Selected Topics in Complex Analysis

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1 Preliminaries

1.1 Topology of the Complex Plane

Before we proceed, we introduce some useful terminology in topology.

Definition 1.1. *We use $\mathbb{C}$ to denote the complex plane $\{x + \mathrm{i}y : x, y \in \mathbb{R}\}$.*

(i) Let $z_0 \in \mathbb{C}$, and $r > 0$. We use $B(z_0, r)$ to denote the open disc of radius r centered at z_0:

$$B(z_0, r) = \{z \in \mathbb{R} : |z - z_0| < \delta\}.$$

Any set $U \subset \mathbb{C}$ that contains an open disc centered at z_0 is called a neighborhood of z_0.

(ii) A subset $U \subset \mathbb{C}$ is said to be open, if for all $z \in U$, there exists $\delta > 0$ such that $D(z, \delta) \subset U$. A subset $D \subset \mathbb{C}$ is said to be closed if its complement $D^c = \mathbb{C} \backslash D$ is open. Particularly, the complex plane $\mathbb{C}$ and the empty set $\emptyset$ are both open and closed.

(iii) A point $z \in \mathbb{C}$ is said to be a limit point (or an accumulation point) of a set $E \subset \mathbb{C}$, if

$$B(z, \delta) \cap E \backslash \{z\} \neq \emptyset$$

for all $\delta > 0$. In other words, z is a limit point of E if each neighborhood of z contains a point of E that is not z. Note that z is not required to be a point of E.

(iv) A point $z \in \mathbb{C}$ is said to be an interior point of a set $E \subset \mathbb{C}$, if there exists $\delta > 0$ such that $B(z, \delta) \subset E$. In other words, z is an interior point of E if E is a neighborhood of z.

(v) The closure of a set $E \subset \mathbb{C}$ is the set of all limit point of E, denoted by $\overline{E}$:

$$\overline{E} = \{z \in E : \forall \delta > 0,\ B(z, \delta) \cap E \backslash \{z\} \neq \emptyset\}.$$

The interior of a set $E \subset \mathbb{C}$ is the set of all interior point of E, denoted by $\mathring{E}$:

$$\mathring{E} = \{z \in E : \exists \delta > 0,\ B(z, \delta) \subset E\}.$$

(vi) The boundary of a set $E \subset \mathbb{C}$ is $\partial E = \overline{E} \cap \overline{E^c}$.

(vii) A set $K \subset \mathbb{C}$ is said to be compact, if every open cover of K has a finite subcover. In other words, for any collection of open sets $\{U_\alpha\}_{\alpha \in J}$ with $K \subset \bigcup_{\alpha \in J} U_\alpha$, there exist finitely many indices $\alpha_1, \cdots, \alpha_n \in J$ such that $K \subset \bigcup_{j=1}^n U_{\alpha_j}$.

(viii) A set $C \subset \mathbb{C}$ is said to be connected, if there do not exist two disjoint nonempty open sets A and B such that $C \subset A \cup B$, and neither A nor B contains C.

(ix) Given $z_1, z_2 \in \mathbb{C}$, we use $[z_1, z_2]$ to denote the line segments on $\mathbb{C}$ with endpoints z_1 and z_2. A polygonal line is a finite union of line segments of the form $[z_0, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{n-1}, z_n]$.

(x) If any two points of a set $E \subset \mathbb{C}$ can be connected by a polygonal line contained in E, then E is said to be polygonally connected.

Since the complex plane $\mathbb{C}$ is homeomorphic to the Euclidean space $\mathbb{R}^2$, some standard results of the topology of Euclidean spaces follows.

Proposition 1.2. *Let E be a subset of $\mathbb{C}$.*

(i) E is closed if and only if E contains all its limit point.

(ii) The closure $\overline{E}$ is closed, and it is the minimal closed set that contains E. In other word, for any closed set $D \supset E$, we have $D \supset \overline{E}$.

(iii) The interior $\mathring{E}$ is open, and it is the maximal open set that is contained E. In other word, for any open set $U \subset E$, we have $U \subset \mathring{E}$.

(iv) E *is compact if and only if it is closed and bounded in* $\mathbb{C}$.

(v) If E *is polygonally connected, then it is connected.*

(vi) If E *is a connected open set, then* E *is polygonally connected.*

Proof. Here we only provide a proof of (vi), since it is related to our later discussion. Take $z_0 \in E$. Let A be the set of all points of E that can be polygonally connected to z_0 in E, and let B be the set of all points of E that cannot be polygonally connected to z_0 in E.

We claim that A is an open set. For every $z \in A$, we can choose an open disc centered at z and contained in E. Since z is polygonally connected to z_0, so is every point in the open disc by joining a line segment connecting the point and the center z. Similarly, B is also an open set.

Finally, we point out that, since A contains z_0, and the connected set E is a disjoint union of open sets A and B, the set B must be empty. Therefore, every point in E is polygonally connected to z_0, and every two point in E is polygonally connected by joining two polygonal lines intersecting at z_0. $\square$

Remark. According to this proof, we can even make our statement stronger. For any two points in a connected open domain E, they can be connected by a polygonal line, of which any two successive vertices represent the endpoints of a horizontal or vertical segment.

To end this section, we see a useful application of polygonal connectedness. A complex-valued function $f : \mathbb{C} \to \mathbb{C}$ can be split into real and imaginary parts:

$$f(z) = f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y),$$

where u, v are both real-valued functions on $\mathbb{R}^2$.

Proposition 1.3. *If the function* $u(x, y)$ *has partial derivatives* u_x *and* u_y *that vanish at every point of a connected open domain* U*, then* u *is a constant in* U*.*

Proof. Let $(x, y), (\widetilde{x}, \widetilde{y}) \in U$, so they can be connected by a polygonal path that is contained in U. Any two successive vertices of the path represent the endpoints of a horizontal or vertical segment. Hence, by the Lagrange mean-value theorem for one real variable, the change in u between these vertices is given by the value of a partial derivative of u somewhere between the endpoints times the difference in the non-identical coordinates of the endpoints. Since, however, u_x and u_y vanish identically in U, the change in u is 0 between each pair of successive vertices. Therefore $u(x, y) = u(\widetilde{x}, \widetilde{y})$. $\square$

In complex analysis, we are often interested in the sets that have no disjoint parts and no holes that go completely through it. These sets are called simply connected sets. A formal definition is given below.

Definition 1.4 (Simply connected domains)**.** *Let* $U \subset \mathbb{C}$ *be a connected open set. If for each point* $z_0 \in U^c$ *and each* $\epsilon > 0$*, there exists a continuous mapping* $\gamma : [0, \infty) \to \mathbb{C}$ *such that (i)* $d(\gamma(t), U^c) < \epsilon$ *for all* $t \geq 0$*, (ii)* $\gamma(0) = z_0$*, and (iii)* $\lim_{t\to\infty} |\gamma(t)| = \infty$*, then* E *is said to be simply connected.*

Remark. Let R be a rectangle in $\mathbb{C}$. *If* U *is a simply connected open set, and* $\partial R \subset U$*, then* $R \subset U$*.* To verify this, assume $z_0 \in R$ is not in U. Then any path γ from z_0 to ∞ intersects ∂R at some $\gamma(t) \in \partial R$. Since $\gamma(t) \in U$, there exists $B(\gamma(t), \delta) \subset U$, and $d\gamma(t)z, U^c) \geq \delta$. This contradicts (i)!

1.2 Differentiability and Holomorphicity

Similar to the derivative for real-valued functions, we can define the derivative for complex-valued functions $f : \mathbb{C} \to \mathbb{C}$, which is given by the limit

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}. \tag{1.1}$$

Note that in this definition, the quantity h is complex. If the limit (1.1) exists at some point $z \in \mathbb{C}$, we say that f is *(complex) differentiable* at z, and the limit is called the *derivative* of f at z.

Theorem 1.5 (Cauchy-Riemann equation). *Let $f : \mathbb{C} \to \mathbb{C}$, and let $f(x+\mathrm{i}y) = u(x,y)+\mathrm{i}v(x,y)$, where u and v are real and imaginary parts of f, respectively. If f is differentiable at $z \in \mathbb{C}$, then the partial derivatives u_x, u_y, v_x, v_y exists at z, and they satisfy the Cauchy-Riemann equation:*

$$\begin{cases} u_x - v_y = 0, \\ v_x + u_y = 0. \end{cases} \tag{1.2}$$

Proof. Fix any $z = x + \mathrm{i}y \in \mathbb{C}$. Since f is differentiable at z, the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. We consider the limit along the real and imaginary directions:

$$\begin{aligned} \lim_{\mathbb{R}\ni h\to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{\mathbb{R}\ni h\to 0} \frac{u(x+h,y) - u(x,y) + \mathrm{i}\,(v(x+h,y) - v(x,y))}{h} = u_x + \mathrm{i}v_x, \\ \lim_{\mathbb{R}\ni h\to 0} \frac{f(z+\mathrm{i}h) - f(z)}{\mathrm{i}h} &= \lim_{\mathbb{R}\ni h\to 0} \frac{u(x,y+h) - u(x,y) + \mathrm{i}\,(v(x,y+h) - v(x,y))}{\mathrm{i}h} = v_y - \mathrm{i}u_y. \end{aligned}$$

The two limits should be equal, and the result follows. $\square$

Remark. The Cauchy-Riemann equation is not a sufficient condition for differentiability. Here consider

$$f(z) = f(x,y) = \begin{cases} \dfrac{xy(x+\mathrm{i}y)}{x^2+y^2}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Since $f = 0$ on coordinate axes, we have $u_x(0,0) + \mathrm{i}v_x(0,0) = v_y(0,0) - \mathrm{i}u_y(0,0) = 0$, which is the Cauchy-Riemann equation at $z = 0$. However, on the line $y = \alpha x$, we have

$$\lim_{\mathbb{R}\ni x\to 0} \frac{f(x+\mathrm{i}\alpha x) - f(0)}{x + \mathrm{i}\alpha x} = \frac{\alpha}{1+\alpha^2},$$

which depends on α. Hence the limit at $z = 0$ does not exist. In the next theorem, we provide a practical sufficient condition for differentiability.

