Partial Differential Equations

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0 Notations

Throughout this book, we assume that U is an open subset of $\mathbb{R}^n$. Given a function $u : U \to \mathbb{R}$, we write $u(x) = u(x_1, \cdots, x_n)$ for $x \in U$. For $i \in [n]$, we write

$$\partial_{x_i} u(x) = \frac{\partial u}{\partial x_i}(x) = \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h}, \quad x \in U$$

for the partial derivative with respect to variable x_i, provided the limit exists. Partial derivatives of higher orders are similarly defined. If $u : U \to \mathbb{R}$ is twice differentiable, we write $\nabla u : \mathbb{R}^n \to \mathbb{R}^n$ and $\nabla^2 u : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ for its *gradient* and *Hessian matrix*, respectively:

$$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), \cdots, \frac{\partial u}{\partial x_n}(x) \right), \qquad \nabla^2 u(x) = \begin{pmatrix} \partial^2_{x_1} u(x) & \partial_{x_1}\partial_{x_2} u(x) & \cdots & \partial_{x_1}\partial_{x_n} u(x) \\ \partial_{x_2}\partial_{x_1} u(x) & \partial^2_{x_2} u(x) & \cdots & \partial_{x_2}\partial_{x_n} u(x) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_n}\partial_{x_1} u(x) & \partial_{x_n}\partial_{x_2} u(x) & \cdots & \partial^2_{x_n} u(x) \end{pmatrix}.$$

The Laplacian Δu of u is defined as the trace of Hessian matrix:

$$\Delta u(x) = \mathrm{tr}(\nabla^2 u(x)) = \frac{\partial^2 u}{\partial x_1^2}(x) + \cdots + \frac{\partial^2 u}{\partial x_n^2}(x).$$

Now we introduce the multi-index notation. A vector $\alpha = (\alpha_1, \cdots, \alpha_n)$ consists of nonnegative integers is called a *multi-index of order* $|\alpha| = \alpha_1 + \cdots + \alpha_n$, Given this multi-index α, we define

$$\partial^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha^n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha^n} u(x).$$

If K is a nonnegative integer, we write

$$\partial^k u(x) := \{\partial^\alpha u(x) : |\alpha| = k\}$$

for the set of all partial derivatives of order k, and define

$$\|\partial^k u\|_{L^p(U)} = \left(\sum_{\alpha:|\alpha|=k} \|\partial^\alpha u\|^p_{L^p(U)} \right)^{1/p}, \quad |\partial^k u| = \|\partial^k u\|_{L^2(U)} = \left(\sum_{\alpha:|\alpha|=k} |\partial^\alpha u|^2 \right)^{1/2}.$$

Furthermore, we replace the symbol ∂ by D when we refer to weak derivatives:

$$\int_U u \partial^\alpha \phi \, dm = (-1)^{|\alpha|} \int_U (D^\alpha u) \phi \, dm, \quad \forall \phi \in C_c^\infty(U),$$

and

$$D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\}, \quad \|D^k u\|_{L^p(U)} = \left(\sum_{\alpha:|\alpha|=k} \|D^\alpha u\|^p_{L^p(U)} \right)^{1/p}, \quad |D^k u| = \left(\sum_{\alpha:|\alpha|=k} |D^\alpha u|^2 \right)^{1/2}.$$

1 Convolution and Smoothing

1.1 Convolution

In this section we first deal with functions on $\mathbb{R}^n$. If a function f is defined on $U \subset \mathbb{R}^n$, we can replace it by its natural zero extension $f : \mathbb{R}^n \to \mathbb{R}$ which assigns $f(x) = 0$ for $x \notin U$.

Definition 1.1 (Convolution)**.** *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable functions. Define the bad set as*

$$E(f,g) := \left\{ x \in \mathbb{R}^n : \int_{\mathbb{R}^n} |f(x-y)g(y)|\, dy = \infty \right\}.$$

The convolution of f and g is the function $f * g : \mathbb{R}^n \to \mathbb{R}$ defined by

$$(f*g)(x) = \begin{cases} \int_{\mathbb{R}^n} f(x-y)g(y)\, dy, & x \notin E(f,g), \\ 0, & x \in E(f,g). \end{cases}$$

Remark. Define $F : \mathbb{R}^{2n} \to \mathbb{R}, (x,y) \mapsto f(x)$ and $G : \mathbb{R}^{2n} \to \mathbb{R}, (x,y) \mapsto g(y)$. Then both F and G are measurable functions on $\mathbb{R}^{2n}$, as well as their product $F \cdot G : (x,y) \mapsto f(x)g(y)$. Given linear transformation $T(x,y) = (x-y, y)$, the composition $H = (F \cdot G) \circ T : (x,y) \mapsto f(x-y)g(y)$ is measurable. By Tonelli's theorem, the function $x \mapsto \int_{\mathbb{R}^n} |H(x,y)|\, dy$ is measurable, and $E(f,g)$ is a Lebesgue measurable set.

Clearly, the convolution operation is both commutative and associative, i.e. $f * g = g * f$, and $(f*g)*h = f*(g*h)$. Furthermore, the distributivity of convolution with respect to functional addition immediately follows, i.e. $f * (g+h) = f*g + f*h$.

Proposition 1.2 (Properties of convolution)**.** *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable functions.*

(i) If $f, g \in L^1(\mathbb{R}^n)$, then the bad set $E(f,g)$ is of measure zero. Moreover, $f * g \in L^1(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^m} (f*g)\, dm = \int_{\mathbb{R}^n} f\, dm \int_{\mathbb{R}^n} g\, dm. \tag{1.1}$$

(ii) If $f \in C_0(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in C_0(\mathbb{R}^n)$.

(iii) If $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in L^p(\mathbb{R}^n)$, and

$$\|f*g\|_p \le \|f\|_p \|g\|_1.$$

Proof. (i) Define the measurable function $H(x,y) \mapsto f(x-y)g(y)$ on $\mathbb{R}^{2n}$. By Tonelli's theorem,

$$\int_{\mathbb{R}^{2n}} |H|\, dm = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|\, |g(y)|\, dx \right) dy = \|f\|_1 \|g\|_1.$$

Hence $H : \mathbb{R}^{2n} \to \mathbb{R}$ is integrable. By Fubini's theorem, for a.e. $x \in \mathbb{R}^n$, $y \mapsto H(x,y)$ is integrable, hence $m(E(f,g)) = 0$. Furthermore, the function $f * g : x \mapsto \int_{\mathbb{R}^n} H(x,y)\, dy$ is also integrable, that is, $f * g \in L^1(\mathbb{R}^n)$. The equation (1.1) follows from Fubini's theorem.

(ii) Given $\epsilon > 0$. By uniform continuity of f, there exists $\eta > 0$ such that $|f(x) - f(x')| < \epsilon / \|g\|_1$ for all $|x - x'| < \eta$, . As a result, for all $x, x' \in \mathbb{R}^n$ such that $|x - x'| < \eta$, we have

$$|(f*g)(x) - (f*g)(x')| \le \int_{\mathbb{R}^n} |f(x-y) - f(x'-y)|\, |g(y)|\, dy < \epsilon.$$

(iii) is a special case of the following proposition. □

Proposition 1.3 (Young's convolution inequality)**.** *Given $r \in [1,\infty]$ and Hölder r-conjugates $p,q \in [1,\infty]$, i.e. $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then the bad set $E(f,g)$ is of measure zero, and we have*

$$\|f * g\|_r \le \|f\|_p \|g\|_q.$$

Remark. Note that $r = \frac{pq}{p+q-pq} \ge 1 \;\Leftrightarrow\; \frac{pq}{p+q} \ge \frac{1}{2} \;\Leftrightarrow\; p \ge \frac{q}{2q-1} \;\Leftrightarrow\; q \ge \frac{p}{2p-1}$, and $r < \infty \;\Leftrightarrow\; p+q > pq \;\Leftrightarrow\; p < \frac{q}{q-1} \;\Leftrightarrow\; q < \frac{p}{p-1}$.

Proof. We first bound $f * g$. By applying generalized Hölder's inequality on $\frac{1}{r} + \frac{r-p}{pr} + \frac{r-q}{qr} = 1$, we have

$$\begin{aligned}
|(f*g)(x)| &\le \int_{\mathbb{R}^n} |f(x-y)|\,|g(y)|\;dy \\
&= \int_{\mathbb{R}^n} \left(|f(x-y)|^p |g(y)|^q\right)^{1/r} |f(x-y)|^{\frac{r-p}{r}} |g(y)|^{\frac{r-q}{r}}\;dy \\
&\le \left(\int_{\mathbb{R}^n} |f(x-y)|^p |g(y)|^q\,dy\right)^{1/r} \left(\int_{\mathbb{R}^n} |f(x-y)|^p\,dy\right)^{\frac{r-p}{pr}} \left(\int_{\mathbb{R}^n} |g(y)|^q\,dy\right)^{\frac{r-q}{qr}} \\
&= \left(\int_{\mathbb{R}^n} |f(x-y)|^p |g(y)|^q\,dy\right)^{1/r} \|f\|_p^{\frac{r-p}{r}} \,\|g\|_q^{\frac{r-q}{r}}\,.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |f(x-y)|\,|g(y)|\;dy\right)^r dx &\le \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)|^p |g(y)|^q\,dy\,dx\right) \|f\|_p^{r-p}\,\|g\|_q^{r-q} \\
&\le \|f\|_p^{r-p}\,\|g\|_q^{r-q} \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |f(x-y)|^p\,dx\right)|g(y)|^q\,dy = \|f\|_p^r\,\|g\|_q^r\,,
\end{aligned}$$

where we use Fubini's theorem in the second inequality. From the last display, we have $m(E(f,g)) = 0$, and $\|f * g\|_r \le \|f\|_p \|g\|_q$. ◻

Remark. If $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, and $g \in L^q(\mathbb{R}^n)$ is compactly supported, then $f * g \in L^r_{\text{loc}}(\mathbb{R}^n)$.

Review: Compact supported functions. Let X be a topological space. The support of function $f : X \to \mathbb{R}$ is defined as the closure of the set of all points in X not mapped to zero by f:

$$\operatorname{supp} f = \overline{\{x \in X : f(x) \ne 0\}} = \overline{\{f \ne 0\}}.$$

If the support of f is compact in X, f is said to be *compactly supported.* Following this definition, any function defined on a closed interval $[a,b]$ can be extended to a compactly supported function on $\mathbb{R}$.

The set of all continuous compactly supported functions on X is denoted by $C_c(X)$. If $f \in C_c(X)$, then f is uniformly continuous on $\operatorname{supp} f$. Note that $f = 0$ outside $\operatorname{supp} f$, we have that f is uniformly continuous on X, which implies $C_c(X) \subset C_0(X)$. Furthermore, by extreme value theorem, f has maximum and minimum on $\operatorname{supp} f$, which implies that f is uniformly bounded on X, i.e. $\max_{x\in X} |f(x)| < \infty$.

Let $(X, \mathscr{A}, \mu)$ be a measure space where X is a topological space. Following the discussion above, we have $C_c^\infty(X) \subset L^\infty(X,\mathscr{A},\mu)$ since every $f \in C_c^\infty(X)$ satisfies $\|f\|_\infty = \max_{x\in X} |f(x)| \le \infty$. Furthermore, if every compact set in X has finite measure, i.e. $\mu(K) < \infty$ for all compact $K \subset X$, then the compactly supported function are always p-integrable:

$$\|f\|_p = \left(\int_X |f|^p d\mu\right)^{1/p} = \left(\int_{\operatorname{supp} f} |f|^p dm\right)^{1/p} \le \mu(\operatorname{supp} f)^{1/p} \|f\|_\infty < \infty.$$

Proposition 1.4 (Convolution of compactly supported functions). *Let $f, g : \mathbb{R}^n \to \mathbb{R}$.*

(i) If $f, g \in L^1(\mathbb{R}^n)$, then $\operatorname{supp}(f * g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g} := \overline{\{x + y : x \in \operatorname{supp} f, y \in \operatorname{supp} g\}}$. Furthermore, if both f and g are compactly supported on $\mathbb{R}$, then $f * g$ is also compactly supported. In this case, $\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g$.

(ii) Let $1 \le p \le \infty$, and let $k \in \mathbb{N}_0$. If $f \in C_c^k(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then $f * g \in C_0^k(\mathbb{R}^n)$. Furthermore, differentiation commutes with convolution, i.e.,

$$\partial^\alpha (f * g) = \partial^\alpha f * g, \qquad \forall |\alpha| \le k,$$

(iii) Let $1 \le p \le \infty$. If $f \in C_c^\infty(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then $f * g \in C_0^\infty(\mathbb{R}^n)$. Similarly, differentiation commutes with convolution, i.e., $\partial^\alpha (f * g) = \partial^\alpha f * g$ for multi-indices α.

Remark. Combining (i) and (ii)/(iii), we obtain a useful conclusion. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. If $f \in C_c^k(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ is compactly supported, then $f * g \in C_c^k(\mathbb{R}^n)$.

Proof. (i) Let $f, g \in L^1(\mathbb{R}^n)$, and take any $x \in \mathbb{R}^n$. Then

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y)\, dy = \int_{(x - \operatorname{supp} f) \cap \operatorname{supp} g} f(x - y) g(y)\, dy.$$

For $x \notin \operatorname{supp} f + \operatorname{supp} g$, we have $(x - \operatorname{supp} f) \cap \operatorname{supp} g = \emptyset$, which implies $(f * g)(x) = 0$. Hence

$$(f * g)(x) \ne 0 \ \Rightarrow x \in \operatorname{supp} f + \operatorname{supp} g \ \Rightarrow \ \operatorname{supp}(f * g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}.$$

If $f, g \in C_c(\mathbb{R}^n)$, then $\operatorname{supp} f$ and $\operatorname{supp} g$ are compact in $\mathbb{R}^n$. Define $\phi(x, y) = x + y$, which is a continuous map on $\mathbb{R}^{2n}$. Then $\operatorname{supp} f + \operatorname{supp} g = \phi(\operatorname{supp} f \times \operatorname{supp} g)$ is also compact. Consequently, $\operatorname{supp} f + \operatorname{supp} g$ is closed, and its closed subset $\operatorname{supp}(f * g)$ is also compact. which implies $f * g \in C_c(\mathbb{R}^n)$.

(ii) *Step I:* We first show the case $k = 0$. Let $q = p/(p-1)$. Note that f is continuous and compact supported, then $m(\operatorname{supp} f) < \infty$, f is uniformly continuous, and $\|f\|_\infty = \max_{x \in \operatorname{supp} f} |f(x)| < \infty$. By Hölder's inequality, for all $x \in \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} |f(x - y)|\, |g(y)|\, dy \le \|f\|_q \|g\|_p \le m\big(\operatorname{supp} f\big)^{1/q} \|f\|_\infty \|g\|_p < \infty.$$

Then $f * g$ is well-defined on $\mathbb{R}^n$. To show uniform continuity of $f * g$, we fix $\epsilon > 0$ and let η be such that $|x - x'| < \eta$ implies $|f(x) - f(x')| < \epsilon$. Then

$$\begin{aligned} |(f * g)(x) - (f * g)(x')| &= \left| \int_{\mathbb{R}^n} [f(x - y) - f(x' - y)]\, g(y)\, dy \right| \\ &\le m\big(\operatorname{supp} f\big)^{1/q} \|g\|_p\, \epsilon. \end{aligned}$$

Step II: We prove the case $k = 1$. It suffices to show the interchangeability of derivative and integral. Given any quantity $h > 0$, we have

$$\frac{(f * g)(x + h e_i) - (f * g)(x)}{h} = \int_{\mathbb{R}^n} \frac{f(x + h e_i - y) - f(x - y)}{h} g(y)\, dy. \tag{1.2}$$

Since $f \in C_c^1(\mathbb{R}^n)$, by Lagrange's mean value theorem, there exists $\xi \in [0, 1]$ such that

$$\left| \frac{f(x + h e_i - y) - f(x - y)}{h} \right| = |\partial_{x_i} f(x + \xi h e_i - y)|, \tag{1.3}$$

Note that $\partial_{x_i} f$ is also continuous and compactly supported on $\mathbb{R}^n$, the RHS of (1.3) is bounded by $\|\partial_{x_i} f\|_\infty$, and the integrand in (1.2) is dominated by an integrable function $\|\partial_{x_i} f\|_\infty g$. Using Lebesgue's dominate convergence theorem, we have

$$\lim_{h\to 0}\int_{\mathbb{R}^n}\frac{f(x+he_i-y)-f(x-y)}{h}g(y)\,dy=\int_{\mathbb{R}^n}\frac{\partial f}{\partial x_i}(x-y)g(y)\,dy.$$

Therefore $\partial_{x_i}(f*g)=\partial_{x_i}f*g$. Since $\partial_{x_i}f\in C_c(\mathbb{R}^n)$, we have $\partial_{x_i}(f*g)\in C_0(\mathbb{R}^n)$, and $f*g\in C_0^1(\mathbb{R}^n)$.

Step III: Use induction. Suppose our conclusion holds for $C_c^{k-1}(\mathbb{R}^n)$. For each $f\in C_c^k(\mathbb{R}^n)\subset C_c^{k-1}(\mathbb{R}^n)$, $\partial^{k-1}f\subset C_c^1(\mathbb{R}^n)$. By Step II, for any $|\alpha|=k-1$,

$$\partial^{\alpha+e_i}(f*g)=\partial_{x_i}(\partial^\alpha(f*g))=\partial_{x_i}(\partial^\alpha f*g)=(\partial^{\alpha+e_i}f)*g,$$

which is uniformly continuous on $\mathbb{R}^n$. Hence $f*g\in C_c^k(\mathbb{R}^n)$.

(iii) Note that $C_c^\infty(\mathbb{R}^n)=\bigcap_{k=0}^\infty C_c^k(\mathbb{R}^n)$, we have $\partial^\alpha(f*g)=\partial^\alpha f*g$ for all $\alpha\in\mathbb{N}_0^n$. Following Step II, $\partial^\alpha f\in C_c(\mathbb{R}^n)$ implies $\partial^\alpha(f*g)\in C_0(\mathbb{R}^n)$ for all $\alpha\in\mathbb{N}_0^n$. Hence $f*g\in\bigcap_{k=0}^\infty C_0^k(\mathbb{R}^n)=C_0^\infty(\mathbb{R}^n)$. $\square$

Review: Translation operators. Let X be a vector space, let Y^X be the set of functions $f:X\to Y$, and let s be a vector in X. The *translation operator* $\tau_s:Y^X\to Y^X$ is defined as

$$(\tau_s f)(x)=f(x-s),\ \forall f\in Y^X.$$

Proposition 1.5. *Let $1\le p<\infty$. For any $f\in C_c(\mathbb{R}^n)$,*

$$\lim_{s\to 0}\|\tau_s f-f\|_p=0. \tag{1.4}$$

Proof. Let $f\in C_c(\mathbb{R}^n)$, and let B_1 be the closed unit ball in $\mathbb{R}^n$. The collection of functions $\{\tau_s f:|s|\le 1\}$ has a common support

$$K=\bigcup_{|s|\le 1}\operatorname{supp}(\tau_s f)=\operatorname{supp}f+B_1=\{x+y:x\in\operatorname{supp}f,y\in B_1\}=\phi(\operatorname{supp}f\times B_1),$$

which is compact as the image of a compact set under a continuous map $\phi:\mathbb{R}^{2n}\to\mathbb{R}^n,(x,y)\mapsto x+y$.

