Lecture Notes for Functional Analysis (MATH130011)

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Contents

0 Notations

m **and** m_d : The Lebesgue measures on $\mathbb{R}$ and $\mathbb{R}^d$

$f \underset{m}{\doteq} g$: $f = g$ almost everywhere in sense of the Lebesgue measure

$f_n \rightrightarrows f$: The sequence (f_n) of functions converges uniformly to f

$C(X)$ **or** $C^0(X)$: The set of all continuous real-valued functions on X

$C^k(X)$: The set of all k-differentiable real-valued functions on X with the k-th derivative continuous

$C^\infty(X)$: The set of all infinitely differentiable real-valued functions on X

$C_0(X)$: The set of all uniformly continuous real-valued functions on X

$C_c(X)$: The set of all compactly supported continuous real-valued functions on X

$\operatorname{supp} f$: The support $\overline{\{f \neq 0\}}$ of f, which is the smallest closed set containing all points not mapped to zero

$\operatorname{ess\,sup}$: The essential supremum

$\amalg$: The union of disjoint sets

1 Metric Spaces

1.1 Metric Spaces

Definition 1.1 (Metric spaces). Let X be a nonempty set. A map $d : X \times X \to \mathbb{R}_+$ is said to be a *metric* on X, if it satisfies the following conditions:

(i) (Positive-definiteness). For each pair of points x, y of X, $d(x, y) \geq 0$; $d(x, y) = 0$ if and only if $x = y$.

(ii) (Symmetry). For each pair of points x, y of X, $d(x, y) = d(y, x)$.

(iii) (Triangle inequality). For any $x, y, z \in X$,

$$d(x, y) + d(y, z) \geq d(x, z).$$

The set together with the metric (X, d) is called a *metric space*.

Remark. The metric $d : X \times X \to \mathbb{R}_+$ is a continuous map. To see this, we fix $\epsilon > 0$ and let $(x_0, y_0) \in X \times X$. Then for all $(x, y) \in O(x_0, \epsilon/3) \times O(y_0, \epsilon/3)$, we have $|d(x, y) - d(x_0, y_0)| \leq d(x, x_0) + d(y, y_0) \leq 2\epsilon/3 < \epsilon$.

Example 1.2. The following are some instances of metric spaces.

(i) On the real line $\mathbb{R}$, define $d(x, y) = |x - y|,\ x, y \in \mathbb{R}$. Then $(\mathbb{R}, d)$ is a metric space.

(ii) On the n-dimensional real space $\mathbb{R}^n$, for two points $\mathbf{x} = (x_1, \cdots, x_n), \mathbf{y} = (y_1, \cdots, y_n)$, define

$$\rho_p(\mathbf{x}, \mathbf{y}) \left(\sum_{i=1}^{n} |x_n - y_n|^p \right)^{1/p}, \ \rho_\infty(\mathbf{x}, \mathbf{y}) = \max_{i \in [n]} |x_n - y_n|.$$

Then for every $1 \leq p < \infty$, $(\mathbb{R}^n, \rho_p)$ is a metric space. To check this, we only need to verify the triangle inequality, which is a special case of the Minkowski's inequality. Also, $(\mathbb{R}, \rho_\infty)$ is a metric space.

(iii) (Discrete space). On a nonempty set X, define the discrete metric

$$d_0(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Then (X, d_0) becomes a metric space called *discrete space*.

(iv) (Subspace). Let (X, d) be a metric space, and let A be a nonempty subset of X. We define on A the *restricted metric* $d_A(x, y) = d(x, y)$ for each pair of points x, y in A. Then (A, d_A) is a metric space, and we call it a *subspace* of X.

(v) Let (X, d) be a metric space. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that (a) f is well-defined on $[0, \infty)$; (b) f is non-decreasing on $[0, \infty)$, strictly increasing at 0, and $f(0) = 0$; and (c) f is concave on $[0, \infty)$, i.e. for all $x, y \in [0, \infty)$ and all $\alpha \in [0, 1]$,

$$f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha) f(y). \tag{1.1}$$

Then the composition

$$d_f(x, y) = f\left(d(x, y)\right), \ x, y \in X$$

is a metric on X. Moreover, it induces the same topology on X as d does.

Proof. To check that d_f is a metric on X, it suffices to show the triangle inequality. Given $x, y, z \in X$, we want to show

$$f(d(x,y)) + f(d(y,z)) \geq f(d(x,z)). \tag{1.2}$$

We show that for all $s, t \geq 0$, $f(s) + f(t) \geq f(s+t)$, which implies (1.2). Without loss of generality, we assume $s, t > 0$. Then

$$\begin{aligned} f(s) + f(t) &= f\left(\frac{s}{s+t} \cdot (s+t) + \frac{t}{s+t} \cdot 0\right) + f\left(\frac{t}{s+t} \cdot (s+t) + \frac{s}{s+t} \cdot 0\right) \\ &\overset{(1.1)}{\geq} \frac{s}{s+t} f(s+t) + \frac{t}{s+t} f(0) + \frac{t}{s+t} f(s+t) + \frac{s}{s+t} f(0) \\ &= f(s+t). \end{aligned}$$

Since f is strictly increasing at 0, there exists some $\delta > 0$ such that f is strictly increasing on $(0, \delta)$. Given x_0 be a point of X, let $O_d(x_0, r) := \{x \in X : d(x, x_0) < r\}$ be the open ball of radius r centered at x_0. When $f(r) < \delta$, we have $O_d(x_0, r) = O_{d_f}(x_0, f(r)) := \{x \in X : d_f(x, x_0) < f(r)\}$.
To show that d_f induces the same topology on X as d does, note that the collection

$$\left\{O_d(x, r) : x \in X, r < f^{-1}\left(\frac{\delta}{2}\right)\right\}$$

is a basis for the topology on X induced by d, which coincides with the basis

$$\left\{O_{d_f}(x, r) : x \in X, r < \frac{\delta}{2}\right\}$$

for the topology induced by f. □

When $f(t) = \min\{t, 1\}$, we obtain the standard bounded metric $\bar{d}(x, y) = \min\{d(x, y), 1\}$ on X.

Definition 1.3 (Limit). Let (X, d) be a metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points of X. Let $p \in X$. If for each $\epsilon > 0$, there exists a positive integer N such that $d(x_n, p) < \epsilon$ for all $n \geq N$, then we say that the sequence $\{x_n\}_{n=1}^{\infty}$ *converges to* p, or that p is the *limit* of $\{x_n\}_{n=1}^{\infty}$. We write $x_n \to p$, or

$$\lim_{n \to \infty} x_n = p.$$

Remark. By definition, convergence in metric space (X, d) equals convergence in the metric topology induced by d. Then if two metrics, for example, d and d_f in Example 1.2 (v), induce the same topology, we can establish the equivalence of convergence in the two corresponding metric spaces. The uniqueness of the limit is ensured by the following lemma.

Lemma 1.4 (The uniqueness of limit). Let (X, d) be a metric space, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points of X. If $x_n \to x$, and $x_n \to y$, then $x = y$.

Proof. By the properties, we have for all $n \in \mathbb{N}$ that

$$0 \leq d(x, y) \leq d(x_n, x) + d(x_n, y).$$

Let $n \to \infty$, we have $d(x, y) = 0$, hence $x = y$. □

Now we introduce the definition of complete metric spaces.

Definition 1.5 (Cauchy sequences and completeness). Let (X, d) be a metric space. A sequence $\{x_n\}_{n=1}^{\infty}$ of points of X is said to be a *Cauchy sequence* if for any $\epsilon > 0$, there exists N such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$. If every Cauchy sequence in (X, d) converges to some point of X, then (X, d) is said to be a *complete metric space*.

Remark. For the metric space $(\mathbb{R}, d)$ where $d(x, y) = |x - y|$, the statement of completeness is in fact the Cauchy's criterion for convergence.

Lemma 1.6. Let (X, d) be a metric space. If (X, d) is complete, A is a closed subspace of X, and d_A is the restricted metric of A, i.e. $d_A(x, y) = d(x, y)\ \forall x, y \in A$, then (A, d_A) is a complete metric space.

Proof. Let (x_n) be a Cauchy sequence in A under d_A. Then (x_n) is also a Cauchy sequence in X under d, and it converges to some $x \in X$. By definition, any neighborhood U of x contains infinitely many points of (x_n). Hence x is a limit point of A. Since A is closed, $x \in A$, and (x_n) converges with respect to d_A. □

Now we introduce a criterion for a metric space to be complete.

Lemma 1.7 (Subsequence criterion). A metric space (X, d) is complete if every Cauchy sequence in X has a convergent subsequence.

Proof. Let (x_n) be a Cauchy sequence in X, and let (x_{n_k}) be a convergent subsequence of (x_n). Fix $\epsilon > 0$. We first choose a positive integer N such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon/2$.

Suppose that the subsequence (x_{n_k}) converges to $x \in X$. We choose a sufficiently large integer K so that $n_K \geq N$ and $k \geq K$ implies $d(x_{n_k}, x) < \epsilon/2$. Then for any $n \geq N$, we have

$$d(x_n, x) \leq d(x_{n_k}, x_n) + d(x_{n_k}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since ϵ is arbitrarily chosen, (x_n) converges to x. □

Example 1.8 (Metrization of pointwise convergence). Let $\mathbb{R}^\infty = \{\mathbf{x} = (x_1, x_2, \cdots) : x_n \in \mathbb{R}\ \forall n \in \mathbb{N}\}$ be the set of all real sequences. We define the metric

$$d(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|},\ \mathbf{x} = (x_1, x_2, \cdots),\ \mathbf{y} = (y_1, y_2, \cdots).$$

Then $(\mathbb{R}^\infty, d)$ is a metric space. Furthermore, convergence of sequence $\left\{\mathbf{x}^{(k)}\right\}$ to $\mathbf{x}$ in metric space (X, d) is equivalent to pointwise convergence (or coordinate-wise convergence), i.e. $\lim_{k\to\infty} x_n^{(k)} = x_n$ for all $n \in \mathbb{N}$.

Proof. "$\Leftarrow$": If $\mathbf{x}^{(k)}$ converges to $\mathbf{x}$ pointwise, then for any $\epsilon > 0$, we choose a positive integer N_ϵ such that

$$N_\epsilon > \frac{1 + \log(1/\epsilon)}{\log 2}.$$

Then we have for all $k \in \mathbb{N}$ that

$$\sum_{n=N_\epsilon+1}^{\infty} \frac{1}{2^n} \cdot \frac{\left|x_n^{(k)} - x\right|}{1 + \left|x_n^{(k)} - x\right|} \leq \sum_{n=N_\epsilon+1}^{\infty} \frac{1}{2^n} < \frac{\epsilon}{2}. \tag{1.3}$$

Moreover, for each $n = 1, \cdots, N_\epsilon$, we can choose K_n such that $|x_n^{(k)} - x_n| < \epsilon/N_\epsilon$ for all $k \geq K_n$. Let K be the largest of K_n. Then for all $k \geq K$, we have

$$\sum_{n=1}^{N_\epsilon} \frac{1}{2^n} \cdot \frac{|x_n^{(k)} - x|}{1 + |x_n^{(k)} - x|} \leq \sum_{n=1}^{N_\epsilon} \frac{1}{2} |x_n^{(k)} - x| < \frac{\epsilon}{2}. \tag{1.4}$$

Combining (1.3) and (1.4), we conclude that $\mathbf{x}^{(k)}$ converges to $\mathbf{x}$ under d.

"$\Rightarrow$": For any $n \in \mathbb{N}$ and sufficiently large k, note that

$$|x_n^{(k)} - x_n| \leq \frac{2^n \cdot d(\mathbf{x}^{(k)}, \mathbf{x})}{1 - 2^n \cdot d(\mathbf{x}^{(k)}, \mathbf{x})} \to 0. \qquad \square$$

Example 1.9 (Metrization of convergence in measure). Let $\mathcal{G}$ be the set of all Lebesgue measurable functions on $[a, b]$ that is bounded almost everywhere. We define an equivalence relation $\sim$ on $\mathcal{G}$ as follows: $f \sim g$ if $f = g$ almost everywhere. Let $G = \mathcal{G}/\sim$. For $f, g \in G$, define

$$d(f, g) = \int_{[a,b]} \frac{|f(t) - g(t)|}{1 + |f(t) - g(t)|} dm(t).$$

Then (G, d) is a metric space. Furthermore, convergence of sequence (f_n) to f in metric space (G, d) is equivalent to convergence in measure, i.e. $m(|f_n - f| \geq \epsilon) \to 0$ for all $\epsilon > 0$.

Proof. "$\Leftarrow$": Given $\epsilon > 0$, define

$$E_n = \left\{ x \in [a, b] : |f_n(x) - f(x)| \geq \frac{\epsilon}{2(b-a)} \right\}. \tag{1.5}$$

Then there exists N such that $m(E_n) < \epsilon/2$ for all $n \geq N$. As a result, for all $n \geq N$, we have

$$\begin{aligned} d(f_n, f) &= \int_{[a,b]\setminus E_n} \frac{|f(t) - g(t)|}{1 + |f(t) - g(t)|} dm(t) + \int_{E_n} \frac{|f(t) - g(t)|}{1 + |f(t) - g(t)|} dm(t) \\ &\leq \int_{[a,b]\setminus E_n} |f(t) - g(t)| dm(t) + \int_{E_n} dm(t) \\ &\leq (b-a) \cdot \frac{\epsilon}{2(b-a)} + m(E_n) < \epsilon. \end{aligned}$$

"$\Rightarrow$": For any $\epsilon > 0$, we have

$$\begin{aligned} m(|f_n - f| \geq \epsilon) &= m\left(\frac{|f_n - f|}{1 + |f_n - f|} \geq \frac{\epsilon}{1+\epsilon} \right) \\ &\leq \frac{1+\epsilon}{\epsilon} \int_{[a,b]} \frac{|f_n - f|}{1 + |f_n - f|} dm \to 0. \end{aligned} \tag{1.6}$$

Hence f_n converges in measure to f. $\square$

1.2 Banach Spaces and Hilbert Spaces

1.2.1 The Hamel Basis

Definition 1.10 (Vector spaces, linearly independent subsets). A *vector space* over a scalar field $\mathbb{F}$ is a non-empty set X together with a binary operation $+ : X \times X \to X$ called *vector addition*, and a binary function $\mathbb{F} \times X \to X$ called *scalar multiplication*. Let x, y, z be any elements of X, and α, β be any scalar in $\mathbb{F}$. A vector space satisfies the following axioms:

(i) (Associativity of vector addition). $(x + y) + z = x + (y + z)$.

(ii) (Commutativity of vector addition). $x + y = y + x$.

(iii) (Identity element of vector addition). There exists an element $\mathbf{0} \in X$ called the *zero vector* such that $x + \mathbf{0} = x$ for all $x \in X$.

(iv) (Inverse elements of vector addition). For each $x \in X$, There exists an element $-x \in X$ called the *additive inverse* of x such that $x + (-x) = 0$.

(v) (Compatibility of scalar multiplication with field multiplication). $\alpha(\beta x) = (\alpha\beta)x$.

(vi) (Identity element of scalar multiplication). $1x = x$, where 1 is the multiplicative identity in $\mathbb{F}$.

(vii) (Distributivity of scalar multiplication with respect to vector addition). $\alpha(x + y) = \alpha x + \alpha y$.

(vii) (Distributivity of scalar multiplication with respect to field addition). $(\alpha + \beta)x = \alpha x + \beta x$.

A finite subset $\{x_1, \cdots, x_n\}$ of X is said to be *linearly independent*, if it satisfies following: $\sum_{i=1}^n \alpha_i x_i = \mathbf{0}$ if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, i.e. there exists no nontrivial linear combination of $x_1, \cdots, x_n$ that equals the zero vector. An infinite subset A of X is said to be *linearly independent*, if every nonempty finite subset of A is linearly independent.

Remark. When the field $\mathbb{F}$ is chosen to be the real field $\mathbb{R}$ (or the complex field $\mathbb{C}$), we say that X is a real vector space (or a complex vector space).

Definition 1.11 (Basis). Let X be a vector space. A collection B of vectors in X is said to be a *basis* of X, if B is linearly independent, and every vector $x \in X$ can be obtained as a linear combination of vectors in B.

A natural question arises: does every vector space has a basis?

Theorem 1.12 (Hamel basis). Let X be a vector space. Let A be a linearly independent subset of X. Then there exists a *maximal linearly independent subset* B of X such that $A \subset B$, and there exists no linearly independent subset of X that includes B properly. Furthermore, B is a basis of X, called a *Hamel basis*.

Proof. We use Zorn's lemma: Suppose a partially ordered set P has the property that every totally ordered subset of P has an upper bound in P, then P has at least one maximal element.

Let $\mathscr{C}$ be the set of all linearly independent subsets of X that contains A. We order the elements of $\mathscr{C}$ by proper inclusion. For any totally ordered subset $\{A_\lambda, \lambda \in \Lambda\}$ of $\mathscr{C}$, where Λ is an index set, the union

$$C = \bigcup_{\lambda \in \Lambda} A_\lambda$$

is an upper bound of $\{A_\lambda, \lambda \in \Lambda\}$. We verify that $C \in \mathscr{C}$. Clearly, $A \subset C$, then we show that C is linearly independent. For any finite subset $\{x_1, \cdots, x_n\}$ of C, there exists $A_{\lambda_i} \ni x_i$ for each i. Since $A_{\lambda_1}, \cdots, A_{\lambda_n}$ are totally ordered, we can find A_{λ_k} that contains all of them. Hence $\{x_1, \cdots, x_n\}$ as a finite subset of the linearly independent subset A_{λ_k} is linearly independent, and C is linearly independent.

By Zorn's lemma, there exists a maximal linearly independent subset B in $\mathscr{C}$, and B is a basis of X. In fact, if B is not a basis for X, we can choose $x \in X$ not lying in the span of B. Then $B \cup \{x\}$ is an linearly independent subset of X, which contradicts the maximality of B! ☐

1.2.2 Normed Spaces and Banach Spaces

Definition 1.13 (Normed spaces). A *seminorm* on a real (or complex) vector space X is a function $\|\cdot\|:X\to\mathbb{R}_+$ satisfying the following conditions:

(i) (Positive semi-definiteness). For all $x\in X$, $\|x\|\geq 0$;

(ii) (Homogeneity). For all $\alpha\in\mathbb{R}$ (or $\mathbb{C}$) and all $x\in X$, $\|\alpha x\|=|\alpha|\|x\|$;

(iii) (Triangle inequality). For all $x,y\in X$, $\|x+y\|\leq\|x\|+\|y\|$.

A norm on X is a seminorm $\|\cdot\|$ that satisfies the following: $\|x\|=0$ only if $x=0$. A vector space together with a norm $(X,\|\cdot\|)$ is called a *normed vector space*, or briefly, a *normed space*.

Remark. A norm $\|\cdot\|$ on a vector space X automatically induces a metric on X defined as $d(x,y)=\|x-y\|$. By equipping a norm, we introduce a topological structure to a vector space, which is an algebraic structure.

The norm $\|\cdot\|$ is a continuous map in space $(X,\|\cdot\|)$, which is implied by the triangle inequality. To see this, fix $\epsilon>0$ and $x_0\in X$. Then for all $x\in O(x_0,\epsilon)$, we have

$$|\|x\|-\|x_0\||\leq\|x-x_0\|<\epsilon,$$

which meets the definition of continuity.

Example 1.14. Following are some instances for normed spaces.

(i) Let $C([a,b])$ be the set of all real-valued continuous functions on $[a,b]$. Define

$$\|f\|_\infty=\max_{x\in[a,b]}|f(x)|,\ f\in C([a,b]).$$

Then $(C([a,b]),\|\cdot\|_\infty)$ is a normed space.

(ii) Let k be a positive integer. Let $C^k([a,b])$ be the set of all functions f on $[a,b]$ such that f is k-differentiable, and the k-th derivative $f^{(k)}$ is continuous. Define

$$\|f\|_{k,\infty}=\max_{0\leq j\leq k}\max_{x\in[a,b]}|f^{(j)}(x)|,\ f\in C^k([a,b]).$$

Then $(C([a,b]),\|\cdot\|_\infty)$ is a normed space.

(iii) Let $(X,\mathscr{A},\mu)$ be a measurable space. For $1<p<\infty$, define $\mathcal{L}^p(X,\mathscr{A},\mu)$ to be the set of all measurable functions f such that $|f|^p$ is integrable, i.e. $\int_X|f|^pd\mu<\infty$. We define

$$\|f\|_p=\left(\int_X|f|^pd\mu\right)^{1/p},\ f\in\mathcal{L}^p(X,\mathscr{A},\mu).$$

Then $\|\cdot\|_p$ is a seminorm on $\mathcal{L}^p(X,\mathscr{A},\mu)$. To check this, it suffices to prove the following two inequalities.

- (Hölder's inquality). For all $p,q>1$ with $p^{-1}+q^{-1}=1$, it holds

$$\int|fg|\,d\mu\leq\|f\|_p\|g\|_q.$$

Proof. Without loss of generality, suppose $\|f\|_p=\|g\|_q=1$. We use Young's inequality:

$$\underbrace{\log\left(\frac{a^p}{p}+\frac{b^q}{q}\right)\geq\frac{1}{p}\log(a^p)+\frac{1}{q}\log(b^q)}_{\text{Concavity of the logarithmic function}}\ \Rightarrow\ ab\leq\frac{a^p}{p}+\frac{b^q}{q}.$$

Then we have

$$|f(x)g(x)| \le \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q} \overset{\text{Integration}}{\Rightarrow} \int |fg|\, d\mu \le \frac{\|f\|^p}{p} + \frac{\|g\|^q}{q} = 1,$$

which concludes the proof. $\square$

- (Minkowski's inquality). For all $p \ge 1$, we have

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

Proof. We only prove the case $p > 1$. Let $q = \frac{p}{p-1}$, then

$$\begin{aligned}
\|f+g\|_p^p &\le \int_X |f| \cdot |f+g|^{p-1}\, d\mu + \int_X |g| \cdot |f+g|^{p-1}\, d\mu \\
&\le (\|f\|_p + \|g\|_p) \left(\int_X |f+g|^{(p-1)q} d\mu \right)^{1/q} && \text{(By Hölder's inquality)} \\
&\le (\|f\|_p + \|g\|_p) \cdot \|f+g\|_p^{p/q}.
\end{aligned}$$

Note that $p - p/q = 1$, then we conclude the proof. $\square$

(iv) Let $f \sim g \overset{\text{def.}}{\Leftrightarrow} f \underset{\mu}{\doteq} g$ be a equivalence relation on $\mathscr{L}^p(X, \mathscr{A}, \mu)$. We define the L^p space as $L^p(X, \mathscr{A}, \mu) = \mathscr{L}^p(X, \mathscr{A}, \mu)/\sim$, and maintain the norm $\|[f]\|_p = \|f\|_p$. This is a well-defined norm, since $\|f\|_p = \|g\|_p$ if $f \sim g$. For simplicity, we drop the brackets and use f to denote its corresponding equivalence class $[f]$ in $L^p(X, \mathscr{A}, \mu)$. Then the space $(L^p(X, \mathscr{A}, \mu), \|\cdot\|_p)$ is a normed space.

(v) Let $p = \infty$ in (ii), then we obtain the set of *essentially bounded* functions on X, which is

$$\mathscr{L}^\infty(X, \mathscr{A}, \mu) = \{f : X \to \mathbb{R} \mid \exists M > 0,\ \mu(|f| \ge M) = 0\}.$$

The seminorm $\|\cdot\|_\infty$ on $\mathscr{L}^\infty(X)$ is the essential supremum:

$$\|f\|_\infty = \operatorname{ess\,sup} |f| := \inf_{\mu(E)=0} \sup_{x \in X \setminus E} |f(x)|.$$

Also, we define $L^\infty(X, \mathscr{A}, \mu) = \mathscr{L}^\infty(X, \mathscr{A}, \mu)/\sim$. Then $(L^\infty(X, \mathscr{A}, \mu), \|\cdot\|_\infty)$ is a normed space.

Definition 1.15 (Banach spaces). Let $(X, \|\cdot\|)$ be a normed space. If X is complete given the metric induced by $\|\cdot\|$, then $(X, \|\cdot\|)$ is said to be a *Banach space*.

Remark. A Banach space is a complete normed space. Let (x_n) be a Cauchy sequence in a Banach space $(X, \|\cdot\|)$, i.e. $\|x_n - x_m\| \to 0$ as $n, m \to \infty$, then (x_n) converges to some point of X.

As a result of Lemma 1.6, a closed subspace A of a Banach space $(X, \|\cdot\|)$ is also a Banach space under the restricted norm. Note that when we use the term "subspace" in discussions of vector spaces, we refer to a vector subspace.

Following are some instances of Banach spaces.

Example 1.16. Recall Example 1.14 (i) and (ii).

(i) The normed space $(C([a,b]), \|\cdot\|_\infty)$ is a Banach space;

(ii) For each $k \in \mathbb{N}$, the normed space $(C^k([a,b]), \|\cdot\|_{k,\infty})$ is a Banach space.

Proof. (i) We pick a Cauchy sequence f_n in $C([a,b])$, i.e. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \epsilon$ for all $n, m \geq N$. Then for each $x \in [a,b]$, $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$, which converges to some $f(x) \in \mathbb{R}$ by completeness of real numbers. Thus we obtain a function f on $[a,b]$.

Now we prove that f is continuous. Fix $\epsilon > 0$. Then for all $x \in X$, we have

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \epsilon, \ \forall n, m \geq N.$$

Let $m \to \infty$, then we get $|f_n(x) - f(x)| \leq \epsilon$ for all $x \in X$. Hence f_n converges to f uniformly. Since f_n is continuous on $[a,b]$, so is f.

(ii) We first prove the case $k = 1$. We pick a Cauchy sequence f_n in $C^1([a,b])$. Then both f_n and f_n' are Cauchy sequences in $(C([a,b]), \|\cdot\|_\infty)$, which converge uniformly to some continuous functions f and g, respectively, by (i). We need to show that f is differentiable, and that g is the derivative of f.

By fundamental theorem of calculus, we have

$$f_n(x) - f_n(a) = \int_a^x f_n'(t)\,dt, \ \forall n \in \mathbb{N}.$$

Let $\epsilon > 0$ be given. Since f_n' converges to g uniformly on $[a,b]$, there exists N such that $|f_n'(x) - g(x)| < \epsilon$ for all $n \geq N$ and $x \in [a,b]$. Hence

$$\left| \int_a^x f_n'(t)\,dt - \int_a^x g(t)\,dt \right| \leq \int_a^x |f_n'(t) - g(t)|\,dt \leq \epsilon(x-a), \ \forall n \geq N.$$

Hence $\int_a^x f_n'(t)\,dt \to \int_a^x g(t)\,dt$. As a result,

$$f(x) - f(a) = \lim_{n\to\infty} (f_n(x) - f_n(a)) = \lim_{n\to\infty} \int_a^x f_n'(t)\,dt = \int_a^x g(t)\,dt,$$

which implies $f' = g$.

For the general case $k \in \mathbb{N}$, let f_n be a Cauchy sequence in $C^k([a,b])$. Similar to the above procedure, we can show that the sequence $f_n^{(j)}$ converges to some continuous function uniformly on $[a,b]$ for each $j = 0, 1, \cdots, k$, and that $\lim_{n\to\infty} f_n^{(j)}$ is the derivative of $\lim_{n\to\infty} f_n^{(j-1)}$. ☐

Example 1.17 (Riesz-Fisher). Let $(X, \mathscr{A}, \mu)$ be measure space. Then for each $1 \leq p < \infty$, the space $(L^p(X, \mathscr{A}, \mu), \|\cdot\|_p)$ is a Banach space.

Proof. We pick a Cauchy sequence f_n in $L^p(X, \mathscr{A}, \mu)$, i.e. $\forall \epsilon > 0$, $\exists N$ such that $\|f_n - f_m\|_p < \epsilon$ for all $n, m \geq N$. By Chebyshev's inequality, for any $\eta > 0$, we have

$$\mu\left(|f_n - f_m| \geq \eta\right) \leq \frac{1}{\eta^p} \int_X |f_n - f_m|^p d\mu = \frac{1}{\eta^p} \|f_n - f_m\|_p^p.$$

Hence f_n is a Cauchy sequence in measure. Starting from $k = 1$, we choose an integer $n_k > n_{k-1}$ such that $\mu(|f_n - f_m| \geq 2^{-k}) < 2^{-k}$ for all $n, m \geq n_k$. Then we obtain a subsequence f_{n_k} such that

$$\mu(E_k) < 2^{-k}, \text{ where } E_k = \left\{ |f_{n_{k+1}} - f_{n_k}| \geq 2^{-k} \right\}.$$

Let $F_N = \bigcup_{k=N}^\infty E_k$, and $E = \bigcap_{N=1}^\infty F_N$. Then $\mu(F_N) < 2^{-N+1}$, and $\mu(E) = 0$. For every $x \in X \backslash E$, there exists N such that $x \notin F_N$. Then for all $k > N$, $|f_{n_{k+1}}(x) - f_{n_k}(x)| < 2^{-k}$. and $|f_{n_l}(x) - f_{n_k}(x)| < 2^{-k+1}$ for all $l > k > N$. Hence $f_{n_k}(x)$ is a Cauchy sequence, which converges to some $f(x) \in \mathbb{R}$. Define $f(E) = \{0\}$, then the subsequence f_{n_k} converges to f almost everywhere.

Fix $\epsilon > 0$, and find N such that $\|f_{n_k} - f_m\|_p < \epsilon$ for all $n_k, m \geq N$. Given $m \geq N$, apply Fatou's lemma:

$$\int_X |f - f_m|^p \, d\mu = \int_X \lim_{k\to\infty} |f_{n_k} - f_m|^p \, d\mu \leq \liminf_{k\to\infty} \|f_{n_k} - f_m\|_p^p < \epsilon^p. \tag{1.7}$$

Hence $f - f_m \in L^p(X, \mathscr{A}, \mu)$, and $f = (f - f_m) + f_m \in L^p(X, \mathscr{A}, \mu)$. Furthermore, since ϵ is arbitrary, we have $\|f - f_n\|_p^p \to 0$, i.e. f is the limit of f_n in $L^p(X, \mathscr{A}, \mu)$. $\square$

Remark. In this example, we also prove that every Cauchy sequence f_n in measure has a subsequence f_{n_k} that converges almost everywhere. In fact, we can prove that f_n converges in measure. In the above proof, we have for all $x \in X \backslash F_k$ that

$$|f(x) - f_{n_k}(x)| \leq \sum_{j=k}^{\infty} |f_{n_{j+1}}(x) - f_{n_j}(x)| < 2^{-k+1}.$$

Fix $\epsilon > 0$, and choose N such that $2^{-N+1} \leq \epsilon$. Then for all $k \geq N$, we have

$$\mu(|f_{n_k} - f| \geq \epsilon) \leq \mu(|f_{n_k} - f| \geq 2^{-k+1}) \leq \mu(F_k) < 2^{-k+1} \to 0.$$

Hence f_{n_k} converges in measure to f. Now given $\eta > 0$ and $\epsilon > 0$, choose K such that $\mu(|f_{n_k} - f| \geq \eta/2) < \epsilon/2$ for all $k \geq K$, and N such that $\mu(|f_n - f_m| \geq \eta/2) < \epsilon/2$ for all $n, m \geq N$. Then for all $n \geq \max\{n_K, N\}$, choose k such that $n_k \geq n$, we have

$$\begin{aligned} \mu(|f_n - f| \geq \eta) &\leq \mu(|f_{n_k} - f_n| + |f_{n_k} - f| \geq \eta) \\ &\leq \mu\left(|f_{n_k} - f_n| \geq \frac{\eta}{2}\right) + \mu\left(|f_{n_k} - f| \geq \frac{\eta}{2}\right) < \epsilon. \end{aligned}$$

Therefore f_n converges in measure.

Example 1.18. Let $(X, \mathscr{A}, \mu)$ be measure space. Then the space $(L^\infty(X, \mathscr{A}, \mu), \|\cdot\|_\infty)$ is a Banach space.

Proof. We pick a Cauchy sequence f_n in $L^\infty(X, \mathscr{A}, \mu)$, i.e. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \epsilon$ for all $n, m \geq N$. Now for each pair $m, n \in \mathbb{N}$, we define the set $E_{m,n}$ of measure zero as

$$E_{m,n} = \{x \in X : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}, \ \mu(E_{m,n}) = 0.$$

Then the union

$$E = \bigcup_{n,m\in\mathbb{N}} E_{m,n} \quad \text{of countably many sets of measure zero also has measure zero.}$$

For each $x \in X \backslash E$, $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$, $f_n(x)$ is a Cauchy sequence in $\mathbb{R}$, which converges to some $f(x) = \lim_{n\to\infty} f_n(x) \in \mathbb{R}$. Now fix $\epsilon > 0$. By defining $f(E) = \{0\}$, we obtain a function on $f : X \to \mathbb{R}$.

Now fix $\epsilon > 0$, and choose N such that $\|f_n - f_m\|_\infty < \epsilon$ for all $n, m \geq N$. Then for all $x \in X \backslash E$, it holds

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \epsilon, \ \forall n, m \geq N$$

Let $m \to \infty$, we have $|f_n(x) - f(x)| \leq \epsilon$ for all $x \in X \backslash E$ and $n \geq N$. Then $\|f_n - f\|_\infty \leq \epsilon$. Moreover,

$$\sup_{x\in X\backslash E} |f(x)| \leq \sup_{x\in X\backslash E} |f(x) - f_n(x)| + \sup_{x\in X\backslash E} |f_n(x)| \leq \epsilon + \|f_n\|_\infty < \infty.$$

Hence $f \in L^\infty(X, \mathscr{A}, \mu)$. Since ϵ is arbitrary, $\|f_n - f\|_\infty \to 0$, and f_n converges to f in $L^\infty(X, \mathscr{A}, \mu)$. $\square$

1.2.3 Inner Product Spaces and Hilbert Spaces

Definition 1.19 (Inner product spaces). Let H be a real (or complex) vector space. A *semi-inner product* on H is defined as a function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ (or $\mathbb{C}$) satisfying the following conditions:

(i) (Positive semi-definiteness). $\langle x, x \rangle \geq 0$ for all $x \in H$;

(ii) (Linearity for the first variable). For all $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$) and all $x, y, z \in H$,

$$\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle;$$

(iii) (Conjugate symmetry). $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in H$.

Furthermore, if $\langle \cdot, \cdot \rangle$ satisfies positive-definiteness, i.e. $\langle x, x \rangle = 0$ only if $x = 0$, then it becomes an *inner product* on H. A real (or complex) vector space H equipped with an inner product $\langle \cdot, \cdot \rangle$ is called a real (or complex) *inner product space*, or a *pre-Hilbert space*.

Remark. If H is a real inner product space, we can drop the conjugate in (iii) and obtain the linearity for both variables. If H is complex, by (ii) and (iii), we have anti-linearity for the second variable:

$$\langle z, \alpha x + \beta y \rangle = \overline{\alpha} \langle z, x \rangle + \overline{\beta} \langle z, y \rangle.$$

Example 1.20. Following are some instances for inner product spaces.

(i) Let $\mathbf{x} = (x_1, \cdots, x_n), \mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{C}^n$. Define

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{n} x_j \overline{y}_j.$$

Then $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{C}^n$.

(ii) Let $(X, \mathscr{A}, \mu)$ be a measure space. For $f, g \in L^2(X, \mathscr{A}, \mu)$, define

$$\langle f, g \rangle = \int_X f(x) \overline{g(x)} \, d\mu(x).$$

Then $\langle \cdot, \cdot \rangle$ is an inner product on $(X, \mathscr{A}, \mu)$.

Lemma 1.21 (Cauchy-Schwarz inequality). Let $\langle \cdot, \cdot \rangle$ be a semi-inner product on a vector space H. Then for all $x, y \in H$, it holds

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle.$$

Proof. Let $x, y \in H$. Then for all $t \in \mathbb{R}$ (or $\mathbb{C}$),

$$0 \leq \langle x + ty, x + ty \rangle = \langle x, x \rangle + 2\mathrm{Re}(t \langle y, x \rangle) + |t|^2 \langle y, y \rangle.$$

If $\langle y, y \rangle \neq 0$, set $t = -\frac{\langle x, y \rangle}{\langle y, y \rangle}$. Then

$$\langle x, x \rangle - 2 \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} + \frac{|\langle x, y \rangle|^2}{\langle y, y \rangle} \geq 0 \;\Rightarrow\; |\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle.$$

If $\langle y, y \rangle \geq 0$, set $t = -\frac{1}{2} \beta \langle x, y \rangle$, where $\beta > 0$. Then

$$\langle x, x \rangle - \beta |\langle y, x \rangle|^2 \geq 0, \; \forall \beta > 0,$$

which implies $\langle x, y \rangle = 0$. Since x is arbitrary, we have $\langle x, y \rangle = 0$ for all $x \in H$. $\square$

Lemma 1.22 (Induced norm). Let $\langle \cdot, \cdot \rangle$ be an inner product on H. Define $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in H$, then $\| \cdot \|$ is a norm on H.

Proof. Check the four properties in Definition 1.13. $\square$

Remark. Following Lemma 1.22, we can rewrite Cauchy-Schwarz inequality (Lemma 1.21) as

$$|\langle x, y \rangle| \le \|x\| \, \|y\| \, .$$

Using this inequality, we can obtain continuity of inner products.

Lemma 1.23 (Continuity of inner products). Let $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ (or $\mathbb{C}$) be an inner product on H. Then $\langle \cdot, \cdot \rangle$ is a continuous map.

Proof. Let (x_n) and (y_n) be sequences of points of H that converge to $x \in H$ and $y \in H$, respectively. Then

$$\begin{aligned} |\langle x_n, y_n \rangle - \langle x, y \rangle| &\le |\langle x_n, y_n \rangle - \langle x, y_n \rangle| + |\langle x, y_n \rangle - \langle x, y \rangle| \\ &\le \|x_n - x\| \, \|y_n\| + \|x\| \, \|y_n - y\| \to 0. \end{aligned}$$

Thus we complete the proof. $\square$

It is seen that in a vector space, an inner product automatically determines a norm. Conversely, if a norm is induced by an inner product, we can also recover the inner product from the norm.

Lemma 1.24 (Polarization identity). Let H be an inner product space.

(i) If H is real, then for all $x, y \in H$,

$$\langle x, y \rangle = \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 \right); \tag{1.8}$$

(ii) If H is complex, then for all $x, y \in H$,

$$\langle x, y \rangle = \frac{1}{4} \left(\|x + y\|^2 + \mathrm{i}\|x + \mathrm{i}y\|^2 - \|x - y\|^2 - \mathrm{i}\|x - \mathrm{i}y\|^2 \right) = \frac{1}{4} \sum_{k=0}^{3} \mathrm{i}^k \|x + \mathrm{i}^k y\|^2. \tag{1.9}$$

Proof. By direct calculation. $\square$

We also introduce a necessary and sufficient condition for a norm to be induced by an inner product.

Lemma 1.25 (Parallelogram law). Let $(X, \| \cdot \|)$ be a normed space. Then $\| \cdot \|$ is induced by an inner product on X if and only if the parallelogram law holds for $\| \cdot \|$:

$$\|x + y\|^2 + \|x - y\|^2 = 2 \left(\|x\|^2 + \|y\|^2 \right). \tag{1.10}$$

Proof. "$\Rightarrow$": By direct calculation.

"$\Leftarrow$": We use the polarization identity (Lemma 1.24) to define a binary operation $\langle \cdot, \cdot \rangle$ on X, and verify that $\langle \cdot, \cdot \rangle$ is an inner product. We work with the complex case, and define $\langle \cdot, \cdot \rangle$ by (1.9). Let $x, y \in X$. Then we can obtain positive definiteness and conjugate symmetry:

$$\langle x, x \rangle = \frac{1}{4} \left(\|2x\|^2 + \mathrm{i}\|(1 + \mathrm{i})x\|^2 - \|0x\|^2 - \mathrm{i}\|(1 - \mathrm{i})x\|^2 \right) = \|x\|^2,$$

$$
\begin{aligned}
\overline{\langle y, x\rangle} &= \frac{1}{4}\left(\|y+x\|^2 - \mathrm{i}\|y+\mathrm{i}x\|^2 - \|y-x\|^2 + \mathrm{i}\|y-\mathrm{i}x\|^2\right) \\
&= \frac{1}{4}\left(\|x+y\|^2 - \mathrm{i}\|x-\mathrm{i}y\|^2 - \|x-y\|^2 + \mathrm{i}\|x+\mathrm{i}y\|^2\right) = \langle x, y\rangle.
\end{aligned}
$$

Now we verify the additivity. For $x, y, z \in X$, by (1.10), we have

$$
\begin{aligned}
\|x+y+z\|^2 &= \frac{1}{2}\left(\|(x+z)+y\|^2 + \|(y+z)+x\|^2\right) \\
&= \|x+z\|^2 + \|y\|^2 - \frac{1}{2}\|x+z-y\|^2 + \|y+z\|^2 + \|x\|^2 - \frac{1}{2}\|y+z-x\|^2. \qquad (1.11)
\end{aligned}
$$

Replace z by $-z$ in (1.11), then we have

$$
\|x+y+z\|^2 - \|x+y-z\|^2 = \|x+z\|^2 - \|x-z\|^2 - \|y+z\|^2 - \|y-z\|^2 \tag{1.12}
$$

Replace z by $\mathrm{i}z$ in (1.12), then we have

$$
\|x+y+\mathrm{i}z\|^2 - \|x+y-\mathrm{i}z\|^2 = \|x+\mathrm{i}z\|^2 - \|x-\mathrm{i}z\|^2 - \|y+\mathrm{i}z\|^2 - \|y-\mathrm{i}z\|^2 \tag{1.13}
$$

Combining (1.12) and (1.13), we obtain

$$
\begin{aligned}
\langle x+y, z\rangle &= \frac{1}{4}\left(\|x+y+z\|^2 - \|x+y-z\|^2 + \mathrm{i}\|x+y+\mathrm{i}z\|^2 - \mathrm{i}\|x+y-\mathrm{i}z\|^2\right) \\
&= \frac{1}{4}\sum_{k=1}^{3}\mathrm{i}^k\|x+\mathrm{i}^k z\|^2 + \frac{1}{4}\sum_{k=1}^{3}\mathrm{i}^k\|y+\mathrm{i}^k z\|^2 = \langle x, z\rangle + \langle y, z\rangle.
\end{aligned}
$$

Now it remains to show the scalar multiplicativity. Given the additivity, we have that for every $n, m \in \mathbb{N}$,

$$
\langle nx, z\rangle = \underbrace{\langle x, z\rangle + \cdots + \langle x, z\rangle}_{n} = n\langle x, z\rangle \;\Rightarrow\; \langle m^{-1}x, z\rangle = \frac{1}{m}\langle x, z\rangle \;\Rightarrow\; \left\langle \frac{n}{m}x, z\right\rangle = \frac{n}{m}\langle x, z\rangle .
$$

Clearly, we have $\langle \mathrm{i}x, z\rangle = \mathrm{i}\langle x, z\rangle$. Then for every $\lambda \in \mathbb{Q} + \mathrm{i}\mathbb{Q} = \{p + \mathrm{i}q : p, q \in \mathbb{Q}\}$, we have $\langle \lambda x, z\rangle = \lambda\langle x, z\rangle$.

Next we prove the Cauchy-Schwarz inequality. For $x, z \in X$ and $\lambda \in \mathbb{Q} + \mathrm{i}\mathbb{Q}$,

$$
\begin{aligned}
0 \le \langle x+\lambda z, x+\lambda z\rangle\langle z, z\rangle &= \|x\|^2\|z\|^2 + 2\mathrm{Re}\left(\lambda\langle z, x\rangle\|z\|^2\right) + |\lambda|^2\|z\|^4 \\
&= \|x\|^2\|z\|^2 - |\langle x, z\rangle|^2 + \left|\lambda\|z\|^2 - \langle x, z\rangle\right|^2,
\end{aligned}
$$

which implies

$$
|\langle x, z\rangle|^2 - \|x\|^2\|z\|^2 \le \inf_{\lambda \in \mathbb{Q}+\mathrm{i}\mathbb{Q}} \left|\lambda\|z\|^2 - \langle x, z\rangle\right|^2 = 0. \tag{1.14}
$$

Now fix $\alpha \in \mathbb{C} = \mathbb{R} + \mathrm{i}\mathbb{R}$. For all $\lambda \in \mathbb{Q} + \mathrm{i}\mathbb{Q}$, we have

$$
|\langle \alpha x, z\rangle - \alpha\langle x, z\rangle| = |\langle(\alpha-\lambda)x, z\rangle - (\alpha-\lambda)\langle x, z\rangle| \le 2\,|\alpha-\lambda|\,\|x\|\,\|z\|,
$$

where the inequality follows from (1.14). By taking infimum of the right hand side, which is zero, we have $\langle \alpha x, z\rangle = \alpha\langle x, z\rangle$. Then we complete the proof. □

Review. Let H be an inner product space, then we obtain a norm $\|\cdot\|$ on H by defining $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in H$. Following this, a metric d is determined by $d(x, y) = \|x - y\|$ for all $x, y \in H$. This metric automatically induces a metric topology on H for which the basis is the collection of all open balls in H.

Definition 1.26 (Hilbert spaces). Let H be an inner product space. If H is complete under the metric induced by its inner product, then H is said to be a *Hilbert space*.

Remark. In other words, a complete inner product space is Hilbert. That is, every Cauchy sequence in H, in sense of the induced norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$, converges in H.

Now we introduce the definition of orthogonality in inner product spaces.

Definition 1.27 (Orthogonality). Let H be a inner product space.

(i) Let x and y be two vectors in H. Then x is said to be *orthogonal to y* if $\langle x, y \rangle = 0$, and we write $x \perp y$. By direct calculation, we have the Pythagorean theorem for $x \perp y$:

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

(ii) Let $\mathscr{H}$ be a collection of non-zero vectors in H. If for each pair of distinct vectors $x \neq y$ in $\mathscr{H}$, we have $x \perp y$, then $\mathscr{H}$ is said to be an *orthogonal system*.

(iii) Furthermore, if $\|x\| = 1$ for all $x \in \mathscr{H}$, then $\mathscr{H}$ is said to be an *orthonormal system*.

(iv) Let $\mathscr{H}$ be an orthonormal system in H. Then the set of numbers

$$\{\langle x, e \rangle, e \in \mathscr{H}\}$$

is said to be the *Fourier coefficients* of x relative to $\mathscr{H}$. If $e \in \mathscr{H}$, then $\langle x, e \rangle$ is called the Fourier coefficient of x relative to e.

Example 1.28 Following are some examples of orthogonal families.

(i) Consider the n-dimensional Euclidean space $\mathbb{R}^n$. The vectors

$$e_1 = (1, 0, \cdots, 0),\ e_2 = (0, 1, \cdots, 0),\ \cdots,\ e_n = (0, 0, \cdots, 1)$$

form an orthonormal system on $\mathbb{R}^n$.

(ii) Consider the space $L^2([0, 2\pi])$ of real-valued square-integrable functions on $[0, 2\pi]$. Define inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x)g(x)\,dx,\ f, g \in L^2([0, 2\pi]).$$

The functions

$$\left\{1, \sqrt{2}\cos x, \sqrt{2}\sin x, \sqrt{2}\cos 2x, \sqrt{2}\sin 2x, \cdots, \sqrt{2}\cos nx, \sqrt{2}\sin nx, \cdots\right\}$$

form an orthonormal system on $L^2([0, 2\pi])$. Furthermore, for a function $f \in L^2([0, 2\pi])$, the Fourier coefficients are

$$\begin{aligned}
a_0 &= \langle f, 1 \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\,dt, \\
a_n &= \langle f, \sqrt{2}\cos nx \rangle = \frac{1}{\sqrt{2}\pi} \int_0^{2\pi} f(t)\cos nx\,dt,\ n \geq 1, \\
b_n &= \langle f, \sqrt{2}\sin nx \rangle = \frac{1}{\sqrt{2}\pi} \int_0^{2\pi} f(t)\sin nx\,dt,\ n \geq 1.
\end{aligned}$$

(iii) Consider the space $L^2([0, 2\pi], \mathbb{C})$ of complex-valued square-integrable functions on $[0, 2\pi]$. Define inner product

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)g(x)\,dx,\ \ f, g \in L^2([0, 2\pi], \mathbb{C}).$$

The functions $\{\mathrm{e}^{\mathrm{i}nx}, n \in \mathbb{Z}\}$ form an orthonormal basis on $L^2([0, 2\pi], \mathbb{C})$. Furthermore, the Fourier coefficients of $f \in L^2([0, 2\pi], \mathbb{C})$ are

$$c_n = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}nx} f(x)\,dx,\ \ n \in \mathbb{Z}.$$

Review: Summation over arbitrary index sets. Let Λ be an index set, and let $\{c_\lambda : \lambda \in \Lambda\}$ be a collection such that $c_\lambda \geq 0$ for all $\lambda \geq \Lambda$. We pick a set $\mathscr{F}(\Lambda) = \{F \subset \Lambda : F \text{ is finite}\}$, and we define a preorder on $\mathscr{F}$ by inclusion: $F_1 \preceq F_2 \overset{\text{def.}}{\Leftrightarrow} F_1 \subset F_2$. Then $\mathscr{F}$ becomes a directed set since every pair F_1, F_2 of elements of $\mathscr{F}$ has an upper bound $F_1 \cup F_2 \in \mathscr{F}$. The general definition of the summation $\sum_{\lambda\in\Lambda} c_\lambda$ is given by the following limit, provided it exists:

$$\sum_{\lambda\in\Lambda} c_\lambda = \lim_{F\in\mathscr{F}(\Lambda)} \sum_{\lambda\in F} c_\lambda.$$

That is,

$$\sum_{\lambda\in\Lambda} c_\lambda = c \ \Leftrightarrow\ \forall\epsilon > 0,\ \exists F_0 \in \mathscr{F}(\Lambda) \text{ such that } \forall F \in \mathscr{F}(\Lambda) \text{ and } F \supset F_0,\ \left|\sum_{\lambda\in F} c_\lambda - c\right| \leq \epsilon.$$

Claim 1.29. If $\sum_{\lambda\in\Lambda} c_\lambda$ converges, then $\{c_\lambda : \lambda \in \Lambda\}$ has at most countably many non-zeros.

Proof. Let $\sum_{\lambda\in\Lambda} c_\lambda = c$. For all $n \in \mathbb{N}$, consider the set

$$F_n := \left\{\lambda \in \Lambda : c_\lambda \geq \frac{1}{n}\right\},$$

Then $F_1 \subset F_2 \subset \cdots \subset F_n \subset F_{n+1} \subset \cdots$ form a chain on $\mathscr{F}$, and $\sum_{\lambda\in F_n} c_\lambda$ is increasing. Moreover,

$$c \geq \sum_{\lambda\in F_n} c_\lambda \geq \frac{1}{n}|F_n| \ \Rightarrow\ |F_n| \leq nc < \infty.$$

Note that the set of all non-zero elements is given by

$$\{\lambda \in \Lambda : c_\lambda \neq 0\} = \bigcup_{n=1}^{\infty} F_n,$$

which is at most countable. □

Theorem 1.29 (Bessel's inequality). Let H be an inner product space, and let $\mathscr{H} = \{e_\lambda : \lambda \in \Lambda\}$ be an orthonormal system on H. Then for all $x \in H$,

$$\sum_{\lambda\in\Lambda} |\langle x, e_\lambda\rangle|^2 \leq \|x\|^2.$$

Proof. Let F be a finite subset of Λ. Consider

$$x = \underbrace{\left(x - \sum_{\lambda \in F} \langle x, e_\lambda \rangle e_\lambda \right)}_{=:y} + \underbrace{\sum_{\lambda \in F} \langle x, e_\lambda \rangle e_\lambda}_{=:z},$$

we have

$$\langle y, z \rangle = \sum_{\lambda \in F} |\langle x, e_\lambda \rangle|^2 - \sum_{\lambda \in F} \sum_{\nu \in F} \langle x, e_\lambda \rangle \overline{\langle x, e_\nu \rangle} \underbrace{\langle e_\lambda, e_\nu \rangle}_{=\delta_{\lambda\nu}} = 0.$$

By Pythagorean theorem, $\|x\|^2 = \|y\|^2 + \|z\|^2 \geq \|z\|^2$, that is,

$$\sum_{\lambda \in F} |\langle x, e_\lambda \rangle|^2 \leq \|x\|^2.$$

By Claim 1.29, the set $F_n = \left\{ \lambda \in \Lambda : |\langle x, e_\lambda \rangle| \geq n^{-1} \right\}$ has no more than $n^2 \|x\|^2$ elements, and the set of nonzero Fourier coefficients $F_\infty = \bigcup_{n=1}^\infty F_n = \{\lambda \in \Lambda : \langle x, e_\lambda \rangle \neq 0\}$ is at most countable. Hence

$$\sum_{\lambda \in \Lambda} |\langle x, e_\lambda \rangle|^2 = \sum_{\lambda \in F_\infty} |\langle x, e_\lambda \rangle|^2 = \lim_{n \to \infty} \sum_{\lambda \in F_n} |\langle x, e_\lambda \rangle|^2 \leq \|x\|^2,$$

which is the desired result. $\square$

Corollary 1.30. Let $\{e_n, n \in \mathbb{N}\}$ be an orthonormal system on H. Then for all $x \in H$,

$$\lim_{n \to \infty} \langle x, e_n \rangle = 0.$$

Remark. Now we let H be a Hilbert space. Fix $x \in H$, we proved that $\{e_\lambda \in \mathscr{H} : \langle x, e_\lambda \rangle \neq 0\}$ is at most countable. If it is countable, we can write it as a sequence $\{e_{\lambda_1}, e_{\lambda_2}, \cdots, e_{\lambda_n}, \cdots\}$. According to Bessel's inequality, we have $\sum_{n=1}^\infty |\langle x, e_{\lambda_n} \rangle|^2 < \infty$. Then for $m, n \in \mathbb{N}$,

$$\left\| \sum_{k=m+1}^{n} \langle x, e_{\lambda_k} \rangle e_{\lambda_k} \right\|^2 = \sum_{k=m+1}^{n} |\langle x, e_{\lambda_k} \rangle|^2 \to 0 \quad \text{as} \quad n, m \to \infty.$$

Thus we obtain a Cauchy sequence $\{\sum_{k=1}^n \langle x, e_{\lambda_k} \rangle e_{\lambda_k}\}_{n=1}^\infty$ in H, which converges to some vector y in H. Intuitively, the vector does not depend on our choice of permutation $\{\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots\}$.