Theorem 1.6. *Let $f : \mathbb{C} \to \mathbb{C}$, and let $f(x+\mathrm{i}y) = u(x,y) + \mathrm{i}v(x,y)$, where u and v are real and imaginary parts of f, respectively. If the partial derivatives u_x, u_y, v_x, v_y exist in a neighborhood of z and are continuous at z, and they satisfy the Cauchy-Riemann equation (1.2).*

Proof. Let $z = x + \mathrm{i}y \in \mathbb{C}$ and $h = \xi + \mathrm{i}\eta \in \mathbb{C}$. We want to show that

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = u_x + \mathrm{i}v_x$$

We decompose the difference of f as follows:

$$f(z+h)-f(z)=u(x+\xi,y+\eta)-u(x,y)+\mathrm{i}v(x+\xi,y+\eta)-\mathrm{i}v(x,y)$$

By mean-value theorem, there exist $\theta_1,\theta_2,\theta_3,\theta_4\in(0,1)$ such that

$$\begin{aligned}
u(x+\xi,y+\eta)-u(x,y)&=u(x+\xi,y+\eta)-u(x+\xi,y)+u(x+\xi,y)-u(x,y)\\
&=\eta u_y(x+\xi,y+\theta_1\eta)+\xi u_x(x+\theta_2\xi,y),\\
v(x+\xi,y+\eta)-v(x,y)&=v(x+\xi,y+\eta)-v(x+\xi,y)+v(x+\xi,y)-v(x,y)\\
&=\eta v_y(x+\xi,y+\theta_3\eta)+\xi v_x(x+\theta_4\xi,y).
\end{aligned}$$

By continuity of partial derivatives,

$$\begin{aligned}
\frac{f(z+h)-f(z)}{h}&=\frac{\eta}{\xi+\mathrm{i}\eta}\left[u_y(x+\xi,y+\theta_1\eta)+\mathrm{i}v_y(x+\xi,y+\theta_3\eta)\right]+\frac{\xi}{\xi+\mathrm{i}\eta}\left[u_x(x+\theta_2\xi,y)+\mathrm{i}v_x(x+\theta_4\xi,y)\right]\\
&=\frac{\eta}{\xi+\mathrm{i}\eta}\left[u_y(x,y)+\mathrm{i}v_y(x,y)+\epsilon_1\right]+\frac{\xi}{\xi+\mathrm{i}\eta}\left[u_x(x,y)+\mathrm{i}v_x(x,y)+\epsilon_2\right],
\end{aligned}$$

where ϵ_1 and ϵ_2 are quantities converging to 0 as $h\to 0$. By Cauchy-Riemann equation at $z=x+\mathrm{i}y$,

$$\begin{aligned}
\frac{f(z+h)-f(z)}{h}&=\frac{\mathrm{i}\eta}{\xi+\mathrm{i}\eta}\left[\mathrm{i}v_x(x,y)+u_x(x,y)-\mathrm{i}\epsilon_1\right]+\frac{\xi}{\xi+\mathrm{i}\eta}\left[u_x(x,y)+\mathrm{i}v_x(x,y)+\epsilon_2\right]\\
&=u_x(x,y)+\mathrm{i}v_x(x,y)+\frac{\eta\epsilon_1+\xi\epsilon_2}{\xi+\mathrm{i}\eta},
\end{aligned}$$

Since

$$\left|\frac{\eta\epsilon_1+\xi\epsilon_2}{\xi+\mathrm{i}\eta}\right|\le|\epsilon_1|+|\epsilon_2|\to 0$$

as $h\to 0$, the result follows. ☐

To end this section, we introduce some useful definitions.

Definition 1.7. *Let $z_0\in\mathbb{C}$ be a point, and let $U\subset\mathbb{C}$ be an open set. Let $f:\mathbb{C}\to\mathbb{C}$ be a complex function.*

(i) If f is differentiable in a neighborhood of a point z_0, then f is said to be <u>holomorphic</u> at z_0;

(ii) If f is everywhere differentiable in U, then f is said to be <u>holomorphic</u> in U;

(iii) If f is holomorphic in $\mathbb{C}$, then f is said to be <u>entire</u>.

Proposition 1.8. *Let $f:U\to\mathbb{C}$ be a holomorphic function in an open connected region U.*

(i) Let $f=u+\mathrm{i}v$. If u is a constant in D, so is f.

(ii) If $|f|$ is a constant in D, so is f.

Proof. (i) This result follows from Cauchy-Riemann equation and the polygonal connectedness of U.

(ii) Let $f=u+\mathrm{i}v$. It suffices to consider the case $|f|=\sqrt{u^2+v^2}\equiv c>0$. By taking derivatives on both sides of $u^2+v^2\equiv c$ in U, we obtain

$$uu_x+vv_x=0,\quad uu_y+vv_y=0.$$

By applying the Cauchy-Riemann equation twice, we have

$$0=u(uu_x+vv_x)=u^2u_x+uvv_x=u^2u_x-vuu_y=u^2u_x+v^2v_y=(u^2+v^2)u_x.$$

Hence $u_x\equiv 0$ in U. Similarly, we can show $u_y\equiv 0$ in U. Then the result follows from (i). ☐

1.3 Complex Power Series and Analyticity

Analytic Polynomial. A polynomial $P(x, y)$ is said to be an *analytic polynomial*, if there exists complex numbers $c_0, c_1, \cdots, c_n \in \mathbb{C}$ such that

$$P(x, y) = c_0 + c_1(x + iy) + c_2(x + iy)^2 + \cdots + c_n(x + iy)^n.$$

We then say that P is a polynomial in the complex variable $z \in \mathbb{C}$, and write

$$P(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n.$$

It is direct to verify that a polynomial $P(x, y) = u(x, y) + iv(x, y)$ is analytic if and only if it satisfies the Cauchy-Riemann equation (1.2).

A power series of z is given by an "infinite polynomial".

Definition 1.9 (Complex power series)**.** *A power series in z is an infinite series of the form*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n,$$

where $c_0, c_1, \cdots \in \mathbb{C}$.

Naturally, we are interested in the domain where a power series converges.

Theorem 1.10. *Suppose $\limsup_{n\to\infty} |c_n|^{1/n} = L$.*

(i) If $L = 0$, the power series $\sum_{n=0}^{\infty} c_n z^n$ converges for all $z \in \mathbb{C}$. In this case, $R = \infty$ is called the radius of convergence of the power series.

(ii) If $L = \infty$, the power series $\sum_{n=0}^{\infty} c_n z^n$ converges if and only if $z = 0$. In this case, $R = 0$ is called the radius of convergence of the power series.

(iii) If $0 < L < \infty$, set $R = \frac{1}{L}$. Then the power series $\sum_{n=0}^{\infty} c_k z^k$ converges for all $|z| < R$, and diverges for all $|z| > R$. In this case, R is called the radius of convergence of the power series.

(iv) When the radius of convergence $R > 0$, the power series $\sum_{n=0}^{\infty} c_n z^n$ converges uniformly on the closed disc $\overline{B(0, r)}$ for each $0 < r < R$.

Proof. (i) When $L = 0$, we have $\limsup_{n\to\infty} |c_n|^{1/n}|z| = 0$ for all $z \in \mathbb{C}$. Hence for each $z \in \mathbb{C}$, there exists some N such that $|c_n z^n| < 2^{-n}$ for all $n \geq N$. Therefore the series converges by Cauchy's criterion.

(ii) When $L = \infty$ and $z \neq 0$, we have $|c_n|^{1/n} > 1/|z|$ for infinitely many $n \in \mathbb{N}$. Then the terms of the series do not approach 0 as $n \to \infty$, and the series diverges.

(iii) Assume $0 < L < \infty$, and $R = 1/L$. If $|z| < R$, we set $|z| = R(1 - 2\epsilon)$ for some $\epsilon > 0$. Then

$$\limsup_{n\to\infty} |c_n|^{1/n}|z| < 1 - \epsilon,$$

and we have $|c_n z^n| < (1 - \epsilon)^n,\ n \geq N$ beginning from some N. On the other hand, when $|z| > R$, we have $\limsup_{n\to\infty} |c_n|^{1/n}|z| > 1$, and there exists infinitely many terms greater than 1 in the series.

(iv) The case $R = \infty$ is clear if we can prove the case $R < \infty$. When $R < \infty$, for all $|z| \leq r < R$, we have

$$\limsup_{n\to\infty} |c_n|^{1/n}|z| \leq \frac{r}{R} < 1 - \frac{R - r}{2R} := 1 - \epsilon.$$

Hence we have $|c_n z^n| < (1 - \epsilon)^n,\ n \geq N$ beginning from some N. For $k \geq N$, the remainder $\sum_{n=k+1}^{\infty} c_n z^n$ is uniformly controlled by $(1 - \epsilon)^k/\epsilon$ on $\overline{B(0, r)}$. Hence the convergence is uniform. $\square$

We can write the derivative of a power series into termwise differentiation.

Theorem 1.11. *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be a power series with the radius $R > 0$ of convergence. Then f is holomorphic in $B(0, R)$, and*

$$f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1}, \quad |z| < R.$$

Furthermore, the above series has the same radius of convergence as f.

The following corollary is easily obtained by applying Theorem 1.11 recursively.

Corollary 1.12. *Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be a power series with the radius $R > 0$ of convergence. Then f is infinitely differentiable in $B(0, R)$, and*

$$c_n = \frac{f^{(n)}(0)}{n!}, \quad \forall n \in \mathbb{N}_0.$$

We now introduce the definition of analytic functions in the complex plane.

Definition 1.13 (Analyticity)**.** *Let $f : U \to \mathbb{C}$ be a complex function on an open set U, and let $z_0 \in U$. The function f is said to be analytic at point z_0, if there exists complex coefficients $c_0, c_1, \cdots \in \mathbb{C}$ such that*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

in a neighborhood of z_0. In other words, f is analytic at z_0 if it equals a power series near z_0.

Remark. By definition, if f is analytic at z_0, it is also holomorphic at z_0. In later discussion, we will show that the two properties are equivalent.

Theorem 1.14 (Uniqueness theorem for power series)**.** *We have the following result:*

(i) If a power series equals zero at all the points of a set with a point at the origin, the power series is identically zero.

(ii) If two power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ converge and agree on a set of points with an accumulation point at the origin, then $a_n = b_n$ for all n.

2 Cauchy's Integral Theorem and its Consequences

2.1 Complex Line Integral

In this section, we deal with the integral on the complex plane $\mathbb{C}$.

Definition 2.1 (Smooth curves). *Let $x = x(t)$ and $y = y(t)$ be continuous real-valued functions on $[a, b]$. If we use these equations as the real and imaginary parts in $z = x + \mathrm{i}y$, we can parameterize the points z on a curve C by means of a complex-valued function of a real-variable t:*

$$C = \{z(t) = x(t) + \mathrm{i}y(t),\ a \le t \le b\}.$$

(i) The curve C parameterized by $z(t)$ is said to be differentiable, if both x and y are continuously differentiable on $[a, b]$. We define the derivative of $z(t)$ by

$$z'(t) = x'(t) + \mathrm{i}y'(t), \quad a \le t \le b.$$

Furthermore, if $z'(t) \neq 0$ for all $a < t < b$, then C is said to be smooth.

(ii) The a curve C is said to be piecewise smooth, if C can be obtained by joining finitely many smooth curves. Formally, if C is piecewise smooth, we can find a partition $a = t_0 < t_1 < \cdots < t_n = b$ such that $z(t)$ is continuous on $[a, b]$, is continuously differentiable on each sub-interval $[t_{j-1}, t_j]$, and $z'(t) \neq 0$ except at finitely many points.

