A Derivation of Non-central Chi-square Density

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Lemma 1. Suppose $X \sim N(\mu, 1)$. Then the characteristic function of X^2 is

$$h(t;\mu) = \frac{\exp\left(\frac{\mathrm{i}\mu^2 t}{1-2\mathrm{i}t}\right)}{(1-2\mathrm{i}t)^{1/2}},\ t \in \mathbb{R}. \tag{1}$$

Proof. By definition, the characteristic function of X^2 is

$$\begin{aligned}
h(t) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}tX^2}\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{\mathrm{i}tx^2 - \frac{(x-\mu)^2}{2}\right\} \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\left[(1-2\mathrm{i}t)^{1/2}x - \frac{\mu}{(1-2\mathrm{i}t)^{1/2}}\right]^2 + \frac{\mathrm{i}\mu^2 t}{1-2\mathrm{i}t}\right\} \mathrm{d}x \\
&= \exp\left(\frac{\mathrm{i}\mu^2 t}{1-2\mathrm{i}t}\right) \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\left[(1-2\mathrm{i}t)^{1/2}x - \frac{\mu}{(1-2\mathrm{i}t)^{1/2}}\right]^2\right\} \mathrm{d}x}_{\text{(a)}},
\end{aligned} \tag{2}$$

where the term (a) is $(1-2\mathrm{i}t)^{-1/2}$ as the integral of a shifted Gaussian density. □

Lemma 2. Suppose V is a non-central chi-square variable with degree of freedom $p > 0$ and non-centrality parameter $\lambda > 0$. Then V can be represented as

$$V = Z_1^2 + Z_2^2 + \cdots + Z_p^2,\ Z_1 \sim N(\sqrt{\lambda}, 1),\ Z_2, \cdots, Z_p \sim N(0,1), \tag{3}$$

where $Z_1, \cdots, Z_p$ are independent.

Proof. Let $X_i \sim N(\mu_i, 1), i = 1, \cdots, p$, with $\lambda = \mu_1^2 + \cdots + \mu_p^2 > 0$. Denote by $\mathbf{X}$ the random vector composed of $X_1, \cdots, X_p$. By definition,

$$V = X_1^2 + X_2^2 + \cdots + X_p^2 = \|\mathbf{X}\|_2^2,\ \mathbf{X} \sim N(\boldsymbol{\mu}, \mathbf{I}_p), \tag{4}$$

where $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_p)^\top \in \mathbb{R}^p$. Then we can expand $\lambda^{-1/2}\boldsymbol{\mu}$ to an orthogonal matrix $\mathbf{Q} = \begin{bmatrix} \lambda^{-1/2}\boldsymbol{\mu}^\top \\ * \end{bmatrix}$ of which the rows form an orthonormal basis on $\mathbb{R}^p$. Let $\mathbf{Z} = \mathbf{QX}$. Then

$$Z_1^2 + Z_2^2 + \cdots + Z_p^2 = \|\mathbf{Z}\|_2^2 = \|\mathbf{QX}\|_2^2 = \|\mathbf{X}\|_2^2 = V. \tag{5}$$

Moreover, $Z_1, Z_2, \cdots, Z_p$ are independent Gaussian variables characterized by (4). □

Lemma 3 (Convolution theorem). Let X, Y be independent random variables. Then the characteristic function of $X + Y$ is the pointwise product of the characteristic functions of X and Y.

Theorem 4 (Density of non-central chi-square distributions). Suppose V is a non-central chi-square variable with degree of freedom $p > 0$ and non-centrality parameter $\lambda > 0$. Then the probability density function of V is

$$f_{\mathrm{NC}}(v; p, \lambda) = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-\lambda/2}(\lambda/2)^k}{k!} f_{\chi^2}(v; p + 2k), \tag{6}$$

where $f_{\chi^2}(\cdot; p + 2k)$ is the probability density function of $\chi^2(p + 2k)$:

$$f_{\chi^2}(v; p + 2k) = \frac{v^{p/2+k-1}}{2^{p/2+k}\Gamma(p/2 + k)} \mathrm{e}^{-v/2}. \tag{7}$$

Proof. We use the representation of V given by Lemma 2. Then applying Lemmas 1 and 3 yields the characteristic function of V :

$$\varphi_V(t) = h(t; \sqrt{\lambda}) \prod_{i=2}^{p} h(t; 0) = \frac{\exp\left(\frac{\mathrm{i}\lambda t}{1-2\mathrm{i}t}\right)}{(1 - 2\mathrm{i}t)^{p/2}},\ t \in \mathbb{R}. \tag{8}$$

We can expand the numerator as follows:

$$\exp\left(\frac{\mathrm{i}\lambda t}{1 - 2\mathrm{i}t}\right) = \mathrm{e}^{-\lambda/2} \exp\left(\frac{\lambda/2}{1 - 2\mathrm{i}t}\right) = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-\lambda/2}}{k!} \left(\frac{\lambda/2}{1 - 2\mathrm{i}t}\right)^k. \tag{9}$$

Then

$$\varphi_V(t) = \sum_{k=1}^{\infty} \frac{\mathrm{e}^{-\lambda/2}(\lambda/2)^k}{k!} \underbrace{\frac{1}{(1 - 2\mathrm{i}t)^{p/2+k}}}_{\text{CF of } \chi^2(p+2k)}. \tag{10}$$

Applying Fourier transform on both sides of (10) yields the result of (6). $\square$

Remark. This theorem proposes another method of generating non-central chi-square variables. Fix $p, \lambda > 0$.

- Generate $k \sim \mathrm{Poisson}(\lambda/2)$.
- Generate i.i.d. $X_1, \cdots, X_{p+2k} \sim N(0, 1)$, and set $V = X_1^2 + \cdots + X_{p+2k}^2$.

Then V is a non-central chi-square variable with degree of freedom p and non-centrality parameter λ.