Let $\{e_{\sigma_1}, e_{\sigma_2}, \cdots, e_{\sigma_n}, \cdots\}$ be another permutation of $\{e_\lambda \in \mathscr{H} : \langle x, e_\lambda \rangle \neq 0\}$. Following the above procedure, $\{\sum_{k=1}^n \langle x, e_{\sigma_k} \rangle e_{\sigma_k}\}_{n=1}^\infty$ is a Cauchy sequence in H, which converges to some $y' \in H$. We fix $\epsilon > 0$, and choose N such that $\sum_{n=N+1}^\infty |\langle x, e_{\lambda_n} \rangle|^2 < \epsilon^2/4$. Since $\{e_{\sigma_1}, e_{\sigma_2}, \cdots, e_{\sigma_n}, \cdots\} = \{e_{\lambda_1}, e_{\lambda_2}, \cdots, e_{\lambda_n}, \cdots\}$, there exists $M \geq N$ such that $\Lambda_N := \{\lambda_1, \cdots, \lambda_N\} \subset \{\sigma_1, \cdots, \sigma_M\}$. Then

$$\begin{aligned}
\left\| y - \sum_{m=1}^{M} \langle x, e_{\sigma_m} \rangle e_{\sigma_m} \right\| &\leq \left\| y - \sum_{n=1}^{N} \langle x, e_{\lambda_n} \rangle e_{\lambda_n} \right\| + \left\| \sum_{m=1}^{M} \langle x, e_{\sigma_m} \rangle e_{\sigma_m} - \sum_{n=1}^{N} \langle x, e_{\lambda_n} \rangle e_{\lambda_n} \right\| \\
&= \sqrt{\sum_{n=N+1}^{\infty} |\langle x, e_{\lambda_n} \rangle|^2} + \sqrt{\sum_{m=1, \sigma_m \notin \Lambda_N}^{M} |\langle x, e_{\sigma_m} \rangle|^2} \\
&\leq 2 \sqrt{\sum_{n=N+1}^{\infty} |\langle x, e_{\lambda_n} \rangle|^2} < \epsilon.
\end{aligned}$$

Let $M \to \infty$, then $\|y - y'\| < \epsilon$. Since ϵ is arbitrary, we have $\|y - y'\| = 0$, which implies $y = y'$. As a result, we can define

$$\sum_{\lambda \in \Lambda} \langle x, e_\lambda \rangle e_\lambda := \lim_{n\to\infty} \sum_{k=1}^{n} \langle x, e_{\lambda_k} \rangle e_{\lambda_k}.$$

By definition, we have

$$\left\| \sum_{\lambda \in \Lambda} \langle x, e_\lambda \rangle e_\lambda \right\| = \lim_{n\to\infty} \sum_{k=1}^{n} |\langle x, e_{\lambda_k} \rangle|^2 = \sum_{\lambda \in \Lambda} |\langle x, e_\lambda \rangle|^2 .$$

We use

$$\operatorname{span}\{e_\lambda, \lambda \in \Lambda\} := \left\{ \sum_{k=1}^{n} \alpha_k e_{\lambda_k} : n \in \mathbb{N},\ \alpha_1, \cdots, \alpha_n \in \mathbb{C},\ \lambda_1, \cdots, \lambda_n \in \Lambda \right\}$$

to denote the vector space spanned by orthonormal system $\{e_\lambda, \lambda \in \Lambda\}$, and use $\overline{\operatorname{span}}\{e_\lambda, \lambda \in \Lambda\}$ to denote its closure. By the above discussion, $\sum_{\lambda\in\Lambda} \langle x, e_\lambda \rangle e_\lambda \in \overline{\operatorname{span}}\{e_\lambda, \lambda \in \Lambda\}$.

Theorem 1.31 (Orthonormal basis). Let H be a Hilbert space, and let $\mathscr{H} = \{e_\lambda : \lambda \in \Lambda\}$ be an orthonormal system on H. The following are equivalent:

(i) For all $x \in H$, $x = \sum_{\lambda\in\Lambda} \langle x, e_\lambda \rangle e_\lambda$;

(ii) $\overline{\operatorname{span}}\{e_\lambda, \lambda \in \Lambda\} = H$;

(iii) For $x \in H$, $x \perp e_\lambda$ for all $\lambda \in \Lambda$ only if $x = 0$;

(iv) (Parseval equality). For all $x \in H$, $\|x\|^2 = \sum_{\lambda\in\Lambda} |\langle x, e_\lambda \rangle|^2$.

If $\mathscr{H}$ satisfies the above conditions, then $\mathscr{H}$ is said to be an *orthonormal basis* of H.

Proof. (i) $\Rightarrow$ (ii): Clearly, $\overline{\operatorname{span}}\{e_\lambda, \lambda \in \Lambda\} \subset H$. The other direction follows from the above Remark.

(ii) $\Rightarrow$ (iii): Let $x \in H$ be such that $\langle x, e_\lambda \rangle = 0$ for all $\lambda \in \Lambda$. Since $x \in H = \overline{\operatorname{span}}\{e_\lambda, \lambda \in \Lambda\}$, there exists sequence x_n of vectors in $\operatorname{span}\{e_\lambda, \lambda \in \Lambda\}$ such that $x_n \to x$. By continuity of inner product,

$$\langle x, x \rangle = \lim_{n\to\infty} \langle x, x_n \rangle = 0.$$

(iii) $\Rightarrow$ (i): Given $x \in H$, let $y = \sum_{\lambda\in\Lambda} \langle x, e_\lambda \rangle e_\lambda$. Then $x - y \perp e_\lambda$ for all $\lambda \in \Lambda$, which implies $x - y = 0$.

(i) $\Leftrightarrow$ (iv): We only prove (iv) $\Rightarrow$ (i), the other direction is clear. Given $x \in H$, let $y = \sum_{\lambda\in\Lambda} \langle x, e_\lambda \rangle e_\lambda$. Then $\langle x - y, y \rangle = 0$. By Pythagorean theorem, $\|x - y\|^2 = \|x\|^2 - \|y\|^2 = 0$, which implies $x = y$. $\square$

Following are some examples for orthonormal basis.

Example 1.32. Recall Example 1.28 (ii). The set

$$\mathscr{H} = \left\{ 1, \sqrt{2}\cos x, \sqrt{2}\sin x, \sqrt{2}\cos 2x, \sqrt{2}\sin 2x, \cdots, \sqrt{2}\cos nx, \sqrt{2}\sin nx, \cdots \right\}$$

is an orthonormal basis of $L^2([0, 2\pi])$.

Proof. Following Theorem 1.31, it suffices to show that $\overline{\operatorname{span}}\,\mathscr{H} = L^2([0, 2\pi])$. Denote by

$$C_c^\infty(0, 2\pi) := \left\{ f \in C([0, 2\pi]) : f \text{ is smooth}, \overline{\{f \neq 0\}} \subset (0, 2\pi) \right\}$$

the set of all smooth functions that have compact support in $(0, 2\pi)$. Clearly, $C_c^\infty(0,2\pi) \subset L^2([0,2\pi])$. Furthermore, for any $f \in C_c^\infty(0,2\pi)$, the Fourier coefficients are given by

$$\begin{aligned} a_n(f) &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\sqrt{2}\cos nt\,dt = -\frac{1}{\sqrt{2}\pi}\int_0^{2\pi} f''(t)\cos nt\,dt \;\Rightarrow\; |a_n(f)| \le \frac{\|f''\|_\infty}{n^2}, \\ b_n(f) &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\sqrt{2}\sin nt\,dt = -\frac{1}{\sqrt{2}\pi}\int_0^{2\pi} f''(t)\sin nt\,dt, \;\Rightarrow\; |b_n(f)| \le \frac{\|f''\|_\infty}{n^2}. \end{aligned}$$

Then the partial sum

$$(S_nf)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt + \sum_{k=1}^n \left(a_n(f)\sqrt{2}\cos nx + b_n(f)\sqrt{2}\sin nx\right)$$

converges uniformly on $[0, 2\pi]$. By Dini-Lipschitz criterion, S_nf converges pointwise, and $\|f - S_nf\|_2 \to 0$. Hence $f \in \overline{\operatorname{span}}\,\mathscr{H}$, and $L^2([0,2\pi]) = \overline{C_c^\infty(0,2\pi)} \subset \overline{\operatorname{span}}\,\mathscr{H}$. $\square$

The following theorem reveals the existence of an orthonormal basis for a Hilbert space.

Theorem 1.33. Suppose A is an orthonormal system on a Hilbert space H. Then there exists an orthonormal basis of $\mathscr{H}$ such that $\mathscr{H} \supset A$. In other words, A can be expanded to an orthonormal basis of H.

Proof. As you can imagine, we use Zorn's lemma. Denote

$$\mathcal{F} = \{B : B \text{ is an orthonormal system on } H,\ B \supset A\},$$

and order the elements of $\mathcal{F}$ by inclusion: $B \preceq B'$ if $B \subset B'$. Let $\mathcal{M} = \{B_\lambda, \lambda \in \Lambda\}$ be a totally ordered subset of $\mathcal{F}$. Then the union

$$B = \bigcup_{\lambda\in\Lambda} B_\lambda$$

is an orthonormal system on H. To see this, choose distinct $x, y \in B$, and assume $f \in B_{\lambda_1}, g \in B_{\lambda_2}$. Since $\mathcal{M}$ is totally ordered, we have either $B_{\lambda_1} \subset B_{\lambda_2}$ or $B_{\lambda_1} \supset B_{\lambda_2}$, which implies that f and g belongs to the same orthonormal system. Clearly, $B \supset A$. Then B is an upper bound of $\mathcal{M}$ in $\mathcal{F}$, and we can apply Zorn's lemma.

Let $\mathscr{H}$ be a maximal element in $\mathcal{F}$, then $\mathscr{H}$ is an orthonormal basis. Otherwise, there exists $x \in H\backslash\{0\}$ such that $\langle x, e_\lambda\rangle = 0$ for all $e_\lambda \in \mathscr{H}$, which implies $\mathscr{H} \cup \{x/\|x\|\} \in \mathcal{F}$, contradicting the maximality of $\mathscr{H}$! $\square$

1.2.4 The Projection Theorem

Review. Let (X, d) be a metric space, and let A be a subset of X. The distance from a point $x \in X$ to A is defined as

$$d(x, A) = \inf_{a\in A} d(x, a).$$

The function $d(\cdot, A) : X \to \mathbb{R}_+$ is continuous. To see this, fix $\epsilon > 0$ and $x_0 \in X$. Then there exists $a \in A$ such that $d(x_0, a) < d(x_0, A) + \epsilon/2$. Once $d(x, x_0) < \epsilon/2$, we have

$$d(x, A) \le d(x, a) \le d(x_0, x) + d(x_0, a) < d(x_0, A) + \epsilon.$$

Similarly, $d(x_0, A) < d(x, A) + \epsilon$. Hence $x \mapsto d(x, A)$ is continuous.

A *projection* of x on A is defined as a point $a_0 \in A$ such that $d(x, a_0) = d(x, A)$. In other words,

$$d(x, a_0) = \min_{a \in A} d(x, a).$$

The existence of projection is not ensured. For example, consider the point $x = -1$ and the open interval $(0, 1)$ in Euclidean space $\mathbb{R}$. Also, a point has possibly more than one projections on a set. For example, consider the point $z = 0$ and the unit circle $\mathbb{T} = \{e^{i\theta}, \theta \in [0, 2\pi)\}$ in the complex plane $\mathbb{C}$.

In this section, we discuss the projection in context of Hilbert spaces.

Definition 1.34 (Convex sets). A subset C of a vector space X is said to be *convex*, if for all $x, y \in C$ and all $t \in [0, 1]$, $tx + (1-t)y \in C$.

Theorem 1.35. Let H be a Hilbert space, and let M be a closed convex subset of H. Then for all $x \in H$, there exists a unique $x_0 \in M$ such that $\|x - x_0\| = d(x, M) := \inf_{y \in M} \|x - y\|$.

Proof. Choose a sequence (x_n) of points of M such that $\|x - x_n\| \to d(x, M)$. By the parallelogram law,

$$2\|x - x_n\|^2 + 2\|x - x_m\|^2 = 4\left\|x - \frac{x_n + x_m}{2}\right\|^2 + \|x_n - x_m\|^2, \ \forall n, m \in \mathbb{N}.$$

Then $0 \le \|x_n - x_m\|^2 \le 2\|x - x_n\|^2 + 2\|x - x_m\|^2 - 4d(x, M)^2 \to 0$, and (x_n) is a Cauchy sequence. By completeness of M, which is a closed subset of a complete space H, the sequence (x_n) converges to some $x_0 \in M$, and $\|x - x_0\| = \lim_{n\to\infty} \|x - x_n\| = d(x, M)$.

To prove the uniqueness, suppose $x_0' \in M$ also satisfies the condition. Then

$$0 \le \|x_0' - x_0\|^2 = 2\left(\|x - x_0\|^2 + \|x - x_0'\|^2\right) - 4\left\|x - \frac{x_0 + x_0'}{2}\right\|^2 \le 0.$$

Hence $\|x_0' - x_0\| = 0$, $x_0 = x_0'$. $\square$

Theorem 1.36 (Projection theorem). Let M be a closed subspace of a Hilbert space H. Then for all $x \in H$, there exists unique $x_0 \in M$ such that $\|x - x_0\| = d(x, M)$, and $x - x_0 \perp M$.

Proof. Following Theorem 1.35, it remains to show that $x - x_0 \perp M$. Given $y \in M$, the vector $x_0 + ty$ lies in M for all $t \in \mathbb{R}$ (or $\mathbb{C}$). Then

$$d(x, M)^2 \le \|x - x_0 - ty\|^2 = \|x - x_0\|^2 + |t|^2\|y\|^2 - 2\mathrm{Re}\left(t\langle y, x - x_0\rangle\right).$$

Let $t = \lambda\langle x - x_0, y\rangle$, then we have $2\lambda\left|\langle x - x_0, y\rangle\right|^2 \le \lambda^2\|y\|^2\left|\langle x - x_0, y\rangle\right|^2$ for all $\lambda \in \mathbb{R}$, which holds only if $\langle x - x_0, y\rangle = 0$. Therefore $x - x_0 \perp M$. $\square$

We also have another version of projection theorem.

Theorem 1.37 (Projection theorem). Let M be a closed subspace of a Hilbert space H. Then for all $x \in H$, there exists unique $x_0 \in M$ and $x_1 \perp M$ such that $x = x_0 + x_1$. Furthermore, $\|x - x_0\| = d(x, M)$.

Proof. We first prove the existence of x_0 and x_1. Note that M is closed in H, M is also a Hilbert space. By Theorem 1.33, there exists an orthonormal basis $\{e_\lambda, \lambda \in \Lambda_1\}$ of M, which can be expanded to an orthonormal basis $\{e_\lambda, \lambda \in \Lambda_2\}$ of H such that $\Lambda_2 \supset \Lambda_1$. By Theorem 1.35,

$$x = \sum_{\lambda \in \Lambda_2} \langle x, e_\lambda \rangle e_\lambda = \underbrace{\sum_{\lambda \in \Lambda_1} \langle x, e_\lambda \rangle e_\lambda}_{=: x_0 \in M} + \underbrace{\sum_{\lambda \in \Lambda_2 \backslash \Lambda_1} \langle x, e_\lambda \rangle e_\lambda}_{=: x_1 \perp M}.$$

For the uniqueness, suppose $x = y_0 + y_1$, where $y_0 \in M$ and $y_1 \perp M$. Then $x_0 - y_0 = y_1 - x_1$, and

$$\|x_0 - y_0\|^2 = \langle \underbrace{x_0 - y_0}_{\in M}, \underbrace{y_1 - x_1}_{\perp M} \rangle = 0.$$

Hence $x_0 = y_0$, and $x_1 = y_1$. ☐

Remark. Let M be a subspace of a Hilbert space H. We define the *orthogonal complement* of M of H as the set $M^\perp$ of all vectors in H that are orthogonal to every vector in M:

$$M^\perp = \{x \in H : x \perp M\}.$$

By continuity of inner product, $M^\perp$ is closed: Given a limit point x of $M^\perp$, we can find a sequence x_n in $M^\perp$ that converges to x. Then for each $y \in M$, $\langle x, y \rangle = \lim_{n\to\infty} \langle x_n, y \rangle = 0$. As a result, $M^\perp$ is complete.

If M is a closed subspace of H. Following the above proof, we can show that $M^\perp = \overline{\operatorname{span}} \{e_\lambda, \lambda \in \Lambda_2 \backslash \Lambda_1\}$. By Theorem 1.31, it suffices to show that for $y \in M^\perp$, $y = \sum_{\lambda \in \Lambda_2 \backslash \Lambda_1} \langle y, e_\lambda \rangle e_\lambda$. This is clear, because $\langle y, e_\lambda \rangle = 0$ for all $\lambda \in \Lambda_1$, and $y = \sum_{\lambda \in \Lambda_2} \langle y, e_\lambda \rangle e_\lambda$. Following Theorem 1.37, every vector $x \in H$ can be uniquely decomposed as $x = x_0 + x_1$, where $x_0 \in M$ and $x_1 \in M^\perp$. That is, the Hilbert space H admits the direct sum $H = M \oplus M^\perp$.

Corollary 1.38. Let M be a closed subspace of a Hilbert space H. If $M \neq H$, then $M^\perp \neq \{0\}$.

Proof. Let $x \in H$ be a vector that does not lie in M. For the decomposition $x = x_0 + x_1$, where $x_0 \in M$ and $x_1 \in M^\perp$, we have $x \neq x_0$. Hence $x_1 \neq 0$. ☐

Corollary 1.39. Let M be a subspace of a Hilbert space H. Then $\overline{M} = \left(M^\perp\right)^\perp$, and $M^\perp = \left(\overline{M}\right)^\perp$. Furthermore, $M^\perp = \{0\}$ if and only if M is dense in H.

Proof. Clearly, $M \subset \left(M^\perp\right)^\perp$: Let $x \in M$. Then

$$\langle x, y \rangle = 0, \ \forall y \in M^\perp \ \Rightarrow \ x \in \left(M^\perp\right)^\perp.$$

Since $\left(M^\perp\right)^\perp$ is closed, we have $\overline{M} \subset \left(M^\perp\right)^\perp$. If $\overline{M}$ is a proper subspace of $\left(M^\perp\right)^\perp$, there exists nonzero $x \in \left(M^\perp\right)^\perp \cap \overline{M}^\perp \subset \left(M^\perp\right)^\perp \cap M^\perp$. Then $x \perp x$, contradicting $x \neq 0$! Hence $\overline{M} = \left(M^\perp\right)^\perp$.

Apply this to $M^\perp$, we have $M^\perp = \left((M^\perp)^\perp\right)^\perp = \left(\overline{M}\right)^\perp$.

If M is dense in H, then $M^\perp = H^\perp = \{0\}$. Conversely, if $M^\perp = \{0\}$, then $\overline{M} = \left(M^\perp\right)^\perp = H$. ☐

Remark. For general subspace M of a Hilbert space H, we have $H = \overline{M} \oplus M^\perp$.

1.3 Density and Separability

1.3.1 Dense sets

Definition 1.40 (Density). Let X be a topological space. Let A and B be subsets of X. Then A is said to be *dense* in B if $B \subset \overline{A}$.

Remark. The condition of density can be described as follows: A is dense in B, if for all $x \in B$ and all $\epsilon > 0$, there exists $a \in A$ such that $d(x, a) < \epsilon$.

By definition, density is transitive: If A is dense in B and B is dense in C, then A is dense in C.

Example 1.41. Following are some instances for dense sets:

(i) The set of rational numbers $\mathbb{Q}$ is dense in the real line $\mathbb{R}$.

(ii) (Stone-Weierstrass). The set of all polynomial functions $P([a, b])$ on closed interval $[a, b]$ is dense in the space $(C([a, b]), \|\cdot\|_\infty)$ of continuous functions on $[a, b]$.

Example 1.42. Let $(X, \mathscr{A}, \mu)$ be a measure space. The set of all simple functions

$$\mathcal{S} = \left\{ \sum_{k=1}^{n} c_k \chi_{A_k} : n \in \mathbb{N},\ c_1, \cdots, c_n \in \mathbb{R},\ A_1, \cdots, A_n \in \mathscr{A} \right\}$$

is dense in $L^p(X, \mathscr{A}, \mu)$, where $1 \le p < \infty$.

Proof. (i) We first consider bounded measurable functions that vanish outside a set A with finite measure. Choose $f \in L^p(X, \mathscr{A}, \mu)$ such that $|f| \le M$ for some $M > 0$, and $\{f \ne 0\} \subset A$ for some $\mu(A) < \infty$. For $n \in \mathbb{N}$, we divide $[-M, M]$ into intervals of length not greater than n^{-1}:

$$-M = y_0 < y_1 < \cdots < y_m = M + \frac{1}{2n}.$$

Define $E_k = \{x \in A : f(x) \in [y_{k-1}, y_k)\}$. The function $f_n = \sum_{k=1}^{m} y_k \chi_{E_k}$ is simple, and $|f - f_n| < n^{-1}$. Note that f is defined on a set A with finite measure,

$$0 \le \|f - f_n\|_p^p = \int_X |f - f_n|^p d\mu \le \frac{1}{n^p} \mu(A) \to 0.$$

(ii) We then consider unbounded measurable functions that vanish outside a set A with finite measure. Choose $f \in L^p(X, \mathscr{A}, \mu)$ such that $\{f \ne 0\} \subset A$ for some $\mu(A) < \infty$. Define the M-truncated function as

$$[f]_M(x) = \begin{cases} M,\ f(x) > M, \\ f(x),\ -M \le f(x) \le M, \\ -M,\ f(x) < -M. \end{cases}$$

By monotone convergence theorem,

$$\int_X |f|^p \, d\mu = \lim_{M \to \infty} \int_X |[f]_M|^p \, d\mu.$$

Given $\epsilon > 0$, we choose M_ϵ such that

$$\int_X |f - [f]_{M_\epsilon}|^p \, d\mu \le \int_X |f|^p \, d\mu - \int_X |[f]_{M_\epsilon}|^p \, d\mu < \frac{\epsilon^p}{2^p}.$$

By (i), there exists simple function $g \in \mathcal{S}$ such that $\int_X |[f]_{M_\epsilon} - g|^p \, d\mu < 2^{-p}\epsilon^p$. Hence

$$\int_X |f-g|^p \, d\mu \le 2^{p-1}\left(\int_X |f-[f]_{M_\epsilon}|^p \, d\mu + \int_X |[f]_{M_\epsilon} - g|^p \, d\mu\right) < \epsilon^p.$$

(iii) Now we prove the general case. Let $f \in L^p(X, \mathscr{A}, \mu)$ and $\epsilon > 0$ be given. For $n \in \mathbb{N}$, we define the level set $F_n = \{x \in X : |f|^p > n^{-1}\}$. Then $\mu(F_n) \le n\|f\|_p^p < \infty$, and $\{f \neq 0\} = \bigcup_{n=1}^\infty F_n$.

Consider the sequence $f\chi_{F_n}$, which converges to f pointwise. By monotone convergence theorem,

$$\int |f|^p \, d\mu = \lim_{n\to\infty} \int |f\chi_{F_n}|^p \, d\mu.$$

Hence there exists N_ϵ such that for all $n \ge N_\epsilon$,

$$\int_X |f - f\chi_{F_n}|^p \, d\mu \le \int_X |f|^p \, d\mu - \int_X |f\chi_{F_n}|^p \, d\mu < \frac{\epsilon^p}{2^p}.$$

By (ii), there exists simple function $h \in \mathcal{S}$ such that $\int_X |f\chi_{F_n} - h|^p \, d\mu < 2^{-p}\epsilon^p$, which implies

$$\int_X |f-h|^p \, d\mu \le 2^{p-1}\left(\int_X |f - f\chi_{F_n}|^p \, d\mu + \int_X |f\chi_{F_n} - h|^p \, d\mu\right) < \epsilon^p.$$

Then we conclude the proof. □

Example 1.43. Following Example 1.42, the set $\mathcal{S}$ of all simple functions is dense in $L^\infty(X, \mathscr{A}, \mu)$.

Proof. Let $f \in L^\infty(X, \mathscr{A}, \mu)$. Then the bad set $E = \{x \in X : f(x) > \|f\|_\infty\}$ has zero measure. For $n \in \mathbb{N}$, we divide $[-\|f\|_\infty, \|f\|_\infty]$ into intervals of length not greater than n^{-1}:

$$-\|f\|_\infty = y_0 < y_1 < \cdots < y_m = \|f\|_\infty + \frac{1}{2n}.$$

Define $A_k = \{x \in X : f(x) \in [y_{k-1}, y_k)\}$. Then the function $f_n = \sum_{k=1}^m y_k \chi_{A_k} \in \mathcal{S}$ satisfies

$$\sup_{x \in X\setminus E} |f(x) - f_n(x)| \le \frac{1}{n} \;\Rightarrow\; 0 \le \|f - f_n\|_\infty \le \frac{1}{n} \to 0.$$

Hence $\mathcal{S}$ is dense in $L^\infty(X, \mathscr{A}, \mu)$, as desired. □

Example 1.44. Let $(X, \mathscr{A}, \mu)$ be a measure space where X is a locally compact normal space, and μ is a Radon measure on the Borel sets of X. The set $C(X, \mathbb{R})$ of all real-valued continuous functions on X is dense in $L^p(X, \mathscr{A}, \mu)$, where $1 \le p < \infty$. For instance, the space $C(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

Proof. Let $f \in L^p(X, \mathscr{A}, \mu)$, and let $\epsilon > 0$. By monotone convergence theorem, the sequence of n-truncated functions $[f]_n : x \mapsto \min\{\max\{f(x), -n\}, n\}$ satisfies

$$\lim_{n\to\infty} \|f - [f]_n\|_p^p = \int_X \lim_{n\to\infty} |f - [f]_n|^p \, d\mu = 0.$$

Then we can choose some N such that $\|f - [f]_N\|_p^p \le 2^{-p}\epsilon^p$.

By Lusin's theorem, there exists a closed set C in X such that $\mu(X\setminus C) < \epsilon^p/(4N)^p$, and the restriction $[f]_N|_C : C \to [-N, N]$ is continuous. Using Tietze extension theorem, we can extend $[f]_N|_C : C \to [-N, N]$

to a continuous function $g : X \to [-N, N]$ on X that agrees with $[f]_N|_C$ on C. As a result,

$$\begin{aligned}\|f-g\|_p^p &\le 2^{p-1}\left(\|f-[f]_N\|_p^p + \|[f]_N - g\|_p^p\right)\\ &\le 2^{p-1}\left(\|f-[f]_N\|_p^p + \int_{X\setminus C} |[f]_N - g|^p \; d\mu\right) \le 2^{p-1}\left(\|f-[f]_N\|_p^p + \mu(X\setminus C)\cdot(2N)^p\right) < \epsilon^p.\end{aligned}$$

Therefore $C(X,\mathbb{R})$ is dense in $L^p(X,\mathscr{A},\mu)$, as desired. $\square$

Review: Convolution. Let $f, g : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable functions. Define the bad set as

$$E(f,g) := \left\{x \in \mathbb{R} : \int_{\mathbb{R}} |f(x-y)g(y)|\, dy = \infty\right\}.$$

The *convolution* of f and g is the function $f * g : \mathbb{R} \to \mathbb{R}$ defined by

$$(f*g)(x) = \begin{cases} \int_{\mathbb{R}} f(x-y)g(y)\, dy, & x \notin E(f,g), \\ 0, & x \in E(f,g). \end{cases}$$

Clearly, the convolution operation is commutative and associative, i.e. $f*g = g*f$, and $(f*g)*h = f*(g*h)$. Furthermore, the distributivity of convolution with respect to functional addition immediately follows.

Proposition 1.45 (Properties of convolution). Let $f, g : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable functions.

(i) If $f, g \in L^1(\mathbb{R})$, then $\mu(E(f,g)) = 0$, $f * g \in L^1(\mathbb{R})$, and

$$\int_{\mathbb{R}} (f*g)\, dm = \int_{\mathbb{R}} f\, dm \int_{\mathbb{R}} g\, dm. \tag{1.15}$$

(ii) If $f \in C_0(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then $f * g \in C_0(\mathbb{R})$.

Proof. (i) Define $F : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto f(x)$ and $G : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto g(y)$. Then for all $\alpha \in \mathbb{R}$, both $F^{-1}((\alpha,\infty)) = f^{-1}((\alpha,\infty)) \times \mathbb{R}$ and $G^{-1}((\alpha,\infty)) = \mathbb{R} \times g^{-1}((\alpha,\infty))$ are Lebesgue measurable sets in $\mathbb{R}$, which implies that both F and G are measurable, as well as their product $F \cdot G : (x,y) \mapsto f(x)g(y)$. Let $T(x,y) = (x-y, y)$ be a linear transformation. Then the composition $H = (F\cdot G) \circ T : (x,y) \mapsto f(x-y)g(y)$ is measurable. By Tonelli's theorem,

$$\int_{\mathbb{R}^2} |H|\, dm_2 = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x-y)|\, |g(y)|\, dx\right) dy = \|f\|_1 \|g\|_1.$$

Hence $H : \mathbb{R}^2 \to \mathbb{R}$ is integrable. By Fubini's theorem, for a.e. $x \in \mathbb{R}$, $y \mapsto H(x,y)$ is integrable, hence $\mu(E(f,g)) = 0$. Furthermore, the function $f * g : x \mapsto \int_{\mathbb{R}} H(x,y)\, dy$ is also integrable, that is, $f * g \in L^1(\mathbb{R})$. The equation (1.15) follows from Fubini's theorem.

(ii) Given $\epsilon > 0$. By uniform continuity of f, there exists $\eta > 0$ such that $|f(x) - f(x')| < \epsilon/\|g\|_1$ for all $|x - x'| < \eta$, . As a result, we have

$$\begin{aligned}|(f*g)(x) - (f*g)(x')| &= \left|\int_{\mathbb{R}} [f(x-y) - f(x'-y)]\, g(y)\, dy\right| \\ &\le \int_{\mathbb{R}} |f(x-y) - f(x'-y)|\, |g(y)|\, dy < \epsilon\end{aligned}$$

for all $x, x' \in \mathbb{R}$ such that $|x - x'| < \eta$. $\square$

Review: Compact supported functions. Let X be a topological space. The support of function $f : X \to \mathbb{R}$ is defined as the closure of the set of all points in X not mapped to zero by f:

$$\operatorname{supp} f = \overline{\{x \in X : f(x) \neq 0\}} = \overline{\{f \neq 0\}}.$$

If the support of f is compact in X, f is said to be *compactly supported*. Following this definition, any function defined on a closed interval $[a, b]$ can be extended to a compactly supported function on $\mathbb{R}$.

The set of all continuous compactly supported functions on X is denoted by $C_c(X)$. If $f \in C_c(X)$, then f is uniformly continuous on $\operatorname{supp} f$. Note that $f = 0$ outside $\operatorname{supp} f$, we have that f is uniformly continuous on X, which implies $C_c(X) \subset C_0(X)$. Furthermore, by extreme value theorem, f has maximum and minimum on $\operatorname{supp} f$, which implies that f is uniformly bounded on X, i.e. $\max_{x\in X} |f(x)| < \infty$.

Let $(X, \mathscr{A}, \mu)$ be a measure space where X is a topological space. Following the discussion above, we have $C_c^\infty(X) \subset L^\infty(X, \mathscr{A}, \mu)$ since every $f \in C_c^\infty(X)$ satisfies $\|f\|_\infty = \max_{x\in X} |f(x)| \leq \infty$. Furthermore, if every compact set in X has finite measure, i.e. $\mu(K) < \infty$ for all compact $K \subset X$, then the compactly supported function are always integrable:

$$\|f\|_p = \left(\int_X |f|^p d\mu\right)^{1/p} = \left(\int_{\operatorname{supp} f} |f|^p dm\right)^{1/p} \leq \mu(\operatorname{supp} f)^{1/p}\|f\| < \infty.$$

Hence $C_c(X) \subset L^p(X)$ for all $1 \leq p \leq \infty$.

Proposition 1.46 (Convolution of compactly supported functions). Let $f, g : \mathbb{R} \to \mathbb{R}$.

(i) If $f, g \in L^1(\mathbb{R})$, then $\operatorname{supp}(f * g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g} := \overline{\{x + y : x \in \operatorname{supp} f, y \in \operatorname{supp} g\}}$. Furthermore, if both f and g are compactly supported on $\mathbb{R}$, then $f * g$ is also compactly supported. In this case, $\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g$.

(ii) Let $1 \leq p \leq \infty$, and let $k \in \mathbb{N}_0$. If $f \in C_c^k(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, then $f * g \in C_0^k(\mathbb{R})$. Furthermore, differentiation commutes with convolution, i.e.,

$$D^j(f * g) = D^j f * g, \ j = 0, 1, \cdots, k,$$

where $D^j f = f^{(j)}$ stands for the j-th derivative.

(iii) Let $1 \leq p \leq \infty$. If $f \in C_c^\infty(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, then $f * g \in C_0^\infty(\mathbb{R})$. Similarly, differentiation commutes with convolution, i.e., $D^k(f * g) = D^k f * g$ for all $k \in \mathbb{N}_0$.

Remark. Combining (ii) and (iii), we obtain a useful conclusion stated as follows: Let $1 \leq p \leq \infty$ and $k \in \mathbb{N}_0 \cup \{\infty\}$. If $f \in C_c^k(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ is compactly supported, then $f * g \in C_c^k(\mathbb{R})$.

Proof of Proposition 1.46. (i) Let $f, g \in L^1(\mathbb{R})$. Take any $x \in \mathbb{R}$. Note that

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y)\, dy = \int_{(x-\operatorname{supp} f)\cap \operatorname{supp} g} f(x - y)g(y)\, dy.$$

For $x \notin \operatorname{supp} f + \operatorname{supp} g$, we have $(x - \operatorname{supp} f) \cap \operatorname{supp} g = \emptyset$, which implies $(f * g)(x) = 0$. Hence

$$(f * g)(x) \neq 0 \ \Rightarrow x \in \operatorname{supp} f + \operatorname{supp} g \ \Rightarrow \ \operatorname{supp}(f * g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}.$$

If $f, g \in C_c(\mathbb{R})$, then $\operatorname{supp} f$ and $\operatorname{supp} g$ are compact in $\mathbb{R}$. Define $\phi(x, y) = x + y$, which is a continuous map on $\mathbb{R}^2$. Then $\operatorname{supp} f + \operatorname{supp} g = \phi(\operatorname{supp} f \times \operatorname{supp} g)$ is also compact. Hence $\operatorname{supp} f + \operatorname{supp} g$ is closed, and $\operatorname{supp}(f * g)$ as a closed subset is also compact, which implies $f * g \in C_c(\mathbb{R})$.

(ii) *Step I:* We first show the case $k = 0$. Let $q = p/(p-1)$. Note that f is continuous and compact supported, then $m(\operatorname{supp} f) < \infty$, f is uniformly continuous, and $\|f\|_\infty = \max_{x\in\operatorname{supp} f} |f(x)| < \infty$. By Hölder's inequality, for all $x \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} |f(x-y)|\,|g(y)|\,dy \le \|f\|_q \|g\|_p \le m\big(\operatorname{supp} f\big)^{1/q} \|f\|_\infty \|g\|_p < \infty.$$

Then $f * g$ is well-defined on $\mathbb{R}$. To show the uniform continuity of $f * g$, we fix $\epsilon > 0$ and let η be such that $|x - x'| < \eta$ implies $|f(x) - f(x')| < \epsilon$. Then

$$\begin{aligned}|(f*g)(x) - (f*g)(x')| &= \left|\int_{\mathbb{R}} [f(x-y) - f(x'-y)]\, g(y)\,dy\right| \\ &\le m\big(\operatorname{supp} f\big)^{1/q} \|g\|_p\, \epsilon.\end{aligned}$$

Step II: We prove the case $k = 1$. It suffices to show the interchangeability of derivative and integral. Given any quantity $\delta \to 0$, we have

$$\frac{(f*g)(x+\delta) - (f*g)x}{\delta} = \int_{\mathbb{R}} \frac{f(x+\delta-y) - f(x-y)}{\delta} g(y)\,dy. \tag{1.16}$$

Since $f \in C_c^1(\mathbb{R})$, by Lagrange's mean value theorem, there exists $\xi \in [0,1]$ such that

$$\left|\frac{f(x+\delta-y) - f(x-y)}{\delta}\right| = |f'(x+\xi\delta - y)|, \tag{1.17}$$

Note that f' is also continuous and compactly supported on $\mathbb{R}$, the RHS of (1.17) is bounded by $\|f'\|_\infty < \infty$, and the integrand in (1.16) is dominated by an integrable function $\|f'\|_\infty g$. Using Lebesgue's dominate convergence theorem, we have

$$\lim_{\delta\to 0} \int_{\mathbb{R}} \frac{f(x+\delta-y) - f(x-y)}{\delta} g(y)\,dy = \int_{\mathbb{R}} f'(x-y) g(y)\,dy.$$

Therefore $(f*g)' = f' * g$. Since $f' \in C_c(\mathbb{R})$, we have $(f*g)' \in C_0(\mathbb{R})$, and $f * g \in C_0^1(\mathbb{R})$.

Step III: Use induction. Suppose our conclusion holds for $C_c^{k-1}(\mathbb{R})$. For each $f \in C_c^k(\mathbb{R}) \subset C_c^{k-1}(\mathbb{R})$, $D^{k-1} f \in C_c^1(\mathbb{R})$. By Step II, we have

$$D^k(f*g) = D(D^{k-1}(f*g)) = D(D^{k-1}f * g) = (D^k f) * g,$$

which is uniformly continuous on $\mathbb{R}$. Hence $f * g \in C_c^k(\mathbb{R})$.

(iii) Note that $C_c^\infty(\mathbb{R}) = \bigcap_{k=0}^\infty C_c^k(\mathbb{R})$, we have $D^k(f*g) = D^k f * g$ for all $k \in \mathbb{N}_0$. Following Step II, $D^k f \in C_c(\mathbb{R})$ implies $D^k(f*g) \in C_0(\mathbb{R})$ for all $k \in \mathbb{N}_0$. Hence $f * g \in \bigcap_{k=0}^\infty C_0^k(\mathbb{R}) = C_0^\infty(\mathbb{R})$. ☐

Review: Translation operators. Let X be a vector space, let Y^X be the set of functions $f : X \to Y$, and let s be a vector X. The *translation operator* $\tau_s : Y^X \to Y^X$ is defined as

$$(\tau_s f)(x) = f(x-s),\ \forall f \in Y^X.$$

Proposition 1.47. Let $1 \le p < \infty$. For any $f \in C^c(\mathbb{R})$,

$$\lim_{s\to 0} \|\tau_s f - f\|_p \to 0. \tag{1.18}$$

Proof. Let $f \in C^c(\mathbb{R})$. The collection of functions $\{\tau_s f : |s| \le 1\}$ has a common support

$$K = \bigcup_{s\in[-1,1]} \operatorname{supp}(\tau_s f) = \operatorname{supp} f + [-1,1] = \{x + y : x \in \operatorname{supp} f, y \in [-1,1]\} = \phi(\operatorname{supp} f \times [-1,1]),$$

which is compact as the image of a compact set under a continuous map $\phi : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x + y$.

By uniform continuity of f, given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $|x - y| < \delta$. Then for any $s < |\min(\delta, 1)|$, we have

$$\|\tau_s f - f\|_p^p = \int_K |f(x - s) - f(x)|^p dx \le \mu(K)\, \epsilon^p.$$

Since $\mu(K) < \infty$, and ϵ is arbitrary, we conclude that $\|\tau_s f - f\|_p \to 0$ as $s \to 0$. $\square$

Example 1.48. For $1 \le p < \infty$, $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

Proof. Let f be a compactly supported function in $L^p(\mathbb{R})$. Then We choose a function $\phi \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \phi\, dm = 1$, for example,

$$\psi(t) = \exp\left(\frac{1}{t^2 - 1}\right) \chi_{[-1,1]}(t),\ \ \phi(x) = \frac{\psi(x)}{\int_{-1}^1 \psi(t)\, dt},$$

and define $\phi_\epsilon(x) = \frac{1}{\epsilon}\phi\left(\frac{x}{\epsilon}\right)$ for $\epsilon > 0$. By Proposition 1.46, $f * \phi_\epsilon \in C^\infty(\mathbb{R})$, and

$$\begin{aligned}
\int_{\mathbb{R}} |(f * \phi_\epsilon)(x) - f(x)|^p\, dx &= \int_{\mathbb{R}} \left| \int_{[-\epsilon,\epsilon]} (f(x - y) - f(x))\phi_\epsilon(y)\, dy \right|^p dx \\
&\le \int_{\mathbb{R}} \int_{[-\epsilon,\epsilon]} |(f(x - y) - f(x))|^p\, \phi_\epsilon(y)\, dydx \qquad \text{(By Jensen's inequality)} \\
&= \int_{[-\epsilon,\epsilon]} \phi_\epsilon(y) \|\tau_y f - f\|_p^p\, dy \\
&\le \sup_{y\in[-\epsilon,\epsilon]} \|\tau_y f - f\|_p^p.
\end{aligned}$$

which converges to 0 as $\epsilon \to 0$ by Proposition 1.47. By monotone convergence theorem, any $g \in L^p(\mathbb{R})$ is approximated by $g\chi_{[-n,n]}, n \in \mathbb{N}$:

$$\|g - g\chi_{[-n,n]}\|_p^p = \int_{\mathbb{R}} |g|^p\, dm - \int_{\mathbb{R}} |g\chi_{[-n,n]}|^p\, dm \to 0,\ \ n \to \infty.$$

Note that $g_{[-n,n]} =: f$ is compactly supported, the result follows. $\square$

Remark. In fact, the limit (1.18) in Proposition 1.47 remains zero for all $f \in L^p(\mathbb{R})$. Fix $\epsilon > 0$, there exists $g \in C_c^\infty(\mathbb{R})$ such that $\|f - g\|_\infty < \epsilon/3$ by Example 1.48. Choose δ such that $\|\tau_s g - g\|_p < \epsilon/3$ for all $|s| < \delta$. Then for all $\in (-\delta, \delta)$,

$$\|\tau_s f - f\|_p \le \|\tau_s f - \tau_s g\|_p + \|\tau_s g - g\|_p + \|g - f\|_p = 2\|f - g\| + \|\tau_s g - g\|_p < \epsilon.$$

Similarly, we have the following conclusion similar to Example 1.48.

Example 1.49. Denote by $C_c^\infty(a, b)$ the set of functions $f : [a, b] \to \mathbb{R}$ such that f is smooth and compactly supported in (a, b), i.e. $\operatorname{supp} f \subset (a, b)$. Then $C_c^\infty(a, b)$ is dense in $L^p([a, b])$, where $1 \le p < \infty$.

1.3.2 Separable sets

Definition 1.50 (Separability). Let X be a topological space. Then X is said to be *separable* if it has a countable dense subset.

Example 1.51. Following are some instances for separable spaces.

(i) The space $\mathbb{R}^n$ is separable, since $\mathbb{Q}^n$ is a countable dense subset.

(ii) The spaces $C([a,b])$ and $L^p([a,b]), 1 \leq p < \infty$ are separable: The set $P([a,b])$ of all polynomials on $[a,b]$ is dense in $C([a,b])$, and the set of all polynomials with rational coefficients is dense in $P([a,b])$.

(iii) If (X,d) is separable, so is (A,d), where $A \subset X$.

Proof of (iii). Let $\mathcal{D} = \{x_n, n \in \mathbb{N}\}$ be a countable dense subset of X. Then we have $X \subset O(x_n, \epsilon)$ for all $\epsilon > 0$. For every $n,k \in \mathbb{N}$, choose arbitrary $y_{n,k} \in A \cap O(x_n, 1/k)$ provided it is not empty. Given $y \in A$ and $\epsilon > 0$, we choose an integer $k > 2/\epsilon$. By density of $\mathcal{D}$, there exists $x_n \in \mathcal{D}$ such that $d(y,x_n) < \epsilon/2$. Moreover, $A \cap O(x_n, 1/k)$ is not empty since it contains y. Then

$$d(y, y_{n,k}) \leq d(y, x_n) + d(x_n, y_{n,k}) < \frac{\epsilon}{2} + \frac{1}{k} < \epsilon.$$

Hence $\{y_{n,k} : n,k \in \mathbb{N}\}$ is dense in A. ☐

Example 1.52. Let $U \subset \mathbb{R}$ be Lebesgue measurable with $\mu(U) > 0$. If $1 \leq p < \infty$, then $L^p(U)$ is separable.

Proof. Consider the set of countably many functions in $L^p(U)$:

$$\mathcal{D} := \left\{ \sum_{j=1}^{n} c_j \chi_{(a_j,b_j)\cap U} : n \in \mathbb{N}, a_1, b_1, c_1, \cdots, a_n, b_n, c_n \in \mathbb{Q} \right\}$$

For any $f \in L^p(\mathbb{R})$, approximate it with functions in $\mathcal{D}$ as follows: (i) By Example 1.42, approximate f by a simple function $\varphi = \sum_{i=1}^m r_i \chi_{A_i}$, with $m(A_i) < \infty$ for each i. (ii) By Littlewood's first principle, we can approximate each Lebesgue measurable set A_i with a finite collection of disjoint open intervals $\{(s_{ij}, t_{ij})\}_{j=1}^{n_i}$. Then we obtain a simple function $\phi = \sum_{i=1}^m \sum_{j=1}^{n_i} r_i \chi_{(s_{ij},t_{ij})}$ near to ψ; (iii) Approximate ϕ by rational coefficients and endpoints.

According to the above procedure, $\mathcal{D}$ is dense in $L^p(U)$. ☐

Remark. The space $L^\infty(\mathbb{R})$ is not separable. To see this, consider the set $A = \{\chi_{(-\infty,t]}, t \in \mathbb{R}\}$. For any two distinct functions f and g in A, we have $\|f - g\|_\infty = 1$. Then any proper subset of A is not dense in A, and A is not separable. As a result, $L^\infty(\mathbb{R})$ is not separable.

1.4 Completeness

1.4.1 Complete metric spaces

Lemma 1.53. The following statements are true.

(i) (Lemma 1.6) A closed subspace of a complete metric space is complete.

(ii) (Lemma 1.7, subsequence criterion) A metric space (X, d) is complete if every Cauchy sequence in X has a convergent subsequence.

(iii) If A is a dense subset of a metric space (X, d), and every Cauchy sequence in A converges to some point of X, then (X, d) is complete.

Proof of (iii). Let (x_n) be a Cauchy sequence in X. Since $\overline{A} = X$, there exists find $a_n \in A$ such that $d(a_n, x_n) < 1/n$ for each $n \in \mathbb{N}$. Fix $\epsilon > 0$. Then there exists N such that $d(x_n, x_m) < \epsilon/3$ for all $n, m \geq N$. By setting $n, m \geq \max\{N, 3\epsilon^{-1}\}$, we have

$$d(a_n, a_m) \leq d(a_n, x_n) + d(x_n, x_m) + d(x_m, a_m) < \epsilon.$$

Hence (a_n) is a Cauchy sequence in A, and it converges to some $x \in X$. Since $d(x_n, a_n) \to 0$, and $d(a_n, x) \to 0$, we have $d(x_n, x) \to 0$, which concludes the proof. □

Example 1.54 (Quotient spaces). Let M be a subspace of a vector X. For $x, y \in X$, define $x \sim y$ if and only if $x - y \in M$. Then $\sim$ is an equivalence relation on X. We define the *quotient space* X/M as

$$X/M := X/\sim = \{[x] : x \in X\}, \text{ where } [x] := \{x + y : y \in M\} \text{ is an equivalence class.}$$

The *quotient map* is defined as $\pi : X \to X/M, x \mapsto x$. Clearly, X/M forms a vector space, if we set $[x] + [y] = [x + y]$ and $\alpha[x] = [\alpha x]$, where $x, y \in X$ and $\alpha \in \mathbb{R}$ (or $\mathbb{C}$), and let $[0]$ be the zero element.

If X is a normed space and M is a closed subspace of X, then we define a norm $\|\cdot\|$ on X/M by

$$\|[x]\| = d(x, M) = \inf_{y \in M} \|x - y\|.$$

It is easy to verify that $\|\cdot\|$ satisfy the conditions in Definition 1.13. Moreover, $\|\cdot\|$ is well-defined, because $x \sim y$ implies $\|[x]\| = \|[y]\|$.

Note that we require M to be closed. Otherwise, there exists $x \in X \backslash M$ such that x is a limit point of M, and there exists a sequence (x_n) of points of M such that $x_n \to x$. As a result, $\|[x]\| = \inf_{y \in M} \|x - y\| = 0$. However $[x] \neq [0]$, a contradiction! In this case, $\|\cdot\|$ is merely a seminorm on X/M.

Claim. If $(X, \|\cdot\|)$ is a Banach space, so is $(X/M, \|\cdot\|)$.

Proof. Let $([x_n])$ be a Cauchy sequence of points of X/M. Then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|[x_n] - [x_m]\| = \inf_{y \in M} \|x_n - x_m - y\| < \epsilon$ for all $n, m \geq N$. We choose a subsequence n_k such that

$$\inf_{y \in M} \|x_{n_{k+1}} - x_{n_k} - y\| < 2^{-k}, \ k \in \mathbb{N}.$$

Then there exists $y_k \in M$ such that $\|x_{n_{k+1}} - x_{n_k} - y_k\| < 2^{-k}$. We define another sequence $\left(x'_{n_k}\right)$ by

$$x'_{n_1} = x_{n_1}, \ x'_{n_2} = x_{n_2} - y_1, \ \cdots, \ x'_{n_k} = x_{n_k} + \sum_{j=1}^{k-1} (-1)^{k-j} y_j, \ \cdots.$$

By definition, $\left\|x'_{n_{k+1}} - x'_{n_k}\right\| < 2^{-k}$, and $x'_{n_k} - x_{n_k} \in M$, which implies $[x'_{n_k}] = [x_{n_k}]$. Then $\left(x'_{n_k}\right)$ is a Cauchy sequence in Banach space $(X, \|\cdot\|)$, which converges to some $x' \in X$.

As a result, the subsequence $[x_{n_k}]$ converges to $[x'] \in X/M$:

$$0 \le \|[x_{n_k}] - [x']\| = \left\|[x'_{n_k}] - [x']\right\| = \inf_{y \in M} \left\|x'_{n_k} - x' - y\right\| \le \left\|x'_{n_k} - x'\right\| \to 0.$$

By subsequence criterion (Lemma 1.53), X/M is a Banach space. $\square$

Example 1.55 (Functions of bounded variation). Let $V([a,b])$ be the set of all functions $f : [a,b] \to \mathbb{R}$ of bounded variation, i.e., the total variation of f on $[a,b]$ is bounded:

$$V_a^b(f) := \sup \left\{ \sum_{i=1}^n |f(x_i) - f(x_{i-1})| : n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b \right\} < \infty.$$

For all $f \in V([a,b])$, we define the norm

$$\|f\| = |f(a)| + V_a^b(f).$$

Then $(V([a,b]), \|\cdot\|)$ is a normed space.

Let $V_0([a,b])$ be the subspace of $V([a,b])$, which consists of all functions $f : [a,b] \to \mathbb{R}$ such that f is of bounded variation, $f(a) = 0$ and that f is right-continuous on (a,b). We continue to use the norm $\|\cdot\|$ on $V([a,b])$, which becomes $\|f\| = V_a^b(f)$ for $f \in V_0([a,b])$. Then $(V_0([a,b]), \|\cdot\|)$ is also a normed space.

Claim. $V([a,b])$ and $V_0([a,b])$ are Banach spaces.

Proof. (i) We first show that $V([a,b])$ is Banach. Let (f_n) be a Cauchy sequence in $V([a,b])$, i.e. for all $\epsilon > 0$, there exists N such that $|f_n(a) - f_m(a)| + V_a^b(f_n - f_m) < \epsilon$ for all $n, m \ge N$. Given $x \in [a,b]$,

$$\begin{aligned} |f_n(x) - f_m(x)| &\le |f_n(a) - f_m(a)| + |(f_n(x) - f_m(x)) - (f_n(a) - f_m(a))| \\ &\le |f_n(a) - f_m(a)| + V_a^b(f_n - f_m). \end{aligned}$$

Then $f_n(x)$ is a Cauchy sequence, which converges to some $f(x) \in \mathbb{R}$. Hence we obtain a function f on $[a,b]$ to which f_n converges pointwise.