By uniform continuity of f, given $\epsilon>0$, there exists $\delta>0$ such that $|f(x)-f(y)|<\epsilon$ for all $|x-y|<\delta$. Then for any s with $|s|<|\min(\delta,1)|$, we have

$$\|\tau_s f-f\|_p^p=\int_K|f(x-s)-f(x)|^p dx\le\mu(K)\,\epsilon^p.$$

Since $\mu(K)<\infty$, and ϵ is arbitrary, we conclude that $\|\tau_s f-f\|_p\to 0$ as $s\to 0$. $\square$

Review: Mollifier. A *mollifier* on $\mathbb{R}^n$ is a symmetric function $\eta\in C_c^\infty(\mathbb{R}^n)$ supported on the closed unit ball $B_1=\{x\in\mathbb{R}^n:|x|\le 1\}$ with $\int_{\mathbb{R}^n}\eta\,dm=1$. For example, the *standard mollifier* is defined as

$$\eta(x)=\frac{1}{Z}\exp\left(\frac{1}{|x|^2-1}\right)\chi_{B_1}(x),\quad \textit{where}\ \ Z=\int_{|t|\le 1}\exp\left(\frac{1}{|t|^2-1}\right)\,dt.$$

For each $\epsilon>0$, we set

$$\eta_\epsilon(x)=\frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right)\quad\Rightarrow\quad\int_{\mathbb{R}^n}\eta_\epsilon(x)\,dx=1,\ \operatorname{supp}(\eta_\epsilon)\subset B(0,\epsilon).$$

Now we provide an important approximation result using compactly supported smooth functions.

Proposition 1.6. *For $1 \le p < \infty$, $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.*

Proof. Let f be a compactly supported function in $L^p(\mathbb{R}^n)$. We choose a mollifier $\eta \in C_c^\infty(\mathbb{R}^n)$, and define $\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right)$ for $\epsilon > 0$. By Proposition 1.4, $f * \eta_\epsilon \in C^\infty(\mathbb{R})$, and

$$\begin{aligned}\int_{\mathbb{R}^n} |(f*\eta_\epsilon)(x) - f(x)|^p\,dx &= \int_{\mathbb{R}^n}\left|\int_{|y|\le\epsilon}(f(x-y)-f(x))\eta_\epsilon(y)\,dy\right|^p dx \\ &\le \int_{\mathbb{R}^n}\int_{|y|\le\epsilon}|f(x-y)-f(x)|^p\,\eta_\epsilon(y)\,dydx \qquad \text{(By Jensen's inequality)} \\ &= \int_{|y|\le\epsilon}\eta_\epsilon(y)\|\tau_y f - f\|_p^p\,dy \\ &\le \sup_{y:|y|\le\epsilon}\|\tau_y f - f\|_p^p.\end{aligned}$$

which converges to 0 as $\epsilon \to 0$ by Proposition 1.5. By monotone convergence theorem, any $g \in L^p(\mathbb{R})$ is approximated by $g\chi_{[-n,n]}, n \in \mathbb{N}$:

$$\|g - g\chi_{[-n,n]}\|_p^p = \int_{\mathbb{R}}|g|^p\,dm - \int_{\mathbb{R}}|g\chi_{[-n,n]}|^p\,dm \to 0,\ n\to\infty.$$

Note that $g_{[-n,n]} =: f$ is compactly supported, the result follows. $\square$

Application I: continuity of translation operators in L^p-spaces. The limit (1.4) in Proposition 1.5 remains zero for all $f \in L^p(\mathbb{R})$. We fix $\epsilon > 0$, so there exists $g \in C_c^\infty(\mathbb{R})$ such that $\|f - g\|_\infty < \epsilon/3$ by Proposition 1.6. Choose δ such that $\|\tau_s g - g\|_p < \epsilon/3$ for all $|s| < \delta$. Then for all $|s| < \delta$,

$$\|\tau_s f - f\|_p \le \|\tau_s f - \tau_s g\|_p + \|\tau_s g - g\|_p + \|g - f\|_p = 2\|f - g\| + \|\tau_s g - g\|_p < \epsilon.$$

Application II: uniform continuity of convolution. Let $\frac{1}{p} + \frac{1}{q} = 1$ be Hölder conjugates. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in C_0(\mathbb{R}^n)$. Given $\epsilon > 0$, we choose $\delta > 0$ such that $\|\tau_s f - f\|_p < \epsilon/\|g\|_q$ for all $|s| \le \delta$. Then one have

$$|(f*g)(x-s) - (f*g)(x)| \le \int_{\mathbb{R}^n}|f(x-s-y) - f(x-y)|\,|g(y)|\,dy \le \|\tau_s f - f\|_p\|g\|_q < \epsilon$$

for all $x \in \mathbb{R}^n$ and all $|s| < \delta$. Clearly, $f * g$ is uniformly continuous on $\mathbb{R}^n$.

Application III: uniform continuity of convolution on bounded sets. If $f \in L^p(\mathbb{R}^n)$ is compactly supported, and $g \in L^q_{\text{loc}}(\mathbb{R}^n)$, we have $f * g \in C(\overline{\mathbb{R}}^n)$. We fix $\epsilon > 0$ and $R > 0$, choose $r > 0$ such that $\operatorname{supp} f \subset B(0,r)$, and choose $\delta > 0$ such that $\|\tau_s f - f\|_p < \epsilon/\|g\chi_{B(0,R+r)}\|_q$ for all $|s| < \delta$. Then

$$|(f*g)(x) - (f*g)(x')| \le \int_{B(0,R+r)}|f(x-y) - f(x'-y)|\,|g(y)|\,dy \le \|\tau_{x-x'}f - f\|_p\|g\chi_{B(0,R+r)}\|_q < \epsilon$$

for all $|x|, |x'| < R$ with $|x - x'| < \delta$. Hence $f * g$ is uniformly continuous on the open ball $O(0,R)$.

In addition, if $f \in C_c^\infty(\mathbb{R}^n)$ and $g \in L^1_{\text{loc}}(\mathbb{R}^n)$, we have $f * g \in C^\infty(\overline{\mathbb{R}}^n)$. This result can be shown by adapting the proof of Proposition 1.4.

1.2 Local Mollification

In this section we study the approximation of locally integrable functions. Our discussion is based on an open region $U \subset \mathbb{R}^n$. Given any $\epsilon > 0$, we define

$$U^\epsilon = \{x \in U : d(x, \partial U) > \epsilon\}.$$

Since U is open, U^ϵ is nonempty for sufficiently small $\epsilon > 0$. In addition, the continuity of $d(\cdot, \partial U)$ implies that U^ϵ is also an open region. Using this notation, we can extend a locally integrable function on U to $\mathbb{R}^n$: given a function $u \in L^1_{\text{loc}}(U)$, define the *zero ϵ-extension* $\overline{u}^{(\epsilon)} : \mathbb{R}^n \to \mathbb{R}$ of $u : U \to \mathbb{R}$ for $\epsilon > 0$ as follows:

$$\overline{u}^{(\epsilon)} := u\chi_{\overline{U}^\epsilon} \quad \Rightarrow \quad \overline{u}^{(\epsilon)} \in L^1_{\text{loc}}(\mathbb{R}^n).$$

Definition 1.7 (Mollification). *Given $u \in L^1_{\text{loc}}(U)$, define its mollification by*

$$u^\epsilon := \eta_\epsilon * \overline{u}^{(\epsilon)} \subset C^\infty(\mathbb{R}^n) \subset C^\infty(U).$$

Clearly, $u^\epsilon = 0$ outside U, and the bad set $E := E(\eta_\epsilon, \overline{u}^{(\epsilon)})$ is of measure zero by Proposition 1.2. Furthermore, the value of this mollification inside $\overline{U}^{2\epsilon}$ is given by

$$u^\epsilon(x) = \begin{cases} \int_{B(x,\epsilon)} \eta_\epsilon(x-y)u(y)\,dy = \int_{B(0,1)} \eta(z)u(x+\epsilon z)\,dz, & x \in \overline{U}^{2\epsilon} \backslash E, \\ 0, & x \in \overline{U}^{2\epsilon} \cap E. \end{cases} \tag{1.5}$$

Proposition 1.8 (Properties of mollification). *Let $u \in L^1_{\text{loc}}(U)$.*

(i) $u^\epsilon \to u$ a.e. on U as $\epsilon \to 0$.

(ii) If $u \in C(U)$, then $u^\epsilon \to u$ uniformly on compact subsets $K \subset U$.

(iii) If $1 \le p < \infty$ and $u \in L^p_{\text{loc}}(U)$, then $u^\epsilon \to u$ in $L^p_{\text{loc}}(U)$.

Proof. (i) According to Lebesgue's differentiation theorem, we have

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B(x,r)} |u(y) - u(x)|\,dy = 0$$

for *a.e.* $x \in U$. Since $x \in \overline{U}^{2\epsilon}$ for sufficiently small $\epsilon > 0$, we have

$$\begin{aligned} \lim_{\epsilon\to 0} |u^\epsilon(x) - u(x)| &\le \lim_{\epsilon\to 0} \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u(y) - u(x)|\,dy \\ &\le \lim_{\epsilon\to 0} \frac{1}{\epsilon^n} \int_{B(x,\epsilon)} \|\eta\|_\infty |u(y) - u(x)|\,dy = 0, \quad \textit{for a.e. } x \in U. \end{aligned}$$

Consequently, we have $u^\epsilon \to u$ *a.e.* on U as $\epsilon \to 0$.

(ii) Given $K \subset U$, choose $\delta > 0$ sufficiently small such that $K \subset \overline{U}^{2\delta}$. Since u is a continuous function, the bad set $E(\eta_\epsilon, \overline{u}^{(\epsilon)})$ is empty. Then for all $\epsilon \in (0, \delta]$, one have

$$\begin{aligned} \sup_{x\in K} |u^\epsilon(x) - u(x)| &= \sup_{x\in K} \left| \int_{B(0,1)} \eta(z)\left(u(x+\epsilon z) - u(x)\right) dz \right| \\ &\le \sup_{x\in K} \sup_{z\in B(0,1)} |u(x+\epsilon z) - u(x)| \end{aligned}$$

Since $x, x+\epsilon z \in \overline{U}^\delta$, we have $|u(x+\epsilon z) - u(x)| \rightrightarrows 0$ by uniform continuity of u on $\overline{U}^\delta$.

(iii) Given any pre-compact set $V \Subset U$, we first choose a pre-compact subset W of U such that $V \Subset W \Subset U$.

We claim that, for sufficiently small $\epsilon > 0$, we have $\|u^\epsilon\|_{L^p(V)} \le \|u\|_{L^p(W)}$. To this end, we note that

$$\begin{aligned}|u^\epsilon(x)| = \left|\int_{B(x,\epsilon)} \eta_\epsilon(x-y)u(y)\,dy\right| &\le \int_{B(x,\epsilon)} \eta_\epsilon(x-y)^{1-1/p}\eta_\epsilon(x-y)^{1/p}|u(y)|\,dy \\ &\le \underbrace{\left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)\,dy\right)^{1-1/p}}_{=1} \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u(y)|^p\,dy\right)^{1/p}.\end{aligned}$$

We choose $\epsilon > 0$ such that $V \subset \overline{W}^\epsilon$. Then

$$\|u^\epsilon\|^p_{L^p(V)} \le \int_V \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u(y)|^p\,dy\right)dx \le \int_W \left(\int_{B(y,\epsilon)} \eta_\epsilon(x-y)\,dx\right)|u(y)|^p\,dy = \|u\|^p_{L^p(W)}.$$

Now we fix $\delta > 0$, and choose $g \in C(W)$ such that $\|f-g\|_{L^p(W)} < \delta/2$. Then

$$\begin{aligned}\|f^\epsilon - f\|_{L^p(V)} &\le \|f^\epsilon - g^\epsilon\|_{L^p(V)} + \|g^\epsilon - g\|_{L^p(V)} + \|g-f\|_{L^p(V)} \\ &\le \|g^\epsilon - g\|_{L^p(V)} + 2\|g-f\|_{L^p(W)} \le \|g^\epsilon - g\|_{L^p(V)} + \delta.\end{aligned}$$

By (ii), $g^\epsilon \rightrightarrows g$ on V as $\epsilon \to 0$, hence $\limsup_{\epsilon\to 0}\|f^\epsilon - f\|_{L^p(V)} \le \delta$. □

Now we provide an application of mollification.

Lemma 1.9. *If $v \in L^1_{\mathrm{loc}}(U)$, and*

$$\int_U v\phi\,dm = 0 \quad \forall \phi \in C_c^\infty(U), \tag{1.6}$$

then $v = 0$ a.e..

Proof. Let K be a compact subset of U, and choose $\varphi \in C_c^\infty(U)$ such that $0 \le \varphi \le 1$, and $\varphi \equiv 1$ on K. [We will show the existence of such function in Lemma 1.10.] By assumption (1.5), we have

$$(\eta_\epsilon * v_\varphi)(x) = \int_{\mathbb{R}^n} \eta_\epsilon(x-y)\varphi(y)v(y)\,dy = \int_U \underbrace{\eta_\epsilon(x-y)\varphi(y)}_{\phi_{\epsilon,x}(y)} v(y)\,dy = 0,$$

since $\phi_{\epsilon,x}(\cdot) = \eta_\epsilon(x-\cdot)\varphi(\cdot) \in C_c^\infty(U)$. By letting $\epsilon \to 0$, we obtain the limit $\eta_\epsilon * v_\varphi \overset{L_1}{\to} \varphi v = 0$ *a.e.*. Consequently, we have $v = 0$ *a.e.* on all compact subsets K of U.

Define $K_r = \{x \in \mathbb{R}^n : d(x, U^c) \ge 2/r \text{ and } |x| \le r\}$. Then $K_r \subset U$ is compact, and $U = \bigcup_{r=1}^\infty K_r$. Since $v = 0$ *a.e.* on all K_r, we have

$$m(\{v=0\}) = m\left(\bigcup_{r=1}^\infty K_r \cap \{v=0\}\right) = \lim_{r\to\infty} m(K_r \cap \{v=0\}) = 0.$$

Hence $v = 0$ *a.e.* on U. □

Remark. Due to the property (1.5), the functions in the class $C_c^\infty(U)$ of compactly supported smooth functions are also called *test functions*.

1.3 Application: Smooth Partition of Unity

In this, section we employ the mollification approach to construct partitions of unity. These technical results are later used to obtain global properties from local ones.

Lemma 1.10. *Let U be an open subset of $\mathbb{R}^n$, and let K be a compact subset of U. Then there exists a function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on K, and $\operatorname{supp}\varphi \subset U$.*

Proof. Given $\epsilon > 0$, we define

$$K_\epsilon := \{x \in \mathbb{R}^n : d(x, K) \le \epsilon\}.$$

Choose $\epsilon > 0$ so small that $K_{3\epsilon} \subset U$, and let $\varphi = \eta_\epsilon * \chi_{K_{2\epsilon}}$. By properties of convolution, $\varphi \in C_c^\infty(\mathbb{R}^n)$, $0 \le \varphi \le 1$, $\varphi \equiv 1$ on K, and $\operatorname{supp}\varphi \subset \overline{\operatorname{supp}\eta_\epsilon + K_{2\epsilon}} \subset K_{3\epsilon} \subset U$, □

Next we introduce a technical lemma in topology, which asserts that we are able to "shrink" a finite open cover of a closed subset of $\mathbb{R}^n$.

Lemma 1.11. *Let $U \subset \mathbb{R}^n$, and let $\{U_i\}_{i=1}^N$ be a collection of open subsets of $\mathbb{R}^n$ such that $\overline{U} \subset \bigcup_{i=1}^N U_i$. Then there exists a collection $\{V_i\}_{i=1}^N$ of open subsets of $\mathbb{R}^n$ such that $\overline{V_i} \subset U_i$, $i = 1, \cdots, N$ and $\overline{U} \subset \bigcup_{i=1}^N V_i$.*

Proof. We proceed by substituting the elements of the cover of $\overline{U}$ one by one. Let $A_1 = \overline{U} \backslash (U_2 \cup \cdots \cup U_N)$. Then A_1 is a closed set contained in U_1. By normality of $\mathbb{R}^n$, we can choose an open set V_1 containing A_1 such that $\overline{V}_1 \subset U_1$. Then we obtain a cover $\{V_1, U_2, \cdots, U_N\}$ of $\overline{U}$.

At the k^{th} step, we are given open sets $V_1, \cdots, V_{k-1}$ such that $\{V_1, \cdots, V_{k-1}, U_k, \cdots, U_N\}$ covers U. We let $A_k = \overline{U} \backslash (V_1 \cup \cdots \cup V_{k-1} \cup U_{k+1} \cup \cdots \cup U_N)$, and choose an open set V_k such that $A_k \subset V_k \subset \overline{V}_k \subset U_k$. Then $\{V_1, \cdots, V_k, U_{k+1}, \cdots, U_N\}$ is also an open cover of $\overline{U}$. At the n^{th} step, our result is proved. □

Remark. In addition, if U is bounded, we may assume that each U_i is bounded. As a result, we can obtain a shrunk open cover $\{V_i\}_{i=1}^N$ of $\overline{U}$ such that $V_i \Subset U_i$. In other words, each $\overline{V}_i$ is a compact set.

Theorem 1.12 (Partition of unity)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$, and let $(V_i)_{i=1}^N$ be a collection of open sets in $\mathbb{R}^n$ such that $U \Subset \bigcup_{i=1}^N V_i$. Then there exists a family of smooth functions $(\psi_i)_{i=1}^N : \mathbb{R}^n \to [0,1]$ such that $\psi_i \in C_c^\infty(V_i)$ for all $i = 1, \cdots, N$, and $\sum_{i=1}^N \psi_i \equiv 1$ on U.*

Remark. The family $(\psi_i)_{i=1}^N$ is called a *smooth partition of unity subordinate to the open sets* $(V_i)_{i=1}^N$.

