}$ and $B = X \backslash \overline{A}$ are open. Define $f = \mathbb{1}_A - \mathbb{1}_B : X \to \{-1, 1\}$, then f is continuous and onto. $\square$

Theorem 3.3 (Connectedness of the real line). The real line $\mathbb{R}$ is a connected space.

Proof. We argue that $\mathbb{R}$ by checking the condition (ii) in Proposition 3.2. Let $\mathbb{R} = A \cup B$ be a partition of $\mathbb{R}$, i.e. A and B are nonempty, and $A \cap B = \emptyset$. Choose $a \in A, b \in B$, and without loss of generality suppose $a < b$. Then $\{x \in A : x < b\}$ is nonempty. Let $s = \sup\{x \in A : x < b\}$. By the very definition of supremum, we have $s \in \overline{A}$. If $s \notin B$, then $s \in \mathbb{R} \backslash B = A$, and $s < b$. Moreover, $(s, b]$ lies in B, then s is a limit point of B, showing $s \in \overline{B}$. Therefore s lies either in $\overline{A} \cup B$ or in $A \cup \overline{B}$. $\square$

Theorem 3.4. (Connected subsets of the real line). A nonempty subset of $\mathbb{R}$ is connected if and only if it is an interval. (Note that any single point $a \in \mathbb{R}$ is also an interval $[a, a]$.)

Proof. Akin to the proof of Theorem 3.3, we can show that any interval is connected. If a nonempty set A is not an interval, then we can find $a < p < b$ such that $p \notin A$ and $a, b \in A$. Let $B = \{x \in A : x < p\}$, then both B and $A \backslash B$ are nonempty. Since $p \notin A$, we have $B \in (-\infty, p)$ and $A \backslash B \in (p, \infty)$. Hence $\overline{B} \cup (A \backslash B) = B \cup \overline{(A \backslash B)} = \emptyset$, showing A is not connected. $\square$

Theorem 3.5 (Connected dense set). Let X be a topological space and let Y be a subspace of X. If Y is connected, and if Y is dense in X, then X is connected.

Proof. Let A be a nonempty subset of X which is both open and closed. Since Y is dense in X, Y intersects every nonempty open subset of X, $A \cap Y$ is nonempty. Note that $A \cap Y$ is both open and closed in Y, and Y is connected, we have $A \cap Y = Y$, i.e., $Y \subseteq A$. Therefore $X = \overline{Y} \subseteq \overline{A} = A \subseteq X$, meaning $A = X$. $\square$

Corollary 3.6. Let X be a topological space and let Y be a connected subspace of X. If $Y \subseteq Z \subseteq \overline{Y}$, then Z is connected. Particularly, the closure $\overline{Y}$ of a connected subspace Y is connected.

Proof. By Proposition 1.26 (i), Y is dense in Z. Applying Theorem 3.5 yields the wanted result. $\square$

Lemma 3.7. If topological space $X = A \cup B$, where A and B are disjoint open sets, and if Y is a connected subspace of X, then Y lies entirely within either A or B.

Proof. We observe that both $A \cap Y$ and $B \cap Y$ are open sets in Y, and they forms a partition of Y. Since Y is connected, at least one of them should be empty. $\square$

Theorem 3.8 (Union of connected subspaces). Let $\mathscr{X} = \{X_\alpha, \alpha \in J\}$ be a collection of connected subspaces of X such that $\bigcap_{\alpha \in J} X_\alpha \neq \emptyset$. Then $\bigcup_{\alpha \in J} X_\alpha$ is connected.

Proof. Let $p \in \bigcap_{\alpha \in J} X_\alpha$, and $Y := \bigcup_{\alpha \in J} X_\alpha = A \cup B$, where A and B are disjoint open sets in Y. Then p is in one of A and B. Without loss of generality, let $p \in A$. For each $\alpha \in J$, $X_\alpha \ni p \in A$. Since X_α is connected, by Lemma 3.7, $X_\alpha \subseteq A$. Hence $A = Y$ and $B = \emptyset$. $\square$

Now we introduce the concepts of product space and box topology. Let $\mathscr{X} = \{X_\alpha, \alpha \in J\}$ be a collection of indexed sets. The *cartesian product* of this indexed collection, denoted by $\prod_{\alpha \in J} X_\alpha$, is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ such that $x_\alpha \in X_\alpha$ for each $\alpha \in J$. Equivalently, it is the set of all functions $\mathbf{x} : J \to \bigcup_{a \in J} X_\alpha$ such that $\mathbf{x}(\alpha) \in X_\alpha$ for each $\alpha \in J$.

Definition 3.9 (Box topology on a product space). We take as a basis for a topology on the product space $\prod_{\alpha \in J} X_\alpha$ the collection of the sets of the form $\prod_{\alpha \in J} O_\alpha$, with O_α open in X_α for each $\alpha \in J$. The topology generated by this basis is called the *box topology*.

Remark. To check the basis we choose is valid, we use Theorem 1.18. The first condition is satisfied because $\prod_{\alpha \in J} X_\alpha$ is itself a basis element. The second condition is satisfied because the intersection of any two basis elements is another basis element:

$$\left(\prod_{\alpha \in J} U_\alpha\right) \cap \left(\prod_{\alpha \in J} Y_\alpha\right) = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha).$$

Theorem 3.10. The cartesian product of finitely many connected spaces is connected.

Proof. It suffices to show that the cartesian product of two connected spaces is connected. Let X and Y be two connected topological space. For each $x \in \mathcal{X}$, $\{x\} \times Y$, being homeomorphic to Y, is connected (we will interpret this in Corollary 3.15). Similarly, $X \times \{y\}$ is connected for each $y \in Y$. By Theorem 3.8, the cross-shaped set $C_{x,y} := (\{x\} \times Y) \cup (X \times \{y\})$ is connected. Fix $(x_0, y_0) \in X \times Y$. Then $X \times Y = \bigcup_{y \in Y} C_{x_0,y}$, and $(x_0, y_0) \in \bigcap_{y \in Y} C_{x_0,y}$. Again by Theorem 3.8, the product space $X \times Y$ is connected. $\square$

3.2 Path-connected Spaces

Definition 3.11 (Path-connected space). A topological space X is *path-connected* if for each $a, b \in X$, there exists a *path* in X from a to b, that is, a continuous function $f : [0, 1] \to X$ with $f(0) = a$ and $f(1) = b$.

Lemma 3.12. A path-connected space is connected.

Proof. Let X be a path-connected space. If X is not connected, there exists a partition $X = A \cup B$ such that A and B are disjoint nonempty open sets in X. Choose $a \in A, b \in B$, then there exists a continuous function $f : [0, 1] \mapsto X$ such that $f(0) = a$ and $f(1) = b$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty disjoint open sets in $[0, 1]$ whose unions are $[0, 1]$, contradicting the connectedness of $[0, 1]$. □

Remark. A connected space is not necessarily path-connected. We will give a counterexample afterwards. Theorem 3.13 establishes a relation between connected sets and path-connected sets in Euclidean spaces.

Theorem 3.13. Any connected open set in a euclidean space is path-connected.