Let $a = x_0 < x_1 < \cdots < x_k = b$ be any partition of $[a,b]$. Then we have

$$\begin{aligned} \sum_{j=1}^k |f(x_j) - f(x_{j-1})| &\le \sum_{j=1}^k |f(x_j) - f_n(x_j)| + \sum_{j=1}^k |f_n(x_j) - f_n(x_{j-1})| + \sum_{j=1}^k |f_n(x_{j-1}) - f(x_{j-1})| \\ &\le \underbrace{\sum_{j=1}^k |f(x_j) - f_n(x_j)| + \sum_{j=1}^k |f_n(x_{j-1}) - f(x_{j-1})|}_{\text{(a)}} + \underbrace{V_a^b(f_n)}_{\text{(b)}}. \end{aligned}$$

The term (a) converges to zero, since f_n converges to f pointwise. Hence it suffices to bound term (b). Note that (f_n) is a Cauchy sequence, there exists N such that $\|f_n - f_m\| < 1$ for all $n, m \ge N$. Then the sequence is uniformly bounded by $M = \{\|f_1\|, \cdots, \|f_{N-1}\|, 1 + \|f_N\|\}$, and $V_a^b(f_n) \le \|f_n\| \le M$ for all $n \in \mathbb{N}$. Since the partition $a = x_0 < x_1 < \cdots < x_n = b$ is arbitrary, the total variation of f is also bounded by M.

Now it remains to show $\|f - f_n\| \to 0$. Note that $f_n(a) \to f(a)$, we need to show $V_a^n(f - f_n) \to 0$. Given

$\epsilon > 0$, we choose N such that $\|f_n - f_m\| < \epsilon$ for all $n \geq N$. Then

$$\sum_{j=1}^{k} |(f_m - f_n)(x_j) - (f_m - f_n)(x_{j-1})| < \epsilon$$

holds for all partition $a = x_0 < x_1 < \cdots < x_n = b$ and all $n, m \geq N$. Let $m \to \infty$. Since f_n converges to f pointwise, we have $\|f - f_n\| < \epsilon$, as desired.

(ii) To show $V_0([a, b])$ is Banach, it suffices to show that $V_0([a, b])$ is a closed subspace of $V([a, b])$. Let f_n be a sequence of functions in $V_0([a, b])$ that converges to $f \in V([a, b])$ in sense that $\|f_n - f\| \to 0$. It suffices to show that f is right-continuous.

Let $x \in (a, b)$ and $\epsilon > 0$ be given. Then there exists N such that $\|f - f_N\| < \epsilon/3$, which implies

$$\begin{aligned}
|f(x+h) - f(x)| &\leq |f(x+h) - f_N(x+h)| + |f_N(x+h) - f_N(x)| + |f_N(x) - f(x)| \\
&\leq |f_N(x+h) - f_N(x)| + 2\|f_N - f\| \\
&< |f_N(x+h) - f_N(x)| + 2\epsilon/3.
\end{aligned}$$

Moreover, by right continuity of f_N, there exists $\delta > 0$ such that $|f_N(x+h) - f_N(x)| < \epsilon/3$ for all $h \in (0.\delta)$. Hence $|f(x+h) - f(x)| < \epsilon$ for all $h \in (0, \delta)$. As a result, $\lim_{h \to 0^+} |f(x+h) - f(x)| = 0$, which implies the right continuity of f. $\square$

Theorem 1.56. Let $(X, \|\cdot\|)$ be a finite-dimensional normed space. Then X is complete.

Proof. Suppose $\dim X = n$. Choose a basis of $X : e_1, \cdots, e_n$. We claim that there exists $c_1, c_2 > 0$ such that for all $x = \sum_{i=1}^n x_i e_i \in X$,

$$c_1 \left(\sum_{i=1}^{n} x_i^2 \right) \leq \|x\| \leq c_2 \left(\sum_{i=1}^{n} x_i^2 \right).$$

We consider the unit sphere

$$S^{n-1} = \left\{ \sum_{i=1}^{n} x_i e_i : x_1^2 + \cdots + x_n^2 = 1 \right\}$$

in $\mathbb{R}^n$, and the map $f : S^{n-1} \to \mathbb{R}$, $(x_1, \cdots, x_n) \mapsto \|\sum_{i=1}^n x_i e_i\|$, which is continuous. By compactness of S^{n-1}, there exists $c_1, c_2 > 0$ such that $f(S^{n-1}) \subset [c_1, c_2]$. By homogeneity of norm, the claim is satisfied.

As a result, any sequence in X converges relative to $\|\cdot\|_2$ also converges relative to $\|\cdot\|$. Since the space $(\mathbb{R}^n, \|\cdot\|_2)$ is complete, $(X, \|\cdot\|)$ is also complete. $\square$

Corollary 1.57. Let L be a finite-dimensional subspace of a normed space X. Then L is closed in X.

Example 1.58. The space $(C([0,1]), \|\cdot\|_1)$, which is a subspace of $L^1([0,1])$, is not complete. Define

$$f_n(x) = \begin{cases} 1, \ 0 \leq x \leq \frac{1}{2}, \\ 1 - n(x - 1/2), \ \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{n}, \\ 0, \ \frac{1}{2} + \frac{1}{n} \leq x \leq 1, \end{cases}$$

which converges to $\chi_{[0,1/2]}$ pointwise. As a result, $\|f_n - \chi_{[0,1/2]}\| = 1/2n \to 0$. Thus we obtain a Cauchy sequence in $C([0,1])$ that does not converges in $(C([0,1]), \|\cdot\|_1)$.

Now we introduce the nested sequence theorem.

Theorem 1.59 (Nested sphere theorem). Let (X, d) be a complete metric space. Let

$$B_n = \{x \in X : d(x, x_n) \leq \epsilon_n\}$$

be a sequence of monotone decreasing closed spheres: $B_1 \supset B_2 \supset \cdots \supset B_n \supset B_{n+1} \supset \cdots$. If $\lim_{n\to\infty} \epsilon_n = 0$, then there exists a unique $\xi \in \bigcap_{n=1}^{\infty} B_n$.

Proof. For any $n \geq m$, $x_n \in B_n \subset B_m$, then $d(x_n, x_m) \leq \epsilon_n$. Since $\lim_{n\to\infty} \epsilon_n = 0$, (x_n) is a Cauchy sequence in X, which converges to some x by completeness of X. Let $m \to \infty$, we have $d(x, x_n) \leq \epsilon_n$, which implies $x \in B_n$ for all $n \in \mathbb{N}$. Hence $x \in \bigcup_{n=1}^{\infty} B_n$.

To show uniqueness, let $y \in \bigcup_{n=1}^{\infty} B_n$. Then $x, y \in B_n$ for all $n \in \mathbb{N}$, and $d(x, y) \leq 2\epsilon_n \to 0$. □

The depiction of nested sequence also implies completeness of the corresponding metric space.

Theorem 1.60. Let (X, d) be a metric space in which the nested sphere theorem (Theorem 1.59) holds. Then (X, d) is complete.

Proof. Let (x_n) be a Cauchy sequence in X, we choose a subsequence (x_{n_k}) such that $d(x_{n_k}, x_{n_{k+1}}) \leq 2^{-k}$. Then for all $m \geq k$, $d(x_{n_k}, x_{n_m}) \leq 2^{-k+1}$. We choose sequence of closed sphere $B_k = B(x_{n_k}, 2^{-k+1})$, then we have $B_1 \supset B_2 \supset \cdots$ and $\lim_{k\to\infty} 2^{-k+1} = 0$. As a result, there exists a unique $x \in \bigcup_{n=1}^{\infty} B_n$ to which (x_{n_k}) converges. □

1.4.2 Completion

We consider the procedure from incomplete to complete space.

Definition 1.61 (Completion). Let (X, d) be a metric space. A complete metric space $(Y, \tilde{d})$ is said to be a *completion* of (X, d), if there exists an injective mapping $\iota : X \to Y$ such that (i) ι is isometric, i.e. $\tilde{d}(\iota(x), \iota(x')) = d(x, x')$ for any pair $x, x' \in X$, and (ii) $\overline{\iota(X)} = Y$. In this case, ι is called an *imbedding.*

The following theorem states that every incomplete metric space has at least one completion.

Theorem 1.61 (Existence of a completion). Let (X, d) be a metric space. Then there exists a completion of (X, d). Namely, there exists an isometric imbedding from X to a complete metric space.

Proof. We construct a complete metric space which consists of equivalence classes of Cauchy sequences in X.

Step I: Let Y' be the set of all Cauchy sequences $\mathbf{x} = (x_1, x_2, \cdots)$ in X. Let $d'(\mathbf{x}, \mathbf{y}) := \lim_{n\to\infty} d(x_n, y_n)$. Then d' is a pseudometric on Y', that is, $d' : Y' \times Y' \to \mathbb{R}_+$ satisfies symmetry and triangle inequality.

Step II: Define a relation $\sim$ on Y': for $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$ in Y',

$$\mathbf{x} \sim \mathbf{y} \overset{\text{def.}}{\Leftrightarrow} \lim_{n\to\infty} d(x_n, y_n) = 0.$$

It is clear that $\sim$ is an equivalence relation on Y', i.e., $\sim$ has reflexivity, symmetry and transitivity. Let $\widetilde{Y} = Y'/\sim$ be the set of equivalence classes on Y', and define $\tilde{d} : \widetilde{Y} \times \widetilde{Y} \to \mathbb{R}_+$ as

$$\tilde{d}([\mathbf{x}], [\mathbf{y}]) = \lim_{n\to\infty} d(x_n, y_n).$$

Note that $\tilde{d}([\mathbf{x}], [\mathbf{y}]) = d'(\mathbf{x}, \mathbf{y})$. Following Step I, $\tilde{d}$ is a metric on $\widetilde{Y}$.

Step III: Define $\iota : X \to \widetilde{Y}, x \mapsto [(x, x, \cdots)]$, which maps a point of X to an equivalence class of a constant sequence. Clearly, $\tilde{d}(\iota(x), \iota(y)) = d(x, y)$, which implies that ι is an isometric imbedding.

Now we show that $\overline{\iota(X)} = \widetilde{Y}$. Given any Cauchy sequence $\mathbf{x} = (x_n) \in Y'$, we have

$$\lim_{n\to\infty} \tilde{d}(\iota(x_n), [\mathbf{x}]) = \lim_{n,m\to\infty} d(x_n, x_m) = 0,$$

which implies $[\mathbf{x}] \in \overline{\iota(X)}$. Since $\mathbf{x}$ is arbitrary, we have $\overline{\iota(X)} = \widetilde{Y}$.

Step IV: It remains to show the completeness of $(\widetilde{Y}, \tilde{d})$. By Lemma 1.53 (iii), it suffices to show that every Cauchy sequence in $\iota(X)$ converges in $\widetilde{Y}$.

Let $\{[\mathbf{x}^{(n)}]\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\iota(X)$, where $\mathbf{x}^{(n)} = (x_n, x_n, \cdots)$ for each $n \in \mathbb{N}$. By definition, $\tilde{d}([\mathbf{x}^{(n)}], [\mathbf{x}^{(m)}]) = d(x_n, x_m)$, which implies that $\mathbf{x} = (x_n)$ is a Cauchy sequence in X. Moreover,

$$\lim_{n\to\infty} \tilde{d}\left([\mathbf{x}^{(n)}], [\mathbf{x}]\right) = \lim_{n\to\infty} \left[\lim_{k\to\infty} d(x_n, x_k)\right] = 0,$$

which implies $[\mathbf{x}^{(n)}] \to [\mathbf{x}] \in \widetilde{Y}$. Therefore we obtain a completion of X. □

By construction, we showed that every metric space has at least one completion. Naturally, we wonder if the completion is unique. We have the following theorem.

Theorem 1.62 (Uniqueness of the completion). The completion of a metric space (X, d) is uniquely determined up to an isometry. Namely, if $\iota_1 : X \to Y_1 = \overline{\iota_1(X)}$ and $\iota_2 : X \to Y_2 = \overline{\iota_2(X)}$ are two isometric imbeddings from X to a complete metric space, then there exists an isometric bijection from Y_1 to Y_2.

Proof. Step I: Define map $\phi_0 : \iota_1(X) \to \iota_2(X)$, $\iota_1(x) \mapsto \iota_2(x)$, which is bijective and isometric from $\iota_1(X)$ to $\iota_2(X)$. We extend ϕ_0 to $\phi : Y_1 \to Y_2$ as follows: Given $y_1 \in Y_1$, choose a sequence (x_n) of points of X such that $d_{Y_1}(\iota_1(x_n), y_1) \to 0$, which is feasible because $Y_1 = \overline{\iota_1(X)}$, and define

$$\phi(y_1) = \lim_{n\to\infty} \phi_0(\iota_1(x_n)) = \lim_{n\to\infty} \iota_2(x_n).$$

Step II: check that ϕ is well-defined. Since $(\iota_1(x_n))$ converges to $y_1 \in Y_1$, it is a Cauchy sequence in Y_1. Note that ι_1 and ι_2 are isometric, (x_n) is a Cauchy sequence in X, and $(\iota_2(x_n))$ is a Cauchy sequence in Y_2. By completeness of Y_2, $(\iota_2(x_n))$ converges to some point y_2 of Y_2.

Suppose (x_n') is another sequence of points of X such that $\iota_1(x_n') \to y_1$. Repeat the above procedure, there exists $y_2' \in Y_2$ such that $\iota_2(x_n') \to y_2'$. Moreover,

$$d_{Y_2}(y_2, y_2') = \lim_{n,m\to\infty} d_{Y_2}(\iota_2(x_n), \iota_2(x_m')) = \lim_{n,m\to\infty} d(x_n, x_m') = \lim_{n,m\to\infty} d_{Y_1}(\iota_1(x_n), \iota_1(x_m')) = d_{Y_1}(y_1, y_1) = 0.$$

Hence $y_2 = y_2'$. Therefore, $\phi : Y_1 \to Y_2$ is well-defined and agrees with $\iota_2 \circ \iota_1^{-1}$ on $\iota_1(X)$.

Step III: It remains to show that ϕ is isometric. Given $y, y' \in Y_1$, we choose two sequences (x_n) and (x_n') from X such that $\iota_1(x_n) \to y$ and $\iota_1(x_n') \to y'$. Then we have

$$d_{Y_2}(\phi(y), \phi(y')) = \lim_{n,m\to\infty} d_{Y_2}(\iota_2(x_n), \iota_2(x_m')) = \lim_{n,m\to\infty} d(x_n, x_m') = \lim_{n,m\to\infty} d_{Y_1}(\iota_1(x_n), \iota_1(x_m')) = d_{Y_1}(y, y').$$

Hence ϕ is an isometric bijection from Y_1 to Y_2. □

Remark. Combining Theorem 1.61 and Theorem 1.62, we conclude that for every metric space, there exists a unique completion up to an isometry.

1.4.3 Contraction mappings and Banach Fixed Point Theorem

Review: Newton's method. To solve a equation $f(x) = 0, x \in [a, b]$, where f is differentiable, define

$$T(x) = x - \frac{f(x)}{f'(x)}.$$

We choose $x_0 \in [a, b]$, and update $x_{n+1} = T(x_n)$. In appropriate conditions, $x_n \to x$ such that $f(x) = 0$.

Review: Picard's method for ordinary differential equations. To solve the ODE

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y), \\ y(x_0) = y_0, \end{cases} \quad \Rightarrow \; y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\mathrm{d}s. \tag{1.19}$$

For appropriate f, let $\varphi_0(x) \equiv y_0$. Update:

$$\varphi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(s, \phi_n(x))\mathrm{d}s.$$

Then $\varphi_n \rightrightarrows \varphi$, where φ is the solution of ODE (1.19).

Definition 1.63 (Fixed points). Let X be a nonempty set, and let $\phi : X \to X$. If there exists $x^* \in X$ such that $\phi(x^*) = x^*$, then x^* is said to be a *fixed point* of X.

Definition 1.64 (Contraction mappings). Let (X, d) be a metric space, and let $\phi : X \to X$. If there exists $\gamma \in (0, 1)$ such that $d(\phi(x), \phi(y)) < \gamma d(x, y)$ for all $x, y \in X$, then ϕ is said to be a *contraction mapping* on X.

Lemma 1.65. A contraction mapping is continuous.

Proof. Given $x_n \to x$, $d(\phi(x_n), \phi(x)) \leq \gamma d(x_n, x) \to 0$ as $n \to \infty$. □

Theorem 1.66 (Banach fixed point theorem). Let (X, d) be a complete metric space. Let $\phi : X \to X$ be a contraction mapping on X. Then ϕ has a unique fixed point.

Proof. Choose $x_0 \in X$. Let $x_n = \phi(x_{n-1}) = \phi^n(x_0)$ for all $n \in \mathbb{N}$. We claim that (x_n) is a Cauchy sequence in X. For all $n \in \mathbb{N}$,

$$d(x_{n+1}, x_n) = d\left(\phi(x_n), \phi(x_{n-1})\right) < \gamma d(x_n, x_{n-1}) < \cdots < \gamma^n d(x_1, x_0).$$

For any $n, p \in \mathbb{N}$, by triangle inequality,

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_{n+p}, d_{n+p-1}) + d(x_{n+p-1}, d_{n+p-2}) + \cdots + d(x_{n+1}, d_n) \\ &\leq \left(\gamma^{n+p-1} + \gamma^{n+p-2} + \cdots + \gamma^n\right) d(x_1, x_0) < \frac{\gamma^n}{1-\gamma} d(x_1, x_0). \end{aligned} \tag{1.20}$$

Then (x_n) is a Cauchy sequence, which converges to some $x^* \in X$. Let $p \to \infty$ in (1.20), then

$$d(x^*, x_n) \leq \frac{\gamma^n}{1-\gamma} d(x_1, x_0).$$

As a result, x is a fixed point of ϕ:

$$\phi(x^*) = \phi\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} x_{n+1} = x^*.$$

To show the uniqueness, let x' be a fixed point of ϕ. Then

$$d(x', x^*) = d(\phi(x'), \phi(x^*)) < \gamma d(x', \phi(x^*)),$$

which implies $d(x', x^*) = 0$, and $x' = x^*$. □

Remark. If X is not complete, Banach fixed point theorem does not hold. Consider

$$\phi : (0,\infty) \to (0,\infty),\ x \mapsto \gamma x,\ \text{where } 0 < \gamma < 1.$$

Furthermore, if $\gamma = 1$ in Definition 1.64, Banach fixed point theorem does not hold. Consider

$$\phi : [0,\infty) \mapsto [0,\infty),\ x \mapsto x + \frac{1}{1+x}.$$

Theorem 1.67 (Generalization). Let (X, d) be a complete metric space, and let $\phi : X \to X$ is a map on X. If there exists $n \in \mathbb{N}$ such that ϕ^n is a contraction mapping, then ϕ has a unique fixed point.

Proof. Denote $\psi = \phi^n$. By Theorem 1.66, ψ has a unique fixed point $x^* \in X$. Also,

$$\phi(x^*) = \phi\left(\psi(x^*)\right) = \phi^n(x^*) = \psi\left(\phi(x^*)\right)$$

is a fixed point of ψ, which implies $\phi(x^*) = x^*$. Hence x^* is a fixed point of ψ. To show the uniqueness, let x' be a fixed point of ϕ. Then x' is a fixed point of ψ, which implies $x' = x^*$. □

We present some applications of Banach fixed point theorem.

Example 1.68 (Implicit function theorem). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous on $\mathcal{D} = [a,b] \times \mathbb{R}$, and there exists $m < M$ such that $0 < m \le D_y f(x,y) \le M$ for all $(x,y) \in \mathcal{D}$. Then there exists unique continuous $\varphi \in C([a,b])$ such that $f(x,\varphi(x)) = 0$.

Proof. Define $A : C([a,b]) \to C([a,b])$ in $(C([a,b]).\|\cdot\|)$ by

$$(A\varphi)(x) := \varphi(x) - \frac{1}{M} f(x,\varphi(x)).$$

Then A is a contraction on $(C([a,b]).\|\cdot\|)$:

$$\begin{aligned}
\|A\varphi_1 - A\varphi_2\|_\infty &= \sup_{x\in[a,b]} |A\varphi_1(x) - A\varphi_2(x)| \\
&= \sup_{x\in[a,b]} \left|\varphi_1(x) - \varphi_2(x) - \frac{1}{M}\left(f(x,\varphi_2(x)) - f(x,\varphi_1(x))\right)\right| \\
&= \sup_{x\in[a,b]} \left|1 - \frac{1}{M} D_y f(x,\xi_x)\right| |\varphi_1(x) - \varphi_2(x)| && \text{(By mean value theorem)} \\
&\le \left(1 - \frac{m}{M}\right) \|\varphi_2 - \varphi_1\|_\infty.
\end{aligned}$$

By Theorem 1.66, there exists unique $\varphi \in C([a,b])$ such that $A\varphi = \varphi$, which implies $f(x,\varphi(x)) = 0$. □

Example 1.69 (Volterra integral equation). Suppose $f \in C([a,b])$, and $K \in C(\mathcal{D})$, where

$$\mathcal{D} = \left\{(x,y) \in \mathbb{R}^2 : a \le x \le b, a \le y \le x\right\}.$$

Then for all $\lambda \in \mathbb{R}$, the Volterra integral equation

$$\varphi(x) = f(x) + \lambda \int_a^x K(x,y)\varphi(y)\,dy$$

has a unique continuous solution in $C([a,b])$.

Proof. Set $M = \sup_{x,y\in\mathcal{D}} |K(x,y)|$. Define $B : C([a,b]) \to C([a,b])$ by

$$(B\varphi)(x) = f(x) + \lambda \int_a^x K(x,y)\varphi(y)\,dy.$$

For all $\varphi_1, \varphi_2 \in C([a,b])$ and all $x \in [a,b]$,

$$\begin{aligned}
|(B\varphi_2)(x) - (B\varphi_1)(x)| &= |\lambda| \left|\int_a^x K(x,y)(\varphi_2(y) - \varphi_1(y))\,dy\right| \\
&\le M(x-a)\,|\lambda|\,\|\varphi_2 - \varphi_1\|_\infty,
\end{aligned}$$

$$\begin{aligned}
\left|(B^2\varphi_2)(x) - (B^2\varphi_1)(x)\right| &= |\lambda| \left|\int_a^x K(x,y)\left[(B\varphi_2)(y) - (B\varphi_1)(y)\right]\,dy\right| \\
&\le |\lambda| \left|\int_a^x M \cdot M(y-a)\,|\lambda|\,\|\varphi_2 - \varphi_1\|_\infty\,dy\right| \\
&\le \frac{1}{2} M^2 (x-a)^2 \,|\lambda|^2\, \|\varphi_2 - \varphi_1\|_\infty,
\end{aligned}$$

$$\cdots,$$

by induction,

$$|(B^n\varphi_2)(x) - (B^n\varphi_1)(x)| = \frac{1}{n!} M^n (x-a)^n \,|\lambda|^n\, \|\varphi_2 - \varphi_1\|_\infty.$$

for all $n \in \mathbb{N}$. Then for efficiently large n,

$$\frac{1}{n!} M^n (b-a)^n \,|\lambda|^n < 1,$$

and B^n is a contraction mapping. By Theorem 1.67, B has a unique fixed point φ^*, which is the solution of Volterra integral equation. $\square$

1.5 Compactness and Sequential Compactness

Review: Compactness and sequential compactness. Given a subset A of a topological space X, A is said to be *compact* if every open cover of A has a finite subcover, and A is said to be *sequentially compact* if every sequence of points of A has a subsequence that converges to some point of A.

In a metric space X, a subset A is compact if and only if it is sequentially compact.

Review: Relatively compactness and relative sequential compactness. Let (X, d) be a metric space, and let A be a subset of X.

(i) A is said to be *relatively compact* (or *precompact*) if its closure is compact;

(ii) A is said to be *relatively sequentially compact*, if for every sequence $(x_n) \subset A$ there exists a subsequence that converges to some $x \in X$. (Clearly, $x \in \overline{A}$.)

The following proposition reveals the equivalence of these two definitions.

Proposition 1.70. Let (X, d) be a metric space. Let $A \subset X$. Then A is relatively sequentially compact if and only if A is relatively compact.

Proof. Suppose that $\overline{A}$ is compact. Then $\overline{A}$ is also sequentially compact by Theorem 4.36, and the relative sequential compactness of A is clear.

Conversely, suppose that A is relatively sequentially compact. We show that $\overline{A}$ is sequentially compact. Let (x_n) be a sequence of points of $\overline{A}$. For every $n \in \mathbb{N}$, since $x_n \in \overline{A}$, we can choose $y_n \in A$ such that $d(x_n, y_n) < 1/n$. By relative sequential compactness of A, there is a subsequence (y_{n_k}) with $y_{n_k} \to y \in \overline{A}$.

Fix $\epsilon > 0$. We first choose $K_1 \in \mathbb{N}$ such that $d(y_{n_k}, y) < \epsilon/2$ for all $k \geq K_1$. Then we choose $K_2 \in \mathbb{N}$ such that $n_k > 2/\epsilon$ for all $k \geq K_2$, which implies $d(x_{n_k}, y_{n_k}) < \epsilon/2$. Hence $d(x_{n_k}, y) < \epsilon$ for all $k \geq \max\{K_1, K_2\}$, and the subsequence (x_{n_k}) converges to $y \in \overline{A}$ as $k \to \infty$. Therefore $\overline{A}$ is sequentially compact. $\square$

Review: Totally bounded sets. Let (X, d) be a metric space, and let A be a subset of X.

- A is said to be *bounded*, if its diameter $\operatorname{diam}(A) := \sup_{x,x' \in A} d(x, x')$ is finite.
- Given $\epsilon > 0$, an *ϵ-cover* of A is a collection of open balls of radius ϵ of which the union covers A. An ϵ-net is the centers of balls in an ϵ-cover.
- If for all $\epsilon > 0$, A has a finite ϵ-net, then A is said to be *totally bounded*.

Let A be a totally bounded set. Clearly, $\overline{A}$ is also totally bounded. Fix $\epsilon > 0$, we first cover A by finitely many open balls $O(x_j, \epsilon/2),\ j = 1, \cdots, n$. For any $y \in \overline{A}$, there exists $x \in A$ such that $d(x, y) < \epsilon/2$. As a result, we can cover $\overline{A}$ by expanding the radii of the balls to ϵ.

The following theorem reveals the equivalence between totally bounded sets and relatively compact sets in metric spaces.

Theorem 1.71 (Hausdorff). Let X be a metric space. Let $A \subset X$.

(i) If A is relatively compact, then A is totally bounded.

(ii) If X is complete and A is totally bounded, then A is relatively compact.

Proof. (i) Consider a cover of $\overline{A}$ consists of open ϵ-balls, the conclusion is clear by finding a finite subcover.

(ii) We shall prove that $\overline{A}$ is sequentially compact. Let (x_n) be a sequence of points, it suffices to construct a subsequence of (x_n) that is a Cauchy sequence, which converges by completeness of $\overline{A}$.

We first cover $\overline{A}$ by finitely many 1-balls. At least one of these balls, denoted by O_1, contains infinitely many elements of (x_n). We denote by $J_1 = \{n \in \mathbb{N} : x_n \in O_1\}$ the index set of these elements.

Next, cover $\overline{A}$ by finitely may 1/2-balls. Since J_1 is infinite, at least one of these balls, denoted by O_2, contains infinitely many elements of $\{x_n : n \in J_1\}$. Similarly, let $J_2 = \{n \in J_1 : x_n \in O_2\}$. By repeating this procedure, we obtain a finite cover of $\overline{A}$ by $1/k$-balls and an infinite index set $J_k = \{n \in J_{k-1} : x_n \in O_k\}$ for arbitrarily large k, with $J_k \subset J_{k-1} \subset \cdots \subset J_1$.

Choose $n_1 \in J_1$. Given n_{k-1}, choose $n_k \in J_k$ with $n_k > n_{k-1}$, which is feasible because J_k is infinite. For any $l, m \geq k$, $n_l, n_m \in J_k$, and $x_{n_l}, x_{n_m} \in O_k$, implying $d(x_{n_l}, x_{n_m}) < 2/k$. Hence the subsequence (x_{n_k}) is a Cauchy sequence, as desired. □

Theorem 1.72. A metric space X is complete if every totally bounded set in X is relatively compact.

Proof. Let (x_n) be a Cauchy sequence in X. Given $\epsilon > 0$, there exists N such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$. Hence $O(x_k, \epsilon),\ k = 1, \cdots, N$ is an ϵ-net of $\{x_n, n \in \mathbb{N}\}$.

By Theorem 1.71, every Cauchy sequence (x_n) in X is relatively sequentially compact, and the completeness follows from subsequence criterion (Lemma 1.53). □

Lemma 1.73. Any relatively compact set is separable.

Proof. Let A be a relatively compact subset of a metric space X. Since A is totally bounded, we can cover it by finitely many 1-balls. We denote the centers of these balls by C_1. Similarly, we can cover A by finitely many $1/n$-balls for arbitrarily large $n \in \mathbb{N}$, and extract their centers C_n. Then $\bigcup_{n=1}^\infty C_n$, being the union of countably many finite sets, is a countable dense set in X. □

Example 1.74. We know that in a finite-dimensional space $\mathbb{R}^n$, the closed unit ball

$$B = \left\{(x_1, \cdots, x_n) : x_1^2 + \cdots + x_n^2 \leq 1\right\}$$

is compact. However, this does not hold when the dimension becomes infinite.

We give a counterexample here. Let $l^2 := \left\{\mathbf{x} = (x_1, x_2, \cdots) : \sum_{n=1}^\infty |x_n|^2 < \infty\right\}$ be the set of square-summable real sequences. Then l^2 is a Banach space under norm

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{k=1}^\infty |x_k|^2}.$$

Consider the closed unit ball

$$B = \left\{\mathbf{x} = (x_1, x_2, \cdots) \in l^2 : \sum_{k=1}^\infty |x_k|^2 \leq 1\right\}.$$

We let $\mathbf{e}_k = (0, \cdots, 0, \underset{k\text{-th}}{1}, 0, \cdots)$ be the unit vector whose k-th element is 1, and choose open balls

$$O = \{\mathbf{x} \in l^2 : \|\mathbf{x}\|_2 < 1\}, \quad O_k = \left\{\mathbf{x} \in l^2 : \|\mathbf{x} - \mathbf{e}_k\|^2 < 1/2\right\},\ k \in \mathbb{N}.$$

And for each $\mathbf{x} \in E := B \backslash (O \cup (\bigcup_{k=1}^\infty O_k))$, define $O_\mathbf{x} = \{\mathbf{y} \in l^2 : \|\mathbf{y} - \mathbf{x}\|_2 < 1/2\}$. Then $\mathbf{e}_k \notin O_\mathbf{x}$ for each k. As a result, we obtain an open cover

$$O \cup \left(\bigcup_{k=1}^\infty O_k\right) \cup \left(\bigcup_{\mathbf{x} \in E} O_\mathbf{x}\right)$$

of B. Moreover, every open ball in this cover contains at most one $\mathbf{e}_k$.

Lemma 1.75 (F.Riesz). Let X be a normed space, and let A be a closed proper subspace of X, i.e. $A \neq X$. Then for all $0 < \epsilon < 1$, there exists unit vector $x_0 \in X$ such that $d(x_0, A) > \epsilon$.

Proof. Choose $\bar{x} \in X \backslash A$. Since A is a closed subspace, $d(\bar{x}, A) > 0$, and there exists $x' \in A$ such that $\|\bar{x} - x'\| < d(\bar{x}, A)/\epsilon$. We define

$$x_0 = \frac{\bar{x} - x'}{\|\bar{x} - x'\|} \notin A.$$

For any $x \in A$, we have

$$x - x_0 = x - \frac{\bar{x} - x'}{\|\bar{x} - x'\|} = \underbrace{x + \frac{x'}{\|\bar{x} - x'\|}}_{\in A} - \underbrace{\frac{\bar{x}}{\|\bar{x} - x'\|}}_{\notin A}$$

$$\|x - x_0\| \geq d\left(\frac{\bar{x}}{\|\bar{x} - x'\|}, A\right) = \frac{d(\bar{x}, A)}{\|\bar{x} - x'\|} > \epsilon.$$

Since x is arbitrary, the result follows. □

Theorem 1.76. Let X be a normed space. If X is infinite-dimensional, then the closed unit ball in X is not compact.

Proof. Choose $x_1 \in X$ with $\|x_1\| = 1$, and let $A_1 = \operatorname{span}\{x_1\}$. By Corollary 1.57 and Lemma 1.75, choose $x_2 \in X$ such that $\|x_2\| = 1$ and $d(x_2, A_1) > 1/2$, and let $A_2 = \operatorname{span}\{x_1, x_2\} \neq X$. Repeat this procedure, we obtain a sequence (x_n) of unit vectors in X. For any $m < n$, we have

$$x_m \in A_{n-1},\ x_n \notin A_{n-1},\ d(x_n, x_m) > 1/2.$$

To obtain an open cover of the closed unit ball in X, we take open balls $O = O(0, 1)$, $O_n = O(x_n, 1/2)$ for every $n \in \mathbb{N}$, and $O_x = O(x, 1/2)$ for every $x \notin O \cup \bigcup_{n=1}^n O_n$. Each of them contains at most one element of $\{x_n, n \in \mathbb{N}\}$. Hence there exists no finite subcover. □

It is seen that bounded closed sets in infinite-dimensional spaces are not always compact. We are going to discuss some instances for compact sets in infinite-dimensional spaces.

Review: Equicontinuity. Let X be a topological space and let (Y, d) be a metric space. Let $\mathcal{F}$ be a collection of functions $X \to Y$. Given $x_0 \in X$, $\mathcal{F}$ is said to be *equicontinuous at* x_0 if for each $\epsilon > 0$, there exists a neighborhood U of x_0 such that $d(f(x), f(x_0)) < \epsilon$ for all $x \in U$ and all $f \in \mathcal{F}$. If $\mathcal{F}$ is equicontinuous at all $x \in X$, then f is said to be *equicontinuous*.

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $\mathcal{F}$ be a collection of functions $X \to Y$. $\mathcal{F}$ is said to be *uniformly equicontinuous* if for each $\epsilon > 0$, there exists $\delta > 0$ such that $d(f(x), f(x')) < \epsilon$ for all $x, x' \in X$ such that $d(x, x') < \delta$ and all $f \in \mathcal{F}$.

Theorem 1.77 (Arzelà-Ascoli). Give $C([a, b])$ the uniform topology (induced by $\|\cdot\|_\infty$). A subset $\mathcal{F}$ of $C([a, b])$ is relatively compact if and only if it is bounded and uniformly equicontinuous.

Proof. "If" case: Suppose $\mathcal{F}$ is bounded and uniformly equicontinuous. By Theorem 1.71, it suffices to show $\mathcal{F}$ is totally bounded. Given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(x')| < \epsilon/3$ for all $|x - x'| < \delta$. We first choose a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that $|x_j - x_{j-1}| < \delta$ for all $j = 1, \cdots, n$.

Since $\mathcal{F}$ is bounded, there exists K such that $\max_{x\in[a,b]}|f(x)| \le K$ for all $f \in \mathcal{F}$, and

$$A = \{(f(x_0), f(x_1), \cdots, f(x_n)) : f \in \mathcal{F}\} \subset \mathbb{R}^{n+1}$$

is a bounded set. Note that A is finite-dimensional, it is totally bounded. Then there exist $f_1, \cdots, f_k \in \mathcal{F}$ that form a $\epsilon/3$-net of A.

We claim that $\{f_1, \cdots, f_k\}$ is an ϵ-net of $\mathcal{F}$: for any $x \in [a,b]$, it lies in some $[x_{j-1}, x_j]$; for any $f \in \mathcal{F}$, there exists $p \in \{1, \cdots, k\}$ such that $\{f(x_j), j = 0, 1, \cdots, n\}$ lies in the $\epsilon/3$-ball centered at $\{f_p(x_j), j = 0, 1, \cdots, n\}$.

$$|f(x) - f_p(x)| \le |f(x) - f(x_j)| + |f(x_j) - f_p(x_j)| + |f_p(x_j) - f_p(x)| < \epsilon.$$

As a result, $\{f_1, \cdots, f_k\}$ is a ϵ-net of $\mathcal{F}$, as desired.

"Only if" case: Suppose $\mathcal{F}$ is relatively compact, it is totally bounded. Given $\epsilon > 0$, let $\mathcal{N} = \{f_1, \cdots, f_k\}$ be an $\epsilon/3$-net of $\mathcal{F}$ in $C([a,b])$. By compactness of $[a,b]$, any f_i is uniformly continuous on $[a,b]$, and we choose $\delta_i > 0$ such that $|f(x) - f(x')| < \epsilon/3$ for all $|x - x'| < \delta$. Let $\delta = \min_{i\in\{1,\cdots,k\}} \delta_i$. Then for any $f \in \mathcal{F}$,

$$|f(x_1) - f(x_2)| \le |f(x_1) - f_p(x_1)| + |f_p(x_1) - f_p(x_2)| + |f_p(x_2) - f(x_2)| < \epsilon, \ \exists f_p \in \mathcal{N}$$

Hence $\mathcal{F}$ is uniformly equicontinuous, as desired. □

Theorem 1.78 (Kolmogorov-Riesz-Fréchet). Let $1 \le p < \infty$. Let $\mathcal{F}$ be a subset of $L^p(\mathbb{R})$. Then $\mathcal{F}$ is relatively compact if and only if the following conditions hold:

(i) (Bounded). there exists $M > 0$ such that $\sup_{f\in\mathcal{F}} \|f\|_p \le M$.

(ii) (Equitight). For all $\epsilon > 0$, there exists $r > 0$ such that

$$\int_{|x|\ge r} |f(x)|^p dm(x) < \epsilon^p, \ \forall f \in \mathcal{F}.$$

(iii) (L^p-Equicontinuous). For all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|\tau_h f - f\|_p < \epsilon, \ \forall f \in \mathcal{F}, \ h \in (0, \delta).$$

Proof. "If" case: We first suppose the functions $\mathcal{F}$ are supported on $[a,b]$. For any $\delta > 0$, define

$$f^\delta(x) := \frac{1}{2\delta}(f * \chi_{[-\delta,\delta]})(x) = \frac{1}{2\delta}\int_{[-\delta,\delta]} f(x-y)\, dm(y).$$

For any $f \in \mathcal{F}$, by Jensen's inequality, we have

$$\left|f^\delta(x) - f(x)\right|^p = \left|\frac{1}{2\delta}\int_{[-\delta,\delta]} (f(x-y) - f(x))\, dm(y)\right|^p \le \frac{1}{2\delta}\int_{[-\delta,\delta]} |f(x-y) - f(x)|^p \ dm(y)$$

Given $\epsilon > 0$, we choose δ that satisfies condition (iii):

$$\int_{\mathbb{R}} |f^\delta(x) - f(x)|^p\, dm(x) \le \frac{1}{2\delta}\int_{\mathbb{R}}\int_{[-\delta,\delta]} |f(x-y) - f(x)|^p\, dm(y) = \frac{1}{2\delta}\int_{[-\delta,\delta]} \|\tau_y f - f\|_p^p\, dm(y) \le \epsilon^p.$$

We still denote by f^δ the restriction of f^δ on $[a,b]$. Let $\mathcal{F}^\delta = \{f^\delta : f \in \mathcal{F}\}$. It is bounded and equicontinuous on $[a,b]$: For any $f^\delta \in \mathcal{F}$ and any $x \in [a,b]$,

$$|f^\delta(x)| \leq \frac{1}{2\delta}\int_{[-\delta,\delta]} |f(x-y)|\, dm(y) \leq \left(\frac{1}{2\delta}\int_{\mathbb{R}} |f(x-y)|^p\, dm(y)\right)^{1/p} \leq (2\delta)^{-1/p} M.$$

To show equicontinuity, fix $\epsilon' > 0$, and choose δ' such that $\|\tau_h f - f\| < (2\delta)^{1/p}\epsilon'$. Then for any $|x_1 - x_2| < \delta'$,

$$\begin{aligned}\left|f^\delta(x_1) - f^\delta(x_2)\right| &\leq \left(\frac{1}{2\delta}\int_{[-\delta,\delta]} |f(x_1-y) - f(x_2-y)|^p \ dm(y)\right)^{1/p} \\ &\leq \left(\frac{1}{2\delta}\int_{\mathbb{R}} |f(t + x_1 - x_2) - f(t)|^p \ dm(t)\right)^{1/p} \leq \epsilon'.\end{aligned}$$

Hence $\mathcal{F}^\delta$ is a bounded and equicontinuous set in $(C([a,b]), \|\cdot\|_\infty)$. Using Arzelà-Ascoli theorem, we choose an ϵ-net $\mathcal{N}^\delta = \{f_1^\delta, \cdots, f_k^\delta\}$ of $\mathcal{F}^\delta$. Then $\mathcal{N}^\delta$ is an $(b-a)^{1/p}\epsilon$-net in $L^p([a,b])$. Let $\mathcal{N} = \{f : f^\delta \in \mathcal{N}^\delta\}$, then $\mathcal{N}$ is a $\left((b-a)^{1/p} + 2\right)\epsilon$-net of $\mathcal{F}$ in $L^p([a,b])$: For any $f \in \mathcal{F}$, choose the closest $f_i \in \mathcal{N}$, then

$$\|f - f_i\|_p \leq \|f - f^\delta\|_p + \|f^\delta - f_i^\delta\|_p + \|f_i^\delta - f_i\|_p < (b-a)^{1/p}\epsilon + 2\epsilon.$$

Since ϵ is arbitrary, $\mathcal{F}$ is totally bounded.

Now we consider general $f \in L^p(\mathbb{R})$. For any $\epsilon > 0$, by equitightness, choose $R > 0$ such that

$$\int_{|x|\geq R} |f(x)|^p\, dm(x) < \left(\frac{\epsilon}{3}\right)^p.$$

Then $\mathcal{F}_R = \{f\chi_{[-R,R]} : f \in \mathcal{F}\}$ is also a totally bounded set in $L^p([-R,R])$. Let

$$\mathcal{N}_R = \left\{f_1\chi_{[-R,R]}, \cdots, f_k\chi_{[-R,R]}\right\}$$

be an $\epsilon/3$-net of $\mathcal{F}_R$ in $L^p([-R,R])$, then $\mathcal{N} = \{f : f\chi_{[-R,R]} \in \mathcal{N}_R\}$ is an ϵ-net in $L^p(\mathbb{R})$:

$$\|f - f_i\|_p = \|f - f\chi_{[-R,R]}\|_p + \|f - f_i\|_p + \|f_i - f_i\chi_{[-R,R]}\|_p < \epsilon.$$

"Only if" case: Let $\mathcal{F}$ be relatively compact. Then $\mathcal{F}$ is totally bounded. Given $\epsilon > 0$, choose an $\epsilon/3$-net $\mathcal{N} = \{f_1, \cdots, f_k\}$ of $\mathcal{F}$ in $L^p(\mathbb{R})$. For each $f_i \in \mathcal{N}$, by Example 1.48 Remark, there exists δ_i such that $\|\tau_h f_i - f_i\|_p < \epsilon/3$ for all $|h| < \delta_i$. Let $\delta = \min_{i\in\{1,\cdots,k\}} \delta_i$. For any $f \in \mathcal{F}$, there exists $f_i \in \mathcal{N}$ such that

$$\|\tau_h f - f\|_p \leq \|\tau_h f - \tau_h f_i\|_p + \|\tau_h f_i - f_i\|_p + \|f_i - f\|_p < \epsilon,\ \forall |h| < \delta. \tag{iii}$$

To show equitightness (ii), choose r_i such that $\int_{|x|\geq r_i} |f_i(x)|^p\, dm(x) < 2^{-p}\epsilon^p$. Let $r = \max_{i\in\{1,\cdots,k\}} r_i$, then

$$\begin{aligned}\left(\int_{|x|\geq r} |f(x)|^p\, dm(x)\right)^{1/p} &= \left(\int_{|x|\geq r} |f(x) - f_i(x)|^p\, dm(x)\right)^{1/p} + \left(\int_{|x|\geq r} |f_i(x)|^p\, dm(x)\right)^{1/p} \\ &\leq \|f - f_i\|_p + \left(\int_{|x|\geq r} |f_i(x)|^p\, dm(x)\right)^{1/p} < \epsilon.\end{aligned}$$

Thus we complete the proof. □

Remark. In fact, condition (i) in Theorem 1.78 is not required. Conditions (ii) and (iii) sufficiently imply relative compactness of $\mathcal{F}$.

2 Linear Functionals

2.1 Linear Operators and Linear Functionals

In this section we investigate linear operators and linear functionals on general vector spaces.

Definition 2.1 (Linear operators). Let X and Y be two real (or complex) vector spaces. A map $T : X \to Y$ is said to be a *linear operator* from X to Y if for all $x, x' \in X$ and all $\alpha \in \mathbb{R}$ (or $\mathbb{C}$), it holds

$$T(x + x') = Tx + Tx' \quad \text{and} \quad T(\alpha x) = \alpha(Tx).$$

The space X is said to be the *domain* of T, which is also denoted by $\mathfrak{D}(T)$. The image $T(X) \subset Y$ is said to be the *range* of T, denoted by $\mathfrak{R}(T)$. The subspace $T^{-1}(\{0\}) = \{x \in X : T(x) = 0\}$ is said to be the *kernel* (or the *null space*) of T, denoted by $\ker T$.

Example 2.2. Following are some examples of linear operators.

(i) (Matrices). Let $\{e_1, \cdots, e_n\}$ be a basis of $\mathbb{R}^n$, and let $\{f_1, \cdots, f_m\}$ be a basis of $\mathbb{R}^m$. Consider the operator $A : \mathbb{R}^n \to \mathbb{R}^m$ corresponding to a $m \times n$ matrix (a_{ij}),

$$x = \sum_{j=1}^{n} x_j e_j \ \mapsto \ y = Ax = \sum_{i=1}^{n} y_i f_i, \quad x_1, \cdots, x_n, y_1, \cdots, y_m \in \mathbb{R},$$

where $y_i = \sum_{j=1}^{m} a_{ij} x_j$. Then A is a linear operator. With the bases $\{e_1, \cdots, e_n\}$ and $\{f_1, \cdots, f_m\}$ chosen, A is uniquely determined by matrix (a_{ij}). Hence we also write $A = (a_{ij})$.

(ii) (Differentiation). Consider the differentiation operator $D : C^1([a, b]) \to C([a, b])$:

$$Df(x) = \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}, \ x \in [a, b].$$

This is a linear operator on $C^1([a, b])$.

(iii) (Fredholm integral operator). $T : L^2([a, b]) \to L^2([a, b])$,

$$(Tf)(x) = \int_a^b K(x, y) f(y)\, dy,$$

where K is a continuous function on rectangle $[a, b] \times [a, b]$.

Theorem 2.3. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be normed spaces. Let $T : X \to Y$ be a linear operator. The following are equivalent:

(i) T is a continuous operator; (ii) T is continuous at 0;

(iii) T is bounded, i.e. there exists $M > 0$ such that $\|Tx\| \le M\|x\|$ for all $x \in X$.

Proof. (i) $\Rightarrow$ (ii): Clearly.

(ii) $\Rightarrow$ (iii): By assumption, there exists $\delta > 0$ such that $\|Tx\| < 1$ for all $\|x\| < \delta$. Then for all $x \in X \backslash \{0\}$,

$$\|Tx\| = \left\| \frac{2\|x\|}{\delta} T\left(\frac{\delta x}{2\|x\|}\right) \right\| \le \frac{2}{\delta}\|x\|. \tag{2.1}$$

Clearly, (2.1) is true for $x = 0$. Setting $M = 2\delta^{-1}$ complete the proof.

(iii) $\Rightarrow$ (i): For $x, x' \in X$, we have $\|Tx - Tx'\| \le M\|x - x'\|$. Hence T is continuous at each $x \in X$. $\square$

Remark. By Theorem 2.3, we know that a linear operator is bounded if and only if it is continuous. Here we present an example of unbounded linear operators. Consider $T : C^1([0,1]) \to \mathbb{R}$,

$$Tf = \frac{d}{dx} f(x)\bigg|_{x=\frac{1}{2}}.$$

We define sequence $f_n(x) = \frac{1}{n}\sin(2n\pi x)$, then $Tf_n = f_n'(1/2) = (-1)^n\, 2\pi$. While $f_n \to 0$ relative to the supremum norm $\|\cdot\|_\infty$, the image sequence Tf_n diverges.

Definition 2.4 (Operator norm). Let T be a bounded operator from normed space X into normed space Y. The norm of T is defined as

$$\|T\| = \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}.$$

Clearly, $\|Tx\| \leq \|T\|\,\|x\|$ for all bounded operators T and all $x \in X$. Furthermore, by linearity of T, we have

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|\leq 1} \|Tx\| = \sup_{\|x\|<1} \|Tx\|.$$

The last equality holds because $\|Tx\| = \sup_{0<\alpha<1} \|T(\alpha x)\|$ when we fix $\|x\| = 1$.

Example 2.5. Consider operator T on $L^1([a,b])$:

$$(Tf)(x) = \int_a^x f(t)\,dt.$$

(i) If T is viewed as $L^1([a,b]) \to C([a,b])$, then $\|T\| = 1$;
(ii) If T is viewed as $L^1([a,b]) \to L^1([a,b])$, then $\|T\| = b-a$.

Proof. (i) By definition,

$$\|T\| = \sup_{f\in L^1([a,b])\setminus\{0\}} \frac{\|Tf\|_\infty}{\|f\|_1}.$$

For any $f \in L^1([a,b])$ with $f \neq 0$,

$$\frac{\|Tf\|_\infty}{\|f\|_1} = \sup_{x\in[a,b]} \frac{\int_a^x f(t)\,dt}{\int_a^b |f(t)|\,dt} \leq \sup_{x\in[a,b]} \frac{\int_a^x |f(t)|\,dt}{\int_a^b |f(t)|\,dt} = 1.$$

If f is nonnegative on $[a,b]$, the equality holds. Hence $\|T\| = 1$.

(ii) By definition,

$$\|T\| = \sup_{f\in L^1([a,b])\setminus\{0\}} \frac{\|Tf\|_1}{\|f\|_1}.$$

For any $f \in L^1([a,b])$ with $f \neq 0$,

$$\frac{\|Tf\|_1}{\|f\|_1} = \frac{\int_a^b \left|\int_a^x f(t)\,dt\right| dx}{\int_a^b |f(t)|\,dt} \leq \frac{\int_a^b \int_a^x |f(t)|\,dt\,dx}{\int_a^b |f(t)|\,dt} \leq \frac{\int_a^b \int_a^b |f(t)|\,dt\,dx}{\int_a^b |f(t)|\,dt} = b-a.$$

Hence $\|T\| \leq b-a$.

For the other side, define $f_n = n\chi_{[a,a+n^{-1}(b-a)]}$, which is supported on closed interval $[a, a+n^{-1}(b-a)]$. Then

$$\|T\| \geq \frac{\|Tf_n\|_1}{\|f_n\|_1} = \frac{\int_a^{a+n^{-1}(b-a)}(x-a)\,dx + \frac{n-1}{n}(b-a)^2}{(b-a)} = \left(1 - \frac{1}{n} + \frac{1}{2n^2}\right)(b-a),\ \forall n \in \mathbb{N}.$$

Let $n \to \infty$, we have $\|T\| \geq b-a$. □

Example 2.6. Let $g \in L^\infty([a,b])$. Define $T : L^1([a,b]) \to L^1([a,b])$ as

$$(Tf)(x) = f(x)g(x),\ x \in [a,b].$$

Then we have

$$\|T\| = \sup_{f \in L^1([a,b])\setminus\{0\}} \frac{\|Tf\|_1}{\|f\|_1}.$$

For any $f \in L^1([a,b])$, we have

$$\frac{\|Tf\|_1}{\|f\|_1} = \frac{\int_a^b |f(x)g(x)|\,dx}{\int_a^b |f(x)|\,dx} \leq \|g\|_\infty.$$

Hence $\|T\| \leq \|g\|_\infty$. Furthermore, we define $E_n = \{x \in X : g(x) \geq \|g\|_\infty - 1/n\}$. By definition of essential supremum $\|\cdot\|_\infty$, $m(E_n) \neq 0$ for all $n \in \mathbb{N}$. Then

$$\|T\| \geq \frac{\|T\chi_{E_n}\|_1}{\|\chi_{E_n}\|_1} \geq \frac{(\|g\|_\infty - n^{-1})m(E_n)}{m(E_n)} = \|g\|_\infty - \frac{1}{n},\ \forall n \in \mathbb{N}.$$

Let $n \to \infty$, then $\|T\| \geq \|g\|_\infty$. Therefore $\|T\| = \|g\|_\infty$.

Now we investigate the kernel of bounded linear operators.

Proposition 2.7. Let X and Y be normed spaces. Let $T : X \to Y$ be a bounded linear operator. Then $\ker T$ is closed in X. The operator $\tilde{T} : X/\ker T \to \mathfrak{R}(T),\ [x] \mapsto Tx$, which is induced by T, is a bijection. Furthermore, $\|\tilde{T}\| = \|T\|$.