Proof. By Lemma 1.11, we take a collection $(K_i)_{i=1}^N$ of compact subsets of $\mathbb{R}^n$ such that $K_i \subset V_i$, $i = 1, \cdots, N$ and $\overline{U} \subset \bigcup_{i=1}^N K_i$. By Lemma 1.10, for each $i = 1, \cdots, N$, there exists a smooth function $\varphi_i : \mathbb{R}^n \to [0,1]$ such that $\varphi \equiv 1$ on K_i, and $\operatorname{supp}\varphi_i \subset V_i$. We then define

$$\psi_1 = \varphi_1,\ \psi_2 = (1-\varphi_1)\varphi_2,\ \cdots,\ \psi_N = (1-\varphi_N)\cdots(1-\varphi_{N-1})\varphi_N.$$

Then $0 \le \psi_i \le 1$, and $\psi_i \in C_c^\infty(V_i)$ for all $i = 1, \cdots, N$. Furthermore,

$$1 - \sum_{i=1}^N \psi_i = (1-\varphi_1)(1-\varphi_2)\cdots(1-\varphi_N).$$

For each point $x \in U \subset \bigcup_{i=1}^N K_i$, at least one factor $(1-\varphi_i)$ vanishes, and we have $\sum_{i=1}^N \psi_i \equiv 1$ on U. □

2 Sobolev Spaces

2.1 Hölder Spaces

Assume that $U \subset \mathbb{R}^n$ is open and $\gamma \in (0,1]$. A function $u : U \to \mathbb{R}$ is said to be *Hölder continuous with exponent* γ, if there exists some constant $C > 0$ such that

$$|u(x) - u(y)| \le C|x-y|^\gamma, \quad \forall x, y \in U.$$

In this section, we first discuss the Hölder spaces, which contain functions with some nice properties.

Definition 2.1 (Hölder spaces). *Let $U \subset \mathbb{R}^n$ be open, and $0 < \gamma \le 1$. If $u : U \to \mathbb{R}$ is a bounded and Hölder continuous function, we define*

$$\|u\|_{C(\overline{U})} := \sup_{x \in U} |u(x)|, \quad [u]_{C^{0,\gamma}(\overline{U})} = \sup_{x,y \in U : x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\gamma},$$

where $[\cdot]_{C^{0,\gamma}(\overline{U})}$ is the $\underline{\gamma^{th}\text{-Hölder seminorm}}$. The $\underline{\gamma^{th}\text{-Hölder norm}}$ is defined as

$$\|u\|_{C^{0,\gamma}(\overline{U})} = \|u\|_{C(\overline{U})} + [u]_{C^{0,\gamma}(\overline{U})}.$$

Let $k \in \mathbb{N}_0$. The $\underline{\text{Hölder space}}$ $C^{k,\gamma}(\overline{U})$ consists of all functions $u \in C^k(\overline{U})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\overline{U})} := \sum_{\alpha : |\alpha| \le k} \|\partial^\alpha u\|_{C(\overline{U})} + \sum_{\alpha : |\alpha| = k} [\partial^\alpha u]_{C^{0,\gamma}(\overline{U})}$$

is finite. In other words, $C^{k,\gamma}(\overline{U})$ contains all k-times continuously differentiable functions whose k^{th}-partial derivatives are bounded and Hölder continuous with exponent γ.

Remark. One can easily check that $C^{k,\gamma}(\overline{U})$ is a vector space, and $\|\cdot\|_{C^{k,\gamma}(\overline{U})}$ is a norm on $C^{k,\gamma}(\overline{U})$.

Theorem 2.2. *The Hölder space $C^{k,\gamma}(\overline{U})$ is a Banach space.*

Proof. It suffices to show completeness of $C^{k,\gamma}(\overline{U})$ under the norm $\|\cdot\| = \|\cdot\|_{C^{k,\gamma}(\overline{U})}$. Let (u_l) be a Cauchy sequence in $C^{k,\gamma}(\overline{U})$, i.e. $\|u_l - u_m\| \to 0$ as $i, j \to \infty$. By completeness of $C(\overline{U})$, (u_l) converges uniformly to some $u \in C(\overline{U})$, and for each $|\alpha| \le k$, the sequence $(\partial^\alpha u_l)$ converges uniformly to some function $u^{(\alpha)} \in C(\overline{U})$. Consequently, we have $\partial^\alpha u_l \to \partial^\alpha u = u^{(\alpha)}$ for all $|\alpha| \le k$, and $u \in C^k(\overline{U})$.

Now it remains to discuss Hölder continuity. Since (u_l) is a Cauchy sequence, there exists $M > 0$ such that $\sup_{l \in \mathbb{N}} \|u_l\| \le M$. For all $|\alpha| = k$,

$$\frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\gamma} \le \frac{|\partial^\alpha u(x) - \partial^\alpha u_l(x)|}{|x-y|^\gamma} + \underbrace{\frac{|\partial^\alpha u_l(x) - \partial^\alpha u_l(y)|}{|x-y|^\gamma}}_{\le M} + \frac{|\partial^\alpha u_l(y) - \partial^\alpha u(y)|}{|x-y|^\gamma}.$$

Since $\partial^\alpha u_l \rightrightarrows \partial^\alpha u$, the first and third terms in the last display converges to zero for all $x, y \in U$. Hence $\partial^\alpha u$ is Hölder continuous with exponent γ. Furthermore,

$$\frac{|\partial^\alpha (u_l - u)(x) - \partial^\alpha (u_l - u)(y)|}{|x-y|^\gamma} = \lim_{m \to \infty} \frac{|\partial^\alpha (u_l - u_m)(x) - \partial^\alpha (u_l - u_m)(y)|}{|x-y|^\gamma} \le \lim_{m \to \infty} [\partial^\alpha (u_l - u_m)]_{C^{0,\gamma}(\overline{U})}$$

Since the last bound does not depend on $x, y \in U$, we can obtain $[\partial^\alpha (u_l - u)]_{C^{0,\gamma}(\overline{U})} \to 0$ by letting $l \to \infty$. Hence $\|u_l - u\| \to 0$ as $l \to \infty$. $\square$

2.2 Weak Derivatives

Review: Integration by Parts. Let $U \subset \mathbb{R}^n$ be a open and bounded region with C^1 boundary. According to the divergence theorem, for each vector field $\mathbf{u} \in C^1(\overline{U}, \mathbb{R}^n)$, we have

$$\int_U (\nabla \cdot \mathbf{u})\, dx = \int_{\partial U} \mathbf{u} \cdot \nu \, dS,$$

where $\nu : \partial\Omega \to \mathbb{R}^n$ is the outward pointing normal vector field. For $u \in C^1(\overline{U})$, we set $\mathbf{u} = ue_i$. Then

$$\int_U \frac{\partial u}{\partial x_i}\, dx = \int_{\partial U} u\nu^i \, dS, \quad i = 1, \cdots, n.$$

Now assume we are given a function $u \in C^1(U)$. If $\phi \in C^\infty(U)$, we apply the above formula to $u\phi$:

$$\int_U u \frac{\partial \phi}{\partial x_i}\, dx = -\int_U \frac{\partial u}{\partial x_i} \phi \, dx, \quad i = 1, \cdots, n.$$

More generally, if $k \in \mathbb{N}$, $u \in C^k(U)$, and α is a multi-index with $|\alpha| = k$, then

$$\int_U u(\partial^\alpha \phi)\, dx = (-1)^{|\alpha|} \int_U (\partial^\alpha u)\phi \, dx.$$

This formula gives rise to the definition of weak derivatives.

Definition 2.3 (Weak derivatives)**.** *Assume that $u, v \in L^1_{\text{loc}}(U)$ and α is a multi-index. Then v is said to be the* <u>*α^{th}-weak partial derivative of u*</u>*, written $\partial^\alpha u = v$, if*

$$\int_U u\partial^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v\phi \, dx.$$

for all test functions $\phi \in C_c^\infty(U)$.

Remark. Suppose both v and $\widetilde{v}$ are α^{th}-weak partial derivatives of u. By applying Lemma 1.9 on $v - \widetilde{v}$, one can show that the α^{th}-weak partial derivative of u is uniquely defined up to a set of measure zero.

Example. Consider the function $u(x) = |x|$, which is in $L^1_{\text{loc}}(\mathbb{R})$. Then the weak derivative of u on $\mathbb{R}$ is

$$v(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0. \end{cases}$$

This is easy to verify. Given any test functions $\phi \in C_\infty^c(\mathbb{R})$, let $\operatorname{supp} \phi \subset [-M, M]$. Then we have

$$\begin{aligned} \int_{\mathbb{R}} u(x)\phi'(x)\, dx &= \int_0^M x \, d\phi(x) - \int_{-M}^0 x \, d\phi(x) \\ &= -\int_0^M \phi(x)\, dx + \int_{-M}^0 \phi(x)\, dx = -\int_{\mathbb{R}} v(x)\phi(x)\, dx. \end{aligned}$$

However, the function $v \in L^1_{\text{loc}}(\mathbb{R})$ has no weak derivative. We argue by contradiction, and assume that there exists $w \in L^1_{\text{loc}}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} v(x)\phi'(x)\, dx = -\int_{\mathbb{R}} w(x)\phi(x)\, dx, \quad \forall \phi \in C_c'(\mathbb{R}).$$

Then we have

$$\int_{\mathbb{R}} w(x)\phi(x)\,dx = -\int_{\mathbb{R}} v(x)\phi'(x)\,dx = -\int_0^{\infty} \phi'(x)\,dx + \int_{-\infty}^{0} \phi'(x)\,dx = 2\phi(0).$$

Now we choose a sequence $\phi_m(x) = \exp\left(\frac{1}{|mx|^2-1}\right)\chi_{(-\frac{1}{m},\frac{1}{m})}$ in $C_c'(\mathbb{R})$, which satisfies $\phi_m \to \mathrm{e}^{-1}\chi_{\{0\}}$. If we replace ϕ by ϕ_m in the last display and let $m \to \infty$, the LHS and RHS converges to different values, a contradiction! Hence v is not weakly differentiable.

Now we discuss the equivalence of weak and partial derivatives of differentiable functions.

Lemma 2.4. *Suppose a continuous function $u : U \to \mathbb{R}$ is weakly differentiable, and the weak derivatives $D^{e_1}u, \cdots, D^{e_n}u$ are also continuous (thus unique). Then $u \in C^1(U)$, and the weak derivatives coincide with the partial ones, in symbols $(\partial^{e_1}u, \cdots, \partial^{e_n}u) = (D^{e_1}u, \cdots, D^{e_n}u)$.*

Proof. Since differentiation is a local problem, we fix any pre-compact set $V \Subset U$ and choose $\epsilon > 0$ such that $V \subset \overline{U}^{2\epsilon}$. Then the value of the mollification u^ϵ inside $\overline{U}^{2\epsilon}$ is given by (1.6). For each $x \in U^{2\epsilon}$, we have

$$\begin{aligned}(\partial^{e_i}u^\epsilon)(x) = (\partial^{e_i}\eta_\epsilon * u)(x) &= \int_{B(x,\epsilon)} (\partial_x^{e_i}\eta_\epsilon)(x-y)u(y)\,dy \\ &= -\int_{B(x,\epsilon)} (\partial_y^{e_i}\eta_\epsilon)(x-y)u(y)\,dy \\ &= \int_{B(x,\epsilon)} \eta_\epsilon(x-y)(D^{e_i}u)(y)\,dy = (\eta_\epsilon * D^{e_i}u)(x).\end{aligned}$$

By Proposition 1.8, $\epsilon \to 0$ gives uniform convergences $u^\epsilon \rightrightarrows u$ and $\partial^{e_i}u^\delta = \eta_\epsilon * D^{e_i}u \rightrightarrows D^{e_i}u$ on the compact set $\overline{V}$. Moreover, for any $x \in V$ and any $|h| > 0$ such that $x + he_i \in V$,

$$u(x+he_i) - u(x) = \lim_{\epsilon\to 0}\left(u^\epsilon(x+he_i) - u^\epsilon(x)\right) = \lim_{\epsilon\to 0}\int_0^h (\partial^{e_i}u^\epsilon)(x+te_i)\,dt = \int_0^h (D^{e_i}u)(x+te_i)\,dt.$$

By continuity of $D^{e_i}u$, we have $\partial_{e_i}u(x) = D^{e_i}u(x)$ for all $x \in V$. Hence $u \in C^1(V)$. Since the pre-compact set V is arbitrary, we have $u \in C^1(U)$. $\square$

Remark. In fact, this proof also provide an approximation approach of weak derivatives. If a function $u : U \to \mathbb{R}$ has weak derivative $D^\alpha u$, we choose any $V \Subset W \Subset U^{2\epsilon}$. Then for each $x \in U^{2\epsilon}$,

$$\begin{aligned}(\partial^\alpha u^\epsilon)(x) = (\partial^\alpha\eta_\epsilon * u)(x) &= \int_{B(x,\epsilon)} (\partial_x^\alpha\eta_\epsilon)(x-y)u(y)\,dy = (-1)^{|\alpha|}\int_{B(x,\epsilon)} (\partial_y^\alpha\eta_\epsilon)(x-y)u(y)\,dy \\ &= \int_{B(x,\epsilon)} \eta_\epsilon(x-y)(D^\alpha u)(y)\,dy = (\eta_\epsilon * D^\alpha u)(x).\end{aligned}$$

Hence $\partial^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u = (D^\alpha u)^\epsilon$ on $W \subset U^{2\epsilon}$. Since $D^\alpha u \in L^1_{\mathrm{loc}}(U) \subset L^1_{\mathrm{loc}}(W)$, by Proposition 1.8, $\partial^\alpha u^\epsilon \to D^\alpha u$ in $L^1(V)$ as $\epsilon \to 0$. Furthermore, since $V \Subset U$ is arbitrary, we have

$$\partial^\alpha u^\epsilon \to D^\alpha u \quad in \quad L^1_{\mathrm{loc}}(U) \quad as \quad \epsilon \to 0.$$

This gives rise to the following approximation theorem.

Theorem 2.5. *A function $u \in L^1_{\mathrm{loc}}(U)$ is weakly differentiable in U if and only if there is a sequence of functions $u_m \in C^\infty(U)$ such that $u_m \to u$ and $\partial^\alpha u_m \to v$ in $L^1_{\mathrm{loc}}(U)$. In that case the weak derivative of u is given by $v = D^\alpha u \in L^1_{\mathrm{loc}}(U)$.*

Proof. If u is weakly differentiable in U, we can construct a desired sequence by mollification, as is discussed in the preceding Remark. Conversely, given such a sequence (u_m), we have

$$\left|\int_U u_m\phi\,dm - \int_U u\phi\,dm\right| = \left|\int_{\operatorname{supp}\phi}(u_m-u)\phi\,dm\right| \le \|\phi\|_\infty \int_{\operatorname{supp}\phi}|u_m-u|\,dm \to 0, \quad \forall\phi\in C_c^\infty(U).$$

Consequently, the L^1_{loc}-convergence of u_m and $\partial^\alpha u_m$ implies

$$\int_U u\partial^\alpha\phi\,dm = \lim_{n\to\infty}\int_U u_m\partial^\alpha\phi\,dm = \lim_{n\to\infty}(-1)^{|\alpha|}\int_U(\partial^\alpha u_m)\phi\,dm = (-1)^{|\alpha|}\int_U v\phi\,dm.$$

Therefore, u is weakly differentiable, and $v = D^\alpha u$. □

Next we introduce some properties of weak derivatives.

Proposition 2.6. *Let U be an open subset of $\mathbb{R}^n$, and $u\in L^1_{\mathrm{loc}}(U)$.*

(i) (Higher order derivatives). Assume that the weak derivatives $D^\alpha u$ and $D^\beta u$ exist for multi-indices $\alpha,\beta\in\mathbb{N}_0^n$. Then if any one of the weak derivatives $D^\alpha(D^\beta u), D^\beta(D^\alpha u), D^{\alpha+\beta}u$ exists, all three weak derivatives exist and are equal.

(ii) (Leibniz product rule). Assume that $v\in C^\infty(U)$. If $u\in L^1_{\mathrm{loc}}(U)$ is weakly differentiable, so is $u\psi$, and

$$D^{e_i}(u\psi) = u\partial^{e_i}\psi + (D^{e_i}u)\psi, \quad i=1,\cdots,n. \tag{2.1}$$

More generally, if the weak derivative $D^\alpha u$ exists for $\alpha\in\mathbb{N}_0^n$, then

$$D^\alpha(u\psi) = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^\beta u\,\partial^{\alpha-\beta}\psi. \tag{2.2}$$

(iii) (Chain rule). Assume that $F\in C^1(\mathbb{R})$, and its derivative $F'\in L^\infty(\mathbb{R})$ is bounded. If $u\in L^1_{\mathrm{loc}}(U)$ is weakly differentiable, so is $F\circ u$, and

$$D^{e_i}(F\circ u) = F'(u)D^{e_i}u, \quad i=1,\cdots,n.$$

Proof. (i) Using the existence of $D^\alpha u$ and the fact that $\partial^\beta\phi\in C_c^\infty(U)$ for all $\phi\in C_c^\infty(U)$, one have

$$\int_U D^\alpha u\,\partial^\beta\phi\,dm = (-1)^{|\alpha|}\int_U u\partial^{\alpha+\beta}\phi\,dm.$$

Hence $D^{\alpha+\beta}u$ exists if and only if $D^\beta(D^\alpha u)$ exists, and $D^\beta(D^\alpha u) = D^{\alpha+\beta}u$ in the weak sense. A symmetric argument holds with α and β exchanged.

