Lecture Notes for Information Theory (ECE 587/STA 563)

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Contents

1 Measure of Information

Throughout this section, we assume that all random variables we study are discrete variables. We use capital letters like X, Y, Z to denote random variables, and their probability mass functions $p_X(x), p_Y(y), p_Z(z)$. For simplicity, we drop the subscripts and use the shorthand $p(x)$, $p(y)$, $p(z)$ instead. We use calligraphy letters like $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ to denote the finite support of random variables.

1.1 Entropy and Conditional Entropy

Definition 1.1 (Entropy)**.** *Let X be a random variable supported on a finite state space $\mathcal{X}$, with probability mass function $p(x)$. The entropy of X is a function of the distribution $p(x)$:*

$$H(X) := \sum_{x\in\mathcal{X}} p(x)\log\frac{1}{p(x)} = -\mathbb{E}\left[\log p(X)\right].$$

Likewise, for a collection $X_1, \cdots, X_n$ of random variables, the (joint) entropy of $X_1, \cdots, X_n$ is defined as the entropy of the random vector $(X_1, \cdots, X_n)$:

$$H(X_1, \cdots, X_n) = \sum_{x_1\in\mathcal{X}_1,\cdots,x_n\in\mathcal{X}_n} p(x_1, \cdots, x_n)\log\frac{1}{p(x_1, \cdots, x_n)}.$$

Remark I. The entropy provides a measure of uncertainty of random variables. We also frequently use the *binary entropy function* $h : [0,1] \to \mathbb{R}_+$, which is defined as the entropy of a Bernoulli variable:

$$h(\alpha) = H(\text{Bernoulli}(\alpha)) = -\alpha\log\alpha - (1-\alpha)\log(1-\alpha), \quad \alpha\in[0,1]$$

with the convention $0\log 0 = 0$.

Remark II. Given any base $b > 0$, we define the entropy of X under base b to be

$$H_b(X) = \sum_{x\in\mathcal{X}} p(x)\log_b\frac{1}{p(x)} = H(X)\log_b \mathrm{e}.$$

Clearly we have $H(X) = H_{\mathrm{e}}(X)$. Another commonly used entropy is the bit entropy, in which the base $b = 2$:

$$H_2(X) = \sum_{x\in\mathcal{X}} p(x)\log_2\frac{1}{p(x)} = H(X)\log_2 \mathrm{e}.$$

Proposition 1.2. *We have the following estimate for the entropy of a random variable X:*

$$0 \le H(X) \le \log|\mathcal{X}|.$$

Proof. The lower bound follows from the definition of entropy. For the upper bound, note that

$$\begin{aligned}\sum_{x\in\mathcal{X}} p(x)\log\frac{1}{p(x)} &= \sum_{x\in\mathcal{X}} p(x)\log\frac{|\mathcal{X}|}{p(x)|\mathcal{X}|} = \log|\mathcal{X}| + \sum_{x\in\mathcal{X}} p(x)\log\frac{1}{p(x)|\mathcal{X}|} \\ &\le \log|\mathcal{X}| + \sum_{x\in\mathcal{X}} p(x)\left(\frac{1}{p(x)|\mathcal{X}|} - 1\right) = \log|\mathcal{X}|.\end{aligned}$$

Then we complete the proof. $\square$

Remark. If $|\mathcal{X}| = \infty$, the entropy of a random variable can be ∞. For example, let $A = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$, which is less than infinity. Define random variable X by

$$\mathbb{P}(X = n) = \frac{1}{An(\log n)^2}, \quad n = 2, 3, \cdots .$$

Then

$$H(X) \geq \int_2^{\infty} \frac{\log A}{x \log x} \, dx = \infty.$$

We may also wonder the uncertainty of a random variable when given potentially relevant observation.

Definition 1.3 (Conditional Entropy)**.** *Let X and Y be two random variables in the same probability space. The entropy of Y conditioned on the event $X = x$ is a function of the conditional distribution $p(y|x)$:*

$$H(Y|X = x) := \sum_{y \in \mathcal{Y}} p(y|x) \log \frac{1}{p(y|x)} = \mathbb{E}\left[\log \frac{1}{p(Y|x)} \,\middle|\, X = x \right].$$

The conditional entropy of Y given X is a function of the joint distribution $p(x, y)$:

$$H(Y|X) := \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y) \log \frac{1}{p(y|x)} = \mathbb{E}\left[\log \frac{1}{p(Y|X)} \right].$$

Remark. Note that $H(Y|X)$ is a deterministic quantity rather than a random variable. In fact, we have

$$H(Y|X) = \sum_{x \in \mathcal{X}} p(x) H(Y|X = x).$$

Next, we study the relation between joint entropy and conditional entropy.

Proposition 1.4 (Chain rule for entropy)**.** *The joint entropy of X and Y has the following decomposition:*

$$H(X, Y) = H(Y|X) + H(X). \tag{1.1}$$

More generally,

$$H(X_1, X_2, \cdots, X_n) = H(X_1) + H(X_2|X_1) + H(X_3|X_2, X_1) + \cdots + H(X_n|X_{n-1}, \cdots, X_1). \tag{1.2}$$

Proof. We first verify the bivariate case (1.1):

$$\begin{aligned} H(Y|X) + H(X) &= \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y) \log \frac{1}{p(y|x)} + \sum_{x \in \mathcal{X}} p(x) \log \frac{1}{p(x)} \\ &= \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y) \log \frac{1}{p(y|x)} + \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y) \log \frac{1}{p(x)} \\ &= \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y) \log \frac{1}{p(x, y)} = H(X, Y). \end{aligned}$$

The general case (1.2) follows from mathematical induction. ◻

Remark. The equality (1.1) also implies the chain rule for conditional entropy:

$$H(X, Y|Z) = H(X|Y, Z) + H(Y|Z)$$

Finally, we introduce an important property of entropy as the function of distribution.

Theorem 1.5 (Concavity of entropy)**.** *Let p and q be two probability distributions that are supported in a common space $\mathcal{X}$. Then for all $0 \le \lambda \le 1$, we have*

$$H(\lambda p + (1-\lambda)q) \ge \lambda H(p) + (1-\lambda)H(q). \tag{1.3}$$

Proof. We simply employ the estimate $\log t \le t - 1$ on $\lambda H(p) + (1-\lambda)H(q) - H(\lambda p + (1-\lambda)q)$:

$$\begin{aligned}
&\lambda \sum_{x\in\mathcal{X}} p(x)\log\frac{1}{p(x)} + (1-\lambda)\sum_{x\in\mathcal{X}} q(x)\log\frac{1}{q(x)} - \sum_{x\in\mathcal{X}} (\lambda p(x) + (1-\lambda)q(x))\log\frac{1}{\lambda p(x) + (1-\lambda)q(x)} \\
&= \lambda \sum_{x\in\mathcal{X}} p(x)\log\frac{\lambda p(x) + (1-\lambda)q(x)}{p(x)} + (1-\lambda)\sum_{x\in\mathcal{X}} q(x)\log\frac{\lambda p(x) + (1-\lambda)q(x)}{q(x)} \\
&\le \lambda \sum_{x\in\mathcal{X}} (\lambda p(x) + (1-\lambda)q(x) - p(x)) + (1-\lambda)\sum_{x\in\mathcal{X}} (\lambda p(x) + (1-\lambda)q(x) - q(x)) = 0.
\end{aligned}$$

Then the result follows. $\square$

Remark. Using the concavity, we can interpret why a transfer of probability that makes the distribution more uniform increases the entropy. We consider the following transformation:

$$(p_1, \cdots, p_i, \cdots, p_j, \cdots, p_m) \to \left(p_1, \cdots, \frac{p_i + p_j}{2}, \cdots, \frac{p_i + p_j}{2}, \cdots, p_m\right), \quad p_1 + \cdots + p_m = 1.$$

Let $p = (p_1, \cdots, p_i, \cdots, p_j, \cdots, p_m)$, and let $q = (p_1, \cdots, p_j, \cdots, p_i, \cdots, p_m)$ be the probability vector with i-th and j-th elements exchanged. Then

$$H\left(\frac{p+q}{2}\right) \ge \frac{1}{2}H(p) + \frac{1}{2}H(q) = H(p).$$

1.2 Mutual Information

Definition 1.6 (Mutual information)**.** *Let X and Y be two discrete random variables in the same probability space. The mutual information of X and Y is defined as*

$$I(X;Y) = \sum_{x\in\mathcal{X}, y\in\mathcal{Y}} p(x,y)\log\frac{p(x,y)}{p(x)p(y)}.$$

Proposition 1.7 (Properties of mutual information)**.** *Let X and Y be two discrete random variables.*

(i) (Symmetry). $I(X;Y) = I(Y;X)$.

(ii) (Reduction). $I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$.

(iii) (Measure of dependency). $I(X;Y) \ge 0$, and the equality holds if and only if X and Y are independent.

Proof. The assertion (i) follows from definition, and the second from direct calculation. Now we verify (iii):

$$\sum_{x\in\mathcal{X}, y\in\mathcal{Y}} p(x,y)\log\frac{p(x,y)}{p(x)p(y)} \ge \sum_{x\in\mathcal{X}, y\in\mathcal{Y}} p(x,y)\left(1 - \frac{p(x)p(y)}{p(x,y)}\right) = 0.$$

Clearly, the equality holds if and only if $p(x,y) = p(x)p(y)$ for every $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. $\square$

Remark. Combining (ii) and (iii), we know that *conditioning does not increase entropy*:

$$H(X|Y) \le H(X), \quad \textit{and} \quad H(Y|X) \le H(Y).$$

An alternative proof of Theorem 1.5. Let $X_1 \sim p$ and $X_2 \sim q$ be two independent random variables, and let $Z \sim \text{Bernoulli}(\lambda)$. Define

$$Y = X_1 \mathbb{1}_{\{Z=1\}} + X_2 \mathbb{1}_{\{Z=0\}}.$$

Then $Y \sim \lambda p + (1-\lambda)q$, and

$$H(Y) \geq H(Y|Z) = \lambda H(Y|Z=1) + (1-\lambda)H(Y|Z=0) = \lambda H(X_1) + (1-\lambda)H(X_2).$$

This is in fact the equality (1.3).

Definition 1.8. *Let X, Y and Z be discrete random variables in the same probability space. The conditional mutual information of X and Y given Z is defined as*

$$I(X;Y|Z) = \sum_{x\in\mathcal{X}, y\in\mathcal{Y}, z\in\mathcal{Z}} p(x,y,z) \log \frac{p(x,y|z)}{p(x|z)p(y|z)}.$$

Similar to Proposition 1.7, conditional mutual information has the following properties.