Proof. Note that $\{0\}$ is closed in Y, and $T : X \to Y$ is continuous, the kernel

$$\ker T = T^{-1}(\{0\})$$

is closed in X by continuity. To verify that $\tilde{T}$ is a bijection from $X/\ker T$, note that $\ker \tilde{T} = \{[0]\}$, and that for any $y \in \mathfrak{R}(T)$, there exists $x \in X$ such that $Tx = y$, which implies $\tilde{T}([x]) = y$. Finally, we determine the norm of $\tilde{T}$. Without loss of generality, assume $\ker T \neq X$. Since the quotient map $\pi : X \mapsto X/\ker T, x \mapsto [x]$ projects unit ball $\{x \in X : \|x\| < 1\}$ onto $\{[x] : x \in X, \|[x]\| < 1\}$, we have

$$\|\tilde{T}\| = \sup_{\|[x]\|<1} \|\tilde{T}([x])\| = \sup_{\|x\|<1} \|Tx\| = \|T\|,$$

which completes the proof. □

Definition 2.8 (Spaces of bounded linear operators). Let X and Y be normed spaces. Define

$$\mathfrak{B}(X,Y) = \{T: \ T \text{ is a bounded linear operator from } X \text{ into } Y\}.$$

And we define addition and scalar multiplication on $\mathfrak{B}(X,Y)$ by

$$\begin{aligned}
&(S+T)(x) = Sx + Tx, \ S,T \in \mathfrak{B}(X,Y)\\
&(\alpha T)(x) = \alpha Tx, \ T \in \mathfrak{B}(X,Y), \ \alpha \text{ is a number.}
\end{aligned}$$

Then

$$\begin{aligned}
&\|T\| = 0 \Rightarrow \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} = 0 \ \Rightarrow \ Tx = 0 \text{ for all } x \in X \ \Rightarrow \ T \equiv 0;\\
&\|(S+T)x\| \leq \|Sx\| + \|Tx\| \leq (\|S\| + \|T\|)\|x\| \ \Rightarrow \ S+T \in \mathfrak{B}(X,Y), \ \|S+T\| \leq \|S\| + \|T\|;\\
&\sup_{x\neq 0} \frac{\|(\alpha T)x\|}{\|x\|} = \sup_{x\neq 0} \frac{|\alpha|\|Tx\|}{\|x\|} = |\alpha| \sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} \Rightarrow \ \alpha T \in \mathfrak{B}(X,Y), \ \|\alpha T\| = |\alpha| \, \|T\| \, .
\end{aligned}$$

Hence $(\mathfrak{B}(X,Y), \|\cdot\|)$ is a normed space, where $\|\cdot\|$ is the operator norm.

We introduce another normed space Z, and define the multiplication operation by composition:

$$(S \circ T)(x) = S(Tx), \ S \in \mathfrak{B}(Y,Z), \ T \in \mathfrak{B}(X,Y).$$

Then we have

$$\|(S \circ T)(x)\| \leq \|S\| \, \|Tx\| \leq \|S\| \, \|T\| \, \|x\| \, , \ \forall x \in X.$$

As a result, $S \circ T \in \mathfrak{B}(X,Z)$, and $\|S \circ T\| \leq \|S\| \, \|T\|$.

Specifically, we write $\mathfrak{B}(X) = \mathfrak{B}(X,X)$. By the above discussion, $\mathfrak{B}(X)$ forms an algebra, given the above multiplication.

Lemma 2.9. Let X and Y be normed spaces. If Y is a Banach space, so is $\mathfrak{B}(X,Y)$.

Proof. Let (T_n) be a Cauchy sequence in $\mathfrak{B}(X,Y)$. For each $x \in X$,

$$\|T_n x - T_m x\| \leq \|T_n - T_m\| \, \|x\| \, .$$

Then $(T_n x)$ is a Cauchy sequence in Y, which converges to some point of Y, denoted by Tx. Hence we obtain an operator $T : X \to Y$ to which T_n converges pointwise.

The linearity of T follows from T_n:

$$\begin{aligned}
\|Tx + Ty - T(x+y)\| &= \left\| \lim_{n\to\infty} (T_n x + T_n y - T_n(x+y)) \right\| = 0,\\
\|\alpha Tx - T(\alpha x)\| &= \left\| \lim_{n\to\infty} (\alpha T_n x - T_n(\alpha x)) \right\| = 0.
\end{aligned}$$

Choose N such that $\|T_n - T_m\| < 1$ for all $n, m \geq N$. Then for all $x \in X$ and all $n \in \mathbb{N}$,

$$\|T_n x\| \leq \|T_n\| \, \|x\| \leq \max\{\|T_1\|, \cdots, \|T_{N-1}\|, \|T_N\| + 1\} \, \|x\|.$$

Hence $\|Tx\| \leq \max\{\|T_1\|, \cdots, \|T_{N-1}\|, \|T_N\| + 1\} \, \|x\|$ for all $x \in X$, and $T \in \mathfrak{B}(X,Y)$.

It remains to show that $\|T_n - T\| \to 0$. Given $\epsilon > 0$, choose N_ϵ such that $\|T_n - T_m\| < \epsilon$ for all $n, m \geq N_\epsilon$. Then for all $n \geq N_\epsilon$,

$$\|(T_n - T)x\| = \lim_{m\to\infty} \|(T_m - T_n)x\| \leq \lim_{m\to\infty} \|T_m - T_n\| \, \|x\| < \epsilon\|x\|, \ \forall x \in X \ \Rightarrow \ \|T_n - T\| \leq \epsilon\|x\|.$$

Hence T_n converges to T relative to operator norm. □

Remark. In Lemma 2.9, we do not require the completeness of domain X.

Let X be a vector space over field $\mathbb{F}$. Then a linear operator $f : X \to \mathbb{F}$ is said to be a *linear functional* on X. The space of bounded linear functionals $\mathfrak{B}(X, \mathbb{F})$ is said to be the *dual space* of X, denoted by X^*.

Lemma 2.10. Let X and Y be two finite-dimensional normed spaces over $\mathbb{R}$ (or $\mathbb{C}$). Then any linear operator $T : X \to Y$ is bounded.

Proof. Use matrix representation of linear operators and equivalence of norms in finite-dimensional spaces. □

Theorem 2.11. Let X be a normed space. A linear operator $f : X \to Y$ is bounded if and only if its kernel $\ker f$ is closed.

Proof. Following Proposition 2.7, it remains to show sufficiency. Assume $\ker f$ is a closed subspace of X. According to Example 1.54, the quotient space

$$X/\ker f := \{[x] = \{x + y, y \in \ker f\}\}$$

has an immediate induced norm

$$\|[x]\| := d(x, \ker f) = \inf_{y\in\ker f} \|x - y\|, \ x \in X.$$

We define $\tilde{f} : X/\ker f \to \mathbb{R}$ (or $\mathbb{C}$), $[x] \mapsto f(x)$. Since $\dim(X/\ker f) = \dim \mathfrak{R}(f) \leq 1$, by Lemma 2.10, f is bounded. Furthermore, f is the composition

$$x \in X \ \stackrel{\pi}{\to} \ [x] \in X/\ker f \ \stackrel{\tilde{f}}{\to} \ f(x).$$

Then for all $x \in X$,

$$|f(x)| \leq \left\|\tilde{f}\right\| \|\pi(x)\| \leq \left\|\tilde{f}\right\| \|x\|$$

Hence f is bounded. □

Remark. Let X be a normed space over $\mathbb{R}$ (or $\mathbb{C}$). By Definition 2.8, the dual space X^* is a vector space equipped with a natural norm

$$\|f\| = \sup_{x\neq 0} \frac{|f(x)|}{\|x\|}, \ f \in X^*.$$

Furthermore, X^* is a Banach space by Lemma 2.9, even though X is not complete.

2.1.1 Riesz Representation Theorem

Now we discuss linear functionals on Hilbert spaces. We first introduce the definition of isomorphism, which allows us to connect an abstract normed space to a specific one.

Definition 2.12 (Isomorphism). Let X and Y be normed spaces, and U is an operator from X into Y.

(i) If $\|Ux\| = \|x\|$ for all $x \in X$, then U is said to be a norm-preserving operator.

(ii) If $U : X \to Y$ is linear, norm-preserving and surjective, then U is said to be an *isomorphism*. If there exists an isomorphism between X and Y, we say X and Y are *isomorphic*, and we write $X \cong Y$.

Remark. Isomorphism is an important tool when we investigate dual spaces. The dual space X^*, which consists of bounded linear functionals on a normed space X, can be intractable. Hopefully, we can find a specific normed space Y that is isomorphic to X^*. Then every bounded linear functional on X is uniquely determined by some element y of Y.

Revisit: Example 1.54. Let H be a Hilbert space, and let M be a closed subspace of H. Then the quotient space H/M is isomorphic to the orthogonal complement $M^\perp$.

Proof. For each $x \in X$, by Theorem 1.37, there exists unique $x_0 \in M$ and $x_1 \in M^\perp$ such that $x = x_0 + x_1$. Then we can define $U : X \to M$ by $U(x) = x_1$. Furthermore, for all $y \in M$, we have $x + y = (x_0 + y) + x_1$, where $x_0 + y \in M$ and $x_1 \in M^\perp$, which implies $U(x + y) = x_1$. As a result, the induced operator

$$\widetilde{U} : X/M \to M^\perp,\ [x] \mapsto U(x)$$

is well-defined. Clearly, $\widetilde{U}$ is a linear operator, and U is norm-preserving:

$$\left\|\widetilde{U}([x])\right\| = \|Ux\| = \|x_1\| = \inf_{y\in M} \|x_1 - y\| = \inf_{y\in M} \|x_1 + x_0 - y\| = \|[x]\|.$$

Furthermore, for any $x_1 \in M^\perp$, $\widetilde{U}([x_1]) = U(x_1) = x_1$. Hence $\widetilde{U}$ is surjective. As a result, $\widetilde{U}$ is an isomorphism, and $H/M \cong M^\perp$. ☐

Review. Consider the finite-dimensional euclidean space $\mathbb{R}^n$ equipped with the standard inner product. We choose an orthonormal basis $\{e_1, \cdots, e_n\}$ of $\mathbb{R}^n$. Then for every linear functional f on $\mathbb{R}^n$, we have

$$f(x) = f\left(\sum_{j=1}^n \langle x, e_j\rangle\, e_j\right) = \sum_{j=1}^n \langle x, e_j\rangle\, f(e_j),\ \forall x \in \mathbb{R}^n.$$

It is seen that f is uniquely determined by tuple $(f(x_1), \cdots, f(x_n)) \in \mathbb{R}^n$. Similarly, every tuple $(f_1, \cdots, f_n) \in \mathbb{R}^n$ induces a linear functional $f(x) = \sum_{j=1}^n f_j\, \langle x, e_j\rangle$. Furthermore,

$$\|f\| = \sup_{x\neq 0} \frac{|f(x)|}{\|x\|} = \sup_{x\neq 0} \frac{\left|\sum_{j=1}^n \langle x, e_j\rangle f(e_j)\right|}{\sqrt{\sum_{j=1}^n |\langle x, e_j\rangle|^2}} = \sqrt{\sum_{j=1}^n |f(e_j)|^2}.$$

Hence we have $(\mathbb{R}^n)^* \cong \mathbb{R}^n$, which is a standard conclusion in linear algebra.

For general Hilbert spaces, we have the following important theorem.

Theorem 2.13 (Riesz representation theorem). Let H be a Hilbert space. Let $F \in H^*$, i.e. F is a bounded linear functional on H. Then there exists uniquely $y \in H$ such that $F(x) = \langle x, y\rangle$ for all $x \in X$, and $\|F\| = \|y\|$.

Proof. If $F = 0$, then $y = 0$. We assume $F \neq 0$. Then $\ker F$ is a closed subspace of H, and there exists $z \notin \ker F$ such that $z \perp \ker F$. We set $z_0 = z/F(z) \in H$, then $z_0 \perp \ker F$, and $F(z_0) = 1$.

For each $x \in H$, we have

$$F(x - F(x)z_0) = F(x) - F(x)F(z_0) = 0.$$

Hence we have $x - F(x)z_0 \in \ker F$, and $\langle x, z_0\rangle = F(x)\langle z_0, z_0\rangle$. Setting $y = \frac{z_0}{\langle z_0, z_0\rangle}$ yields the desired result. Uniqueness is clear, and $\|F\| = \|y\|$. $\square$

Remark. By Theorem 2.13, we have $H^* \cong H$. Then every bounded linear functional F on H corresponds to a unique vector $y \in H$, and we can write F as $F_y = \langle \cdot, y\rangle$.

Definition 2.14 (Sesquilinear forms). Let X be a complex vector space. Let $\varphi : X \times X \to \mathbb{C}$.

(i) If for all $x, y, z \in H$ and all $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned}\varphi(\alpha x + \beta y, z) &= \alpha\varphi(x, z) + \beta\varphi(y, z),\\ \varphi(z, \alpha x + \beta y) &= \overline{\alpha}\varphi(z, x) + \overline{\beta}\varphi(z, y),\end{aligned}$$

then φ is said to be a *sesquilinear form* on X. If there exists $M > 0$ such that

$$|\varphi(x, y)| \leq M\|x\|\|y\|, \ \forall x, y \in X,$$

then φ is said to be a bounded sesquilinear form on X, and we define norm of φ by

$$\|\varphi\| = \sup_{\|x\|=\|y\|=1} |\varphi(x, y)|.$$

(ii) If φ is a sesquilinear functional on X and $\varphi(x, y) = \overline{\varphi(y, x)}$ for all $x, y \in X$, then φ is said to be a *Hermitian form* on X.

Remark. Let T be a bounded linear operator on a Hilbert space H. Clearly, the map $\varphi(x, y) := \langle Tx, y\rangle$ induced by T is a sesquilinear form on H. Moreover,

$$\begin{aligned}|\varphi(x, y)| = \langle Tx, y\rangle \leq \|Tx\|\|y\| \leq \|T\|\|x\|\|y\|, \ \forall x, y \in H \ &\Rightarrow \ \|\varphi\| \leq \|T\|,\\ \|T\| = \sup_{\|x\|=1}\|Tx\| = \sup_{\|x\|=\|y\|=1}|\langle Tx, y\rangle| = \sup_{\|x\|=\|y\|=1}|\varphi(x, y)| \ &\Rightarrow \ \|\varphi\| \geq \|T\|.\end{aligned}$$

Therefore $\|\varphi\| = \|T\|$. Furthermore, every bounded sesquilinear form on H uniquely determines a bounded linear operator T on H, as stated below.

Theorem 2.15 (Riesz). Let H be a Hilbert space, and let φ be a bounded sesquilinear form on H. Then there exists uniquely $T \in \mathfrak{B}(H)$ such that $\varphi(x, y) = \langle Tx, y\rangle$ for all $x, y \in H$, and $\|T\| = \|\varphi\|$.

Proof. Fix $x \in H$. By definition of bounded sesquilinear form, $\varphi_x(\cdot) = \overline{\varphi(x, \cdot)}$ is a bounded linear functional on H, with $\|\varphi_x\| \leq \|\varphi\|\|x\|$. By Riesz representation theorem (Theorem 2.13), there exists $z \in H$ such that

$$\varphi(x, y) = \overline{\varphi_x(y)} = \overline{\langle y, z\rangle} = \langle z, y\rangle, \ \forall y \in H,$$

and $\|z\| = \|\varphi_x\| \leq \|\varphi\| \, \|x\|$. We denote by $Tx = z$ the uniquely determined Riesz vector of φ_x. Then T is an operator on X. Clearly, T is linear, and $\|T\| \leq \|\varphi\|$. $\square$

Lemma 2.16. Let H be a complex Hilbert space, and let φ be a bounded sesquilinear form on H.

(i) φ is Hermitian if and only if $\varphi(x,x) \in \mathbb{R}$ for all $x \in H$;

(ii) If φ is Hermitian, and there exists $M > 0$ such that $|\varphi(x,x)| \leq M\|x\|^2$ for all $x \in H$, then φ is bounded, and $\|\varphi\| \leq M$.

Proof. (i) The necessity is clear. To show sufficiency, use the polarization identity:

$$\varphi(x,y) = \frac{1}{4}\sum_{k=0}^{3} \mathrm{i}^k \varphi\left(x + \mathrm{i}^k y, x + \mathrm{i}^k y\right), \ \forall x, y \in H.$$

Then we can verify that $\varphi(x,y) = \overline{\varphi(y,x)}$.

(ii) Assume $\varphi(x,y) \neq 0$. Let $\lambda = \frac{\overline{\varphi(x,y)}}{|\varphi(x,y)|}$, then $\varphi(\lambda x, y) \in \mathbb{R}$, and

$$\begin{aligned}
|\varphi(x,y)| = \varphi(\lambda x, y) &= \frac{1}{4}\left[\varphi(x+y, x+y) - \varphi(x-y, x-y)\right] \\
&\leq \frac{M}{4}\left[\|x+y\|^2 + \|x-y\|^2\right] \\
&= \frac{M}{2}\left(\|x\|^2 + \|y\|^2\right).
\end{aligned}$$

Whenever $\|x\| = \|y\| = 1$, we have $|\varphi(x,y)| \leq M$. Hence φ is bounded, and $\|\varphi\| \leq M$. $\square$

Remark. Let H be a complex Hilbert space. A linear operator $T : X \to X$ is said to be an *Hermitian* operator if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$. By Theorem 2.15, every bounded Hermitian form on H is uniquely induced by a bounded Hermitian operator $T \in \mathfrak{B}(H)$. Furthermore,

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

2.2 Hahn-Banach Theorem

Definition 2.17 (Linear extension). Let L be a real vector space, and let L_0 be a subspace of L. Given a linear functional f_0 on L_0, a linear functional $f : L \to \mathbb{R}$, is said to be a *(linear) extension of* f_0, if $f|_{L_0} = f_0$, i.e. $f(x) = f_0(x)$ for all $x \in L_0$.

Remark. Let $\{b_\lambda, \lambda \in \Lambda_0\}$ be a Hamel basis of L_0, we can expand it to a Hamel basis $\{b_\lambda, \lambda \in \Lambda\}$ on L, where $\Lambda_0 \subset \Lambda$. Given a linear functional $f_0 : L_0 \to \mathbb{R}$, we maintain $f(b_\lambda) = f_0(b_\lambda)$ for all $\lambda \in \Lambda_0$, and set $f(b_\lambda) = 0$ for $\lambda \notin \Lambda_0$. Then we obtain a trivial extension of f_0 on L.

Definition 2.18 (Sublinear functional). Let L be a real vector space. A function $p : L \to \mathbb{R}$ is said to be a *sublinear functional* on L, if

$$p(x+y) \le p(x) + p(y),\ \forall x, y \in L; \quad p(\lambda x) = \lambda p(x),\ \forall x \in L,\ \lambda \ge 0.$$

Theorem 2.19 (Hahn-Banach, real version). Let p be a sublinear functional on a real vector space L. Let $L_0 \subset L$ be a subspace. Suppose f_0 is a linear functional on L_0 that is subject to p, i.e. $f_0(x) \le p(x)$ for all $x \in L_0$. Then there exists an extension $f : L \to \mathbb{R}$ of f_0 that is subject to p.

Proof. Step I: Suppose $L_0 \ne L$, and choose $z \in L \backslash L_0$. We claim that f_0 can be extended to a linear functional f_1 on space $L_1 = \operatorname{span}\{L_0, z\}$ such that f_1 is subject to p.

For any $tz + x \in L_1$, where $t \in \mathbb{R}$ and $x \in L_0$, if f_1 is an extension of f_0, then

$$f_1(tz + x) = t f_1(z) + f_0(x).$$

We need to determine $f_1(z)$, denoted by c. To ensure that f is subject to p, we require

$$f_1(tz + x) = tc + f_0(x) \le p(tz + x),\ \forall x \in L_0, t \in \mathbb{R}.$$

If $t > 0$, then

$$\begin{aligned} tc + f_0(x) \le p(tz + x),\ \forall x \in L_0, t > 0 \ &\Leftrightarrow\ c \le p\left(z + \frac{x}{t}\right) - f_0\left(\frac{x}{t}\right),\ \forall x \in L_0, t > 0 \\ &\Leftrightarrow\ c \le p(z + y') - f_0(y'),\ \forall y' \in L_0. \end{aligned} \tag{2.2}$$

If $t < 0$, then

$$\begin{aligned} tc + f_0(x) \le p(tz + x),\ \forall x \in L_0, t < 0 \ &\Leftrightarrow\ c \ge -p\left(-z - \frac{x}{t}\right) - f_0\left(\frac{x}{t}\right),\ \forall x \in L_0, t < 0 \\ &\Leftrightarrow\ c \ge -p(-z - y'') - f_0(y''),\ \forall y'' \in L_0. \end{aligned} \tag{2.3}$$

By (2.2) and (2.3), to choose an appropriate c, it suffices to show

$$\sup_{y'' \in L_0} -p(-z - y'') - f_0(y'') \le \inf_{y' \in L_0} p(z + y') - f_0(y'). \tag{2.4}$$

For any $y', y'' \in L_0$, we have $y' - y'' \in L_0$. Then

$$p(z + y') - f_0(y') + p(-z - y'') + f_0(y'') \ge p(y' - y'') - f_0(y' - y'') \ge 0,$$

which implies (2.4). Then we choose an appropriate c and set $f_1(z) = c$.

Step II: Use Zorn's lemma. Let $\mathscr{H}$ be the set of all extensions of f_0:

$$\mathscr{H} = \{(f, Y) : L_0 \subset Y \subset L \text{ is a subspace of } L;\ f \text{ is a linear functional on } Y \text{ such that } f|_{L_0} = f_0, f \leq p\}$$

We define a partial order on $\mathscr{H}$: $(f_1, Y_1) \preceq (f_2, Y_2)$ if $Y_1 \subset Y_2$ and $f_2|_{Y_1} = f_1$. Let $C = \{(f_\lambda, Y_\lambda), \lambda \in \Lambda\}$ be a chain in $\mathscr{H}$, and let $Y = \bigcup_{\lambda\in\Lambda} Y_\lambda$. Then Y is a subspace of L: for any $x_1, x_2 \in Y$, there exists $\lambda_1, \lambda_2 \in \Lambda$ such that $x_1 \in Y_{\lambda_1}$ and $x_2 \in Y_{\lambda_2}$. Furthermore, one of Y_{λ_1}, Y_{λ_2} contains the other because C is a chain, then both x_1 and x_2 belong to Y_{λ_2}, without loss of generality. Hence $\alpha x_1 + \beta x_2 \in Y_{\lambda_2} \subset Y$ for $\alpha, \beta \in \mathbb{R}$.

For any $x \in Y$, there exists $\lambda \in \Lambda$ such that $x \in Y_\lambda$. Then we define $f(x) = f_\lambda(x)$. Note that $f(x)$ is well-defined: if x belongs both $Y_{\lambda_1} \cap Y_{\lambda_2}$, f_{λ_1} and f_{λ_2} will agree on x, since C is a chain. Similar arguments also show that f is linear, subjected to p, and that it is an extension of f_λ for all $\lambda \in \Lambda$. Then we obtain an upper bound (f, Y) of C in $\mathscr{H}$, and we can apply Zorn's lemma.

Now $\mathscr{H}$ has a maximal element (f, Y). It remains to show $Y = L$; suppose not. By Step I, there exists subspace $Y_1 \supsetneq Y$, and f can be extended to $f_1 : Y_1 \to \mathbb{R}$, contradicting the maximality of (f, Y)! ☐

The following theorem is a corollary of Theorem 2.19.

Theorem 2.20 (Hahn-Banach, real version). Let L_0 be a subspace of a real normed space L. If f_0 is a bounded linear functional on L_0, then there exists an extension $f : L \to \mathbb{R}$ of f_0 such that $\|f\| = \|f_0\|$.

Proof. Let $p(x) = \|f_0\|\,\|x\|$ for all $x \in L$, p is a sublinear functional to which f_0 is subject. By Theorem 2.19, there exists an extension $f : L \to \mathbb{R}$ of f_0 with $f \leq p$. Then

$$\|f\| = \sup_{x\in L\backslash\{0\}} \frac{|f(x)|}{\|x\|} \geq \sup_{x\in L_0\backslash\{0\}} \frac{|f(x)|}{\|x\|} = \sup_{x\in L_0\backslash\{0\}} \frac{|f_0(x)|}{\|x\|} = \|f_0\|;$$

$$-\|f_0\|\,\|x\| = -p(-x) \leq |f(x)| \leq p(x) = \|f_0\|\,\|x\|,\ \forall x \in L \ \Rightarrow\ \|f\| \leq \|f_0\|.$$

Hence $\|f\| = \|f_0\|$. ☐

Now we consider the complex case. Let X be a complex vector space, and let f be a complex linear functional on X. For each $x \in X$, $f(x) = \mathrm{Re} f(x) + \mathrm{i}\,\mathrm{Im} f(x)$. We denote $f_R = \mathrm{Re} f$, $f_I = \mathrm{Im} f$.

If f is $\mathbb{C}$-linear, then for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{R}$,

$$\begin{cases} f_R(x+y) = f_R(x) + f_R(y),\ f_I(x+y) = f_I(x) + f_I(y), \\ f_R\left((\alpha + \mathrm{i}\beta)x\right) = \alpha f_R(x) - \beta f_I(x), \\ f_I\left((\alpha + \mathrm{i}\beta)x\right) = \beta f_R(x) + \alpha f_I(x). \end{cases}$$

Hence the following are equivalent: (i) $f = f_R + \mathrm{i} f_I$ is $\mathbb{C}$-linear; (ii) f_R and f_I are $\mathbb{R}$-linear, and $f_R(\mathrm{i}x) = -f_I(x)$ for all $x \in X$; (iii) f_R and f_I are $\mathbb{R}$-linear, and $f_R(x) = f_I(\mathrm{i}x)$ for all $x \in X$.

By (ii), f is uniquely determined by its real part f_R: $f(x) = f_R(x) - \mathrm{i} f_R(\mathrm{i}x)$. By (iii), f is uniquely determined by its imaginary part f_I: $f(x) = f_I(\mathrm{i}x) + \mathrm{i} f_I(x)$.

Theorem 2.21 (Hahn-Banach, complex version). Let L_0 be a subspace of a complex vector space L. Let p be a seminorm on L. Suppose f_0 is a linear functional on L_0 such that $|f_0(x)| \leq p(x)$ for all $x \in L_0$. Then there exists a linear functional $f : L \to \mathbb{R}$ such that $f|_{L_0} = f_0$ and $|f(x)| \leq p(x)$ for all $x \in L$.

Proof. We first view L and L_0 as $\mathbb{R}$-vector spaces, denoted by $L_\mathbb{R}$ and $L_{0\mathbb{R}}$, respectively. Then p is a sublinear functional on $L_{0\mathbb{R}}$, and $f_{0R} = \mathrm{Re} f_0$ satisfies

$$f_{0R}(x) \leq |f_0(x)| \leq p(x), \ \forall x \in L_{0\mathbb{R}}.$$

By Theorem 2.19, there exists $\mathbb{R}$-linear functional $f_R : L_\mathbb{R} \to \mathbb{R}$ such that $f_R|_{L_{0\mathbb{R}}} = f_{0R}$, and $f_R(x) \leq p(x)$ for all $x \in L_\mathbb{R}$. Let $f(x) = f_R(x) - \mathrm{i} f_R(\mathrm{i}x)$ for all $x \in L_\mathbb{R}$. Then f is $\mathbb{C}$-linear on L, and f extends f_0.

For any $x \in L$, denote $\theta = \mathrm{Arg}\, f(x)$. Then

$$|f(x)| = \mathrm{e}^{-\mathrm{i}\theta} f(x) = \underbrace{f\left(\mathrm{e}^{-\mathrm{i}\theta} x\right)}_{\in \mathbb{R}} = f_R\left(\mathrm{e}^{-\mathrm{i}\theta} x\right) \leq p\left(\mathrm{e}^{-\mathrm{i}\theta} x\right) \leq \left|\mathrm{e}^{-\mathrm{i}\theta}\right| p(x) = p(x).$$

Thus we complete the proof. □

Now we introduce some useful corollaries of Hahn-Banach theorem.

Corollary 2.22. Let X be a normed space.

(i) For each $x_0 \in X \backslash \{0\}$, there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x_0) = \|x_0\|$.

(ii) For each $x_1, x_2 \in X$ such that $x_1 \neq x_2$, there exists $f \in X^*$ such that $f(x_1) \neq f(x_2)$.

(iii) For all $x \in X$,

$$\|x\| = \max_{f \in X^*, \|f\|=1} |f(x)|.$$

Proof. (i) Consider the subspace $Y = \mathrm{span}\{x_0\} = \mathbb{C}x_0 = \{\alpha x_0 : \alpha \in \mathbb{C}\}$. Define $f_0 : Y \to \mathbb{C}, \ \alpha x_0 \mapsto \alpha \|x_0\|$. Then $\|f_0\| = 1$. By Hahn-Banach theorem, there exists an extension $f \in X^*$ such that $\|f\| = \|f_0\| = 1$, and $f|_Y = f_0$. Hence $f(x_0) = \|x_0\|$.

(ii) Apply (i) to $x_0 = x_1 - x_2$.

(iii) Clearly, for all $f \in X^*$ such that $\|f\| = 1$, we have $|f(x)| \leq \|x\|$. By (i), there exists $f \in X^*$ such that $\|f\| = 1$ and $|f(x)| = \|x\|$. □

Corollary 2.23. Let M be a closed subspace of a normed space X. For all $x \in X \backslash M$, there exists $f \in X^*$ such that $\|f\| = 1$, $f(M) = \{0\}$, and $f(x) = d(x, M)$.

Proof. Let $X_0 = \mathrm{span}\{M, x\}$. For any $y = m + \lambda x \in X_0$, where $m \in M$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), define

$$f_0 : X_0 \to \mathbb{R} \text{ (or } \mathbb{C}), \ m + \lambda x \to \lambda d(x, M).$$

Then f_0 is a linear functional on X_0, and

$$\|f_0\| = \sup_{m+\lambda x \neq 0} \frac{|\lambda|\, d(x, M)}{\|m + \lambda x\|} = \sup_{m' \in M} \frac{d(x, M)}{\|m' + x\|} = \frac{d(x, M)}{\inf_{m' \in M} \|m' + x\|} = 1.$$

By Hahn-Banach theorem, there exists an extension f of f_0 on X such that $\|f\| = \|f_0\| = 1$, $f(M) = \{0\}$, and $f(x) = d(x, M)$. Note that we require M to be closed. Otherwise, let x be a limit point of M not lying in M. Then $d(x, M) = 0$, and $f_0 \equiv 0$ on X_0. □

Corollary 2.24. Let M be a subset of a normed space X. Let $x \in X$. Then $x \in \overline{\mathrm{span}}(M)$ if and only if $f(x) = 0$ for all $f \in X^*$ such that $f(M) = \{0\}$.

Proof. "⇒": Clearly. "⇐": Argue by contradiction. By Corollary 2.23, if $x \notin \overline{\operatorname{span}}(M)$, there exists $f \in X^*$ such that $f(x) = d(x, \overline{\operatorname{span}}(M)) > 0$ and $f(M) = \{0\}$. □

Now we introduce the generalization of orthogonal complements in Banach spaces.

Definition 2.25 (Annihilators and pre-annihilators). Let X be a normed space.

(i) For a subset $M \subset X$, the *annihilator* of M is defined as

$$M^\perp = \{f \in X^* : f(x) = 0,\ \forall x \in M\}.$$

(ii) For a subset $N \subset X^*$, the *pre-annihilator* of N is defined as

$${}^\perp N = \{x \in X : f(x) = 0,\ \forall f \in N\}.$$

Clearly, $M^\perp$ is a closed subspace of X^*, and ${}^\perp N$ is a closed subspace of X.

Remark. By definition, $\overline{M}^\perp \subset M^\perp$. For each $x \in \overline{M}$, there exists sequences (x_n) of points of M such that $x_n \to x$. If $f \in M^\perp$, by continuity of f, we have $f(x) = \lim_{n\to\infty} f(x_n) = 0$. Hence $\overline{M}^\perp = M^\perp$.

Similarly, for every $f \in \overline{N}$, there exists sequences (f_n) of points of N such that $f_n \to f$, which implies $f_n(x) \to f(x)$ for all $x \in X$. If $x \in {}^\perp N$, then $f(x) = 0$. Hence ${}^\perp\overline{N} = {}^\perp N$.

Theorem 2.26. Let M be a closed subspace of a normed space X.

(i) For all $m^* \in M^*$, there exists a norm-preserving extension x^* of M^* on X. We define

$$\sigma(m^*) = [x^*],$$

where $[x^*]$ is the equivalence class of x^* in quotient space $X^*/M^\perp$. Then map $\sigma : M^* \to X^*/M^\perp$ is a well-defined norm-preserving isomorphism.

(ii) Let $\pi : X \to X/M,\ x \mapsto [x]$ be the quotient map. For all $f \in (X/M)^*$, define

$$\tau(f) = f \circ \pi.$$

Then $\tau : (X/M)^* \to M^\perp$ is a norm-preserving isomorphism.

Proof. (i) We first check that the map σ is well-defined. Let $x^*, y^* \in X^*$ be two extensions of m^* on X. Then $x^*m = y^*m = m^*m$ for all $m \in M$, and $x^* - y^* \in M^\perp$. Hence $[x^*] = [y^*]$ in $X/M^\perp$.

Clearly, σ is linear: if $x_1^*, x_2^* \in X^*$ are extensions of $m_1^*, m_2^* \in M^*$, respectively, then $\alpha x_1^* + \beta x_2^*$ is an extension of $\alpha m_1^* + \beta m_2^*$. Also, σ is norm-preserving: $\|\sigma(m^*)\| = \|[x^*]\| \le \|x^*\| = \|m^*\|$, and

$$y^*|_M = m^*,\ \forall y^* \in [x^*] \ \Rightarrow\ \|y^*\| \ge \|m^*\|,\ \forall y^* \in [x^*] \ \Rightarrow\ \|[x^*]\| = \inf\{\|y^*\| :\ y^* \in [x^*]\} \ge \|m^*\|.$$

For any $[x^*] \in X^*/M^\perp$, $\sigma(x^*|_M) = [x^*]$, which implies σ is surjective. As a result, $\sigma : M^* \to X^*/M^\perp$ is a norm-preserving isomorphism, and $M^* \cong X^*/M^\perp$.

(ii) Clearly, τ is linear. To show that τ is norm-preserving, note that for all $f \in (X/M)^*$,

$$\|\tau(f)\| = \sup_{y \in X\setminus M} \frac{f([y])}{\|y\|} = \sup_{x \in X\setminus M, y \in [x]} \frac{f([y])}{\|y\|} = \sup_{x \in X\setminus M} \frac{f([x])}{\inf_{y\in[x]}\|y\|} = \sup_{x \in X\setminus M} \frac{f([x])}{\|[x]\|} = \|f\|.$$

Finally, for each $g \in M^\perp$, we define $\tilde{g} : X/M \to \mathbb{R},\ [x] \mapsto g(x)$. Then $g = \tilde{g} \circ \pi = \tau(\tilde{g})$, and τ is surjective. Therefore, $\tau : (X/M)^* \to M^\perp$ is a norm-preserving isomorphism, and $(X/M)^* \cong M^\perp$. □

Finally, we present an interesting application of Hahn-Banach theorem.

Example 2.27. We wish to find a finitely additive translation-invariant probability measure μ on $\mathbb{R}$, such that μ is defined on all subsets of $\mathbb{R}$, and

$$\mu(\alpha + A) = \mu(A) \quad \text{for all } A \subset \mathbb{R} \text{ and all } \alpha \in \mathbb{R}, \text{ where } \alpha + A = \{x + \alpha : x \in A\}.$$

Let $B(\mathbb{R})$ be the set of all bounded $\mathbb{R}$-valued functions on $\mathbb{R}$, and define $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$ for all $f \in B(\mathbb{R})$. Clearly, $\|\cdot\|_\infty$ is a norm on $B(\mathbb{R})$, and $(B(\mathbb{R}), \|\cdot\|_\infty)$ is a Banach space.

Claim. We define $p : B(\mathbb{R}) \to \mathbb{R}$ as follows: for all $f \in B(\mathbb{R})$,

$$p(f) = \inf_{n\in\mathbb{N}, \alpha_1,\cdots,\alpha_n\in\mathbb{R}} \left\{ N(f; \alpha_1, \cdots, \alpha_n) := \sup_{s\in\mathbb{R}} \frac{1}{n} \sum_{k=1}^{n} f(s + \alpha_k) \right\}.$$

Then p is a sublinear functional on $\mathbb{R}$.

Proof. Clearly, for all $\lambda \geq 0$, $p(\lambda f) = \lambda p(f)$. For all $f, g \in B(\mathbb{R})$ and all $n \in \mathbb{N}$, $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$,

$$N(f + g; \alpha_1, \cdots, \alpha_n) = \sup_{s\in\mathbb{R}} \frac{1}{n} \sum_{k=1}^{n} \big[f(s + \alpha_k) + g(s + \alpha_k)\big] \leq N(f; \alpha_1, \cdots, \alpha_n) + N(g; \alpha_1, \cdots, \alpha_n).$$

Hence $p(f + g) \leq p(f) + p(g)$. □

Theorem. There exists a linear functional $\nu : B(\mathbb{R}) \to \mathbb{R}$ such that (i) $\nu(\mathbf{1}) = 1$, and (ii) $\nu(\tau_\alpha f) = \nu(f)$ for all $\alpha \in \mathbb{R}$, where τ_α is the translation operator $(\tau_\alpha f)(x) = f(x - \alpha)$.

Proof. We first consider the linear functional $\nu_0(\alpha\mathbf{1}) = \alpha$ on subspace $\mathbb{R} \cdot \mathbf{1}$, which satisfies $\nu_0 = p$. By Hahn-Banach theorem, there exists linear functional ν on $B(\mathbb{R})$ extending ν_0, with $\nu \leq p$.

For all $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$, we have

$$p(\tau_\alpha f - f) \leq N(\tau_\alpha f - f; \alpha, \cdots, n\alpha) = \sup_{s\in\mathbb{R}} \frac{1}{n} \left[f(s) - f(s + n\alpha)\right] \leq \frac{2}{n}\|f\|_\infty.$$

Let $n \to \infty$, then $p(\tau_\alpha f - f) \leq 0$, and $\nu(\tau_\alpha f - f) \leq 0$. Analogously, $\nu(f - \tau_\alpha f) \leq p(f - \tau_\alpha f) \leq 0$. Therefore $\nu(\tau_\alpha f) = \nu(f)$ for all $\alpha \in \mathbb{R}$, completing the proof. □

For any $A \in \mathbb{R}$, we define $\mu(A) = \nu(\chi_A)$. Then μ is the desired probability measure on $\mathbb{R}$.

2.3 Hyperplane Separation Theorem

Definition 2.28 (Hyperplane). Let f be a nonzero linear functional on a vector space X. Let c be a constant. The set

$$M_c = \{f(x) = c\} = \{x \in X : f(x) = c\}$$

is said to be a *hyperplane* in X.

Definition 2.29 (Separation). Suppose X is a real vector space, M and N are subsets of X, and f is a linear functional on X.

(i) If there exists $c \in \mathbb{R}$ such that $f(x) \geq c$ for all $x \in M$, and $f(y) \leq c$ for all $y \in N$, then f is said to *separate* M and N. In other words, $\sup_{y \in N} f(y) \leq \inf_{x \in M} f(x)$.

(ii) If $\sup_{y \in N} f(y) < \inf_{x \in M} f(x)$, then f is said to *strictly separate* M and N.

Remark. By definition, the following are equivalent:

(i) f separates M and N;

(ii) f separates $M - N$ and $\{0\}$, where $M - N := \{x - y : x \in M,\ y \in N\}$;

(iii) f separates $M - x$ and $N - x$ for all $x \in X$, where $M - x = \{y - x : y \in M\}$.

Lemma 2.30. Let M be a convex set in a normed space X. Then for all $x \in M$, all $y \in \mathring{M}$, and all $t \in (0, 1)$,

$$(1 - t)x + ty \in \mathring{M}.$$

Proof. Let $x \in M$ and $y \in \mathring{M}$. Then there exists $\epsilon > 0$ such that $O(y, \epsilon) \subset M$. Let $0 < t < 1$. Then

$$O((1 - t)x + ty, t\epsilon) = \{(1 - t)x + tz : z \in O(y, \epsilon)\} \subset M.$$

Hence $(1 - t)x + ty \in \mathring{M}$. ☐

Now we introduce Minkowski functional theory, which connects convexity to seminorm.

Proposition 2.31. Let p be a seminorm on a vector space X. Then for all $c > 0$, the set

$$M = \{x \in X : p(x) \leq c\}$$

satisfies the following: (i) $0 \in M$; (ii) M is convex; (iii) M is balanced: $\alpha M \subset M$ for all $|\alpha| = 1$; (iv) M is absorbing: for all $x \in X$, there exists $\alpha > 0$ such that $x \in \alpha M$; (v) The seminorm p can be recovered by

$$p(x) = \inf_{\alpha > 0,\, x \in \alpha M} c\alpha.$$

Proof. The properties (i), (ii), (iii) and (iv) are clear. It remains to prove (iv). For all $x \in X$,

$$x \in \alpha M \ \Leftrightarrow\ p(\alpha^{-1}x) \leq c \ \Leftrightarrow\ p(x) \leq c\alpha \quad \Rightarrow \quad p(x) \leq \inf_{\alpha > 0,\, x \in \alpha M} c\alpha$$

Conversely, if $p(x) \neq 0$, then $\frac{cx}{p(x)} \in M$. Set $\alpha = \frac{p(x)}{c}$, then $p(x) = c\alpha$, and

$$p(x) = \min_{\alpha > 0,\, x \in \alpha M} c\alpha.$$

If $p(x) = 0$, then $x \in \alpha M$ for all $\alpha > 0$, and $\inf_{\alpha > 0} c\alpha = 0$. ☐

Let M be an absorbing convex set in vector space X. By definition, for all $x \in X$, there exists $\alpha > 0$ such that $x \in \alpha M$. Hence $0 \in M$, and M is *star-shaped*: whenever $x \in M$, we have $tx \in M$ for all $t \in [0,1]$. As a result, if $x \in \alpha M$ for $x \in X$ and $\alpha > 0$, then $x \in \beta M$ for all $\beta > \alpha$.

Lemma 2.32 (Minkowski functionals). Let M be an absorbing convex subset of a vector space X. For each $x \in X$, define

$$p_M(x) = \inf_{\alpha>0,\, x\in\alpha M} \alpha.$$

Then $p_M : X \to \mathbb{R}_+$ is called the *Minkowski functional* of M. It satisfies the following:

(i) p_M is a sublinear functional on X;

(ii) If M is balanced, then p_M is a seminorm on X.

Proof. (i) Clearly, $0 \in M$, and $p_M(\lambda x) = \lambda p_M(x)$ for all $\lambda > 0$. It remains to verify subadditivity.

For all $x, y \in X$ and all $\epsilon > 0$, by definition,

$$\frac{x}{p(x)+\epsilon} \in M,\ \frac{y}{p(y)+\epsilon} \in M.$$

Note that M is convex, we have

$$\frac{x+y}{p(x)+p(y)+2\epsilon} = \frac{p(x)+\epsilon}{p(x)+p(y)+2\epsilon}\cdot\frac{x}{p(x)+\epsilon} + \frac{p(y)+\epsilon}{p(x)+p(y)+2\epsilon}\cdot\frac{y}{p(y)+\epsilon} \in M.$$

Therefore $p(x)+p(y)+2\epsilon \geq p(x+y)$. Since $\epsilon > 0$ is arbitrary, the subadditivity of p follows.

(ii) It remains to show homogeneity. When $\lambda \neq 0$,

$$\begin{aligned} p_M(\lambda x) = \inf\{\alpha : \alpha > 0, \lambda x \in \alpha M\} &= \inf\left\{\alpha : \alpha > 0, \lambda\alpha^{-1}x \in M\right\} \\ &= \inf\left\{\alpha : \alpha > 0, |\lambda|\alpha^{-1}x \in M\right\} \\ &= |\lambda| \inf\{\alpha : \alpha > 0, x \in \alpha M\} = |\lambda| p_M(x). \end{aligned}$$

The third equality holds because M is balanced, which implies $\lambda\alpha^{-1}x \in M$ if and only if $|\lambda|\alpha x \in M$. □

Lemma 2.33. Let X be a vector space, and let M be an absorbing convex subset of X. Then

$$\{x \in X : p_M(x) < 1\} \subset M \subset \{x \in X : p_M(x) \leq 1\}.$$

Proof. For the first inclusion, let x be a point of X such that $p_M(x) < 1$. Then there exists $\alpha \in (0,1)$ such that $\alpha^{-1}x \in M$. Since M is star-shaped, $x \in M$.

For the second inclusion, we have $p_M(x) = \inf\{\alpha : \alpha > 0, x \in \alpha M\} \leq 1$ for all $x \in M$. □

Remark. Let X be a vector space. We consider the sets

$$\mathcal{P} = \{p : p \text{ is a seminorm on } X\} \quad \text{and} \quad \mathscr{M} = \{M \subset X :\ M \text{ is a balanced absorbing convex set in } X\}.$$

By Proposition 2.31 and Lemma 2.32, we define maps

$$\Phi : \mathcal{P} \to \mathscr{M},\ p \mapsto \{x \in X : p(x) \leq 1\} \quad \text{and} \quad \Psi : \mathscr{M} \to \mathcal{P},\ M \mapsto p_M,$$

where p_M is the Minkowski functional of M. By Proposition 2.31 (v), we have $\Psi \circ \Phi(p) = p$ for all $p \in \mathcal{P}$.

However, the equality $\Phi \circ \Psi(M) = M$ does not hold for all $M \in \mathscr{M}$. To see a counterexample, let X be the Euclidean space $\mathbb{R}^n$, and let M be the unit open ball $\{x \in X : \|x\|_2 < 1\}$. Then

$$p_M(x) = \inf_{\alpha>0,\, x\in\alpha M} \alpha = \|x\|_2,$$

and $\Phi \circ \Psi(M) = \{x \in X : \|x\|_2 \le 1\}$ is the unit closed ball. By Lemma 2.33, when p_M is given, we can only determine M between a lower bound $\{p_M(x) < 1\}$ and an upper bound $\{p_M(x) \le 1\}$.

Lemma 2.34. Let M be an absorbing convex set in a vector space X. Let $x \in X$. The following are equivalent: (i) $p_M(x) < 1$; (ii) For all $y \in X$, there exists $\epsilon_y > 0$ such that $x + ty \in M$ for all $|t| < \epsilon_y$.

Proof. (i) $\Rightarrow$ (ii): By definition, for every $y \in X$, there exists $\lambda_y > 0$ such that $\lambda_y^{-1} y \in M$ and $-\lambda_y^{-1} y \in M$. Since $p_M(x) < 1$, there exists $0 < \alpha < 1$ such that $\alpha^{-1}x \in M$. Then

$$x + (1-\alpha)\lambda_y^{-1} y \in M,\ x - (1-\alpha)\lambda_y^{-1} y \in M.$$

Note that $x \in M$, we have $x + ty \in M$ for all $|t| < \epsilon_y := (1-\alpha)\lambda_y^{-1}$.

(ii) $\Rightarrow$ (i): We prove the contrapositive. Assume that $p_M(x) \ge 1$, then $\alpha^{-1}x \notin M$ for all $0 < \alpha < 1$. Therefore, $x + \epsilon x \notin M$ for all $\epsilon > 0$, contradicting (ii). $\square$

The following criterion to determine continuity of a Minkowski functional is useful.

Lemma 2.35. Let M be an absorbing convex set in a normed space X. The Minkowski functional p_M is continuous if and only if $0 \in \mathring{M}$.

Proof. "$\Rightarrow$": By continuity, $\{p_M(x) < 1\} = p_M^{-1}((-\infty, 1))$ is an open set containing 0 and contained in M.

"$\Leftarrow$": Let $O(0, \delta)$ be an open ball contained in M. Then $p(x) \le 1$ for all $\|x\| \le \delta$. Let $x_0 \in X$ and $\epsilon > 0$ be given. For all $x \in O(x_0, \delta\epsilon/2)$, since p is a sublinear functional, we have

$$p_M(x) - p_M(x_0) \le p_M(x - x_0) \le \frac{\epsilon}{2} < \epsilon.$$

A similar statement holds for $p_M(x_0) - p_M(x)$. Since x_0 is arbitrary, $p_M(x)$ is continuous on X. $\square$

Lemma 2.36. Let M be an absorbing convex set in a normed space X such that $0 \in \mathring{M}$, and let $y \in X$. If $p_M(y) < 1$, then $y \in \mathring{M}$. (Contrapositive: If $y \notin \overline{M}$, then $p_M(y) = 1$.)

Proof. If $p_M(y) < 1$, then there exists $\lambda > 1$ such that $\lambda y \in M$. Since $0 \in \mathring{M}$, by Lemma 2.30, $y \in \mathring{M}$. $\square$

Remark. By Lemma 2.33 and Lemma 2.36, if $y \in \partial M$, where $\partial M = \overline{M} \cap \overline{(X \backslash M)}$ is the frontier ∂M of an absorbing convex set M, then $p_M(y) = 1$.

Now we discuss the separation of convex sets.

Lemma 2.37 (Separation of a convex set and a one-point set). Let A be a convex subset of a real normed space X such that $\mathring{A} \neq \emptyset$. Let $y_0 \notin \mathring{A}$. Then there exists nonzero $f \in X^*$ that separates A and $\{y_0\}$.

Proof. If $a \in \mathring{A}$, we separate $A - a$ and $\{y_0 - a\}$. Without loss of generality, we suppose $0 \in \mathring{A}$. Then there exists open ball $O(0, \epsilon) \subset A$, and A is absorbing. By Lemma 2.35, the Minkowski functional p_A is a continuous sublinear functional on X. By Lemma 2.33, $p_A(x) \le 1$ for all $x \in A$. By Lemma 2.36, $p_A(y_0) \ge 1$.

Let $X_0 = \text{span}\{y_0\} = \mathbb{R}\cdot y_0$. Then $f_0 : ty_0 \mapsto tp_A(y_0)$ is a linear functional on X_0 that is subject to p_A:

$$tp_A(y_0) = p_A(ty_0),\ t \geq 0; \quad tp_A(y_0) < 0 \leq p_A(ty_0),\ t < 0.$$

By Hahn-Banach theorem, there exists an extension f on X such that $f|_{X_0} = f_0$ and $f \leq p_A$. To show that $f \in X^*$, note that the continuity of f follows from p_A:

$$|f(x) - f(x')| = |f(x - x')| \leq p_A(x - x')$$

Furthermore, f separates A and $\{y_0\}$: $f(x) \leq p_A(x) \leq 1$ for all $x \in A$, and $f(y_0) = p_A(y_0) \geq 1$. ☐

Remark. In Lemma 2.37, if A is an open convex set, and $y_0 \notin A$, then

$$f(x) < \sup_{y\in A} f(y) \leq f(y_0),\ \forall x \in A.$$

For the first equality, let $O(x,\epsilon)$ be an open ball contained in A. Since f is nonzero, there exists $f(z) > 0$. Then $x + \frac{\epsilon z}{2\|z\|} \in A$, and $f(x) < f\left(x + \frac{\epsilon z}{2\|z\|}\right)$.

Theorem 2.38 (Hyperplane separation theorem). Let M and N be convex sets in a real normed space X. If $\mathring{M} \neq \emptyset$, and $\mathring{M} \cap N = \emptyset$, then there exists nonzero $f \in X^*$ that separates M and N.

Proof. By Lemma 2.30, $\mathring{M}$ is also a convex set. Define

$$A = \mathring{M} - N = \bigcup_{y\in N} (\mathring{M} - y) = \{x - y : x \in \mathring{M}, y \in N\}.$$

Then A is an open and convex set in X, and $0 \notin A$. By Lemma 2.37, there exists nonzero $f \in X^*$ that separates A and $\{0\}$. Then f^* separates $\mathring{M}$ and N: $\sup_{x\in\mathring{M}} f(x) \leq \inf_{y\in N} f(y)$.

Given $x \in M$, we fix some $z \in \mathring{M}$. Then $(1-t)x + tz \in \mathring{M}$ for all $t \in (0,1)$. Since f is continuous,

$$f(x) = \lim_{t\to 0^+} f\left((1-t)x + tz\right) \leq \sup_{x\in\mathring{M}} f(x) \leq \inf_{y\in N} f(y).$$

Since $x \in M$ is arbitrary, f also separates M and N. ☐

Corollary 2.39 (Hyperplane separation theorem). Let M and N be disjoint closed convex sets in a normed space X. Then there exists nonzero $f \in X^*$ that strictly separates M and N.

Proof. Since M and N are closed disjoint sets, $d(M,N) = \inf_{x\in M, y\in N} \|x - y\| > 0$. Let

$$\widetilde{N} = \bigcup_{x\in N} O\left(x, \frac{d(M,N)}{3}\right) = \left\{x \in X : d(x,N) < \frac{d(M,N)}{3}\right\}.$$

Then $\widetilde{N}$ is an open set in X that is disjoint from M. Furthermore, for all $x, y \in \widetilde{N}$, there exists $z_x \in M$ and $z_y \in M$ such that $\|x - z_x\| < \frac{1}{3}d(M,N)$ and $\|y - z_y\| < \frac{1}{3}d(M,N)$. For all $t \in (0,1)$,

$$d\left((1-t)x + ty, N\right) \leq \|(1-t)x + ty - (1-t)z_x - tz_y\| \leq (1-t)\|x - z_x\| + t\|y - z_y\| < \frac{1}{3}d(M,N).$$

Hence $\widetilde{N}$ is convex.

By Theorem 2.38, there exists nonzero $f \in X^*$ that separates M and $\widetilde{N}$:

$$\sup_{x\in M} f(x) \le \beta \le \inf_{y\in \widetilde{N}} f(y),\ \beta \in \mathbb{R}.$$

Let $y \in N$ and $z \in O\left(0, \frac{d(M,N)}{3}\right)$, then $y - z \in \widetilde{N}$. Since $f(y) = f(y-z) + f(z) \ge \beta + f(z)$ for all $z \in O\left(0, \frac{d(M,N)}{3}\right)$, we have

$$f(y) \ge \beta + \sup_{z\in O(0,d(M,N)/3)} f(z) = \beta + \frac{d(M,N)\|f\|}{3},\ \forall y \in N.$$

As a result, f strictly separates M and N. $\square$

2.4 Dual Spaces

2.4.1 Dual Spaces of $L^p(X, \mathscr{A}, \mu)$ and $C([a,b])$

Example 2.40. Let $(X, \mathscr{A}, \mu)$ be a measure space, and let $1 \le p < \infty$. Given $g \in L^q(X, \mathscr{A}, \mu)$, where q is the conjugate of p. (That is, $p^{-1} + q^{-1} = 1$ if $p > 1$, and $q = \infty$ if $p = 1$.) We define a linear functional $T : L^p(X, \mathscr{A}, \mu) \to \mathbb{R}$ by

$$T(f) = \int_X fg\,d\mu,\ \forall f \in L^p(X, \mathscr{A}, \mu).$$

By Hölder's inequality, $T(f) \le \|f\|_p\|g\|_q$, which implies the continuity of T. Furthermore, $\|T\| = \|g\|_q$.