(iii) The curve C is said to be simple if z injective, i.e. $s \neq t$ implies $z(s) \neq z(t)$.

(iv) The curve C is said to be closed if $z(a) = z(b)$.

(v) A simple closed curve C is said to be a contour (or a Jordan curve). By Jordan curve theorem, a contour always divide the complex plane $\mathbb{C}$ into two connected open components. One of these components is bounded and simply connected, called the interior of C, and the other component is unbounded, called the exterior of C. Furthermore, the contour C is the boundary of each component.

Definition 2.2 (Complex integral). *Let $f(t) = u(t) + \mathrm{i}v(t)$ be a continuous complex valued function of the real variable $a \le t \le b$, where u and v are both real-valued. Define*

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + \mathrm{i}\int_a^b v(t)\,dt.$$

Let C be a smooth curve given by $z(t)$, $a \le t \le b$, and suppose $f : \mathbb{C} \to \mathbb{C}$ is continuous at all the points $z(t)$. Then, the integral of f along C (or round C, if C is a contour) is

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

Remark. By this definition, the linearity of complex line integral follows from the real case. Furthermore, the value of the integral is independent of the particular parameterization. To see this, we consider two particular smooth curves

$$C_1 : z(t),\ a \le t \le b \quad \text{and} \quad C_2 : \omega(t),\ c \le t \le d.$$

We assume that there exists a bijective C^1-mapping $\lambda : [a, b] \to [c, d]$ such that $c = \lambda(a)$, $d = \lambda(b)$, $\lambda'(t) > 0$ for all $t \in [a, b]$, and $z(t) = \omega(\lambda(t))$. Then C_1 and C_2 are *smoothly equivalent* in the complex plane $\mathbb{C}$, and

$$\int_{C_1} f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt = \int_a^b f(\omega(\lambda(t)))\omega'(\lambda(t))\lambda'(t)\,dt \overset{s=\lambda(t)}{=} \int_c^d f(\omega(s))\omega'(s)\,ds = \int_{C_2} f(z)\,dz.$$

Change the direction. The line integral depends not only on the set of points of the curve, but also the direction of the curve. Given a curve C parameterized by $z(t),\ a \le t \le b$, the curve $-C$ along the opposite direction is given by $z(b+a-t),\ a \le t \le b$. Then

$$\int_{-C} f(z)\,dz = \int_a^b f(z(b+a-t))\,(-z'(b+a-t))\,dt = -\int_a^b f(z(s))z'(s)\,ds = -\int_C f(z)\,dz,$$

where we change the variable $s = b+a-t$ in the second identity.

Estimation. Now we aim to derive a bound for the complex line integral. Before we proceed, we point out that a smooth curve $C = \{z(t) : a \le t \le b\}$ is *rectifiable*, and its length is given by

$$L = \int_C |dz| = \int_a^b |z'(t)|\,dt = \sup_{a=t_0<t_1<\cdots<t_n=b} \sum_{j=1}^n |z(t_i) - z(t_{i-1})|.$$

We will use this definition to estimate a complex integral.

Lemma 2.3. *Let $F : [a,b] \to \mathbb{C}$ be a complex function that is continuous at each point of C. Then*

$$\left| \int_a^b F(t)\,dt \right| \le \int_a^b |F(t)|\,dt.$$

Proof. Assume that $\int_a^b F(z)\,dz = r\mathrm{e}^{\mathrm{i}\theta}$, where $r = \left|\int_a^b F(z)\,dz\right|$ and $\theta \in [0, 2\pi)$. Then

$$r = \int_a^b \mathrm{e}^{-\mathrm{i}\theta} F(t)\,dt = \int_a^b \mathrm{Re}\left(\mathrm{e}^{-\mathrm{i}\theta} F(t)\right)dt \le \int_a^b \left|\mathrm{e}^{-\mathrm{i}\theta} F(t)\right| dt = \int_a^b |F(t)|\,dt.$$

Thus we complete the proof. □

Theorem 2.4 (M-L formula)**.** *Let $C = \{z(t) : a \le t \le b\}$ be a piecewise smooth curve of length L. Let f be a complex function that is continuous at each point of C, and $|f| \le M$ on C. Then*

$$\left| \int_C f(z)\,dz \right| \le ML.$$

Proof. According to the previous lemma, we have

$$\left| \int_C f(z)\,dz \right| = \left| \int_a^b f(z(t))z'(t)\,dt \right| \le \int_a^b |f(z(t))|\,|z'(t)|\,dt \le M \int_a^b |z'(t)|\,dt = ML.$$

Thus we complete the proof. □

Fundamental theorem of complex line integrals. For complex functions, the Newton-Leibniz formula also holds. This formula is really helpful. It removes the dependence of the integral value on the integral path, so the integral value only relies on the initial and terminal points.

Theorem 2.5 (Fundamental theorem of complex line integrals)**.** *Let $C = \{z(t) : a \le t \le b\}$ be a piecewise smooth curve. Let F be a complex function defined in an open set containing C. Suppose that F is holomorphic at each point of C, and the derivative $f(z) = F'(z)$ is continuous at each point of C. Then*

$$\int_C f(z)\,dz = F(z(b)) - F(z(a)).$$

Proof. Let $G(t) = F(z(t)),\ a \le t \le b$. By the chain rule, the derivative of G is

$$G'(t) = \lim_{\mathbb{R}\ni h\to 0} \frac{G(t+h)-G(t)}{h} = \lim_{\mathbb{R}\ni h\to 0} \frac{F(z(t+h))-F(z(t))}{z(t+h)-z(t)} \cdot \frac{z(t+h)-z(t)}{h} = F'(z(t))z'(t).$$

Then

$$\int_C f(z)\,dz = \int_a^b F'(z(t))z'(t)\,dt = \int_a^b G'(t)\,dt = G(b) - G(a) = F(z(b)) - F(z(a)),$$

where the third equality holds by apply Newton-Leibniz formula on both real and imaginary parts. □

Uniform Convergence. In the real case, one can interchange the integral and the function limit when the function sequence to be integrated is uniformly convergent. This result also applies to the complex line integral. A function sequence (f_n) *converges uniformly* to f on a set $U \subset \mathbb{C}$, if

$$\lim_{n\to\infty} \sup_{z\in U} |f_n(z) - f(z)| = 0.$$

We have the following theorem.

Theorem 2.6. *Let U be an open domain, and let (f_n) be a sequence of continuous functions which converges uniformly on U. For any piecewise smooth curve $C = \{z(t) : a \le t \le b\}$ in U,*

$$\lim_{n\to\infty} \int_C f_n(z)\,dz = \int_C \lim_{n\to\infty} f_n(z)\,dz.$$

Proof. Let f be the uniform limit of (f_n) on U, which is also continuous. By M-L formula [Theorem 2.4],

$$\left| \int_C f(z)\,dz - \int_C f_n(z)\,dz \right| \le \sup_{z\in U} |f(z) - f_n(z)| \int_C |dz| \to 0.$$

Thus we complete the proof. □

2.2 Cauchy-Goursat Theorem

From now on, we assume that all the curves and contours we study are piecewise smooth. Unless otherwise specified, we also assume that all contour integrals are taken in the counterclockwise direction, which is consistent with the unit circle $\{e^{i\theta} : 0 \le \theta \le 2\pi\}$.

We study the contour integral of holomorphic functions in this section. Before we proceed, we first study a special case of the contour integral, where the contour is assumed to be the boundary of a rectangle.

Lemma 2.7. *Let Γ be the boundary of a rectangle R. Let f be an affine function defined in an open domain U containing R, i.e. f is of the form $f(z) = \alpha + \beta z$. Then*

$$\int_\Gamma f(z)\,dz = 0.$$

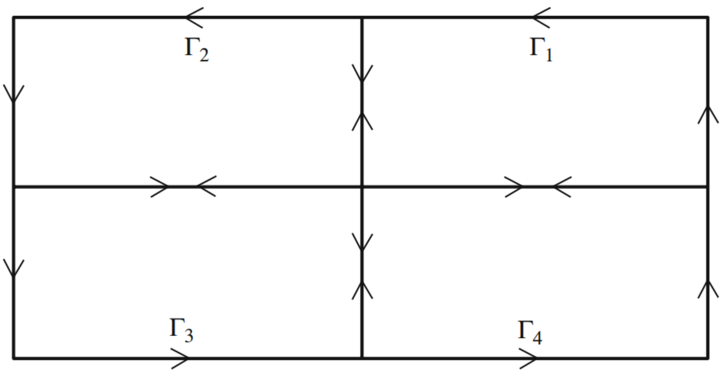
Proof. Note that f is everywhere the derivative of an entire function $F(z) = \alpha z + \frac{1}{2}\beta z^2$, and that Γ is a closed curve. Assume $\Gamma = \{z(t) : a \le t \le b\}$, so $z(a) = z(b)$. The result immediately follows from Theorem 2.5. □

Lemma 2.8. *Let Γ be the boundary of a rectangle R. Let f be a holomorphic function defined in an open domain U containing R. Then*

$$\int_\Gamma f(z)\,dz = 0.$$

Proof. We write $I = \left|\int_\Gamma f(z)\,dz\right|$.To show that $I = 0$, we use the method of continued bisection. That is, we split the rectangle R into four congruent subrectangles by bisecting each of the sides. We denote by $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ the boundaries of the four rectangles. Since the integral on the interior lines cancels out when integrating along opposite directions, we have

$$\int_\Gamma f(z)\,dz = \sum_{i=1}^{4} \int_{\Gamma_i} f(z)\,dz.$$



Hence for at least one $\Gamma_i,\ 1 \le i \le 4$, denoted by $\Gamma^{(1)}$,

$$\left|\int_{\Gamma^{(1)}} f(z)\,dz\right| \ge \frac{I}{4}.$$

Let $R^{(1)}$ be the rectangle bounded by $\Gamma^{(1)}$. We repeat this procedure by dividing $R^{(n)}$ into four congruent subrectangles. Then we obtain a nested sequence

$$R^{(1)} \supset R^{(2)} \supset R^{(3)} \supset \cdots$$

with their boundaries $\Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}, \cdots$. This sequence satisfies $\operatorname{diam} R^{(n+1)} = \frac{1}{2} \operatorname{diam} R^{(n)}$, and

$$\left|\int_{\Gamma^{(n)}} f(z)\,dz\right| \ge \frac{I}{4^n}. \tag{2.1}$$

Take $z_0 \in \bigcap_{n=1}^{\infty} R^{(n)}$, which is nonempty by the nested interval theorem. Since f is holomorphic at z_0,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0).$$

By Lemma 2.7, we have

$$\begin{aligned}
\int_{\Gamma^{(n)}} f(z)\,dz &= \int_{\Gamma^{(n)}} \left(f(z_0) + f'(z_0)(z - z_0) + \left(\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right)(z - z_0) \right) dz \\
&= \int_{\Gamma^{(n)}} \left(\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right)(z - z_0)\,dz.
\end{aligned}$$

We assume that the largest side of the original boundary Γ has length ℓ. Then

$$\int_{\Gamma^{(n)}} |dz| \le \frac{4\ell}{2^n}, \quad \textit{and} \quad |z - z_0| \le \frac{\sqrt{2}\ell}{2^n}, \quad \forall z \in \Gamma^{(n)}.$$

We fix $\epsilon > 0$, and choose N so that

$$|z - z_0| \leq \frac{\sqrt{2}\ell}{2^N} \quad \Rightarrow \quad \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| \leq \epsilon.$$

Then for all $n \geq N$, by M-L formula [Theorem 2.4],

$$\left|\int_{\Gamma^{(n)}} f(z)\,dz\right| \leq \frac{4\sqrt{2}\ell^2}{4^n}\epsilon. \tag{2.2}$$

Combining (2.1) and (2.2), we have

$$I \leq 4\sqrt{2}\ell^2\epsilon.$$

Since $\epsilon > 0$ can be chosen arbitrarily small, we have $I = 0$. □

We use this lemma to show that any holomorphic function is the derivative of another one.