Proposition 1.9 (Properties of conditional mutual information)**.** *Let X, Y and Z be discrete random variables in the same probability space.*

(i) (Symmetry). $I(X;Y|Z) = I(Y;X|Z)$.

(ii) (Reduction). $I(X;Y|Z) = H(X|Z) - H(X|Y,Z) = H(Y|Z) - H(Y|X,Z)$.

(iii) (Measure of dependency). $I(X;Y|Z) \geq 0$, and the equality holds if and only if X and Y are conditionally independent on Z.

By direct calculation and induction, we also have the following chain rule for mutual information.

Proposition 1.10 (Chain rule for mutual information)**.** *The mutual information $I(X;Y,Z)$ has the following decomposition:*

$$I(X;Y,Z) = I(X;Z) + I(X;Y|Z).$$

More generally,

$$I(X;Y_1,Y_2,\cdots,Y_n) = I(X;Y_1) + I(X;Y_2|Y_1) + I(X;Y_3|Y_2,Y_1)\cdots + I(X;Y_n|Y_{n-1},\cdots,Y_1).$$

We can use this rule to derive the data processing inequality for Markov chains.

Definition 1.11 (Markov chain)**.** *Random variables X, Y and Z are said to form a Markov chain, written $X \to Y \to Z$, if X and Z are conditionally independent on Y:*

$$p(x,z|y) = p(x|y)p(z|y).$$

Particularly, if $Z = g(Y)$ is a function of Y, then $X \to Y \to Z$.

The following theorem asserts that no manipulation of Y can increase the mutual information.

Theorem 1.12 (Data processing inequality)**.** *If $X \to Y \to Z$, then*

$$I(X;Y) \geq I(X;Z).$$

Particularly, for any function g defined on $\mathcal{Y}$, we have

$$I(X;Y) \geq I(X;g(Y)).$$

Proof. By chain rule, we have that

$$I(X;Y) + I(X;Z|Y) = I(X;Y,Z) = I(X;Z) + I(X;Y|Z).$$

Since $X \perp\!\!\!\perp Z \,|\, Y$, we have $I(X;Z|Y) = 0$. Since $I(X;Y|Z) \geq 0$, the result follows. $\square$

Remark. By Proposition 1.7, we also have $H(X|Z) \geq H(X|Y)$ when $X \to Y \to Z$.

Next, we introduce an alternative definition of mutual information.

Definition 1.13 (Kullback-Leibler divergence/relative entropy)**.** *Let p and q be two probability distributions such that $\mathcal{X} = \operatorname{supp} q \supset \operatorname{supp} p$. The Kullback-Leibler divergence of q from p is defined as*

$$D(p\|q) := \sum_{x\in\mathcal{X}} p(x) \log \frac{p(x)}{q(x)} = \mathbb{E}_{X\sim p}\left[\log \frac{p(X)}{q(X)}\right].$$

This is also known as the relative entropy.

Remark. By definition, we have

$$D(p\|q) = \sum_{x\in\mathcal{X}} p(x) \log \frac{p(x)}{q(x)} \geq \sum_{x\in\mathcal{X}} p(x) \left(1 - \frac{q(x)}{p(x)}\right) = 0.$$

Therefore, $D(p\|q) \geq 0$, and the equality holds if and only if $p = q$. Moreover, by definition, we have the following result:

$$I(X;Y) = D(p_{X,Y}\|p_X p_Y) = \mathbb{E}_{X\sim p_X}\left[D(p_{Y|X}\|p_Y)\right].$$

In other words, the mutual information of X and Y is the relative entropy of their marginal product $p_X p_Y$ from their joint distribution $p_{X,Y}$.

Application: Misclassification Rate. To end this section, we introduce a useful application of mutual information. We discuss the estimation of a discrete random variable X from an observation Y. To deal with this problem, we construct a function $\phi : \mathcal{Y} \to \mathcal{X}$. The probability of error of the estimator $\widehat{X} = \phi(Y)$ is

$$p_e = \mathbb{P}(\widehat{X} \neq X).$$

The following Fano's inequality provide a lower bound of the error rate p_e.

Theorem 1.14 (Fano's inequality)**.** *For any estimator $\widehat{X}$ of X such that $X \to Y \to \widehat{X}$, we have*

$$p_e \geq \frac{H(X|Y) - \log 2}{\log |\mathcal{X}|}.$$

Proof. Let $B = \mathbb{1}_{\{X=\widehat{X}\}}$, which is a Bernoulli variable with parameter p_e. By the chain rule, the conditional entropy of (B, X) given $\widehat{X}$ is

$$H(B|\widehat{X}) + H(X|B,\widehat{X}) = H(B,X|\widehat{X}) = H(X|\widehat{X}) + H(B|X,\widehat{X}).$$

Now we analyze the four terms in the equality.

(i) Since conditioning does not increase entropy, $H(B|\widehat{X}) \leq H(B) = h(p_e)$.

(ii) The conditional entropy $H(X|B,\widehat{X})$ has the following estimate:

$$
\begin{aligned}
H(X|B,\widehat{X}) &= \sum_{b\in\{0,1\}} \sum_{x\in\mathcal{X}} \sum_{\widehat{x}\in\mathcal{X}} \mathbb{P}(B=b, X=x, \widehat{X}=\widehat{x}) \log \frac{1}{\mathbb{P}(X=x|B=b,\widehat{X}=\widehat{x})} \\
&= \sum_{x\in\mathcal{X}} \sum_{\widehat{x}\in\mathcal{X}} \mathbb{P}(B=0, X=x, \widehat{X}=\widehat{x}) \log \frac{1}{\mathbb{P}(X=x|B=0,\widehat{X}=\widehat{x})} \\
&= \sum_{\widehat{x}\in\mathcal{X}} \mathbb{P}(B=0,\widehat{X}=\widehat{x}) \underbrace{\sum_{x\in\mathcal{X}} \mathbb{P}(X=x|B=0,\widehat{X}=\widehat{x}) \log \frac{1}{\mathbb{P}(X=x|B=0,\widehat{X}=\widehat{x})}}_{\leq \log|\mathcal{X}|} \\
&\leq p_e \log|\mathcal{X}|.
\end{aligned}
$$

(iii) Since $X \to Y \to \widehat{X}$, the data processing inequality implies $H(X|\widehat{X}) \geq H(X|Y)$.

(iv) Since B is a function of X and $\widehat{X}$, we have $H(B|X,\widehat{X}) = 0$.

Combining these estimates, we obtain

$$
H(X|Y) \leq h(p_e) + p_e \log|\mathcal{X}| \leq \log 2 + p_e \log|\mathcal{X}|.
$$

Then we complete the proof. □

1.3 The Typical Set and Asymptotic Equipartition Property

In this section, we investigate a sequence of i.i.d. copies $X_1, X_2, \cdots$ of a random variable $X \sim p(x)$ with finite support $\mathcal{X}$. We write for a random vector of length n and its realization

$$
X_{1:n} = (X_1, \cdots, X_n), \quad x_{1:n} = (x_1, \cdots, x_n).
$$

The joint distribution of $X_{1:n}$ is given by

$$
p(x_{1:n}) = \mathbb{P}\left(X_{1:n} = x_{1:n}\right) = \prod_{i=1}^{n} p(x_i).
$$

In this section, our key task is to find a confidence set $A \subset \mathcal{X}^n$ that contains our observation $X_{1:n}$ with a high probability. Formally, we require

$$
\mathbb{P}(X_{1:n} \in A) \geq 1 - \epsilon,
$$

where $\epsilon > 0$ is an arbitrarily given small quantity.

Typical Sets. We first propose an idea of constructing high probability sets. Let $g : \mathcal{X} \to \mathbb{R}$ be a function such that $\mathbb{E}|g(X)| < \infty$. By the weak law of large number, for each $\epsilon > 0$, there exists $N_\epsilon > 0$ such that

$$
\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} g(X_i) - \mathbb{E}[g(X)]\right| \leq \epsilon\right) \geq 1 - \epsilon \quad \forall n \geq N_\epsilon.
$$

Consequently, almost all probability mass is concentrated on the following set A:

$$
A = \left\{ x_{1:n} \in \mathcal{X}^n : \mathbb{E}\left[g(X)\right] - \epsilon \leq \frac{1}{n}\sum_{i=1}^{n} g(x_i) \leq \mathbb{E}\left[g(X)\right] + \epsilon \right\}.
$$

In the last display, the constraint can be equivalently expressed as

$$2^{-n(\mathbb{E}[g(X)]+\epsilon)} \leq 2^{-\sum_{i=1}^{n} g(x_i)} \leq 2^{-n(\mathbb{E}[g(X)]-\epsilon)}.$$

The construction of typical sets follows by plugging in $g(x) = \log_2 \frac{1}{p(x)}$.

Definition 1.15. *The ϵ-typical set is defined by*

$$A_\epsilon^{(n)} = \left\{ x_{1:n} \in \mathcal{X}^n : 2^{-n(H_2(X)+\epsilon)} \leq p(x_{1:n}) \leq 2^{-n(H_2(X)-\epsilon)} \right\},$$

or equivalently, the set of all tuples $x_{1:n} \in \mathcal{X}^n$ obeying

$$H_2(X) - \epsilon \leq -\frac{1}{n} \log_2 p(x_{1:n}) \leq H_2(X) + \epsilon.$$

Clearly, there exists N_ϵ such that $A_\epsilon^{(n)}$ contains $X_{1:n}$ with probability at least $1-\epsilon$ whenever $n > N_\epsilon$.

Size of Typical Sets. When n increased, the number of possible realizations of $X_{1:n}$ would rise very quickly, which is $|\mathcal{X}|^n$. The idea of typical sets is to concentrate the probability mass of $X_{1:n}$ on a smaller set $A_\epsilon^{(n)}$:

$$A_\epsilon^{(n)} = \left\{ x_{1:n} \in \mathcal{X}^n : 2^{-n(H_2(X)+\epsilon)} \leq p(x_{1:n}) \leq 2^{-n(H_2(X)-\epsilon)} \right\}.$$

In this set, all tuples have roughly the same probability mass. This is know as the *Asymptotic Equipartition property* (AEP). Here is an intuition of this typical set:

- For the low probability tuples $p(x_{1:n}) < 2^{-n(H_2(X)+\epsilon)}$, they are too unlikely to matter;
- For the high probability tuples $p(x_{1:n}) > 2^{-n(H_2(X)-\epsilon)}$, they are too few to matter;
- Therefore, we exclude those unimportant tuples and retain only the average probability tuples.