Proof. Consider euclidean space $\mathbb{R}^n$. Let X be a connected open set in $\mathbb{R}^n$ and fix $x \in X$. Let $U(x)$ be the set of all points in X that can be joined to x by a path in X. By construction, $U(x)$ is path-connected. It suffices to show $U(x) = X$. Let $y \in U(x)$ and choose an open ball $O(y, \epsilon)$ that lies entirely in X. Then we can join z to x whenever $z \in O(y, \epsilon)$. Hence $O(y, \epsilon) \subseteq U(x)$, and $U(x)$ is open in X. Also, $X \backslash U(x) = \bigcup_{y \in X \backslash U(x)} U(y)$ as the union of a collection of open sets is open, then $U(x)$ is closed. Recall that X is connected, $U(x) = X$. □

Theorem 3.14 (The continuous image of connected/path-connected sets). Let function $f : X \to Y$ be continuous and onto. (i) If X is connected, so is Y; (ii) If X is path-connected, so is Y.

Proof. (i) Let $Y = A \cup B$, where A and B are disjoint open set in Y. Then $X = f^{-1}(A) \cup f^{-1}(B)$, with $f^{-1}(A)$ and $f^{-1}(B)$ being disjoint and open in X. Since X is connected, one of $f^{-1}(A)$ and $f^{-1}(B)$ is empty. f is onto, hence one of A and B is empty.

(ii) For each $a, b \in Y$, choose $u \in f^{-1}(\{a\})$ and $v \in f^{-1}(\{b\})$, whose nonemptiness is ensured by the surjectivity of f. Since X is path-connected, we can find a path g in X from u to v. Since the composition preserves continuity, $f \circ g$ is a path in Y from a to b. □

Theorem 3.14 immediately implies the following conclusion.

Corollary 3.15. If $h : X \to Y$ is a homeomorphism, then X is connected (path-connected) if and only if Y is connected (path-connected). In other words, connectedness (path-connectedness) is a topological property.

3.3 Local Connectedness and Local Path-connectedness

Definition 3.16 (Locally connected sets and locally path-connected sets). A topological space X is said to be *locally connected at x* if for every neighborhood U of x, there is a connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is said simply to be *locally connected*.

Similarly, a topological space X is said to be *locally path-connected at x* if for every neighborhood U of x, there is a path-connected neighborhood V of x contained in U. If X is locally path-connected at each of its points, then it is said to be *locally path-connected*.

To characterize local connectedness and local path-connectedness, we need to introduce the concepts of components and path components.

Definition 3.17 (Components and path components). Let X be a topological space, and define an equivalence relation on X by letting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence classes are called the *(connected) components* of X.

Define another equivalence relation on X by letting $x \sim y$ if there is a path in X joining x and y. The equivalence classes are called the *path components* of X.

Remark. We need to verify the validity of the equivalence relations we define. For the first statement, the symmetry and reflexivity is clear, and the transitivity follows from Theorem 3.8.

For the second statement, the symmetry holds because when $f : [0,1] \to X$ is a path from x to y then $g : t \mapsto f(1-t)$ is a path from y to x, and the reflexivity follows from the continuity of constant functions. For the transitivity, suppose $f : [0,1] \to X$ is a path from x to y, and $g : [0,1] \to X$ a path from y to z. Then we can construct a path from x to z by $h : t \mapsto f(2t)\mathbb{1}_{\{0 \le t \le 1/2\}} + g(2t-1)\mathbb{1}_{\{1/2 < t \le 1\}}$.

Theorem 3.18 (Component decomposition). The components of X are disjoint connected subspaces of X whose union is X, and each nonempty connected subspace of X lies in one of them.

The path-components of X are disjoint path-connected subspaces of X whose union is X, and each nonempty path-connected subspace of X lies in one of them.

Proof. We first prove the first statement. Being equivalence classes, the components of X are disjoint and their union is X. For each connected subspace A of X, if there exists $p_1, p_2 \in A$ such that $p_1 \in C_1$ and $p_2 \in C_2$, where both C_1 and C_2 are components of X, then $C_1 = C_2$ because $p_1 \sim p_2$. Hence A intersects with only one component of X, and it must lie entirely in that component.

It remains to show that each component C is connected. To argue this, choose $x_0 \in C$, for each $x \in C$, $x \sim x_0$, and there exists a connected subspace containing x_0 and x. By the result just proved, $A_x \subseteq C$, and $C = \bigcup_{x \in C} A_x$ is connected by Theorem 3.8.

For the second statements, we make a slight modification on Theorem 3.8: for a collection of path-connected subspaces $\{X_\alpha, \alpha \in J\}$ in X, if $\exists p \in \bigcap_{\alpha \in J} X_\alpha$, then we can construct a path between any two points in $\bigcup_{\alpha \in J} X_\alpha$ that meets p. $\square$

Remark. By Theorem 3.18, we can set that the components (path-components) are the collection of maximal connected (path-connected) subspaces of a topological space.

Theorem 3.19. A topological space X is locally connected if and only if for every open set U of X, each component of U is open in X. Similarly, X is locally path-connected if and only if for every open set U of X, each path component of U is open in X.

Proof. We only prove the first statement, since the proof of the second is parallel. Suppose X is locally connected and U is an open set in X. If C is a component of U and $x \in C$, then we can choose some $V \subseteq U$ such that V is a connected neighborhood of x. By Theorem 3.18, $V \subseteq C$, and C is open in X.

Conversely, given $x \in X$ and a neighborhood U of x (without loss of generality suppose it is open), let C be the component of U that contains x. Then C is a connected neighborhood of x if C is open. Since $C \subseteq U$, X is locally connected at x. $\square$

Theorem 3.20. Let X be a topological space. Then every path component of X lies in a component of X. If X is locally path-connected, then the components and the path components of X are the same.

Proof. Let C be a component of X and $x \in C$. Suppose P is the path component containing x. Since P is connected, $P \subseteq C$. It remains to show $P = C$ if X is locally path-connected.

Assume $P \subsetneq C$. Denote by Q the union of all the path components of X that is different from P and meets C. Then $C = P \cup Q$. By Theorem 3.19, each path component of X is open in X. Then P and Q are disjoint nonempty open sets whose union is C, contradicting the connectedness of C. $\square$

The following statements immediately follows from Theorem 3.20.

Corollary 3.21. If a topological space X is connected and locally path-connected, then X is path-connected.