Theorem 2.9 (Integral theorem)**.** *Let U be a simply connected open domain. If $f : U \to \mathbb{C}$ is a holomorphic function in U, then f is everywhere the derivative of another holomorphic function in U. That is, there exists a holomorphic function $F : U \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in U$.*

Proof. We may assume without loss of generality that $0 \in U$. Define $F(z)$ as

$$F(z) = \int_{\Gamma(z)} f(\zeta)\,d\zeta, \quad z \in U$$

where $\Gamma(z)$ is a polygonal line contained in U, starting from 0 and terminating at z, and every line segment is either horizontal or vertical. In fact, the value of this integral is independent of the choice of the particular path $\Gamma(z)$, because the difference of this integral along any two such paths can be represented as a contour integral round the boundaries of finitely many rectangles contained in U (because U is simply connected). Since f is holomorphic through these rectangles, by Lemma 2.8, the boundary integral always cancels out.

Fix $z \in U$. For sufficiently small $|h| > 0$, we have

$$F(z + h) - F(z) = \int_z^{z+h} f(\zeta)\,d\zeta$$

Where $\int_z^{z+h}$ is the line integral along the segments $[z, z + \operatorname{Re} h] \cup [z + \operatorname{Re} h, z + h]$. Note that the line integral along line segments $[z, z + \operatorname{Re} h] \cup [z + \operatorname{Re} h, z + h]$ is h, we have

$$\frac{F(z + h) - F(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h} (f(\zeta) - f(z))\,d\zeta. \tag{2.3}$$

We then fix $\epsilon > 0$, and choose $\delta > 0$ such that $|f(\zeta) - f(z)| < \epsilon$ for all $|\zeta - z| < \delta$. According to the M-L formula [Theorem 2.4], whenever $|h| < \delta$,

$$\left|\frac{1}{h}\int_z^{z+h} (f(\zeta) - f(z))\,d\zeta\right| \leq \frac{1}{|h|} \cdot 2|h|\epsilon = 2\epsilon.$$

Since $\epsilon > 0$ is arbitrarily chosen, 2.3 converges to 0 when $h \to 0$. Therefore $f(z) = F'(z)$. □

This result immediately gives the Cauchy's integral theorem for holomorphic functions.

Theorem 2.10 (Cauchy-Goursat)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic function in a simply connected open domain U. Let $C = \{z(t) : a \le t \le b\}$ be a piecewise smooth contour in U. Then*

$$\int_C f(z)\,dz = 0.$$

Proof. By integral theorem [Theorem 2.9], we can find a holomorphic function $F : U \to \mathbb{R}$ whose derivative is f everywhere in U. Then

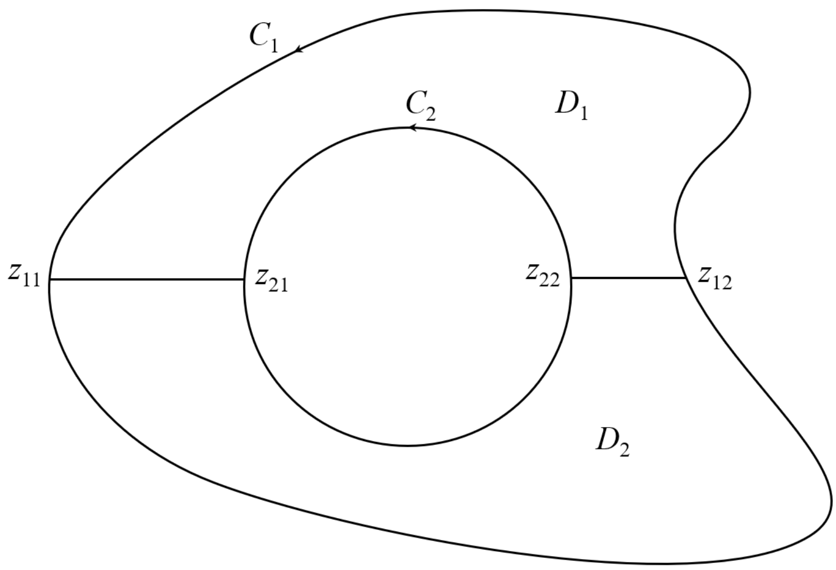
$$\int_C f(z)\,dz = F(z(b)) - F(z(a))$$

by Theorem 2.5. Since C is closed, $z(a) = z(b)$, and the value of the integral is 0. □

Remark. According to our proof, to cancel out a contour integral, we only require that f is the derivative of a holomorphic function inside a simply connected open domain containing the contour C.

Theorem 2.11 (Deformation principle)**.** *Let C_1 and C_2 be contours, with C_2 lying wholly inside C_1, and suppose that f is holomorphic in an open domain containing the closed domain between C_1 and C_2. Then*

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$



Proof. We join C_1 and C_2 by two segments $[z_{11}, z_{21}]$ and $[z_{22}, z_{12}]$, as is shown in the figure. Then we obtain two contours on the upper domain D_1 and lower domain D_2, respectively, and the integrals of f round these two contours are both 0. We add up these two integrals, of which the part on the segments is canceled out:

$$\int_{\partial D_1} f(z)\,dz + \int_{\partial D_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0.$$

In the last display, changing the direction of $-C_2$ and rearranging complete our proof. □

Remark. In the deformation principle, we require f to be holomorphic near the inner contour C_2, but we allow f to be not holomorphic inside the inner contour.

2.3 Cauchy's Integral Formula

An important integral. To begin this section, we first compute an important contour integral

$$\int_C (z-z_0)^n \, dz,$$

where C is a piecewise smooth contour going around a fixed point $z_0 \in \mathbb{C}$, and $n \in \mathbb{Z}$.

- *Case I: $n \geq 0$.* In this case, $(z-z_0)^n$ is an analytic polynomial, which is entire. Then the value of the integral is 0 by Cauchy-Goursat theorem [Theorem 2.10].
- *Case II: $n = -1$.* We use the deformation principle to compute this integral. Since $\frac{1}{z-z_0}$ is holomorphic on the whole complex plane $\mathbb{C}$ except at z_0, we choose a circle $\partial B(z_0, \epsilon)$ that is lying wholly inside C, which is parameterized by $\{z_0 + \epsilon e^{i\theta} : 0 \leq \theta \leq 2\pi\}$:

$$\int_C \frac{dz}{z-z_0} = \int_{\partial B(z_0,\epsilon)} \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{i\epsilon e^{i\theta} d\theta}{\epsilon e^{i\theta}} = 2\pi i.$$

- *Case II: $n \leq -2$.* In this case, $(z-z_0)^n$ is the derivative of the function $\frac{(z-z_0)^{n+1}}{n+1}$, which is holomorphic on the whole on the whole complex plane $\mathbb{C}$ except at z_0. Since C is closed, the value of the integral is 0 by the fundamental theorem of complex line integral [Theorem 2.5].

We then summarize our result below:

$$\int_C (z-z_0)^n \, dz = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1. \end{cases}$$

The Cauchy's integral formula is motivated by this example.

Theorem 2.12 (Cauchy's integral formula)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic function in a simply connected open domain U. Let C be any piecewise smooth contour in U that goes around some fixed point $z_0 \in U$. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} \, dz. \tag{2.4}$$

Proof. Since $\frac{f(z)}{z-z_0}$ is holomorphic in $U \backslash \{z_0\}$, by the deformation principle [Theorem 2.11], we may change C to any circle $\partial B(z_0, \epsilon)$ of radius $\epsilon > 0$ and centered at z_0:

$$\begin{aligned} \int_C \frac{f(z)}{z-z_0} \, dz &= \int_{\partial B(z_0,\epsilon)} \frac{f(z)}{z-z_0} \, dz \\ &= \int_{\partial B(z_0,\epsilon)} \frac{f(z_0) + f'(z_0)(z-z_0) + (f(z) - f(z_0) - f'(z_0)(z-z_0))}{z-z_0} \, dz \\ &= \underbrace{\int_{\partial B(z_0,\epsilon)} \frac{f(z_0)}{z-z_0} \, dz}_{=2\pi i f(z_0)} + \underbrace{\int_{\partial B(z_0,\epsilon)} f'(z_0) \, dz}_{=0} + \underbrace{\int_{\partial B(z_0,\epsilon)} \left(\frac{f(z) - f(z_0)}{z-z_0} - f'(z_0) \right) dz}_{\textit{remainder}}. \end{aligned}$$

By M-L formula [Theorem 2.4], the remainder satisfies

$$\left| \int_C \frac{f(z)}{z-z_0} \, dz - 2\pi i f(z_0) \right| = \left| \int_{\partial B(z_0,\epsilon)} \left(\frac{f(z) - f(z_0)}{z-z_0} - f'(z_0) \right) dz \right| \leq 2\pi\epsilon \sup_{|z-z_0| \leq \epsilon} \left| \frac{f(z) - f(z_0)}{z-z_0} - f'(z_0) \right|.$$

The remainder can be dominated by arbitrarily small quantity as $\epsilon \to 0$. Then (2.4) follows. $\square$

We have similar results for higher order derivatives.

Theorem 2.13 (Generalized Cauchy's integral formula)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic function in a simply connected open domain U. Then f is infinitely differentiable in U, and*

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz, \tag{2.5}$$

where C is any piecewise smooth contour in U that goes around some fixed point $z_0 \in U$.