We now study the size of the reduced set.

Proposition 1.16. *Let $A_\epsilon^{(n)}$ be the ϵ-typical set for $X_{1:n}$. Then there exists $N_\epsilon > 0$ such that*

$$\mathbb{P}\left(X_{1:n} \in A_\epsilon^{(n)}\right) \geq 1 - \epsilon, \quad \forall n \geq N_\epsilon.$$

Furthermore, the upper bound of the typical set is given by

$$\left|A_\epsilon^{(n)}\right| \leq 2^{n(H_2(X)+\epsilon)}, \quad \forall n \geq 1;$$

and the lower bound of the typical set is given by

$$\left|A_\epsilon^{(n)}\right| \geq (1-\epsilon) 2^{n(H_2(X)-\epsilon)}, \quad \forall n \geq N_\epsilon.$$

Proof. For the upper bound, note that

$$1 = \sum_{x_{1:n} \in \mathcal{X}^n} p(x_{1:n}) \geq \sum_{x_{1:n} \in A_\epsilon^{(n)}} p(x_{1:n}) \geq \left|A_\epsilon^{(n)}\right| 2^{-n(H_2(X)+\epsilon)}.$$

For the lower bound, when $n \geq N_\epsilon$, we have

$$1 - \epsilon \leq \mathbb{P}\left(X_{1:n} \in A_\epsilon^{(n)}\right) = \sum_{x_{1:n} \in A_\epsilon^{(n)}} p(x_{1:n}) \leq \left|A_\epsilon^{(n)}\right| 2^{-n(H_2(X)-\epsilon)}.$$

Rearranging each inequality completes the proof. $\square$

1.4 Entropy Rates

In this section, we study a discrete-time stochastic process $X = (X_t)_{t\in\mathbb{N}}$, where each X_t is a random variable in a finite range $\mathcal{X}$. These random variables do not need to be i.i.d..

Definition 1.17. *Let $X = (X_t)_{t\in\mathbb{N}}$ be a stochastic process.*

(i) Average entropy per symbol

$$H(X) = \lim_{n\to\infty} \frac{H(X_{1:n})}{n}$$

(ii) The k-th order entropy

$$H^k(X) = H(X_k|X_{k-1}, \cdots, X_1)$$

(iii) Rate of information innovation

$$H^\infty(X) = \lim_{k\to\infty} H^k(X) = \lim_{k\to\infty} H(X_k|X_{k-1}, \cdots, X_1)$$

Remark. If $X = (X_t)_{t\in\mathbb{N}}$ is an i.i.d. sequence, we have

$$H(X) = H^\infty(X) = H(X_1).$$

Stationarity. Recall that a stochastic process $X = (X_t)_{t\in\mathbb{N}}$ is said to be *(strongly) stationary* if

$$\mathbb{P}(X_1 = x_1, \cdots, X_n = x_n) = \mathbb{P}(X_{k+1} = x_1, \cdots, X_{n+k} = x_n)$$

for every $n \in \mathbb{N}$, every lapse $k \in \mathbb{N}$ and all $x_1, \cdots, x_n \in \mathcal{X}$.

Theorem 1.18. *For a stationary process $X = (X_t)_{t\in\mathbb{N}}$,*

$$H(X) = H^\infty(X).$$

Proof. We first prove the existence of rate of information innovation. By stationarity,

$$H^n(X) = H(X_n|X_{n-1}, \cdots, X_2, X_1) \leq H(X_n|X_{n-1}, \cdots, X_2) = H(X_{n-1}|X_{n-2}, \cdots, X_1)$$

Therefore, $H(X_n|X_{n-1}, \cdots, X_1)$ is decreasing in n. Since conditional entropy is nonnegative, the monotone sequence converges: $H^n \searrow H^\infty$. Next, by the chain rule of entropy,

$$\frac{1}{n} H(X_1, \cdots, X_n) = \frac{1}{n} \sum_{i=1}^{n} H(X_i|X_{i-1}, \cdots, X_1).$$

The right-hand side of the last display, which is a Cesàro mean, has the same limit as $H(X_n|X_{n-1}, \cdots, X_1)$, which is $H^\infty(X)$. Since the limit of the left-hand side is the average entropy per symbol, the result follows. □

Kolmogorov extension. If $(X_t)_{t\in\mathbb{N}}$ is a stationary process, then all finite-dimensional marginal distributions of this process are determined. By Kolmogorov extension theorem, we can extend the index of this process to the integer set $\mathbb{Z}$ and obtain a stationary process $(X_t)_{t\in\mathbb{Z}}$. We write for the past history

$$X_{\leq 0} = (X_t)_{t\in -\mathbb{N}_0} = (X_0, X_{-1}, X_{-2}, \cdots).$$

Furthermore, we can define the conditional p.m.f. of X_1 given $X_{\le 0}$:

$$\begin{aligned} p(x_1|X_{\le 0}) = \mathbb{E}\left[\mathbb{1}_{\{X_1=x_1\}}|X_{\le 0}\right] &= \lim_{n\to\infty}\left[\mathbb{1}_{\{X_1=x_1\}}|X_0, X_{-1}, \cdots, X_{-n}\right] \\ &= \lim_{n\to\infty} p(x_1|X_0, X_{-1}, \cdots, X_{-n}). \end{aligned}$$

Here the convergence holds both in L^1 and almost surely, since the sequence we take limit of is a uniformly integrable martingale. Furthermore, by Lebesgue's dominated convergence theorem,

$$\mathbb{E}\left[-\log p(X_1|X_{\le 0})\right] = \lim_{n\to\infty} H^k(X) = H^\infty(X).$$

Ergodicity. Let $(\Omega, \mathscr{F}, P)$ be a measure space. A measurable mapping $T : (\Omega, \mathscr{F}) \to (\Omega, \mathscr{F})$ is said to be *ergodic*, if every set $A \in \mathscr{F}$ such that $TA = A$ *a.e.* satisfies $P(A) = 0$ or $P(A) = 1$. We let T play a role of time shift. The stochastic process $X = (X_t)_{t\in\mathbb{N}}$ is said to be an *ergodic* process, where $X_t(\omega) = X_0(T^t\omega)$ for all $t \in \mathbb{N}$ and $X_0 : \Omega \to \mathcal{X}$ is a random variable.

According to *Birkhoff's ergodic theorem*, the strong law of large numbers holds for a stationary ergodic process $X = (X_t)_{t\in\mathbb{N}}$:

$$\overline{X}_n := \frac{1}{n}\sum_{k=1}^n X_k \to \mu = \mathbb{E}X_1, \quad a.s..$$

Lemma 1.19. *For the process $(X_t)_{t\in\mathbb{Z}}$, define the k-th order Markov approximation by*

$$p^k(X_{1:n}) = p(X_{1:k}) \prod_{j=k+1}^n p(X_j|X_{j-1}, \cdots, X_{j-k}).$$

If $(X_t)_{t\in\mathbb{Z}}$ is a stationary ergodic process,

$$\frac{1}{n}\log\frac{1}{p^k(X_{1:n})} \to H^k(X) \ a.s., \quad \text{and} \quad \frac{1}{n}\log\frac{1}{p(X_{1:n}|X_{\le 0})} \to H^\infty(X) \ a.s..$$

Proof. Since $(X_t)_{t\in\mathbb{Z}}$ is an ergodic process, so is the process $Y_t = f(X_{\le t})$, where f is any measurable function. Then both $\log p(X_n|X_{n-1}, \cdots, X_{n-k})$ and $\log p(X_n|X_{\le n-1})$ are stationary ergodic processes on $n \in \mathbb{N}$. By Birkhoff's ergodic theorem, we have

$$\begin{aligned} &\frac{1}{n}\log\frac{1}{p^k(X_{1:n})} = \frac{1}{n}\log\frac{1}{p(X_{1:k})} + \frac{1}{n}\sum_{j=k+1}^n \log\frac{1}{p(X_j|X_{j-1}, \cdots, X_{j-k})} \to 0 + H^k(X), \ a.s., \\ &\frac{1}{n}\log\frac{1}{p(X_{1:n}|X_{\le 0})} = \frac{1}{n}\sum_{j=1}^n \log\frac{1}{p(X_j|X_{\le j-1})} \to H^\infty(X), \ a.s.. \end{aligned}$$

Then we complete the proof. ☐

Lemma 1.20 (Sandwich)**.** *Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary ergodic process. Then*

$$\limsup_{n\to\infty}\frac{1}{n}\log\frac{p^k(X_{1:n})}{p(X_{1:n})} \le 0 \ a.s., \quad \limsup_{n\to\infty}\frac{1}{n}\log\frac{p(X_{1:n})}{p(X_{1:n}|X_{\le 0})} \le 0 \ a.s..$$

Proof. Let A be the support set of $p(x_{1:n})$. Then

$$\mathbb{E}\left[\frac{p^k(X_{1:n})}{p(X_{1:n})}\right] = \sum_{x_{1:n}\in A}\frac{p^k(x_{1:n})}{p(x_{1:n})}p(x_{1:n}) = \sum_{x_{1:n}\in A} p^k(x_{1:n}) \le \sum_{x_{1:n}\in\mathcal{X}^n} p^k(x_{1:n}) = 1.$$

By Markov's inequality, we have

$$\mathbb{P}\left(\frac{1}{n}\log\frac{p^k(X_{1:n})}{p(X_{1:n})} \geq \frac{2\log n}{n}\right) = \mathbb{P}\left(\frac{p^k(X_{1:n})}{p(X_{1:n})} \geq n^2\right) \leq \frac{1}{n^2}$$

By Borel-Cantelli Lemma, since $\sum_{n=1}^{\infty} n^{-2} < \infty$, the events

$$\left\{\frac{1}{n}\log\frac{p^k(X_{1:n})}{p(X_{1:n})} \geq \frac{2\log n}{n}, \quad n \in \mathbb{N}\right\}$$

happens finitely many times with probability 1, which proves the first result. On the other hand, let $B(X_{\leq 0})$ be the support set of $p(x_{1:n}|X_{\leq 0})$. Then

$$\mathbb{E}\left[\frac{p(X_{1:n})}{p(X_{1:n}|X_{\leq 0})}\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{p(X_{1:n})}{p(X_{1:n}|X_{\leq 0})}\middle| X_{\leq 0}\right]\right] = \mathbb{E}\left[\sum_{x_{1:n}\in B(X_{\leq 0})} p(X_{1:n})\right] \leq 1.$$

The second result then follows from a similar procedure. $\square$

Now we point out that, the Asymptotic Equilibrium property holds not only for i.i.d. sequences, but also for stationary ergodic processes.