Remark. As an end of this section, let's see an example of connected space that is not path-connected. Consider the following closed set in euclidean space $\mathbb{R}^2$:

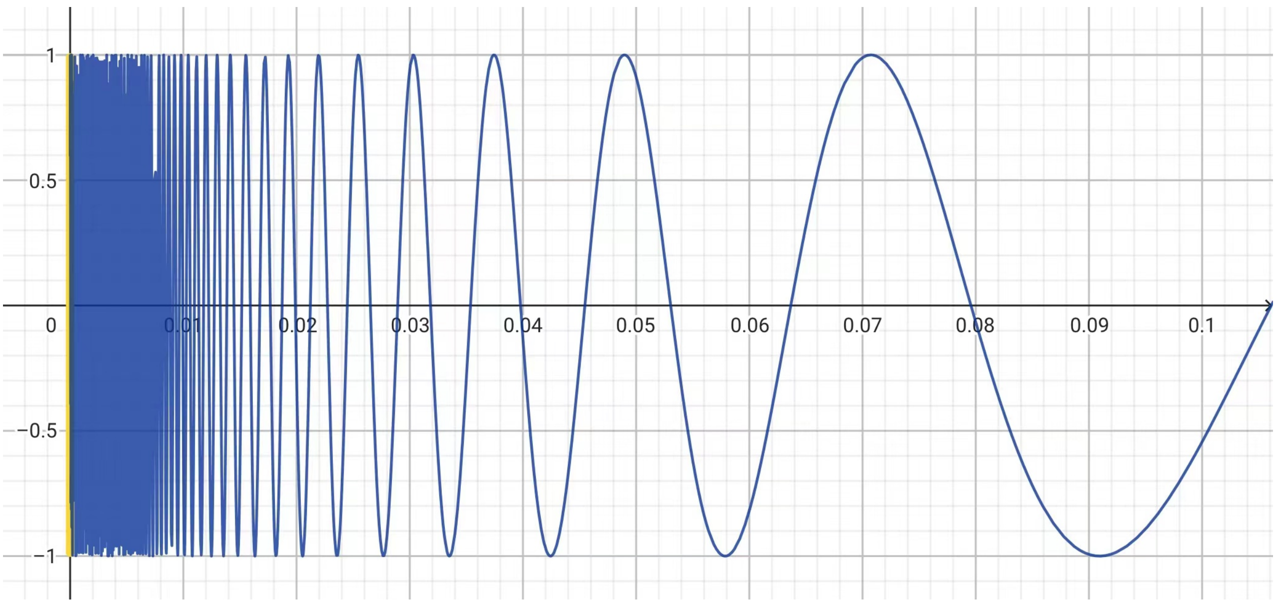
$$A = \overline{\left\{(x, y) \in \mathbb{R}^2 : y = \sin\frac{1}{x}, x > 0\right\}}$$

The line segment $L = \{(0, y) : -1 \leq y \leq 1\}$ lies in A.

Consider the function $f : \mathbb{R} \to \mathbb{R}^2, x \mapsto (x, \sin\frac{1}{x})$, which is continuous. Then $f((0, \infty))$ as the image of a connected set $(0, \infty)$ is connected, and $A = \overline{f((0, \infty))}$ as the closure is also connected.

Let $f : [0, 1] \to X$ be a path starting at a point in L. Then $f^{-1}(L)$ is closed since f is continuous. If we can show $f^{-1}(L)$ is open, then $f^{-1}(L) = [0, 1]$ because $[0, 1]$ is connected and $f^{-1}(L)$ is nonempty. Hence $f([0, 1]) \in L$, and there is no path joining a point in A to a point in B.

Fix $t \in f^{-1}(L)$, and choose an open ball $U = O(f(t), \epsilon)$ in $\mathbb{R}^2$. Then $U \cap A$ has infinitely many path components including $U \cap L$. Since f is continuous, $f^{-1}(U) \ni t$ is an open set in $[0, 1]$. Then there exists an interval $I \subseteq f^{-1}(U)$ such that I is open in $[0, 1]$ and $I \ni t$. Note that I is path-connected, then $f(I)$, being path-connected, lies in $U \cap L$ by Theorem 3.18. Therefore $I \subseteq f^{-1}(L)$, and t is an interior point of $f^{-1}(L)$. Since t is arbitrarily chosen, $f^{-1}(L)$ is an open set in $[0, 1]$, which concludes our proof.



4 Compactness

4.1 Compact Sets

Definition 4.1 (Cover). A collection $\mathscr{A} = \{A_\alpha, \alpha \in J\}$ of subsets of a topological space X is said to be a *cover of* X (or briefly, a *cover*), if $X = \bigcup_{\alpha\in J} A_\alpha$. It is called an *open cover* of X if its elements are open sets in X. A subcollection of a cover $\mathscr{A}$ whose union is equal to X is called a *subcover*.

Definition 4.2 (Compact sets). Let X be a topological space. X is said to be *compact*, if every open cover of X has a finite subcover.

Remark. If Y is a subspace of X, a collection $\mathscr{A} = \{A_\alpha, \alpha \in J\}$ of (open) subsets of X is said to be a cover (an open cover) of Y if the union of its elements contains Y. By definition, we can verify that a subspace Y of X is compact if and only if every open cover of Y contains a finite subcover of Y.

Lemma 4.3. Every closed subspace of a compact space is compact.

Proof. Let X be a compact space and let K be a closed subspace of X. Then for every open cover $\mathscr{A}$ of K, then $\mathscr{B} = \mathscr{A} \cup \{X\backslash K\}$ forms an open cover of X. By the very definition of compactness, $\mathscr{B}$ contains a finite subcover $\mathscr{B}'$ of X. Since $X\backslash K$ does not intersect K, we can remove $X\backslash K$ from $\mathscr{B}'$ if required, resulting in a finite subcover $\mathscr{A}' \subset \mathscr{A}$ of K. $\square$

Theorem 4.4 (Continuity and compactness). The continuous image of a compact space is compact.

Proof. Let X be a compact space and let $f : X \to Y$ be a continuous function. Let $\mathscr{A}$ be an open cover of $f(X)$ in Y. By continuity, $\{f^{-1}(A) : A \in \mathscr{A}\}$ is an open cover of X, from which we can find finite many $f^{-1}(A_1), \cdots, f^{-1}(A_n)$ that cover X. Then $A_1, \cdots, A_n \in \mathscr{A}$ form a finite subcover of $f(X)$. $\square$

Now we investigate the compactness of product spaces.

Lemma 4.5 (Tube lemma). Let X and Y be two topological spaces, and let Y be compact. For every $x \in X$, if an open set O in $X \times Y$ contains $\{x\} \times Y$, then there exists a neighborhood U_x of x such that $U_x \times Y \subseteq O$.

Proof. Fix $x \in X$, and let O be an open set in $X \times Y$ containing slice $\{x\} \times Y$. By the property of box topology, we can find an collection of basis sets in $X \times Y$ whose elements are in form of $U \times V$, where U and V are open sets in X and Y, respectively, and whose union is $X \times Y$. Since the space $\{x\} \times Y$ is compact (because Y is compact, and $\{x\}$ itself is open in the subspace topology), then we can cover it with finitely many basis sets $U_1 \times V_1, \cdots, U_n \times V_n$.

Let $U_x = \bigcap_{j=1}^n U_j$, then U_x is an open neighborhood of x in X. Then for each $(x', y') \in U_x \times Y$, y' must lie in some V_j and x' lies in $U_x \subseteq U_j$. Hence $U_x \times Y$ is contained in the union of the basis set, which is O. $\square$

Theorem 4.6 (Product of compact spaces). The product of finitely many compact spaces is compact.