Proof. Since U is open, we can find an open disc $B(z_0, \epsilon) \subset U$. We fix C to be its boundary $|z - z_0| = \epsilon$. Then for all z with $|z - z_0| = \epsilon$ and $\omega \in B(z_0, \frac{\epsilon}{2})$, we have the expansion

$$\frac{1}{z - \omega} = \frac{1}{(z - z_0)\left(1 - \frac{\omega - z_0}{z - z_0}\right)} = \sum_{n=0}^{\infty} \frac{(\omega - z_0)^n}{(z - z_0)^{n+1}}. \tag{2.6}$$

This series converges uniformly on $B(z_0, \frac{\epsilon}{2})$, since for all $n \in \mathbb{N}$,

$$\left| \sum_{n=N+1}^{\infty} \frac{(\omega - z_0)^n}{(z - z_0)^{n+1}} \right| < \frac{1}{2^N \epsilon}, \quad \forall \omega \in B\left(z_0, \frac{\epsilon}{2}\right).$$

Then we apply Theorem 2.6 to interchange the infinite sum and the integration:

$$\begin{aligned} f(\omega) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{z - \omega} \, dz &= \frac{1}{2\pi \mathrm{i}} \int_C \sum_{n=0}^{\infty} \frac{f(z)(\omega - z_0)^n}{(z - z_0)^{n+1}} \, dz \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \right) (\omega - z_0)^n, \quad \omega \in B\left(z_0, \frac{\epsilon}{2}\right). \end{aligned}$$

Therefore, f can be written into a power series in a neighborhood $B(z_0, \frac{\epsilon}{2})$. Furthermore, the coefficients are in fact independent of our choice of C by the deformation principle, since C lies in U and goes around z_0. Hence f is infinitely differentiable in this neighborhood, and

$$f^{(k)}(\omega) = \sum_{n=k}^{\infty} \left(\frac{n(n-1)\cdots(n-k+1)}{2\pi \mathrm{i}} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \right) (\omega - z_0)^{n-k}, \quad \omega \in B\left(z_0, \frac{\epsilon}{2}\right).$$

Taking $\omega = z_0$, we obtain (2.5), which completes the proof. □

The proof above also establishes the analyticity of holomorphic functions. We can even modify our result to make it a little stronger.

Corollary 2.14. *Let $f : U \to \mathbb{C}$ be a holomorphic function in a simply connected open domain U. Then f is analytic in U. Furthermore, for each $z_0 \in U$, f can be represented as the local Taylor series near z_0:*

$$f(\omega) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (\omega - z_0)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \right) (\omega - z_0)^n, \quad \omega \in B(z_0, \epsilon) \subset U,$$

where $B(z_0, \epsilon)$ is the largest open disc centered at z_0 and contained in U.

Proof. Since $\omega \in B(z_0, \epsilon)$, there exists $\theta > 0$ such that $\omega \in B(z_0, (1 - \theta)\epsilon)$. Then we fix C to be the circle $|z - z_0| = (1 - \frac{\theta}{2})\epsilon$. With this choice, the power series (2.6) still converges uniformly on $B(z_0, (1 - \theta)\epsilon)$, and the remainder totally follows from the previous proof. □

Remark. If $f : \mathbb{C} \to \mathbb{C}$ is entire, then f is globally equal to its Taylor series at any point $z_0 \in \mathbb{C}$.

2.4 Liouville's Theorem and the Fundamental Theorem of Algebra

Theorem 2.15 (Liouville's Theorem)**.** *A bounded entire function is a constant.*

Proof. Let $f:\mathbb{C}\to\mathbb{C}$ be a entire function bounded by some $M>0$. For each $z_1, z_2\in\mathbb{C}$, we take any circle $C=\{z:|z|=R\}$ with $R>\max\{|z_1|,|z_2|\}$. By Cauchy's integral formula,

$$f(z_1)-f(z_2)=\frac{1}{2\pi \mathrm{i}}\int_C \frac{f(z)}{z-z_1}\,dz-\frac{1}{2\pi \mathrm{i}}\int_C \frac{f(z)}{z-z_2}\,dz=\frac{1}{2\pi \mathrm{i}}\int_C \frac{f(z)(z_2-z_1)}{(z-z_1)(z-z_2)}\,dz.$$

By M-L formula [Theorem 2.4],

$$|f(z_1)-f(z_2)|=\frac{1}{2\pi}\left|\int_C \frac{f(z)(z_2-z_1)}{(z-z_1)(z-z_2)}\,dz\right|\le \frac{MR\,|z_2-z_1|}{(R-|z_1|)(R-|z_2|)}\to 0,\quad R\to\infty.$$

Hence $f(z_1)=f(z_2)$ for all $z_1,z_2\in\mathbb{C}$, and f is a constant on $\mathbb{C}$. ☐

Theorem 2.16 (Extended Liouville's Theorem)**.** *If $f:\mathbb{C}\to\mathbb{C}$ is an entire function, and if for some integer $k\ge 0$, f grows no faster than $|z|^k$, that is, there exist constants $A, B>0$ such that*

$$|f(z)|\le A+B|z|^k,\quad \forall z\in\mathbb{C},$$

then f is a polynomial of degree at most k.

Proof. The case $k=0$ is the original Liouville theorem, and we use induction to prove the general case. We define a new function $g:\mathbb{C}\to\mathbb{C}$ as follows:

$$g(z)=\begin{cases}\frac{f(z)-f(0)}{z}, & z\neq 0\\ f'(0), & z=0\end{cases}\qquad\Rightarrow\qquad g(z)=\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}z^{k-1},\quad z\in\mathbb{C}.$$

Clearly, g is also an entire function. We pick any $R>0$. Then $g(z)$ is bounded in the compact disc $|z|\le R$, and its absolute value grows no faster than $|z|^{k-1}$ as $z\to\infty$. Hence there exists $C, D>0$ such that

$$|g(z)|\le C+D|z|^{k-1}.$$

According to the induction hypothesis, g is polynomial of degree at most $k-1$. Therefore f is a polynomial of degree at most k. ☐

We then use Liouville's theorem to prove the fundamental theorem of algebra.

Lemma 2.17. *Every non-constant polynomial with complex coefficients has a zero in $\mathbb{C}$.*

Proof. Let $P(z)$ be a non-constant polynomial. If $P(z)$ has no zero in $\mathbb{C}$, then $\frac{1}{P(z)}$ is an entire function. Since $P(z)$ is non-constant, $P(z)\to\infty$ as $z\to\infty$, and $\frac{1}{P(z)}$ is bounded in $\mathbb{C}$. By Liouville's theorem, $\frac{1}{P(z)}$ is a constant, and so is $P(z)$, which is a contradiction! ☐

Theorem 2.18 (Fundamental theorem of algebra)**.** *If $P(z)$ is a polynomial of degree $n\ge 1$ with complex coefficients, then there exist complex numbers A and $z_1,\cdots,z_n$ such that*

$$P(z)=A(z-z_1)(z-z_2)\cdots(z-z_n),\quad z\in\mathbb{C}.$$

Proof. The case $n=1$ is clear, and the general case follows by induction. By the previous lemma, any polynomial $P(z)$ of degree n has a zero $z_n\in\mathbb{C}$, and the function $Q(z)=\frac{P(z)}{z-z_n}$ grows no faster than $|z|^{n-1}$. By the extended Liouville's theorem, $Q(z)$ is a polynomial of degree $k-1$, and the result follows. ☐

Remark. According to the fundamental algebra, we can write a polynomial $P(z)$ of degree n to the form

$$P(z) = A(z - z_1)^{m_1}(z - z_2)^{m_2} \cdots (z - z_k)^{m_k},$$

where $z_1, z_2, \cdots, z_k \in \mathbb{C}$ are zeroes of $P(z)$ and are mutually distinct, and $m_1, \cdots, m_k$ are positive integers such that $m_1 + \cdots + m_k = n$. The number m_j is called the *multiplicity* of the zero z_j. It is easy to see that z_j is a zero of multiplicity m_j if and only if

$$P(z_j) = P'(z_j) = P''(z_j) = \cdots = P^{(m_j-1)}(z_j) = 0, \quad P^{(m_j)}(z_j) \neq 0. \tag{2.7}$$

Factorization of polynomials with real coefficients. We let $P(z)$ be a polynomial of degree n with real coefficients. In this case, we have $P(\overline{z}) = \overline{P(z)}$ for all $z \in \mathbb{C}$. Consequently, if $\omega \in \mathbb{C}$ is a zero of $P(z)$, so is $\overline{\omega}$. Furthermore, the conjugate zeroes have the same multiplicity by the condition (2.7). Therefore, there exists an integer $k \leq \frac{n}{2}$, non-real zeroes $\omega_1, \omega_2, \cdots, \omega_k \in \mathbb{C}$ and real zeroes $c_1, \cdots, c_{n-2k} \in \mathbb{R}$ such that

$$P(z) = A(z - \omega_1)(z - \overline{\omega_1})(z - \omega_2)(z - \overline{\omega_2}) \cdots (z - \omega_k)(z - \overline{\omega_k})(z - c_1)(z - c_2) \cdots (z - c_{n-2k}),$$

Note that $(z - \omega_j)(z - \overline{\omega_j})$ is a quadratic polynomial with real coefficients:

$$(z - \omega_j)(z - \overline{\omega_j}) = z^2 - \alpha_j z + \beta_j, \quad \textit{where} \ \ \alpha_j = 2 \operatorname{Re} \omega_j \ \ \textit{and} \ \ \beta_j = |\omega_j|^2.$$

Therefore, we have the following factorization of $P(z)$:

$$P(z) = A \prod_{j=1}^{k} (z^2 - \alpha_j z + \beta_j) \prod_{p=1}^{n-2k} (z - c_p),$$

which consists of only linear and quadratic polynomials with real coefficients.

We finally introduce a result concerning both the zeroes of a polynomial and of its derivative.

Theorem 2.19 (Gauss-Lucas theorem)**.** *The zeroes of the derivative of any polynomial lie within the convex hull of the zeroes of the polynomial.*

Proof. Let $P(z)$ be a polynomial of degree n with zeroes $z_1, \cdots, z_n \in \mathbb{C}$. Then

$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n}, \quad \forall z \notin \{z_1, z_2, \cdots, z_n\}.$$

Assume $\omega \in \mathbb{C}$ is a zero of $P'(z)$. We may assume $\omega \in \{z_1, \cdots, z_n\}$, otherwise the result is clear. Then

$$0 = \overline{\left(\frac{P'(\omega)}{P(\omega)} \right)} = \frac{\omega - z_1}{|\omega - z_1|^2} + \frac{\omega - z_2}{|\omega - z_2|^2} + \cdots + \frac{\omega - z_n}{|\omega - z_n|^2}.$$

Hence

$$\omega = \sum_{j=1}^{n} \lambda_j z_j, \quad \textit{where} \ \ \lambda_j = \frac{|\omega - z_j|^{-2}}{\sum_{k=1}^{n} |\omega - z_k|^{-2}} > 0, \ \ \textit{and} \ \ \lambda_1 + \cdots + \lambda_n = 1.$$

Then we conclude the proof. □

Remark. By induction, for a polynomial $P(z)$ of degree n, the zeroes of the derivative $P^{(k)}(z)$ of any order $k = 1, 2, \cdots, n-1$ lie in the convex hull of the zeroes of $P(z)$.