Theorem 1.21 (Shannon-McMillan-Breiman)**.** *Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary ergodic process. Then*

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{p(X_{1:n})} = H^\infty(X).$$

Proof. By Lemmas 1.19 and 1.20, almost surely,

$$\begin{aligned}
\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{p(X_{1:n})} &\leq \liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{p^k(X_{1:n})} = H^k(X),\\
\liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{p(X_{1:n})} &\geq \limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{p(X_{1:n}|X_{\leq 0})} = H^\infty(X).
\end{aligned}$$

Therefore, for all $k \in \mathbb{N}$, we have

$$H^\infty(X) \leq \liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{p(X_{1:n})} \leq \limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{p(X_{1:n})} \leq H^k(X).$$

Since X is stationary, $H^k(X) \searrow H^\infty(X)$ as $k \to \infty$. Hence $\frac{1}{n}\log\frac{1}{p(X_{1:n})} \xrightarrow{a.s.} H^\infty(X)$. $\square$

Remark. An example for stationary ergodic process is the irreducible and aperiodic Markov chain.

2 Lossless Compression

In this section, we study the problem of lossless coding. To begin with, we have a source alphabet $\mathcal{X}$ and a D-ary alphabet $\{0, 1, \cdots, D-1\}$. Our key goal is to transform a string of $\mathcal{X}$ to a string of $\mathcal{D}$.

- A *source code* is a mapping $C : \mathcal{X} \to \mathcal{D}^*$, where $\mathcal{D}$ is a D-ary alphabet $\{0, 1, \cdots, D-1\}$, and

$$\mathcal{D}^* = \bigcup_{n=1}^{\infty} \mathcal{D}^n.$$

  The elements of $C(\mathcal{X})$ are called *codewords*. For every symbol $x \in \mathcal{X}$, we denote by $\ell(x)$ the length of the codeword $C(x)$ associated with x.
- A source code $C : \mathcal{X} \to \mathcal{D}^*$ is said to be *nonsingular* if it is injective.
- The *extension* $C^* : \mathcal{X}^* \to \mathcal{D}^*$ of a source code C is the mapping from finite length strings of $\mathcal{X}$ to finite length strings of $\mathcal{D}$:

$$C^*(x_1 x_2 \cdots x_n) = C(x_1)C(x_2)\cdots C(x_n).$$

- A source code $C : \mathcal{X} \to \mathcal{D}^*$ is said to be *uniquely decodable* if its extension C^* is injective.
- A source code $C : \mathcal{X} \to \mathcal{D}^*$ is said to be *instantaneous* (or *prefix-free*) if no codeword of C is prefixed by any other codeword.
- We have the inclusions: *nonsingular codes* $\supset$ *uniquely decodable codes* $\supset$ *instantaneous codes*.

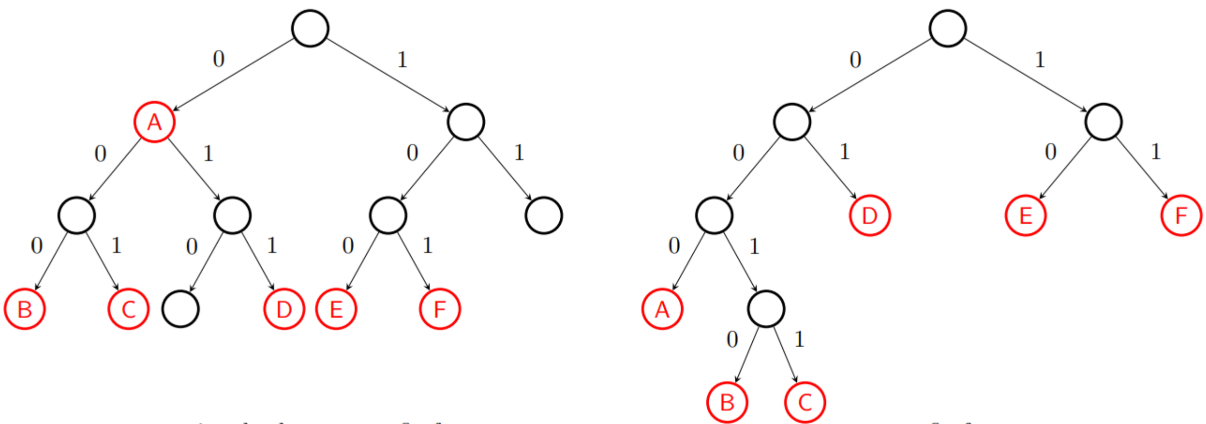
In general, some nice properties of a code are wanted:

- it is uniquely decodable;
- it is prefix free, so one can decode a string instantaneously while reading;
- it is efficient, i.e. given the distribution p of letters $\mathcal{X}$ in a string, we would like to minimize the average codeword length:

$$\mathbb{E}\left[\ell(X)\right] = \sum_{x \in \mathcal{X}} p(x)\ell(x).$$

2.1 Kraft-McMillan Inequality

Tree representation. A D-ary code $C : \mathcal{X} \to \mathcal{D}$ can be represented as a D-ary tree that consists of a root with branches, nodes and leaves. The root and every node has exactly D children, with each branch labeled by a letter in $\mathcal{D}$. Starting from the root, each vertex is uniquely associated with a string $d \in \mathcal{D}^*$, specified by the path from the root to itself. Some examples of binary trees are given below.



nonsingular but not prefix-free   prefix-free

We can determine whether a code is instantaneous right away by looking at its tree.

Proposition 2.1. *A code $C : \mathcal{X} \to \mathcal{D}^*$ is instantaneous if and only if all its codeword are leaves.*

Proof. If $C : \mathcal{X} \to \mathcal{D}^*$ is an instantaneous code, then each of its codeword has no descendant in the tree, which is a leaf; conversely, if each codeword of C is a leaf in the tree, it has no ancestor which is also a codeword, and C is instantaneous. $\square$

Using the tree representation, we can show a property which characterizes the instantaneous codes.

Theorem 2.2 (Kraft's inequality)**.** *Let $\ell : \mathcal{X} \to \mathbb{N}$ be a length function. Then ℓ is the length function of an instantaneous code if and only if it satisfies Kraft's inequality:*

$$\sum_{x \in \mathcal{X}} D^{-\ell(x)} \leq 1. \tag{2.1}$$

Proof. We first prove necessity. Let ℓ is the length function of an instantaneous code C, and let L be the depth of the tree. Then every codeword $C(x)$ at depth $\ell(x)$ prunes away $D^{L-\ell(x)}$ leaves from the complete tree of depth L. Since there are no more than D^L leaves in the complete tree, we have

$$\sum_{x \in \mathcal{X}} D^{L-\ell(x)} \leq D^L \quad \Rightarrow \quad \sum_{x \in \mathcal{X}} D^{-\ell(x)} \leq 1.$$

Now we prove the sufficiency. To this end, we prove the following argument: at every step $k \in \mathbb{N}$, after all codewords of length $\ell(x) < k$ have been assigned, there is enough room left at the depth k for the codewords of length $\ell(x) = k$. More explicitly, we want to show

$$D^k - \sum_{x \in \mathcal{X}:\ell(x)<k} D^{k-\ell(x)} \geq \left|C^{-1}(\mathcal{D}^k)\right|, \quad \forall 1 \leq k \leq L.$$

Note that

$$\left|C^{-1}(\mathcal{D}^k)\right| = \sum_{x \in \mathcal{X}:\ell(x)=k} D^{k-\ell(x)}.$$

Then our conclusion holds if

$$\sum_{x \in \mathcal{X}:\ell(x)\leq k} D^{-\ell(x)} \leq 1, \quad \forall k \in \mathbb{N}.$$

Clearly this is valid by Kraft's inequality (2.1). $\square$

The Kraft's inequality is also a necessary condition for a code to be uniquely decodable.

Theorem 2.3 (McMillan)**.** *Every uniquely decodable code $C : \mathcal{X} \to \mathcal{D}^*$ satisfies Kraft's inequality (2.1).*

Proof. Let $C : \mathcal{X} \to \mathcal{D}^*$ be a uniquely decodable code, and let $L = \max_{x\in\mathcal{X}} \ell(x)$, where ℓ is the length function of C. Then for a source string $x_{1:n}$, the length of the extended codeword $C^*(x_{1:n})$ is given by

$$\ell^*(x_{1:n}) = \sum_{i=1}^{n} \ell(x_i) \leq nL.$$

Let N_k be the number of source strings of length n with $\ell^*(x_{1:n}) = k$. Since C is uniquely decodable, the source strings with codewords of length k are no more than D-ary strings of length k, i.e. $N_k \leq D^k$. Then

$$\sum_{x_{1:n}\in\mathcal{X}^n} D^{-\ell^*(x_{1:n})} = \sum_{k=1}^{nL} N_k D^{-k} \leq \sum_{k=1}^{nL} D^k D^{-k} \leq nL.$$

On the other hand,

$$\sum_{x_{1:n}\in\mathcal{X}^n} D^{-\ell^*(x_{1:n})} = \sum_{x_1\in\mathcal{X}}\sum_{x_2\in\mathcal{X}}\cdots\sum_{x_n\in\mathcal{X}} D^{-\ell(x_1)}D^{-\ell(x_2)}\cdots D^{-\ell(x_n)}$$
$$= \sum_{x_1\in\mathcal{X}} D^{-\ell(x_1)}\sum_{x_2\in\mathcal{X}} D^{-\ell(x_2)}\cdots\sum_{x_n\in\mathcal{X}} D^{-\ell(x_n)} = \left(\sum_{x\in\mathcal{X}} D^{-\ell(x)}\right)^n.$$

Therefore, we have

$$\sum_{x\in\mathcal{X}} D^{-\ell(x)} \le \inf_{n\in\mathbb{N}} \sqrt[n]{nL} = 1.$$

Then we complete the proof. $\square$

Remark. To summarize, the Kraft's inequality (2.1) is a

- sufficient condition for the existence of an instantaneous code;
- necessary condition for a code to be uniquelt decodable.

2.2 Fundamental Limits of Compression

In this section, we study the limits of lossless compression. Given a source distribution p on $\mathcal{X}$, we want to minimize the average codeword length of our code. By Kraft-McMillan inequality, the search for optimal code can be expressed as the following optimization problem:

$$\min_{l:\mathcal{X}\to\mathbb{N}} \sum_{x\in\mathcal{X}} p(x)\ell(x) \quad \textit{subject to} \quad \sum_{x\in\mathcal{X}} D^{-\ell(x)} \le 1.$$

Following is a fundamental result of lossless compression.