(ii) For any $\phi\in C_c^\infty(U)$, the function $\psi\phi\in C_c^\infty(U)$, and

$$\int_U(D^{e_i}u)\psi\phi\,dm = -\int_U u\partial^{e_i}(\psi\phi)\,dm = -\int_U u(\partial^{e_i}\psi)\phi\,dm - \int_U u\psi\partial^{e_i}\phi\,dm.$$

By definition, we have $D^{e_i}(u\psi) = (D^{e_i}u)\psi + u\partial^{e_i}\psi$, which is the case $\alpha=e_i$ of (2.2). Now we prove the general case by induction. Suppose formula (2.2) is valid for all multi-indices $\beta<\alpha$. We choose $\alpha=\beta+e_i$ for some $|\beta|=|\alpha|-1$ and $i\in[n]$. Then for any $\phi\in C_c^\infty(U)$, by the assumption of induction, we have

$$\int_U u\psi\partial^\alpha\phi\,dm = \int_U u\psi\partial^\beta(\partial^{e_i}\phi)\,dm = (-1)^{|\beta|}\int_U\sum_{\gamma\le\beta}\binom{\beta}{\gamma}D^\gamma u\,\partial^{\beta-\gamma}\psi\,\partial^{e_i}\phi\,dm.$$

Using the product rule, we have

$$\begin{aligned}
\int_U u\psi\partial^\alpha\phi\,dm &= (-1)^{|\beta|+1}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\int_U D^{e_i}(D^\gamma u\,\partial^{\beta-\gamma}\psi)\phi\,dm\\
&= (-1)^{|\alpha|}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\int_U \left(D^{\gamma+e_i}u\,\partial^{\alpha-\gamma-e_i}\psi + D^\gamma u\,\partial^{\alpha-\gamma}\psi\right)\phi\,dm\\
&= (-1)^{|\alpha|}\sum_{\gamma\le\beta+e_i}\int_U\left(\binom{\beta}{\gamma-e_i}D^\gamma u\,\partial^{\alpha-\gamma}\psi + \binom{\beta}{\gamma}D^\gamma u\,\partial^{\alpha-\gamma}\psi\right)\phi\,dm\\
&= (-1)^{|\alpha|}\int_U\left(\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}D^\gamma u\,\partial^{\alpha-\gamma}\psi\right)\phi\,dm.
\end{aligned}$$

(iii) Since $F' \in L^\infty(\mathbb{R})$, the function F is globally Lipschitz, and we suppose $|F(t)-F(s)| \le L|t-s|$. By Theorem 2.5, we choose a sequence $u_m \in C^\infty(U)$ such that $u_m \to u$ and $\partial^{e_i}u_m \to \partial^{e_i}u$ in $L^1_{\text{loc}}(U)$. Let $v = F\circ u$, and $v_m = F\circ u_m \in C^1(U)$, with $\partial^{e_i}v_m = F'(u_m)\partial^{e_i}u_m \in C(U)$. If $V \Subset U$, then

$$\int_V |v_m - v|\,dm = \int_V |F(u_m)-F(u)|\,dm \le L\int_V |u_m-u|\,dm \to 0 \quad as \quad n\to\infty.$$

Furthermore, for the partial derivatives, we have

$$\begin{aligned}
\int_V |\partial^{e_i}v_m - F'(u)D^{e_i}u|\,dm &= \int_V |F'(u_m)\partial^{e_i}u_m - F'(u)D^{e_i}u|\,dm\\
&\le \int_V |F'(u_m)|\,|\partial^{e_i}u_m - D^{e_i}u|\;dm + \int_V |F'(u_m)-F'(u)|\,|D^{e_i}u|\;dm\\
&\le L\int_V |\partial^{e_i}u_m - D^{e_i}u|\;dm + \int_V \underbrace{|F'(u_m)-F'(u)|\,|D^{e_i}u|}_{\le 2L|D_i^e u|\in L^1(V)}\,dm.
\end{aligned}$$

Using the fact that $\partial_i^e u_m \to D^{e_i}u$ in $L^1_{\text{loc}}(U)$ and the Dominated Convergence Theorem, the last display converges to zero. Since $V \Subset U$ is arbitrary, we have $v_m \to v$ and $\partial^{e_i}v_m \to F'(u)D^{e_i}u$ in $L^1_{\text{loc}}(U)$. Again by Theorem 2.5, we have $D^{e_i}(F\circ u) = D^{e_i}v = F'(u)D^{e_i}u$. $\square$

Remark. Using a similar approximation argument applied in the proof of (iii), we can show that the product rule (2.1) holds for all $\psi \in C^1(U)$ and all weakly differentiable $u \in L^1_{\text{loc}}(U)$.

2.3 Sobolev Spaces and Approximation

Sobolev spaces consist of functions whose weak derivatives belong to L^p. These spaces provide one of the most useful settings for the analysis of PDEs.

Definition 2.7 (Sobolev spaces)**.** *Let U be an open subset of $\mathbb{R}^n$, $k\in\mathbb{N}$, and $1\le p\le\infty$. The Sobolev space $W^{k,p}(U)$ consists of all locally integrable functions $u : U\to\mathbb{R}$ such that for each multi-index α with $|\alpha|\le k$, the weak derivative $D^\alpha u$ exists and belongs to $L^p(U)$. We identify functions in $W^{k,p}(U)$ which agree a.e., and define the norm of $u\in W^{k,p}(U)$ to be*

$$\|u\|_{W^{k,p}(U)} := \begin{cases}\left(\sum_{|\alpha|\le k}\int_U |D^\alpha u|^p\,dm\right)^{1/p}, & 1\le p\le\infty,\\ \max_{|\alpha|\le k}\operatorname{ess\,sup}_U |D^\alpha u|, & p=\infty.\end{cases}$$

We write $H^k(U) = W^{k,2}(U)$, where we define the inner product $\langle u,v\rangle_{H^k(U)} := \sum_{|\alpha|\le k}\int_U D^\alpha u\,D^\alpha v\,dm.$

Remark I. We need to check that $\|\cdot\|_{W^{k,p}(U)}$ is a norm on $W^{k,p}(U)$. Nonnegativeness and homogeneity of $\|\cdot\|_{W^{k,p}(U)}$ are clear, and the triangle inequality is also clear when $p=\infty$. Hence we only verify the triangle inequality in the case $1 \le p \le \infty$. By Minkowski's inequality,

$$\begin{aligned}\|u+v\|_{W^{k,p}(U)} &= \left(\sum_{|\alpha|\le k} \|D^\alpha u + D^\alpha v\|_{L^p(U)}^p\right)^{1/p} \le \left(\sum_{|\alpha|\le k} \left(\|D^\alpha u\|_{L^p(U)} + \|D^\alpha v\|_{L^p(U)}\right)^p\right)^{1/p} \\ &\le \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(U)}^p\right)^{1/p} + \left(\sum_{|\alpha|\le k} \|D^\alpha v\|_{L^p(U)}^p\right)^{1/p} = \|u\|_{W^{k,p}(U)} + \|v\|_{W^{k,p}(U)}.\end{aligned}$$

Remark II. Corresponding to Proposition 2.6, the following properties of Sobolev spaces holds:

(i) If $k \le l$, then $W^{k,p}(U) \subset W^{l,p}(U)$. If $u \in W^{k,p}(U)$, then $D^\alpha u \in W^{k-|\alpha|,p}(U)$ for all $|\alpha| \le k$.

(ii) If $u \in W^{k,p}(U)$ and $\psi \in C^\infty(U)$, then $u\psi \in W^{k,p}(U)$;

(iii) If $u \in W^{1,p}(U)$ and $F \in C^1(\mathbb{R})$, then $F \circ u \in W^{1,p}(U)$.

The Sobolev spaces have a nice structure.

Theorem 2.8. *For each $k \in \mathbb{N}$ and $1 \le p \le \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach space.*

Proof. We need to show that $W^{k,p}(U)$ is complete. Let $(u_m)_{m=1}^\infty$ be a Cauchy sequence in $W^{k,p}(U)$. Then for each $|\alpha| \le k$, $(D^\alpha u_m)_{m=1}^\infty$ is a Cauchy sequence in $L^p(U)$. By completeness of $L^p(U)$, there exists $u^{(\alpha)} \in L^p(U)$ such that $D^\alpha u_m \to u^{(\alpha)}$ in $L^p(U)$ for each $|\alpha| \le k$, and in particular $u_m \to u$ in $L^p(U)$ when $\alpha = 0$.

Clearly, if we can show that $u \in W^{k,p}(U)$ and $D^\alpha u = u^{(\alpha)}$ for all $|\alpha| \le k$, the result follows. To this end, we let $q = \frac{p}{p-1}$ be the Hölder conjugate, and fix any $\phi \in C_c^\infty(U)$. By Hölder's inequality,

$$\left|\int_U (u_m - u)\partial^\alpha \phi \, dx\right| \le \|u_m - u\|_{L^p(U)} \|\partial\phi\|_{L^q(U)} \to 0, \quad \text{and} \tag{2.3}$$

$$\left|\int_U (D^\alpha u_m - u^{(\alpha)})\phi \, dx\right| \le \|D^\alpha u_m - u^{(\alpha)}\|_{L^p(U)} \|\phi\|_{L^q(U)} \to 0. \tag{2.4}$$

These two limits imply the interchangeability of the limit and the integral:

$$\int_U u\partial^\alpha \phi \, dx = \lim_{m\to\infty} \int_U u_m \partial^\alpha \phi \, dx = (-1)^{|\alpha|} \lim_{m\to\infty} \int_U D^\alpha u_m \phi \, dx = (-1)^{|\alpha|} \int_U u^{(\alpha)} \phi \, dx.$$

Hence our assertion is valid. Since $D^\alpha u_m \to D^\alpha u$ in $L^p(U)$ for all $|\alpha| \le k$, we have $u_m \to u$ in $W^{k,p}(U)$. $\square$

Definition 2.9 (Local Sobolev spaces)**.** *Let U be an open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$, and $1 \le p \le \infty$. The local Sobolev space $W^{k,p}_{\mathrm{loc}}(U)$ consists of all locally integrable functions $u : U \to \mathbb{R}$ whose restriction to any pre-compact $V \Subset U$ lies in $W^{k,p}(V)$, i.e.*

$$W^{k,p}_{\mathrm{loc}}(U) = \left\{u \in L^1_{\mathrm{loc}}(U) : \forall V \Subset U,\ u|_V \in W^{k,p}(V)\right\}.$$

We say a sequence of functions $u_m \in W^{k,p}_{\mathrm{loc}}(U)$ converges to u in $W^{k,p}_{\mathrm{loc}}(U)$ if $\|u_m - u\|_{W^{k,p}(V)} \to 0$ as $m \to \infty$ for all pre-compact $V \Subset U$.

Remark. To summarize, for $k \in N$ and $1 \le p \le \infty$, there are the, in general strict, inclusions

$$\begin{array}{ccccc} L^p(U) & \subset & L^p_{\mathrm{loc}}(U) & \subset & L^1_{\mathrm{loc}}(U) \\ \cup & & \cup & & \cup \\ W^{k,p}(U) & \subset & W^{k,p}_{\mathrm{loc}}(U) & \subset & W^{k,1}_{\mathrm{loc}}(U) \end{array}$$

Next we are going to discuss approximation of Sobolev functions.

Theorem 2.10 (Local approximation by smooth functions)**.** *Assume $1 \le p < \infty$. For each $u \in W^{k,p}(U)$, the function $u^\epsilon = \eta_\epsilon * \overline{u}^{(\epsilon)} \in C^\infty(U)$ for each $\epsilon > 0$, and $u^\epsilon \to u$ in $W^{k,p}_{\mathrm{loc}}(U)$ as $\epsilon \to 0$.*

Proof. According to Proposition 1.8 and the Remark under Lemma 2.4, $u^\epsilon \to u$ and $D^\alpha u^\epsilon \to D^\alpha u$ in $L^p(V)$ as $\epsilon \to 0$ for all $|\alpha| \le k$ and all pre-compact $V \Subset U$. Then

$$\|u^\epsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha|\le k} \|D^\alpha u^\epsilon - D^\alpha u\|^p_{L^p(V)} \to 0 \quad as \quad \epsilon \to 0. \tag{2.5}$$

Hence $u^\epsilon \to u$ in $W^{k,p}_{\mathrm{loc}}(U)$ as $\epsilon \to 0$. $\square$

Remark. If $U = \mathbb{R}^n$, the convergence (2.5) remains valid by Proposition 1.5 when we replace V by $\mathbb{R}^n$. Consequently, $C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ for $k \in \mathbb{N}$ and $1 \le p < \infty$. Now we assume $u \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$, and choose $\phi \in C^\infty_c(\mathbb{R}^n)$ such that $\phi(x) = 1$ for $|x| \le 1$ and $\phi(x) = 0$ for $|x| \ge 2$. Let $\phi_R = \phi(x/R)$. Then $u^R := \phi_R u \in C^\infty_c(\mathbb{R}^n)$, and by Leibniz rule, we have

$$D^\alpha u^R = \phi_R D^\alpha u + \frac{1}{R} h_R \to D^\alpha u, \quad as \quad R \to \infty,$$

where h_R is bounded in L^p uniformly in R. Hence $u^R \to u$ in $W^{k,p}(\mathbb{R}^n)$ as $R \to \infty$. Therefore, the space $C^\infty_c(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ for $k \in \mathbb{N}$ and $1 \le p < \infty$.

We denote by $W^{k,p}_0(U)$ the closure of $C^\infty_c(U)$ in $W^{k,p}(U)$:

$$W^{k,p}_0(U) := \overline{C^\infty_c(U)}^{\|\cdot\|_{W^{k,p}(U)}}$$

For the case $U = \mathbb{R}^n$, we have the result $W^{k,p}_0(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$. However, we do not have a similar global approximation conclusion for general $U \subset \mathbb{R}^n$.

Theorem 2.11 (Global approximation by smooth functions on bounded domains)**.** *Assume that $U \subset \mathbb{R}^n$ is open and bounded, and $1 \le p < \infty$. Then for each $u \in W^{k,p}(U)$, there exists a sequence of functions $u_m \in C^\infty(U) \cap W^{k,p}(\mathbb{R}^n)$ such that $u_m \to u$ in $W^{k,p}(U)$ as $m \to \infty$.*

Proof. We write $U_r = \{x \in U : d(x, \partial U) > 1/r\}$, and $V_r := U_{r+3} \backslash \overline{U}_{r+1}$, where $r = 1, 2, \cdots$. Take any open $V_0 \Subset U_4$ such that $U = \bigcup_{r=0}^\infty V_r$, and choose a smooth partition of unity $\phi_r : U \to [0,1]$ subordinate to $(V_r)_{r=0}^\infty$:

$$\phi_r \in C^\infty_c(V_r), \quad \sum_{r=0}^\infty \phi_r = 1 \ on\ U.$$

Then for any $u \in W^{k,p}(U)$, we have $\phi_r u \in W^{k,p}(U)$ and $\mathrm{supp}(\phi_r u) \Subset V_r$. Now fix $\delta > 0$, and choose $\epsilon_r > 0$ so small that $u^r = \eta * (\phi_r u)$ satisfies

$$\|u^r - \phi_r u\|_{W^{k,p}(U)} \le \frac{\delta}{2^{r+1}}, \ r = 0, 1, 2, \cdots; \quad \mathrm{supp}\, u^r \subset U_{r+4} \backslash \overline{U}_r, \ r = 1, 2, \cdots.$$

Let $v = \sum_{r=0}^\infty u^r$. Then $v \in C^\infty(U)$, since for each open set $V \Subset U$ there are at most finitely many nonzero terms in the sum. Furthermore,

$$\|v - u\|_{W^{k,p}(V)} \le \sum_{r=0}^\infty \|u^r - \phi_r u\|_{W^{k,p}(U)} \le \delta \sum_{r=1}^\infty \frac{1}{2^{r+1}} = \delta.$$

Taking the supremum over open sets $V \Subset U$, we conclude that $\|v - u\|_{W^{k,p}(U)} \le \delta$. $\square$

Now we discuss the approximation of Sobolev functions even up to the boundary of domain U. To prepare, we introduce some regularity conditions on boundaries.

Definition 2.12 (Regularity of boundaries). *For a pre-compact $U \Subset \mathbb{R}^n$, its boundary ∂U is said to be Lipschitz, if for each $x^0 \in \partial U$, there exists a radius $r > 0$ and a Lipschitz continuous map $\gamma : \Omega \to \mathbb{R}$, defined on an open set $\Omega \subset \mathbb{R}^{n-1}$ with Lipschitz constant, say L_γ, such that, after possibly relabeling and reorienting some coordinate axes, (i) the part of the boundary ∂U inside the closed ball $B(x^0, r)$ is the graph of γ, and (ii) the part of U inside the closed ball $B(x^0, r)$ is of the simple form*

$$U \cap B(x^0, r) = \left\{x \in B(x^0, r) : x_n > \gamma(x_1, \cdots, x_n)\right\}.$$

In addition, for any $k \in \mathbb{N} \cup \{\infty\}$, ∂U is said to be C^k if $\gamma \in C^k(\Omega)$.

Remark. By compactness of ∂U, we can choose finitely many tuples $(x^0_1, r_1, \gamma_1), \cdots, (x^0_N, r_N, \gamma_N)$ such that the open balls $B^0(x^0_1, r_1), \cdots .B^0(x^0_N, r_N)$ cover ∂U. Consequently, the Lipschitz maps γ we choose are *uniformly Lipschitz*. In other words, for all $x^0 \in \partial U$, the map γ we choose to describe the local geometry of ∂U has Lipschitz constant smaller than $\gamma := \max_{1 \le j \le N} \gamma_j$.

In a domain U whose boundary ∂U is Lipschitz, we can approximate a Sobolev function using functions smooth up to the boundary, i.e. the functions in $C^\infty(\overline{U})$.

Theorem 2.13 (Global approximation by functions smooth up to the boundary of Lipschitz domains). *Assume that $U \subset \mathbb{R}^n$ is open and bounded, ∂U is Lipschitz, and $1 \le p < \infty$. Then for each $u \in W^{k,p}(U)$, there exists a sequence of functions $u_m \in C^\infty(\overline{U})$ such that $u_m \to u$ in $W^{k,p}(U)$ as $m \to \infty$.*

Proof. Step I: In this step, we construct a space for mollification within U. Given $x^0 \in \partial U$, we pick a radius $r > 0$ and a Lipschitz map γ whose graph is part of ∂U inside $B(x^0, r)$. Define the closed horizontal double cone $\widetilde{C}_0$ and open upward cone C_0:

$$\widetilde{C}_0 = \{(x', x_n) \in \mathbb{R}^n : |x_n| \le L|x'|\}, \quad C_0 = \{(x', x_n) \in \mathbb{R}^n : x_n > L|x'|\}.$$

Then for any $y \in \partial U$, the translated horizontal double cone $\widetilde{C}_y = y + \widetilde{C}_0$ contains $\partial U \cap B(y, r(y))$, and the translated open upward cone $C_y = y + C_0$ lies in U within some radius $r(y)$ from y.

Let $V = U \cap B^0(x^0, r/2)$. For any $x \in V$, define the upward shifted point

$$x^\epsilon := x + \epsilon\lambda e_n, \quad x \in V,\ \epsilon > 0,$$

where $\lambda > \sqrt{1 + L^2}$ is so large that the ball $B(x^\epsilon, \epsilon)$ lies in the upward cone $C_{\tilde{x}}$ for all $0 < \epsilon < 1$, where $\tilde{x} \in \partial U \cap B(x_0, r/2)$ shares the same horizontal coordinates with x. Moreover, for all $\epsilon > 0$ sufficiently small, the family $B(x^\epsilon, \epsilon)$ is located near x, hence in the open neighborhood $W := U \cap B^0(x^0, r)$ for all $x \in V$.