2.5 The Converse of Cauchy's Integral Theorem: Morera's Theorem

The Cauchy's theorem asserts that the integral of any holomorphic function in a simple connected open domain round any contour inside the domain is zero. The following result, due to Morera, gives a converse statement.

Theorem 2.20 (Morera's theorem)**.** *Let $f : U \to \mathbb{R}$ be a continuous function on an open set U. If*

$$\int_C f(z)\,dz = 0$$

for all contours C in U, then f is holomorphic in U.

Proof. We fix $z_0 \in U$, and choose an open disc $B(z_0, \delta) \subset U$. We then define the primitive

$$F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta,$$

where $\int_{z_0}^{z}$ denote the integral along the path $[z_0, z_0 + \mathrm{Re}(z - z_0)] \cup [z_0 + \mathrm{Re}(z - z_0), z]$. Like in the proof of Theorem 2.9, we consider a difference quotient of F and that fact that $\int_\Gamma f(z)\,dz = 0$ around any rectangle in $B(z_0, \delta)$, we may conclude that

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h} (f(\zeta) - f(z))d\zeta \to 0, \quad as \;\; h \to 0.$$

Hence F is holomorphic in $B(z_0, \delta)$, and $f = F'$, which can be represented by a power series in $B(z_0, \delta)$, is holomorphic at z_0. Finally, since $z_0 \in U$ is arbitrary, f is holomorphic in U. □

Remark. We can make the condition slightly weaker by letting C to be any rectangular boundary inside U with sides parallel to the real and imaginary axes.

Compact convergence. In some cases, we may concern whether the limit of a sequence of holomorphic functions is holomorphic. To answer this question, we need to introduce a new convergence mode. Let (f_n) be a sequence of functions defined on a topological space U. We say that (f_n) *converges compactly to* f, if $f_n \to f$ uniformly on each compact subset K of U. That is,

$$\lim_{n\to\infty} \sup_{z\in K} |f_n(z) - f(z)| = 0, \quad \forall \text{ compact } K \subset U.$$

Theorem 2.21. *Let (f_n) be a sequence of holomorphic functions in an open domain U such that $f_n \to f$ compactly. Then f is also holomorphic in U.*

Proof. For each point z_0, f is the uniform limit of the sequence (f_n) on a compact disc $\overline{B(z_0, \delta)}$ about z_0. Hence f is continuous in U. Furthermore, for any rectangular boundary $\Gamma \subset U$, since Γ is compact, f_n converges uniformly to f on Γ. By Theorem 2.6,

$$\int_\Gamma f(z)\,dz = \int_\Gamma \lim_{n\to\infty} f(z)\,dz = \lim_{n\to\infty}\int_\Gamma f(z)\,dz = 0.$$

Hence, by Morera's theorem, f is holomorphic in U. □

Theorem 2.22. *Suppose f is a continuous function in an open set U, and f is holomorphic there except possibly at the points of a line segment L. Then f is holomorphic throughout U.*

Proof. We may assume that the line segment L lies on the real axis. Otherwise, our proof applies to the function $g(z) = f(T^{-1}z)$, where $Tz = \frac{z-z_0}{z_1-z_0}$ is a linear transformation that maps any segment $[z_0, z_1]$ to the compact interval $[0,1] \subset \mathbb{R}$, and the holomorphicity of f in U is equivalent to the holomorphicity of g in TU.

Since holomorphicity is a local property, we may also assume that U is an open disc. We prove that $\int_\Gamma f(z)\,dz = 0$ for any rectangular boundary $\Gamma \subset U$ with sides parallel to the real and imaginary axes.

(i) When L does not coincide the rectangular bounded by Γ, we have $\int_\Gamma f(z)\,dz$ by Cauchy-Goursat theorem.

(ii) When L coincides with the bottom (top) side of Γ, we slightly shift up (down) the bottom (top) side by a small quantity $\epsilon > 0$ and get a boundary Γ_ϵ, which reduces to case (i). By boundedness and uniform continuity of f on compact sets,

$$\lim_{\epsilon\to 0}\left(\int_\epsilon^0 f(a+\mathrm{i}y)\,dy + \int_a^b f(x+\mathrm{i}\epsilon)\,dx + \int_0^\epsilon f(b+\mathrm{i}y)\,dy\right) = 0 + \int_a^b f(x)\,dx + 0 = \int_a^b f(x)\,dx.$$

for all $[a,b] \subset U$. Then

$$\int_\Gamma f(z)\,dz = \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} f(z)\,dz = 0.$$

(iii) When L is surrounded by Γ, split Γ by the real line to Γ_1 and Γ_2. Then the integral of Γ_1 and Γ_2 reduces to the case (ii), and

$$\int_\Gamma f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz = 0.$$

Finally, by Morera's theorem, f is holomorphic in U. ☐

Theorem 2.23 (Schwarz Reflection Principle)**.** *Suppose f is holomorphic in a simply connected open domain U that is contained in either the upper or lower half plane and whose boundary contains a segment $[a,b]$ on the real axis, and suppose $f(z) \in \mathbb{R}$ for real z. Then we can define a holomorphic extension g of f to the domain $U \cup (a,b) \cup U^*$ that is symmetric with respect to the real axis by setting*

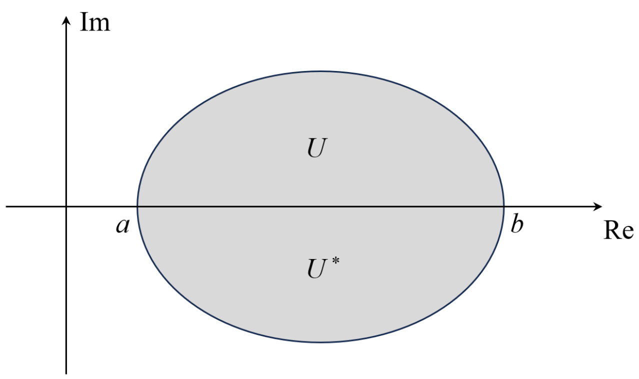
$$g(z) = \begin{cases} f(z), & z \in U \cup (a,b), \\ \overline{f(\overline{z})}, & z \in U^*, \end{cases}$$

where $U^* = \{\overline{z} : z \in U\}$.

Proof. If $z \in U^*$, we choose a small quantity h with $z+h \in U^*$, then

$$\lim_{h\to 0}\frac{g(z+h)-g(z)}{h} = \lim_{h\to 0}\frac{\overline{f(\overline{z}+\overline{h}) - f(\overline{z})}}{h} = \lim_{h\to 0}\overline{\left[\frac{f(\overline{z}+\overline{h}) - f(\overline{z})}{\overline{h}}\right]} = \overline{f'(\overline{z})}.$$

Hence g is holomorphic both in U and U^*. Since $f(z) \in \mathbb{R}$ for all real z, the function f is continuous in the domain $U \cup (a,b) \cup U^*$. Then f is also holomorphic in this domain by employing Theorem 2.22. ☐



3 Singularities of Analytic Functions and Calculus of Residues

3.1 Branch Cuts

Motivation. We consider the logarithm function and the root function in a complex plane. Given a complex number $re^{\mathrm{i}\theta}$, the solutions of the equation $\mathrm{e}^z = re^{\mathrm{i}\theta}$ are

$$z = \log r + \mathrm{i}(\theta + 2k\pi), \quad \textit{where } k \in \mathbb{Z},$$

and the solutions of the equation $z^n = re^{\mathrm{i}\theta}$ $(n = 2, 3, \cdots)$ are

$$z = r^{1/n}\mathrm{e}^{\mathrm{i}\frac{\theta+2k\pi}{n}}, \quad \textit{where } k = 0, 1, \cdots, n-1.$$

These functions are multi-valued in the complex case, which map a number $z \in \mathbb{C}$ to a subset of $\mathbb{C}$. The difficulty comes from the fact that $z \mapsto \mathrm{e}^z$ and $z \mapsto z^n$ are no longer bijective. In the last two cases, each k correspond to a *branch* of the multi-valued function.

A *branch cut* is a curve in the complex plane such that it is possible to define a single analytic branch of a multi-valued function on the plane minus that curve.

Definition 3.1. *Let U be a simply connected open domain. A function $f : U \to \mathbb{C}$ is said to be an analytic branch of* $\log(z)$*, if*

(i) f is analytic in U, and

(ii) f is an inverse of the exponential function there, i.e. $\mathrm{e}^{f(z)} = z$.

Remark. When $z \in \mathbb{C}\backslash\{0\}$, the *principal value* of $\log z$ is chosen to be the logarithm whose imaginary part lies in the interval $(-\pi, \pi]$, called the *principle argument of* z, written $\mathrm{Arg}(z)$:

$$\mathrm{Log}(z) = \log|z| + \mathrm{i}\,\mathrm{Arg}(z), \quad z \in \mathbb{C}\backslash\{0\}.$$

We choose the branch cut to be $(-\infty, 0]$. On the simply connected open domain $U = \mathbb{C}\backslash(-\infty, 0]$, the principal logarithm $z \mapsto \mathrm{Log}(z)$ is continuous, because both the modulus $|z+h| - |z|$ and the principal argument $\mathrm{Arg}(z+h) - \mathrm{Arg}(z)$ changes little within a small region $|h| < \epsilon$ for fixed $z \in U$. Then

$$\lim_{h\to 0} \frac{\mathrm{Log}(z+h) - \mathrm{Log}(z)}{h} = \left(\lim_{h\to 0} \frac{\mathrm{e}^{\mathrm{Log}(z+h)} - \mathrm{e}^{\mathrm{Log}(z)}}{\mathrm{Log}(z+h) - \mathrm{Log}(z)} \right)^{-1} = \mathrm{e}^{-\mathrm{Log}(z)} = \frac{1}{z}.$$

Therefore $\mathrm{Log}(z)$ is analytic in $\mathbb{C}\backslash(-\infty, 0]$. This is a useful analytical branch of $\log(z)$.

Theorem 3.2. *Let $U \subset \mathbb{C}$ be a simply connected domain such that $0 \notin U$. We choose $z_0 \in U$, fix a value of* $\log z_0$ *and define*

$$f(z) = \log z_0 + \int_{z_0}^{z} \frac{d\zeta}{\zeta}.$$

Then f is an analytic branch of $\log(z)$ *in U.*

Proof. Since $1/\zeta$ is a holomorphic function of ζ in the simply connected open domain U, by Theorem 2.9, it is the derivative of some primitive holomorphic function in U, and the integral along any two paths from z_0 to z has the same value. Then $f'(z) = z^{-1}$ on U, and f is holomorphic, hence analytic.