Theorem 2.4. *For any source distribution $X \sim p$ on $\mathcal{X}$, the expected length $\mathbb{E}[\ell(X)]$ of an optimal uniquely decodable D-ary code satisfies*

$$\frac{H(X)}{\log D} \le \mathbb{E}\left[\ell(X)\right] < \frac{H(X)}{\log D} + 1. \tag{2.2}$$

Proof. UPPER BOUND. By Theorem 2.2, it suffices to construct a length function $\ell : \mathcal{X} \to \mathbb{N}$ that satisfies both the Kraft's inequality and the second (strict) inequality given in (2.2). Consider Shannon's length function:

$$\ell(x) = \left\lceil \log_D \frac{1}{p(x)} \right\rceil, \quad x \in \mathcal{X}, \tag{2.3}$$

Since

$$\sum_{x\in\mathcal{X}} D^{-\ell(x)} \le \sum_{x\in\mathcal{X}} D^{\log_D p(x)} = \sum_{x\in\mathcal{X}} p(x) = 1,$$

there exists an instantaneous code $C : \mathcal{X} \to \mathcal{D}^*$ whose length function is ℓ. On the other hand,

$$\mathbb{E}[\ell(X)] = \sum_{x\in\mathcal{X}} p(x)\ell(x) < \sum_{x\in\mathcal{X}} p(x)\left(\log_D \frac{1}{p(x)} + 1\right) = \frac{H(X)}{\log D} + 1.$$

Hence the upper bound holds.

Lower bound. We consider the following relaxed optimization problem:

$$\min_{l:\mathcal{X}\to\mathbb{R}} \sum_{x\in\mathcal{X}} p(x)\ell(x) \quad \textit{subject to} \quad \sum_{x\in\mathcal{X}} D^{-\ell(x)} \le 1.$$

Note that the range of ℓ is $\mathbb{R}_+$. The Lagrange function is

$$L(l,\lambda) = \sum_{x\in\mathcal{X}} p(x)\ell(x) + \lambda\left(\sum_{x\in\mathcal{X}} D^{-\ell(x)} - 1\right),$$

with KKT conditions

$$\begin{cases} \frac{\partial L}{\partial l(x)} = p(x) - \lambda D^{-l(x)}\log D = 0, \\ \lambda \ge 0,\ \sum_{x\in\mathcal{X}} D^{-\ell(x)} - 1 \le 0, \\ \lambda\left(\sum_{x\in\mathcal{X}} D^{-\ell(x)} - 1\right) = 0. \end{cases}$$

The optimal solution is given by

$$\lambda = \frac{1}{\log D}, \quad l(x) = \log_D \frac{\lambda \log D}{p(x)} = \log_D \frac{1}{p(x)},\ x \in \mathcal{X},$$

and the optimal value is

$$\sum_{x\in\mathcal{X}} p(x)\ell(x) = \sum_{x\in\mathcal{X}} p(x)\log_D \frac{1}{p(x)} = \frac{H(X)}{\log D}. \tag{2.4}$$

Since our problem is relaxed, the primal problem (2.3) has optimal value no less than (2.4). Hence the lower bound holds for all uniquely decodable codes. □

Remark. In fact, we proved the existence of an *instantaneous* code with

$$\mathbb{E}\left[\ell(X)\right] < \frac{H(X)}{\log D} + 1.$$

Coding over blocks. Using integer codeword lengths may lead to waste of memory. To overcome this effect, we consider coding over blocks of input symbols. If the input data $X_1, X_2, \cdots$ is an i.i.d. sequence of symbols, we partition it into blocks of size n and create a new source $\widetilde{X}_1, \widetilde{X}_2, \cdots$, where

$$\widetilde{X}_1 = (X_1, \cdots, X_n),\ \widetilde{X}_2 = (X_{n+1}, \cdots, X_{2n}),\ \cdots,\ \widetilde{X}_k = (X_{(k-1)n+1}, \cdots, X_{kn}),\ \cdots.$$

Consequently, every vector $\widetilde{X}_k$ can be viewed as a symbol from the alphabet $\widetilde{\mathcal{X}} = \mathcal{X}^n$, and we can find an optimal code $\widetilde{C} : \widetilde{X} \to \mathcal{D}$, whose length function ℓ satisfies

$$\frac{H(\widetilde{X})}{\log D} \le \mathbb{E}\left[\ell(\widetilde{X})\right] \le \frac{H(\widetilde{X})}{\log D} + 1.$$

Note that $H(\widetilde{X}) = nH(X)$, the average codeword length per symbol (in $\mathcal{X}$) satisfies

$$\frac{H(X)}{\log D} \le \frac{1}{n}\mathbb{E}\left[\ell(\widetilde{X})\right] < \frac{H(X)}{\log D} + \frac{1}{n}.$$

As the block size n increases, the integer effect becomes negligible. However, we also introduce delay in our system and increase the complexity of our code.

2.3 Shannon-Fano-Elias Coding

In this section, we introduce a specific coding approach that is near-optimal.

Midpoints of CDF. Without loss of generality, we assume that the source alphabet is $\mathcal{X} = \{1, 2, \cdots, m\}$, and $p(1) \geq p(2) \geq \cdots \geq p(m)$. The cumulative distribution function of p is

$$F(r) = \sum_{j=1}^{m} \mathbb{1}_{\{j \leq r\}} p(j), \quad r \in \mathbb{R}.$$

We define $\overline{F}(x)$ to be the midpoint of the interval $[F(x-1), F(x))$:

$$\overline{F}(x) = \sum_{j=1}^{x-1} p(j) + \frac{p(x)}{2}, \quad x = 1, \cdots, m.$$

Then $\overline{F}(x)$ is a real number in $(0, 1)$ that uniquely identifies $x \in \mathcal{X}$.

***D*-ary expansion and truncation.** The D-ary expansion of a real number $\overline{F}(x) \in (0, 1)$ is given by

$$\overline{F}(x) = (0.z_1 z_2 \cdots)_D = \sum_{k=1}^{\infty} z_k D^{-k} = z_1 D^{-1} + z_2 D^{-2} + \cdots, \quad z_1, z_2, \cdots \in \{0, 1, \cdots, D-1\}.$$

Given a positive integer $\ell \in \mathbb{N}$, one have the ℓ-truncation of the D-ary expansion of $\overline{F}(x)$:

$$C(x) = (0.z_1 z_2 \cdots z_\ell)_D = \sum_{k=1}^{\ell} z_k D^{-k}$$

To ensure that the codeword of x is unique, we let $\overline{F}(x) - C(x) < \frac{p(x)}{2}$, so that

$$C(x-1) \leq \overline{F}(x-1) < F(x-1) < C(x).$$

To this end, we set

$$\ell = \left\lceil \log_D \frac{1}{p(x)} \right\rceil + 1,$$

then

$$\overline{F}(x) - C(x) < D^{-\ell} \leq D^{-\log_D \frac{1}{p(x)} - 1} \leq \frac{p(x)}{D} \leq \frac{p(x)}{2}.$$

Construction of the Shannon-Fano-Elias code. For each $x \in \mathcal{X}$:

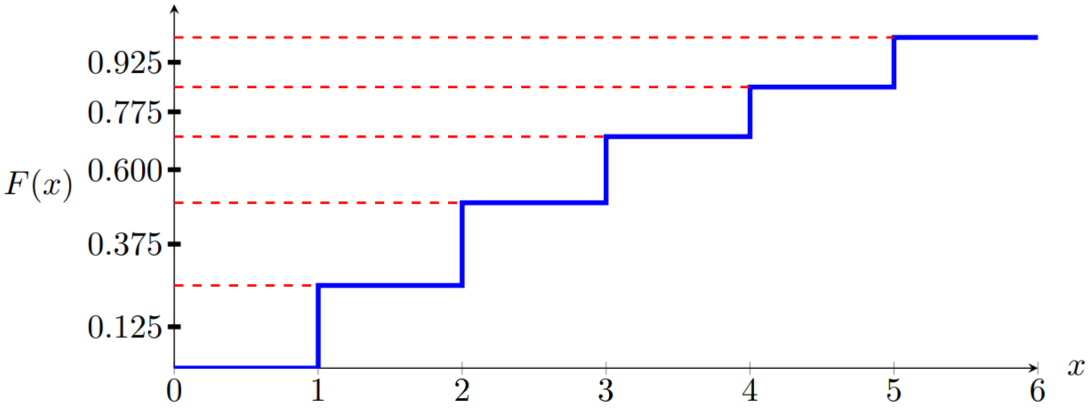
- Let z be the D-ary expansion of x;
- Choose the length of the codeword of x:

$$\ell(x) = \left\lceil \log_D \frac{1}{p(x)} \right\rceil + 1;$$

- Choose the codeword of x to be the first most significant D-ary digits:

$$z = 0.\underbrace{z_1 z_2 \cdots z_{\ell(x)}}_{C(x)} z_{\ell(x)+1} \cdots.$$

An example of binary Shannon-Fano-Elias code. Here we let $\mathcal{X} = \{1,2,3,4,5\}$, and $D = 2$.



x	$p(x)$	$F(x)$	$\overline{F}(x)$	$\overline{F}(x)$ in binary	$\ell(x) = \left\lceil \log_2 \frac{1}{p(x)} \right\rceil + 1$	codeword
1	0.25	0.25	0.125	0.001	3	001
2	0.25	0.5	0.375	0.011	3	011
3	0.2	0.7	0.6	0.10011	4	1001
4	0.15	0.85	0.775	0.1100011	4	1100
5	0.15	1.0	0.925	0.1110110	4	1110

Shannon-Fano-Elias code is instantaneous. If the codeword $C(x) = (0.z_1 \cdots z_{\ell(x)})_D$ is a prefix of another codeword, this codeword lies in the half-open interval

$$\left[(0.z_1 \cdots z_{\ell(x)})_D, (0.z_1 \cdots z_{\ell(x)})_D + \frac{1}{D^{l(x)}} \right).$$

However, a contradiction rises because

$$C(x+1) - C(x) > F(x) - \overline{F}(x) = \frac{p(x)}{2} \geq D^{-l(x)}.$$

Average codeword length. The average codeword length of Shannon-Fano-Elias code is given by

$$\mathbb{E}[\ell(X)] = \sum_{x \in \mathcal{X}} p(x) \left(\left\lceil \log_D \frac{1}{p(x)} \right\rceil + 1 \right),$$

which satisfies

$$\frac{H(X)}{\log D} + 1 \leq \mathbb{E}\left[\ell(X)\right] < \frac{H(X)}{\log D} + 2.$$

It is revealed that the Shannon code is sub-optimal.