Naturally, we wonder if every continuous linear functional on $L^p(X, \mathscr{A}, \mu)$ admits this form. If so, we can determine a unique function $g \in L^q(X, \mathscr{A}, \mu)$ for each $T \in (L^p(X, \mathscr{A}, \mu))^*$, and $\|g\|_q = \|T\|$. As a result, we have $(L^p(X, \mathscr{A}, \mu))^* \cong L^q(X, \mathscr{A}, \mu)$.

Riesz Representation Theorem. Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space. Let $1 \le p < \infty$, and let q be the conjugate of p. Then for any bounded linear functional $T \in (L^p(X, \mathscr{A}, \mu))^*$, there exists a unique $g \in L^q(X, \mathscr{A}, \mu)$ such that

$$T(f) = \int_X fg\,d\mu,\ \forall f \in L^p(X, \mathscr{A}, \mu).$$

Immediately, we have $(L^p(X, \mathscr{A}, \mu))^* \cong L^q(X, \mathscr{A}, \mu)$.

Proof. Step I: We first suppose $\mu(X) < \infty$. Then $\chi_A \in L^p(X, \mathscr{A}, \mu)$, $\forall A \in \mathscr{A}$, and $\nu : \mathscr{A} \to \mathbb{R}, A \mapsto T(\chi_A)$ is well-defined. By continuity of T, ν is a signed measure that is absolutely continuous with respect to μ:

$$\mu(A) = 0 \ \Rightarrow\ 0 \le |\nu(A)| = |T(\chi_A)| \le \|T\|\,\|\chi_A\|_p = 0$$

By Radon-Nikodym theorem, there exists $g \in L^1(X, \mathscr{A}, \mu)$ such that

$$T(\chi_A) = \nu(A) = \int_A g\,d\mu = \int_X g\chi_A\,d\mu,\ \forall A \in \mathscr{A}.$$

As a result, for all simple functions φ on $(X, \mathscr{A}, \mu)$, we have

$$T(\varphi) = \int_X g\varphi\,d\mu. \tag{2.5}$$

Step II: We prove that (2.5) holds for all bounded measurable φ on $(X, \mathscr{A}, \mu)$. Suppose $|\varphi| \leq M$. Then there exists a sequence of simple functions φ_n such that $|\varphi_n| \leq M$ and φ_n converges pointwise to φ. By Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \|\varphi - \varphi_n\|_p^p = \lim_{n\to\infty} \int_X |\varphi - \varphi_n|^p \, d\mu = \int_X \lim_{n\to\infty} |\varphi - \varphi_n|^p \, d\mu = 0.$$

Since T is continuous, and $|g\varphi_n| \leq M|g|$, which is integrable, we have

$$T(\varphi) = \lim_{n\to\infty} T(\varphi_n) = \lim_{n\to\infty} \int_X g\varphi_n \, d\mu = \int_X g\varphi \, d\mu.$$

Step III: We prove that $g \in L^q(X, \mathscr{A}, \mu)$. Suppose $p > 1$ and $q < \infty$. Define sequence

$$g_n(x) = \begin{cases} |g(x)|^{q-1}\mathrm{sgn}\,(g(x)), & \text{if } |g(x)|^q \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

By (2.5), we have

$$\int_X |g_n|^q \, d\mu = T(g_n) \leq \|T\| \, \|g_n\|_p \leq \|T\| \left(\int_X |g_n|^q \, d\mu \right)^{1/p} \quad \Rightarrow \quad \left(\int_X |g_n|^q \, d\mu \right)^{1/q} \leq \|T\|.$$

Let $n \to \infty$, then we have $g \in L^q(X, \mathscr{A}, \mu)$, and $\|g\|_q \leq \|T\|$.

Step IV: We show that $S(f) := T(f) - \int_X fg \, d\mu = 0$ for all $f \in L^p(X, \mathscr{A}, \mu)$. By Hölder's inequality, S is a continuous linear functional on $L^p(X, \mathscr{A}, \mu)$ that vanishes on all bounded measurable functions. Since any $f \in L^p(X, \mathscr{A}, \mu)$ can be approximated by its n-truncations $[f]_n = \min\{n, \max\{-n, f\}\}$ in $\|\cdot\|_p$, the result follows. The uniqueness of g is clear.

Step V: Let $(X, \mathscr{A}, \mu)$ be σ-finite. Write $X = \bigcup_{n=1}^\infty X_n$, where $\mu(X_n) < \infty$ for all $n \in \mathbb{N}$. By Steps I-IV, we can find $g_n \in L^q(X, \mathscr{A}, \mu)$, supported within X_n, such that $T(f) = \int_X fg_n \, d\mu = \int_{X_n} fg_n \, d\mu$ for all $f \in L^p(X, \mathscr{A}, \mu)$, and $\|g_n\|_q \leq \|T\|$. Since g_n's are unique, we assume that $g_{n+1} = g_n$ on X_n.

Let $g(x) = \lim_{n\to\infty} g_n(x)$ for all $x \in X$. Then $|g_n| \nearrow |g|$. By monotone convergence theorem,

$$\int_X |g|^q \, d\mu = \lim_{n\to\infty} \int_X |g_n|^q \, d\mu \leq \|T\|.$$

Hence $g \in L^q(X, \mathscr{A}, \mu)$. For any $f \in L^p(X, \mathscr{A}, \mu)$, $f_n := f\chi_{X_n} \to f$ pointwise, and $|f_n g| \leq |fg|$. By Lebesgue dominated convergence theorem,

$$\int_X fg \, d\mu = \lim_{n\to\infty} \int_X f_n g \, d\mu = \lim_{n\to\infty} \int_{X_n} f_n g_n \, d\mu = \lim_{n\to\infty} T(f_n) = T(f),$$

as desired. Note the last equality follows from continuity of T. $\square$

Remark I. If $p = 1$ and $q = \infty$, we need to modify Step III. Argue by contradiction. If $g \notin L^\infty(X, \mathscr{A}, \mu)$, we have $\mu(E_\epsilon) > 0$ for $E_\epsilon = \{|g| > \|T\| + \epsilon\}$ and all $\epsilon > 0$. Then

$$T(\chi_{E_\epsilon}) = \int_X g\chi_{E_\epsilon} \, d\mu \geq \mu(E_\epsilon) \left(\|T\| + \epsilon \right).$$

Meanwhile, $|T(\chi_{E_\epsilon})| \leq \|T\| \, \|\chi_{E_\epsilon}\|_1 = \mu(E_\epsilon) \, \|T\|$, a contradiction! Hence $g \in L^\infty(X, \mathscr{A}, \mu)$, and $\|g\|_\infty \leq \|T\|$.

Remark II. If $p > 1$, we can drop the requirement of σ-finiteness. Let $(X, \mathscr{A}, \mu)$ be any measure space, and let $E \subset X$ be σ-finite. Then there exists a unique $g_E \in L^q(X, \mathscr{A}, \mu)$, vanishing outside E, such that

$$T(f) = \int_X f g_E \, d\mu, \ \forall f \in L^p(X, \mathscr{A}, \mu) \text{ vanishing outside } E, \text{ and } \|g_E\|_q \le \|T\|.$$

By uniqueness of g_E, for any $A \subset E$, $g_A = g_E$ almost everywhere on A. Define $\nu(E) = \int_X |g_E|^q \, d\mu$ for every σ-finite set E in X. Then ν is a measure such that $\nu \ll \mu$, and $\nu(A) \le \mu(E) < \|T\|^q$ for all $A \subset E$.

Let $M = \sup\{\nu(E) : E \text{ is } \sigma\text{-finite}\}$, and $\{E_n, n \in \mathbb{N}\}$ a sequence of sets such that $\lim_{n\to\infty} \nu(E_n) = M$. Then $H := \bigcup_{n=1}^\infty E_n$ is σ-finite, and $\nu(H) = M$. For any σ-finite set $F \supset H$, $g_F = g_H$ a.e. on H, and

$$\int_X |g_F|^q \, d\mu = \nu(F) \le \nu(H) = \int_X |g_H|^q \, d\mu.$$

Hence $g_F = 0$ a.e. on $F \backslash H$. Let $g = g_H$, we have $g \in L^q(X, \mathscr{A}, \mu)$, and $g_F = g$ a.e. for all σ-finite set $F \supset H$.

Given $f \in L^p(X, \mathscr{A}, \mu)$, let $E = \{x \in X : f(x) \ne 0\}$. Then $E = \bigcup_{n=1}^\infty \{|f| > 1/n\}$ is σ-finite. As a result,

$$T(f) = \int_E f g_E \, d\mu = \int_X f g \, d\mu.$$

Review: Lebesgue-Stieltjes Measure. Let $\Omega = [a, b]$ be a closed interval on $\mathbb{R}$. Then the collection of sets $\mathcal{E} = \{(u, v] : a \le u \le v \le b\}$ is a ring: (i) $\emptyset \in \mathcal{E}$; (ii) $\forall A, B \in \mathcal{E}$, $A \cap B \in \mathcal{E}$; (iii) $\forall A, B \in \mathcal{E}$, $A \backslash B \in \mathcal{E}$.

Let g be a non-decreasing function in $V_0([a, b])$, that is, g is of bounded variation and right-continuous on $[a, b]$, and $g(a) = 0$. We define a finite additive measure μ_{0g} on $\mathcal{E}$ by $\mu_{0g}((u, v]) = g(v) - g(u)$, and extend it to a pre-measure on the algebra $\mathcal{A}$ generated by $\mathcal{E}$ by setting $\mu_{0g}(\{a\}) = 0$. This pre-measure gives rise to an outer measure:

$$\mu_g^*(E) = \inf\left\{\sum_{n=1}^\infty \mu_{0g}(A_n) : \{A_n, n \in \mathbb{N}\} \subset \mathcal{A}, \bigcup_{n=1}^\infty A_n \supset E\right\}, \ \forall E \in 2^\Omega.$$

By Carathéodory extension theorem, μ_g^* is a measure on $\{A \subset \Omega : \mu_g^*(A) = \mu_g^*(A \cap E) + \mu_g^*(A \backslash E), \forall E \subset \Omega\}$, which is a σ-algebra that contains all Borel sets $\mathscr{B}([a, b])$. Furthermore, the restriction $\mu_g = \mu_g^*|_{\mathscr{B}(\mathbb{R})}$ is the unique extension of μ_{0g} on $\mathscr{B}(\mathbb{R})$: $\mu_g((u, v]) = g(v) - g(u)$ for all $a \le u \le v \le b$.

Generally, let $g \in V_0([a, b])$ be given. Then

$$v_g(x) := V_a^x(g) = \sup\left\{\sum_{j=1}^n |g(x_j) - g(t_{j-1})| : n \in \mathbb{N}, \ a = t_0 < t_1 < \cdots < t_n = x\right\}$$

is a monotone non-decreasing function in $V_0([a, b])$. Clearly, $v_g - g \in V_0([a, b])$, and for all $a \le x < y \le b$,

$$g(y) - g(x) \le V_x^y(g) \le V_a^y(g) - V_a^x(g) = v_g(y) - v_g(x).$$

Hence $v_g - g$ is also non-decreasing. As a result, we can find a signed Borel measure $\mu_g := \mu_{v_g} - \mu_{v_g - g}$ such that $\mu_g((u, v]) = g(v) - g(u)$ for all $a \le u \le v \le b$. Moreover, if ν_g is another such extension on $\mathscr{B}([a, b])$, then $\mathcal{F} = \{E \subset [a, b] : \mu_g(E) = \nu_g(E)\}$ is a λ-system that contains $\mathcal{E}$, which is a π-system. By Dynkin's π-λ theorem, we have $\mathscr{B}([a, b]) \subset \sigma(\mathcal{E}) \subset \mathcal{F}$. Therefore $\nu_g = \mu_g$, and the extension μ_g is unique. We call this unique signed Borel measure μ_g the *Lebesgue-Stieltjes measure* of g.

Theorem 2.41. Define the space of all finite signed Borel measures on $[a,b]$ by

$$M([a,b]) = \{\mu : \mu \text{ is a finite signed Borel measure on } [a,b]\},$$

and define a norm on $M([a,b])$ by

$$\|\mu\| = \sup\left\{\sum_{j=1}^{n} |\mu(E_i)| : E_j \in \mathscr{B}([a,b]), \coprod_{i=1}^{n} E_i = [a,b]\right\}, \ \forall \mu \in M([a,b]).$$

Then $V_0([a,b]) \cong M([a,b])$.

Proof. Define $U : V_0([a,b]) \to M([a,b]),\ g \mapsto \mu_g$, where μ_g is the Lebesgue-Stieltjes measure of $g \in V_0([a,b])$. Clearly, U is a linear map. For all $a = t_0 < t_1 < \cdots < t_n = b$, we have

$$\sum_{j=1}^{n} |g(t_j) - g(t_{j-1})| = \sum_{j=1}^{n} |\mu_g\left((t_j, t_{j-1}]\right)| \le \|\mu_g\|,$$

which implies $\|g\| \le \|\mu_g\|$. Furthermore, $g_1 = \frac{1}{2}(v_g - g)$ and $g_2 = \frac{1}{2}(v_g + g)$ are non-decreasing, then

$$\|\mu_g\| = \|\mu_{g_2} - \mu_{g_1}\| \le \|\mu_{g_1} + \mu_{g_2}\| = \|\mu_{v_g}\| = v_g(b) = V_a^b(g) = \|g\|.$$

The inequality holds because $|\mu_{g_1}(E) - \mu_{g_2}(E)| \le \mu_{g_1}(E) + \mu_{g_2}(E)$ for all $E \in \mathscr{B}([a,b])$.

Finally, for each $\mu \in M([a,b])$, let $g(x) := \mu((a,x])$, $x \in [a,b]$. Clearly, $V_a^b g \le \|\mu_g\|$. Moreover, for every sequence $\epsilon_n \searrow 0$,

$$\lim_{n\to\infty} \mu((a, x+\epsilon_n]) = \mu\left(\bigcap_{n=1}^{\infty} (a, x+\epsilon_n]\right) = \mu((a,x]).$$

Hence g is right-continuous, and μ is the Lebesgue-Stieltjes measure of $g \in V_0([a,b])$, which implies surjectivity of the map U. Therefore, $U : V_0([a,b]) \to M([a,b])$ is a norm-preserving isomorphism, as desired. □

Example 2.42 (Dual spaces of $C([a,b])$) Given a function $g \in V_0([a,b])$, we define the *Lebesgue-Stieltjes integral* of $\varphi \in C([a,b])$ relative to g as

$$F_g(\varphi) = \int_a^b \varphi(t)\, dg(t) := \int_{[a,b]} \varphi\, d\mu_g$$

Clearly, F_g is a linear functional on $C([a,b])$. Moreover,

$$|F_g(\varphi)| \le \int_a^b |\varphi(t)|\, |dg|(t) \le \|\varphi\|_\infty \|g\|,$$

and the equality holds if $\varphi = \chi_P - \chi_N$, where $P \amalg N = [a,b]$ is a Hahn decomposition for μ_g. As a result, we have $F_g \in (C([a,b]))^*$, and $\|F_g\| = \|g\| = V_a^b(g)$. Naturally, we wonder if every continuous linear functional on $C([a,b])$ is determined by Lebesgue-Stieltjes integration, which implies $(C([a,b]))^* \cong V_0([a,b])$.

Riesz Representation Theorem. For all $F \in (C([a,b]))^*$, there exists a unique $g \in V_0([a,b])$ such that

$$F(\varphi) = F_g(\varphi) = \int_a^b \varphi(t)\, dg(t), \ \forall \varphi \in C([a,b]), \text{ and } \|F\| = \|g\| = V_a^b(g).$$

Proof. *Step I:* We first view $C([a,b])$ as a subspace of $(B([a,b]), \|\cdot\|_\infty)$, which is the space of all bounded functions on $[a,b]$, and $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$ for all $f \in B([a,b])$. By Hahn-Banach theorem, there exists an extension $F_B : B([a,b]) \to \mathbb{R}$ of F such that $F_B|_{C([a,b])} = F$ and $\|F_B\| = \|F\|$.

Step II: Let $h(t) := F_B(\chi_{[a,t]})$ for all $t \in [a,b]$. We first show that $h \in V([a,b])$. For each partition $a = t_0 < t_1 < \cdots < t_n = b$, let $\epsilon_j = \operatorname{sgn}\left[h(t_j) - h(t_{j-1})\right],\ t = 1, \cdots, n$. Then

$$\begin{aligned}\sum_{j=1}^{n} |h(t_j) - h(t_{j-1})| &= \sum_{j=1}^{n} \epsilon_j \left[h(t_j) - h(t_{j-1})\right] \\ &= F_B\left(\sum_{j=1}^{n} \epsilon_j \chi_{(t_{j-1},t_j]}\right) \leq \|F_B\| \left\|\sum_{j=1}^{n} \epsilon_j \chi_{(t_{j-1},t_j]}\right\|_\infty \leq \|F\|.\end{aligned}$$

Hence $h \in V([a,b])$, and $V_a^b(h) \leq \|F\|$. Clearly, $h(a) = 0$. Now we define

$$g(x) = \begin{cases} \lim_{\epsilon\to 0^+} h(x+\epsilon), & x \in (a,b), \\ h(x), & x \in \{a,b\}. \end{cases}$$

Then g is right-continuous, and $V_a^b(g) \leq V_a^b(h)$.

Step III: We prove that

$$F_B(\varphi) = \int_a^b \varphi(t)\, dg(t),\ \forall \varphi \in C([a,b]).$$

Fix $\varphi \in C([a,b])$, and choose partitions $a = x_0^{(k)} < x_1^{(k)} < \cdots < x_{n_k}^{(k)} = b$ such that

- $\left\{x_j^{(k)},\ j = 1, \cdots, n_k - 1\right\}$ are continuous points of h; and
- $\lim_{k\to\infty} \max_{1\leq j\leq n_k} \left[x_j^{(k)} - x_{j-1}^{(k)}\right] = 0$.

We can always choose such partitions on $[a,b]$, because a function h of bounded variation on $[a,b]$ has at most countably many discontinuous points. Define

$$\varphi_k(t) = \sum_{j=1}^{n_k} \varphi(x_j^{(k)}) \left(\chi_{[a,x_j^{(k)}]}(t) - \chi_{[a,x_{j-1}^{(k)}]}(t)\right)$$

Then $\varphi_k \in B([a,b])$, and $\lim_{k\to\infty} \|\varphi_k - \varphi\|_\infty = 0$. Furthermore,

$$\begin{aligned}F_B(\varphi_k) &= \sum_{j=1}^{n_k} \varphi(x_j^{(k)}) \left[h(x_j^{(k)}) - h(x_{j-1}^{(k)})\right] \\ &= \sum_{j=1}^{n_k} \varphi(x_j^{(k)}) \left[g(x_j^{(k)}) - g(x_{j-1}^{(k)})\right] \\ &= \int_a^b \sum_{j=1}^{n_k} \varphi(x_j^{(k)}) \left(\chi_{[a,x_j^{(k)}]}(t) - \chi_{[a,x_{j-1}^{(k)}]}(t)\right) dg(t) = \int_a^b \varphi_k(t)\, dg(t).\end{aligned}$$

Note that $|\varphi_k| \leq \|\varphi\|_\infty$. By Lebesgue dominated convergence theorem,

$$F_B(\varphi) = \lim_{k\to\infty} F_B(\varphi_k) = \int_a^b \lim_{n\to\infty} \varphi_k(t)\, dg(t) = \int_a^b \varphi(t)\, dg(t).$$

Then $F(\varphi) = F_B(\varphi) = F_g(\varphi)$ for all $\varphi \in C([a,b])$. Clearly, $\|F\| = \|F_g\| = V_a^b(g)$, and g is unique. $\square$

Remark. By Theorem 2.41, $V_0([a,b]) \cong M([a,b])$. Then $(C([a,b]))^* \cong M([a,b])$: for all bounded linear functional F on $C([a,b])$, there exists a unique finite signed measure μ on $([a,b], \mathscr{B}([a,b]))$ such that

$$F(\varphi) = \int_{[a,b]} \varphi \, d\mu, \ \forall \varphi \in C([a,b]).$$

2.4.2 Reflexive Spaces

Let X be a normed space. The dual space X^* is the set of all bounded linear functionals on X. The *bidual space* X^{**}, is the set of all bounded linear functionals on X^*.

Definition 2.43 (Canonical maps). Let x be a normed space. Given $x \in X$, define $x^{**} : X^* \to \mathbb{R}$ as

$$x^{**}(f) = f(x), \ \forall f \in X^*.$$

Then x^{**} is a linear functional on X^*, and $|x^{**}(f)| \leq \|f\| \, \|x\|$. Hence we have $x^{**} \in X^{**}$, and $\|x^{**}\| \leq \|x\|$. We define the *canonical map* $J : X \to X^{**}$ as $J(x) = x^{**}$.

Lemma 2.44. Let X be a normed space, and let $J : X \to X^{**}$ be the canonical map. Then J is a norm-preserving linear operator.

Proof. The linearity is clear: $(\alpha x + \beta y)^{**} = \alpha x^{**} + \beta y^{**}$ for all $x, y \in X$, $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$). To show that J is norm preserving, it suffices to show $\|x^{**}\| \geq \|x\|$ for all $x \in X$. By Corollary 2.22, there exists $f_0 \in X^*$ such that $\|f_0\| = 1$ and $f_0(x) = \|x\|$. Then $\|x^{**}\| \geq x^{**}(f_0) = f_0(x) = \|x\|$. ☐

Definition 2.45 (Reflexive spaces). Let X be a normed space. X is said to be a *reflexive space*, if the canonical map $J : X \to X^{**}$ is an isomorphism. In this case, $X^{**} \cong X$.

Remark. (i) If X is reflexive, so is X^*: $(X^*)^{**} = (X^{**})^* = X^*$.

(ii) By definition, X^{**} is complete. If X is not complete, then the closure of JX in X^{**} automatically gives a completion of X.

Example 2.46. Let $(X, \mathscr{A}, \mu)$ be a measure space. By Example 2.42, for $1 < p < \infty$, $L^p(X, \mathscr{A}, \mu)$ is reflexive.

Theorem 2.47. Let M be a closed subspace of a reflexive space X. Then M and X/M are reflexive.

Proof. (i) To prove that M is reflexive, it suffices to show the canonical map $J_M : M \to M^{**}$ is surjective.

Let $m^{**} \in M^{**}$. Define $x^{**}(f) = m^{**}(f|_M)$ for all $f \in X^*$. Then $x^{**} \in X^{**}$, and there exists $x \in X$ such that $J(x) = x^{**}$. If $x \notin M$, by Corollary 2.23, there exists $f \in X^*$ such that $f(x) \neq 0$, $f(M) = \{0\}$. However, $f(x) = x^{**}(f) = m^{**}(f|_M) = 0$, a contradiction! Hence $x \in M$.

We want $J_M(x) = m^{**}$, which completes the proof. For each $g \in M^*$, by Hahn-Banach theorem, there exists an extension $f \in X^*$ of g. Hence $g(x) = f(x) = x^{**}(f) = m^{**}(f|_M) = m^{**}(g)$, and $m^{**} = J(x)$.

(ii) We show the canonical map $J_{X/M} : X/M \to (X/M)^{**}$ is surjective. Given $[x]^{**} \in (X/M)^{**}$, we define $y^{**}(f \circ \pi) = [x]^{**}(f)$ for all $f \in (X/M)^*$, where $\pi : X \to X/M$ is the quotient map. Then y^{**} is a bounded linear functional on $(X/M)^* \circ \pi := \{f \circ \pi : f \in (X/M)^*\}$, a subspace of X^*. By Hahn-Banach theorem, there exists an extension $x^{**} \in X^{**}$ of y^{**}. Since X is reflexive, there exists $x \in X$ such that $J(x) = x^{**}$.

It remains to show $J_{X/M}([x]) = [x]^{**}$, which completes the proof: For all $f \in (X/M)^*$,

$$f([x]) = (f \circ \pi)(x) = x^{**}(f \circ \pi) = y^{**}(f \circ \pi) = [x]^{**}(f).$$ ☐

Lemma 2.48. Let X be a reflexive space. Then for all $f \in X^*$, there exists $x \in X$ such that $\|x\| = 1$ and $f(x) = \|f\|$.

Proof. By Corollary 2.22 (i), for all $f \in X^*$, let $x^{**} \in X^{**}$ be such that $\|x^{**}\| = 1$ and $x^{**}(f) = \|f\|$. By reflexivity, choosing x such that $J(x) = x^{**}$ completes the proof. $\square$

We also have the following conclusion similar to Corollary 1.39.

Theorem 2.49. Let X be a normed space.

(i) Let M be a subspace of X. Then ${}^{\perp}(M^{\perp}) = \overline{M}$;

(ii) Let G be a subspace of X^*. If X is reflexive, then $({}^{\perp}G)^{\perp} = \overline{G}$.

Proof. (i) Let $x \in M$. Then $f(x) = 0$ for all $f \in M^{\perp}$, which implies $x \in {}^{\perp}(M^{\perp})$. Since ${}^{\perp}(M^{\perp})$ is a closed subspace of X, $\overline{M} \subset {}^{\perp}(M^{\perp})$. If $\overline{M} \neq {}^{\perp}(M^{\perp})$, choose $x \in {}^{\perp}(M^{\perp}) \backslash \overline{M}$. By Corollary 2.23, there exists $f \in X^*$ such that $f(\overline{M}) = \{0\}$ and $f(x) \neq 0$. Then $f \in M^{\perp}$. However, $x \in {}^{\perp}(M^{\perp})$ implies $f(x) = 0$, a contradiction!

(ii) Clearly, $\overline{G} \subset ({}^{\perp}G)^{\perp}$. If $\overline{G} \neq ({}^{\perp}G)^{\perp}$, choose $g \in ({}^{\perp}G)^{\perp} \backslash \overline{G}$. There exists $x^{**} \in X^{**}$ such that $x^{**}\left(\overline{G}\right) = \{0\}$ and $x^{**}(g) \neq 0$. By reflexivity, choose $x = J^{-1}(x^{**}) \in X$. Then $x \in \overline{G}^{\perp} \subset G^{\perp}$, and $g(x) \neq 0$. However $g \in ({}^{\perp}G)^{\perp}$, which implies $g(x) = 0$, a contradiction! $\square$

Finally we see some examples of normed spaces that are not reflexive.

Lemma 2.50. Let X be a normed space. If X^* is separable, so is X.

Proof. If X^* is separable, choose a dense sequence $\{f_n\}$ in X^* in the unit sphere $S_{X^*} = \{f \in X^* : \|f\| = 1\}$. For each $n \in \mathbb{N}$, there exists $x_n \in S_X = \{x \in X : \|x\| = 1\}$ such that $|f_n(x_n)| > 1/2$. Denote by $X_0 := \overline{\operatorname{span}}\,\{x_n, n \in \mathbb{N}\}$ the closed subspace spanned by $\{x_n\}$. We prove $X_0 = X$.

Argue by contradiction. If $x \in X \backslash X_0$, then there exists $f \in X^*$ such that $\|f\| = 1$, $f(X_0) = \{0\}$ and $f(x) \neq 0$. Then for all $n \in \mathbb{N}$,

$$\|f_n - f\| = |f_n(x_n) - f(x_n)| = |f_n(x_n)| > \frac{1}{2},$$

contradicting the density of $\{f_n\}$ in S_{X^*}. Hence $X_0 = X$. Furthermore, $\{qx_n, q \in \mathbb{Q}, n \in \mathbb{N}\}$ is a countable dense subset of X_0, as desired. $\square$

Example 2.51. The space $L^1([a, b])$ is not reflexive.

Proof. If $L^1([a, b])$ is reflexive, then $(L^1([a, b]))^{**} = L^1([a, b])$. Since $L^1([a, b])$ is separable, by Lemma 2.50, $L^{\infty}([a, b]) = (L^1([a, b]))^*$ is separable. Recall that every countable subset of $\{\chi_{[a,t]}, t \in [a, b]\}$ is not dense in itself, giving rise to a contradiction! $\square$

2.5 Weak and Weak-* Topologies

Pointwise convergence of sequences can be topologized in function spaces.

Definition 2.52 (Weak topology and weak-* topology). Let X be a normed space.

(i) Given a point x_0 of X, finitely many $f_1, \cdots, f_n \in X^*$ and $\epsilon_1, \cdots, \epsilon_n > 0$, define

$$U_{f_1,\cdots,f_n}^{\epsilon_1,\cdots,\epsilon_n}(x_0) := \{x \in X : |f_1(x - x_0)| < \epsilon_1, \cdots, |f_n(x - x_0)| < \epsilon_n\}$$

The collection of sets $\left\{U_{f_1,\cdots,f_n}^{\epsilon_1,\cdots,\epsilon_n}(x) : x \in X,\ n \in \mathbb{N},\ f_1, \cdots, f_n \in X^*,\ \epsilon_1, \cdots, \epsilon_n > 0\right\}$ forms a basis for a topology on X, which is called the *weak topology.*

(ii) Given a point f_0 of X^*, finitely many $x_1, \cdots, x_n \in X$ and $\epsilon_1, \cdots, \epsilon_n > 0$, define

$$U_{x_1,\cdots,x_n}^{\epsilon_1,\cdots,\epsilon_n}(f_0) := \{f \in X^* : |(f - f_0)(x_1)| < \epsilon_1, \cdots, |(f - f_0)(x_n)| < \epsilon_n\}$$

The collection of sets $\left\{U_{x_1,\cdots,x_n}^{\epsilon_1,\cdots,\epsilon_n}(f) : f \in X^*,\ n \in \mathbb{N},\ x_1, \cdots, x_n \in X,\ \epsilon_1, \cdots, \epsilon_n > 0\right\}$ forms a basis for a topology on X^*, which is called the *weak-* topology.*

Remark. We obtain three topologies on the dual space X^*: the norm topology, the weak topology, and the weak-* topology. By definition, the weak-* topology is in fact the product topology (or the point-open topology) on $\mathbb{C}^X$ (the space of all complex-valued functionals on X) restricted to X^*. Furthermore, if X is reflexive, its weak and weak-* topologies coincide.

Definition 2.53 (Weak and weak-* convergence). Let X be a normed space. Let $x \in X$ and $f \in X^*$.

(i) A sequence (x_n) of points of X is said to *converges to x in the weak topology on X*, if $f(x_n) \to f(x)$ for all $f \in X^*$. We write $x_n \xrightarrow{w} x$.

(i) A sequence (f_n) of points of X^* is said to *converges to f in the weak-* topology on X^**, if $f_n(x) \to f(x)$ for all $x \in X$. We write $f_n \xrightarrow{w^*} f$.

Theorem 2.54 (Banach-Alaoglu). Let X be a normed space. The unit closed ball $B^* = \{f \in X^* : \|f\| \leq 1\}$ is compact in the weak-* topology on X^*.

Proof. The weak-* topology on X^* is the same as the product topology on $\mathbb{C}^X$ restricted to X^*. Then they also coincide on $B^* \subset X^*$. Hence we can prove that B^* is compact in the product topology.

For each $x \in X$, we define the closed disc $D_x = \{z \in \mathbb{C} : |z| \leq \|x\|\}$. Then D_x is compact in $\mathbb{C}$. By Tychonoff theorem,

$$D = \prod_{x \in X} D_x$$

is a compact topological space under the product topology. Furthermore, every element $f \in D$ is a $\mathbb{C}$-valued functional on X such that $|f(x)| \leq \|x\|$ for each $x \in X$. Clearly, $B^* \subset D$.

It suffices to show B^* is closed in D (given the product topology), which implies compactness of B^*. Let $\{f_\lambda, \lambda \in \Lambda\}$ be a net in B^* such that $f_\lambda \to f \in D$. Since the projection maps π_x are continuous in the product topology, we have

$$f_\lambda(x) = \pi_x(f_\lambda) \to \pi_x(f) = f(x),\ \forall x \in X.$$

Then $f(\alpha x + \beta y) = \lim_\lambda f_\lambda(\alpha x + \beta y) = \lim_\lambda(\alpha f_\lambda(x) + \beta f_\lambda(y)) = \alpha f(x) + \beta f(y)$ for all $\alpha, \beta \in \mathbb{C}$ and all $x, y \in X$, and f is linear. By $f \in D$, we have $\|f\| \leq 1$. Therefore $f \in B^*$, and B^* is closed in D, as desired. $\square$

Remark. Let X be a normed space. If X^* is infinite-dimensional, we know that the closed unit ball in X^* is not compact in the norm topology. However, by Banach-Alaoglu theorem, it is compact in the weak-* topology. This is one important reason why we introduce the weak-* topology.

We also have another version of Banach-Alaoglu theorem on separable normed spaces, which is similar to the Bolzano-Weierstrass theorem for $\mathbb{R}$.

Theorem 2.55 (Banach-Alaoglu). Let X be a separable normed space. If $\{f_n\}$ is a bounded sequence of points of X^*, then there exists a subsequence $\{f_{n_k}\}$ that converges in the weak-* topology on X^*.

Proof. By boundedness, $\{f_n\}$ is contained in some closed ball $B_M^* := \{f \in X^* : \|f\| \leq M\}$. By Theorem 2.54, B_M^* is compact, hence limit-point compact in the weak-* topology on X^*. Then $\{f_n\}$, being an infinite subset of B_M^*, has at least one limit point $f_0 \in B_M^*$.

We want to find a subsequence $f_{n_k} \xrightarrow{w^*} f_0$. Note that X is separable, we choose its dense subset $\{x_n, n \in \mathbb{N}\}$. For each $k \in \mathbb{N}$, define

$$U_k = \left\{ f \in X^* : |f(x_1) - f_0(x_1)| < \frac{1}{k}, \cdots, |f(x_1) - f_0(x_k)| < \frac{1}{k} \right\}.$$

Since f_0 is a limit point of $\{f_n\}$, we can choose a subsequence f_{n_k} such that $f_{n_k} \in U_k$. For all $x \in X$ and all $\epsilon > 0$, there exists x_p such that $\|x - x_p\| < \epsilon/(3M)$. Once $k \geq \max\{p, 3/\epsilon\}$, we have

$$\begin{aligned} |f_0(x) - f_{n_k}(x)| &\leq |f_0(x) - f_0(x_p)| + |f_0(x_p) - f_{n_k}(x_p)| + |f_{n_k}(x_p) - f_{n_k}(x)| \\ &\leq 2M\|x - x_p\| + \frac{1}{k} < \epsilon. \end{aligned}$$

Hence we have $f_{n_k} \xrightarrow{w^*} f_0$, as desired. $\square$

Corollary 2.56. Let X be a separable reflexive space. Then any bounded sequence $\{x_n\}$ of points of X has weakly convergent subsequence.

Proof. Since $X^{**} = X$ is separable, X^* is also separable by Lemma 2.50. Then the bounded sequence $\{Jx_n\}$ has a subsequence $\{Jx_{n_k}\}$ that converges in the weak-* topology on X^{**} by Theorem 2.55. According to Definition 2.53, $\{x_{n_k}\}$ converges in the weak topology on X. $\square$

Finally, we consider the relation between weak convergence and convergence in norm.

Theorem 2.57 (Mazur). Let X be a real normed space, and let (x_n) be a sequence of points of X that converges to $x \in X$ in the weak topology. Then there exists sequence (y_n) such that every y_n is a convex combination of finitely many $x_{n_1}, \cdots, x_{n_k}$, and that $y_n \xrightarrow{\|\cdot\|} x$. Equivalently, $x \in \overline{\operatorname{co}(\{x_n, n \in \mathbb{N}\})}$.

Proof. Argue by contradiction. Let $M := \operatorname{co}(\{x_n, n \in \mathbb{N}\})$ be the convex hull of (x_n). Using hyperplane separation theorem, if $x \notin \overline{M}$, there exists $f \in X^*$ and $c \in \mathbb{R}$ such that $f(y) < c < f(x)$ for all $y \in \overline{M}$. As a result, $f(x_n) < c < f(x)$, contradicting $x_n \xrightarrow{w} x$! Therefore $x \in \overline{M}$, as desired. $\square$

Corollary 2.58. Let M be a convex set in a normed space X. Then the closures of M in the norm topology and in the weak topology coincide.

3 Bounded Linear Operators

3.1 Baire Category Theorem

Definition 3.1 (Nowhere dense sets/rare sets). Let A be a subset of a topological space X. If the closure of A does not contain any nonempty open subset of X, that is, $\overline{A}$ has no interior point, then A is said to be a *nowhere dense* set (or a *rare* set) in X.

Remark. By definition, A is nowhere dense if and only if $\overline{A}$ is nowhere dense.

Lemma 3.2. Let A be a subset of a topological space X. Then A is rare if and only if $X \backslash \overline{A}$ is dense in X.

Proof. $X \backslash \overline{A}$ *is dense in* $X \Leftrightarrow X \backslash \overline{A}$ *intersects every open set in* $X \Leftrightarrow \overline{A}$ *is rare in* X. $\square$

Example 3.3. Following are some instances for nowhere dense sets.

(i) The set of integers $\mathbb{Z}$ is rare in $\mathbb{R}$. The set $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is rare in $[0, 1]$.

(ii) Let X be a normed space. Let Y be a proper subspace of X. Then Y is rare in X. (If not, there exists an open neighborhood of 0 contained in Y. Then Y is absorbing, and $Y = X$, a contradiction!)

(iii) A Cantor set C is obtained by repeatedly removing the open middle third from a collection of line segments, starting from the unit interval $[0, 1]$:

$$C_1 = [0, 1] \ \rightarrow \ C_2 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \ \rightarrow \ C_3 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \ \rightarrow \ \cdots .$$

The Cantor set $C = \bigcup_{n=1}^{\infty} C_n$ is a rare set in $\mathbb{R}$. To see this, let $x \in C$. Then any open interval of form $(x - \epsilon, x + \epsilon)$ is not contained in C, because the length of subintervals in C_n with $n > \frac{\log \epsilon}{\log 2}$ is less than $2^{-n+1} < 2\epsilon$. (Note that C_n has 2^{n-1} subintervals of the same length.)

(iv) A Smith-Volterra-Cantor set is also obtained by removing certain intervals from $[0, 1]$. For instance, we start from $[0, 1]$ and remove the middle $1/4^n$ from the remaining 2^{n-1} intervals at the n-th step:

$$K_1 = [0, 1] \ \rightarrow \ K_2 = \left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right] \ \rightarrow \ K_3 = \left[0, \frac{5}{32}\right] \cup \left[\frac{7}{32}, \frac{3}{8}\right] \cup \left[\frac{5}{8}, \frac{25}{32}\right] \cup \left[\frac{27}{32}, 1\right] \ \rightarrow \ \cdots .$$

Similar to (iii), the Smith-Volterra-Cantor set $K = \bigcup_{n=1}^{\infty} K_n$ is a nowhere dense set in $\mathbb{R}$.

Note that at the n-th step, we remove subintervals of length $4^{-n} \times 2^{n-1} = 2^{-n-1}$ in total. Then the Lebesgue measure of K_n is

$$m(K_n) = 1 - \sum_{k=1}^{n-1} \frac{1}{2^{k+1}} = \frac{1}{2} + \frac{1}{2^n}.$$

Hence $m(K) = 1/2 > 0$. It is seen that K is a rare set with positive Lebesgue measure.

Definition 3.4 (Baire spaces). A topological space X is said to be a *Baire space* if the following condition holds: given any countable collection $\{C_n, n \in \mathbb{N}\}$ of closed nowhere dense subsets of X, their union $\bigcup_{n=1}^{\infty} C_n$ is also nowhere dense in X.

Remark. Let $\{A_n, n \in \mathbb{N}\}$ be a collection of nowhere dense subsets of a Baire space X. Then the union $\bigcup_{n=1}^{\infty} A_n$, being a subset of $\bigcup_{n=1}^{\infty} \overline{A_n}$, is also nowhere dense in X. Therefore we can drop the requirement of "closed sets" in Definition 3.4. Applying Lemma 3.2, we obtain an equivalent definition of Baire spaces:

A topological space X is a Baire space if and only if the following condition holds: given any countable collection $\{U_n, n \in \mathbb{N}\}$ of open dense subsets of X, their intersection $\bigcap_{n=1}^{\infty} U_n$ is also dense in X.

Definition 3.5 (René-Louis Baire categories). Let A be a subset of a topological space X. Then A is said to be of the *first category* if it is contained in the union of a countable collection of nowhere dense sets in X; otherwise, it is said to be of the *second category.*

Lemma 3.6. A space X is a Baire space if and only if every open subset of X is of the second category.

Proof. If X is a Baire space, then the union of every countable collection of nowhere dense subsets of X is nowhere dense, which is impossible to contain X. Hence X is of the second category.

If X is not a Baire space, let $\{C_n, n \in \mathbb{N}\}$ be a collection of nowhere dense sets in X such that the union $\bigcup_{n=1}^{\infty} C_n$ contains some open set U in X. Then U is of the first category. $\square$

Remark. A space of second category is not necessarily a Baire space. Consider $Y = X \cup \mathbb{Q}$, where $X = [0, 1]$. Since X is of the second category, so is Y. However, $\bigcap_{q \in \mathbb{Q}} Y \backslash \{q\}$ is a countable intersection of open dense sets that is not dense.

Theorem 3.7 (Baire category theorem). A complete metric space X is a Baire space.

Proof. Let $\{C_n, n \in \mathbb{N}\}$ be a collection of closed nowhere dense sets in X. Given an open set U in X, we prove that there exists $x \in U$ such that $x \notin \bigcup_{n=1}^{\infty} C_n$. This implies $\bigcup_{n=1}^{\infty} C_n$ is nowhere dense.

We first consider A_1. By hypothesis, A_1 does not contain U. Then we choose $x_1 \in U \backslash A_1$. Since A_1 is closed, we choose $0 < \epsilon_1 < 1$ such that $U_1 = O(x_1, \epsilon_1)$ satisfies

$$\overline{U_1} \subset U \text{ and } \overline{U_1} \cap A_1 = \emptyset.$$

Now consider $n \geq 2$. With the open set U_{n-1} given, we choose $x_n \in U_{n-1} \backslash A_n$, and choose $0 < \epsilon_n < 1/n$ such that $U_n = O(x_n, \epsilon_n)$ satisfies

$$\overline{U_n} \subset U_{n-1} \text{ and } \overline{U_n} \cap A_n = \emptyset.$$

Since X is complete, by Theorem 1.59, the nested sequence $\overline{U_1} \supset \overline{U_2} \supset \cdots$ admits a unique $x \in \bigcup_{n=1}^{\infty} \overline{U_n}$. Then $X \notin A_n$ for all $n \in \mathbb{N}$, as desired. $\square$

Remark. Let $\{U_n, n \in \mathbb{N}\}$ be a collection of open dense subsets of a complete metric space X. According to Theorem 3.7, their intersection $\bigcap_{n=1}^{\infty} U_n$ is dense in X. Furthermore, we can prove that $\bigcap_{n=1}^{\infty} U_n$ is of the second category. Otherwise, there exists a collection $\{E_k, k \in \mathbb{N}\}$ of closed nowhere dense sets in X such that $\bigcap_{n=1}^{\infty} U_n \subset \bigcup_{k=1}^{\infty} E_k$, which implies $\left(\bigcap_{n=1}^{\infty} U_n\right) \cap \left(\bigcap_{k=1}^{\infty} X \backslash E_k\right) = \emptyset$. However, this is a dense subset of X by the conclusion we proved.

Example 3.8. Following are some instances for spaces of the first category and of the second category.

(i) The set of integers $\mathbb{Z}$ is a Baire space itself: Only $\emptyset$ is nowhere dense in X, because every subset of $\mathbb{N}$ is open. Nevertheless, $\mathbb{Z}$ is of the first category in $\mathbb{R}$.

(ii) The set of rationals $\mathbb{Q}$ is not a Baire space. It is of the first category in $\mathbb{R}$.

(iii) The set of irrationals $\mathbb{R} \backslash \mathbb{Q}$ is of the second category. Otherwise, there exist countably many nowhere dense sets $\{A_n\}$ such that $\mathbb{R} \backslash \mathbb{Q} = \bigcup_{n=1}^{\infty} A_n$. Then $\mathbb{R} = \left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{q \in \mathbb{Q}} \{q\}\right)$ is of the first category, a contradiction to Baire category theorem!

(iv) The unit closed interval $[0,1]$ is of the second category by Baire category theorem. Then it is uncountable. Otherwise, it is of the first category.

(v) Choose a collection of open subsets

$$E_k = \bigcup_{r_n \in \mathbb{Q}} \left(r_n - \frac{1}{k2^n}, r_n + \frac{1}{k2^n} \right), \ k \in \mathbb{N}.$$

By Baire category theorem, $\bigcap_{k=1}^{\infty} E_k$ is a dense set in X of the second category. Since $\mathbb{Q}$ is of the first category, we have $Q \subsetneq \bigcap_{k=1}^{\infty} E_k$. Then

$$0 \leq m(\mathbb{Q}) \leq m\left(\bigcup_{n=1}^{\infty} E_k\right) = \lim_{k\to\infty} m(E_k) = 0.$$

It is seen that $\bigcap_{n=1}^{\infty} E_k$ is a set of the second category with Lebesgue measure zero.

(vi) According to Example 3.3, the Smith-Volterra-Cantor K is a set of the first category with $m(K) > 0$.

Review: Weierstrass function. Karl Weierstrass has given a construction of continuous but nowhere differentiable functions. Let $a \in (0,1)$, and let b be an odd integer such that $ab > 1 + \frac{3\pi}{2}$. We define function

$$f(x) = \sum_{n=0}^{\infty} a^n \cos\left(b^n \pi x\right), \ x \in \mathbb{R}.$$

By Weierstrass M-test, the partial sum given by f converges uniformly, hence f is continuous. Interestingly, f is nowhere differentiable on $\mathbb{R}$. Fix $x_0 \in \mathbb{R}$. By definition, we need to argue that the limit

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

does not exist. In particular, we show that the difference quotient oscillates drastically as x approaches x_0.

We first construct two sequences (y_m) and (z_m) that approach x_0 from below and above, respectively. For each $m \in \mathbb{N}$, we choose an integer α_m close to $b^m x_0$. To be specific, let α_m be such that

$$x_m := b^m x_0 - \alpha_m \in \left(-\frac{1}{2}, \frac{1}{2}\right], \ \alpha_m \in \mathbb{Z}.$$

And choose y_m and z_m as follows:

$$y_m = x_0 - \frac{1+x_m}{b^m} = \frac{\alpha_m - 1}{b^m}, \quad z_m = x_0 + \frac{1-x_m}{b^m} = \frac{\alpha_m + 1}{b^m}.$$

The difference quotient at y_m is

$$\begin{aligned}
\frac{f(y_m)-f(x_0)}{y_m - x_0} &= \frac{\sum_{n=0}^{\infty} a^n \left[\cos(b^n \pi y_m) - \cos\left(b^n \pi x_0\right)\right]}{y_m - x_0} \\
&= \sum_{n=0}^{m-1} a^n \frac{\cos(b^n \pi y_m) - \cos\left(b^n \pi x_0\right)}{y_m - x_0} + \sum_{n=0}^{\infty} a^{n+m} \frac{\cos(b^{n+m} \pi y_m) - \cos\left(b^{n+m} \pi x_0\right)}{y_m - x_0} \\
&= \underbrace{-\sum_{n=0}^{m-1} \pi (ab)^n \frac{\sin\left(\frac{b^n \pi (y_m + x_0)}{2}\right) \sin\left(\frac{b^n \pi (y_m - x_0)}{2}\right)}{\frac{b^n \pi (y_m - x_0)}{2}}}_{=:S_1} + \underbrace{\sum_{n=0}^{\infty} a^{n+m} \frac{(-1)^{\alpha_m}\left(1 + \cos\left(b^n \pi x_m\right)\right)}{\frac{1+x_m}{b^m}}}_{=:S_2},
\end{aligned}$$

We can easily bound S_1:

$$|S_1| = \sum_{n=0}^{m-1} \pi(ab)^n \left|\sin\left(\frac{b^n\pi(y_m+x_0)}{2}\right)\right| \left|\frac{\sin\left(\frac{b^n\pi(y_m-x_0)}{2}\right)}{\frac{b^n\pi(y_m-x_0)}{2}}\right| \leq \sum_{n=0}^{m-1} \pi(ab)^n = \pi\frac{(ab)^m-1}{ab-1}.$$

Hence there exists $|\xi| < 1$ such that $S_1 = \xi\pi\frac{(ab)^m}{ab-1}$. For S_2, note that there exists $\eta \geq 1$ such that

$$\sum_{n=0}^{\infty} a^n \frac{(1+\cos(b^n\pi x_m))}{1+x_m} \geq \frac{(1+\cos(\pi x_m))}{1+x_m} \geq \frac{2}{3},$$

where the inequality follows from $-1/2 \leq x_m < 1/2$. Then there exists $\eta \geq \frac{2}{3}$ such that $S_2 = (-1)^{\alpha_m}(ab)^m\eta$. As a result,

$$\left|\frac{f(y_m)-f(x_0)}{y_m-x_0}\right| = |S_1+S_2| = \left|(-1)^{\alpha_m}(ab)^m\eta\left(1-(-1)^{\alpha_m}\frac{\xi\pi}{\eta(ab-1)}\right)\right| \geq \frac{2}{3}(ab)^m\left(1-\frac{3\pi}{2(ab-1)}\right) \to \infty.$$

A similar statement also holds for (z_m). Therefore, f is not differentiable at x_0. Since x_0 is arbitrary, f is nowhere differentiable on $\mathbb{R}$.

Example 3.9 (The set of continuous and nowhere differentiable functions). The construction of continuous but nowhere differentiable functions given by Weierstrass is non-trivial. Interestingly, we can argue that these "strange" functions are very rich in the space of continuous functions.

Consider the space $C([0,1])$ of continuous functions on $[0,1]$. The set of all continuous and nowhere differentiable functions on $[0,1]$ is of the second category in $C([0,1])$. In a nutshell, there exists a large amount of continuous and nowhere differentiable functions. To see this, we define a collection of subsets

$$F_N = \left\{f \in C([0,1]) : \ \forall x \in [0,1], \ \exists y \in [0,1] \text{ such that } \left|\frac{f(x)-f(y)}{x-y}\right| > N\right\}, \ N \in \mathbb{N}.$$

Claim 1. We first claim that F_N is open. Let (f_n) be a sequence of functions in

$$F_N^c = \left\{f \in C([0,1]) : \exists x \in [0,1] \text{ such that } \forall y \in [0,1], \ \left|\frac{f(x)-f(y)}{x-y}\right| \leq N\right\}$$

such that f_n converges uniformly to $f \in C([0,1])$. We choose $x_n \in [0,1]$ to be such that

$$\left|\frac{f_n(y)-f_n(x_n)}{y-x_n}\right| \leq N$$

for all $y \in [0,1]$. There exists convergent subsequence $x_{n_k} \to x \in [0,1]$. Then for all $y \in [0,1]$,

$$\begin{aligned}
|f(x)-f(y)| &\leq |f(x)-f_{n_k}(x)| + |f_{n_k}(x)-f_{n_k}(x_{n_k})| + |f_{n_k}(x_{n_k})-f_{n_k}(y)| + |f_{n_k}(y)-f(y)| \\
&\leq \|f-f_{n_k}\|_\infty + N|x-x_{n_k}| + N|x_{n_k}-y| + \|f-f_{n_k}\|_\infty \\
&\leq 2\|f-f_{n_k}\|_\infty + 2N|x-x_{n_k}| + N|x-y|.
\end{aligned}$$

Let $k \to \infty$, then we have $f \in F_N^c$. Hence F_N^c is closed, and F_N is open.

Claim 2. We then claim that F_N is dense in $C([0,1])$.

Proof. Given $f \in C([0,1])$ and $\epsilon > 0$, we wish to find $g \in F_N$ such that $\|f-g\|_\infty < \epsilon$. By uniform continuity of f, we choose $\delta > 0$ such that $|f(x)-f(y)| < \epsilon/5$ for all $|x-y| < \delta$.

Let $n > 1/\delta$, and divide $[0,1]$ into n subintervals $\left[\frac{k-1}{n}, \frac{k}{n}\right]$, within each the amplitude of f is less than $\epsilon/5$. Then let $M > \frac{5N}{n\epsilon}$, and partition $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ into M subintervals again:

$$\frac{k-1}{n} = x_{(k-1)M} < \frac{(k-1)M+1}{nM} = x_{(k-1)M+1} < \cdots < \frac{kM-1}{nM} = x_{kM-1} < \frac{k}{n} = x_{kM}.$$

For $p = 0, 1, \cdots, nM$, we define $g(x_p) = f(x_p) + (-1)^p\, \epsilon/5$ at $x_p \in [0,1]$, and connect points $(x_{p-1}, g(x_{p-1}))$ and $(x_p, g(x_p))$ by line segment. Then we obtain a piecewise-linear function $g \in C([0,1])$.