Proof. It suffices to show the product of two compact spaces X and Y is compact. Let $\mathscr{A}$ be an open cover of $X \times Y$. Then for each $x \in X$, the slice $\{x\} \times Y$ is compact, and there exist finitely many $A_1^x, \cdots, A_{n_x}^x \in \mathscr{A}$ with $\bigcup_{j=1}^{n_x} A_{n_x}^x \supseteq \{x\} \times Y$. By Lemma 4.5, there exists an open neighborhood U_x of x such that $U_x \times Y$ is covered by finitely many elements of $\mathscr{A}$. Noticing that $\{U_x, x \in X\}$ is an open cover of compact space X, there exists finitely many $U_x \times Y$ that covers $X \times Y$, with each $U_x \times Y$ covered by finitely many elements of $\mathscr{A}$. Then $X \times Y$ is covered by finitely many elements of $\mathscr{A}$. $\square$

Lemma 4.7 (Projection). For two topological spaces X and Y, define $\pi_1 : X \times Y \to X, (x, y) \mapsto x$.

(i) π_1 is an open map, that is, it carries open sets to open sets.

(ii) If Y is compact, then π_1 is a closed map, that is, it carries closed sets to closed sets.

Proof. (i) Let O be an open set in $X \times Y$. Then for any $x \in \pi_1(O)$, we can find some $(x, y) \in O$. By the property of box topology, we can find a basis set $U \times V$ such that $(x, y) \in U \times V \subseteq O$, where U and V are open sets in X and Y, respectively. As a result, $x \in U \subseteq \pi_1(O)$.

(ii) Let C be a closed sets in $X \times Y$, where Y is compact. We are about to show $X \backslash \pi_1(C)$ is open. Take $x \notin \pi_1(C)$. The slice $\{x\} \times Y$ is disjoint from C. Since Y is compact, by Lemma 4.5, there exists a neighborhood $U_x \ni x$ such that $U_x \times Y \subseteq (X \times Y) \backslash C$. Therefore U_x is a neighborhood of x which is disjoint from $\pi_1(C)$, completing the proof. □

Now we investigate the compact sets in euclidean spaces.

Theorem 4.8. A closed interval $[a, b]$ is compact.

Proof. The case $a = b$ is trivial, so we may assume $a < b$.

Step I: Let $\mathscr{A}$ be an open cover of $[a, b]$. We first prove that if $x \in [a, b] \backslash \{b\}$, then $\exists y > x$ of $[a, b]$ such that $[x, y]$ can be covered by at finitely many elements of $\mathscr{A}$. Choose $A \in \mathscr{A}$ such that $A \ni x$. Since $x \neq b$ and A is open, A contains an interval of the form $[a, c)$ for some $c \in [a, b]$. Choose $y \in (x, c)$, then $[x, y]$ is covered by a single element of $\mathscr{A}$.

Step II: Let C be the set of all points $y > a$ of $[a, b]$ such that $[a, y]$ can be covered by finitely many elements of $\mathscr{A}$. By our conclusion in Step I, C is nonempty. Let c be the **least upper bound** of C, then $a < c \leq b$. We show that $c \in C$. Choose $B \in \mathscr{A}$ such that $B \ni c$. B is open, so it contains an interval of the form $(d, c]$ for some $d \in [a, b]$. If $c \notin C$, then there exists $z \in C$ lying in (d, c), otherwise d would be an upper bound of C smaller than c. Since $z \in C$, $[a, z]$ is able to be covered by finitely many elements of $\mathscr{A}$, so is $[a, z] \cup [z, c]$, contradicting $c \notin C$!

Step III: It remains to show $c = b$, which completes our proof. Assume $c < b$, then applying Step I can we find some $y > c$ in $[a, b]$ such that $[c, y]$ is covered by finitely many elements of $\mathscr{A}$. So is $[a, y] = [a, c] \cup [c, y]$. However this means $C \ni y > c$, another contradiction! □

Theorem 4.9 (Heine-Borel). A subspace of an euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. *"If" part:* Provided K is bounded in $\mathbb{R}^n$, we can find a cell $[-b, b]^n \supseteq K$, which is compact by Theorems 4.6 and 4.8. By Lemma 4.3, if K is closed, then it is compact.

"Only if" part: Suppose K is compact. The collection of centered open balls $\{O(0, n), n = 1, 2, \cdots\}$ is a cover of $\mathbb{R}^n$, so there exist finitely many balls that cover K. Then K must be contained in an open ball of finite radius, and it suffices to show that K is closed.

Argue by contraction. Assume x is a limit point of K that is not contained in K. Then we can construct an open cover $\left\{\mathbb{R} \backslash \overline{O(x, n^{-1})}, n = 1, 2, \cdots\right\}$ of K. Since x is a limit point, any $O(x, n^{-1})$ contains at least one point in K, and we cannot find a finite subcover of K from our construction. □

Theorem 4.10 (Extreme value theorem). Let $X \to \mathbb{R}$ be a continuous function. If X is compact, then there exists points $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for every $x \in X$.

Proof. By the continuity of f, the image $f(X)$ is compact in $\mathbb{R}$. Then it suffices to show that $f(X)$ has a largest element M and a smallest element m. Argue by contradiction. Suppose $f(X)$ has no largest element. Then $\{(-\infty, a) : a \in f(X)\}$ is an open cover of $f(X)$ because for every $x \in X$ there exists $a > x$ in X. However, every finite subcollection $\{(-\infty, a_j) : j = 1, \cdots, n\}$ does not cover X because there exists $b > \max_{j=1,\cdots,n} a_j$ in X, contradicting the compactness of $f(X)$. Similarly we can prove that $f(X)$ has a smallest element. $\square$

Now we introduce the concept of uniform continuity.

Definition 4.11 (Uniform continuity). Let (X, d_X) and (Y, d_Y) be two metric spaces. A function $f : X \to Y$ is said to be *uniformly continuous*, if given any $\epsilon > 0$, $\exists \delta > 0$ such that for every $x, x' \in X$, $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \epsilon$.

By definition, a uniformly continuous function must be continuous, but the converse is not true: in the definition of uniformly continuity, the choice of δ only depends on ϵ but not on location x. The following Theorem 4.13 tells us in what case does a continuous function become uniformly continuous. We first introduce a technical lemma. For a bounded subset B of a metric space (X, d), we denote by $D_B = \sup_{x,x' \in B} d(x, x')$ the diameter of B.

Lemma 4.12 (Lebesgue number lemma). Let $\mathscr{A}$ be an open cover of a compact metric space (X, d). Then there exists a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathscr{A}$ that contains it. The number δ is called a *Lebesgue number* for the cover $\mathscr{A}$.

Proof. If $X \in \mathscr{A}$, then any positive number is a Lebesgue number of $\mathscr{A}$. So we assume $X \notin \mathscr{A}$. By compactness of X, there exist finite many $A_1, \cdots, A_n \in \mathscr{A}$ whose union contains X, and we set $C_j = X \backslash A_j$ for $j = 1, \cdots, n$. Define $f : X \to \mathbb{R}, x \mapsto \sum_{j=1}^{n} d(x, C_j)$, which is a continuous function by Lemma 2.8. By Theorem 4.10, there exists $x_0 \in X$ such that $f(x) \geq f(x_0) := \delta$ for all $x \in X$. Since $A_1, \cdots, A_n$ is an open cover of X, there exists $\epsilon > 0$ such that the open ball $O(x_0, \epsilon)$ lies in some A_j. Then $d(x_0, C_j) \geq \epsilon$, and $\delta = f(x) \geq \epsilon/n > 0$.