Improvement: Shannon Code. We consider

$$F(x) = \sum_{j=1}^{x-1} p(x), \quad \ell(x) = \left\lceil \log \frac{1}{p(x)} \right\rceil.$$

We choose the codeword $c(x)$ to be the $\ell(x)$-truncation of the D-ary expansion of $F(x)$.

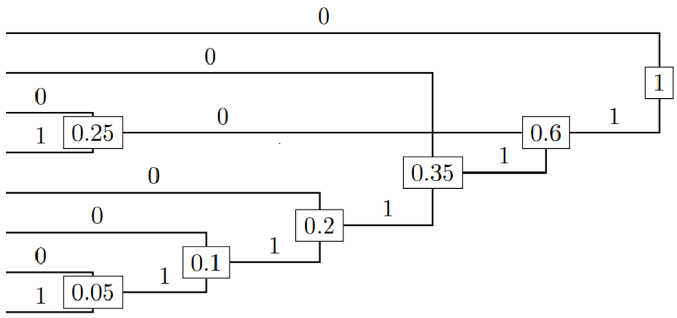
2.4 Huffman Coding

The search for binary optimal code was discovered by David Huffman (1952).

Construction of Huffman tree. The construction procedure is greedy.

- Take the two least probable symbols, which will be assigned the longest codewords having equal lengths and differing only at the last digit;
- Merge these two symbols into a new symbol with combined probability mass and repeat.

Codeword	x	$p(x)$
0	1	0.40
110	2	0.15
100	3	0.15
101	4	0.10
1110	5	0.10
11110	6	0.05
111110	7	0.04
111111	8	0.01



Optimality of Huffman code. Let $\mathcal{X} = \{1, 2, \cdots, m\}$. Without loss of generality, assume probabilities are in descending order $p(1) \geq p(2) \geq \cdots \geq p(m)$. We prove the optimality of Huffman code through three step.

Lemma 2.5. *In an optimal code, shorter codewords are assigned larger probabilities, i.e.* $p(i) > p(j)$ *implies* $\ell(i) \leq \ell(j)$.

Proof. Argue by contradiction. If there exists $i, j \in \mathcal{X}$ with $\ell(i) \leq \ell(j)$ and $p(i) > p(j)$, then we can exchange these codewords and reduce the expected length. Hence the code is not optimal. $\square$

Lemma 2.6. *There exists an optimal code for which the codewords assigned to the smallest probabilities are siblings, i.e., they have the same length and differ only in the last symbol.*

Proof. Consider any optimal code. By Lemma 2.5, the codeword $C(m)$ has the longest length. Assume for the sake of contradiction, its sibling is not a codeword. Then the expected length can be decreased by moving $C(m)$ to its parent. Thus, the code is not optimal and a contradiction is reached.

Now, we know the sibling of $C(m)$ is a codeword. If it is $C(m-1)$, we are done. If it is some $C(i)$ for $i \neq m-1$ and the code is optimal, by Lemma 2.5, we have $p(i) = p(m-1)$. Therefore, $C(i)$ and $C(m-1)$ can be exchanged without changing expected length. $\square$

Theorem 2.7 (Optimality of Huffman coding)**.** *Huffman's coding algorithm produces an optimal code tree.*

Proof. Let ℓ be the length function of the optimal code. By Lemmas 2.5 and 2.6, $C(m)$ and $C(m-1)$ are siblings and the longest codewords. Then we merge the two symbols and let $\widetilde{p}_1 \geq \cdots \geq \widetilde{p}_{m-1}$ denote the reordered probabilities after merging $p(m)$ and $p(m-1)$, and denote by $\widetilde{C}_1, \cdots, \widetilde{C}_{m-1}$ the corresponding codewords. The reduced length function $\widetilde{\ell}$ satisfies

$$\mathbb{E}\left[\ell(X)\right] = \mathbb{E}\left[\widetilde{\ell}(\widetilde{X})\right] + \mathbb{P}\left(\ell(X) \neq \widetilde{\ell}(\widetilde{X})\right) = \mathbb{E}\left[\widetilde{\ell}(\widetilde{X})\right] + p(m-1) + p(m).$$

Hence ℓ is the length function of an optimal code if and only if $\widetilde{\ell}$ is the length function of an optimal code for the reduced alphabet. The problem then is reduced to finding an optimal code tree for $\widetilde{p}_1 \geq \cdots \geq \widetilde{p}_{m-1}$. Repeat the merging procedure above for m times, and the result follows. $\square$

2.5 Coding with Unknown Distributions

Given a distribution $X \sim p$, it is possible to construct a code that achieves the optimal expected length. However, we do not know what to do when the distribution p is unknown. In this section, we suppose that X is drawn from some distribution p_θ parameterized by an unknown parameter $\theta \in \Theta$.

Definition 2.8 (Redundancy)**.** *The redundancy of coding a distribution p with respect to the optimal code for a distribution q, i.e. $\ell(x) = -\log q(x)$, is given by*

$$R(p, q) = \sum_{x \in \mathcal{X}} p(x)\ell(x) - H(p) = \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)} = D(p||q).$$

Given a family of distributions $\{p_\theta\}_{\theta \in \Theta}$, the minimax redundancy is

$$R^* = \min_q \max_{\theta \in \Theta} R(p_\theta, q).$$

Remark. Intuitively, the distribution q leading to a code that minimizes the maximum redundancy is the distribution at the center of the "information ball" of radius R^*. Therefore, by constructing an optimal code based on q, we can reduce the redundancy in the worst case.

Lemma 2.9. *We impose a prior distribution π on Θ. Then*

$$\max_{\theta \in \Theta} R(p_\theta, q) = \max_\pi \sum_{\theta \in \Theta} \pi(\theta) R(p_\theta, q).$$

Proof. On the one hand,

$$\max_{\theta \in \Theta} R(p_\theta, q) = \max_{\theta_0 \in \Theta} \sum_{\theta \in \Theta} \delta_{\theta_0}(\theta) R(p_\theta, q) \le \max_\pi \sum_{\theta \in \Theta} \pi(\theta) R(p_\theta, q).$$

On the other hand, if $\theta^* \in \Theta$ maximizes $R(p_\theta, q)$, one have

$$\sum_{\theta \in \Theta} \pi(\theta) R(p_\theta, q) \le \sum_{\theta \in \Theta} \pi(\theta) R(p_{\theta^*}, q) = R(p_{\theta^*}, q) = \max_{\theta \in \Theta} R(p_\theta, q), \quad \forall \pi \in \Delta(\Theta).$$

Then we complete the proof. ☐

We also introduce another technical theorem.

Theorem 2.10 (Minimax theorem)**.** *If $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is a continuous function that is convex in the first variable and concave in the second variable. If both $\mathcal{X}$ and $\mathcal{Y}$ are convex compact sets, then*

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y) = \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} f(x, y).$$

Remark. To develop the following theorem, we use the joint convexity of Kullback-Leibler divergence:

$$D((1-\lambda)p_0 + \lambda p_1 || (1-\lambda)q_0 + \lambda q_1) \le (1-\lambda) D(p_0||q_0) + \lambda D(p_1||q_1).$$

Theorem 2.11. *The minimax redundancy is the maximum mutual information between θ and X:*

$$R^* = \max_\pi I(\theta; X),$$

where $\pi(\theta)$ is the prior distribution of the parameter θ, and $X|\theta \sim p_\theta(x)$.

Proof. Using Lemma 2.9 and Theorem 2.10, we reformulate the optimization problem:

$$R^* = \min_q \max_{\theta \in \Theta} R(p_\theta, q) = \min_q \max_\pi \sum_{\theta \in \Theta} \pi(\theta) R(p_\theta, q) = \max_\pi \min_q \sum_{\theta \in \Theta} \pi(\theta) R(p_\theta, q). \tag{2.5}$$

We write

$$q_\pi(x) = \sum_{\theta \in \Theta} \pi(\theta) p_\theta(x).$$

Then

$$\begin{aligned}
\sum_{\theta \in \Theta} \pi(\theta) R(p_\theta, q) &= \sum_{\theta \in \Theta} \pi(\theta) D(p_\theta || q) - D(q_\pi || q) + D(q_\pi || q) \\
&= \sum_{\theta \in \Theta} \sum_{x \in \mathcal{X}} \pi(\theta) p_\theta(x) \log \frac{p_\theta(x)}{q(x)} - \sum_{x \in \mathcal{X}} \sum_{\theta \in \Theta} \pi(\theta) p_\theta(x) \log \frac{q_\pi(x)}{q(x)} + D(q_\pi || q) \\
&= \sum_{\theta \in \Theta} \sum_{x \in \mathcal{X}} \pi(\theta) p_\theta(x) \log \frac{p_\theta(x)}{q_\pi(x)} + D(q_\pi || q)
\end{aligned}$$

Since the first term does not depends on q, the last display reaches its minimum if and only if $q = q_\pi$:

$$\begin{aligned}
\min_q \sum_{\theta \in \Theta} \pi(\theta) R(p_\theta, q) &= \sum_{\theta \in \Theta} \sum_{x \in \mathcal{X}} \pi(\theta) p_\theta(x) \log \frac{p_\theta(x)}{q_\pi(x)} \\
&= \sum_{\theta \in \Theta} \sum_{x \in \mathcal{X}} \pi(\theta) p_\theta(x) \log \frac{\pi(\theta) p_\theta(x)}{\pi(\theta) q_\pi(x)} = I(\theta; X),
\end{aligned}$$

where $\pi(\theta) p_\theta(x)$ is the joint distribution of θ and X, and $q_\pi(x)$ is the marginal distribution of X. Plugging in this expression to (2.5) completes the proof. $\square$

3 Channel Coding

Motivation. In a communication situation, we often have two primary goals:

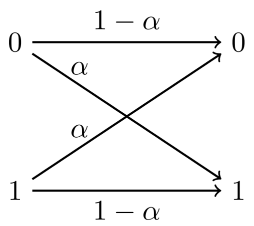
- *Reliability.* The received message should be equal to the transmitted message in most cases. In other words, we wish to reduce the error probability:

$$P_e = \mathbb{P}\left(\textit{received message} \neq \textit{transmitted message}\right).$$

- *Efficiency.* The message should be transmitted as quickly as possible. In other words, we wish to send as much information as possible in a unit time:

$$R = \textit{average number of information bits transmitted per unit time.}$$

However, these two goals often conflict with each other. We use the Binary Symmetric Channel (BSC) to interpret this. Suppose that we want to send a bit $W \in \{0,1\}$. A binary symmetric channel has a binary input $X \in \{0,1\}$ and a binary output $Y \in \{0,1\}$. While sending a bit, it flips the bit with probability α:



To reduce the error probability, we use the channel multiple times. Assume that each use of the channel consumes a unit time, and the channel is memoryless, i.e., given the input, the outputs of the channel are conditionally independent. We encode the bit using a repetition code:

$$W = 0 \quad \Rightarrow \quad X_{1:n} = \underbrace{00\cdots 0}_{n}, \qquad W = 1 \quad \Rightarrow \quad X_{1:n} = \underbrace{11\cdots 1}_{n}.$$

Given the output $Y_{1:n}$, we decode the bit using the maximum likelihood rule:

$$\widehat{W} = \begin{cases} 0, & \textit{if there are more 0's observed in } Y_{1:n} \textit{ than 1's}, \\ 1, & \textit{otherwise}. \end{cases}$$

As the uses n of channel increases, the error probability decreases, but the bit the channel transmitted every unit time $R = 1/n$ also decreases. Hence a tradeoff between reliability and efficiency is required.