Now we consider the function $g(z) = z\mathrm{e}^{-f(z)}$. Since $g'(z) = \mathrm{e}^{-f(z)} - zf'(z)\mathrm{e}^{-f(z)} = 0$, the function g is a constant in U, and $g(z) = g(z_0) = z_0\mathrm{e}^{-f(z_0)} = 1$. Hence $\mathrm{e}^{f(z)} = z$, completing the proof. □

Remark. The principal logarithm $z \mapsto \mathrm{Log}(z)$ corresponds to $z_0 = 1, \log 1 = 0$ and $U = \mathbb{C}\backslash(-\infty, 0]$.

3.2 Isolated Singularities

In this section, we study the behavior of an analytic function in the neighborhood of an isolated singularity. We call a set of the form $U\backslash\{z_0\}$ a *deleted neighborhood* of z_0, where U is a neighborhood of z_0.

Definition 3.3. *A function f is said to have an isolated singularity at z_0 if f is holomorphic in a deleted neighborhood $U\backslash\{z_0\}$, but is not holomorphic at z_0.*

Remark. According to Theorem 2.22, f must be *discontinuous* at an isolated singularity.

Definition 3.4 (Classification of isolated singularities)**.** *Assume that f has a singularity at z_0.*

(i) If there exists a function g such that $f(z) = g(z)$ for all z in some deleted neighborhood of z_0, we say f has a removable singularity at z_0.

(ii) If there exists analytic functions $A(z)$ and $B(z)$ such that $A(z_0) \neq 0$, $B(z_0) = 0$ and $f(z) = \frac{A(z)}{B(z)}$ for all z in some deleted neighborhood of z_0, we say f has a pole at z_0.
In addition, if $B(z)$ has a zero of multiplicity k at z_0, the pole at z_0 is said to be of order k. A pole of order 1 (resp. 2, 3) is called a simple (resp. double, triple) pole.

(iii) If f has neither a removable singularity nor a pole at z_0, we say f has an essential singularity at z_0.

Now we discuss the properties of each class of singularity.

Theorem 3.5 (Riemann's principle of removable singularities)**.** *If f has an isolated singularity at z_0 and if $\lim_{z\to z_0}(z - z_0)f(z) = 0$, then the singularity is removable.*

Proof. Define the function $h(z) = (z - z_0)f(z)$ in an appropriate deleted neighborhood of z_0, which is also analytic. If we add a continuous extension $h(z_0) = 0$, by Theorem 2.22, h is also analytic at z_0. Then the function $g(z) = \frac{h(z)}{z-z_0}$ is analytic at z_0 and agrees with f in a deleted neighborhood of f. □

Remark. According to this theorem, if f is bounded in a deleted neighborhood of an isolated singularity, then the singularity is removable.

Theorem 3.6. *If f has an isolated singularity at z_0 and if there exists a positive integer k such that*

$$\lim_{z\to z_0} (z - z_0)^k f(z) \neq 0, \quad \textit{but} \quad \lim_{z\to z_0} (z - z_0)^{k+1} f(z) = 0$$

then the singularity at z_0 is a pole of order k.

Proof. We set $g(z) = (z - z_0)^{k+1}f(z)$ in an appropriate deleted neighborhood of z_0, and set $g(z_0) = 0$. By Theorem 2.22, g is continuous and analytic at z_0. Likewise, the function $A(z) = \frac{g(z)}{z-z_0} = (z - z_0)^k f(z)$ is also analytic at z_0, and $A(z_0) \neq 0$ by hypothesis. Then we conclude the proof by setting $B(z) = (z - z_0)^k$ in the previous Definition 3.4(ii) of poles. □

Remark. Combining Theorems 3.5 and 3.6, we conclude that $f(z)$ has a pole of order k at z_0 if and only if $(z - z_0)^k f(z)$ has a removable singularity at z_0.

Theorem 3.7 (Casorati-Weierstrass Theorem)**.** *If f has an essential singularity at z_0 and if $D = U\backslash\{z_0\}$ is a deleted neighborhood of z_0, then the range $R = \{f(z) : z \in D\}$ is dense in the complex plane.*

Proof. Argue by contradiction. If there exists some open disc $B(\omega, \delta)$ not intersecting R, then $|f(z) - \omega| \geq \delta$, and $\frac{1}{|f(z)-\omega|} \leq \frac{1}{\delta}$ throughout D. By Riemann's principle of removable singularities, $\frac{1}{f(z)-\omega}$ has at worst a removable singularity at z_0, and $\frac{1}{f(z)-\omega} = g(z)$ on D for some g that is analytic at z_0. Consequently, we have $f(z) = \omega + \frac{1}{g(z)}$ near z_0. Therefore, f has either a removable singularity (if $g(z_0) \neq 0$) or a pole (if $g(z_0) = 0$) at z_0, contradicting the fact that f has an essential singularity at z_0! □

3.3 Laurent Series

In the previous sections, we see that an analytic function in an open disc can be represented there by its Taylor series. For analytic functions in an *annulus* $\{z \in \mathbb{C} : R_1 < |z| < R_2\}$, we have a similar representation, which is in the form of a two-sided power series, which is called the *Laurent series*.

Theorem 3.8. *The series* $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ *is convergent in the domain*

$$D = \{z \in \mathbb{C} : |z| < R_2 \text{ and } |z| > R_1\},$$

where

$$R_2 = \left(\limsup_{n\to\infty} |c_n|^{1/n}\right)^{-1}, \quad \text{and} \quad R_1 = \limsup_{n\to\infty} |c_{-n}|^{1/n}.$$

If $R_1 < R_2$, $D = \{z \in \mathbb{C} : R_1 < |z| < R_2\}$ *is an annulus and* f *is analytic in* D.

Proof. Define power series $g(z) = \sum_{n=1}^{\infty} c_{-n} z^n$, and $f_2(z) = \sum_{n=0}^{\infty} c_n z^n$. By Theorem 1.10, g is convergent in the domain $\{z \in \mathbb{C} : |z| < 1/R_1\}$, and f_2 is convergent in the domain $\{z \in \mathbb{C} : |z| < R_2\}$ are convergent. Furthermore, by Theorem 1.11, they are both analytic in their respective domains. Let $f_1(z) = g(1/z)$, which is convergent and analytic in the domain $\{z \in \mathbb{C} : |z| > R_1\}$. Hence $f = f_1 + f_2$ is convergent in D, and is analytic when D is nonempty. $\square$

Theorem 3.9. *If* f *is analytic in the annulus* $D = \{z \in \mathbb{C} : R_1 < |z| < R_2\}$, *then* f *has an expansion of the following form throughout* D:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n. \tag{3.1}$$

Furthermore, in this form, the coefficients c_n *are uniquely given by*

$$c_n = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{z^{n+1}}\, dz, \quad n \in \mathbb{Z},$$

where C *is any circle in* D *centered at* 0. *This expansion is called the* *Laurent series* *of* f *about* 0.

Proof. Choose $R_1 < r_1 < r_2 < R_2$, and let C_1 and C_2 be the circles centered at 0 of radii r_1 and r_2, respectively. Fix $z \in D$ with $r_1 < |z| < r_2$. Since f is analytic in D, so is the function $g(\omega) = \frac{f(\omega)-f(z)}{\omega - z}$ with $g(z) = f'(z)$. By the deformation principle [Theorem 2.11],

$$\int_{C_1} \frac{f(\omega) - f(z)}{\omega - z}\, d\omega = \int_{C_2} \frac{f(\omega) - f(z)}{\omega - z}\, d\omega, \quad z \in D. \tag{3.2}$$

Since z lies inside C_2 and outside C_1,

$$\int_{C_1} \frac{f(z)}{\omega - z} = 0, \quad \int_{C_2} \frac{f(z)}{\omega - z} = 2\pi \mathrm{i} f(z).$$

Therefore, we rearrange (3.2) and then use the power series $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$, $|t| < 1$:

$$\begin{aligned} 2\pi \mathrm{i} f(z) &= \int_{C_2} \frac{f(\omega)}{\omega - z}\, d\omega - \int_{C_1} \frac{f(\omega)}{\omega - z}\, d\omega = \int_{C_2} \frac{f(\omega)}{\omega\left(1 - \frac{z}{\omega}\right)}\, d\omega + \int_{C_1} \frac{f(\omega)}{z\left(1 - \frac{\omega}{z}\right)}\, d\omega \\ &= \int_{C_2} \sum_{n=0}^{\infty} f(\omega) \frac{z^n}{\omega^{n+1}}\, d\omega + \int_{C_1} \sum_{n=0}^{\infty} f(\omega) \frac{\omega^n}{z^{n+1}}\, d\omega. \end{aligned}$$

Note that $\sum_{n=0}^{\infty} f(\omega)\frac{z^n}{\omega^{n+1}}$ converges uniformly on the circle C_2, and $\sum_{n=0}^{\infty} f(\omega)\frac{\omega^n}{z^{n+1}}$ converges uniformly on the circle C_1. Therefore, we interchange the infinite sum and the integral by Theorem 2.6:

$$2\pi \mathrm{i} f(z) = \sum_{n=0}^{\infty} \left(\int_{C_2} \frac{f(\omega)}{\omega^{n+1}} \, d\omega \right) z^n + \sum_{n=0}^{\infty} \left(\int_{C_1} \omega^n f(\omega) \, d\omega \right) \frac{1}{z^{n+1}}.$$

Again, by the deformation principle, we can switch the integral paths C_1 and C_2 to any circle C in D centered at 0 without changing the integral value:

$$2\pi \mathrm{i} f(z) = \sum_{n=-\infty}^{\infty} \left(\int_{C} \frac{f(\omega)}{\omega^{n+1}} \, d\omega \right) z^n.$$

This ensures the existence of the expansion in form (3.1). It remains to prove the uniqueness of this expansion. If the expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ converges in D, it converges uniformly on any circle C lying in D centered at 0. Hence for all $k \in \mathbb{Z}$,

$$\int_C \frac{f(z)}{z^{k+1}} \, dz = \sum_{n=-\infty}^{\infty} c_n \left(\int_C z^{n-k-1} \, dz \right) = 2c_k \pi \mathrm{i},$$

where all terms in the infinite sum vanish except $n - k - 1 = -1$. This proves uniqueness. □

Remark. For any point $z_0 \in \mathbb{C}$, if f is analytic in the annulus $D = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$, we can apply the previous result to the function $g(z) = f(z + z_0)$. This gives the Laurent series of f about z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad R_1 < |z| < R_2,$$

where the coefficients are

$$c_n = \frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz, \quad n \in \mathbb{Z},$$

and $C = \partial B(z_0, R)$ is any circle of radius $R_1 < R < R_2$ centered at z_0. Furthermore, an isolated singularity at z_0 corresponds to case of $R_1 = 0$. We formally state the result below.