We show that $g \in F_N$. For each $x \in [0,1]$, choose $x \in [x_{kM+j-1}, x_{kM+j}]$. Then

$$\begin{aligned}\left|\frac{g(x_{kM+j}) - g(x)}{x_{kM+j} - x}\right| &= \frac{\left|f(x_{kM+j}) + (-1)^{kM+j}\epsilon/5 - f(x_{kM+j-1}) - (-1)^{kM+j-1}\epsilon/5\right|}{(nM)^{-1}} \\ &\geq \frac{2\epsilon/5 - |f(x_{kM+j}) - f(x_{kM+j-1})|}{(nM)^{-1}} > N.\end{aligned}$$

Then $g \in F_N$. Furthermore,

$$\begin{aligned}|f(x) - g(x)| &\leq |f(x) - f(x_{kM+j})| + |f(x_{kM+j}) - g(x_{kM+j})| + \underbrace{|g(x_{kM+j}) - g(x)|}_{\leq |g(x_{kM+j}) - g(x_{kM+j-1})|} \\ &\leq |f(x) - f(x_{kM+j})| + |f(x_{kM+j}) - g(x_{kM+j})| + |f(x_{kM+j}) - f(x_{kM+j-1})| + \frac{2\epsilon}{5} < \epsilon.\end{aligned}$$

Hence $\|f - g\|_\infty < \epsilon$. As a result, F_N is dense in $C([0,1])$. $\square$

Claim 3. Finally, we claim that the set of all continuous and nowhere differentiable functions in $C([0,1])$ is of the second category.

If $f \in C([0,1])$ is differentiable at some $x \in [0,1]$, then we choose $\delta > 0$ such that

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq 1 + |f'(x)|$$

for all $|x - y| < \delta$. For $|x - y| \geq \delta$, we have

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq \frac{2\|f\|_\infty}{\delta}.$$

Therefore, if we choose

$$N > \max\left\{1 + |f'(x)|, \frac{2\|f\|_\infty}{\delta}\right\},$$

then we have $f \notin F_N$. Hence every function in $\bigcap_{n=1}^\infty F_N$ is continuous and nowhere differentiable. Moreover, $\bigcap_{n=1}^\infty F_N$ is of the second category by Baire category theorem.

3.2 Banach Bounded Inverse, Open Mapping & Closed Graph Theorems

Definition 3.10 (Invertible bounded linear operators). Let X and Y be normed spaces. Let $T \in \mathfrak{B}(X,Y)$. Then T is said to be *invertible* if $T : X \to Y$ is bijective and $T^{-1} \in \mathfrak{B}(Y,X)$.

Remark. By definition,the operator $T \in \mathfrak{B}(X,Y)$ is invertible if and only if there exists $S \in \mathfrak{B}(Y.X)$ such that $S \circ T = I_X$ and $T \circ S = I_Y$, where I_X and T_Y are identity operators in X and Y. In this case, $T^{-1} := S$ is said to be the *inverse* of T.

Example 3.11. Let X be a finite-dimensional normed space, and let $T \in \mathfrak{B}(X)$. Then T is injective if and only if it is surjective. In this case, $T^{-1} \in \mathfrak{B}(X)$.

However, for infinite-dimensional spaces, the case becomes complicated. Let $X = l^2(\mathbb{N})$ be the space of square-summable sequences. Define the left-shift and right-shift operators on X as follows:

$$S : (x_1, x_2, \cdots) \to (x_2, x_3, \cdots), \quad T : (x_1, x_2, \cdots) \to (0, x_1, x_2, \cdots).$$

Then $S \cdot T = I_X$, but $T \circ S \neq I_X$. Hence T is injective but not surjective, and S is surjective but not injective.

In Definition 3.10, the boundedness of T^{-1} is required. Here is an example of bounded linear operators that are bijective but not invertible. Let $X = C([a,b])$, and let $Y = \{f \in C([a,b]) : f(a) = 0 \text{ and } f' \in C([a,b])\}$ be a subspace of X. Define $T : X \to Y$ as

$$(Tf)(x) = \int_a^x f(t)\,dt, \ x \in [a,b].$$

By definition, $\|T\| \leq b - a$. Then $T \in \mathfrak{B}(X,Y)$, and $T : X \to Y$ is bijective. However, the inverse of T is the differential operator: $(T^{-1}\varphi)(x) = \frac{d}{dx}\varphi(x)$, which is not bounded.

The reason that $T : X \to Y$ is not invertible is that Y is not complete. In general, we have the following important theorem about invertible operators.

Theorem 3.12 (Banach bounded inverse theorem). Let X and Y be Banach spaces. If $T \in \mathfrak{B}(X,Y)$ is a bijection from X onto Y, then T is invertible, that is, $T^{-1} \in \mathfrak{B}(Y,X)$.

The proof of Theorem 3.12 uses the open mapping theorem.

Theorem 3.13 (Open mapping theorem). Let X and Y be Banach spaces, and let $T \in \mathfrak{B}(X,Y)$. If $T : X \to Y$ is surjective, then T is an *open mapping*, i.e. for all open $U \subset X$, its image TU is open in Y.

Proof of Theorem 3.12. By definition, if $T : X \to Y$ is an open mapping, then $T^{-1} : Y \to X$ is continuous. □

Before proving Theorem 3.13, we introduce some notations. Let A be a subset of a vector space X. Let $x \in X$, and let α be a number. Then $x + A = \{x + y : y \in A\}$, $\alpha A = \{\alpha x : x \in A\}$.

Proof of Theorem 3.13. Given an open set $G \subset X$, we prove that TG is open in Y. That is, for any point x of G, Tx is an interior point of TG. We use O_X and B_X to denote open and closed balls in X.

Step I: We prove that there exists $\delta > 0$ such that $TB_X(0,1)$ is dense in $O_Y(0,\delta)$.

Since $X = \bigcup_{n=1}^{\infty} B_X(0,n)$, we have $Y = TX = \bigcup_{n=1}^{\infty} TB_X(0,n)$. By completeness of Y, it is of the second category. Hence there exists $TB_X(0,N)$ that is not nowhere dense. As a result, there exists $y_0 \in Y$ and $\eta > 0$ such that $TB_X(0,N)$ is dense in $O_Y(y_0,\eta)$.

Let $Tx_0 = y_0$. If $M = N + \|x_0\|$, then $TB_X(0,M) \supset TB_X(0,N) - Tx_0$ is dense in $O_Y(0,\eta)$.

Step II: We prove that $TB_X(0,1) \supset O_Y\left(0, \frac{\delta}{2}\right)$.

Choose $y_0 \in O_Y(0,\delta)$. By Step II, choose $x_1 \in B_X(0,1)$ such that $\|y_0 - Tx_1\| < \delta/2$, which implies $y_1 = y_0 - Tx_1 \in O_Y(0,\delta/2)$. By induction, with $y_{n-1} \in O_Y\left(0, 2^{1-n}\delta\right)$ given, choose $x_n \in B_X(0, 2^{1-n})$ such that $y_n = y_{n-1} - Tx_n \in O_Y(0, 2^{-n}\delta)$. Therefore

$$\|y_0 - T(x_1 + \cdots + x_n)\| < \frac{\delta}{2^n}.$$

Since $x_n \in B_X(0, 2^{1-n})$, by completeness of X, let $x_0 = \sum_{n=1}^{\infty} x_n \in X$. Then $\|x_0\| \leq \sum_{n=1}^{\infty} \|x_n\| \leq 2$. By continuity of T, we have $Tx_0 = y_0$. Then $TB_X(0,2) \supset O_Y(0,\delta)$. The result follows from linearity of T.

Step III: Since G is open, for all $x \in G$, there exists $O_X(0,b_x) \subset G$. If $\alpha < b_x$, $B_X(x,\alpha) \subset G$. Then

$$TB_X(x,\alpha) = Tx + \alpha TB_X(0,1) \subset TG.$$

By Step II, we have $O_Y(Tx, \frac{\alpha\delta}{2}) \subset TG$. ☐

Remark. Analogously, an operator $T: X \to Y$ is said to be a *closed mapping*, if for all closed subset $G \subset X$, its image TG is closed in Y.

A surjection $T \in \mathfrak{B}(X,Y)$ is an open mapping, but it need not to be a closed mapping. For instance, consider the projection map $\pi_1 : \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto x$. The set $G = \{(x,y) : xy = 1\}$ is a closed set in $\mathbb{R}^2$, but its image $TG = \mathbb{R}\backslash\{0\}$ is not closed in $\mathbb{R}$.

Following are applications of bounded inverse theorem and open mapping theorem.

Theorem 3.14 (Equivalence of norms). Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms on a vector space X. If both $(X, \|\cdot\|_a)$ and $(X, \|\cdot\|_b)$ are complete, and there exists $c > 0$ such that $\|x\|_b \leq c\|x\|_a$ for all $x \in X$, then $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent.

Proof. We show that there exists $c' > 0$ such that $\|x\|_a \leq c'\|x\|_b$. Consider the identity map

$$Id_X : (X, \|\cdot\|_a) \to (X, \|\cdot\|_b).$$

Then $\|Id_X\| \leq c$. By Theorem 3.12, Id_X^{-1} is also bounded, and the result follows. ☐

Theorem 3.15. Let X and Y be Banach spaces, and $T \in \mathfrak{B}(X,Y)$ is injective. Then $\mathfrak{R}(T)$ is closed if and only if T is bounded from below, that is, there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in X$.

Proof. If $\mathfrak{R}(T) \subset Y$ is closed, then $\mathfrak{R}(T)$ is complete. By Theorem 3.12, $T : X \to \mathfrak{R}(T)$ has bounded inverse.

Conversely, let y_n be a sequence in $\mathfrak{R}(T)$ that converges to y. It suffices to show $y \in \mathfrak{R}(T)$. By injectivity of T, choose $x_n = T^{-1}y_n$. Then $\|x_n - x_m\| \leq \frac{1}{c}\|y_n - y_m\|$, and (x_n) is a Cauchy sequence, which converges to some $x \in X$. By continuity of T, we have $Tx = \lim_{n\to\infty} Tx_n = y$. ☐

Example 3.16. Given $\varphi \in L^\infty([0,1])$, define linear operator M_φ on $L^1([0,1])$ by

$$(M_\varphi f)(t) = \varphi(t)f(t),\ f \in L^1([0,1]),\ t \in [0,1].$$

Then $\|M_\varphi\| = \|\varphi\|_\infty$, and M_φ is invertible if and only if there exists $c > 0$ such that $|\varphi| \geq c$ a.e. on $[0,1]$.

Proof. If M_φ is invertible, then $\|f\|_1 \leq \|M_\varphi^{-1}\|\,\|M_\varphi f\|_1$ for all $f \in L^1([0,1])$. If $m(|\varphi| < \|M_\varphi^{-1}\|^{-1}) > 0$, we derive a contradiction by setting $f = \chi_{\{|\varphi| < \|M_\varphi^{-1}\|^{-1}\}}$. Hence $|\varphi| \geq c := \|M_\varphi^{-1}\|^{-1}$ a.e. on $[0,1]$.

Conversely, if there exists $c > 0$ such that $|\varphi| \geq c$ a.e. on $[0,1]$, we can recover M_φ^{-1} by

$$(M_\varphi^{-1} f) = \begin{cases} \frac{f(t)}{\varphi(t)}, & \varphi(t) \neq 0, \\ 0, & \varphi(t) = 0. \end{cases}$$

Clearly, $\|M_\varphi^{-1} f\|_1 \leq \frac{1}{c}\|f\|_1$ for all $f \in L^1([0,1])$, which implies $M_\varphi^{-1} \in \mathfrak{B}(L^1([0,1]))$, and $\|M_\varphi^{-1}\| \leq \frac{1}{c}$.

An alternative proof is based on Theorem 3.12, where we prove that T is a bijection on $L^1([0,1])$. □

Now we introduce the definition of graphs.

Definition 3.17 (Graphs). Let $T : \mathfrak{D}(T) \to Y$ be a map. The set

$$\mathrm{Gr}\,(T) = \{(x, Tx) : x \in \mathfrak{D}(T)\},$$

which is a subset of $\mathfrak{D}(T) \times Y$, is called the *graph* of T.

Remark. Let (X, d_X) and (Y, d_Y) be metric spaces. We can define a product metric d on $X \times Y$ as

$$d((x,y),(x',y')) = \sqrt{d_X(x,x')^2 + d_Y(y,y')^2}.$$

In fact, the topology that d induces is the product topology on $X \times Y$. That is, every basis element of this topology is of the form $U \times V$, where U and V are open subsets of X and Y, respectively.

Let $T : \mathfrak{D}(T) \to Y$ be a map, where $\mathfrak{D}(T) \subset X$. If $\mathrm{Gr}\,(T)$ is closed in $X \times Y$ (given the product topology), then T is said to have a closed graph.

Lemma 3.18. Let X and Y be two topological spaces, and let Y be Hausdorff. Let $T : \mathfrak{D}(T) \to Y$ be a continuous operator, and $\mathfrak{D}(T) \subset X$. If $\mathfrak{D}(T)$ is closed, then T has closed graph.

Proof. Let (x_0, y_0) be a limit point of $\mathrm{Gr}\,(T) = \{(x, Tx) : x \in \mathfrak{D}(T)\}$. Then every neighborhood of (x_0, y_0) has at least one point of $\mathrm{Gr}\,(T)$, and x_0 is a limit point of $\mathfrak{D}(T)$. Since $\mathfrak{D}(T)$ is closed, $x_0 \in \mathfrak{D}(T)$.

If $y_0 \neq Tx_0$, there exists disjoint open subsets U and V of Y that contains y_0 and Tx_0, respectively. By continuity of T, the set $T^{-1}V \times U$ is an open neighborhood of (x_0, y_0) in $X \times Y$. However, it does not contain any point of $\mathrm{Gr}\,(T)$, contradicting the fact that (x_0, y_0) is a limit point of $\mathrm{Gr}\,(T)$. Hence $y_0 = Tx_0$. □

Example 3.19. (Differential operator). We define $T : C^1([a,b]) \to C([a,b])$ to be the differential operator:

$$(Tf)(t) = f'(t),\ f \in C^1([a,b]),\ t \in [a,b].$$

Give $C^1([a,b])$ the supremum norm on $C([a,b])$. Clearly, T is unbounded: if $f_n(T) = \sin(nt) \in C([0,2\pi])$, so that $\|f_n\| = 1$, then

$$\|Tf_n\| = \sup_{t \in [0,2\pi]} n \cos nt = n \to \infty.$$

Interestingly, T has a closed graph. To see this, let (f_n) be a sequence in $C^1([a,b])$ such that $f_n \rightrightarrows f$ and $f_n' \rightrightarrows g$. The uniform convergence of derivatives implies $f \in C^1([a,b])$ and $f' = g$.

Theorem 3.20 (Closed graph theorem). Let X and Y be Banach spaces. Let T be a linear operator from $\mathfrak{D}(T)$ into Y, where $\mathfrak{D}(T)$ is a closed subspace of X. Then T is continuous if and only if it has a closed graph.

Proof. Following Lemma 3.18, it remains to show the continuity of linear operator T with a closed graph.

Define norm $\|(x,y)\| = \sqrt{\|x\|^2 + \|y\|^2}$ on $X \times Y$. Since X and Y are Banach spaces, $X \times Y$ becomes a Banach space under norm $\|\cdot\|$. Clearly, the closed subspaces $\mathfrak{D}(T)$ and $\mathrm{Gr}\,(T)$ are also Banach spaces.

Define $\pi : \mathrm{Gr}\,(T) \to \mathfrak{D}(T),\ (x, Tx) \mapsto x$. Then $\|\pi\| \leq 1$, and π is bounded. By Theorem 3.12, the inverse $\pi^{-1} : x \mapsto (x, Tx)$ is also bounded, and

$$\|Tx\| \leq \|(x,Tx)\| = \|\pi^{-1}(x)\| \leq \|\pi^{-1}\|\,\|x\|,\ \forall x \in \mathfrak{D}(T).$$

Hence $T \in \mathfrak{B}(\mathfrak{D}(T), Y)$ is continuous. □

Following are applications of closed graph theorem.

Example 3.21. Let f be a measurable function on $[0,1]$. If $fg \in L^1([0,1])$ for all $g \in L^2([0,1])$, then $f \in L^2([0,1])$.

Proof. Define $T : L^2([0,1]) \to L^1([0,1]),\ g \mapsto fg$. We show that T has closed graph.

Let g_n be a sequence in $L^2([0,1])$ such that $\|g_n - g\|_2 \to 0$ and $\|fg_n - h\|_1 \to 0$. By Chebyshev inequality, for all $\sigma > 0$,

$$m(|g_n - g| \geq \sigma) \leq \frac{1}{\sigma^2}\|g_n - g\|_2^2, \quad m(|fg_n - h| \geq \sigma) \leq \frac{1}{\sigma}\|fg - h\|_1.$$

Hence $g_n \to g$, and $fg_n \to h$ in Lebesgue measure, and there exists subsequence (g_{n_k}) such that $g_{n_k} \to g$ a.e., and $fg_{n_k} \to h$ a.e.. Then $fg = h$, and T has closed graph.

By Theorem 3.20, T is continuous. Choose $f\chi_{\{|f|<n\}} \in L^2([0,1])$, then

$$\|f\chi_{\{|f|<n\}}\|_2 = \frac{\|f^2\chi_{\{|f|<n\}}\|_1}{\|f\chi_{\{|f|<n\}}\|_2} \leq \|T\|.$$

Let $n \to \infty$, we have $\|f\|_2 \leq \|T\|$. Hence $f \in L^2([0,1])$. □

Example 3.22. Let X and Y be Banach spaces. Let T be an linear operator from X into Y. If $f \circ T \in X^*$ for all $f \in Y^*$, then $T \in \mathfrak{B}(X,Y)$.

Proof. Following Theorem 3.20, we show that T has a closed graph. Let (x_n) be a sequence of points of X such that $x_n \to x \in X$, and $Tx_n \to y \in Y$. We need to show that $y = Tx$.

By continuity of $f \in Y^*$, we have $f(Tx_n) \to f(y)$. By continuity of $f \circ T$, $f(Tx_n) \to f(Tx)$. Then for $f \in Y^*$, $f(y) = f(Tx)$. If $y \neq Tx$, by Hahn-Banach theorem, there exists $f_0 \in Y^*$ such that $f(y) \neq f(Tx)$, a contradiction! Hence $y = Tx$. □

3.3 Banach-Steinhaus Theorem

Theorem 3.23 (Banach-Steinhaus theorem/uniform boundedness principle). Let X be a Banach space, and Y a normed space. Let $\{T_\lambda\}_{\lambda\in\Lambda} \subset \mathfrak{B}(X,Y)$. If $\sup_{\lambda\in\Lambda}\|T_\lambda x\| < \infty$ for all $x \in X$, then $\sup_{\lambda\in\Lambda}\|T_\lambda\| < \infty$.

Proof. Since $\sup_{\lambda\in\Lambda}\|T_\lambda x\| < \infty$ for all $x \in X$, we define a norm on X by

$$\|x\|_1 := \max\left\{\|x\|,\ \sup_{\lambda\in\Lambda}\|T_\lambda x\|\right\}.$$

We claim that $(X, \|\cdot\|_1)$ is a Banach space. Let (x_n) be a Cauchy sequence in $(X, \|\cdot\|_1)$: for all $\epsilon > 0$, there exists N such that $\|x_n - x_m\|_1 < \epsilon/2$ for all $n, m \geq N$. Since $\|x\| \leq \|x\|_1$ for all $x \in X$, (x_n) is also a Cauchy sequence in $(X, \|\cdot\|)$. Hence there exists $x_0 \in X$ such that $\lim_{n\to\infty}\|x_n - x_0\| = 0$.

Fix $\epsilon > 0$. Since (x_n) is Cauchy relative to $\|\cdot\|_1$, there exists N such that for all $\lambda \in \Lambda$ and all $n, m \geq N$,

$$\|T_\lambda(x_n - x_m)\| < \frac{\epsilon}{2}. \tag{3.1}$$

Let $m \to \infty$ in (3.1), we have $\|T_\lambda(x_n - x_0)\| \leq \epsilon/2$ for all $\lambda \in \Lambda$ and all $n \geq N$, which implies

$$\sup_{\lambda\in\Lambda}\|T_\lambda(x_n - x_0)\| \leq \frac{\epsilon}{2} < \epsilon,\ \forall n \geq N.$$

Hence $\lim_{n\to\infty}\|x_n - x_0\|_1 = 0$, and $(X, \|\cdot\|_1)$ is complete. By Theorem 3.14, there exists $c > 0$ such that $\|x\|_1 \leq c\|x\|$ for all $x \in X$. As a result, $\sup_{\lambda\in\Lambda}\|T_\lambda x\| \leq c\|x\|$ for all $x \in X$, and $\sup_{\lambda\in\Lambda}\|T_\lambda\| \leq c$. ☐

Remark. The hypothesis of completeness of X cannot be removed. Following is a simple counterexample.

Consider the space $(C_c(\mathbb{R}), \|\cdot\|_\infty)$, where $C_c(\mathbb{R})$ is the set of compactly supported continuous functions on $\mathbb{R}$, and $\|f\|_\infty = \sup_{t\in\mathbb{R}}|f(t)|$ for all $f \in C_c(\mathbb{R})$. Clearly, $(C_c(\mathbb{R}), \|\cdot\|_\infty)$ is not a Banach space, because the Cauchy sequence $f_n(t) = \exp(-t^2)\chi_{[-n,n]}(t)$ does not converges in $C_c(\mathbb{R})$.

We define operators T_n for all $n \in \mathbb{N}$ as follows:

$$(T_n f)(t) = t\chi_{[-n,n]}(t)f(t),\ \forall f \in C_c(\mathbb{R}).$$

Then for all $f \in C_c(\mathbb{R})$, $t \mapsto tf(t)$ is also compactly supported and continuous. Then we have

$$\sup_{n\in\mathbb{N}}\|T_n f\| = \sup_{t\in\mathbb{R}} tf(t) < \infty.$$

However, $\|T_n\| = n$, which implies that $\{T_n\}_{n\in\mathbb{N}}$ is not uniformly bounded.

Theorem 3.24 (Banach-Steinhaus, Baire category version). Let X be a Banach space, and let Y be a normed space. Let $\{T_\lambda\}_{\lambda\in\Lambda} \subset \mathfrak{B}(X,Y)$. If the set

$$R := \left\{x \in X : \sup_{\lambda\in\Lambda}\|T_\lambda x\| < \infty\right\}$$

is of the second category in X, then $\sup_{\lambda\in\Lambda}\|T_\lambda\| < \infty$.

Proof. Define $p : X \to [0,\infty],\ x \mapsto \sup_{\lambda\in\Lambda}\|T_\lambda x\|$. Then p is a seminorm on X, and

$$R = \{x \in X : p(x) < \infty\} = \bigcup_{k=1}^\infty \underbrace{\{x \in X : p(x) \leq k\}}_{=:X_k} = \bigcup_{k=1}^\infty \bigcap_{\lambda\in\Lambda}\{x \in X : \|T_\lambda x\| \leq k\}$$

By continuity of T_λ, $\{x \in X : \|T_\lambda x\| \leq k\}$ is closed in X, and X_k is closed for all $k \in \mathbb{N}$. Since R is of the second category, there exists X_k that is not nowhere dense. Moreover, there exists $O(x_0, \epsilon)$ contained in X_k.

Let $N = k + p(x_0)$. Then $O(0, \epsilon) \subset X_N$, which implies

$$p\left(\frac{\epsilon x}{2\|x\|}\right) \leq N, \ \forall x \in X \ \Rightarrow \ p(x) \leq \frac{2N}{\epsilon}\|x\|, \ \forall x \in X.$$

As a result, $\sup_{\lambda \in \Lambda} \|T_\lambda\| \leq \frac{2N}{\epsilon}$, as desired. □

Example 3.25. By Hölder's inequality, we know that $L^2([0,1]) \subset L^1([0,1])$. In fact, $L^2([0,1])$ is a first-category subset of of $L^1([0,1])$.

Proof. Let $f_n = n\chi_{[0,n^{-3}]}$ for all $n \in \mathbb{N}$. Define $F_n \in (L^1([0,1]))^*$ as

$$F_n(g) = \int_{[0,1]} f_n g \, dm, \ \forall g \in L^1([0,1]).$$

Then $\|F_n\| = \|f_n\|_\infty = n$. Meanwhile, for all $h \in L^2([0,1])$,

$$F_n(h) \leq \|f_n\|_2 \, \|h\|_2 = \frac{1}{\sqrt{n}}\|h\|_2 \leq \|h\|_2 < \infty, \ \forall n \in \mathbb{N}.$$

If $L^2([0,1])$ is of the second category, Theorem 3.24 implies that $\sup_{n \in \mathbb{N}} \|F_n\| < \infty$, a contradiction! □

Another version of Banach-Steinhaus theorem is based on countable collections of operators.

Theorem 3.26 (Banach-Steinhaus). Let X be a Banach space, and Y a normed space. Let (T_n) be a sequence of operators in $\mathfrak{B}(X,Y)$. If $(T_n x)$ converges in Y for all $x \in X$, then there exists $T \in \mathfrak{B}(X,Y)$ such that $\lim_{n\to\infty} T_n x = Tx$, and $\|T\| \leq \liminf_{n\to\infty} \|T_n\|$.

Proof. Define $T : X \to Y, \ x \mapsto \lim_{n\to\infty} T_n x$. The linearity of T follows immediately from (T_n). Clearly, $\{x \in X : \sup_{n\in\mathbb{N}} \|T_n x\|\} = X$ is complete, hence of the second category. By Theorem 3.24, $\sup_{n\in\mathbb{N}} \|T_\lambda\| < \infty$. Furthermore,

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| = \liminf_{n\to\infty} \|T_n x\| \leq \liminf_{n\to\infty} \|T_n\| \, \|x\|, \ \forall x \in X.$$

Hence $\|T\| \leq \liminf_{n\to\infty} \|T_n\|$, as desired. □

Example 3.27. Let $1 \leq p \leq \infty$. Let f be a Lebesgue measurable function on $[a,b]$. If $fg \in L^1([a,b])$ for all $g \in L^p([a,b])$, then $f \in L^q([a,b])$, where $p^{-1} + q^{-1} = 1$.

Proof. Choose $g = \chi_{[a,b]}$, we know that $f \in L^1([a,b])$. For all $n \in \mathbb{N}$, define sequence of bounded functions on $[a,b]$ by $f_n = f\chi_{|f|<n}$. Then $f_n \to f$ a.e. on $[a,b]$. We then define

$$F_n(g) = \int_{[a,b]} f_n g \, dm, \ g \in L^p([a,b]).$$

By Hölder's inequality, $F_n \in (L^p([a,b]))^*$, and $\|F_n\| = \|f_n\|_q$. Since $|f_n g| \leq |fg|$, the Lebesgue dominated convergence theorem implies

$$\lim_{n\to\infty} F_n(g) = \lim_{n\to\infty} \int_{[a,b]} f_n g \, dm = \int_{[a,b]} fg \, dm, \ \forall g \in L^p([a,b]).$$

Define linear functional $F(g) = \int_{[a,b]} fg\, dm$ on $L^p([a,b])$. Using Theorem 3.26, we have $F \in (L^p([a,b]))^*$, and $\|f\|_q = \|F\| \le \liminf_{n\to\infty} \|F_n\|$, which implies $f \in L^q([a,b])$. $\square$

Analogous to the weak topology in dual spaces, we also use some weaker convergences in the space of operators in lieu of convergence in operator norm.

Definition 3.28 Let X and Y be normed spaces. Let (T_n) be a sequence of operators in $\mathfrak{B}(X,Y)$, and $T \in \mathfrak{B}(X,Y)$.

(i) (Convergence in strong operator topology, SOT). (T_n) is said to *converges to T in strong operator topology*, if $\lim_{n\to\infty} \|T_n x - Tx\| = 0$ for all $x \in X$. We write $T_n \overset{SOT}{\to} T$.

(ii) (Convergence in Weak operator topology, WOT). (T_n) is said to *converges to T in weak operator topology*, if $T_n x \overset{w}{\to} Tx$ for all $x \in X$, namely, $f(T_n x) \to f(Tx)$ for all $f \in Y^*$ and all $x \in X$. We write $T_n \overset{WOT}{\to} T$.

Remark. Clearly, $\|Tx\| \le \|T\|\,\|x\|$, and $\|f(Tx)\| \le \|f\|\,\|Tx\|$. Therefore,

$$T_n \overset{\|\cdot\|}{\to} T \;\Rightarrow\; T_n \overset{SOT}{\to} T \;\Rightarrow\; T_n \overset{WOT}{\to} T.$$

The converse does not hold in general. Let $X = Y = l^2(\mathbb{N})$, and consider the left-shift operator S and right-shift operator T.

(i) $S^n \overset{SOT}{\to} 0$, but $\|S^n\| = 1$ for all $n \in \mathbb{N}$.

(ii) $T^n \overset{WOT}{\to} 0$, but $\|T^n x\| = \|x\|$ for all $x \in X$ and $n \in \mathbb{N}$, namely, T^n does not converges to 0 in strong operator topology.

Theorem 3.29 (Banach-Steinhaus). Let X and Y be Banach spaces. Let (T_n) be a sequence of operators in $\mathfrak{B}(X,Y)$ such that $T_n \overset{WOT}{\to} T$, where $T \in \mathfrak{B}(X,Y)$. Then $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$.

Proof. By convergence in weak operator topology, $f(T_n x) \to f(Tx)$ for all $x \in X$ and all $f \in Y^*$. Then

$$R_x = \left\{ f \in Y^* : \sup_{n\in\mathbb{N}} |f(T_n x)| < \infty \right\} = Y^*$$

is of the second category. By Theorem 3.24, $\sup_{n\in\mathbb{N}} \|T_n x\| < \infty$ for all $x \in X$. Again by Theorem 3.23, $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$. $\square$

3.4 Adjoint Operators

3.4.1 Adjoint Operators in Normed Spaces

Definition 3.30 (Adjoint operators/conjugate operators). Let X and Y be normed spaces, and $T \in \mathfrak{B}(X, Y)$. If there exists $T^* \in \mathfrak{B}(Y^*, X^*)$ such that $(T^* f)(x) = f(Tx)$ for all $x \in X$ and $f \in Y^*$, then T^* is said to be the *adjoint (operator)/conjugate operator* of T.

Example 3.31 Following are some instances for adjoint operators.

(i) Let X be an n-dimensional normed space, and $\{e_1, \cdots, e_n\}$ a basis of X. Let Y be an m-dimensional normed space, and $\{f_1, \cdots, f_m\}$ a basis of Y. Then any linear operator $T : X \to Y$ is determined by an m-by-n matrix $A = (a_{ij})_{m\times n}$:

$$Te_j = \sum_{k=1}^{m} a_{kj} f_k \overset{def}{\Leftrightarrow} T(e_1, \cdots, e_n) = (f_1, \cdots, f_m) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Let $e_j^* \in X^*$ be such that $e_j^* e_k = \delta_{jk}$ for $k = 1, \cdots, n$. Then $\{e_1^*, \cdots, e_n^*\}$ is a basis of X^*. Similarly we choose a dual basis $\{f_1^*, \cdots, f_m^*\}$ for Y^*. Then

$$(T^* f_i^*)\, e_j = f_i^*(Te_j) = \sum_{k=1}^{n} a_{kj} f_i^* f_k = a_{ij} \;\Rightarrow\; T^*(f_1^*, \cdots, f_m^*) = (e_1^*, \cdots, e_n^*) \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}.$$

It is seen that under the dual basis, the adjoint of T is the matrix transpose.

(ii) We define an operator T on $L^1([a, b])$ by

$$(Tf)(x) = \int_a^x f(t)\, dt,\; \forall f \in L^1([a, b]).$$

We view T as an operator from $L^1([a, b])$ into $C([a, b])$. Let's find its adjoint $T^* : V_0([a, b]) \to L^\infty([a, b])$, such that for all $\varphi \in V_0([a, b])$ and all $f \in L^1([a, b])$,

$$(T^*\varphi)(f) = \langle Tf, \varphi\rangle = \int_a^b (Tf)(t)\, d\varphi(t) = \int_a^b \int_a^t f(s)\, ds\, d\varphi(t).$$

By Fubini's theorem, since the mapping $(s, t) \mapsto f(s)$ lies in $L^1([a, b] \times [a, b])$, we have

$$\int_a^b \int_a^t f(s)\, ds\, d\varphi(t) = \int_a^b f(s) \left(\int_s^b d\varphi(t) \right) ds.$$

Therefore $(T^*\varphi)(s) = \int_s^b d\varphi(t)$, $\forall \varphi \in V_0([a, b])$.

(iii) We view T as an operator from $L^1([a, b])$ into $L^1([a, b])$. Let's find its adjoint $T^* : L^\infty([a, b]) \to L^\infty([a, b])$, such that for all $g \in L^\infty([a, b])$ and all $f \in L^1([a, b])$,

$$(T^*g)(f) = \langle Tf, g\rangle = \int_a^b (Tf)(t) g(t)\, dt = \int_a^b \int_a^t f(s) g(t)\, ds\, dt = \int_a^b f(s) \left(\int_s^b g(t)\, dt \right) ds.$$

Therefore $(T^*g)(s) = \int_s^b g(t)\, dt$, $\forall g \in L^\infty([a, b])$.

Theorem 3.32 (Properties of adjoint operators). Let X, Y and Z be normed spaces.

(i) For all $T \in \mathfrak{B}(X,Y)$, its has a unique adjoint $T^* \in \mathfrak{B}(Y^*, X^*)$.

(ii) The mapping $T \mapsto T^*$, $\mathfrak{B}(X,Y) \to \mathfrak{B}(Y^*, X^*)$ is linear and norm-preserving.

(iii) $Id_X^* = Id_{X^*}$.

(iv) For all $T \in \mathfrak{B}(X,Y)$ and $S \in \mathfrak{B}(Y,Z)$, $(ST)^* = T^*S^*$.

(v) Whenever $T \in \mathfrak{B}(X,Y)$ is invertible, so is T^*. Moreover,

$$(T^*)^{-1} = (T^{-1})^*.$$

(vi) For all $T \in \mathfrak{B}(X,Y)$,

$$\ker(T^*) = \mathfrak{R}(T)^\perp, \ \ker(T) = {}^\perp\mathfrak{R}(T^*).$$

As a result, $\overline{\mathfrak{R}(T)} = {}^\perp \ker(T^*)$.

(vii) View X and Y as subspaces of X^{**} and Y^{**}, respectively. Then for all $T \in \mathfrak{B}(X,Y)$, the biconjugate $T^{**} := (T^*)^* \in \mathfrak{B}(X^{**}, Y^{**})$, and $T^{**}|_X = T$.

Proof. (i) Clearly, $|f(Tx)| \le \|f\| \, \|T\| \, \|x\|$ for all $f \in Y^*$ and $x \in X$. Given $f \in Y^*$, define

$$(T^*f)(x) = f(Tx), \ \forall x \in X.$$

Then $T^*f \in X^*$, and $\|T^*f\| \le \|f\| \, \|T\|$. Furthermore, $T^* \in \mathfrak{B}(Y^*, X^*)$, and $\|T^*\| \le \|T\|$.

(ii) The linearity of $T \mapsto T^*$ is clear. Following (i), it remains to show $\|T^*\| \ge \|T\|$: if $T \ne 0$,

$$\|Tx\| = \sup_{f\in Y^*,\, \|f\|=1} |f(Tx)| = \sup_{f\in Y^*,\, \|f\|=1} |(T^*f)(x)| \le \sup_{f\in Y^*,\, \|f\|=1} \|T^*f\| \, \|x\| \le \|T^*\| \, \|x\| \, .$$

(iii) By definition, for all $f \in X^*$,

$$(Id_X^* f)(x) = f(Id_X(x)) = f(x), \ \forall x \in X.$$

(iv) For all $f \in Z^*$, we have

$$\left((ST)^*f\right)(x) = f\left(S(Tx)\right) = (S^*f)(Tx) = (T^*(S^*f))(x), \ \forall x \in X.$$

(v) Let $V = T^{-1}$. Then $VT = Id_X$, and $TV = Id_Y$. By (iii) and (iv), we have

$$T^*V^* = (VT)^* = Id_{X^*}, \ V^*T^* = (TV)^* = Id_{Y^*}.$$

(vi) By definition, we have

$$\begin{aligned} &y^* \in \ker(T^*) \ \Leftrightarrow \ T^*y^* = 0 \ \Leftrightarrow \ 0 = (T^*y^*)(x) = y^*(Tx), \ \forall x \in X \ \Leftrightarrow \ y^* \in \mathfrak{R}(T)^\perp. \\ &x \in \ker(T) \ \Leftrightarrow \ Tx = 0 \ \Leftrightarrow \ 0 = y^*(Tx) = (T^*y^*)(x), \ \forall y^* \in Y^* \ \Leftrightarrow \ x \in {}^\perp\mathfrak{R}(T^*). \end{aligned}$$

By Theorem 2.49, $\overline{\mathfrak{R}(T)} = {}^\perp(\mathfrak{R}(T)^\perp) = {}^\perp \ker(T^*)$.

(vii) For all $x \in X$, let $x^{**} = J_X(x)$, where $J_X : X \to X^{**}$ is the canonical map. Then

$$(T^{**}x^{**})f = x^{**}(T^*f) = (T^*f)(x) = f(Tx), \ \forall f \in Y^*.$$

Then $T^{**}x^{**} = (Tx)^{**}$, which is the embedding of Tx into Y^{**}. As a result, $T^{**}|_X = T$. $\square$

Remark. We have an immediate corollary of Theorem 3.32 (vi): Let X and Y be normed spaces, and $T \in \mathfrak{B}(X, Y)$. (i) $\mathfrak{R}(T)$ is dense in Y if and only if T^* is injective; (ii) If $\mathfrak{R}(T^*)$ is dense in X^*, then T is injective. Furthermore, we have the following conclusion.

Lemma 3.33. Let X and Y be Banach spaces, and $T \in \mathfrak{B}(X, Y)$. If T^* is injective, and $\mathfrak{R}(T^*)$ is closed, then T is surjective.

Proof. If $\mathfrak{R}(T^*)$ is closed, then $T^* : Y^* \to \mathfrak{R}(T^*)$ is a bijection between Banach spaces. By bounded inverse theorem, there exists $\delta > 0$ such that $\|T^* y^*\| \geq \delta \|y^*\|$ for all $y^* \in Y^*$. We claim that $O_Y(0, \delta) \subset \overline{TB_X(0, 1)}$. Then akin to Step II in the proof of open mapping theorem (Theorem 3.13), we have $O_Y\big(0, \frac{\delta}{2}\big) \subset TB_X(0, 1)$. Hence T is surjective.

Let's prove our claim. If there exists $y_0 \in Y$ such that $\|y\| < \delta$ and $y_0 \notin \overline{TB(0, 1)}$. Since $\overline{TB_X(0, 1)}$ is a closed convex subset of Y, by hyperplane separation theorem, there exists $f \in Y^*$ such that $f(y_0) > 1$ and $f(y) \leq 1$ for all $y \in \overline{TB(0, 1)}$. Then for all $x \in B_X(0, 1)$,

$$|(T^* f)(x)| = |f(Tx)| \leq 1,$$

which implies $\|T^* f\| \leq 1$. However,

$$\|f\| \geq \frac{|f(y_0)|}{\|y_0\|} > \frac{1}{\delta},$$

which implies $\|T^* f\| \geq \delta \|f\| > 1 \geq \|T^* f\|$, a contradiction! Hence $O_Y(0, \delta) \subset \overline{TB_X(0, 1)}$. $\square$

Now we introduce the closed range theorem.

Theorem 3.34 (Closed range theorem). Let X and Y be Banach spaces, and $T \in \mathfrak{B}(X, Y)$. The following are equivalent: (i) $\mathfrak{R}(T)$ is closed; (ii) $\mathfrak{R}(T^*)$ is closed; (iii) $\mathfrak{R}(T) = {}^{\perp}\ker(T^*)$; (iv) $\mathfrak{R}(T^*) = \ker(T)^{\perp}$.

Proof. (i) $\Rightarrow$ (iii) is clear by Theorem 3.32 (vi); (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii) are trivial.

Now we prove (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv). We first decompose T as

$$T = \iota \circ \widetilde{T} \circ \pi : \ X \xrightarrow{\pi} X/\ker(T) \xrightarrow{\widetilde{T}} \overline{\mathfrak{R}(T)} \overset{\iota}{\hookrightarrow} Y,$$

where π is the quotient map from X onto $X/\ker(T)$, $\widetilde{T}[x] = Tx$ is the induced map from $X/\ker(T)$ to $\overline{\mathfrak{R}(T)}$, and $\iota : \overline{\mathfrak{R}(T)} \to Y$ is the identity embedding. Correspondingly, we decompose T^* as

$$T^* = \pi^* \circ \widetilde{T}^* \circ \iota^* : X^* \xleftarrow{\pi^*} (X/\ker(T))^* \xleftarrow{\widetilde{T}^*} \left(\overline{\mathfrak{R}(T)}\right)^* \xleftarrow{\iota^*} Y.$$

We first check π^*. For all $g \in (X/\ker(T))^*$ and all $x \in X$, we have $(\pi^* g)(x) = g(\pi(x))$. By Theorem 2.26, $\pi^* : g \mapsto g \circ \pi$ is a norm-preserving embedding from $(X/\ker(T))^*$ into X^*, and $\mathfrak{R}(\pi^*) = \ker(T)^{\perp}$.

Next we check ι^*. For all $f \in Y^*$ and all $\xi \in \overline{\mathfrak{R}(T)}$, we have $(\iota^* f)(\xi) = f(\iota(\xi)) = f(\xi)$. It is seen that ι^* is in fact the restriction: $\iota^* f = f|_{\overline{\mathfrak{R}(T)}}$. By Hahn-Banach theorem, ι^* is surjective.

Now we check $\widetilde{T} : X/\ker(T) \to \overline{\mathfrak{R}(T)}$. Clearly, $\mathfrak{R}(\widetilde{T}) = \mathfrak{R}(T)$. Then $\widetilde{T}$ has dense range, and $\widetilde{T}^*$ is injective. If $\mathfrak{R}(T^*)$ is closed, so is $\mathfrak{R}(\widetilde{T}^*) = \mathfrak{R}(T^*) \circ \pi$, because ι^* is surjective and π^* is a norm-preserving embedding. By Lemma 3.33, $\widetilde{T}$ is surjective, and $\mathfrak{R}(T) = \mathfrak{R}(\widetilde{T}) = \overline{\mathfrak{R}(T)}$. Hence (ii) $\Rightarrow$ (iv).

If $\mathfrak{R}(T)$ is closed, $\widetilde{T} : X/\ker(T) \to \overline{\mathfrak{R}(T)}$ is a bijection between Banach spaces. By bounded inverse theorem, $\widetilde{T}$ is invertible, so is $\widetilde{T}^*$. As a result, $\widetilde{T}^* : (\mathfrak{R}(T))^* \to (X/\ker(T))^*$ is a bijection. Since ι^* is surjective, we have $\mathfrak{R}(T^*) = \mathfrak{R}(\pi^*) = \ker(T)^{\perp}$. Hence (i) $\Rightarrow$ (iv). $\square$

Using Theorem 3.34, we obtain the converse of Theorem 3.32 (v) in Banach spaces.

Corollary 3.35. Let X and Y be Banach spaces, and $T \in \mathfrak{B}(X, Y)$. Then T is invertible if and only if T^* is invertible.

Proof. Following Theorem 3.32 (v), it remains to show that T is invertible whenever T^* is invertible. If T^* is invertible, $\mathfrak{R}(T^*)$ is closed. By Theorem 3.34,

$$\ker(T)^{\perp} = \mathfrak{R}(T^*) = X^*,\ \mathfrak{R}(T) = {}^{\perp}\ker(T^*) = {}^{\perp}\{0\} = Y.$$

By Hahn-Banach theorem, $\ker(T) = \{0\}$. Hence $T : X \to Y$ is a bijection. Using the bounded inverse theorem, T is an invertible operator between Banach spaces. □

3.4.2 Adjoint Operators in Hilbert Spaces

Now we discuss adjoint operators in Hilbert spaces. Since any Hilbert space H is isomorphic to its dual space H^*, we can define adjoint operators on primal spaces.

Definition 3.36. (Adjoints). Let H_1 and H_2 be Hilbert spaces. Then for all $T \in \mathfrak{B}(H_1, H_2)$, there exists a unique operator $T^* \in \mathfrak{B}(H_2, H_1)$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle,\ \forall x \in H_1,\ \forall y \in H_2. \tag{3.2}$$

Furthermore, $\|T^*\| = \|T\|$. The operator T^* is said to be the *adjoint (operator)* of T.

Proof. Given $y \in H_2$, we have $|\langle Tx, y\rangle| \leq \|T\| \|x\| \|y\|$ for all $x \in H_1$. Hence the mapping $x \mapsto \langle Tx, y\rangle$ is a bounded linear functional on H_1. By Riesz representation theorem, there exists a unique $\xi_y \in H_1$ such that $\langle Tx, y\rangle = \langle x, \xi_y\rangle$ for all $x \in X$. Define $T^* : H_2 \to H_1,\ y \to \xi_y$. Clearly, T^* is linear and satisfies (3.2). Furthermore, T^* is bounded: $\|\xi_y\|^2 = \langle T\xi_y, y\rangle \leq \|T\| \|\xi_y\| \|y\|$, which implies $\|T^*\| \leq \|T\|$. Furthermore,

$$\|Tx\|^2 = \langle x, T^*Tx\rangle \leq \|x\| \|T^*\| \|Tx\|,\ \forall x \in X,$$

which implies $\|T\| \leq \|T^*\|$. Hence $\|T^*\| = \|T\|$. □

Example 3.37. Following are instances for adjoint operators in Hilbert spaces.

(i) Let H_1 be an n-dimensional Hilbert space, and $\{e_1, \cdots, e_n\}$ an orthonormal basis of H_1. Let H_2 be an m-dimensional normed space, and $\{f_1, \cdots, f_m\}$ an orthonormal basis of H_2. Then any linear operator $T : H_1 \to H_2$ is determined by an m-by-n matrix $A = (a_{ij})_{m\times n}$:

$$Te_i = \sum_{k=1}^{m} a_{ki} f_k \ \overset{def}{\Leftrightarrow}\ T(e_1, \cdots, e_n) = (f_1, \cdots, f_m) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

By definition of orthonormal basis, $a_{ij} = \langle Te_j, f_i\rangle = \langle e_j, T^*f_i\rangle = \overline{\langle T^*f_i, e_j\rangle}$. Hence

$$T^*(f_1, \cdots, f_m) = (e_1, \cdots, e_n) \begin{pmatrix} \overline{a_{11}} & \cdots & \overline{a_{m1}} \\ \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \cdots & \overline{a_{mn}} \end{pmatrix}$$

Under the orthonormal basis, the adjoint of T is its conjugate transpose.

(ii) (Fredholm integral operator). Let $H = L^2([a,b])$, and $K \in L^2([a,b] \times [a,b])$. Define $T_K : H \to H$ by

$$(T_K f)(x) := \int_a^b K(x,y) f(y)\, dy, \ \forall f \in L^2([a,b]),$$

Then

$$\|T_K f\|_2^2 = \int_a^b \left| \int_a^b K(x,y) f(y)\, dy \right|^2 dx \le \int_a^b \int_a^b |K(x,y)|^2 \ dy \, \|f\|_2^2 \, dx = \|K\|_2^2 \, \|f\|_2^2 .$$

Hence $T_K \in \mathfrak{B}(H)$, and $\|T_K\| \le \|K\|_2$. Now let's find the adjoint of T_K. For all $f, g \in L^2([a,b])$, by Fubini's theorem, we have

$$\langle T_K f, g \rangle = \int_a^b \left(\int_a^b f(s) K(t,s)\, ds \right) \overline{g(t)}\, dt = \int_a^b f(s) \left(\int_a^b K(t,s) \overline{g(t)}\, dt \right) ds. \tag{3.3}$$

Since $\langle T_K f, g \rangle = \langle f, T_K^* g \rangle = \int_a^b f(s) \overline{(T_K^* g)(s)}\, ds$, if we define $(T_K^* g)(s) = \int_a^b \overline{K(t,s)} g(t)\, dt$, we have $\langle T_K f, g \rangle = \langle f, T_K^* g \rangle$ for all $f, g \in L^2([a,b])$.

Remark. To apply Fubini's theorem in (3.3), we need to show $K(s,t) f(t) \overline{g(s)} \in L^1([a,b] \times [a,b])$:

$$\int_{[a,b]\times[a,b]} |K(s,t) f(t) g(s)|\, ds\, dt \le \|K\|_2 \sqrt{\int_{[a,b]\times[a,b]} |f(t) g(s)|^2\, ds\, dt} = \|K\|_2 \, \|f\|_2 \, \|g\|_2 .$$

Similar to the conjugate transpose of matrices, we define the conjugate transpose of K by $K^*(s,t) = \overline{K(t,s)}$, where $s, t \in [a,b]$. Then $T_K^* = T_{K^*}$.

Theorem 3.38 (Properties of adjoints in Hilbert spaces). Let H, G and K be Hilbert spaces.

(i) For all $T \in \mathfrak{B}(H,G)$, all $x \in H$ and all $y \in G$, $\langle y, Tx \rangle = \langle T^* y, x \rangle$. As a result, $(T^*)^* = T$.

(ii) For all $T, S \in \mathfrak{B}(H,G)$ and all $\alpha, \beta \in \mathbb{C}$, $(\alpha S + \beta T)^* = \overline{\alpha} S^* + \overline{\beta} T^*$.

(iii) For all $T \in \mathfrak{B}(H,G)$ and all $S \in \mathfrak{B}(G,K)$, $(ST)^* = T^* S^*$.

(iv) For all $T \in \mathfrak{B}(H,G)$, $\|T\|^2 = \|T^* T\| = \|T T^*\|$.

(v) For all $T \in \mathfrak{B}(H,G)$, T is invertible if and only if T^* is invertible. Moreover, $(T^*)^{-1} = (T^{-1})^*$.

(vi) For all $T \in \mathfrak{B}(H,G)$,

$$\ker(T^*) = \mathfrak{R}(T)^\perp, \ \ker(T) = \mathfrak{R}(T^*)^\perp, \ \overline{\mathfrak{R}(T)} = \ker(T^*)^\perp, \ \overline{\mathfrak{R}(T^*)} = \ker(T)^\perp.$$

Proof. (i) For all $T \in \mathfrak{B}(H,G)$, all $x \in H$ and all $y \in G$,

$$\langle y, Tx \rangle = \overline{\langle Tx, y \rangle} = \overline{\langle x, T^* y \rangle} = \langle T^* y, x \rangle.$$

(ii) For all $x \in H$ and all $y \in G$,

$$\langle (\alpha S + \beta T)^* y, x \rangle = \langle y, (\alpha S + \beta T) x \rangle = \overline{\alpha} \langle y, Sx \rangle + \overline{\beta} \langle y, Tx \rangle = \overline{\alpha} \langle S^* y, x \rangle + \overline{\beta} \langle T^* y, x \rangle.$$

(iii) For all $x \in H$ and all $z \in K$,

$$\langle (ST)^* z, x \rangle = \langle z, S(Tx) \rangle = \langle S^* z, Tx \rangle = \langle T^*(S^* z), x \rangle.$$

(iv) Clearly, $\|T^*T\| \le \|T^*\|\,\|T\| = \|T\|^2$. For the other side,

$$\langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \le \|T^*Tx\|\,\|x\| \le \|T^*T\|\,\|x\|^2\,,\ \forall x \in H,\ \Rightarrow\ \|T\|^2 \le \|T^*T\|.$$

Similarly, we have $\|TT^*\| = \|T\|^2$.
(v) Let $V = T^{-1}$. Then $VT = Id_H$, and $TV = Id_G$. By (iii) and (iv), we have

$$T^*V^* = (VT)^* = Id_H,\ V^*T^* = (TV)^* = Id_G.$$

Hence T^* is invertible, and $(T^{-1})^* = (T^*)^{-1}$. If T^* is invertible, by (i), $T = (T^*)^*$ is also invertible.
(vi) By definition, we have

$$y \in \ker(T^*)\ \Leftrightarrow\ T^*y = 0\ \Leftrightarrow\ \langle x, T^*y\rangle = \langle Tx, y\rangle = 0,\ \forall x \in H\ \Leftrightarrow\ y \in \mathfrak{R}(T)^\perp.$$

$$x \in \ker(T)\ \Leftrightarrow\ Tx = 0\ \Leftrightarrow\ \langle Tx, y\rangle = \langle x, T^*y\rangle = 0,\ \forall y \in H\ \Leftrightarrow\ x \in \mathfrak{R}(T^*)^\perp.$$

By Corollary 1.38, $\overline{\mathfrak{R}(T)} = (\mathfrak{R}(T)^\perp)^\perp = \ker(T^*)^\perp$, and $\overline{\mathfrak{R}(T^*)} = (\mathfrak{R}(T^*)^\perp)^\perp = \ker(T)^\perp$. ☐

Now we introduce some special operators, which is the generalization of unitary matrices, Hermitian matrices and normal matrices.

Definition 3.39. Let H be a Hilbert space, and let $T \in \mathfrak{B}(H)$.
(i) (Unitary operators). T is said to be a *unitary operator* if $T^*T = TT^* = Id_H$.
(ii) (Self-adjoint operators). T is said to be a *self-adjoint operator* if $T^* = T$.
(iii) (Normal operators). T is said to be a *normal operator* if $T^*T = TT^*$. By definition, both unitary and self-adjoint operators are unitary.

Example 3.40. Let L be a subspace of a Hilbert space H. Let $P_L : H \to L$ be the projection operator. Then P_L is a self-adjoint operator.