3.1 Set-up of Channel Encoding

In this section, we study the problem of channel coding. Consider the communication over a random channel:

$$\underset{\textit{Message}}{W} \xrightarrow{\textit{Encoder } \mathcal{E}} X_{1:n} \xrightarrow{\textit{Channel}} Y_{1:n} \xrightarrow{\textit{Decoder } \mathcal{D}} \underset{\textit{Estimate}}{\widehat{W}}$$

- The message $W \in \{1, \cdots, M\}$ is one of the possible M numbers that we want to send. We always assume W to be uniformly distributed over all possibilities.
- An (M,n)-coding scheme is an encoder $\mathcal{E} : \{1, \cdots, M\} \to \mathcal{X}^n$ that maps the message M to an n-length string of channel inputs X^n;

- The channel specifies the probabilistic transformation from inputs to outputs:

$$p(y_{1:n}|x_{1:n}) = \mathbb{P}\left(Y_1 = y_1, \cdots, Y_n = y_n | X_1 = x_1, \cdots, X_n = x_n\right).$$

  We are particularly interested in the *discrete memoryless channel (DMC)*, which is specified by

  (i) an input alphabet $\mathcal{X}$,

  (ii) an output alphabet $\mathcal{Y}$, and

  (iii) a conditional probability distribution $p_{Y|X}(y|x)$ such that the outputs between channel uses are conditionally independent given the inputs:

$$p(y_{1:n}|x_{1:n}) = p_{Y|X}(y_1|x_1)\cdots p_{Y|X}(y_n|x_n).$$

- A decoder $\mathcal{D} : \mathcal{Y}^n \to \{1, \cdots, M\}$ maps an n-length string of channel outputs $Y_{1:n}$ to an estimate $\widehat{W}$ of the transmitted message.

Now recall our two primary goals in communication:

- *Reliability.* Assuming that the message W is uniformly distributed over all possibilities, the *conditional error probability* and the *average error probability* are

$$P_e^{(n)}(w) = \mathbb{P}(\widehat{W} \neq w | W = w), \qquad P_e^{(n)} = \mathbb{P}(\widehat{W} \neq W) = \frac{1}{M}\sum_{w=1}^{M} P_e^{(n)}(W).$$

  The maximum error probability is

$$P_{\text{e,max}}^{(n)}(w) = \max_{w\in\{1,\cdots,M\}} P_e^{(n)}(w) = \max_{w\in\{1,\cdots,M\}} \mathbb{P}(\widehat{W} \neq w | W = w).$$

- *Efficiency.* The *rate* R of an (M, n) encoding scheme is

$$R = \frac{\log_2 M}{n} \quad \textit{bits/transmission}.$$

  Alternatively, the number of messages for a given rate R and block-length n is given by $M = 2^{nR}$. To specify a rate R code, we write $(2^{nR}, n)$ instead of (M, n). Particularly, are interested in the case that the error probability becomes negligible as the coding length n goes infinity.

Definition 3.1 (Operational Capacity). *A rate R is achievable for given discrete memoryless channel $p(y|x)$, if there exists a sequence of $(\lceil 2^{nR}\rceil, n)$ coding schemes such that maximum error probability*

$$\lim_{n\to\infty} P_{\text{e,max}}^{(n)} = 0.$$

The operational capacity C_{op} is the supremum over all achievable rates:

$$C_{\text{op}} = \sup\left\{R : R \textit{ is achievable}\right\}.$$

Definition 3.2 (Information Capacity). *The information capacity of a discrete memoryless channel is*

$$C = \sup_{p_X} I(X;Y) = \sup_{p_X} \sum_{x\in\mathcal{X}, y\in\mathcal{Y}} p_X(x) p_{Y|X}(y|x) \log_2 \frac{p_{Y|X}(y|x)}{\sum_{x'\in\mathcal{X}} p_{Y|X}(y|x') p_X(x')}$$

Remark. Since the map $p_X, p_{Y|X} \mapsto I(X,Y)$ is concave about p_X, we can always find a maximizer p_X^* that reaches the supremum: $C = \max_{p_X} I(X;Y)$.

3.2 Shannon's Channel Coding Theorem: Achievability

In the next two sections, we will establish Shannon's channel coding theorem.

Theorem 3.3 (Shannon's channel coding theorem)**.** *The operational capacity of a discrete memoryless channel is equal to the information capacity:*

$$C_{\text{op}} = \sup_{p_X} I(X;Y).$$

Remark. In fact, the channel coding theorem consists of two statements:

- *Achievability.* Every rate $R < C$ is achievable, i.e. there exists a sequence of $(2^{nR}, n)$ coding schemes such that the maximum error probability $P^{(n)}_{\text{e,max}} \to 0$ as $n \to \infty$:

$$R < C \quad \Rightarrow \quad R \textit{ is achievable.}$$

- *Converse.* Any sequence of $(2^{nR}, n)$ coding schemes with the maximum error probability $P^{(n)}_{\text{e,max}} \to 0$ as $n \to \infty$ must satisfy $R \le C$.

$$R \textit{ is achievable} \quad \Rightarrow \quad R \le C.$$

In this section, we are going to establish the achievability part of channel encoding theorem.

Construction of encoder $\mathcal{E}$. A $(2^{nR}, n)$ encoder $\mathcal{E}$ can be represented by a codebook:

$$\mathcal{E} = \begin{pmatrix} x_{1:n}(1) \\ x_{1:n}(2) \\ \vdots \\ x_{1:n}(2^{nR}) \end{pmatrix} = \begin{pmatrix} x_1(1) & x_1(2) & \cdots & x_n(1) \\ x_1(2) & x_2(2) & \cdots & x_n(2) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(2^{nR}) & x_2(2^{nR}) & \cdots & x_n(2^{nR}) \end{pmatrix} \in \mathcal{X}^{2^{nR} \times n}. \tag{3.1}$$

To transmit a message w, the encoder assigns

$$\mathcal{E}(w) = x_{1:n}(w), \quad w \in \left\{1, 2, \cdots, 2^{nR}\right\}.$$

We consider the construction of random encoder. To proceed, we first choose a input distribution p_X. We let each entry in the codebook $\mathcal{E}$ to be drawn from i.i.d. p_X. The probability of generating any particular random codebook (3.1) is then given by

$$p(\mathcal{E}) = \prod_{w}^{2^{nR}} \prod_{i=1}^{n} p_X(x_n(w)).$$

With the codebook $\mathcal{E}$ specified, the conditional distribution of input string $X_{1:n}$ is the

$$p_{X_{1:n}|\mathcal{E}}(x_{1:n}) = \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \mathbb{1}_{\{x_{1:n} = \mathcal{E}(w)\}}, \quad x_{1:n} \in \mathcal{X}^n,$$

and

$$p_{Y_{1:n}|\mathcal{E}}(y_{1:n}) = \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} p_{Y_{1:n}|X_{1:n}}(y_{1:n}|\mathcal{E}(w)), \quad y_{1:n} \in \mathcal{Y}^n.$$

To find the unconditional distribution, note that each row of the codebook has the same distribution:

$$\begin{aligned}
p_{X_{1:n}}(x_{1:n}) &= \prod_{i=1}^{n} p_X(x_i); \\
p_{Y_{1:n}}(y_{1:n}) &= \sum_{x_{1:n}\in\mathcal{X}^n} p_{X_{1:n}}(x_{1:n}) p_{Y_{1:n}|X_{1:n}}(y_{1:n}|x_{1:n}) \\
&= \sum_{x_1\in\mathcal{X}} \sum_{x_2\in\mathcal{X}} \cdots \sum_{x_n\in\mathcal{X}} \prod_{i=1}^{n} p_X(x_i) p_{Y|X}(y_i|x_i) \\
&= \prod_{i=1}^{n} \underbrace{\left(\sum_{x_i\in\mathcal{X}} p_X(x_i) p_{Y|X}(y_i|x_i) \right)}_{p_Y(y_i)} = \prod_{i=1}^{n} p_Y(y_i).
\end{aligned}$$

Since the channel is memoryless, the *information density* of $(X_{1:n}, Y_{1:n})$ can be factorized:

$$\begin{aligned}
i(x_{1:n}; y_{1:n}) &= \log_2 \frac{p_{X_{1:n},Y_{1:n}}(x_{1:n}, y_{1:n})}{p_{X_{1:n}}(x_{1:n}) p_{Y_{1:n}}(y_{1:n})} = \log_2 \frac{p_{Y_{1:n}|X_{1:n}}(y_{1:n}|x_{1:n})}{p_{Y_{1:n}}(y_{1:n})} \\
&= \sum_{k=1}^{n} \log_2 \frac{p_{Y|X}(y_k|x_k)}{p_Y(y_k)} = \sum_{k=1}^{n} i(x_k; y_k).
\end{aligned}$$

These distributions arise from the randomness in both the codebook and the message.