Now we define $u_\epsilon(x) = u(x^\epsilon)$ for all $x \in V$, which is the function u translated a distance $\lambda\epsilon$ in the e_n direction. Write $v^\epsilon = \eta_\epsilon * u_\epsilon$. Then v^ϵ is not only defined on V, because for any $\tilde{x} \in \partial U \cap B(x_0, r/2)$,

$$v^\epsilon(\tilde{x}) = \int_{B(\tilde{x},\epsilon)} \eta_\epsilon(\tilde{x} - y) u_\epsilon(y)\, dy = \int_{B(\tilde{x},\epsilon)} \eta_\epsilon(\tilde{x} - y) u(\underbrace{y + \epsilon\lambda e_n}_{\in B(\tilde{x}+\epsilon\lambda e_n, \epsilon)})\, dy.$$

Since $B(\tilde{x} + \epsilon\lambda e_n, \epsilon) \subset C_{\tilde{x}}$, $v^\epsilon(\tilde{x})$ is well-defined. Consequently, v^ϵ is also defined on a sufficiently small neighborhood of $\tilde{x} \in \partial V \cap \partial U$, and $v^\epsilon \in C^\infty(\overline{V})$.

Step II: We prove that $v^\epsilon \to u$ in $W^{k,p}(V)$. To this end, we take any multi-index $|\alpha| \le k$. Then

$$\begin{aligned}\|\partial^\alpha v^\epsilon - D^\alpha u\|_{L^p(V)} &\le \|\partial^\alpha v^\epsilon - D^\alpha u_\epsilon\|_{L^p(V)} + \|D^\alpha u_\epsilon - D^\alpha u\|_{L^p(V)} \\ &= \|\eta_\epsilon * (D^\alpha u_\epsilon) - D^\alpha u_\epsilon\|_{L^p(V)} + \|D^\alpha u_\epsilon - D^\alpha u\|_{L^p(V)} \\ &\le \|\eta_\epsilon * (D^\alpha u) - D^\alpha u\|_{L^p(\mathbb{R}^n)} + \|D^\alpha u_\epsilon - D^\alpha u\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

The first term vanishes as $\epsilon \to 0$ by Proposition 1.6, and the second term also vanishes by continuity of translation operator in L^p-norm.

Step III: We finally prove the global result via partition of unity. Pick $\delta > 0$. By compactness of ∂U, there exist finitely many points $x_i^0 \in \partial U$, radii $r_i > 0$, corresponding sets $V_i = U \cap B^0(x_i^0, \frac{r_i}{2})$ and functions $v^i \in C^\infty(V_i)$, where $i = 1, \cdots, N$ such that the open balls $B^0(x_i^0, \frac{r_i}{2})$ form a cover of ∂U, and (by Step II)

$$\|v^i - u\|_{W^{k,p}(V_i)} < \delta.$$

Choose $V_0 \Subset U$ such that $(V_i)_{i=0}^N$ is an open cover U, and $v^0 \in C^\infty(\overline{V}_0)$ such that $\|v^0 - u\|_{W^{k,p}(V_0)} < \delta$ by Theorem 2.10. By taking a smooth partition of unity $(\phi_i)_{i=0}^N$ subordinate to the open cover, we construct a smooth function $v = \sum_{i=0}^N \phi_i v_i \in C^\infty(\overline{U})$. Furthermore, for each $|\alpha| \le k$, one have

$$\begin{aligned}\|D^\alpha v - D^\alpha u\|_{L^p(U)} &\le \sum_{i=1}^N \|D^\alpha(\phi_i v_i) - D^\alpha(\phi_i u)\|_{L^p(V_i)} \\ &\le \sum_{i=1}^N \left\| \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \phi_i (D^{\alpha-\beta} v_i - D^{\alpha-\beta} u) \right\|_{L^p(V_i)} \\ &\le C \sum_{i=1}^N \|v_i - u\|_{W^{k,p}(U)} \le C(N+1)\delta\end{aligned}$$

for some constant $C = C(k,p) > 0$. Since $\delta > 0$ can be arbitrarily small, the proof is completed. □

2.4 Absolute Continuity on Lines

In this section, we discuss the relation between the weak partial derivatives and the classical partial derivatives. Throughout this discussion, the absolute continuity of functions restricted to line segments plays an important role. Keep in mind that we identify functions that agree a.e..

Theorem 2.14 (ACL characterization). *Let $1 \le p \le \infty$ and $u \in L^p(U)$. Then $u \in W^{1,p}(U)$ if and only if u has a representative $\overline{u}$ that has the ACL property, i.e. $\overline{u}$ is absolutely continuous on almost all line segments in U parallel to the coordinate axes and whose (classical) partial derivatives exist a.e. and belong to $L^p(U)$. Moreover, the (classical) partial derivatives of $\overline{u}$ agree a.e. with the weak derivatives of u.*

Proof. Step I: We first suppose that $u \in W^{1,p}(U)$, and find its representative $\overline{u}$ having the desired property.

Case I: $1 \le p < \infty$. Write $x \in I$ as $x = (x_{-i}, x_i)$, where

$$x_{-i} \in U_i := \left\{t_{-i}\mathbb{R}^{n-1} : \{(t_{-i}, t_i) : t_i \in \mathbb{R}\} \cap U \neq \emptyset\right\}, \quad \text{and} \quad x_i \in U_{x_{-i}} := \{t_i \in \mathbb{R} : (x_{-i}, t_i) \in U\}$$

By Theorem 2.10, the mollifiers u^ϵ converges to u in $W^{k,p}(V)$ for any $V \Subset U$. By Fubini's theorem,

$$\lim_{\epsilon \to 0} \int_{U_i} \int_{V_{x_{-i}}} \sum_{|\alpha| \le 1} |D^\alpha u^\epsilon(x_{-i}, x_i) - D^\alpha u(x_{-i}, x_i)|^p dx_i \, dx_{-i} = 0.$$

Consequently, we can find a subsequence $\epsilon_l \to 0$ such that

$$\lim_{l\to\infty} \int_{V_{x_{-i}}} \sum_{|\alpha|\le 1} |D^\alpha u^{\epsilon_l}(x_{-i}, x_i) - D^\alpha u(x_{-i}, x_i)|^p dx_i = 0 \quad for \ a.e. \ x_{-i} \in U_i. \tag{2.6}$$

Denote $u_l = u^{\epsilon_l}$, and let $\overline{u} = \lim_{l\to\infty} u_l$. By Proposition 1.8, $\overline{u}$ agrees with u except on a Lebesgue null set $E \subset U$. Again by Fubini's theorem,

$$\int_{U_i} \int_{U_{x_{-i}}} \sum_{|\alpha|=1} |D^\alpha u(x_{-i}, x_i)|^p \, dx_i \, dx_{-i} < \infty, \qquad \int_{U_i} \mathcal{L}^1(\{x_i \in U_{x_{-i}} : (x_{-i}, x_i) \in E\}) \, dx_{-i} = 0.$$

Correspondingly, we may find a set $N_i \subset U_i$ with $\mathcal{L}^{n-1}(N_i) = 0$ such that for all $x_{-i} \in U_i \backslash N_i$,

$$\int_{U_{x_{-i}}} \sum_{|\alpha|=1} |D^\alpha u(x_{-i}, x_i)|^p \, dx_i < \infty, \quad \mathcal{L}^1(\{x_i \in U_{x_{-i}} : (x_{-i}, x_i) \in E\}) = 0.$$

Fix any such x_{-i}, and let $I \subset U_{x_{-i}}$ be a maximal open interval. Fix $t_0 \in I$ with $(x_{-i}, t_0) \in U\backslash E$, and let $t \in I$. Then there exists an open set $V \Subset U$ containing both (x_{-i}, t_0) and (x_{-i}, t). Since $u_l \in C^\infty(V)$, by fundamental theorem of calculus, one have

$$u_l(x_{-i}, t) = u_l(x_{-i}, t_0) + \int_{t_0}^t \partial_{x_i} u_l(x_{-i}, s) \, ds.$$

Since $(x_{-i}, t_0) \in U\backslash E$, we have $u_l(x_{-i}, t_0) \to \overline{u}(x_{-i}, t_0)$. Moreover, by (2.6),

$$\lim_{l\to\infty} \int_{t_0}^t |\partial_{x_i} u_l(x_{-i}, s) - D_{x_i} u(x_{-i}, s)| \, ds = 0.$$

Therefore, once $(x_{-i}, t_0) \in U\backslash E$, which holds for a.e. $t \in I$, we have

$$\overline{u}(x_{-i}, t) = \overline{u}(x_{-i}, t_0) + \int_{t_0}^t \partial_{x_i} u(x_{-i}, s) \, ds.$$

It is seen that the function $\overline{u}(x_{-i}, \cdot)$ is absolutely continuous in I, and $\partial_{x_i}\overline{u} = D_{x_i} u$ for a.e. $t \in I$.

Case II: $p = \infty$. We first consider an open ball $B \Subset U$, and prove that u is Lipschitz in B. Since $u \in W^{1,\infty}(U)$, there exists $M > 0$ such that $\operatorname{ess\,sup}_U |Du| \le M$. Then for all $\epsilon > 0$ small enough,

$$u^\epsilon(x) = (\eta_\epsilon * u)(x) \quad and \quad \partial_{x_i} u^\epsilon(x) = (\eta_\epsilon * D_{x_i} u)(x), \ i = 1, \cdots, n, \quad \forall x \in B.$$

Hence $\|u^\epsilon\|_{L^\infty(B)} \le \|u\|_{L^\infty(B)}$, and $\sup_B |\nabla u^\epsilon| \le \operatorname{ess\,sup}_B \|Du\|_\infty \le M$. This implies that the family (u^ϵ) is uniformly bounded and equicontinuous:

$$|u^\epsilon(x) - u^\epsilon(y)| \le M|x - y|.$$

By Arzelà-Ascoli theorem, we may find a subsequence $\epsilon_l \to 0$ such that $u_l := u^{\epsilon_l}$ converges uniformly to a function $\overline{u} : B \to \mathbb{R}$ as $l \to \infty$, and $|\overline{u}(x) - \overline{u}(y)| \le M|x - y|$. Note $u = \overline{u}$ a.e. in B.

By covering U with countably many balls and applying the standard diagonal trick, we can extend u to a continuous function $\overline{u} : U \to \mathbb{R}$ such that $u = \overline{u}$ a.e..

Now we prove that $\overline{u}$ is Lipschitz on all segments I in U. If I falls in a ball, the result is clear. Otherwise, by compactness of I, we can find finitely many balls B_i covering I and points $x_0, x_1, \cdots, x_N \in U$ such that the segment $I = \{tx_0 + (1-t)x_N : t \in [0, 1]\}$ consists of N subsegments $I_i = \{tx_{i-1} + (1-t)x_i : t \in [0, 1]\} \subset B_i$, where $i = 1, \cdots, N$. For any $x, y \in I$, with $x_{j+1}, x_{j+2}, \cdots, x_k \in \{tx + (1-t)y : t \in [0,1]\}$, we have

$$\begin{aligned}|u(x) - u(y)| &\le |u(x) - u(x_j)| + |u(x_{j+1}) - u(x_j)| + \cdots + |u(x_k) - u(x_{k-1})| + |u(y) - u(x_k)| \\ &\le M|x - x_j| + M|x_j - x_{j-1}| + \cdots + M|x_k - x_{k-1}| + M|y - x_k| = M|x - y|.\end{aligned}$$

Hence $\overline{u}$ is Lipschitz on I. If I is parallel to any coordinate axis, the partial derivative of $\overline{u}$ with respect to the corresponding variable is bounded by M. Hence $\partial_{x_i}\overline{u} \in L^\infty(U)$.

Step II: Conversely, let $\overline{u}$ be the representative of u having the desired property. Fix $i = 1, \cdots, n$ and let $x_{-i} \in U_i$ be such that $\overline{u}(x_{-i}, \cdot)$ is absolutely continuous on every connected component of the open set $U_{x_{-i}}$. Then for every $\phi \in C_c^\infty(U)$, $\overline{u}(x_{-i}, \cdot)\phi(x_{-i}, \cdot)$ is absolutely continuous. By the integration by parts formula,

$$\int_{U_{x_{-i}}} \overline{u}(x_{-i}, t)\partial_{x_i}\phi(x_{-i}, t)\, dt = -\int_{U_{x_{-i}}} \partial_{x_i}\overline{u}(x_{-i}, t)\phi(x_{-i}, t)\, dt,$$

which holds for a.e. $x_{-i} \in U_i$. Integrating over U_i and using Fubini's theorem yields

$$\int_U \overline{u}(x)\partial_{x_i}\phi(x)\, dx = \int_U \partial_{x_i}\overline{u}(x)\phi(x)\, dx.$$

Therefore, $D^{e_i}\overline{u} = \partial^{e_i}\overline{u} \in L^p(U)$ for all $i = 1, \cdots, n$, and $u \in W^{1,p}(U)$. □

Remark. In the case $W^{1,\infty}(U)$, we did not require I to be coordinate-aligned, and the Lipschitz property holds on all line segments. We next introduce a very useful characterization of space $W^{1,\infty}(U)$.

Theorem 2.15. *Let $U \subset \mathbb{R}^n$ be a convex set. Then $C^{0,1}(\overline{U}) = W^{1,\infty}(U)$.*

Proof. Step I: Let $u \in C^{0,1}(\overline{U})$. Then u is Lipschitz on every segment parallel to coordinates axis, with partial derivatives bounded by $[u]_{C^{0,1}(\overline{U})}$. This implies $u \in W^{1,\infty}(U)$.

Step II: Conversely, let $u \in W^{1,\infty}(U)$. According to our construction of $\overline{u}$ in the Step I in the proof of Theorem 2.14, u admits a representative $\overline{u}$ that is Lipschitz on all line segments in U with Lipschitz constant $M \ge \operatorname{ess\,sup}_U |Du|$. Since U is convex, the line segment connecting any two points $x, y \in U$ lies in U, and the global Lipschitzness follows. Noticing that $u \in L^\infty(U)$, we have $u \in C^{0,1}(\overline{U})$. □

3 Extensions and Traces

3.1 Extensions

In this section, we discuss the extension of functions in the Sobolev space. Whereas in the realm of L^p spaces extending an L^p function on a domain $U \subset \mathbb{R}^n$ to all $\mathbb{R}^n$ within L^p is trivial, just extend naturally by zero. This does not work for Sobolev spaces, already not for those of first order $W^{1,p}$. A key point is to jump singularities across ∂ that obstruct existence of weak derivatives. We let $1 \le p \le \infty$ throughout this section.

Theorem 3.1 (Extension). *Assume that $U \Subset \mathbb{R}^n$ is bounded and ∂U is Lipschitz. Then for any bounded open set V that contains the closure of U, in symbols $U \Subset V \Subset \mathbb{R}^n$, there is a bounded linear operator*

$$E : W^{1,p}(U) \to W^{1,p}(V) \hookrightarrow W^{1,p}(\mathbb{R}^n), \quad u \mapsto Eu = \overline{u},$$

such that (i) $\overline{u}|_U = u$ a.e.; (ii) $\overline{u}$ is compactly supported in V; and (iii)

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} = \|\overline{u}\|_{W^{1,p}(V)} \le c\|u\|_{W^{1,p}(U)}, \tag{3.1}$$

where $c > 0$ is a constant depending on n, p, U and V.

Remark. The function $Eu = \overline{u}$ is called an *extension* of u on $\mathbb{R}^n$.

Proof. Step I: In this step, we derive the extension operator in the half ball model. Let $B \subset \mathbb{R}^n$ be the open ball with center x^0 lying in the hyperplane $\{x_n = 0\}$ and of radius r. Define

$$B_+ := B \cap \{x_n > 0\}, \quad B_- := B \cap \{x_n < 0\}.$$

We prove that there exists a linear map

$$E_0 : W^{1,p}(B_+) \to W^{1,p}(B), \quad u \mapsto E_0 u = \overline{u}$$

such that $\overline{u}|_{B+} = u$, and

$$\|\overline{u}\|_{W^{1,p}(B)} \le 16\|u\|_{W^{1,p}(B_+)}. \tag{3.2}$$

Case I: $1 \le p < \infty$. Without loss of generality, we suppose $u \in C^1(\overline{B}_+)$. By Theorem 2.13, the first two spaces in the inclusion $C^\infty(\overline{B}_+) \subset C^1(\overline{B}_+) \subset W^{1,p}(B_+)$ are both dense in $W^{1,p}(B_+)$. Therefore, if we can construct a linear operator $E_0 : C^1(\overline{B}_+) \to C^1(\overline{B})$ satisfying (3.2), then we can extend it to $E_0 : W^{1,p}(B_+) \to W^{1,p}(B)$ by a density argument and completeness of $W^{1,p}(B)$. To this end, we define

$$\overline{u}(x) = \begin{cases} u(x), & x \in \overline{B}_+, \\ -3u(x', -x_n) + 4u(x', -\frac{x_n}{2}), & x = (x', x_n) \in \overline{B}_-. \end{cases}$$

We claim that $\overline{u} \in C^1(\overline{B})$. To check this, we write $u^+ = \overline{u}|_{\overline{B}_+}$ and $u^- = \overline{u}|_{\overline{B}_-}$. Clearly, we have $u^+ = u^-$ on $B \cap \{x_n = 0\}$. Furthermore,

$$\partial_{x_i} u^-(x', x_n) = -3\partial_{x_i} u(x', -x_n) + 4\partial_{x_i} u(x', -\frac{x_n}{2}), \quad i = 1, \cdots, n-1,$$
$$\partial_{x_n} u^-(x', x_n) = 3\partial_{x_n} u(x', -x_n) - 2\partial_{x_n} u(x', -\frac{x_n}{2}).$$

Hence we have $\partial^\alpha u^+ = \partial^\alpha u^-$ along $B \cap \{x_n = 0\}$ for all $|\alpha| \le 1$, and $\overline{u} \in C^1(\overline{B})$.

