Machine Learning HW10

Student ID: 111652015

Problem 1 Consider a forward SDE

$$dx_t = f(x_t, t)\, dt + g(x_t, t) dW_t$$

show that the corresponding probability flow ODE is written as

$$dx_t = \left[f(x_t, t) - \frac{1}{2}\frac{\partial}{\partial x} g^2(x_t, t) - \frac{g^2(x_t, t)}{2}\frac{\partial}{\partial x}\log p(x_t, t) \right] dt$$

Forward SDE and its Fokker–Planck equation. Consider the one–dimensional SDE

$$\mathrm{d}x_t = f(x_t, t)\, \mathrm{d}t + g(x_t, t)\, \mathrm{d}W_t \tag{1}$$

with probability density $p(x, t)$. The corresponding Fokker–Planck equation is

$$\partial_t p \;=\; -\,\partial_x\big(f\, p\big) \;+\; \tfrac{1}{2}\,\partial_x^2\big(g^2\, p\big) \tag{2}$$

Rewrite the diffusion term by the product rule. Using only lecture symbols and the product rule,

$$\partial_x^2\big(g^2 p\big) \;=\; \partial_x\Big((\partial_x g^2)\, p + g^2\, \partial_x p \Big) \tag{3}$$

Substitute (3) into (2) and factor as a single divergence:

$$\begin{aligned} \partial_t p &= -\,\partial_x(fp) + \tfrac{1}{2}\,\partial_x\Big((\partial_x g^2)\, p + g^2\, \partial_x p \Big) \\ &= -\,\partial_x\Big(fp - \tfrac{1}{2}(\partial_x g^2)\, p - \tfrac{1}{2} g^2\, \partial_x p \Big) \end{aligned} \tag{4}$$

Use the identity $\partial_x p = p\, \partial_x \log p$ to obtain

$$\partial_t p \;=\; -\,\partial_x\Big(\big[f - \tfrac{1}{2}\partial_x g^2 - \tfrac{1}{2} g^2\, \partial_x \log p \big]\, p \Big) \tag{5}$$

Probability flow drift and ODE. Equation (5) is a continuity equation $\partial_t p = -\partial_x(v\, p)$ with velocity (drift) field

$$v(x, t) \;=\; f(x, t) - \tfrac{1}{2}\partial_x g^2(x, t) - \tfrac{1}{2} g^2(x, t)\, \partial_x \log p(x, t)$$

Therefore, the deterministic probability flow ODE that transports the same family of densities $\{p(\cdot, t)\}_t$ as the SDE (1) is

$$\boxed{\mathrm{d}x_t = \Big[\, f(x_t, t) - \tfrac{1}{2}\partial_x g^2(x_t, t) - \tfrac{1}{2} g^2(x_t, t)\, \partial_x \log p(x_t, t) \Big]\, \mathrm{d}t} \tag{6}$$

Problem 3 There are unanswered questions during the lecture, and there are likely more questions we haven't covered. Take a moment to think about them and write them down here.

No questions this week