Construction of decoder $\mathcal{D}$. To finish the construction of a coding scheme, we need to find an optimal decoder. To minimize the probability of error, we use a *maximum a posteriori* (MAP) decoder:

$$\begin{aligned}
\mathcal{D}^*(y_{1:n}) &= \operatorname*{argmax}_{w\in\{1,\cdots,2^{nR}\}} p_{W|Y_{1:n}}(w|y_{1:n}) \\
&= \operatorname*{argmax}_{w\in\{1,\cdots,2^{nR}\}} p_W(w) p_{Y_{1:n}|W}(y_{1:n}|w).
\end{aligned}$$

Since the message W is uniform, the MAP decoder is equivalent to the maximum likelihood decoder:

$$\mathcal{D}^*(y_{1:n}) = \operatorname*{argmax}_{w\in\{1,\cdots,2^{nR}\}} p_{Y_{1:n}|W}(y_{1:n}|w).$$

Using the information density, we have

$$\begin{aligned}
\mathcal{D}^*(y_{1:n}) &= \operatorname*{argmax}_{w\in\{1,\cdots,2^{nR}\}} p_{Y_{1:n}|X_{1:n}}(y_{1:n}|x_{1:n}(w)) \\
&= \operatorname*{argmax}_{w\in\{1,\cdots,2^{nR}\}} \frac{p_{Y_{1:n}|X_{1:n}}(y_{1:n}|x_{1:n}(w))}{p_{Y_{1:n}}(y_{1:n})} \\
&= \operatorname*{argmax}_{w\in\{1,\cdots,2^{nR}\}} i(x_{1:n}(w); y_{1:n}).
\end{aligned}$$

To simplify the analysis, we study a sub-optimal thresholding decoder: For a given threshold T_n, we define the decoding rule as follows:

$$\mathcal{D}(y_{1:n}) = \begin{cases} \widehat{w}, & if\ i(x_{1:n}(\widehat{w}); y_{1:n}) > T_n\ and\ i(x_{1:n}(w); y_{1:n}) \le T_n\ for\ all\ w \neq \widehat{w}, \\ 0, & otherwise. \end{cases}$$

Decoding error is uniform. We now analyze the decoding error of our coding scheme. By uniformity of our construction of codebook and the message W,

$$\begin{aligned}
\mathbb{P}(\widehat{W} \neq W) &= \sum_{\mathcal{E}} p(\mathcal{E})\, \mathbb{P}(\widehat{W} \neq W|\mathcal{E}) \\
&= \sum_{\mathcal{E}} p(\mathcal{E}) \sum_{w=1}^{2^{nR}} \frac{1}{2^{nR}} \mathbb{P}(\widehat{W} \neq W|\mathcal{E}, W = w) \\
&= \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \sum_{\mathcal{E}} p(\mathcal{E}) \mathbb{P}(\widehat{W} \neq W|\mathcal{E}, W = w) \\
&= \frac{1}{2^{nR}} \sum_{w=1}^{2^{nR}} \sum_{\mathcal{E}} p(\mathcal{E}) \mathbb{P}(\widehat{W} \neq W|\mathcal{E}, W = 1) \\
&= \sum_{\mathcal{E}} p(\mathcal{E}) \mathbb{P}(\widehat{W} \neq W|\mathcal{E}, W = 1) \\
&= \mathbb{P}(\widehat{W} \neq W|W = 1)
\end{aligned}$$

Therefore, it suffices to control the decoding error conditioned on the event $W = 1$.

Proof of Theorem 3.3 (Achievability). Define events A and B as follows:

$$A_n = \{i(X_{1:n}(1); Y_{1:n}) > T_n\}\,, \quad B_n = \bigcap_{w=2}^{2^{nR}} \{i(X_{1:n}(w); Y_{1:n}) \leq T_n\}\,.$$

Consider the following bound:

$$P(\widehat{W} \neq W|W = 1) = P(A_n^c \cup B_n^c) \leq \mathbb{P}(A_n^c) + \mathbb{P}(B_n^c).$$

Analysis of $\mathbb{P}(A_n^c)$. By construction, the input $X_{1:n}(1)$ and output $Y_{1:n}$ satisfies

$$(X_k(1), Y_k) \overset{\text{i.i.d.}}{\sim} p_X p_{Y|X}.$$

Meanwhile,

$$\mathbb{E}\left[i(X_k(1), Y_k)\right] = \mathbb{E}\left[\log_2 \frac{p_{Y|X}(Y_k|X_k(1))}{p_Y(Y_k)}\right] = I(X;Y), \quad \textit{where} \;\; (X,Y) \sim p_X p_{Y|X}.$$

By strong law of large numbers,

$$\frac{i(X_{1:n}(1); Y_{1:n})}{n} = \frac{1}{n} \sum_{k=1}^{n} i(X_k(1), Y_k) \overset{a.s.}{\to} I(X;Y) \;\; \textit{as} \;\; n \to \infty.$$

Fix any $\epsilon > 0$, and set $T_n = n(I(X;Y) - \epsilon)$. Hence

$$\begin{aligned}
\limsup_{n\to\infty} \mathbb{P}(A_n^c) &= \limsup_{n\to\infty} \mathbb{P}\left(\frac{i(X_{1:n}(1); Y_{1:n})}{n} \leq I(X;Y) - \epsilon\right) \\
&\leq \mathbb{P}\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{\frac{i(X_{1:n}(1); Y_{1:n})}{n} \leq I(X;Y) - \epsilon\right\}\right) \\
&= \mathbb{P}\left(\limsup_{n\to\infty} \frac{i(X_{1:n}(1); Y_{1:n})}{n} \leq I(X;Y) - \epsilon\right) = 0.
\end{aligned}$$

Analysis of $\mathbb{P}(B_n^c)$. By construction, for all $w \neq 1$, $X_{1:n}(w)$ is independent of $X_{1:n}(1)$. Since the output $Y_{1:n}$ is generated from $X_{1:n}(1)$ and $p_{Y|X}$, it is independent of $X_{1:n}(w)$:

$$(X_k(w), Y_k) \overset{\text{i.i.d.}}{\sim} p_X p_Y.$$

Using the Chernoff bound, we have

$$\begin{aligned}
\mathbb{P}\left(i(X_{1:n}(w), Y_{1:n}) > T_n\right) &\leq 2^{-T_n} \mathbb{E}\left[2^{i(X_{1:n}(w);Y_{1:n})}\right] \\
&= 2^{-T_n} \mathbb{E}\left[\frac{p_{X_{1:n},Y_{1:n}}(X_{1:n}(w), Y_{1:n})}{p_{X_{1:n}}(X_{1:n}(w)) p_{Y_{1:n}}(Y_{1:n})}\right] \\
&= 2^{-T_n} \sum_{x_{1:n}\in\mathcal{X}^n} \sum_{y_{1:n}\in\mathcal{Y}^n} p_{X_{1:n}}(x_{1:n}) p_{Y_{1:n}}(y_{1:n}) \frac{p_{X_{1:n},Y_{1:n}}(x_{1:n}, y_{1:n})}{p_{X_{1:n}}(x_{1:n}) p_{Y_{1:n}}(y_{1:n})} \\
&= 2^{-T_n}.
\end{aligned}$$

We then employ a union bound:

$$\begin{aligned}
\mathbb{P}(B_n^c) &= \mathbb{P}\left(\bigcup_{w=2}^{2^{nR}} \{i(X_{1:n}(w), Y_{1:n}) > T_n\}\right) \\
&\leq \sum_{w=2}^{2^{nR}} \mathbb{P}\left(i(X_{1:n}(w), Y_{1:n}) > T_n\right) \\
&\leq 2^{nR-T_n} \\
&= 2^{n(R-I(X;Y)+\epsilon)}.
\end{aligned}$$

Choice of ϵ and p_X. Since $R < C = \sup_{p_X} I(X;Y)$, we choose $\epsilon = \frac{1}{3}(C-R)$, and choose p_X such that

$$I(X;Y) \geq R + 2\epsilon = C - \frac{1}{3}(C-R).$$

Then we have

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(\widehat{W} \neq W | W = 1) &\leq \lim_{n\to\infty} \mathbb{P}(A_n^c) + \lim_{n\to\infty} \mathbb{P}(B_n^c) \\
&\leq \lim_{n\to\infty} 2^{-n\epsilon} = 0.
\end{aligned}$$

Based on our previous discussion, the result follows. □

Remark. Although the theorem shows that there exist good codes with arbitrarily small error probability for long block lengths, it does not provide an approach to construct the optimal codebooks.

If we employed the scheme suggested by the proof and generate a code at random with the appropriate distribution, the code constructed is likely to be good for long block lengths. However, without some structure in the code, it is very difficult to decode (the simple scheme of table lookup requires an exponentially large table). Hence the theorem does not provide a practical coding scheme.

3.3 Shannon's Channel Coding Theorem: Weak Converse

In this section, we prove the converse part of Shannon's channel coding theorem.

Lemma 3.4. *Let $C = \sup_{p_X}(X;Y)$ be the information capacity of a discrete memoryless channel $p_{Y|X}$. For any input distribution $p_{X_{1:n}}(x_{1:n})$, it holds*

$$I(X_{1:n};Y_{1:n}) \le nC.$$

Proof. We decompose the mutual information $I(X_{1:n};Y_{1:n})$ by chain rule:

$$\begin{aligned}
I(X_{1:n};Y_{1:n}) &= H(Y_{1:n}) - H(Y_{1:n}|X_1,\cdots,X_n) \\
&= \sum_{i=1}^{n} H(Y_i|Y_{i-1},\cdots,Y_1) - \sum_{i=1}^{n} H(Y_i|Y_{i-1},\cdots,Y_1,X_1,\cdots,X_n) \\
&= \sum_{i=1}^{n} H(Y_i|Y_{i-1},\cdots,Y_1) - \sum_{i=1}^{n} H(Y_i|X_i) \\
&\le \sum_{i=1}^{n} H(Y_i) - \sum_{i=1}^{n} H(Y_i|X_i) \\
&= \sum_{i=1}^{n} I(X_i;Y_i) \le nC.
\end{aligned}$$

Hence we conclude the proof. □

Proof of Theorem 3.3 (Converse). By Fano's inequality [Theorem 1.14],

$$P_e^{(n)} = \mathbb{P}(\widehat{W} \neq W) \ge \frac{H(W|\widehat{W}) - 1}{\log_2 2^{nR}} = \frac{H(W|\widehat{W}) - 1}{nR}.$$

Since W is uniform over all possibilities,

$$\begin{aligned}
nR = H(W) &= H(W|\widehat{W}) + I(W;\widehat{W}) \\
&= nRP_e^{(n)} + 1 + I(W;\widehat{W}) \\
&\le nRP_e^{(n)} + 1 + I(X_{1:n};Y_{1:n}) && \text{(By data processing inequality)} \\
&\le nRP_e^{(n)} + 1 + nC.
\end{aligned}$$

Therefore,

$$P_e^{(n)} \ge \frac{n(R-C)-1}{nR} \ge 1 - \frac{C}{R}, \quad \forall n \in \mathbb{N}.$$

If $R > C$, the error probability $P_e^{(n)}$ does not converge to 0, and R is not achievable. □