Now we derive the estimate (3.2). By Jensen's inequality,

$$|u^-(x',x_n)|^p \le 2^{p-1}\left(|3u(x',-x_n)|^p + \left|4u(x',-\frac{x_n}{2})\right|^p\right) \le 2^{3p-1}\left(|u(x',-x_n)|^p + \left|u(x',-\frac{x_n}{2})\right|^p\right)$$

Integrate on both sides of the last display, and change the variable x_n:

$$\|u^-\|^p_{L^p(B_-)} \le 2^{3p-1}\|u\|^p_{L^p(B_+)} + 2^{3p}\|u\|^p_{L^p(B_+)} \le 2^{3p+1}\|u\|^p_{L^p(B_+)}.$$

Similarly, we have $\|\partial_{x_i}u^-\|^p_{L^p(B^-)} \le 2^{3p+1}\|\partial_{x_i}u\|^p_{L^p(B^+)}$ for all $i = 1,\cdots,n$. Henceforth,

$$\|\overline{u}\|^p_{W^{1,p}(B)} = \sum_{|\alpha|\le 1}\|\partial^\alpha \overline{u}\|^p_{L^p(B)} = \sum_{|\alpha|\le 1}\left(\|\partial^\alpha u^+\|^p_{L^p(B_+)} + \|\partial^\alpha u^-\|^p_{L^p(B_-)}\right) \le 2^{4p}\|u\|^p_{W^{1,p}(B_+)}.$$

Case II: $p = \infty$. By Theorem 2.15, we have $C^{0,1} = W^{1,\infty}$ for both B_+ and B. We then consider the map E_0 given by simple horizontal reflection:

$$E_0 : C^{0,1}(B_+) \to C^{0,1}(B), \quad u \mapsto \overline{u} : B \ni (x',x_n) \mapsto u(x',|x_n|).$$

Then $\overline{u}$ is indeed Lipschitz with the same Lipschitz constant as u, and

$$\|\overline{u}\|_{W^{1,\infty}(B)} = \max_{|\alpha|\le k} \operatorname{ess\,sup}_B |D^\alpha \overline{u}| = \max_{|\alpha|\le k} \operatorname{ess\,sup}_{B_+} |D^\alpha u| = \|u\|_{W^{1,\infty}(B_+)},$$

Step II: In this step we extend u near $x_0 \in \partial U$. If ∂U is not flat near x^0, we can find a Lipschitz map $\gamma : \mathbb{R}^{n-1} \supset \Omega \to \mathbb{R}$ with Lipschitz constant M whose graph coincides the part of ∂U within a small ball $B(x^0,r)$. Consider the neighborhoods $X = \Omega \times \mathbb{R}$ of $x^0 = (x^0_{-n}, x^0_n)$ and $Y = \Omega \times \mathbb{R}$ of $y^0 = (x^0_{-n}, 0)$. Define

$$\begin{aligned}
&\Phi : X \to Y, \quad x \mapsto \Phi(x) := (x_1,\cdots,x_{n-1}, x_n - \gamma(x_1,\cdots,x_{n-1})),\\
&\Psi : Y \to X, \quad y \mapsto \Psi(y) := (y_1,\cdots,y_{n-1}, y_n + \gamma(y_1,\cdots,y_{n-1})).
\end{aligned}$$

Then $\Phi = \Psi^{-1}$ is a bi-Lipschitz map, since

$$|\Phi(x) - \Phi(z)| \le \sqrt{2(1+M^2)}|x-z| \quad \textit{and} \quad |\Psi(y) - \Psi(z)| \le \sqrt{2(1+M^2)}|y-z|.$$

By definition, Φ flattens ∂U near x^0. By Rademacher's Theorem, the graph map γ is differentiable for a.e. $x_{-n} \in \Omega$. Hence the linearizations of Φ and Ψ exist pointwise a.e. and, furthermore, the Jacobian is triangular with diagonal elements 1. Thus $\det D\Phi = 1 = \det D\Psi$ pointwise a.e..

Now we derive the local extension of $u \in W^{k,p}(U)$ near $x^0 \in \partial D$. Pick a small ball B centered at $y^0 = \Phi(x^0)$ and contained in the open neighborhood $\Phi(B^0(x^0,r))$ of y^0. Let B_+ be the upper open half ball of B, and consider the restriction of u to the open set $V = \Psi(B_+)$. Then $u \in W^{1,p}(V)$.

Next pull back $u : V \to \mathbb{R}$ to the y coordinates to obtain the function $v := u \circ \psi : B_+ \to \mathbb{R}$ which lies in $W^{1,p}(B_+)$ by Proposition, and $\|v\|_{W^{1,p}(B_+)} = \|u\|_{W^{1,p}(V)}$. Then we employ the extension operator constructed in Step I to pick an extension $\overline{v} = E_0 v$ of $v = u \circ \psi$ from the upper half ball B_+ to the whole ball B. The extension of u from $V = \Psi(B_+)$ to $A = \Psi(B)$ is defined by

$$\overline{u} = \overline{v} \circ \Phi \in W^{1,p}(A), \quad \|\overline{u}\|_{W^{1,p}(A)} = \|\overline{v}\|_{W^{1,p}(B)}.$$

According to estimate (3.2), we have

$$\|\overline{u}\|_{W^{1,p}(A)} = \|\overline{v}\|_{W^{1,p}(B)} \le 16\|v\|_{W^{1,p}(B^+)} = 16\|u\|_{W^{1,p}(V)}. \tag{3.3}$$

Step III: In this step, we extend u globally via a finite partition of unity. By Step II and compactness of ∂U, there exist finitely many $x_i^0 \in \partial U$ and local extensions $\overline{u}_i = \overline{v}^i \circ \Phi : A_i \to \mathbb{R}$ covering ∂U, where $i = 1, \cdots, N$. Now we pick $A_0 \Subset U$ such that $U \Subset A := \bigcup_{i=0}^N A_i \Subset \mathbb{R}^n$, and pick a smooth partition of unity $(\phi_i)_{i=0}^N$ subordinate to the open cover $(A_i)_{i=0}^N$ of U. Extend U to A by $\overline{u} = \sum_{i=0}^N \phi_i \overline{u}_i \in W^{1,p}(A)$. We then have the following estimate of $\|\overline{u}\|_{W^{1,p}(A)}$:

$$\begin{aligned}
\|\overline{u}\|_{W^{k,p}(A)} \leq \sum_{i=0}^N \|\phi_i \overline{u}_i\|_{W^{1,p}(A_i)} &\leq \sum_{i=0}^N 2n^{1/p} \|\phi_i\|_{W^{1,\infty}(A_i)} \|\overline{u}_i\|_{W^{k,p}(A_i)} && \text{(By product rule)} \\
&\leq 2n^{1/p} \max_{1 \leq i \leq N} \|\phi_i\|_{W^{1,\infty}(A_i)} \sum_{i=0}^N \|\overline{u}_i\|_{W^{1,p}(A_i)} \\
&\leq \underbrace{32n^{1/p}(1+N) \max_{1 \leq i \leq N} \|\phi_i\|_{W^{1,\infty}(A_i)}}_{=:c} \|u\|_{W^{1,p}(U)}, && \text{(By estimate (3.3))}
\end{aligned}$$

where we use $1/p = 0$ when $p = \infty$. Then c is a constant depending only on n, p and U. Furthermore, the linearity of the mapping $u \mapsto \overline{u}$ follows from E_0 in Step I.

Step IV: Given $u \in W^{1,p}(U)$ and $U \Subset V \Subset \mathbb{R}^n$, we have $U \Subset (V \cap A) \Subset \mathbb{R}^n$. We then pick up a cutoff function $\chi \in C_c^\infty(V \cap A)$ with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on U. Then $\chi \overline{u} \in W^{1,p}(V)$, where $\overline{u}$ constructed in Step III is restricted to V. Furthermore, we have the following estimate for $\|\chi \overline{u}\|_{W^{1,p}(V)}$:

$$\|\chi \overline{u}\|_{W^{1,p}(V)} = \|\chi \overline{u}\|_{W^{1,p}(V \cap A)} \leq \|\chi \overline{u}\|_{W^{1,p}(A)} \leq 2n^{1/p} \|\chi\|_{W^{1,\infty}(A)} \|\overline{u}\|_{W^{k,p}(A)} \leq 2cn^{1/p} \|u\|_{W^{1,p}(U)}.$$

This completes the proof. □

Remark. (i) If $1 \leq p < \infty$, by Theorem 2.11, we can approximate $u \in W^{1,p}(V)$ by a sequence of functions $v_l \in C^\infty(V)$, and $C_c^\infty(V) \ni \chi v_l \to \chi \overline{u}$ in $W^{1,p}(V)$. Consequently, the extension $\overline{u} \in W_0^{1,p}(V)$:

$$E : W^{1,p}(U) \to W_0^{1,p}(V) \hookrightarrow W^{1,p}(\mathbb{R}^n), \quad u \mapsto Eu := \overline{u}.$$

(ii) If $p = \infty$, the constant c in (3.1) is actually independent of n.

Theorem 3.2. *Let U be a bounded, open subset of $\mathbb{R}^n$, and let ∂U be Lipschitz. Then $C^{0,1}(\overline{U}) = W^{1,\infty}(U)$.*

Proof. If $u \in C^{0,1}(\overline{U})$, we can apply Step I in the proof of Theorem 2.15 to argue that $u \in W^{1,\infty}(U)$. Conversely, if $u \in W^{1,\infty}(U)$, we can simply apply Step I in the proof of Theorem 2.15 to the extension Eu of u on $\mathbb{R}^n$, which is a convex set. □

3.2 Traces

4 Sobolev Inequalities

4.1 Sub-dimensional Case $p < n$: Gagliardo-Nirenberg-Sobolev Inequality

In this section, we suppose $1 \le p < n$, and we consider the following basic question: Can we estimate the $L^q(\mathbb{R}^n)$-norm of a smooth, compactly supported function in terms of the $L^p(\mathbb{R}^n)$-norm of its derivative. In other words, we are looking for an estimate of the form

$$\|u\|_{L^q(\mathbb{R}^n)} \le c\|Du\|_{L^p(\mathbb{R}^n)}, \quad u \in C_c^\infty(\mathbb{R}^n). \tag{4.1}$$

A scaling argument. We wonder if the estimate (4.1) holds for any $q \in [1, \infty]$. Take $u \in C_c^\infty(\mathbb{R}^n)$ with $u \not\equiv 0$, and define for $\lambda > 0$ the rescaled function $u_\lambda(x) = u(\lambda x)$. Then

$$Du_\lambda = \lambda(Du)_\lambda.$$

We then obtain

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u_\lambda|^q \, dx\right)^{1/q} = \left(\lambda^{-n} \int |u|^q \, dx\right)^{1/q} = \lambda^{-n/q}\|u\|_{L^q(\mathbb{R}^n)},$$
$$\|Du_\lambda\|_{L^p(\mathbb{R}^n)} = \left(\sum_{|\alpha|=1} \int_{\mathbb{R}^n} |D^\alpha u|^p\right)^{1/p} = \left(\lambda^{p-n} \sum_{|\alpha|=1} \int_{\mathbb{R}^n} |D^\alpha u|^p\right)^{1/p} = \lambda^{1-n/p}\|Du\|_{L^p(\mathbb{R}^n)}.$$

These norms must scale according to the same exponent, otherwise (4.1) is falsified by letting $\lambda \to 0$ or $\lambda \to \infty$. Hence we have $n/p - n/q = 1$, and $q = \frac{np}{n-p}$.

Definition 4.1 (Sobolev conjugate)**.** *If $1 \le p < n$, the Sobolev conjugate of p is*

$$p^* = \frac{np}{n-p}.$$

Note that $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, and $p^* > p$.

We have the following estimate for L^{p^*}-norm f a Sobolev function.

Theorem 4.2 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Assume that $1 \le p < n$. There exists a constant C, depending on p and n only, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)}, \quad \forall u \in C_c^1(\mathbb{R}^n). \tag{4.2}$$

Proof. Step I: We first prove the case $p = 1$. Since u has compact support, we have

$$u(x) = \int_{-\infty}^{x_i} \partial_{x_i} u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) \, dy_i,$$

We denote by $|Du|_1 = |\partial_{x_1} u| + \cdots + |\partial_{x_n} u|$. Then

$$|u(x)| \le \int_{-\infty}^{x_i} |\partial_{x_i} u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)| \, dy_i \le \int_{-\infty}^{\infty} |Du|_1 \, dx_i.$$

Consequently,

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |Du|_1 \, dx_i\right)^{\frac{1}{n-1}}.$$

We integrate both sides of the last display with respect to the variable x_1. By generalized Hölder's inequality,

$$\begin{aligned}\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\,dx_1 &\le \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du|_1\,dx_i\right)^{\frac{1}{n-1}} dx_1 \\ &= \left(\int_{-\infty}^{\infty} |Du|_1\,dx_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du|_1\,dx_i\right)^{\frac{1}{n-1}} dx_1 \\ &\le \left(\int_{-\infty}^{\infty} |Du|_1\,dx_1\right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|_1\,dx_1\,dx_i\right)^{\frac{1}{n-1}}.\end{aligned}$$

Again, we integrate both sides with respect to x_2. By generalized Hölder's inequality,

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\,dx_1\,dx_2 \\ &\le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|_1\,dx_1\,dx_2\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |Du|_1\,dx_1\right)^{\frac{1}{n-1}} \left(\prod_{i=3}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|_1\,dx_1\,dx_i\right)^{\frac{1}{n-1}} dx_2 \\ &\le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|_1\,dx_1\,dx_2\right)^{\frac{2}{n-1}} \left(\prod_{i=3}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|_1\,dx_1\,dx_2\,dx_i\right)^{\frac{1}{n-1}}.\end{aligned}$$

We continue to integrate with respect to $x_3, \cdots, x_n$, and obtain that

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\,dx \le \left(\int_{\mathbb{R}^n} |Du|_1\,dx\right)^{\frac{n}{n-1}}. \tag{4.3}$$

This is indeed the case $p^* = \frac{n}{n-1}$ and $C = 1$ of estimate (4.2).

Step II: Now we consider the case $1 < p < n$. Applying the estimate (4.3) to $v = |u|^\gamma$, where $\gamma > 1$ is to be selected, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\,dx\right)^{\frac{n-1}{n}} &\le \int_{\mathbb{R}^n} \gamma |u|^{\gamma-1} |Du|_1\,dx \\ &\le \gamma \left(\int_{\mathbb{R}^n} |u|^{\frac{(\gamma-1)p}{p-1}}\,dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|_1^p\,dx\right)^{1/p} \\ &\le \gamma \left(\int_{\mathbb{R}^n} |u|^{\frac{(\gamma-1)p}{p-1}}\,dx\right)^{\frac{p-1}{p}} n^{\frac{p-1}{p}} \|Du\|_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{4.4}$$

Now we choose $\gamma > 1$ such that $\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1}$. That is, $\gamma = \frac{(n-1)p}{n-p} = \frac{(n-1)p^*}{n}$. Then (4.4) becomes

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\,dx\right)^{1/p^*} \le \frac{n^{\frac{p-1}{p}}(n-1)p}{n-p} \|Du\|_{L^p(\mathbb{R}^n)},$$

which completes the proof of (4.2). □

Theorem 4.3 (Estimate for $W^{1,p}$ on $\mathbb{R}^n$, $1 \le p < n$)**.** *Assume that $1 \le p \le n$ and $p \le q \le p^*$, and $u \in W^{1,p}(U)$. Then $u \in L^q(U)$, with the estimate*

$$\|u\|_{L^q(\mathbb{R}^n)} \le C \|u\|_{W^{1,p}(\mathbb{R}^n)} \tag{4.5}$$

for some constant C depending only on p, q and n.

Proof. By the Remark under Theorem 2.10, we can find a sequence $u_m \in C_c^\infty(\mathbb{R}^n)$ that converges to u in $W^{1,p}(\mathbb{R}^n)$. According to Theorem 4.2, we have

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \le np^* \|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}, \quad \forall l, m \ge 1.$$

Hence (u_m) is a Cauchy sequence in $L^{p^*}(\mathbb{R}^n)$, and $u_m \to \tilde{u}$ for some $\tilde{u} \in L^{p^*}(\mathbb{R}^n)$. Furthermore, $\tilde{u}$ and u are identified, since we can find a subsequence of (u_m) that converges a.e. to $\tilde{u}$ from L^{p^*} convergence, and to u, from L^p convergence. Hence $u \in L^{p^*}(\mathbb{R}^n)$, and

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le np^* \|Du\|_{L^p(\mathbb{R}^n)}.$$

For the estimate (4.5), the case $q = p$ and $q = p^*$ are clear. If $p < q < p^*$, we choose $0 < \theta < 1$ such that $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*}$. By Hölder's inequality,

$$\int_{\mathbb{R}^n} |u|^q \, dx = \int_{\mathbb{R}^n} |u|^{\theta q} |u|^{(1-\theta)q} \, dx \le \left(\int_{\mathbb{R}^n} |u|^p \, dx \right)^{\frac{\theta q}{p}} \left(\int_{\mathbb{R}^n} |u|^{p^*} \, dx \right)^{\frac{(1-\theta)q}{p}}.$$

Therefore

$$\|u\|_{L^q(\mathbb{R}^n)} \le \|u\|_{L^p(\mathbb{R}^n)}^\theta \|u\|_{L^{p^*}(\mathbb{R}^n)}^{1-\theta} \le (np^*)^{1-\theta} \|u\|_{L^p(\mathbb{R}^n)}^\theta \|Du\|_{L^p(\mathbb{R}^n)}^{1-\theta}.$$

To derive (4.5), we use Jensen's inequality:

$$\theta \log \frac{a^p}{\theta} + (1-\theta) \log \frac{b^p}{1-\theta} \le \log(a^p + b^p) \quad \Rightarrow \quad a^\theta b^{1-\theta} \le \theta^{\frac{\theta}{p}} (1-\theta)^{\frac{1-\theta}{p}} (a^p + b^p)^{1/p}, \qquad \forall a, b > 0.$$

Then we obtain

$$\|u\|_{L^q(\mathbb{R}^n)} \le (np^*)^{1-\theta} \theta^{\frac{\theta}{p}} (1-\theta)^{\frac{1-\theta}{p}} \left(\|u\|_{L^p(\mathbb{R}^n)}^p + \|Du\|_{L^p(\mathbb{R}^n)}^p \right)^{1/p} =: C \|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

This completes the proof of (4.5). □

Now we give a similar estimate of the $W^{1,p}$-norm of a weakly differentiable function on a Lipschitz domain.

Theorem 4.4 (Estimate for $W^{1,p}$ on Lipschitz domains, $1 \le p < n$)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$ and suppose ∂U is Lipschitz. Assume that $1 \le p < n$, and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate*

$$\|u\|_{L^{p^*}(U)} \le C \|u\|_{W^{1,p}(U)}$$

for some constant C depending only on p, n and U.