Now we prove δ is a Lebesgue number of $\mathscr{A}$. Let B be a subset of X of diameter less than δ. Choose any $b \in B$, then $O(b, \delta) \supset B$. Let $m \in \operatorname{argmax}_{j=1,\cdots,n} d(b, C_j)$, then $\delta \leq f(b) \leq d(b, C_m)$. Consequently, $B \subset O(b, \delta) \subset X \backslash C_m = A_m$, completing the proof. $\square$

Theorem 4.13 (Uniform continuity theorem). Let $f : X \to Y$ be a continuous function on a compact metric space (X, d_X) to a metric space (Y, d_Y). Then f is uniformly continuous.

Proof. By the continuity of f, the image $f(X)$ is compact in Y. Fix $\epsilon > 0$, then the collection of open balls $\{O_Y(y, \epsilon/2) : y \in f(X)\}$ covers $f(X)$, and there exist finite many open balls $O_Y(y_1, \epsilon/2), \cdots, O_Y(y_n, \epsilon/2)$ that cover $f(X)$. Take δ to be a Lebesgue number of $\{U_j := f^{-1}O_Y(y_j, \epsilon/2), j = 1, \cdots, n\}$, which is an open cover of X. Then for any $d_X(x, x') < \delta$, $\{x, x'\}$ as a point set of diameter less than δ must lie in some U_j, and $d_Y(f(x), f(x')) \leq d_Y(f(x), y_j) + d_Y(y_j, f(x')) < \epsilon$, completing the proof. $\square$

Theorem 4.14 (Closed set criterion for compactness). Let X be a topological space. Then X is compact if and only if for every collection $\mathscr{C}$ of closed sets in X having the finite intersection property, that is, for every finite subcollection $\{C_1, \cdots, C_n\}$ of $\mathscr{C}$, their intersection $\bigcap_{i=1}^{n} C_n$ is nonempty, the intersection $\bigcap_{C \in \mathscr{C}} C$ of all elements of $\mathscr{C}$ is nonempty.

Proof. Given a collection $\mathscr{A}$ of subsets of X, let $\mathscr{C} = \{X \backslash A : A \in \mathscr{A}\}$ be the collection of their complements. Then the following statements hold:

(i) $\mathscr{A}$ is a collection of open sets in X if and only if $\mathscr{C}$ is a collection of closed sets.
(ii) $\mathscr{A}$ covers X if and only if $\bigcap_{C \in \mathscr{C}} C$ is empty.
(iii) A finite subcollection $\{A_1, \cdots, A_n\} \subset \mathscr{A}$ covers X if and only if the intersection of the finite subcollection $\{C_j = X \backslash A_j : j = 1, \cdots, n\} \subset \mathscr{C}$ is empty.

Then we can derive three equivalent characterizations of compactness:

- Given any collection $\mathscr{A}$ of open sets in X, if $\mathscr{A}$ covers X, then there exists a finite subcollection of $\mathscr{A}$ that covers X.
- Given any collection $\mathscr{A}$ of open sets in X, if no finite subcollection of $\mathscr{A}$ covers X, then $\mathscr{A}$ does not cover X.
- Given any collection $\mathscr{C}$ of closed sets in X, if every finite subcollection of $\mathscr{C}$ has nonempty intersection, then $\bigcap_{C\in\mathscr{C}} C$ is nonempty.

The last statement is the condition of our theorem. □

The following corollary immediately follows from Theorem 4.14.

Corollary 4.15 For a *nested sequence* $C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots$ of nonempty closed sets in a compact space X, the intersection $\bigcap_{n=1}^{\infty} C_n$ is nonempty.

4.2 Hausdorff Spaces

Motivation. One's experience with open and closed sets and limit points in euclidean spaces can be misleading when considering general topological space. For example, in an euclidean space, every single point set $\{x_0\}$ is closed because for every $x \neq x_0$ we can find one of its neighborhood $O(x, \epsilon)$ not containing x_0 when ϵ is sufficiently small. However, this property does not hold for arbitrary topological spaces.

We can also consider the properties of convergent sequences. In a topological space X, a sequence $\{x_n\}_{n=1}^{\infty}$ of points is said to *converge* to a point $x_0 \in X$ if for every neighborhood N of x_0, there exists a positive integer N such that $x_n \in N$ for all $n \geq N$. It is clear that x_0 is a limit point of any set that contains $\{x_n\}_{n=1}^{\infty}$. In euclidean spaces a convergent sequence never converges to more than one point.

On a three-point set $\{a, b, c\}$, consider the topology $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. The one-point set $\{b\}$ is not closed, because its complement $\{a, c\}$ is not open. Also, the sequence defined by $\{x_n = b, n = 1, 2, \cdots\}$ converges not only to b, but also to points a and c since their neighborhoods always contain b.

In this section we consider a special class of topological spaces, which enjoys some nice properties.

Definition 4.16 (Hausdorff spaces/T_2 spaces). A topological space X is called a *Hausdorff* space if for each pair of distinct points $x, y \in X$, there exists a neighborhood U of x and a neighborhood V of y such that U and V are disjoint.

Proposition 4.17 (Properties of Hausdorff spaces). Suppose X is a Hausdorff space.

(i) Every finite point set in X is closed;
(ii) A sequence of points of X converges to at most one point of X;
(iii) The product of two Hausdorff spaces X and Y is a Hausdorff space;
(iv) Any subspace of X is a Hausdorff space.

Proof. (i) Fix $x_0 \in X$. For any $x \neq x_0$ in X, we can find two disjoint neighborhoods U and V of x_0 and x, respectively. Since $x \notin U$, $x \notin \overline{\{x_0\}}$. Consequently, $\{x_0\} = \overline{\{x_0\}}$.

(ii) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points of X that converges to $x \in X$. Then for any $x' \in X$ distinct from x, let $U \ni x$ and $V \ni x'$ be their disjoint neighborhoods. Then there exists infinite many elements of $\{x_n\}_{n=1}^{\infty}$ that do not lies in V.

(iii) For any pair of distinct points $(x_1, y_1), (x_2, y_2)$ in $X \times Y$, we have either $x_1 \neq x_2$ or $y_1 \neq y_2$. Without loss of generality suppose $x_1 \neq x_2$, then there exist distinct neighborhoods $U \ni x_1$ and $V \ni x_2$ in X, and $U \times Y$ and $V \times Y$ are distinct neighborhoods of the two points in $X \times Y$.

(iv) Let A be a subspace of X. For each pair of distinct points $x_1, x_2 \in A \subseteq X$, there exist distinct neighborhoods $U \ni x_1$ and $V \ni x_2$ in X. Then $U \cap A$ and $V \cap A$ are disjoint neighborhoods of x_1 and x_2, respectively, in the subspace topology. □

Now let's investigate the compact sets in Hausdorff spaces.

Lemma 4.18. If K is a compact subspace of a Hausdorff space X, and $x_0 \in X$ is not in K. Then there exists disjoint open sets U and V in X such that $U \ni x_0$ and $V \supseteq K$.