Corollary 3.10. *If f have an isolated singularity at z_0, then f equals to its Laurent series*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi \mathrm{i}} \int_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz \right) (z - z_0)^n$$

in some deleted neighborhood $0 < |z - z_0| < \delta$ *of* z_0*.*

Terminology. The series

$$f_a(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

is called the *analytic part of f at z_0*. The series

$$f_p(z) = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z - z_0)^n}$$

is called the *principal part of f at z_0*. We can use the principle part of the Laurent series to determine the class of isolated singularities.

Theorem 3.11. *Suppose f has an isolated singularity at z_0, and the Laurent series of f about z_0 is*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad 0 < |z - z_0| < \delta.$$

(i) If f has an isolated singularity at z_0, all the coefficients of the principle part are zero.
(ii) If f has a pole of order k at z_0, then $c_{-k} \neq 0$ and $c_{-N} = 0$ for all $N > k$.
(iii) If f has an essential singularity at z_0, then there are infinitely many nonzero terms in its principal part.

Proof. (i) If f has an isolated singularity at z_0, then there exists a one-point modification g of f in an appropriate neighborhood of z_0, and g is analytic at z_0. By the uniqueness argument, the Laurent series of f about z_0 equals the Taylor series of g at z_0.

(ii) If f has a pole of order k at z_0, there exists an analytic function $A(z)$ such that $f(z) = \frac{A(z)}{(z-z_0)^k}$ in an appropriate deleted neighborhood of z_0 and such that $A(z_0) \neq 0$. The result follows by plugging in the Taylor series of $A(z)$ at z_0 and uniqueness of the Laurent series.

(iii) If f had an essential singularity at z_0, and there were only finitely many nonzero terms in its principal part, then $(z - z_0)^N f(z)$ would be analytic for large enough N and hence f would have either a removable singularity or a pole at z_0. ☐

We can use the Laurent series to derive a partial fraction decomposition of any rational function.

Theorem 3.12 (Partial fraction decomposition)**.** *Any proper rational function*

$$R(z) = \frac{P(z)}{Q(z)} = \frac{P(z)}{(z - z_1)^{k_1}(z - z_2)^{k_2} \cdots (z - z_n)^{k_n}},$$

where P and Q are polynomials with $\deg P < \deg Q$ and $z_1, \cdots, z_n \in \mathbb{C}$ are mutually distinct, can be written as a sum of polynomials in $\frac{1}{z-z_j}$, where $j = 1, 2, \cdots, n$. That is, there exist polynomials $P_1, \cdots, P_n$ such that

$$R(z) = P_1\left(\frac{1}{z - z_1}\right) + P_2\left(\frac{1}{z - z_2}\right) + \cdots + P_n\left(\frac{1}{z - z_n}\right).$$

Proof. We may assume $P(z_j) \neq 0$ for all $j = 1, \cdots, n$, otherwise we can eliminate the factor $(z - z_j)$ in both the numerator and the denominator. By definition, $R(z)$ has a pole of order k_1 at z_1. Write

$$R(z) = \sum_{\ell=1}^{k_1} \frac{c_{-\ell}}{(z - z_1)^{\ell}} + \sum_{m=0}^{\infty} c_m (z - z_1)^m = P_1\left(\frac{1}{z - z_1}\right) + A_1(z),$$

where the principal part $P_1(\frac{1}{z-z_1})$ of the Laurent series of R about z_1 is a polynomial in $\frac{1}{z-z_1}$ of order k_1, and A_1 is the analytic part. We can repeat this procedure on the analytic parts $A_1, A_2, \cdots$ to obtain

$$R(z) = P_1\left(\frac{1}{z - z_1}\right) + P_2\left(\frac{1}{z - z_2}\right) + \cdots + P_n\left(\frac{1}{z - z_n}\right) + A_n(z).$$

By construction, A_n is entire because it is analytic at all possible singularities $z_1, \cdots, z_n$. Furthermore, A_n is bounded because the rational function $R(z)$ and all the principal parts approach 0 as $z \to \infty$. Then by Liouville's theorem [Theorem 2.15], $A_n \equiv 0$ on $\mathbb{C}$, and

$$R(z) = P_1\left(\frac{1}{z - z_1}\right) + P_2\left(\frac{1}{z - z_2}\right) + \cdots + P_n\left(\frac{1}{z - z_n}\right).$$

This completes the proof. ☐

3.4 The Residue Theorem

In this section, we generalize the Cauchy-Goursat theorem to functions with isolated singularities.

Definition 3.13 (Residues)**.** *Assume that f has an isolated singularity at z_0, and*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n, \quad 0 < |z-z_0| < \delta$$

for some $\delta > 0$, which is the Laurent series of f about z_0. The coefficient c_{-1} of $\frac{1}{z-z_0}$ in the Laurent series is called the residue of f at z_0, and we write $c_{-1} = \operatorname{Res}(f, z_0)$.

Remark. Here are some general methods for computing residues.

(i) If f has a removable singularity at z_0, then $\operatorname{Res}(f, z_0) = 0$.

(ii) If f has a pole at z_0, we may find analytic functions A and B such that $A(z_0) \neq 0$ and $B(z_0) = 0$, and

$$f(z) = \frac{A(z)}{B(z)}, \quad 0 < |z-z_0| < \delta, \ \exists \delta > 0.$$

If f has a simple pole at z_0, then

$$(z-z_0)f(z) = c_{-1} + c_0(z-z_0) + c_1(z-z_0)^2 + c_2(z-z_0)^3 + \cdots$$

is analytic, and

$$\operatorname{Res}(f, z_0) = c_{-1} = \lim_{z\to z_0} (z-z_0)f(z) = \lim_{z\to z_0} \frac{A(z)}{\frac{B(z)-B(z_0)}{z-z_0}} = \frac{A(z_0)}{B'(z_0)}.$$

(iii) More generally, if f has a pole of order k at z_0, then

$$(z-z_0)^k f(z) = \sum_{n=0}^{\infty} c_{n-k}(z-z_0)^n,$$

$$\frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right] = \sum_{n=0}^{\infty} \frac{(n+k-1)!}{n!} c_{n-1}(z-z_0)^n$$

Hence

$$\operatorname{Res}(f, z_0) = c_{-1} = \frac{1}{(k-1)!} \lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right].$$

(iv) In most cases of higher-order poles, as with essential singularities, the most convenient way to determine the residue is directly from the Laurent expansion.

Examples.

- $\operatorname{Res}\left(\dfrac{1}{z^4-1}, \mathrm{i}\right) = \dfrac{1}{4\mathrm{i}^3} = -\dfrac{\mathrm{i}}{4}.$
- $\mathrm{e}^{1/z} = 1 + \dfrac{1}{z} + \dfrac{1}{2!z^2} + \dfrac{1}{3!z^3} + \cdots$, $\operatorname{Res}(\mathrm{e}^{1/z}, 0) = 1.$
- $\dfrac{1}{z-1}\cos\left(\dfrac{1}{z-1}\right) = \dfrac{1}{z-1} - \dfrac{1}{2!(z-1)^3} + \dfrac{1}{4!(z-1)^5} - \cdots$, $\operatorname{Res}\left(\dfrac{1}{z-1}\cos\left(\dfrac{1}{z-1}\right), 1\right) = \dfrac{1}{2}.$
- $\dfrac{\sin z}{z^4} = \dfrac{1}{z^3} - \dfrac{1}{3!z} + \dfrac{z}{5!} - \dfrac{z^3}{7!} + \cdots$, $\operatorname{Res}\left(\dfrac{\sin z}{z^4}, 0\right) = -\dfrac{1}{6}.$

Theorem 3.14 (Cauchy's residue theorem). *Let U be a simply connected open domain. If f is analytic in U except for finitely many isolated singularities $z_1, \cdots, z_k \in U$, and C is a piecewise smooth contour in U, then*

$$\int_C f(z)\, dz = 2\pi \mathrm{i} \sum_{j=1}^{k} n(C, z_j) \operatorname{Res}(f, z_j), \tag{3.3}$$

where for each $j = 1, 2, \cdots, k$,

$$n(C, z_j) = \begin{cases} 1, & if\ z_j\ is\ enclosed\ by\ C, \\ 0, & otherwise. \end{cases}$$

Proof. We subtract the principal parts from the Laurent series of $f(z)$ about $z_1, \cdots, z_k$:

$$f(z) = \underbrace{P\left(\frac{1}{z - z_1}\right) + P\left(\frac{1}{z - z_2}\right) + \cdots + P\left(\frac{1}{z - z_k}\right)}_{principal\ parts} + \underbrace{A(z)}_{analytic\ part}, \quad z \in U \backslash \{z_1, z_2, \cdots, z_k\}.$$

By Cauchy-Goursat theorem [Theorem 2.10],

$$\int_C A(z)\, dz = 0.$$

Note that the principal part series $P\left(\frac{1}{z-z_j}\right) = \sum_{m=1}^{\infty} c_{-m,j}(z - z_j)^{-m}$ about z_j is analytic in U except at z_j. This series converges uniformly outside any open disc centered at z_j. Hence

$$\int_C P\left(\frac{1}{z - z_j}\right) dz = \sum_{m=1}^{\infty} c_{-m,j} \int_C \frac{dz}{(z - z_j)^m}\, dz = \operatorname{Res}(f, z_j) \int_C \frac{dz}{z - z_j},$$

where the terms in the infinite sum with $m \geq 2$ vanish because $(z - z_j)^{-m}$ is the derivative of the holomorphic function $\frac{1}{1-m}(z - z_j)^{1-m}$. Finally, note that

$$\int_C \frac{dz}{z - z_j} = \begin{cases} 2\pi \mathrm{i}, & if\ z_j\ is\ enclosed\ by\ C, \\ 0, & otherwise. \end{cases}$$

We set $n(C, z_j) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{dz}{z - z_j}$. Hence

$$\int_C f(z)\, dz = \sum_{j=1}^{k} P\left(\frac{1}{z - z_j}\right) dz = \sum_{j=1}^{k} \operatorname{Res}(f, z_j) \int_C \frac{dz}{z - z_j} = 2\pi \mathrm{i} \sum_{j=1}^{k} n(C, z_j) \operatorname{Res}(f, z_j).$$

Thus we complete the proof. □

Remark. Let C be a piecewise smooth contour. If f is analytic in an open domain containing C except for finitely many isolated singularities $z_1, \cdots, z_k$, which are enclosed by C,

$$\int_C f(z)\, dz = 2\pi \mathrm{i} \sum_{j=1}^{k} \operatorname{Res}(f, z_j).$$

Using this formula, we transform the integration into the calculus of residues.

3.5 Applications of the Residue Theorem: Zeroes of Analytic Functions

3.6 Applications of the Residue Theorem: Integration

References

[1] John M. Howie (2003). *Complex Analysis.* Springer, New York.

[2] Joseph Bak and Donald J. Newman (2010). *Complex Analysis, 2nd Edition.* Springer, New York.