Proof. Since ∂U is Lipschitz, by Theorem 3.1, there exists an extension $\overline{u} \in W^{1,p}(\mathbb{R}^n)$ such that $\overline{u} = u$ in U, $\overline{u}$ has compact support in $\mathbb{R}^n$, and

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C_1 \|u\|_{W^{1,p}(U)}, \tag{4.6}$$

where C_1 is a constant depending only on p, n and U. Since $\overline{u}$ has compact support, by the Remark under Theorem 2.10, there exists a sequence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that $u_m \to \overline{u}$ in $W^{1,p}(\mathbb{R}^n)$. By Theorem 4.2, $u_m \to \overline{u}$ in $L^{p^*}(\mathbb{R}^n)$ as well, and $\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \le np^* \|Du_m\|_{L^p(\mathbb{R}^n)}$. Then we have the limiting bound

$$\|u\|_{L^{p^*}(U)} \le \underbrace{\|\overline{u}\|_{L^{p^*}(\mathbb{R}^n)} \le np^* \|D\overline{u}\|_{L^p(\mathbb{R}^n)}}_{m \to \infty} \le np^* \|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \overset{(4.6)}{\le} C_1 np^* \|u\|_{W^{1,p}(U)}.$$

The desired result then follows by letting $C = C_1 np^*$. □

Remark. If U is a bounded, open subset of $\mathbb{R}^n$ and ∂U is Lipschitz, we have

$$W^{1,p}(U) \subset L^{p^*}(U) \subset L^q(U), \quad q \in [1, p^*].$$

by Hölder's inequality $\|u\|_{L^q(U)} \le |U|^{\frac{p^*-q}{p^* q}} \|u\|_{L^{p^*}(U)}$, we have

$$\|u\|_{L^q(U)} \le C\|u\|_{W^{1,p}(U)}, \quad q \in [1, p^*],$$

where C is a constant depending only on p, q, n and U.

Theorem 4.5 (Estimate for $W_0^{1,p}$ on bounded domains, $1 \le p < n$)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$. Assume that $1 \le p < n$, and $u \in W_0^{1,p}(U)$. Then we have the estimate*

$$\|u\|_{L^q(U)} \le C\|Du\|_{L^p(U)} \tag{4.7}$$

for each $q \in [1, p^*]$, with the constant C depending only on p, q, n and U.

Proof. Since $u \in W_0^{1,p}(U)$, there exists a sequence of functions $u_m \in C_c^\infty(U)$ such that $u_m \to u$ in $W^{1,p}(U)$. We the extend each u_m to $\mathbb{R}^n$ by assigning $u_m = 0$ on $\mathbb{R}^n \backslash U$. By letting $m \to \infty$ in the Gagliardo-Nirenberg-Sobolev inequality for u_m, we obtain

$$\|u\|_{L^{p^*}(U)} \le C\|Du\|_{L^p(U)}.$$

Since U is bounded, we have $|U| < \infty$, and the desired result follows from Hölder's inequality. □

Corollary 4.6 (Classical Poincaré's inequality)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$, and $1 \le p \le \infty$. For any $u \in W_0^{1,p}(U)$, we have the estimate*

$$\|u\|_{L^p(U)} \le C\|Du\|_{L^p(U)}, \tag{4.8}$$

where the constant C depending only on p, n and U.

Proof. For $1 \le p < n$, the estimate (4.8) is a special case of (4.7), since $p < p^*$. For $n \le p < \infty$, we choose $1 \le q < n$ such that $q < n \le p < q^* := \frac{nq}{n-q}$. Since $W_0^{1,p}(U) \subset W^{1,q}(U)$, by (4.7), we have

$$\|u\|_{L^p(U)} \le C\|Du\|_{L^q(U)} \le |U|^{\frac{pq}{p-q}} C\|Du\|_{L^p(U)}.$$

Finally, for $p = \infty$, we take a sequence $u_m \in C_c^\infty(U)$ that converges to u in $W^{1,\infty}(U)$. Using the fundamental theorem of calculus, we have

$$\begin{aligned} |u_m(x_1, \cdots, x_n)| &= \left| \int_{-\infty}^{x_i} \partial_{x_i} u_m(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)\, dy_i \right| \\ &\le \int_{-\infty}^{\infty} \|Du_m\|_{L^\infty(U)}\, dx_i \le \operatorname{diam}(U)\, \|Du_m\|_{L^\infty(U)} \end{aligned}$$

By taking the supremum of the left hand side and letting $m \to \infty$ in the last display, we can obtain that $\|u\|_{L^\infty(U)} \le \operatorname{diam}(U)\, \|Du\|_{L^\infty(U)}$. This complete the proof. □

The borderline case: $p = n$. Owing to Theorem 4.5 and the fact that $p^* = \frac{np}{n-p} \to \infty$ as $p \nearrow n$, we might expect $u \in L^\infty(U)$, provided $u \in W^{1,n}(U)$. This is however false if $n > 1$.

As a counterexample, let $U = B^0(0,1)$ be the unit open ball in $\mathbb{R}^n$, where $n > 1$. Then the function $u(x) = \log\log\big(1 + \frac{1}{|x|}\big)$ belongs to $W^{1,n}(U)$, but not to $L^\infty(U)$.

4.2 Super-dimensional Case $p > n$: Morrey's Inequality

In this section, we assume that $n < p \le \infty$. We show that u has a Hölder continuous representative, provided that $u \in W^{1,p}(U)$.

Theorem 4.7 (Morrey's inequality)**.** *Assume that $n < p \le \infty$. There exists a constant C, depending on p and n only, such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}, \quad \forall u \in C^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n), \tag{4.9}$$

where $\gamma = 1 - \frac{n}{p}$.

Proof. Step I: We claim that there exists a constant C_1, depending only on n, such that

$$\frac{1}{\mathcal{L}^n(B(x,r))}\int_{B(x,r)} |u(y) - u(x)|\,dy \le C_1 \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy, \tag{4.10}$$

for each ball $B(x,r)$, where $\mathcal{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$. To this end, take any $|w| = 1$. If $0 < s < r$,

$$|u(x+sw) - u(x)| = \left|\int_0^s \frac{d}{dt}u(x+tw)\,dt\right| = \left|\int_0^s Du(x+tw)\cdot w\,dt\right| \le \int_0^s |Du(x+tw)|\,dt.$$

Integrate with respect to w on $\partial B(0,1)$:

$$\begin{aligned}
\int_{\partial B(0,1)} |u(x+sw) - u(x)|\,dS(w) &\le \int_0^s \int_{\partial B(0,1)} |Du(x+tw)|\,dS(w)\,dt \\
&\overset{y=x+tw}{=} \int_0^s \int_{\partial B(x,t)} \frac{|Du(y)|}{t^{n-1}}\,dS(y)\,dt \\
&\overset{t=|x-y|}{=} \int_{B(x,s)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy = \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy.
\end{aligned}$$

By changing the variable $z = x + sw$ in the left hand side of the last display, we have

$$\int_{\partial B(x,s)} |u(z) - u(x)| dS(z) \le s^{n-1} \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy.$$

Next integrate with respect to s from 0 to r:

$$\int_{B(x,r)} |u(y) - u(x)|\,dy \le \frac{r^n}{n} \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy.$$

This completes the proof of (4.10).

Step II: Fix any $x \in \mathbb{R}^n$. By (4.10) and Hölder's inequality,

$$\begin{aligned}
|u(x)| &\le \frac{1}{\mathcal{L}^n(B(x,1))}\left(\int_{B(x,1)} |u(x) - u(y)|\,dy + \int_{B(x,1)} |u(y)|\,dy\right) \\
&\le C_1 \int_{B(x,1)} \frac{|Du(y)|}{|y-x|^{n-1}}\,dy + \mathcal{L}^n(B(x,1))^{-1/p}\|u\|_{L^p(B(x,1))} \\
&\le C_1 \left(\int_{\mathbb{R}^n} |Du|^p\,dy\right)^{1/p} \left(\int_{B(x,1)} |y-x|^{-\frac{(n-1)p}{p-1}}\,dy\right)^{\frac{p-1}{p}} + \mathcal{L}^n(B(x,1))^{-1/p}\|u\|_{L^p(\mathbb{R}^n)} \\
&\le C\|u\|_{W^{1,p}(\mathbb{R}^n)},
\end{aligned}$$

where $C = C(n,p)$ is a constant. The last estimate holds since $p > n$ implies $(n-1)\frac{p}{p-1} < n$, and

$$\int_{B(x,1)} |y-x|^{-\frac{(n-1)p}{p-1}}\, dy < \infty.$$

Step III: Choose any two points $x, y \in \mathbb{R}^n$, and write $r := |x-y|$. Let $W = B(x,r) \cap B(y,r)$. Then

$$|u(y) - u(x)| \le \frac{1}{\mathcal{L}^n(W)} \left(\int_W |u(x) - u(z)|\, dz + \int_W |u(y) - u(z)|\, dz \right).$$

By estimate (4.10), we have

$$\begin{aligned}
\frac{1}{\mathcal{L}^n(W)} \int_W |u(x) - u(z)|\, dz &\le \frac{\mathcal{L}^n(B(x,r))}{\mathcal{L}^n(W)} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} |u(x) - u(z)|\, dz \\
&\le \frac{C_1 \mathcal{L}^n(B(x,r))}{\mathcal{L}^n(W)} \int_{B(x,r)} \frac{|Du(z)|}{|z-x|^{n-1}}\, dz \\
&\le \frac{C_1 \mathcal{L}^n(B(x,r))}{\mathcal{L}^n(W)} \left(\int_{B(x,r)} |Du|^p\, dz \right)^{1/p} \left(\int_{B(x,r)} \frac{dz}{|z-x|^{\frac{(n-1)p}{p-1}}} \right)^{\frac{p-1}{p}} \\
&\le C_2 \left(r^{n - \frac{(n-1)p}{p-1}} \right)^{\frac{p-1}{p}} \|Du\|_{L^p(B(x,r))} \le C_2 r^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

where C_2 is a constant depending on n and p only. Similarly, we have

$$\frac{1}{\mathcal{L}^n(W)} \int_W |u(x) - u(z)|\, dz \le C_2 r^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}.$$

Consequently,

$$[u]_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} = \sup_{x \ne y} \frac{|u(y) - u(x)|}{|y-x|^{1-\frac{n}{p}}} \le C \|Du\|_{L^p(\mathbb{R}^n)}.$$

This inequality together with (4.2) completes the proof of (4.9). $\square$

Remark. We provide a slight variant of the estimate of $|u(x) - u(y)|$, where $|x-y| \le r$. Since both $B(x,r)$ and $B(y,r)$ are include in the ball $B(x,2r)$, we have

$$|u(y) - u(x)| \le C r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,2r))}$$

for all $u \in C^1(B(x,2r))$, $y \in B(x,r)$ and $n < p < \infty$.

Theorem 4.8 (Estimate for $W^{1,p}$ on Lipschitz domains, $n < p \le \infty$)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose that ∂U is Lipschitz. Assume $n < p \le \infty$ and $u \in W^{1,p}(U)$. Then u has a representative $u^* \in C^{0,\gamma}(\overline{U})$ for $\gamma = 1 - \frac{n}{p}$, with the estimate*

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \le C \|u\|_{W^{1,p}(U)}, \tag{4.11}$$

where the constant C depends on p, n and U only.

Proof. The case $p = \infty$ can be easily adapted from Theorem 3.2. Hence we assume that $n < p < \infty$.

Since ∂U is Lipschitz, by Theorem 3.1, there exists an extension $\overline{u} \in W^{1,p}(\mathbb{R}^n)$ such that $\overline{u} = u$ a.e. in U, $\overline{u}$ has compact support in $\mathbb{R}^n$, and

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C_1 \|u\|_{W^{1,p}(U)}, \tag{4.12}$$

where C_1 is a constant depending only on p, n and U. According to the Remark under Theorem 2.10, we can find a sequence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ converging to $\overline{u}$ in $W^{1,p}(\mathbb{R}^n)$. By Theorem 4.7, (u_m) is also a Cauchy sequence in $C^{1-\frac{n}{p}}(\mathbb{R}^n)$, which converges to some $u^* \in C^{1-\frac{n}{p}}(\mathbb{R}^n)$. Clearly, $u^* = u$ a.e. on U. Furthermore, letting $m \to \infty$ in Morrey's inequality for u_m yields $\|u^*\|_{C^{0,\gamma}(\overline{U})} \leq C\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)}$. Combining this with estimate (4.12) concludes the proof. $\square$

Remark. The preceding proof remains valid if we replace U by $\mathbb{R}^n$ and omit the extension step. We therefore restate our conclusion as follows: Assume $n < p \leq \infty$ and $u \in W^{1,p}(\mathbb{R}^n)$. Then u has a representative $u^* \in C^{0,\gamma}(\mathbb{R}^n)$ for $\gamma = 1 - \frac{n}{p}$, with the estimate

$$\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

where the constant C depends on p and n only.

Now we use the tool of Morrey's inequality to investigate more closely the connections between weak partial derivatives and partial derivatives.

Theorem 4.9 (Super-dimensional differentiability almost everywhere)**.** *Assume that $u \in W^{1,p}_{\mathrm{loc}}(U)$ for some $n < p \leq \infty$. Then u is differentiable a.e. in U, and its gradient equals its weak gradient a.e..*

Proof. We first assume that $n < p < \infty$. We identify u to its continuous version by applying Morrey's inequality on a countable set of balls covering U. For a.e. $x \in U$, by Lebesgue's differentiation theorem,

$$\frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} |Du(x) - Du(z)|^p \, dz \to 0 \quad as \quad r \to 0.$$

We then fix such a point x, and set $v(y) := u(y) - u(x) - Du(x) \cdot (y - x)$. Since the differentiation is a local problem, we choose $B(x, \delta) \subset U$. Then $v \in W^{1,p}(B(x, \delta))$.

By Proposition 1.8 and Theorem 2.10, the mollifications $v^\epsilon \in C^\infty(U)$ converges to v uniformly on $B(x, \delta)$ and in $W^{1,p}(B(x, \delta))$ as $\epsilon \to 0$. According to the remark under Theorem 4.7 and by approximation $\epsilon \to 0$, for each $y \in U$ with $r := |x - y| < \delta/2$, we have Morrey's estimate

$$|v(y) - v(x)| \leq Cr^{1-\frac{n}{p}} \left(\int_{B(x,2r)} |Dv(z)|^p \, dz \right)^{1/p}.$$

Consequently,

$$\begin{aligned} |u(y) - u(x) - Du(x) \cdot (y - x)| &\leq Cr^{1-\frac{n}{p}} \left(\int_{B(x,2r)} |Du(x) - Du(z)|^p \, dz \right)^{1/p} \\ &\leq C'r \left(\frac{1}{\mathcal{L}^n(B(x,2r))} \int_{B(x,2r)} |Du(x) - Du(z)|^p \, dz \right)^{1/p} = o(r) = o(|x - y|). \end{aligned}$$

Hence u is differentiable at x, and its gradient coincides its weak gradient at x. Finally, for the case $p = \infty$, just note that $W^{1,\infty}_{\mathrm{loc}}(U) \subset W^{1,p}_{\mathrm{loc}}(U)$ for all $1 \leq p < \infty$. $\square$

The following theorem is a direct consequence of Theorem 4.9.

Theorem 4.10 (Rademacher's theorem)**.** *Let u be locally Lipschitz continuous in U. Then u is differentiable almost everywhere in U.*

4.3 General Sobolev Inequalities

4.3.1 Sub-dimensional Case: $kp < n$

Theorem 4.11 (General Sobolev inequality, $kp < n$). *Let U be a bounded, open subset of $\mathbb{R}^n$, with a Lipschitz boundary. Assume $u \in W^{k,p}(U)$, and $kp < n$. Then $u \in L^q(U)$, where*

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}, \quad q = \frac{np}{n-kp}.$$

Furthermore, we have the estimate

$$\|u\|_{L^q(U)} \le C\|u\|_{W^{k,p}(U)},$$

where C is a constant depending only on k, p, n and U.

Proof. Step I: For every multi-index $|\alpha| \le k-1$, we have $D^\alpha u \in W^{1,p}(U)$. By Gagliardo-Nirenberg-Sobolev inequality [Theorem 4.4], there exists a constant $C = C(n,p,U) > 0$ depending only on n, p and U, such that

$$\|D^\alpha u\|_{L^{p^*}(U)} \le C\|D^\alpha u\|_{W^{1,p}(U)} \le C\|u\|_{W^{k,p}(U)}.$$

Hence $u \in W^{k-1,p^*}(U)$, where $p < p^* = \frac{np}{n-p} < n$. If $k = 2$, we are done by applying Gagliardo-Nirenberg-Sobolev inequality once again, where $q = p^{**} = \frac{np^*}{n-p^*} = \frac{np}{n-2p}$:

$$\|u\|_{L^{p^{**}}(U)} \le C(n,p^*,U)\|u\|_{W^{1,p^*}(U)} \le C(n,p^*,U)(1+n)C(n,p,U)\|u\|_{W^{2,p}(U)}.$$

Step II: We denote $p_2 = p^{**}$, $p_3 = p^{***}$, and so on. If $k \ge 3$, we can prove by induction such that

$$\begin{aligned}
&\|D^\alpha u\|_{L^{p^{**}}(U)} \le C_2\|D^\alpha u\|_{W^{1,p^*}(U)} \le C_2\|u\|_{W^{k-1,p^*}(U)}, \quad \forall |\alpha| \le k-2, \quad \textit{and} \ \ u \in W^{k-2,p^{**}}(U);\\
&\|D^\alpha u\|_{L^{p^{***}}(U)} \le C_3\|D^\alpha u\|_{W^{1,p^{**}}(U)} \le C_3\|u\|_{W^{k-2,p^{**}}(U)}, \quad \forall |\alpha| \le k-3, \quad \textit{and} \ \ u \in W^{k-3,p^{***}}(U);\\
&\cdots;\\
&\|D^\alpha u\|_{L^{p_{k-1}}(U)} \le C_{k-1}\|D^\alpha u\|_{W^{1,p_{k-2}}(U)} \le C_{k-1}\|u\|_{W^{2,p_{k-2}}(U)}, \quad \forall |\alpha| \le 1, \quad \textit{and}\ u \in W^{1,p_{k-1}}(U).
\end{aligned}$$

Hence $u \in W^{1,p_{k-1}}(U)$. Since $p < p_{k-1} < n$, again by Gagliardo-Nirenberg-Sobolev inequality, we have

$$\begin{aligned}
\|u\|_{L^{p_k}(U)} \le C_k\|u\|_{W^{1,p_{k-1}}(U)} &\le (1+n)C_kC_{k-1}\|u\|_{W^{2,p_{k-2}}(U)}\\
&\le (1+n)\left(1+n+n^2\right)C_kC_{k-1}C_{k-2}\|u\|_{W^{3,p_{k-3}}(U)} \le \cdots\\
&\le (1+n)\left(1+n+n^2\right)\cdots\left(1+n+n^2+\cdots+n^{k-1}\right)C_kC_{k-1}\cdots C_1\|u\|_{W^{k,p}(U)}.
\end{aligned}$$

where $C_1, \cdots, C_k$ are constants depending only on k, n, p and U. This completes the proof. □

Remark. In fact, we have the inclusions

$$W^{k,p}(U) \subset W^{k-1,p^*}(U) \subset W^{k-2,p^{**}}(U) \subset \cdots \subset W^{k-l,q}(U),$$

where $l \in \{0, 1, \cdots, k\}$ and $\frac{1}{q} = \frac{1}{p} - \frac{l}{n}$. Moreover, there exists a constant C depending only on n, p, q, l and U such that

$$\|u\|_{W^{k-l,q}(U)} \le C\|u\|_{W^{k,p}(U)}, \quad \forall u \in W^{k,p}(U).$$

This means that $W^{k,p}(U) \hookrightarrow W^{k-l,q}(U)$ is a continuous embedding, where $q = \frac{np}{n-lp} > p$.