Proof. For each point $y \in K$, we are able to choose two disjoint open neighborhoods $U_y \ni x_0$ and $V_y \ni y$. The collection $\{V_y : y \in K\}$ is an open cover of K, then there exist finitely many $y_1, \cdots, y_n \in K$ such that the $V := \bigcup_{j=1}^n V_{y_j} \supseteq K$. As a result, $U := \bigcap_{j=1}^n U_{y_j}$ is an open neighborhood of x that does not intersect K. □

Theorem 4.19 (Compact sets in Hausdorff spaces). Every compact subspace of a Hausdorff space is closed.

Proof. Let K be a compact subspace of a Hausdorff space X. Lemma 4.18 tells us $X \backslash K$ is an open set, because every $x_0 \in X \backslash K$ lies in the interior of $X \backslash K$. Thus we complete the proof. □

One important use of Theorem 4.19 is as a tool for verifying that a function is a homeomorphism.

Theorem 4.20. Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. We show that images of closed sets of X under f are closed in Y, which implies the continuity of f^{-1}. This is clear: K is closed in $X \Rightarrow K$ is compact $\Rightarrow f(K)$ is compact $\Rightarrow f(K)$ is closed in Y. □

Theorem 4.21 (Closed graph). Let $f : X \to Y$, and define the *graph* of f as $G_f = \{(x, f(x)) : x \in X\}$.

(i) If G_f is closed and Y is compact, then f is continuous.

(ii) If f is continuous and Y is Hausdorff, then G_f is closed.

Proof. (i) Let O be an open set in Y, we need to show $f^{-1}(O)$ is open in X. The intersection $G_f \cap X \times (Y \backslash O)$ is closed in $X \times Y$. By Lemma 4.7 (ii), the compactness of Y implies that $\pi_1(G_f \cap X \times (Y \backslash O)) = f^{-1}(Y \backslash O)$ is closed in X. Then $f^{-1}(O)$ is open in X.

(ii) For any $(x, y) \in (X \times Y) \backslash G_f$, we have $y \neq f(x)$. Since Y is Hausdorff, we can find two disjoint open sets $U \ni y$ and $V \ni f(x)$. Then $(x, y) \in f^{-1}(V) \times U$. Moreover, for any point $(z, f(z)) \in G_f$, if it lies in $f^{-1}(V) \times U$, then $z \in f^{-1}(V)$, however $f(z)$ lies in V which is disjoint from U, a contradiction! Hence $f^{-1}(V) \times U$ is a neighborhood of (x, y) which is disjoint from G_f. □

Now we introduce the definition of isolated points in topological spaces.

Definition 4.22 (Isolated points). Let X be a topological space. An *isolated point* of X is a point x of X such that the one-point set $\{x\}$ is open in X.

Theorem 4.23. Let X be a nonempty compact Hausdorff space. If X has no isolated point, then X is uncountable.

Proof. *Step I:* We first show that for any nonempty open set U in X and any $x \in X$, we can find a nonempty open set V contained in U such that $x \notin \overline{V}$. By our assumption that X has no isolated points, we can always find some $y \in U$ such that $y \neq x$. Since X is Hausdorff, we can find two disjoint open sets $W_y \ni y$ and $W_x \ni x$. Letting $V = W_y \cap U$ yields the desired result.

Step II: It suffices to show that any function $f : \mathbb{N} \to X$ is not surjective. Let $x_n = f(n), n = 1, 2, \cdots$. For $x_1 \in X$, we can find a nonempty open set V_1 such that $x_1 \notin \overline{V_1}$. Then we can iteratively find $V_{n+1} \subset V_n$ such that $x_{n+1} \notin \overline{V_{n+1}}$ for each $n \in \mathbb{N}$. Then we obtain a nested sequence $\overline{V_1} \supset \overline{V_2} \supset \cdots$ of nonempty closed sets in X. By Corollary 4.15, $\bigcap_{n=1}^{\infty} \overline{V_n}$ is nonempty, that is, there exists $x \in \bigcap_{n=1}^{\infty} \overline{V_n} \subset X$ such that $x \notin \{x_n\}_{n=1}^{\infty}$, which concludes the proof. ☐

The uncountability of real numbers immediately follows from Theorem 4.23.

Corollary 4.24. Every closed interval in $\mathbb{R}$ is uncountable.

As supplementary, let's discuss another class of spaces called T_1 spaces. They are weaker than Hausdorff spaces and less commonly used. The proof of Lemma 4.26 can be adapted from Proposition 4.17 (i).

Definition 4.25 (T_1 spaces). A topological space X is called a T_1 *space* if for each pair of distinct points $x, y \in X$ there exists a neighborhood U of x such that $y \notin U$, and a neighborhood V of y such that $x \notin V$.

Lemma 4.26. Let X be a T_1 space. Then every finite point set in X is closed.

Theorem 4.27. Let X be a T_1 space; let A be a subset of X. Then a point $x \in X$ is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Proof. The sufficiency is clear, so we need to prove the necessity. We let x be a limit point of A, and choose an arbitrary neighborhood N of x in X. If N contains only finitely many points $a_1, \cdots, a_n$ of $A \backslash \{x\}$, then $U := N \cap (X \backslash \{a_1, \cdots, a_n\})$ is also a neighborhood of x, since $\{a_1, \cdots, a_n\}$ is closed by Lemma 4.26. However, U as a neighborhood of the limit point x should contains at least one point of $A \backslash \{x\}$, a contradiction! ☐

Remark. By definition, a Hausdorff space must be a T_1 space. but not conversely. As a counterexample, consider the finite complement topology on $\mathbb{N}$: $\mathscr{T} = \{U : U \subseteq \mathbb{N} \text{ and } \mathbb{N} \backslash U \text{ is finite}\} \cup \{\emptyset\}$. This is a T_1 space, because for any distinct $m, n \in \mathbb{N}$, we can choose neighborhoods $N \backslash \{m\}$ and $N \backslash \{n\}$ that separates m and n. However, it is not Hausdorff because any two nonempty open sets are not disjoint!

4.3 Extension: Limit Point Compactness, Countable Compactness and Sequential Compactness

In this section we introduce other formulations of compactness that are commonly used.

Definition 4.28 (Limit point compactness/Fréchet compactness/Bolzano-Weierstrass property). Let X be a topological space. Then X is said to be *limit point compact* if every infinite subset of X has a limit point.

Theorem 4.29 (Compactness implies limit point compactness). A compact space X is limit point compact.

Proof. Let X be a compact space. We prove the contrapositive: if a subset A of X has no limit point, then A is finite. Suppose A has no limit point. Then A is closed because it contains all its limit points (which is the empty set). Since X is compact, A is also compact. Furthermore, for each $a \in A$ we can choose an open neighborhood U_a of a such that $U_a \cap A = \{a\}$. Then space A is covered by open sets $\{U_a, a \in A\}$, and we can find a finite subcollection $\{U_{a_1}, \cdots, U_{a_n}\}$ that contains A. Hence $A = \{a_1, \cdots, a_n\}$. ☐

Remark. Conversely, limit point compactness does not necessarily imply compactness. Consider $A = \{a_1, a_2\}$ with a topology $\{A, \emptyset\}$. Given $\mathbb{N}$ the discrete topology, the space $X = \mathbb{N} \times A$ is limit point compact, because every nonempty subset of X has a limit point. To see this, suppose (a_1, n) lies in a set $U \subseteq X$, then (a_2, n) must be a limit point of U because any neighborhood of (a_2, n) contains (a_1, n). However X is not compact, since the open cover $\{\{n\} \times A, n \in \mathbb{N}\}$ has no finite subcover.