Proof. For all $x, y \in H$, let $x = x_0 + x_1$ and $y = y_0 + y_1$, where $x_0, y_0 \in L$ and $x_1, y_1 \in L^\perp$. Then

$$\langle x, P_L^*y\rangle = \langle P_Lx, y\rangle = \langle x_0, y_0 + y_1\rangle = \langle x_0, y_0\rangle = \langle x_0 + x_1, y_0\rangle = \langle x, P_Ly\rangle,\ \forall x \in H.$$

Hence $P_L^* = P_L$. ☐

Lemma 3.41. Let H be a complex-valued Hilbert space, and $T \in \mathfrak{B}(H)$. Then T is self-adjoint if and only if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$.

Proof. If T is self-adjoint, then

$$\overline{\langle Tx, x\rangle} = \langle x, Tx\rangle = \langle T^*x, x\rangle = \langle Tx, x\rangle,\ \forall x \in X.$$

Conversely, if $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in X$, then

$$\langle T(x+y), x+y\rangle = \langle Tx, x\rangle + \langle Ty, x\rangle + \langle Tx, y\rangle + \langle Ty, y\rangle \in \mathbb{R}, \forall x, y \in H\ \Rightarrow\ \mathrm{Im}(\langle Tx, y\rangle + \langle Ty, x\rangle) = 0$$
$$\langle T(x+\mathrm{i}y), x+\mathrm{i}y\rangle = \langle Tx, x\rangle + \mathrm{i}\langle Ty, x\rangle - \mathrm{i}\langle Tx, y\rangle + \langle Ty, y\rangle \in \mathbb{R}, \forall x, y \in H\ \Rightarrow\ \mathrm{Re}(\langle Tx, y\rangle - \langle Ty, x\rangle) = 0.$$

Therefore $\langle Tx, y\rangle = \overline{\langle Ty, x\rangle} = \langle T^*x, y\rangle$ for all $x, y \in H$, and T is self-adjoint. ☐

Lemma 3.42. Let H be a Hilbert space, and $U \in \mathfrak{B}(H)$. Then U is a unitary operator if and only if U is surjective and norm-preserving.

Proof. If U is unitary, then U is bounded and invertible. Clearly, U is surjective. Furthermore, $\|U\| = \|U^*\| = \sqrt{\|U^*U\|} = 1$.

$$\|x\| = \|U^*Ux\| \leq \|U^*\| \, \|Ux\| = \|Ux\| \leq \|U\| \, \|x\| = \|x\|, \ \forall x \in H.$$

Hence U is norm-preserving, i.e. $\|Ux\| = \|x\|$ for all $x \in H$.

Conversely, suppose $U \in \mathfrak{B}(H)$ is surjective and norm-preserving. We first prove that U is and inner-product-preserving. By polarization identity,

$$\langle Ux, Uy \rangle = \sum_{k=0}^{3} \mathrm{i}^k \|Ux + \mathrm{i}^k Uy\|^2 = \sum_{k=0}^{3} \mathrm{i}^k \|x + \mathrm{i}^k y\|^2 = \langle x, y \rangle, \ \forall x, y \in H.$$

Since U preserves inner product, we have

$$\langle U^*Ux, y \rangle = \langle Ux, Uy \rangle = \langle x, y \rangle, \ \forall x, y \in H.$$

Hence $U^*U = Id_H$. Moreover, for all $y \in H$, by surjectivity of U, there exists $\xi_y \in H$ such that $U\xi_y = y$, and

$$\langle UU^*x, y \rangle = \langle UU^*x, U\xi_y \rangle = \langle U^*x, \xi_y \rangle = \langle x, U\xi_y \rangle = \langle x, y \rangle, \ \forall x, y \in H.$$

Therefore $UU^* = Id_H$, and U is unitary. $\square$

Lemma 3.43. Let H be a Hilbert space, and $T \in \mathfrak{B}(H)$ is a normal operator on H. Then $\ker(T) = \ker(T^*)$, and $\ker(T) \perp \mathfrak{R}(T)$.

Proof. For all $x \in H$,

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle = \langle x, TT^*x \rangle = \langle T^*x, T^*x \rangle = \|T^*x\|^2.$$

Hence $\ker(T) = \ker(T^*)$. The second result follows from $\ker(T^*) = \mathfrak{R}(T)^\perp$. $\square$

4 Spectral Theory

Without specification, the vector spaces we are going to discuss in this section are all complex.

4.1 Resolvent Sets and Spectra

Recall that in matrix theory, the scalar-vector couple (λ, v) is said to be an eigenpair of a matrix A, if $v \neq 0$ and $Av = \lambda v$. λ is said to be an eigenvalue of A, and v is said to be the related eigenvector of λ. Clearly, A is a linear mapping defined on a finite-dimensional space. We can extend this definition to linear operators on general vector spaces.

Definition 4.1. Let X be a complex vector space, and let $T : X \to X$ be a linear operator on X.

(i) (Eigenvalues and eigenvectors). $\lambda \in \mathbb{C}$ is said to be an *eigenvalue* of T, if there exists nonzero vector $x \in X$ such that $Tx = \lambda x$. The vector x is said to be an *eigenvector* of T associated with λ.

(ii) (Eigenspaces). The *eigenspace* (or *characteristic subspace*) of T associated with eigenvalue λ is defined as the set $E_\lambda = \{x \in X : Tx = \lambda x\}$. Clearly, E_λ is a subspace of X. The dimension of E_λ is said to be the *multiplicity* of λ.

Let's find eigenvalues of some linear operators.

Example 4.2. (i) Let $P : H \to M$ be the projection operator onto a subspace M of a Hilbert space H. For all $x \in M$, $Px = x$; and for all $x \in M^\perp$, $Px = 0$. If $\lambda \notin \{0, 1\}$, $P - \lambda\, Id_H$ is invertible. Hence P has eigenvalues 1 and 0, and the corresponding eigenspaces are $E_1 = M$, and $E_0 = M^\perp$.

(ii) Define $T : L^2([a, b]) \to L^2([a, b])$, $(Tf)(x) = \int_a^x f(t)\, dt$, $\forall f \in L^2([a, b])$. We solve the characteristic equation $Tf = \lambda f$ as follows:

- If $\lambda = 0$, then $F(x) = \int_a^x f(t)\, dt = 0$ for all $x \in [a, b]$, which implies $F' = f = 0$ (a.e.) on $[a, b]$.
- If $\lambda \neq 0$, then $f(x) = \frac{1}{\lambda} \int_a^x f(t)\, dt$ for all $x \in [a, b]$, and $f' = \frac{1}{\lambda} f$. By solving the differential equation, $f(t) = C\mathrm{e}^{-t/\lambda}$ for some constant C. Since $f(a) = 0$, we have $C = 0$, and $f \equiv 0$.

Hence T has no eigenvalue.

Remark. By Example 4.2 (ii), we see that linear operators on infinite-dimensional spaces possibly have no eigenvalue. According to Definition 4.1, we can equivalently define eigenvalues of operator T as the numbers $\lambda \in \mathbb{C}$ such that $T - \lambda\, Id_X$ is not injective. If X finite-dimensional, a linear operator $T : X \to X$ is invertible if and only if it is injective. However, it is not the case when the dimension of X becomes infinite. Inspiring by this observation, we introduce the definition of spectra.

Definition 4.3 Let X be a complex normed space, and $T \in \mathfrak{B}(X)$. Let I be the identity operator on X.

(i) (Regular value). Given $\lambda \in \mathbb{C}$, if $T - \lambda I$ is invertible, i.e. $T - \lambda I$ is a bijection $X \to Y$, and the inverse $R_\lambda(T) = (T - \lambda I)^{-1}$ is bounded, then λ is said to be a *regular value* of T. The inverse operator $R_\lambda(T)$ is said to be the *resolvent* of T.

(ii) (Resolvent sets). The *resolvent set* of T is the set of all regular values of T:

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \in \mathfrak{B}(X)\}.$$

(iii) (Spectra). The *spectrum* of T is the complement of the resolvent set: $\sigma(L) = \mathbb{C} \backslash \rho(L)$. In other words, the spectrum $\sigma(T)$ of T is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible.

Proposition 4.4. Let X be a complex Banach space, and $T \in \mathfrak{B}(X)$.

(i) Given a polynomial $p(z) = \sum_{k=0}^{n} a_k z^k$ $(a_n \neq 0)$ defined on C, define $p(T) = \sum_{k=0}^{n} a_k T^k$. Then

$$\sigma(p(T)) = p(\sigma(T)) := \{p(\lambda) : \lambda \in \sigma(T)\}.$$

(ii) If T is invertible, then

$$\sigma(T^{-1}) = \sigma(T)^{-1} := \left\{ \frac{1}{\lambda} : \lambda \in \sigma(T) \right\}.$$

Proof. (i) We first prove a technical lemma: Let $T_1, T_2 \in \mathfrak{B}(X)$, and $T_1T_2 = T_2T_1$. Then T_1T_2 is invertible if and only if both T_1 and T_2 are invertible. If both T_1 and T_2 are invertible, then $T_2^{-1}T_1^{-1} \in \mathfrak{B}(X)$ is the inverse of T_1T_2. Conversely, if T_1T_2 is invertible, then $(T_1T_2)^{-1}T_2 = T_2(T_1T_2)^{-1}$ is the bounded inverse of T_1, and $(T_1T_2)^{-1}T_1 = T_1(T_1T_2)^{-1}$ is the bounded inverse of T_2.

Let $\mu \in \mathbb{C}$, and let $p(z) - \mu = a_n \prod_{k=1}^{n}(z - \lambda_k)$ be the factorization of polynomial $p(z) - \mu$, where $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$. Then $\mu = p(\lambda_k)$ for all $k = 1, \cdots, n$. By above lemma,

$$p(T) - \mu I = a_n \prod_{k=1}^{n} (T - \lambda_k I)$$

is invertible if and only if $T - \lambda_k I$ is invertible for all $k = 1, \cdots, n$. Then

$$\begin{aligned} \mu \in \sigma(p(T)) \;\Leftrightarrow\; p(T) - \mu I \text{ is not invertible } &\Leftrightarrow \text{ there exists } \lambda_k \text{ such that } T - \lambda_k I \text{ is not invertible} \\ &\Leftrightarrow \text{ there exists } \lambda_k \text{ such that } \lambda_k \in \sigma(T) \\ &\Leftrightarrow \text{ there exists } \lambda \in \sigma(T) \text{ such that } p(\lambda) - \mu = 0. \end{aligned}$$

(ii) Clearly, $0 \notin \sigma(T)$. If $\lambda \neq 0$, then $T - \lambda I$ is invertible if and only if $\frac{1}{\lambda} - T^{-1}$ is invertible, as desired. ☐

4.1.1 Classification of Points in the Spectrum

Now we discuss the spectrum of bounded linear operators on Banach spaces.

Definition 4.5. Let X be a Banach space, and $T \in \mathfrak{B}(X)$. Let $\lambda \in \sigma(T)$. Then λ is one of the three cases:

(i) If $T - \lambda I$ is not injective, by definition λ is an eigenvalue of T. The set of all eigenvalues of T is said to be the *point spectrum* of T:

$$\sigma_p(T) = \{\lambda \in \sigma(T) : \ \ker(T - \lambda I) \neq 0\}.$$

(ii) If $T - \lambda I$ is injective but does not have dense range, λ is said to belong to the *residual spectrum* of T:

$$\sigma_r(T) = \left\{ \lambda \in \sigma(T) : \ \ker(T - \lambda I) = 0, \ \overline{\mathfrak{R}(T - \lambda I)} \neq X \right\}.$$

(iii) If $T - \lambda I$ is injective and has dense range, λ is said to belong to the *continuous spectrum* of T:

$$\sigma_c(T) = \left\{ \lambda \in \sigma(T) : \ \ker(T - \lambda I) = 0, \ \overline{\mathfrak{R}(T - \lambda I)} = X \right\}.$$

In this case, $T - \lambda I$ is not bounded from below. In fact, if there exists $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in X$, by Theorem 3.15, $\mathfrak{R}(T - \lambda I)$ is closed. Hence $\mathfrak{R}(T - \lambda I) = \overline{\mathfrak{R}(T - \lambda I)} = X$, and $T - \lambda I \in \mathfrak{B}(X)$ by bounded inverse theorem. But $\lambda \in \sigma(T)$, a contradiction!

Remark. Since X is a Banach space, all bijections on X are invertible. Hence there exists no $\lambda \in \sigma(T)$ such that $T - \lambda I$ is a bijection that has unbounded inverse. As a result, we have the following decomposition of the spectrum of T:

$$\sigma(T) = \sigma_p(T) \amalg \sigma_r(T) \amalg \sigma_c(T).$$

Example 4.6. (i) Let $X = C([0,1])$. Define $T : X \to X$ by

$$(Tf)(x) = xf(x), \ \forall f \in X.$$

Then $T \in \mathfrak{B}(X)$, and $\|T\| = 1$. We find the spectrum of T as follows.

- If $\lambda \notin [0,1]$, then for all $g \in C([0,1])$, the equation $(T - \lambda I)f = g$ has a unique solution $f(x) = \frac{g(x)}{x-\lambda}$, and

$$\frac{\|f\|_\infty}{\|g\|_\infty} = \frac{\sup_{x\in[0,1]} \frac{g(x)}{x-\lambda}}{\sup_{x\in[0,1]} g(x)} \leq \sup_{x\in[0,1]} \frac{1}{x-\lambda} = \max\left\{\frac{1}{1-\lambda}, -\frac{1}{\lambda}\right\} < \infty.$$

  Hence $T - \lambda I$ is invertible, and $\lambda \in \rho(T)$.
- If $\lambda \in [0,1]$, we have $\ker(T - \lambda I) = 0$: If $Tf = 0$, $f(x) = 0$ for all $x \neq \lambda$, and $f \equiv 0$ by continuity. Furthermore, $\mathfrak{R}(T - \lambda I) \subset \{g \in C([a,b]) : g(\lambda) = 0\}$, and

$$\overline{\mathfrak{R}(T - \lambda I)} \subset \overline{\{g \in C([a,b]) : g(\lambda) = 0\}} = \{g \in C([a,b]) : g(\lambda) = 0\} \neq C([a,b]).$$

  Hence $\lambda \in \sigma_r(T)$. To summarize, $\sigma(T) = \sigma_r(T) = [0,1]$.

(ii) We shift to $X = L^2([0,1])$. Still, $T \in \mathfrak{B}(X)$, and $\|T\| = 1$. We find the spectrum of T as follows.

- If $\lambda \notin [0,1]$, similar to (i), $T - \lambda I$ is invertible, and $\lambda \in \rho(T)$.
- If $\lambda \in [0,1]$, we have $\ker(T - \lambda I) = 0$: If $Tf = 0$, then $f = 0$ a.e. on $[0,1]\backslash\{\lambda\}$, and the modification at single point λ does not change $f \in L^2([0,1])$. However, f is not surjective, since $\chi_{[0,1]} \notin \mathfrak{R}(T - \lambda I)$. Given $g \in L^2([0,1])$, we choose the sequence $g_n = g\chi_{\{x:|x-\lambda|>n^{-1}\}}$. By Lebesgue dominated convergence theorem, $\|g_n - g\|_2 \to 0$ as $n \to \infty$. Furthermore, define $f_n(x) = \frac{g_n(x)}{x-\lambda}$ for $x \neq \lambda$ and $f_n(\lambda) = 0$, then

$$\int_{[0,1]} |f_n(x)|^2\,dx \leq \int_{\{x:|x-\lambda|>n^{-1}\}} |ng(x)|^2\,dx = n^2\|g\|_2^2 < \infty.$$

  Hence $f_n \in L^2([0,1])$, and g_n is a sequence in $\mathfrak{R}(T - \lambda I)$ that converges to g. As a result, $\overline{\mathfrak{R}(T - \lambda I)} = L^2([0,1])$, and $\lambda \in \sigma_c(T)$. To summarize, $\sigma(T) = \sigma_c(T) = [0,1]$.

Now we discuss the spectrum of adjoint operators.

Theorem 4.7. Let X be a Banach space, and let $T \in \mathfrak{B}(X)$. Then (i) $\sigma(T) = \sigma(T^*)$; (ii) $\sigma_r(T^*) \subset \sigma_p(T)$, and $\sigma_r(T) \subset \sigma_p(T^*)$; (iii) $\sigma_c(T^*) = \sigma_c(T)$.

Proof. (i) By Corollary 3.35, $T - \lambda I_X$ is invertible if and only if $(T - \lambda I_X)^* = T^* - \lambda I_{X^*}$ is invertible.

(ii) If $\lambda \in \sigma_r(T^*)$, then $\overline{\mathfrak{R}(T^* - \lambda I_{X^*})} \neq X^*$, and

$$\ker(T - \lambda I_X) = {}^\perp\mathfrak{R}(T^* - \lambda I_{X^*}) = {}^\perp\overline{\mathfrak{R}(T^* - \lambda I_{X^*})} \neq \{0\}.$$

Hence $\lambda \in \sigma_p(T)$, and $\sigma_r(T^*) \subset \sigma_p(T)$. Similarly, $\sigma_r(T) \subset \sigma_p(T^*)$.

(iii) Following Theorem 3.32 (vi) and Remark of Definition 2.25,

$$\ker(T^* - \lambda I_{X^*}) = \overline{\mathfrak{R}(T-\lambda I_X)}^{\perp},\ \ker(T-\lambda I_X) = {}^{\perp}\overline{\mathfrak{R}(T^*-\lambda I_{X^*})}.$$

Then

$$\begin{aligned}\lambda \in \sigma_c(T)\ &\Leftrightarrow\ \ker(T-\lambda I_X)=0,\ \overline{\mathfrak{R}(T-\lambda I_X)}=X\\ &\Leftrightarrow\ \overline{\mathfrak{R}(T^*-\lambda I_{X^*})}=X^*,\ \ker(T^*-I_{X^*})=0\ \Leftrightarrow\ \lambda\in\sigma_c(T^*).\end{aligned}$$

Therefore, $\sigma_c(T^*) = \sigma_c(T)$. □

When we discuss adjoints in Hilbert spaces, Theorem 4.7 need modification.

Theorem 4.8. Let H be a Hilbert space, and let $T \in \mathfrak{B}(H)$. Then (i) $\sigma(T^*) = \overline{\sigma(T)} := \left\{\overline{\lambda} : \lambda \in \sigma(T)\right\}$; (ii) $\overline{\sigma_r(T^*)} \subset \sigma_p(T)$, and $\overline{\sigma_r(T)} \subset \sigma_p(T^*)$; (iii) $\sigma_c(T^*) = \overline{\sigma_c(T)}$.

Proof. Similar to Theorem 4.7. Note that in Hilbert space H, $(T-\lambda I)^* = T^* - \overline{\lambda} I$. □

4.1.2 Properties of the Spectrum

Lemma 4.9 (Neumann series). Let X be a Banach space, and let $T \in \mathfrak{B}(X)$. If $\|T\| < 1$, then $I - T$ is invertible, and

$$(I-T)^{-1} = \sum_{k=0}^{\infty} T^k.$$

Proof. We first verify that the limit $\lim_{n\to\infty}\sum_{k=1}^n T^k$ exists. By completeness of $\mathfrak{B}(X)$, it suffices to show that $\left(\sum_{k=1}^n T^k\right)_{n\in\mathbb{N}}$ is a Cauchy sequence:

$$\left\|\sum_{k=m+1}^{n} T^k\right\| \le \sum_{k=m+1}^{n} \|T\|^k = \frac{\|T\|^{m+1}(1-\|T\|^{n-m})}{1-\|T\|},\ \forall n > m.$$

Since $\|T\| < 1$, $\left(\sum_{k=1}^n T^k\right)_{n\in\mathbb{N}}$ is a Cauchy sequence, hence converges in $\mathfrak{B}(X)$. Furthermore,

$$\sum_{k=0}^{\infty} T^k(I-T) = (I-T)\sum_{k=0}^{\infty} T^k = \lim_{n\to\infty}\sum_{k=0}^{n}(I-T)T^k = \lim_{n\to\infty} I - T^{n+1} = I.$$

Hence $(I-T)^{-1} = \sum_{k=0}^{\infty} T^k$. □

Corollary 4.10. Let X be a Banach space, and let $T \in \mathfrak{B}(X)$ be invertible. If $S \in \mathfrak{B}(X)$ satisfies $\|S-T\| < \frac{1}{2\|T^{-1}\|}$, then S is invertible, and $\|S^{-1}-T^{-1}\| \le 2\|T^{-1}\|^2\|T-S\|$.

Proof. Since T is invertible, we have

$$S = T + (S-T) = T\left(I + T^{-1}(S-T)\right),\quad \|T^{-1}(S-T)\| \le \left\|T^{-1}\right\|\|S-T\| < \frac{1}{2}.$$

By Lemma 4.9, $I + T^{-1}(S-T)$ is invertible, hence S is invertible. Furthermore,

$$\left\|S^{-1}\right\| = \left\|\left(I + T^{-1}(S-T)\right)^{-1} T^{-1}\right\| \le \left\|\left(I+T^{-1}(S-T)\right)^{-1}\right\|\left\|T^{-1}\right\| \le 2\left\|T^{-1}\right\|. \tag{4.1}$$

We use (4.1) to bound $\left\|S^{-1}-T^{-1}\right\|$:

$$\left\|S^{-1}-T^{-1}\right\| = \left\|S^{-1}(T-S)T^{-1}\right\| \leq \left\|S^{-1}\right\| \|T-S\| \left\|T^{-1}\right\| \leq 2\left\|T^{-1}\right\|^2 \|T-S\|. \tag{4.2}$$

Remark. Following our discussion, the set of all invertible linear operators in $\mathfrak{B}(H)$ is an open set, and the map $T \mapsto T^{-1}$ is continuous. $\square$

Theorem 4.11. Let X be a Banach space, and $T \in \mathfrak{B}(X)$.

(i) If $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$;

(ii) $\rho(T)$ is open in $\mathbb{C}$;

(iii) $\sigma(T)$ is compact.

Proof. (i) When $|\lambda| > \|T\|$, $T - \lambda I = \lambda\left(\frac{T}{\lambda} - I\right)$ is invertible by Lemma 4.9.

(ii) By Corollary 4.10, if $\lambda \in \rho(T)$, namely, $T - \lambda I$ is invertible, then $T - \mu I$ is invertible for all

$$|\mu - \lambda| < \frac{1}{2\|(T-\lambda I)^{-1}\|}.$$

Hence λ is in the interior of $\rho(T)$. As a result, $\rho(T)$ is open in $\mathbb{C}$.

(iii) By (i) and (ii), $\sigma(T) = \mathbb{C}\backslash\rho(T) \subset \{z \in \mathbb{C} : |z| \leq \|T\|\}$. Then $\sigma(T)$ is a bounded closed subset of $\mathbb{C}$, hence is compact. $\square$

Definition 4.12 (Spectral radii). The *spectral radius* of operator T is defined as

$$r(T) := \sup_{\lambda \in \sigma(T)} |\lambda|.$$

Remark. Following Theorem 4.11, we have $r(T) \leq \|T\|$. Generally, $r(T) = \|T\|$ does not hold.

Example 4.13. (i) Consider the space $\mathbb{C}^2$. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\|A\| = 1$, $\sigma(A) = \{0\}$, and $r(A) = 0$.

(ii) Let l^2 be the space of all square-summable sequences. Define the left-shift and right-shift operators:

$$S : (x_1, x_2, \cdots) \mapsto (x_2, x_3, \cdots), \quad T : (x_1, x_2, \cdots) \mapsto (0, x_1, x_2, \cdots).$$

Choose an orthonormal basis $e_n = (0, \cdots, 0, \underset{n\text{-th}}{1}, 0, \cdots), n \in \mathbb{N}$ of l^2. Then $Te_n = e_{n+1}$, $Se_{n+1} = e_n$, and $Se_1 = 0$. As a result,

$$T\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} x_n e_{n+1}, \quad S\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} x_{n+1} e_n.$$

Then $\|S\| = \|T\| = 1$, $r(S) \leq 1$, and $r(T) \leq 1$. Moreover, we can verify $T = S^*$, which implies $\sigma(T) = \overline{\sigma(S)}$.

Consider the operator $S - \lambda I$ for $|\lambda| \leq 1$. Note that

$$(S - \lambda I)(x_1, x_2, \cdots) = 0 \ \Rightarrow \ x_{n+1} = \lambda x_n, \ \forall n \in \mathbb{N}.$$

If $|\lambda| < 1$, then $(1, \lambda, \lambda^2, \cdots) \in \ker(S)$ for all $|\lambda| < 1$, which implies $\lambda \in \sigma_p(S)$. By Theorem 4.11, the spectrum of S is closed, hence $\sigma(S) = \{z \in \mathbb{C} : |z| \leq 1\}$.

In fact, $\sigma_p(S) = \{z \in \mathbb{C} : |z| < 1\}$ is then open unit disk, and $\sigma_c(S) = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. To see this, let $|\lambda| = 1$, and fix $(y_1, y_2, \cdots) \in l^2$. Given $\epsilon > 0$, choose n such that $\sum_{k=n+1}^{\infty} |y_k|^2 < \epsilon$, and choose the sequence

$$x_k = \begin{cases} \sum_{j=0}^{n-k} \lambda^j y_{k+j}, & k = 1, \cdots, n. \\ 0, & k > n. \end{cases}$$

Then $(x_1, x_2, \cdots) \in l^2$, and $(S - \lambda I)(x_1, x_2, \cdots) = (y_1, \cdots, y_n, 0, \cdots)$. Therefore $(y_1, y_2, \cdots) \in \mathfrak{R}(S)$, and $\lambda \in \sigma_c(S)$.

(iii) Let H be a Hilbert space, and let U be a unitary operator on H. Clearly, $\|U\| = 1$, which implies $\sigma(U) \subset \{z \in \mathbb{C} : |z| \leq 1\}$. Then $\sigma(U^*) = \overline{\sigma(U)} \subset \{z \in \mathbb{C} : |z| \leq 1\}$, and $\sigma(U^{-1}) \subset \{z \in \mathbb{C} : |z| \geq 1\}$. Note that $U^* = U^{-1}$, we conclude that $\sigma(U) \subset \{z \in \mathbb{C} : |z| = 1\}$.

(iv) Let H be a Hilbert space, and let $T \in \mathfrak{B}(H)$ be a self-adjoint operator on H. Then $\sigma(T) = \overline{\sigma(T^*)} = \overline{\sigma(T)}$, which implies $\sigma(T) \subset \mathbb{R}$.

(v) Let H be a Hilbert space, and let $T \in \mathfrak{B}(H)$ be a normal operator on H. If there exists $\lambda \in \sigma_r(T)$, then $\overline{\lambda} \in \sigma_p(T^*)$, and there exists $x \neq 0$ such that $T^*x = \overline{\lambda}x$. By normality of T,

$$\ker(T - \lambda I) = \ker(T^* - \overline{\lambda} I) \neq 0,$$

contradicting $\lambda \in \sigma_r(T)$! Hence $\sigma_r(T) = \emptyset$, and $\sigma(T) = \sigma_p(T) \amalg \sigma_c(T)$.

Example 4.14. Given a Lebesgue measurable function $\varphi \in L^\infty([0,1])$, we define $M_\varphi \in \mathfrak{B}(L^2([0,1]))$ by

$$(M_\varphi f)(x) = \varphi(t) f(t),\ \ f \in L^2([0,1]), t \in [0,1].$$

Clearly, $\|M_\varphi\| \leq \|\varphi\|_\infty$. To prove the other side, note that by choosing $E_\epsilon = \{x \in [0,1] : |\varphi(x)| \geq \|\varphi\|_\infty - \epsilon\}$, we have

$$\|M_\varphi \chi_{E_\epsilon}\|_2 = \sqrt{\int_{E_\epsilon} |\varphi(x)|^2 \, dx} \geq (\|\varphi\|_\infty - \epsilon) \sqrt{\mu(E_\epsilon)} = (\|\varphi\|_\infty - \epsilon) \|\chi_{E_\epsilon}\|_2,\ \ \forall \epsilon > 0.$$

Hence $\|M_\varphi\| = \|\varphi\|_\infty$, and $r(M_\varphi) \leq \|\varphi\|_\infty$. Now we determine the adjoint of M_φ:

$$\langle M_\varphi f, g \rangle = \int_{[0,1]} (\varphi f) \, \overline{g} \, dm = \int_{[0,1]} f \, \overline{(\overline{\varphi} g)} \, dm = \langle f, M_{\overline{\varphi}} g \rangle,\ \ \forall f, g \in L^2([0,1]).$$

Hence $M_\varphi^* = M_{\overline{\varphi}}$, and M_φ is a normal operator on $L^2([0,1])$. As a result, $\sigma(M_\varphi) = \sigma_p(M_\varphi) \amalg \sigma_c(M_\varphi)$.

Now let's find the spectrum of M_φ. We define the essential range of $\varphi \in L^\infty([0,1])$ as

$$\operatorname{ess\,ran} \varphi = \{\lambda \in \mathbb{C} :\ m\left(\{|\varphi(x) - \lambda| < \epsilon\}\right) \neq 0,\ \forall \epsilon > 0\}$$

If $\mu \notin \operatorname{ess\,ran} \varphi$, there exists $\epsilon > 0$ such that $m(E_\mu^\epsilon) = 0$, where $E_\mu^\epsilon = \{x \in [0,1] : |\varphi(x) - \mu| < \epsilon\}$. Let $f \in L^2([0,1])$ be given, we define

$$g_f(x) = \begin{cases} \frac{f(x)}{\varphi(x) - \mu}, & x \notin E_\mu^\epsilon \\ 0, & x \in E_\mu^\epsilon \end{cases} \Rightarrow \int_{[0,1]} |g_f|^2 \, dm \leq \int_{x \notin E_\mu^\epsilon} \left| \frac{f(x)}{\varphi(x) - \mu} \right|^2 dx \leq \frac{\|f\|_2^2}{\epsilon^2} < \infty. \tag{4.3}$$

Define $T : L^2([0,1]) \to L^2([0,1]),\ f \mapsto g_f$. By (4.3), T is linear and bounded: $\|T\| < \epsilon^{-1}$. Furthermore, we have $T(M_\varphi - \mu I) = (M_\varphi - \mu I)T = I$. Therefore, $\mu \in \rho(M_\varphi)$.

Conversely, if $\mu \in \rho(M_\varphi)$, then for all $g \in L^2([0,1])$, there exists $f \in L^2([0,1])$ such that $(M_\varphi - \mu I)f = g$. In other words, $\frac{g}{\varphi-\mu} \in L^2([0,1])$. We prove that $\frac{1}{\varphi-\mu} \in L^\infty([0,1])$. If not, then

$$m\Bigg(\underbrace{\left\{\left|\frac{1}{\varphi-\mu}\right| \geq n\right\}}_{=:E_n}\Bigg) > 0,\ \forall n \in \mathbb{N}.$$

Clearly, $\frac{1}{\varphi-\mu} \in L^2([0,1])$, which implies $\lim_{n\to\infty} m(E_n) = 0$. Then we can choose a subsequence such that $m(E_{n_k} \backslash E_{n_{k+1}}) > 0$, and define $g \in L^2([0,1])$ as follows:

$$g = \sum_{k=1}^\infty \frac{\chi_{E_{n_k}\backslash E_{n_{k+1}}}}{n_k\sqrt{m(E_{n_k}\backslash E_{n_{k+1}})}} \ \Rightarrow\ \int_{[0,1]} |g|^2\, dm \leq \sum_{k=1}^\infty \frac{1}{n_k^2} \leq \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}.$$

However,

$$\int_{[0,1]} \left|\frac{g}{\varphi-\mu}\right|^2 dm = \sum_{k=1}^\infty \frac{\int_{E_{n_k}\backslash E_{n_{k+1}}} \frac{1}{|\varphi-\mu|^2}\, dm}{n_k^2\, m(E_{n_k}\backslash E_{n_{k+1}})} \geq \sum_{k=1}^\infty \frac{n_k^2\, m(E_{n_k}\backslash E_{n_{k+1}})}{n_k^2\, m(E_{n_k}\backslash E_{n_{k+1}})} = \infty,$$

a contradiction! Hence $\frac{1}{\varphi-\mu} \in L^\infty([0,1])$, which implies $\mu \notin \operatorname{ess\,ran}\varphi$. Therefore, $\sigma(M_\varphi) = \operatorname{ess\,ran}\varphi$.

Finally we determine the point spectrum of M_φ. If $\lambda \in \sigma_p(M_\varphi)$, there exists $f \in L^2([0,1])$ such that $m(\{f \neq 0\}) > 0$ and $(M_\varphi - \lambda I)f = 0$, which implies $m(\{x \in [0,1] : \varphi(x) = \lambda\}) \geq m(\{f \neq 0\}) > 0$. Conversely, if $m(\{x \in [0,1] : \varphi(x) = \lambda\}) > 0$, we have $(M_\varphi - \lambda I)\chi_{\{\varphi(x)=\lambda\}} = 0$, which implies $\lambda \in \sigma_p(M_\varphi)$. Hence

$$\sigma_p(M_\varphi) = \{\lambda \in \mathbb{C} : m(\{x \in [0,1] : \varphi(x) = \lambda\} > 0\}.$$

Since $\sigma(M_\varphi) = \sigma_p(M_\varphi) \amalg \sigma_c(M_\varphi)$, we can obtain $\sigma_c(M_\varphi)$ by choose the complement.

Theorem 4.15. Let X be a Banach space, $T \in \mathfrak{B}(X)$. Given $f \in \mathfrak{B}(X)^*$, define $F : \rho(T) \to \mathbb{C}$ by

$$F(\lambda) = f\left((T - \lambda I)^{-1}\right).$$

Then F is analytic on $\rho(T)$.

Proof. For all $\lambda, \mu \in \rho(T)$, we have $(T-\lambda I)^{-1} - (T - \mu I)^{-1} = (\lambda - \mu)(T - \lambda I)^{-1}(T - \mu I)^{-1}$. Then F is differentiable on $\rho(T)$:

$$\lim_{\lambda\to\lambda_0} \frac{f\left((T-\lambda I)^{-1}\right) - f\left((T-\lambda_0 I)^{-1}\right)}{\lambda - \lambda_0} = f\left((T - \lambda_0 I)^{-2}\right).$$

Since $\rho(T)$ is open, F is analytic on $\rho(T)$. □

Corollary 4.16. Let X be a Banach space, and $T \in \mathfrak{B}(X)$. Then $\sigma(T) \neq \emptyset$.

Proof. If $\sigma(T) = \emptyset$, $\rho(T) = \mathbb{C}$. Given $f \in \mathfrak{B}(X)^*$, let $F(\lambda) = f\left((T - \lambda I)^{-1}\right)$. While $|\lambda| > \|T\|$,

$$(T - \lambda I)^{-1} = -\sum_{n=0}^\infty \frac{T^n}{\lambda^{n+1}} \ \Rightarrow\ \left\|(T-\lambda I)^{-1}\right\| \leq \sum_{n=0}^\infty \frac{\|T\|^n}{|\lambda|^{n+1}} \leq \frac{1}{|\lambda| - \|T\|},$$

$$F(\lambda) = f\left((T-\lambda I)^{-1}\right) \le \|f\| \left\|(T-\lambda I)^{-1}\right\| \le \frac{\|f\|}{|\lambda| - \|T\|} \ \Rightarrow \ \lim_{\lambda\to\infty} F(\lambda) = 0.$$

Since F is analytic on $\mathbb{C}$, by Liouville's theorem, $F \equiv 0$. Then for all $\lambda \in \mathbb{C}$, $f\left((T-\lambda I)^{-1}\right) = 0$ for all $f \in \mathfrak{B}(X)^*$. By Hahn-Banach theorem, $(T-\lambda I)^{-1} = 0$, a contradiction! $\square$

It is seen that any bounded linear operator on Banach spaces has non-empty compact spectrum. Using the Laurent series, we obtain the exact formula for spectral radii.

Theorem 4.17 (Gelfand). Let X be a Banach space, and $T \in \mathfrak{B}(X)$. Then

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

Proof. Step I: Let $a = \inf_{n\ge 1} \|T^n\|^{1/n}$. We claim that $\lim_{n\to\infty} \|T^n\|^{1/n} = a$.

By definition, for all $\epsilon > 0$, there exists $m \ge 1$ such that $\|T^m\|^{1/m} < a + \epsilon$. For all $n \in \mathbb{N}$, let $n = km + l$ where $k \in \mathbb{N}_0$ and $l \in \{0, 1, \cdots, m-1\}$. Then

$$\|T^n\|^{1/n} \le \left(\|T^m\|^k \|T\|^l\right)^{1/n} \le (a+\epsilon)^{km/n} \|T\|^{l/n}$$

Let $n \to \infty$, we have $\limsup_{n\to\infty} \|T^n\|^{1/n} \le a + \epsilon$ for all $\epsilon > 0$. Hence

$$a \le \liminf_{n\to\infty} \|T^n\|^{1/n} \le \limsup_{n\to\infty} \|T^n\|^{1/n} \le a.$$

Step II: If $|\lambda| > a$, we have

$$\lim_{n\to\infty} \left(\frac{\|T^n\|}{|\lambda|^{n+1}}\right)^{1/n} = \frac{a}{|\lambda|} < 1.$$

Then $S = -\sum_{n=1}^{\infty} \frac{T^n}{\lambda^{n+1}}$ converges in norm, and $S(T-\lambda I) = (T-\lambda I)S = I$. Hence $\lambda \in \rho(T)$ for all $|\lambda| > a$, which implies $r(T) \le a = \lim_{n\to\infty} \|T^n\|^{1/n}$. Furthermore,

$$(T-\lambda I)^{-1} = -\sum_{n=1}^{\infty} \frac{T^n}{\lambda^{n+1}}, \ |\lambda| > a.$$

Step III: We prove the other side. For all $f \in \mathfrak{B}(X)^*$, use Laurent series:

$$f\left((T-\lambda I)^{-1}\right) = -\sum_{n=1}^{\infty} \frac{f(T^n)}{\lambda^{n+1}}, \ |\lambda| > a. \tag{4.4}$$

By uniqueness of Laurent series, (4.4) holds for all $|\lambda| > r(T)$. Hence for all $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{|f(T^n)|}{(r(T)+\epsilon)^{n+1}} < \infty.$$

Let $U_n = \frac{T^n}{(r(T)+\epsilon)^{n+1}}$. Since $\sup_{n\ge 1} |f(U_n)| < \infty$ holds for all $f \in \mathfrak{B}(X)^*$, by Banach-Steinhaus theorem, there exists $M > 0$ such that $\sup_{n\ge 1} \|U_n\| \le M$. Hence

$$\|T^n\| \le M\left(r(T)+\epsilon\right)^{n+1}$$

for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|T^n\|^{1/n} \le r(T) + \epsilon$. Let $\epsilon \to 0$, we have $\lim_{n\to\infty} \|T^n\|^{1/n} \le r(T)$, as desired. $\square$

Now we show some applications of Gelfand spectral radius theorem.

Corollary 4.18 (F. Riesz). Let S and T be bounded linear operators on a Banach space X.

(i) $r(ST) = r(TS)$.

(ii) if $ST = TS$, then $r(S+T) \le r(S) + r(T)$.

Proof. (i) Using Theorem 4.17, we have

$$r(ST) = \lim_{n\to\infty} \|(ST)^n\|^{1/n} = \lim_{n\to\infty} \|S(TS)^{n-1}T\|^{1/n} = \lim_{n\to\infty} \|S\|^{1/n} \left\|(TS)^{n-1}\right\|^{1/n} \|T\|^{1/n} = r(TS).$$

Similarly, we have $r(TS) \le r(ST)$, which concludes the proof of (i).

(ii) Suppose $ST = TS$. Given $\epsilon > 0$, we choose $M > 0$ such that $\|S^n\|^{1/n} < r(S)+\epsilon$ and $\|T^n\|^{1/n} < r(T)+\epsilon$ for all $n > M$. For sufficiently large n, we have

$$\begin{aligned}
\|(S+T)^n\| &\le \sum_{k=0}^{n} \binom{n}{k} \|S^k\| \, \|T^{n-k}\| \\
&\le \sum_{k=0}^{M} \binom{n}{k} \|S\|^k \left(r(T)+\epsilon\right)^{n-k} + \sum_{k=M+1}^{n-M-1} \binom{n}{k} \left(r(S)+\epsilon\right)^k \left(r(T)+\epsilon\right)^{n-k} \\
&\quad + \sum_{k=n-M}^{n} \binom{n}{k} \left(r(S)+\epsilon\right)^k \|T\|^{n-k} \\
&\le \sum_{k=0}^{M} \binom{n}{k} \left(\frac{\|S\|}{r(S)+\epsilon}\right)^k \left(r(S)+\epsilon\right)^k \left(r(T)+\epsilon\right)^{n-k} + \sum_{k=M+1}^{n-M-1} \binom{n}{k} \left(r(S)+\epsilon\right)^k \left(r(T)+\epsilon\right)^{n-k} \\
&\quad + \sum_{k=n-M}^{n} \left(\frac{\|T\|}{r(T)+\epsilon}\right)^{n-k} \binom{n}{k} \left(r(S)+\epsilon\right)^k \left(r(T)+\epsilon\right)^{n-k} \\
&\le \underbrace{\max\left\{ \max_{0\le k\le M} \left(\frac{\|S\|}{r(S)+\epsilon}\right)^k,\ 1,\ \max_{0\le k\le M} \left(\frac{\|T\|}{r(T)+\epsilon}\right)^k \right\}}_{=:L} \sum_{k=0}^{n} \binom{n}{k} \left(r(S)+\epsilon\right)^k \left(r(T)+\epsilon\right)^{n-k} \\
&= L \left(r(S) + r(T) + 2\epsilon\right)^n ,
\end{aligned}$$

where L is a constant independent of n. Let $n \to \infty$, we have $r(S+T) \le r(S) + r(T) + 2\epsilon$. Since $\epsilon > 0$ is arbitrary, the result follows when $\epsilon \to 0$. $\square$

Remark. (i) In fact, we have $\sigma(ST)\backslash\{0\} = \sigma(TS)\backslash\{0\}$. To see this, note that

$$\begin{aligned}
(I - ST)\left(I + S(I-TS)^{-1}T\right) &= \left(I + S(I-TS)^{-1}T\right)(I - ST) = I, \\
(I - TS)\left(I + T(I-ST)^{-1}S\right) &= \left(I + T(I-ST)^{-1}S\right)(I - TS) = I.
\end{aligned}$$

Hence $I - ST$ is invertible if and only if $I - TS$ is invertible. As a result, for all $\lambda \neq 0$, $ST - \lambda I$ is invertible if and only if $TS - \lambda I$ is invertible.

(ii) The second statement in Corollary 4.18 fails when S and T are not commutable, i.e. $ST \neq TS$. For instance, consider

$$S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which are linear operators on $\mathbb{C}^2$. Then $r(S) = r(T) = 0$, but $r(S+T) = 1$.

Corollary 4.19. Let H be a Hilbert space, and let $T \in \mathfrak{B}(H)$ be a normal operator on H. Then $r(T) = \|T\|$.

Proof. First, let T be self-adjoint. Then $\|T\|^2 = \|T^*T\| = \|T^2\|$, and T^2 is also self-adjoint. By induction,

$$\|T\|^2 = \|T^2\| \;\Rightarrow\; \|T\|^4 = \|T^4\| \;\Rightarrow\; \cdots \;\Rightarrow\; \|T\|^{2^k} = \left\|T^{2^k}\right\|,\ \forall k \in \mathbb{N}.$$

Hence $r(T) = \lim_{k\to\infty} \left\|T^{2^k}\right\|^{1/2^k} = \|T\|$. Now let T be normal. If $(T^n)^*T^n = (T^*T)^n$, then

$$(T^{n+1})^*T^{n+1} = T^*(T^n)^*T^nT = T^*(T^*T)^nT = T^*(TT^*)^nT = (T^*T)^{n+1}.$$

By induction, $(T^n)^*T^n = (T^*T)^n$ for all $n \in \mathbb{N}$. Furthermore, we have

$$r(T)^2 = \lim_{n\to\infty} \|T^n\|^{2/n} = \lim_{n\to\infty} \|(T^n)^*T^n\|^{1/n} = \lim_{n\to\infty} \|(T^*T)^n\|^{1/n} = r(T^*T).$$

Since T^*T is self-adjoint, $r(T^*T) = \|T^*T\| = \|T\|^2$. Hence $r(T) = \|T\|$. $\square$

Example 4.20. Suppose $f \in C([a,b])$, and $K \in C(\mathcal{D})$, where $\mathcal{D} = \left\{(x,y) \in \mathbb{R}^2 : a \le x \le b, a \le y \le x\right\}$. Define $T : C([a,b]) \to C([a,b])$ by

$$(Tf)(x) := \int_a^x K(x,y)f(y)\,dy,\ \forall f \in C([a,b]),$$

Following Example 1.69,

$$|(T^nf)(x)| \le \frac{1}{n!}M^n(x-a)^n\|f\|_\infty,\ \forall x \in [a,b].$$

where $M = \sup_{(x,y)\in\mathcal{D}} |K(x,y)|$. As a result,

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} \le \lim_{n\to\infty} \frac{M(b-a)}{\sqrt[n]{n!}} = 0.$$

Since $\sigma(T) \neq \emptyset$, we have $\sigma(T) = \{0\}$.

4.2 Compact Operators

4.2.1 Finite-rank Operators and Compact Operators

Definition 4.21 (Finite-rank operators). Let X and Y be vector spaces, and let $T : X \to Y$ be a linear operator. T is said to be a *finite-rank operator* if TX is a finite-dimensional subspace of Y.

Remark. By definition, if Y is finite-dimensional, all linear operators from X into Y are of finite rank.

Proposition 4.22. Let X and Y be vector spaces, and let $T : X \to Y$ be a linear operator. Then T is a finite-rank operator if and only if there exist linear functionals $f_1, \cdots, f_n$ on X and linear independent vectors $y_1, \cdots, y_n$ of Y such that

$$Tx = \sum_{j=1}^{n} f_j(x)\, y_j, \ \forall x \in X.$$

Proof. We only show the "only if" case, since the other direction is trivial. Let $T : X \to Y$ be a finite rank operator. We choose a basis $\{y_1, \cdots, y_n\}$ of TX. Then for all $x \in X$, there exist uniquely determined $f_1(x), \cdots, f_n(x) \in \mathbb{F}$ such that $Tx = \sum_{j=1}^{n} f_j(x)\, y_j$. It remains to verify f_j is linear for each j.

Given $x, x' \in X$ and $\alpha, \beta \in \mathbb{F}$, we have

$$T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2 \ \Rightarrow \ \sum_{j=1}^{n} f_j(\alpha x_1 + \beta x_2) y_j = \sum_{j=1}^{n} (\alpha f_j(x_1) + \beta f_j(x_2))\, y_j.$$

Since $y_1, \cdots, y_n$ are linearly independent, $f_j(\alpha x_1 + \beta x_2) = \alpha f_j(x_1) + \beta f_j(x_2)$ for each j. Hence f_j is linear. □

Proposition 4.23. Let X and Y be normed spaces, and let $T : X \to Y$ be a linear operator. Then T is a bounded finite-rank operator if and only if there exist bounded linear functionals $f_1, \cdots, f_n \in X^*$ and linear independent vectors $y_1, \cdots, y_n$ of Y such that

$$Tx = \sum_{j=1}^{n} f_j(x)\, y_j, \ \forall x \in X.$$

Proof. "$\Leftarrow$": Clearly T is of finite rank. Furthermore, $\|Tx\| \le \sum_{j=1}^{n} \|f_j\| \, \|y_j\| \, \|x\|$.

"$\Rightarrow$": By Proposition 4.22, there exist linear functionals $f_1, \cdots, f_n$ on X and points $y_1, \cdots, y_n$ of Y such that $Tx = \sum_{j=1}^{n} f_j(x)\, y_j$ for all $x \in X$. It remains to show f_j is bounded for all $j \in \{1, \cdots, n\}$.

Let $L_{-j} = \operatorname{span}\{y_1, \cdots, y_{j-1}, y_{j+1}, \cdots, y_n\}$. By Hahn-Banach theorem, there exists $f \in Y^*$ such that $f(L_{-j}) = 0$, and $f(y_j) = 1$. Then

$$f(Tx) = f\left(\sum_{j=1}^{n} f_j(x)\, y_j\right) = f_n(x), \ \forall x \in X.$$

As a result, $f_n = f \circ T \in X^*$. □

Example 4.24 (Finite-rank operators on infinite-dimensional spaces). Let X be an infinite-dimensional Banach space, and let $T \in \mathfrak{B}(X)$ be a finite-rank operator. By Proposition 4.23, there exist $\alpha_1, \cdots, \alpha_n \in X^*$ and linear independent vectors $x_1, \cdots, x_n \in X$ such that

$$Tx = \sum_{j=1}^{n} \alpha_j(x)\, x_j, \ \forall x \in X.$$

To find the eigenvalues of T, we solve the equation $Tx = \lambda x$. If $\lambda = 0$, we have $\alpha_1(x) = \cdots = \alpha_n(x) = 0$. For each j, the induced map $\tilde{\alpha}_j : X/\ker\alpha_j \to \mathbb{C},\ [x] \mapsto \alpha_j(x)$ is an injection into $\mathbb{C}$, which implies

$$\operatorname{codim}\ker\alpha_j = \dim(X/\ker\alpha_j) \le 1 \ \Rightarrow\ \operatorname{codim}\left(\bigcap_{j=1}^n \ker\alpha_j\right) \le \sum_{j=1}^n \operatorname{codim}\ker\alpha_j \le n.$$

Hence $\bigcap_{j=1}^n \ker\alpha_j$ is an infinite-dimensional subspace of X. As a result, there exists nonzero $x \in \bigcap_{j=1}^n \ker\alpha_j$ such that $Tx = 0$, which implies $0 \in \sigma_p(T)$.

If $\lambda \ne 0$, we have $x \in \operatorname{span}\{x_1, \cdots, x_n\}$, because

$$x = \frac{1}{\lambda}Tx = \sum_{j=1}^n \beta_j x_j,\ \beta_j = \frac{\alpha_j(x)}{\lambda},\ j = 1, \cdots, n. \tag{4.5}$$

Plugging (4.5) into $Tx = \lambda x$, we have

$$\sum_{j=1}^n \lambda\beta_j x_j = \sum_{k=1}^n \beta_k T x_k = \sum_{k=1}^n\sum_{j=1}^n \beta_k\alpha_j(x_k)x_j \ \Rightarrow\ \lambda\beta_j = \sum_{k=1}^n \alpha_j(x_k)\beta_k.$$

Hence λ is an eigenvalue of matrix $A = (A_{jk})_{n\times n} = (\alpha_j(x_k))_{n\times n}$, and $\beta = (\beta_1, \cdots, \beta_n)^\top$ is the associated eigenvector. Conversely, if (λ, β) is an eigenpair of matrix A, we have

$$T\left(\sum_{j=1}^n \beta_j x_j\right) = \sum_{j=1}^n\sum_{k=1}^n \beta_j\alpha_k(x_j)x_k = (x_1, \cdots, x_n)\, A\beta = \lambda\,(x_1, \cdots, x_n)\,\beta = \lambda\left(\sum_{j=1}^n \beta_j x_j\right).$$

Hence $\sigma_p(T) = \sigma_p(A) \cup \{0\}$.

Definition 4.25 (Compact operators). Let X and Y be normed spaces, and let $T : X \to Y$ be a linear operator. Then T is said to be a *compact operator* if T maps every bounded subset of X to a relatively compact subset of Y, i.e., for all $A \subset X$ such that $\sup_{x\in A}\|x\| < \infty$, $\overline{TA}$ is a compact subset of Y.

Remark. By definition, a compact operator is automatically bounded, since it maps bounded subsets to bounded subsets.

Lemma 4.26. Bounded linear finite-rank operators are compact operators.

Proof. Let X and Y be normed spaces, and let $T \in \mathfrak{B}(X, Y)$ be finite-rank linear operators. Then TX is a finite-dimensional subspace of Y. By Theorem 1.56, TX is complete.

Let A be a bounded subset of X, then TA is a bounded subset of TX. Since TX is finite-dimensional, all bounded subsets of TX are totally bounded, hence relatively compact. □

Example 4.27 (Fredholm integral operators). Given $K \in C([a, b]\times[a, b])$, define the corresponding Fredholm operator $T_K : C([a, b]) \to C([a, b])$ as follows:

$$(T_K\varphi)(x) = \int_a^b K(x, y)\varphi(y)\, dy,\ \forall\varphi \in C([a, b]).$$

Then T_K is a compact operator. To prove this, let $\mathcal{A}$ be a bounded subset of $C([a, b])$. By Arzelà-Ascoli theorem (Theorem 1.77), it suffices to show that $T_K\mathcal{A}$ is bounded and uniformly equicontinuous.

Choose $M > 0$ such that $\|\varphi\|_\infty \le M$ for all $\varphi \in \mathcal{A}$. Then

$$\|T_K\varphi\|_\infty = \sup_{x\in[a,b]} \left| \int_a^b K(x,y)\varphi(y)\,dy \right| \le (b-a)\,\|K\|_\infty\,\|\varphi\|_\infty \le (b-a)M\,\|K\|_\infty\,,\ \forall\varphi\in\mathcal{A}.$$

Hence $T_K\mathcal{A}$ is bounded. Furthermore,

$$\begin{aligned} |(T_K\varphi)(x) - (T_K\varphi)(x')| &= \left| \int_a^b \left(K(x,y) - K(x',y)\right)\varphi(y)\,dy \right| \\ &\le \left| \int_a^b \left(K(x,y) - K(x',y)\right)dy \right| \|\varphi\|_\infty\,. \end{aligned} \tag{4.6}$$

Note that K is uniformly continuous. Given $\epsilon > 0$, choose $\delta > 0$ such that $|K(x,y) - K(x',y)| < \frac{\epsilon}{M(b-a)}$ for all $x, x' \in [a,b]$ such that $|x - x'| < \delta$ and all $y \in [a,b]$. By (4.6), we have $|(T_K\varphi)(x) - (T_K\varphi)(x')| < \epsilon$ for all $\varphi \in \mathcal{A}$. Hence $T_K\mathcal{A}$ is equicontinuous.