4.3.2 Super-dimensional Case: $kp > n$

Theorem 4.12 (General Sobolev inequality, $kp > n$)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$, with a Lipschitz boundary. Assume $u \in W^{k,p}(U)$, and $kp > n$. Then u has a representative $u^* \in C^{k-\lfloor \frac{n}{p} \rfloor -1,\gamma}(\overline{U})$, where*

$$\gamma = \begin{cases} 1 + \left\lfloor \frac{n}{p} \right\rfloor - \frac{n}{p}, & \frac{n}{p} \notin \mathbb{N}, \\ \textit{any } \mu \in (0,1), & \frac{n}{p} \in \mathbb{N}. \end{cases}$$

Furthermore, we have the estimate

$$\|u^*\|_{C^{k-\lfloor \frac{n}{p} \rfloor -1,\gamma}(U)} \leq C\|u\|_{W^{k,p}(U)},$$

where C is a constant depending only on k, p, n, γ and U.

Proof. Case I: $n/p \notin \mathbb{N}$. The key idea is to apply general Sobolev inequality [Theorem 4.11] to the largest sub-dimensional case $lp < n$. Given $lp < n$, we have $u \in W^{k-l,r}(U)$, where $\frac{1}{r} = \frac{1}{p} - \frac{l}{n}$. Choose $l \in \mathbb{N}$ such that $l < \frac{n}{p} < l+1$, that is, $l = \lfloor n/p \rfloor$. Then $r = \frac{np}{n-pl} > n$ is super-dimensional, $k - l \geq 1$, and $D^\alpha u \in W^{1,r}(U)$ admits a representative $(D^\alpha u)^* \in C^{0,\gamma}(\overline{U})$ by Morrey's inequality for each $|\alpha| \leq k-l-1$, where $\gamma = 1 - n/r = 1 + \lfloor n/p \rfloor - n/p$. Furthermore, we have the estimate

$$\|D^\alpha u\|_{C^{0,\gamma}(\overline{U})} \leq C\|D^\alpha u\|_{W^{1,r}(U)} \leq C\|u\|_{W^{k-l,r}(U)},$$

where the constant C only depends on n, p and U. Consequently, $u^* \in C^{k-\lceil \frac{n}{p} \rceil,\gamma}(\overline{U})$, and

$$\|u\|_{C^{k-l-1,\gamma}(\overline{U})} = \sum_{|\alpha| \leq k-l-1} \|D^\alpha u\|_{C(\overline{U})} + \sum_{|\alpha|=k-l-1} [D^\alpha u]_{C^{0,\gamma}(\overline{U})} \leq C'\|u\|_{W^{k-l,r}(U)},$$

where the constant C' only depends on n, p, k and U.

Case II: $n/p \in \mathbb{N}$. To apply general Sobolev inequality [Theorem 4.11] to the sub-dimensional case, we choose $l = \frac{n}{p} - 1 \in \{0, 1, \cdots, k-2\}$. Then $u \in W^{k-l,q}(U)$ for $q = \frac{np}{n-lp} = n$. By Gagliardo-Nirenberg-Sobolev inequality, for all $r \in (n, \infty)$, we have

$$\|D^\alpha u\|_{L^r(U)} \leq C\|D^\alpha u\|_{W^{1,\frac{nr}{n+r}}(U)}, \quad \forall |\alpha| \leq k-l-1 = k - \frac{n}{p},$$

where C is a constant depending only on n, r and U, and $D^\alpha u \in L^r(U)$. By Morrey's inequality, we have $D^\alpha u \in C^{0,1-\frac{n}{r}}(\overline{U})$ for all $|\alpha| \leq k - \frac{n}{p} - 1$ and all $r \in (n, \infty)$. Consequently, $u \in C^{k-\frac{n}{p}-1,\gamma}(\overline{U})$ for all $0 < \gamma < 1$, and we have the estimate

$$\|u\|_{C^{k-\frac{n}{p}-1,\gamma}(\overline{U})} \leq C'\|u\|_{W^{k-l,n}(U)} \leq C''\|u\|_{W^{k,p}(U)},$$

where C' is a constant depending only on k, n, p, γ and U. $\square$

Remark. For the case $p = \infty$, we have the limit conclusion $W^{1,\infty}(U) = C^{0,1}(\overline{U})$ Theorem 3.2 for $k = 1$.

4.3.3 The Borderline Case: $kp = n$

Lemma 4.13. *Let U be a bounded, open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Let*

$$\begin{cases} p = \infty, & n = 1, \\ 1 \leq p < \infty, & n \geq 2. \end{cases}$$

Then $W^{1,n}(U) \subset L^p(U)$, and there exists a constant C, depending on n, p and U only, such that

$$\|u\|_{L^p(U)} \leq C\|u\|_{W^{1,n}(U)}, \quad \forall u \in W^{1,n}(U).$$

Proof. Case I: $n = 1$. If $v \in C_c^\infty(\mathbb{R})$, we have

$$|v(x)| \leq \int_{-\infty}^{\infty} |Du(y)| dy.$$

Hence $\|v\|_{L^\infty(\mathbb{R})} \leq \|Dv\|_{L^1(\mathbb{R})} \leq \|v\|_{W^{1,1}(\mathbb{R})}$. Then for each $u \in W^{1,1}(U)$, extend u to $\overline{u} \in W^{1,1}(\mathbb{R})$ with

$$\|\overline{u}\|_{W^{1,1}(\mathbb{R})} \leq c\|u\|_{W^{1,1}(U)},$$

where c is a constant depending on U only. By approximation $\overline{u}$ with $C_c^\infty(\mathbb{R})$, we have

$$\|u\|_{L^\infty(U)} \leq \|\overline{u}\|_{L^\infty(\mathbb{R})} \leq \|\overline{u}\|_{W^{1,1}(\mathbb{R})} \leq c\|u\|_{W^{1,1}(U)}.$$

Case II: $n \geq 2$. Take $n \leq q < \infty$, and set $\frac{1}{s} = \frac{1}{n} + \frac{1}{q}$. Then $1 \leq s < n$, and $q = \frac{ns}{n-s}$. Since U is bounded, by Hölder's inequality, we have

$$\|u\|_{W^{1,s}(U)} \leq (1+n)^{\frac{1}{n}-\frac{1}{s}} |U|^{\frac{n-s}{ns}} \|u\|_{W^{1,n}(U)}.$$

Since $q = s^* = \frac{ns}{n-s}$, by Theorem 4.4, we can find a constant $C(n, q, U)$ such that

$$\|u\|_{L^q(U)} \leq C(n, q, U)\|u\|_{W^{1,s}(U)} \leq C'(n, q, U)\|u\|_{W^{1,n}(U)}.$$

Since $|U| < \infty$, we have

$$\|u\|_{L^p(U)} \leq C''(n, q, U)\|u\|_{W^{1,n}(U)}$$

for all $1 \leq q \leq p$. Since q can be chosen arbitrarily large, the result follows. $\square$

Remark. The conclusion still holds if $n = 1$ and we replace U by $\mathbb{R}$, where constant C is 1.

Theorem 4.14. *Let U be a bounded, open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Assume $u \in W^{k,p}(U)$, and $kp = n$. Then $u \in L^q(U)$ for all $1 \leq q < \infty$, and we have the estimate*

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)},$$

where C is a constant depending only on k, p, q, n and U.

Proof. Similar to our proof of Theorem 4.12, we have the inclusions

$$W^{k,p}(U) \subset W^{k-1,p^*}(U) \subset W^{k-2,p^{**}}(U) \subset \cdots \subset W^{1,n}(U).$$

The last inclusion holds since $\frac{1}{n} = \frac{1}{p} - \frac{k-1}{n}$. The result then immediately follows from Lemma 4.13. $\square$

4.4 Compact Embeddings: Rellich-Kondrachov Compactness Theorem

The Gagliardo-Nirenberg-Sobolev inequality shows that $W^{1,p}(U)$ is continuously embedded into $L^{p^*}(U)$ in the sub-dimensional case $1 \le p < n$. Next, we are going to demonstrate that $W^{1,p}(U)$ is in fact compactly embedded into the space $L^q(U)$ when $1 \le q < p^*$.

Definition 4.15 (Compact Embedding)**.** *Let X and Y be Banach spaces, and $X \subset Y$. We say X is compactly embedded in Y, written $X \Subset Y$, if the identity operator*

$$\mathrm{Id} : X \to Y, \quad x \mapsto x$$

is continuous and compact, i.e.

(i) there exist some constant c such that $\|x\|_Y \le c\|x\|_X$ for all $x \in X$, and

(ii) each bounded subset of X is precompact in Y.

Remark. Since compactness coincides sequential compactness in metrizable spaces, (ii) equals that *every bounded sequence of points of X has a subsequence converging in Y.*

Theorem 4.16 (Rellich-Kondrachov Compactness Theorem)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Assume $1 \le p < n$. Then*

$$W^{1,p}(U) \Subset L^q(U)$$

for all $1 \le q < p^*$.

Proof. Step I: Assume that $1 \le q < p^*$. Using Gagliardo-Nirenberg-Sobolev inequality [Theorem 4.4], we obtain the continuous embedding $W^{1,p}(U) \hookrightarrow L^q(U)$, with

$$\|u\|_{L^q(U)} \le C\|u\|_{W^{1,p}(U)}$$

for all $u \in W^{1,p}(U)$, where the constant C depending only on n, p, q and U. Then it remains to show that any bounded sequence (u_m) in $W^{1,p}(U)$ has a subsequence (u_{m_l}) converging in $L^q(U)$.

Step II: By extension theorem [3.1], we may assume that every u_m is in $W^{1,p}(\mathbb{R}^n)$ and supported on a precompact set $V \Subset U$, and $\sup_{m\in\mathbb{N}} \|u_m\|_{W^{1,p}(\mathbb{R}^n)} < \infty$.

Then we study the mollifiers $u_m^\epsilon = \eta_\epsilon * u_m$, and we may assume that the support of u_m^ϵ is in V for all $m \in \mathbb{N}$. We first prove that

$$\lim_{\epsilon\to 0} \sup_{m\in\mathbb{N}} \|u_m^\epsilon - u_m\|_{L^q(V)} = 0. \tag{4.13}$$

If u_m is smooth, we have

$$\begin{aligned} u_m^\epsilon(x) - u_m(x) &= \frac{1}{\epsilon^n} \int_{B(x,\epsilon)} \eta\left(\frac{x-z}{\epsilon}\right)(u_m(z) - u_m(x))\, dz \\ &= \int_{B(0,1)} \eta(y)\left(u_m(x-\epsilon y) - u_m(x)\right)\, dy \\ &= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt}\left(u_m(x-\epsilon t y)\right) dt\, dy \\ &= -\epsilon \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x - \epsilon t y)\cdot y\, dt\, dy. \end{aligned}$$

Consequently,

$$\begin{aligned}\|u_m^\epsilon - u_m\|_{L^1(V)} &= \int_V |u_m^\epsilon(x) - u_m(x)|\,dx \\ &\leq \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |Du_m(x - \epsilon t y)|\,dx\,dt\,dy \\ &\leq \epsilon \int_V |Du_m(z)|\,dz = \epsilon \|Du_m\|_{L^1(V)}.\end{aligned}$$

By approximation, this estimate also holds for $u_m \in W^{1,p}(U)$. Since V is bounded, we have

$$\|u_m^\epsilon - u_m\|_{L^1(V)} \leq \epsilon \|Du_m\|_{L^1(V)} \leq \epsilon C \|Du_m\|_{L^p(V)}$$

Note that u_m is bounded in $W^{1,p}(\mathbb{R}^n)$. Then the estimate (4.13) holds when $q = 1$. If $1 < q < p^*$, let $0 < \theta < 1$ be such that

$$\frac{\theta}{1} + \frac{1-\theta}{p^*} = \frac{1}{q}.$$

Akin to the interpolation statement employed in the proof of Theorem 4.3, we have

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta \|u_m^\epsilon - u_m\|_{L^{p^*}(V)}^{1-\theta}.$$

While the first term converges to 0, the estimate (4.13) follows from the boundedness of the second term, by Gagliardo-Nirenberg-Sobolev inequality.

Step III: Fix any $\epsilon > 0$. We verify that $(u_m^\epsilon)_{m=1}^\infty$ satisfies Arzelà-Ascoli criterion: We claim that the sequence $(u_m^\epsilon)_{m=1}^\infty$ is uniformly bounded and uniformly equicontinuous, i.e.

(i) $\sup_{m\in\mathbb{N}} \|u_m^\epsilon\|_\infty < \infty$, and

(ii) for all $\eta > 0$, there exists $\delta > 0$ such that for all $m \in \mathbb{N}$ and all $|x - y| < \delta$, $|u_m^\epsilon(x) - u_m^\epsilon(y)| < \eta$.

To prove the first assertion, note that

$$\begin{aligned}|u_m^\epsilon(x)| &\leq \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u_m(y)|\,dy \leq \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \|u_m\|_{L^1(V)} \\ &\leq \frac{1}{\epsilon^n}\|u_m\|_{L^1(V)} \leq \frac{|V|^{1/p}}{\epsilon^n}\|u_m\|_{L^p(V)}.\end{aligned}$$

Since $(u_m)_{m=1}^\infty$ is bounded in $W^{1,p}(U)$, the first assertion holds. For the second assertion,

$$\begin{aligned}|Du_m^\epsilon(x)| &\leq \int_{B(x,\epsilon)} |D\eta_\epsilon(x-y)|\,|u_m(y)|\,dy \\ &\leq \|D\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)} \|u_m\|_{L^1(V)} \leq \frac{|V|^{1/p}}{\epsilon^{1+n}}\|Du_m\|_{L^p(V)}.\end{aligned}$$

Consequently, we have $\sup_{m\in\mathbb{N}} \|Du_m^\epsilon\|_{L^\infty(V)} < \frac{C}{\epsilon^{1+n}}$ for some constant C depending only on n, p and V, and the second assertion holds. By Arzelà-Ascoli theorem, the sequence $(u_m^\epsilon)_{m=1}^\infty$ has a subsequence $(u_m^j)_{j=1}^\infty$ that converges uniformly on V, and

$$\limsup_{j,k\to\infty} \left\|u_{m_j}^\epsilon - u_{m_k}^\epsilon\right\|_{L^q(V)} = 0. \tag{4.14}$$

Step IV: Fix any $\delta > 0$. By estimate (4.13), we choose $\epsilon > 0$ to so small that

$$\sup_{m\in\mathbb{N}} \|u_m^\epsilon - u_m\|_{L^q(V)} < \frac{\delta}{2}.$$

Combining this bound with (4.14), we obtain

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \le \limsup_{j,k\to\infty} \Big(\|u_{m_j} - u^\epsilon_{m_j}\|_{L^q(V)} + \|u^\epsilon_{m_j} - u^\epsilon_{m_k}\|_{L^q(V)} + \|u^\epsilon_{m_k} - u_{m_k}\|_{L^q(V)} \Big) < \delta,$$

where $(m_j)_{j=1}^\infty$ is the subsequence chosen in Step III, which depends on ϵ. Next, we employ our conclusion on $\delta = 1, \frac{1}{2}, \frac{1}{3}, \cdots$ and use Cantor's standard diagonal statement to extract a subsequence $(m_l)_{l=1}^\infty$ satisfying

$$\limsup_{l,k\to\infty} \|u_{m_l} - u_{m_k}\|_{L^q(V)} = 0.$$

By completeness of the space $L^q(V)$, the result follows. $\square$

For $n < p \le \infty$, we have a similar conclusion following from Morrey's inequality and Arzelà-Ascoli theorem.

Theorem 4.17. *Let U be a bounded, open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Assume $n < p \le \infty$. Then*

$$W^{1,p}(U) \Subset L^q(U)$$

for all $1 \le q \le \infty$.

Proof. By Arzelà-Ascoli theorem, we know that $C^{0,\gamma}(\overline{U}) \Subset C(\overline{U})$ for all $0 < \gamma \le 1$. Let $(u_m)_{m=1}^\infty$ be a bounded sequence in $W^{1,p}(U)$. By Morrey's inequality, (u_m), identified to its Hölder continuous version, is also bounded in $C^{0,1-\frac{n}{p}}(\overline{U})$. Hence there is a subsequence $(u_{m_k})_{k=1}^\infty$ that converges uniformly on U. Since U is bounded, $(u_{m_k})_{k=1}^\infty$ converges in $L^q(U)$ for all $1 \le q \le \infty$, and the result follows. $\square$

For the borderline case $p = n$, we have the following limiting conclusion.

Theorem 4.18. *Let U be a bounded, open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Then*

$$W^{1,n}(U) \Subset L^q(U)$$

for all $1 \le q < \infty$.

Proof. According to Lemma 4.13, the embedding $W^{1,n}(U) \hookrightarrow L^q(U)$ is continuous for all $1 \le q < \infty$. Now take any bounded sequence $(u_m)_{m=1}^\infty$ in $W^{1,n}(U)$. Then for every $1 \le p < n$, since U is bounded, $(u_m)_{m=1}^\infty$ is also bounded in $W^{1,p}(U)$. By Rellich-Kondrachov compactness theorem, for any $1 \le q < p^*$, there exists a subsequence $(u_{m_k})_{k=1}^\infty$ that converges in $L^q(U)$. Since $p^* = \frac{np}{n-p} \to \infty$ as $p \to n$, the result follows. $\square$

Remark. Summarizing Theorems 4.16, 4.17 and 4.18, we have

$$W^{1,p}(U) \Subset L^p(U)$$

for all $1 \le p \le \infty$. Moreover, we have

$$W^{1,p}_0(U) \Subset L^p(U)$$

for all $1 \le p \le \infty$, even if ∂U is not Lipschitz.

4.5 Poincaré's Inequality

Given a bounded, open set $U \subset \mathbb{R}^n$ and a function $u : U \to \mathbb{R}$, we define the *mean value of u in U* as

$$(u)_U = \frac{1}{|U|} \int_U u(x)\, dx.$$

Theorem 4.19 (Poincaré's inequality)**.** *Let U be a bounded, open and connected subset of $\mathbb{R}^n$, with a Lipschitz boundary. Assume $1 \leq p \leq \infty$. Then there exists a constant C, depending only on n, p and U, such that*

$$\|u - (u)_U\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$$

for each $u \in W^{1,p}(U)$.