Definition 4.30 (Countable compactness). A space X is said to be *countably compact* if every countable open cover of X contains a finite subcover of X.

Theorem 4.31. Let X be a topological space. (i) If X is countably compact, then it is limit point compact; (ii) If X is a limit point compact T_1 space, then it is countably compact.

Proof. (i) Let A be an infinite subset of X that has no limit point. We can assume A to be countable because if A has no limit point, so does its countable subsets. Since A has no limit points, A is closed. Moreover, for each $a \in A$ we can find an open set U_a in X with $U_a \cap A = \{a\}$. Then $X \backslash A$ and $\{U_a : a \in A\}$ form a countable open cover of X that has no finite subcover.

(ii) Argue by contradiction. Let $\{A_n\}_{n=1}^{\infty}$ be an countable open cover of X. If there exists no finite subcover, then we choose $x_n \in X \backslash \bigcup_{j=1}^{n} A_n$ for each n. Since X is limit point compact, $B := \{x_n : n \in \mathbb{N}\}$ has a limit point x. Moreover, there exists at least one element A_m that contains x, and $A_m \cap B \subseteq \{x_1, \cdots, x_m\}$. However, A_m as a neighborhood of x contains only finite points of B, contradicting with Theorem 4.27! ☐

Definition 4.32 (Sequential compactness). A topological space X is said to be *sequentially compact* if every sequence of points of X has a convergent subsequence.

Lemma 4.33 (Lebesgue number lemma for sequentially compact metric space). Let $\mathscr{A}$ be an open cover of a sequentially compact metric space (X, d). Then there exists a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathscr{A}$ that contains it.

Proof. Argue by contradiction. Let $\mathscr{A}$ be an open cover of X, we assume that there exists no $\delta > 0$ such that each set of diameter less than δ has an element of $\mathscr{A}$ containing it. Then for each $n \in \mathbb{N}$, there exists a set C_n of diameter less than $1/n$ that is not contained by any element of $\mathscr{A}$. Choose a point $x_n \in C_n$ for each n. Since X is sequentially compact, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ that converges to some $x_\infty \in X$. Since $x_\infty \in A$ for some element A of $\mathscr{A}$, we can choose some $\epsilon > 0$ such that $O(x_\infty, \epsilon) \subseteq A$. Then for a large enough k such that $1/n_k < \epsilon/2$ and $d(x_{n_k}, x_\infty) < \epsilon/2$, $C_{n_k} \subseteq O(x_{n_k}, \epsilon/2) \subseteq O(x_\infty, \epsilon) \subseteq A$, a contradiction! ☐

Definition 4.34 (Totally bounded sets). A metric space (X, d) is said to be *totally bounded*, if for every $\epsilon > 0$, there exists a finite cover of X by open ϵ-balls.

Lemma 4.35. A metric space (X, d) is totally bounded if it is sequentially compact.

Proof. Argue by contradiction. Suppose that $\exists \epsilon > 0$ such that X cannot be covered by finitely many open ϵ-balls. We construct a sequence $\{x_n\}$ as follows. Choose any $x_1 \in X$, then $O(x_1, \epsilon) \subsetneqq X$. For any $n > 1$, choose $x_n \in X \backslash \bigcup_{j=1}^{n-1} O(x_j, \epsilon)$. By construction, $d(x_n, x_j) \geq \epsilon$ for $j = 1, \cdots, n-1$. Then $\{x_n\}$ does not converge in X, contradicting the sequential compactness of X. ☐

Theorem 4.36 (Equivalence of four kinds of compactness in metric spaces). Let (X, d) be a metric space. The following are equivalent: (i) X is compact; (ii) X is limit point compact; (iii) X is countably compact; (iv) X is sequentially compact.

Proof. (i) $\Rightarrow$ (ii): By Theorem 4.29. (i) $\Leftrightarrow$ (iii): A metric space is T_1. Apply Theorem 4.31.

(ii) $\Rightarrow$ (iv): Assume X is limit point compact, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points of X. Consider the set $A = \{x_n : n \in \mathbb{N}\}$. If A is finite, then we can choose infinitely many x_n that coincides with some $x \in A$, which form a convergent subsequence. On the other hand, if A is infinite, then A has a limit point $x \in X$. We can construct a convergent subsequence as follows. We first choose n_1 such that x_{n_1} lies in open ball $O(x, 1)$.

For every $k \geq 2$, we can also find N_k such that $x_n \in O(x, 1/k)$ for all $n \geq N_k$. If $x_{n_{k-1}}$ is given, we can choose $n_k \geq \max\{n_{k-1}, N_k\}$ so that $x_{n_k} \in O(x, 1/k)$. Then $\{x_{n_k}\}_{k=1}^{\infty}$ converges to $x \in X$.

(iv) $\Rightarrow$ (i): Let $\mathscr{A}$ be an open cover of a sequentially compact metric space (X, d). By Lemma 4.33, $\mathscr{A}$ has a Lebesgue number $\delta > 0$. By Lemma 4.35, we can cover X by finitely many open $\delta/3$-balls. Each of these balls has diameter no greater than $2\delta/3$, hence lies in some elements of $\mathscr{A}$. By choosing this elements of $\mathscr{A}$ we immediately obtain a finite subcover of X. $\square$

4.4 Local Compactness and Compactification

Definition 4.37 (Locally compact spaces). A topological space X is said to be *locally compact at x* if there is some compact subspace C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said to be *locally compact*.

Example. (i) The euclidean space $\mathbb{R}^n$ is locally compact.

(ii) The rational numbers $\mathbb{Q}$ as a subspace of $\mathbb{R}$ is not locally compact. For any $q \in Q$, choose an open neighborhood $N_q := \mathbb{Q} \cap O(q, \epsilon)$ of q. Since N_q is countable, denote by $\{q_1, q_2, \cdots\}$ its elements. Then $\mathscr{A} = \{O(q_n, 2^{-n}\epsilon), n = 1, 2, \cdots\}$ is an open cover of N_q. However, any finite subcollection of $\mathscr{A}$, with total length less than 2ϵ, does not cover N_q.

Theorem 4.38. Let X be a topological space. Then X is locally compact Hausdorff if and only if there exists a space Y satisfying the following conditions:

(i) X is a subspace of Y;
(ii) $Y \backslash X$ consists of a single point;
(iii) Y is a compact Hausdorff space.

If Y and Y' are two spaces satisfying these conditions, then there is a homeomorphism of Y with Y' that equals the identity map on X.

Proof. Step I: We first verify the uniqueness. Let Y and Y' be two spaces satisfying (i)-(iii). Define $h : Y \to Y'$ by letting h map the single point p of $Y \backslash X$ to the point q of $Y' \backslash X$, and let h equal the identity on X. We show that if U is open in Y, then $h(U)$ is open in Y'. Symmetry then implies that h is a homeomorphism.