Theorem 4.28. Let X, Y and Z be normed spaces. Denote by $\mathcal{K}(X,Y)$ the set of all compact operators from X into Y. Then:

(i) $\mathcal{K}(X,Y)$ is a linear subspace of $\mathfrak{B}(X,Y)$.

(ii) $\mathfrak{B}(Y,Z)\circ\mathcal{K}(X,Y) \subset \mathcal{K}(X,Z)$, and $\mathcal{K}(X,Y)\circ\mathfrak{B}(Z,X) \subset \mathcal{K}(Z,Y)$.

(iii) If Y is a Banach space, then $\mathcal{K}(X,Y)$ is a closed subspace of Y.

Proof. (i) Let $S, T \in \mathcal{K}(X,Y)$, and $\alpha \in \mathbb{C}$. Clearly, $\alpha S \in \mathcal{K}(X,Y)$. To show that $S + T \in \mathcal{K}(X,Y)$, let A be a bounded subset of X, and choose a sequence (x_n) of points of A. Since TA is relatively compact, we can find a subsequence (x_{n_k}) that Sx_{n_k} converges in Y. Also, we choose a subsequence $(x_{n'_k})$ of (x_{n_k}) such that $Tx_{n'_k}$ converges in Y. Hence $(S+T)\,x_{n'_k}$ converges in Y, and $(S+T)A$ is relatively compact.

(ii) Let $S \in \mathcal{K}(X,Y)$, and $T \in \mathfrak{B}(Y,Z)$. If $A \subset X$ is bounded, then $SA \subset Y$ is relatively compact. Since T is continuous, $T(SA) \subset Z$ is relatively compact. Hence $TS \in \mathcal{K}(X,Z)$.

Now let $S \in \mathfrak{B}(Z,X)$, and $T \in \mathcal{K}(X,Y)$. If $B \subset Z$ is bounded, then $SB \subset Z$ is also bounded, and $T(SB) \subset Y$ is relatively compact. Hence $TS \in \mathcal{K}(Z,Y)$.

(iii) Clearly, $\mathfrak{B}(X,Y)$ is a Banach space. Let $T_n : X \to Y$ be a sequence of compact operators that converges to $T \in \mathfrak{B}(X,Y)$. It suffices to show that $T \in \mathcal{K}(X,Y)$: Let A be a bounded subset of X such that $L = \sup_{x\in A}\|x\| > 0$. We prove that TA is totally bounded.

Given $\epsilon > 0$, we choose $N > 0$ such that $\|T_n - T\| < \frac{\epsilon}{3L}$ for all $n \ge N$. By definition, T_NA is totally bounded, so we choose an $\epsilon/3$-net $\{T_Nx_1, \cdots, T_Nx_m\}$ of T_NA. Then $\{Tx_1, \cdots, Tx_m\}$ is an ϵ-net of TA: for each $x \in A$, choose x_k such that $\|T_Nx - T_Nx_k\| < \epsilon/3$, hence

$$\begin{aligned} \|Tx - Tx_k\| &\le \|Tx - T_Nx\| + \|T_Nx - T_Nx_k\| + \|T_Nx_k - Tx_k\| \\ &\le \|T - T_N\|\,\|x\| + \|T_Nx - T_Nx_k\| + \|T - T_N\|\,\|x_k\| \\ &< \frac{\epsilon}{3L}L + \frac{\epsilon}{3} + \frac{\epsilon}{3L}L = \epsilon \end{aligned}$$

Therefore TA is totally bounded, and T is a compact operator. ☐

Corollary 4.29. Let X be a normed space, let Y be a Banach space, and let $T_n : X \to Y$ be a sequence of bounded finite-rank operators. If $T_n \to T \in \mathfrak{B}(X,Y)$ in norm, T is a compact operator.

Proof. By Theorem 4.28 (iii). ☐

Review: separable Hilbert spaces. Recall that every Hilbert space H has an orthonormal basis $\{e_\lambda, \lambda \in \Lambda\}$ such that $H = \overline{\operatorname{span}}\{e_\lambda, \lambda \in \Lambda\}$. If H is separable, we take a countable dense subset Q of H. For every $x \in Q$, there are at most countably many basis element e_λ such that $\langle x, e_\lambda\rangle \neq 0$. Take $E_x = \{e_\lambda : \langle x, e_\lambda\rangle \neq 0\}$, then $x \in \overline{\operatorname{span}}\, E_x$. Furthermore, $E = \bigcup_{x\in Q} E_x$ is a countable basis of H:

$$Q \subset \overline{\operatorname{span}}\, E \ \Rightarrow\ H = \overline{Q} = \overline{\operatorname{span}}\, E.$$

Therefore, every separable Hilbert space H has a countable basis $\{e_n, n \in \mathbb{N}\}$.

Remark. Let X be a Banach space, and $T \in \mathfrak{B}(X)$. If T is a compact operator, so is T^2. Conversely, even if T^2 is a compact operator, T is possibly not a compact operator.

Here is a counterexample. Let H_1 and H_2 be two infinite-dimensional separable Hilbert spaces. Let $\{e_n, n \in \mathbb{N}\}$ be an orthonormal basis of H_1, and $\{f_n, n \in \mathbb{N}\}$ an orthonormal basis for H_2. Define

$$T = \begin{array}{c} \begin{matrix} H_1 & H_2 \end{matrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{array}, \text{ i.e. } Te_n = 0,\ Tf_n = e_n,\ \forall n \in \mathbb{N}.$$

Clearly, $T^2 = 0$ is a compact operator. However, T maps unit ball in H_2 to unit ball in H_1, which is not relatively compact! Hence T is not a compact operator.

Corollary 4.30. Let H be a separable Hilbert space, and $T \in \mathfrak{B}(H)$. Then T is a compact operator if and only if T is the limit of a sequence of bounded finite-rank operators.

Proof. Following Corollary 4.29, it suffices to show the "only if" case. Let $T \in \mathcal{K}(H)$, and let $\{e_n, n \in \mathbb{N}\}$ be a basis of H. Define P_n to be the projection operator from H into the subspace $\operatorname{span}\{e_1, \cdots, e_n\}$, i.e.

$$P_n x = \sum_{j=1}^n \langle x, e_j\rangle e_j,\ \forall x \in H.$$

Clearly, P_nT is a sequence of bounded finite-rank operators. It remains to show that $P_nT \to T$ in norm. Since T is a compact operator, $TB(0,1)$ is relatively compact, hence totally bounded. Given $\epsilon > 0$, we choose a $\epsilon/2$-net $\{Tx_1, \cdots, Tx_m\}$ of $TB(0,1)$, where $x_1, \cdots, x_m \in B(0,1)$. Then there exists $N > 0$ such that $\|(I - P_n)Tx_j\| < \epsilon/2$ for all $j \in \{1, \cdots, m\}$. Then for each $x \in B(0,1)$, choose x_j such that $\|Tx - Tx_j\| < \epsilon/2$. Once $n \geq N$, we have

$$\begin{aligned} \|(I-P_n)Tx\| &\leq \|(I-P_n)T(x-x_j)\| + \|(I-P_n)Tx_j\| \\ &\leq \|T(x-x_j)\| + \|(I-P_n)Tx_j\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $\|(I - P_n)T\| < \epsilon$ for all $n \geq N$. Since $\epsilon > 0$ is arbitrary, P_nT converges to T in norm. □

Example 4.31 (Fredholm integral operators). Given $K \in L^2([a,b] \times [a,b])$, define the corresponding Fredholm operator $T_K : L^2([a,b]) \to L^2([a,b])$ as follows:

$$(T_K\varphi)(x) = \int_a^b K(x,y)\varphi(y)\,dy,\ \forall \varphi \in L^2([a,b]).$$

Following Example 3.37 (ii), $\|T_K\| \leq \|K\|_2$. Furthermore, T_K is a compact operator.

Proof. We approximate $K \in L^2([a,b] \times [a,b])$ by a sequence of simple functions

$$K_n(s,t) = \sum_{k=1}^{m_n} \alpha_{n,k} \chi_{D_{n,k}},$$

where $D_{n,k} = (a_{n,k}, b_{n,k}) \times (c_{n,k}, d_{n,k})$ is an open rectangle, and $\|T_K - T_{K_n}\| = \|T_{(K-K_n)}\| \le \|K - K_n\|_2 \to 0$, hence $T_{K_n} \to T_K$ in norm. By Corollary 4.29, it suffices to show that every T_{K_n} is of finite rank:

$$(T_{K_n}\varphi)(x) = \int_a^b \sum_{k=1}^{m_n} \alpha_{n,k} \chi_{D_{n,k}}(x,y)\varphi(y)\,dy = \sum_{k=1}^{m_n} \alpha_{n,k} \left(\int_{c_{n,k}}^{d_{n,k}} \varphi(y)\,dy \right) \chi_{(a_{n,k}, b_{n,k})}(x).$$

Hence $\mathfrak{R}(T_{K_n}) \subset \operatorname{span} \left\{ \chi_{(a_{n,k}, b_{n,k})} \right\}_{k=1}^{m_n}$, which is of finite dimension. □

Finally, we discuss the adjoints of compact operators.

Proposition 4.32. Let X and Y be normed spaces, and $T \in \mathcal{K}(X,Y)$. Then TX is a separable subset of Y.

Proof. By definition, $TB(0,n)$ is a relatively compact subset of Y for all $n \in \mathbb{N}$. By Lemma 1.73, $TB(0,n)$ is separable. As a result, $TX = \bigcup_{n=1}^{\infty} TB(0,n)$ is separable. □

Theorem 4.33. Let X and Y be normed spaces, and $T \in \mathfrak{B}(X,Y)$. Let $T^* \in \mathfrak{B}(Y^*, X^*)$ be the adjoint.

(i) If T is a compact operator, so is T^*.

(ii) If Y is a Banach space and T^* is a compact operator, so is T.

Proof. (i) Let (f_n) be a bounded sequence in Y^* such that $\|f_n\| \le M$ for all $n \in \mathbb{N}$. We want to prove that there exists a subsequence (f_{n_k}) such that $(T^* f_{n_k})$ converges in X^*.

Step I: Let $Y_0 = \overline{TX}$, which is a separable subspace of Y. We define $T_0 : X \to Y_0,\ x \to Tx$. Then for all $f \in Y^*$, we have $T^* f = T_0^* f|_{Y_0}$. Hence $(T^* f_{n_k})$ converges in X^* if and only if $(T_0^* f_{n_k}|_{Y_0})$ converges in X^*. Without loss of generality, we suppose Y is separable.

Step II: By Banach-Alaoglu theorem (Theorem 2.55), there exists a subsequence (f_{n_k}) that converges in the weak-* topology on Y^*:

$$\lim_{k\to\infty} f_{n_k}(y) = f(y).$$

Step III: We verify that $(T^* f_{n_k})$ converges in norm. Let $S = \{x \in X : \|x\| = 1\}$ be the unit sphere in X. Then we have

$$\|T^* f_{n_k} - T^* f\| = \sup_{x\in S} |(T^* f_{n_k} - T^* f)(x)| = \sup_{x\in S} |f_{n_k}(Tx) - f(Tx)| = \sup_{y\in TS} |f_{n_k}(y) - f(y)|. \tag{4.7}$$

Since TS is relatively compact, given $\epsilon > 0$, we choose an $\frac{\epsilon}{3M}$-net $\{y_1, \cdots, y_m\}$ of TS. Then for all $y \in TS$, there exists y_j such that $\|y - y_j\| < \frac{\epsilon}{3M}$. Furthermore, we choose $K > 0$ such that $|f_{n_k}(y_j) - f(y_j)| < \epsilon/3$ for all $j \in \{1, \cdots, m\}$ and all $k \ge K$. Then

$$\begin{aligned} |f_{n_k}(y) - f(y)| &\le |f_{n_k}(y) - f_{n_k}(y_j)| + |f_{n_k}(y_j) - f(y_j)| + |f(y_j) - f(y)| \\ &\le \|f_{n_k}\|\,\|y - y_j\| + |f_{n_k}(y_j) - f(y_j)| + \|f\|\,\|y - y_j\| \\ &< \frac{\epsilon}{3M} M + \frac{\epsilon}{3} + \frac{\epsilon}{3M} M = \epsilon,\ \forall y \in TS,\ k \ge K. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, by (4.7), $\|T^* f_{n_k} - T^* f\| \to 0$ as $k \to \infty$.

(ii) Since $T^* \in \mathcal{K}(Y^*, X^*)$, by (i), $T^{**} \in \mathcal{K}(X^{**}, Y^{**})$. We view X and Y as subspaces of X^{**} and Y^{**}, respectively. Then the unit ball $B_X(0,1) \subset B_{X^{**}}(0,1)$ is a bounded subset of X^{**}, and $T^{**}B_X(0,1)$ is relatively compact, hence totally bounded in Y^{**}. By Theorem 3.32 (vii), $TB(0,1) = TB^{**}(0,1)$ is totally bounded in Y^{**}, so is in Y. Since Y is a Banach space, $TB(0,1)$ is relatively compact in Y. □

4.2.2 Spectra of Compact Operators

Theorem 4.34 (Riesz-Schauder). Let X be a Banach space, and $T \in \mathcal{K}(X)$.

(i) If $\dim X = \infty$, $0 \in \sigma(T)$. In other words, T is not invertible.

(ii) If $\lambda \in \sigma(T)\backslash\{0\}$, there exists $x \neq 0$ such that $Tx = \lambda x$. Namely, every nonzero point of $\sigma(T)$ is an eigenvalue of T. Following (i), if $\dim X = \infty$, then $\sigma(T) = \sigma_p(T) \cup \{0\}$.

(iii) If $\lambda \in \sigma(T)\backslash\{0\}$, $\dim\ker(T-\lambda I) < \infty$, i.e. the eigenspace of λ is a finite-dimensional subspace of X.

(iv) Eigenvectors associated with distinct eigenvalues of T are linearly independent.

(v) $\sigma(T)$ has at most one limit point, which would necessarily be zero.

Proof. We leave the proof of (ii) for later.

(i) If T is invertible, $T^{-1} \in \mathfrak{B}(X)$, and $I = T^{-1}T$ is a compact operator on X. Nevertheless, by Theorem 1.76, the unit ball $B(0,1)$ is not relatively compact, a contradiction!

(iii) For every $x_0 \in B_\lambda = \{x \in \ker(T-\lambda I) : \|x\| \leq 1\}$, we have $x_0 = \lambda^{-1}Tx_0 = T(\lambda^{-1}x_0)$. As a result, $B_\lambda \subset TB(0,|\lambda|^{-1})$ is relatively compact. Since B_λ is the unit ball in $\ker(T-\lambda I)$, $\dim\ker(T-\lambda I) < \infty$.

(iv) Let $\lambda_1, \cdots, \lambda_n$ be distinct eigenvalues of T, and let $x_1, \cdots, x_n$ be the associated eigenvectors. Suppose $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$. Then

$$\begin{pmatrix} I & I & \cdots & I \\ \lambda_1 I & \lambda_2 I & \cdots & \lambda_n I \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} I & \lambda_2^{n-1} I & \cdots & \lambda_n^{n-1} I \end{pmatrix} \begin{pmatrix} \alpha_1 x_1 \\ \alpha_2 x_2 \\ \vdots \\ \alpha_n x_n \end{pmatrix} = \begin{pmatrix} \alpha_1 x_1 + \cdots + \alpha_n x_n \\ T(\alpha_1 x_1 + \cdots + \alpha_n x_n) \\ \vdots \\ T^{n-1}(\alpha_1 x_1 + \cdots + \alpha_n x_n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since the Vandermonde matrix is invertible, $\alpha_1 x_1 = \alpha_2 x_2 = \cdots = \alpha_n x_n = 0$.

(v) We prove an equivalent statement: for all $\epsilon > 0$, the set $\{\lambda \in \sigma(T) : |\lambda| \geq \epsilon\}$ is finite. If not, choose a sequence (λ_n) of distinct eigenvalues, and let (x_n) be the sequence of associated eigenvectors. Denote $L_n = \operatorname{span}\{x_1, \cdots, x_n\}$. By Lemma 1.75, there exists sequence (y_n) of unit vectors such that $y_n \in L_n$ and $d(y_n, L_{n-1}) > 1/2$. Note that $y_n - \frac{Ty_n}{\lambda_n} \in L_{n-1}$. Furthermore, once $n > m$,

$$\left\|\frac{Ty_n}{\lambda_n} - \frac{Ty_m}{\lambda_m}\right\| = \Bigg\|y_n - \underbrace{\left(y_n - \frac{Ty_n}{\lambda_n} + \frac{Ty_m}{\lambda_m}\right)}_{\in L_{n-1}}\Bigg\| > \frac{1}{2}.$$

However, $\left\{\frac{Ty_n}{\lambda_n}, n \in \mathbb{N}\right\} \subset TB(0,\epsilon^{-1})$ is relatively sequentially compact, a contradiction! □

Remark. The statement (v) also gives a characterization of the spectrum of compact operator T: $\sigma(T)$ is discrete, i.e. $\sigma(T)$ has at most countably elements:

$$\sigma(T) = \bigcup_{n=1}^{\infty} \left\{\lambda \in \sigma(T) : |\lambda| \geq n^{-1}\right\}.$$

As a result, if T has infinitely many eigenvalues, we can make a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of these eigenvalues, which satisfies $\lim_{n\to\infty}|\lambda_n| = 0$. Clearly, we can permute them in a decreasing order: $|\lambda_1| \geq |\lambda_2| \geq \cdots$.

The proof of Theorem 4.34 (ii) is a bit complicated, which requires some technical lemmas.

Lemma 4.35. Let X be a Banach space, $T \in \mathcal{K}(X)$, and $\lambda \in \mathbb{C}\backslash\{0\}$. If $(T-\lambda I)X = X$, then $\lambda \in \rho(T)$.

Proof. We assume $\dim X = \infty$, since the finite-dimensional case is clear. By bounded inverse theorem, it remains to show $T - \lambda I$ is injective. We choose $L_n = \{x \in X : (T-\lambda I)^n x = 0\} = \ker(T-\lambda I)^n$. Then we obtain a sequence $L_1 \subset L_2 \subset \cdots$ of subspaces of X. We wish to show $L_1 = \{0\}$.

If $L_1 \neq \{0\}$, choose $x_1 \in L_1$ such that $x_1 \neq 0$, and generate a sequence by choosing $(T-\lambda I)x_n = x_{n-1}$. Then $x_n \in L_n \backslash L_{n-1}$. By Lemma 1.75, we can choose a sequence (y_n) of unit vectors such that $y_n \in L_n \backslash L_{n-1}$ and that $d(y_n, L_{n-1}) > 1/2$. Once $p > q$, we have $L_q \subset L_{p-1} \subsetneq L_p$, and

$$\left\|\frac{Ty_p}{\lambda} - \frac{Ty_q}{\lambda}\right\| = \Bigg\|y_p - \underbrace{\left(y_q - \frac{(T-\lambda I)y_p}{\lambda} + \frac{(T-\lambda I)y_q}{\lambda}\right)}_{\in L_{p-1}}\Bigg\| > \frac{1}{2}.$$

However, $\left\{\frac{Ty_n}{\lambda}, n \in \mathbb{N}\right\} \subset TB(0, \lambda^{-1})$ is relatively sequentially compact, a contradiction! $\square$

Lemma 4.36. Let X be a Banach space, $T \in \mathcal{K}(X)$, and $\lambda \in \mathbb{C}\backslash\{0\}$. $\mathfrak{R}(T-\lambda I)$ is a closed subspace of X.

Proof. Let (y_n) be a sequence of points of $\mathfrak{R}(T-\lambda I)$ that converges to $y \in Y$, and choose sequence (x_n) such that $(T-\lambda I)x_n = y_n$ for all $n \in \mathbb{N}$. We need to show $y \in \mathfrak{R}(T-\lambda I)$.

Step I: If (x_n) is a bounded sequence, by compactness of T, there exists subsequence (x_{n_k}) such that (Tx_{n_k}) converges in X. As a result, the subsequence $x_{n_k} = \lambda^{-1}(Tx_{n_k} - y_{n_k})$ also converges. Let $x = \lim_{k\to\infty} x_{n_k}$, then $y = \lim_{k\to\infty} y_{n_k} = (T-\lambda I)x$, which implies $y \in \mathfrak{R}(T-\lambda I)$.

Step II: If (x_n) is not bounded, let $\alpha_n = d(x_n, \ker(T-\lambda I)) > 0$. Then there exists sequence $(w_n) \subset \ker(T-\lambda I)$ such that $\alpha_n \leq \|x_n - w_n\| \leq \left(1+\frac{1}{n}\right)\alpha_n$. Define $x_n' = x_n - w_n$, then $(T-\lambda I)x_n' = y_n$, and $\alpha_n \leq \|x_n'\| \leq \left(1+\frac{1}{n}\right)\alpha_n$. If (α_n) is bounded, so is (x_n'). Back to Step I.

Step III: If (α_n) is not bounded, choose subsequence $a_{n_k} \to \infty$, and let $z_k = \frac{x'_{n_k}}{\|x'_{n_k}\|}$. Then $\|z_k\| = 1$, and $(T-\lambda I)z_k = \frac{y_{n_k}}{\|x'_{n_k}\|} \to 0$. Since T is compact, there exists subsequence $z_{k_l} = \lambda^{-1}\left(Tz_{k_l} - (T-\lambda I)z_{k_l}\right)$ such that (z_{k_l}) converges to some $z \in X$. Clearly, $(T-\lambda I)z = 0$. Furthermore,

$$x_{n_{k_l}} - \underbrace{\left(w_{n_{k_l}} + z\big\|x_{n_{k_l}} - w_{n_{k_l}}\big\|\right)}_{\in \ker(T-\lambda I)} = (z_{k_l} - z)\big\|x_{n_{k_l}} - w_{n_{k_l}}\big\|.$$

As a result, we have

$$\alpha_{n_{k_l}} \leq \big\|z_{k_l} - z\big\|\big\|x_{n_{k_l}} - w_{n_{k_l}}\big\| \leq \big\|z_{k_l} - z\big\|\left(1 + \frac{1}{n_{k_l}}\right)\alpha_{n_{k_l}} \;\Rightarrow\; \big\|z_{k_l} - z\big\| \geq \frac{n_{k_l}}{1+n_{k_l}} \geq \frac{1}{2}.$$

However, $z_{k_l} \to z$, a contradiction! Hence (α_n) is bounded, and $\mathfrak{R}(T-\lambda I)$ is closed. $\square$

Now we are prepared to prove Theorem 4.34 (ii).

Proof of Theorem 4.34 (ii). If $\lambda \in \sigma(T)\backslash\{0\}$ is not an eigenvalue of T, $\ker(T-\lambda I_X) = \{0\}$. By Lemma 4.36 and Theorem 3.34, $\mathfrak{R}(T-\lambda I_X)$ is closed, and

$$\mathfrak{R}(T^* - \lambda I_{X^*}) = \ker(T-\lambda I_X)^\perp = X^*.$$

By Theorem 4.33 and Lemma 4.35, $T^* \in \mathcal{K}(X^*)$, and $\lambda \in \rho(T^*)$. However $\lambda \in \sigma(T) = \sigma(T^*)$, a contradiction! Therefore, $\lambda \in \sigma(T)\backslash\{0\}$ is an eigenvalue of T. $\square$

Theorem 4.37 (Riesz-Schauder). Let X be a Banach space, and $T \in \mathcal{K}(X)$. Let $T^* \in \mathcal{K}(X^*)$ be the adjoint.

(i) $\sigma(T) = \sigma(T^*)$.

(ii) If $\lambda \in \sigma(T)\backslash\{0\}$, then

$$\dim\ker(T-\lambda I_X) = \dim\ker(T^*-\lambda I_{X^*}) = \operatorname{codim}\mathfrak{R}(T-\lambda I_X) = \operatorname{codim}\mathfrak{R}(T^*-\lambda I_{X^*}).$$

(iii) If λ, μ are distinct eigenvalues of T, then $f(x) = 0$ for all $x \in \ker(T-\lambda I_X)$ and all $f \in \ker(T^*-\mu I_{X^*})$.

(iv) If $\lambda \in \sigma(T)\backslash\{0\}$, then

$$\mathfrak{R}(T-\lambda I_X) = {}^{\perp}\ker(T^*-\lambda I_{X^*}), \quad \mathfrak{R}(T^*-\lambda I_{X^*}) = \ker(T-\lambda I_X)^{\perp}.$$

Proof. (ii) We first claim that $\operatorname{codim}\mathfrak{R}(T-\lambda I_X) \le \dim\ker(T-\lambda I_X)$. Clearly, $n = \dim\ker(T-\lambda I_X) > 0$, and we choose a basis $\{x_1, \cdots, x_n\}$ of $\ker(T-\lambda I_X)$. If $\operatorname{codim}\mathfrak{R}(T-\lambda I_X) > n$, there exists $y_1, \cdots, y_{n+1} \in X$ such that $\{[y_1], \cdots, [y_{n+1}]\}$ are linearly independent in $X/\mathfrak{R}(T-\lambda I_X)$.

By Hahn-Banach theorem, there exist $f_1, \cdots, f_n \in X^*$ such that $f_j(x_k) = \delta_{jk}$ for all $j, k \in \{1, \cdots, n\}$. Let

$$Ax = Tx - \sum_{j=1}^{n} f_j(x)\, y_j, \ \forall x \in X.$$

Then $A \in \mathcal{K}(X)$. We will verify that $A - \lambda I_X$ is injective. If $(A-\lambda I_X)x = 0$, then

$$(T-\lambda I_X)x = \sum_{j=1}^{n} f_j(x)\, y_j \ \Rightarrow \ 0 = \sum_{j=1}^{n} f_j(x)[y_j] \ \Rightarrow \ f_1(x) = \cdots = f_n(x) = 0 \ \Rightarrow \ (T-\lambda I_X)x = 0.$$

Let $x = \sum_{k=1}^{n} c_k x_k \in \ker(T-\lambda I_X)$. Then $0 = f_j(x) = c_j \sum_{k=1}^{n} f_j(x_k) = c_j$ for all j, which implies $x = 0$. Hence $A - \lambda I_X$ is injective. Since $\lambda \ne 0$, and $A \in \mathcal{K}(X)$, we have $\lambda \in \rho(A)$. As a result, $A - \lambda I$ is invertible, and there exists $x_{n+1} \in X$ such that $(A-\lambda I_X)x_{n+1} = y_{n+1}$. Then in $X/\mathfrak{R}(T-\lambda I_X)$,

$$[y_{n+1}] = [(A-\lambda I_X)x_{n+1}] = \left[(T-\lambda I_X)x_{n+1} - \sum_{j=1}^{n} f_j(x_{n+1})y_j\right] = -\sum_{j=1}^{n} f_j(x_{n+1})[y_j].$$

However $\{[y_1], \cdots, [y_{n+1}]\}$ are linearly independent in $X/\mathfrak{R}(T-\lambda I_X)$, a contradiction!

Similarly, we know that $\operatorname{codim}\mathfrak{R}(T^*-\lambda I_{X^*}) \le \dim\ker(T^*-\lambda I_{X^*})$. By Lemma 4.36, $T - \lambda I_X$ has closed range. Using Theorem 3.34 and Theorem 2.26, we have

$$\begin{aligned}
\dim\ker(T-\lambda I_X) &\ge \dim X/\mathfrak{R}(T-\lambda I_X) = \dim\left(X/\mathfrak{R}(T-\lambda I_X)\right)^* = \dim\mathfrak{R}(T-\lambda I_X)^{\perp} \\
&= \dim\ker(T^*-\lambda I_{X^*}) \ge \dim X^*/\mathfrak{R}(T^*-\lambda I_{X^*}) = \dim X^*/\ker(T-\lambda I_X)^{\perp} \\
&= \dim\left(\ker(T-\lambda I_X)\right)^* = \dim\ker(T-\lambda I_X). \\
&\Downarrow \\
\dim\ker(T-\lambda I_X) &= \dim\ker(T^*-\lambda I_{X^*}) = \operatorname{codim}\mathfrak{R}(T-\lambda I_X) = \operatorname{codim}\mathfrak{R}(T^*-\lambda I_{X^*}).
\end{aligned}$$

(iii) Let $x \in \ker(T-\lambda I_X)$ and all $f \in \ker(T^*-\mu I_{X^*})$. Since $\mu \ne \mu$, we have

$$Tx = \lambda x, \ T^*f = \mu f \ \Rightarrow \ (\lambda-\mu)f(x) = f(\lambda x) - (\mu f)(x) = f(Tx) - (T^*f)(x) = 0 \ \Rightarrow \ f(x) = 0.$$

(iv) is a corollary of Lemma 4.36 and Theorem 3.34. $\square$

4.3 Compact Self-adjoint Operators

Let T be a compact operator on a Hilbert space. If $T^* = T$, then T is a compact self-adjoint operator. The spectrum of T possesses some nice properties.

Lemma 4.38. Let H be a Hilbert space, and let T be a compact self-adjoint operator on H.
(i) If λ is an eigenvalue of T, then $\lambda \in \mathbb{R}$.
(ii) If λ and μ are distinct eigenvalues of T, the eigenvectors associated with λ and μ are orthogonal.
(iii) $\max_{\lambda\in\sigma_p(T)} |\lambda| = \|T\|$.

Proof. (i) Let $Tx = \lambda x$, where $x \neq 0$. Then

$$\lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle = \langle x, Tx\rangle = \langle x, \lambda x\rangle = \overline{\lambda}\langle x, x\rangle.$$

(ii) By (i), $\lambda, \mu \in \mathbb{R}$. Let $Tx = \lambda x$, and $Ty = \mu y$, where $x, y \neq 0$. Then

$$(\lambda - \mu)\langle x, y\rangle = \langle \lambda x, y\rangle - \langle x, \mu y\rangle = \langle Tx, y\rangle - \langle x, Ty\rangle = 0.$$

(iii) The case $T = 0$ is trivial. If $T \neq 0$, by Corollary 4.19, $r(T) = \|T\| > 0$. By Theorem 4.34 (v), there exists $\lambda_1 \in \sigma(T)$ such that $|\lambda_1| = \max_{\lambda\in\sigma(T)} |\lambda| = r(T) = \|T\|$. Since $\lambda_1 \neq 0$, we have $\lambda_1 \in \sigma_p(T)$. □

Theorem 4.39. Let T be a compact self-adjoint operator on a Hilbert space H. Then there exist eigenvectors $\{e_\lambda, \lambda \in \Lambda\}$ of T that form a orthonormal basis of H.

Proof. Since T is compact, let $\{\lambda_k\}_{k\in J}$ be the nonzero eigenvalues of T, where J is finite or countable. By Riesz-Schauder theorem, $n_k = \dim\ker(T - \lambda_k I) < \infty$ for all $k \in J$, and $\ker(T - \lambda_k I) \perp \ker(T - \lambda_l I)$ for all $k \neq l$. For every $k \in J$, we choose an orthonormal basis $\{e_{k,j}\}_{j=1}^{n_k}$ of $\ker(T - \lambda_k I)$. By Lemma 4.38 (ii), we obtain an orthonormal system on H:

$$\mathscr{F}_1 = \{e_{k,j} : k \in J,\ j = 1, \cdots, n_k\}.$$

If $\mathscr{F}_1^\perp = \{0\}$, then $\mathscr{F}_1$ is an orthonormal basis of H, and the result holds.

If $\mathscr{F}_1^\perp \neq \{0\}$, let $H_1 = \overline{\operatorname{span}}\,\mathscr{F}_1$, and $H_0 = H_1^\perp = \mathscr{F}_1^\perp$. Then $TH_1 = H_1$. Furthermore,

$$\langle x, Ty\rangle = \langle Tx, y\rangle,\ \forall x \in H_1, y \in H_0 \ \Rightarrow\ Ty \in H_0,\ \forall y \in H_0 \ \Rightarrow\ TH_0 \subset H_0.$$

Since H_0 is an invariant subspace of T, we use its restriction $\widetilde{T} = T|_{H_0}$, which is also a compact self-adjoint operator. If $\widetilde{T} \neq 0$, by Lemma 4.38 (iii), $\widetilde{T}$ has at least one nonzero eigenvalue λ_0, with a corresponding eigenvector $x_0 \in H_0\backslash\{0\}$. However, $Tx_0 = \widetilde{T}x_0 = \lambda_0 x_0$, which implies $x_0 \in H_1$, contradicting $x_0 \in H_0$! Therefore $\widetilde{T} = 0$, and H_0 is the eigenspace of $0 \in \sigma_p(T)$. We choose an orthonormal basis $\mathscr{F}_0$ of H_0. Then $\mathscr{F}_1 \cup \mathscr{F}_0$ is the desired orthonormal basis of $H = H_1 \oplus H_0$. □

Remark. If T has only finitely many nonzero eigenvalues $\lambda_1, \cdots, \lambda_n$, then H_1 is finite-dimensional. Let $E_k = \ker(T - \lambda_k I)$ be the eigenspace of λ_k, where $k = 1, \cdots, n$. Then $H_1 = E_1 \oplus \cdots \oplus E_n$. For all $x \in H$, let $x = x_1 + x_0$, where $x_1 \in H_1$ and $x_0 \in H_0$. Then

$$Tx = Tx_1 = T\left(\sum_{k=1}^n P_{E_k} x_1\right) = \sum_{k=1}^n \lambda_k P_{E_k} x_1 = \sum_{k=1}^n \lambda_k P_{E_k} x.$$

Hence we have $T = \sum_{k=1}^n \lambda_k P_{E_k}$. In fact, we can extend this result to the case where T has infinitely many nonzero eigenvalues.

Theorem 4.40. Let T be a compact self-adjoint operator on a Hilbert space H. If T has infinitely many nonzero eigenvalues, we sequentialize them in decreasing order: $|\lambda_1| \geq |\lambda_2| \geq |\lambda_{n-1}| \geq |\lambda_n| \geq \cdots$. Let $E_n = \ker(T - \lambda_n I)$ be the eigenspace of λ_n. Then

$$T = \sum_{n=1}^{\infty} \lambda_n P_{E_n} \quad \text{(convergence in norm).}$$

Proof. For any $n \geq m$, define $S_{n,m} = \sum_{k=m}^{n} \lambda_k P_{E_k}$. Then $\|S_{n,m}\| = |\lambda_m| \to 0$ as $n, m \to 0$. As a result, $\sum_{n=1}^{\infty} \lambda_n P_{E_n}$ converges in norm. Take the orthonormal basis $\mathscr{F}_1 \cup \mathscr{F}_0$ defined in the proof of Theorem 4.39. Since $x = \sum_{e_\lambda \in \mathscr{F}_1 \cup \mathscr{F}_0} \langle x, e_\lambda \rangle e_\lambda$,

$$\begin{aligned} \left\| Tx - \sum_{k=1}^{n} \lambda_k P_{E_k} x \right\|^2 &= \left\| \sum_{k=1}^{\infty} \sum_{j=1}^{n_k} \lambda_k \langle x, e_{k,j} \rangle e_{k,j} - \sum_{k=1}^{n} \lambda_k \sum_{j=1}^{n_k} \langle x, e_{k,j} \rangle e_{k,j} \right\|^2 = \left\| \sum_{k=n+1}^{\infty} \sum_{j=1}^{n_k} \lambda_k \langle x, e_{k,j} \rangle e_{k,j} \right\|^2 \\ &\leq |\lambda_{n+1}|^2 \sum_{k=n+1}^{\infty} \sum_{j=1}^{n_k} | \langle x, e_{k,j} \rangle |^2 \leq |\lambda_{n+1}|^2 \|x\|^2 . \qquad \text{(Bessel's inequality)} \end{aligned}$$

Hence $\|T - \sum_{k=1}^{n} \lambda_k P_{E_k}\| \leq |\lambda_{n+1}| \to 0$, which implies $T = \sum_{n=1}^{\infty} \lambda_n P_{E_n}$. ☐

Example 4.41 (Mercer). Let $K \in C([a,b] \times [a,b])$ be a conjugate symmetric function, i.e. $K(s,t) = \overline{K(t,s)}$ for all $s, t \in [a,b]$. Following Example 3.37 (ii) and Example 4.31, the Fredholm integral operator

$$(T_K f)(s) = \int_a^b K(s,t) f(t)\, dt, \ \forall s \in [a,b], \ f \in L^2([a,b])$$

is a compact self-adjoint operator on $L^2([a,b])$. Using Theorem 4.39, we choose an orthonormal basis $\mathscr{F}_1 \cup \mathscr{F}_0$ of H, where $\mathscr{F}_1$ contains the eigenvectors associated with nonzero eigenvalues of T_K, and $\mathscr{F}_0$, possibly empty, is an orthonormal basis of $\ker(T_K)$. Clearly, $\mathscr{F}_1$ is countable. Since $L^2([a,b])$ is separable, $\mathscr{F}_0$ is chosen to be at most countable. Hence $\mathscr{F}_1 \cup \mathscr{F}_0 = \{\phi_n, n \in \mathbb{N}\}$ is a countable orthonormal basis of $L^2([a,b])$.

Let λ_n be the eigenvalue of T associated with ϕ_n. Without generality, assume $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \geq \cdots$. Note that $\{\lambda_n, n \in \mathbb{N}\}$ may have finitely many nonzero elements. By Theorem 4.40, we have

$$T_K f = \sum_{n=1}^{\infty} \lambda_n \langle f, \phi_n \rangle \phi_n, \ f \in L^2([0,1]).$$

Given $s \in [a,b]$, define $K_s(t) = \overline{K(s,t)}$ for all $t \in [a,b]$. Then

$$\langle K_s, \phi_n \rangle = \int_a^b \overline{K(s,t)\phi_n(t)}\, dt = \overline{(T_K \phi_n)(s)} = \overline{\lambda_n \phi_n(s)}$$

By expanding K_s, we obtain the following representation of K:

$$K(s,t) = \overline{K_s(t)} = \sum_{n=1}^{\infty} \overline{\langle K_s, \phi_n \rangle \phi_n(t)} = \sum_{n=1}^{\infty} \lambda_n \phi_n(s) \overline{\phi_n(t)}. \tag{4.8}$$

Note that $K \in C([a,b] \times [a,b]) \subset L^2([a,b] \times [a,b])$. Using (4.8), we have

$$\int_a^b \int_a^b |K(s,t)|^2\, ds\, dt = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda_m \overline{\lambda_n} \int_a^b \phi_m(s) \overline{\phi_n(s)}\, ds \int_a^b \overline{\phi_m(t)} \phi_n(t)\, dt = \sum_{n=1}^{\infty} |\lambda_n|^2 < \infty.$$

Then back in (4.8), the function series converges in L^2 sense.

4.4 Spectral Measures

4.4.1 Projection Operators and Spectral Measures

Theorem 4.42. Let H be a Hilbert space, and $P \in \mathfrak{B}(H)$. Then P is a projection if and only if P is self-adjoint and idempotent, i.e. $P = P^* = P^2$.

Proof. Clearly, a projection P is idempotent. Following Example 3.40, it is self-adjoint.

Conversely, if P is self-adjoint and idempotent, let $L = \{x \in H : Px = x\}$. We claim that P is a projection onto L. Clearly, L is a closed subspace of H. Then it suffices to prove that $\langle Pv, v - Pv\rangle = 0$ for all $v \in H$: $\langle Pv, v - Pv\rangle = \langle Pv, v\rangle - \langle Pv, Pv\rangle = \langle Pv, v\rangle - \langle P^*Pv, v\rangle = \langle Pv, v\rangle - \langle P^2v, v\rangle = \langle Pv, v\rangle - \langle Pv, v\rangle = 0.$ ☐

We denote by $\mathcal{P}(H)$ the set of all projections on a Hilbert space H, which has some nice structure.

Proposition 4.43. Let $P_M, P_N \in \mathcal{P}(H)$, where M and N are closed subspaces of Hilbert space H.

(i) $P_M P_N \in \mathcal{P}(H)$ if and only if $P_M P_N = P_N P_M$. Furthermore, $P_M P_N = 0$ if and only if $M \perp N$.

(ii) $P_M + P_N \in \mathcal{P}(H)$ if and only if $M \perp N$. If so, $P_M + P_N$ is the projection onto $M \oplus N$.

(iii) $P_M - P_N \in \mathcal{P}(H)$ if and only if $M \supset N$. If so, $P_M - P_N$ is the projection onto $M \ominus N := M \cap N^\perp$.

(iv) Let P_n be a sequence of mutually orthogonal projections onto closed subspaces M_n, i.e. $P_n P_m = 0$ for all $n \neq m$. Then $\sum_{n=1}^\infty P_n \overset{SOT}{\to} P \in \mathcal{P}(H)$, where P is the projection onto $M = \overline{span}\,\{M_n\}_{n=1}^\infty$.

Proof. (i) The first statement is clear, since $P_M P_N = P_N P_M$ if and only if $P_M P_N$ is self-adjoint and idempotent. For the next statement, if $M \perp N$, then $P_N x \in N \subset M^\perp$ for all $x \in H$, which implies $P_M P_N = 0$. Conversely, if $P_M P_N = 0$, then $\langle x, y\rangle = \langle P_M x, P_N y\rangle = \langle x, P_M P_N y\rangle = 0$ for all $x \in M$ and all $y \in N$.

(ii) The sufficiency is clear. For the necessity, if $P_M + P_N \in \mathcal{P}(H)$, then

$$\begin{aligned} P_M + P_N = (P_M + P_N)^2 = P_M + P_N + P_M P_N + P_N P_M \ &\Rightarrow\ P_M P_N + P_N P_M = 0 \\ &\Rightarrow P_M P_N + P_M P_N P_M = 0 = P_M P_N P_M + P_N P_M \\ &\Rightarrow P_M P_N = P_N P_M = 0\ \Rightarrow M \perp N. \end{aligned}$$

(iii) If $N \subset M$, then $P_M P_N = P_N P_M = P_N$, and

$$(P_M - P_N)^2 = P_M + P_N - P_M P_N - P_N P_M = P_M - P_N.$$

Conversely, if $P_M - P_N \in \mathcal{P}(H)$, let $L = \ker(I - P_M + P_N)$. Then $P_M - P_N = P_L$, and (ii) implies $L \perp N$. Hence for all $x \in N$, $0 = P_L x = P_M x - P_N x = P_M x - x$, which implies $x \in M$.

(iv) For all $x \in H$, note that

$$\sum_{k=1}^n \|P_k x\|^2 = \left\|\sum_{k=1}^n P_k x\right\|^2 \leq \|x\|^2.$$

Then $\left(\sum_{k=1}^n P_k x\right)_{n=1}^\infty$ is a Cauchy sequence. Define operator T as the strong operator limit:

$$Px = \lim_{n\to\infty} \sum_{k=1}^n P_n x,\ \forall x \in H.$$

One can easily verify that $P^2 = P$. Furthermore, for all $x, y \in H$,

$$\langle Px, y\rangle = \lim_{n\to\infty} \sum_{k=1}^n \langle P_k x, y\rangle = \lim_{n\to\infty} \sum_{k=1}^n \langle x, P_k y\rangle = \langle x, Py\rangle.$$

Hence P is self-adjoint, and P is a projection operator.

It remains to show $P = P_M$. If $x \in M^\perp$, then $P_n x = 0$ for all $n \in \mathbb{N}$, which implies $Px = 0$. If $x \in M_n$, then $Px = P_n x = x$, and $x \in M$. As a result, $\ker(I - P)$ contains each M, hence contains M. Hence P is the projection onto M. $\square$

Definition 4.44 (Spectral measures). Let $\mathscr{B}(\mathbb{C})$ be the set of all Borel sets in $\mathbb{C}$. A *spectral measure* is a function $E : \mathscr{B}(\mathbb{C}) \to \mathcal{P}(H)$ satisfying the following conditions:

(i) $E(\mathbb{C}) = I$, where I is the identity map on H;

(ii) If $\{B_n\}_{n\in\mathbb{N}}$ is a collection of disjoint Borel sets in $\mathbb{C}$, then

$$E\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} E(B_n) \quad \text{(convergence in SOT)}.$$

Remark. We can apply standard techniques in complex-valued measures to derive many basic facts about spectral measures.

(i) $E(\emptyset) = 0$;

(ii) Following Proposition 4.43 (ii), if B_0 and B_1 are disjoint Borel sets in $\mathbb{C}$, then $E(B_0) \perp E(B_1)$.

(iii) Following Proposition 4.43 (iii), if $B_0 \subset B_1$, then $\|E(B_0)x\| \leq \|E(B_1)x\|$ for all $x \in H$.

(iv) Let B_0 and B_1 be Borel sets in $\mathbb{C}$. Then

$$E(B_0) + E(B_1) = E(B_0 \cup B_1) + E(B_0 \cap B_1).$$

Furthermore, observing that $E(B_0)E(B_0 \cap B_1) = E(B_0 \cap B_1)$ and $E(B_0)E(B_0 \cup B_1) = E(B_0)$, we have $E(B_0)E(B_1) = E(B_0 \cap B_1)$.

Proposition 4.45. Let $E : \mathcal{B}(\mathbb{C}) \to \mathcal{P}(H)$ be a projection-valued function such that for all $x, y \in H$,

$$\left\langle E\left(\bigcup_{n=1}^{\infty} B_n\right) x, y\right\rangle = \sum_{n=1}^{\infty} \langle E(B_n)x, y\rangle,\ \forall \text{ sequence } \{B_n\}_{n=1}^{\infty} \text{ of disjoint Borel sets in } \mathbb{C}$$

and that $E(\mathbb{C}) = 1$. Then E is a spectral measure.

Proof. We need to verify the second property in Definition 4.44. Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of disjoint Borel sets in $\mathbb{C}$. By assumption,

$$\left\langle \sum_{k=1}^{n} E(B_k)x, y\right\rangle = \sum_{k=1}^{n} \langle E(B_k)x, y\rangle = \left\langle E\left(\bigcup_{k=1}^{n} B_k\right) x, y\right\rangle,\ \forall x, y \in H.$$

Then

$$\sum_{k=1}^{n} E(B_k)x = E\left(\bigcup_{k=1}^{n} B_k\right) x,\ \forall x \in H. \tag{4.9}$$

Observing that

$$\sum_{n=1}^{\infty} \|E(B_n)x\|^2 = \sum_{n=1}^{\infty} \langle E(B_n)x, x\rangle = \left\langle E\left(\bigcup_{n=1}^{\infty} B_n\right) x, x\right\rangle = \left\| E\left(\bigcup_{n=1}^{\infty} B_n\right) x\right\|^2,$$

the sequence $x_n = E(B_n)x$ is summable. Let $n \to \infty$ in (4.9), we have $E\left(\bigcup_{n=1}^{\infty} B_n\right) \overset{SOT}{=} \sum_{n=1}^{\infty} E(B_n)$. $\square$

4.4.2 Spectral Integrals and their Associated Operators

Let H be a Hilbert space, and let E be a spectral measure on H. Given two vectors $x, y \in H$, define

$$E^*(B) = \langle E(B)x, y\rangle, \ \forall B \in \mathscr{B}(\mathbb{C}).$$

By definition, E^* is a complex-valued Borel measure on $\mathbb{C}$. Hence for all Borel-measurable function f on $\mathbb{C}$, we can compute its Lebesgue-Stieltjes integral with respect to E^*. For brevity, we denote $E(\lambda)$ by E_λ.

Definition 4.46 (Spectral integral). Let E be a spectral measure on a Hilbert space H. We define the *spectral integral* of the measurable function f with respect to $x, y \in H$ to be the Riemann-Stieltjes integral

$$\int f(\lambda)\, d\langle E_\lambda x, t\rangle,$$

which we sometimes abbreviate $\int f(\lambda)\, dE$.

Definition 4.47 (The spectrum of a spectral measure). Let E be a spectral measure on a Hilbert space H. The *spectrum* of E is defined to be the set

$$\sigma(E) = \mathbb{C} \backslash \left(\bigcup_{\alpha \in J} U_\alpha \right),$$

where the union is taken over all open sets U_α such that $E(U_\alpha) = 0$. We say E is *compact* if $\sigma(E)$ is compact.

Theorem 4.48. Let E be a compact spectral measure on a Hilbert space H. There is a unique normal operator T such that $\int \lambda\, d\langle E_\lambda x, y\rangle = \langle Tx, y\rangle$ for all $x, y \in H$. For the sake of brevity, we write $T = \int \lambda\, dE$.

Proof. Since $\sigma(E)$ is compact, let $M = \max_{\lambda \in \sigma(E)} |\lambda|$. Define $\varphi(x, y) = \int \lambda\, d\langle E_\lambda x, y\rangle$. Clearly, $\varphi : H \times H \to \mathbb{C}$ is a sesquilinear form. Furthermore,

$$|\varphi(x, x)| = \left| \int \lambda\, d\langle E_\lambda x, x\rangle \right| \le M \int |d\langle E_\lambda x, x\rangle| = M \int d\left\|E_\lambda x\right\|^2 \le M \left\|x\right\|^2.$$

Use the parallelogram law,

$$|\varphi(x, y)| \le \frac{1}{4} M \left(\|x + y\|^2 + \|x + \mathrm{i}y\|^2 + \|x - y\|^2 + \|x - \mathrm{i}y\|^2 \right) = M \left(\|x\|^2 + \|y\|^2 \right).$$

Set $\|x\| = \|y\| = 1$, we have $\|\varphi\| \le 2M$. By Theorem 2.15, there exists a unique operator $T \in \mathfrak{B}(H)$ such that $\varphi(x, y) = \langle Tx, y\rangle$ for all $x, y \in H$.

We now show that T is a normal operator. Define $S = \int \overline{\lambda}\, dE$. Then

$$\langle x, Sy\rangle = \overline{\langle Sy, x\rangle} = \overline{\int \overline{\lambda}\, d\langle E_\lambda y, x\rangle} = \int \lambda\, d\langle x, E_\lambda y\rangle = \int \lambda\, d\langle E_\lambda x, y\rangle = \langle Tx, y\rangle.$$

Hence S is the adjoint of T. Furthermore, for all $B \in \mathscr{B}(\mathbb{C})$,

$$\begin{aligned} \langle E(B)x, Ty\rangle &= \overline{\langle Ty, E(B)x\rangle} = \overline{\int \lambda\, d\langle E_\lambda y, E(B)x\rangle} = \overline{\int \lambda\, d\langle E(B)E_\lambda y, x\rangle} \\ &= \overline{\int \lambda\, d\langle E(B \cap \lambda)y, x\rangle} = \overline{\int_B \lambda\, d\langle E_\lambda y, x\rangle} = \int_B \overline{\lambda}\, d\langle x, E_\lambda y\rangle = \int_B \overline{\lambda}\, d\langle E_\lambda x, y\rangle. \end{aligned} \tag{4.10}$$

Given $x, y \in H$, by (4.10), we have

$$\langle STx, y\rangle = \langle Tx, Ty\rangle = \int \lambda \, d\langle E_\lambda x, Ty\rangle = \int \lambda \cdot \overline{\lambda} \, d\langle E_\lambda x, y\rangle = \int |\lambda|^2 \, dE.$$

Similarly we have $\langle TSx, y\rangle = \int |\lambda|^2 \, dE$. Since x and y are arbitrary, T is a normal operator. □

Theorem 4.49. If E is a compact spectral measure on a Hilbert space H, and $T = \int \lambda \, dE$, then $\sigma(E) = \sigma(A)$.

Proof. Assume $\lambda_0 \in \mathbb{C}\backslash\sigma(E)$. By definition, $\mathbb{C}\backslash\sigma(E)$ is open, so there exists $\epsilon > 0$ such that $B(\lambda_0, \epsilon) \subset \mathbb{C}\backslash\sigma(E)$. Then $E(B(\lambda_0, \epsilon)) = 0$, and $T - \lambda_0 I = \int (\lambda - \lambda_0) \, dE$, and

$$\|Tx - \lambda_0 x\|^2 = \int |\lambda - \lambda_0|^2 \, d\langle E_\lambda x, x\rangle = \int_{\mathbb{C}\backslash B(\lambda_0, \epsilon)} |\lambda - \lambda_0|^2 \, d\langle E_\lambda x, x\rangle \geq \epsilon^2 \|x\|^2 .$$

Hence $T - \lambda_0 I$ is bounded from below. Following Theorem 3.15, if we $T_0 = T - \lambda_0 I$ has dense range in H, then $\lambda_0 \in \rho(T)$. Equivalently, we prove $\mathfrak{R}(T - \lambda_0 I)^\perp = 0$: If $x \in \mathfrak{R}(T - \lambda_0 I)^\perp$, then $\langle T_0^* x, y\rangle = \langle x, T_0 y\rangle = 0$ for all $y \in H$, which implies $T_0^* x = 0$. Meanwhile, T_0 is normal by Theorem 4.48, hence $\ker(T_0^*) = \ker(T_0) = 0$. Therefore $x = 0$.

Conversely, assume $\lambda_1 \in \sigma(E)$. Given $\eta > 0$, we have $E(B(\lambda_1, \eta)) \neq 0$. Since $E(B(\lambda_1, \eta))$ must maintain some unit vector $u \in H$, we have

$$\|Tu - \lambda_1 u\|^2 = \int_{B(\lambda_1, \eta)} |\lambda - \lambda_1|^2 \, d\langle E_\lambda u, u\rangle \leq \eta^2 \|u\|^2 .$$

Because $\eta > 0$ is arbitrary, $T - \lambda_1 I$ is not bounded below, hence is not invertible. □