First, consider the case $p \notin U$. Since U is open in Y and $U \subseteq X$, it is open in X. Noticing $X = Y' \backslash \{q\}$ and Y' is Hausdorff, X is open in T'. Then $h(U) = U$ is open in Y'.

Second, let $p \in U$. Since $C = Y \backslash U$ is closed in Y' and Y' is compact, C is compact as a subspace of Y. Since $C \subseteq X$, it is a compact subspace of X. Because X is a subspace of Y', C is also compact in Y'. Since Y' is Hausdorff, C is closed in Y, and $Y' \backslash C = h(U)$ is open in Y'.

Step II: Now we suppose X is a locally compact Hausdorff space X and construct Y. Let us take some object that is not a point of X, denoted by the symbol ∞ for convenience, and adjoin it to X, forming the set $Y = X \cup \{\infty\}$. Inspired by Step I, we topologize Y by defining the collection of open sets of Y to consist of

(i) all sets U that are open in X, and
(ii) all sets of the form $Y \backslash C$, where C is a compact subspace of X.

We first check that such collection is indeed a topology on Y.

- Clearly, $\emptyset$ and Y are open sets of type (i) and (ii), respectively.
- For the intersection condition, let U_1 and U_2 be open sets of X, and let C_1 and C_2 be compact sets in X. Then $U_1 \cap U_2$ is of type (i), $(Y \backslash C_1) \cap (Y \backslash C_2) = Y \backslash (C_1 \cup C_2)$ is of type (ii), and $U \cap (Y \backslash C) = U \backslash (X \backslash C)$ is of type (i) because X is Hausdorff.
- For the union condition, let $\{U_\alpha\}$ be a collection of open sets of X, and let $\{C_\beta\}$ be a collection of compact sets in X. Then $\bigcup_\alpha U_\alpha = U$ is of type (i), $\bigcup_\beta (Y \backslash C_\beta) = Y \backslash \bigcap_\beta C_\beta = Y \backslash C$ is of type (ii), and $U \cup (Y \backslash C) = Y \backslash (C \backslash U)$ is of type (ii) because $C \backslash U$ is a closed subset of compact set C.

Then we need to verify that X is a subspace of Y:

- Given any open set in Y, its intersection with X is open in X. If the open set is of type (i), it is clearly open in X. If it is of type (ii), then $(Y\backslash C) \cap X = X\backslash C$ is open in Hausdorff space X.
- Conversely, given any open set in X, it is a type (i) open set in Y.

Now we show Y is compact. Let $\mathscr{A}$ be an open cover of Y, Then it must contain at least one open set of type (ii), denoted by $Y\backslash C$, to contain ∞. Take all members in $\mathscr{A}$ but $Y\backslash C$ and intersect them with X, we obtain a cover of X. Since C is a compact subspace of X, finitely many of them cover C. Then the corresponding finite collection of elements of $\mathscr{A}$ along with $Y\backslash C$ form a cover of Y.

It remains to show Y is Hausdorff. Let x and y be two elements of Y:

- Both x and y lies in X, which is a clear case since X is Hausdorff.
- Assume $y = \infty$. By the local compactness of X, we can choose a compact set C in X that contains a neighborhood U of x, then U and $Y\backslash C$ are disjoint neighborhoods of x and ∞, respectively, in Y.

Step III: Finally, we prove the converse. Suppose a space Y satisfying conditions (i)–(iii) exists. Then X is Hausdorff, because it is a subspace of Hausdorff space Y. Now fix $x \in X$. Choose disjoint open sets $U \ni x$ and $V \supset \{Y\backslash X\}$ in Y. Then $C = Y\backslash V$ is closed in Y, and is compact. Since C is contained in X, it is also compact in X. Furthermore, it contains a neighborhood U of x. □

Definition 4.39 (Compactification). If Y is a compact Hausdorff space, and X is a proper dense subspace of Y, then Y is said to be a *compactification* of X. If $Y\backslash X$ equals a single point, then Y is called the *one-point compactification* of X.

Remark. By Theorem 4.38, X has a one-point compactification Y if and only if X is a locally compact Hausdorff space that is not itself compact. Moreover, it is uniquely determined up to a homeomorphism.

Example. (i) The one-point compactification of real line $\mathbb{R}$ is homeomorphic to the circle S^1. To see this, we define $f : \mathbb{R} \cup \{\infty\} \to S_1$ as follows:

$$f(t) := \begin{cases} \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right), & t \in \mathbb{R}, \\ (-1, 0), & t = \infty, \end{cases} \quad , \quad f^{-1}(x, y) := \begin{cases} \frac{y}{1+x}, & (x, y) \neq (-1, 0), \\ \infty, & (x, y) = (-1, 0). \end{cases}$$

By construction, $f|_{\mathbb{R}}$ is a homeomorphism between $\mathbb{R}$ and $S_1\backslash\{(-1, 0)\}$. Furthermore, the compactification of $S_1\backslash\{(-1, 0)\}$ is its closure S_1.

(ii) The one-point compactification of $\mathbb{R}^2$ is homeomorphic to the sphere S^2. If $\mathbb{R}^2$ is looked at as the space $\mathbb{C}$ of complex numbers, then $\mathbb{C} \cup \{\infty\}$ is called the Riemann sphere, or the extended complex plane.

Here we also give another formulation of local compactness which aligns with Definition 3.16. The two formulations are equivalent in Hausdorff spaces.

Theorem 4.40 (Another equivalent characterization of local compactness in Hausdorff spaces). Let X be a Hausdorff space. Then X is locally compact if and only if given any $x \in X$ and any neighborhood U of x, there exists a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.

Proof. Clearly this characterization implies local compactness of X, and this direction does not require X to be Hausdorff. We prove the converse.

Suppose X is locally compact and Hausdorff, then we can take the one-point compactification Y of X. For any $x \in X$ and any neighborhood U of x, let $C = Y\backslash U$. Then C is compact since Y is compact Hausdorff and U is open. Applying Lemma 4.18, we can find disjoint open sets V and W in Y such that $V \ni x$ and $W \supseteq C$.

Then $\overline{V}$ is compact because it is a closed subset of a compact space Y, and $\overline{V} \subseteq U$ because V is disjoint from a open set W containing C. $\square$

Corollary 4.41. (i) A closed subspace of a locally compact space is locally compact; (ii) An open subspace of a locally compact Hausdorff space is locally compact.

Proof. (i) Suppose that A is a closed subspace of a locally compact space X. For any $x \in A$, let X be a compact subspace of X that contains a neighborhood U of x. Then $C \cap A$ as a closed subset of C is compact. Furthermore, it contains the neighborhood $U \cap A$ of x in A.

(ii) Now suppose that A is an open subspace of a locally compact Hausdorff space X. Then for any $x \in A$ and any neighborhood U of x in A, U is also a neighborhood of x in X. Hence we find can a neighborhood V of x in X such that $\overline{V}$ is compact and $\overline{V} \subseteq U$. $\square$

Combining Theorem 4.38 and Corollary 4.41 immediately yields the following result.

Corollary 4.42. A subspace X is homeomorphic to an open subspace of a compact Hausdorff subspace if and only if X is locally compact Hausdorff.

References

[1] Armstrong, M.A. (1983). *Basic Topology*. Springer.

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